CHAPTER 17 SECOND-ORDER DIFFERENTIAL EQUATIONS

17.1 SECOND-ORDER LINEAR EQUATIONS

1. $y'' - y' - 12y = 0 \Rightarrow r^2 - r - 12 = 0 \Rightarrow (r-4)(r+3) = 0 \Rightarrow r = 4$ or $r = -3 \Rightarrow y = c_1e^{4x} + c_2e^{-3x}$

2. $3y'' - y' = 0 \Rightarrow 3r^2 - r = 0 \Rightarrow r(3r-1) = 0 \Rightarrow r = 0$ or $r = \frac{1}{3} \Rightarrow y = c_1e^{0\cdot x} + c_2e^{\frac{1}{3}x} \Rightarrow y = c_1 + c_2e^{\frac{1}{3}x}$

3. $y'' + 3y' - 4y = 0 \Rightarrow r^2 + 3r - 4 = 0 \Rightarrow (r+4)(r-1) = 0 \Rightarrow r = -4$ or $r = 1 \Rightarrow y = c_1e^{-4x} + c_2e^{x}$

4. $y'' - 9y = 0 \Rightarrow r^2 - 9 = 0 \Rightarrow (r-3)(r+3) = 0 \Rightarrow r = 3$ or $r = -3 \Rightarrow y = c_1e^{3x} + c_2e^{-3x}$

5. $y'' - 4y = 0 \Rightarrow r^2 - 4 = 0 \Rightarrow (r-2)(r+2) = 0 \Rightarrow r = 2$ or $r = -2 \Rightarrow y = c_1e^{2x} + c_2e^{-2x}$

6. $y'' - 64y = 0 \Rightarrow r^2 - 64 = 0 \Rightarrow (r-8)(r+8) = 0 \Rightarrow r = 8$ or $r = -8 \Rightarrow y = c_1e^{8x} + c_2e^{-8x}$

7. $2y'' - y' - 3y = 0 \Rightarrow 2r^2 - r - 3 = 0 \Rightarrow (2r-3)(r+1) = 0 \Rightarrow r = \frac{3}{2}$ or $r = -1 \Rightarrow y = c_1e^{\frac{3}{2}x} + c_2e^{-x}$

8. $9y'' - y = 0 \Rightarrow 9r^2 - 1 = 0 \Rightarrow (3r-1)(3r+1) = 0 \Rightarrow r = \frac{1}{3}$ or $r = -\frac{1}{3} \Rightarrow y = c_1e^{\frac{1}{3}x} + c_2e^{-\frac{1}{3}x}$

9. $8y'' - 10y' - 3y = 0 \Rightarrow 8r^2 - 10r - 3 = 0 \Rightarrow (4r+1)(2r-3) = 0 \Rightarrow r = -\frac{1}{4}$ or $r = \frac{3}{2} \Rightarrow y = c_1e^{-\frac{1}{4}x} + c_2e^{\frac{3}{2}x}$

10. $3y'' - 20y' + 12y = 0 \Rightarrow 3r^2 - 20r + 12 = 0 \Rightarrow (3r-2)(r-6) = 0 \Rightarrow r = \frac{2}{3}$ or $r = 6 \Rightarrow y = c_1e^{\frac{2}{3}x} + c_2e^{6x}$

11. $y'' + 9y = 0 \Rightarrow r^2 + 9 = 0 \Rightarrow r = 0 \pm 3i \Rightarrow y = e^{0\cdot x}(c_1\cos 3x + c_2\sin 3x) \Rightarrow y = c_1\cos 3x + c_2\sin 3x$

12. $y'' + 4y' + 5y = 0 \Rightarrow r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4\pm\sqrt{4^2-4(1)(5)}}{2(1)} = -2 \pm i \Rightarrow y = e^{-2x}(c_1\cos x + c_2\sin x)$

13. $y'' + 25y = 0 \Rightarrow r^2 + 25 = 0 \Rightarrow r = 0 \pm 5i \Rightarrow y = e^{0\cdot x}(c_1\cos 5x + c_2\sin 5x) \Rightarrow y = c_1\cos 5x + c_2\sin 5x$

14. $y'' + y = 0 \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y = e^{0\cdot x}(c_1\cos x + c_2\sin x) \Rightarrow y = c_1\cos x + c_2\sin x$

15. $y'' - 2y' + 5y = 0 \Rightarrow r^2 - 2r + 5 = 0 \Rightarrow r = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)} = 1 \pm 2i \Rightarrow y = e^{x}(c_1\cos 2x + c_2\sin 2x)$

16. $y'' + 16y = 0 \Rightarrow r^2 + 16 = 0 \Rightarrow r = 0 \pm 4i \Rightarrow y = e^{0\cdot x}(c_1\cos 4x + c_2\sin 4x) \Rightarrow y = c_1\cos 4x + c_2\sin 4x$

17. $y'' + 2y' + 4y = 0 \Rightarrow r^2 + 2r + 4 = 0 \Rightarrow r = \frac{-2\pm\sqrt{2^2-4(1)(4)}}{2(1)} = -1 \pm \sqrt{3}\,i \Rightarrow y = e^{-x}\left(c_1\cos\sqrt{3}\,x + c_2\sin\sqrt{3}\,x\right)$

18. $y'' - 2y' + 3y = 0 \Rightarrow r^2 - 2r + 3 = 0 \Rightarrow r = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(3)}}{2(1)} = 1 \pm \sqrt{2}\,i \Rightarrow y = e^{x}\left(c_1\cos\sqrt{2}\,x + c_2\sin\sqrt{2}\,x\right)$

19. $y'' + 4y' + 9y = 0 \Rightarrow r^2 + 4r + 9 = 0 \Rightarrow r = \frac{-4\pm\sqrt{4^2-4(1)(9)}}{2(1)} = -2 \pm \sqrt{5}\,i \Rightarrow y = e^{-2x}\left(c_1\cos\sqrt{5}\,x + c_2\sin\sqrt{5}\,x\right)$

20. $4y'' - 4y' + 13y = 0 \Rightarrow 4r^2 - 4r + 13 = 0 \Rightarrow r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(13)}}{2(4)} = \frac{1}{2} \pm \sqrt{3}\,i$
$\Rightarrow y = e^{\frac{1}{2}x}\left(c_1 \cos \sqrt{3}\,x + c_2 \sin \sqrt{3}\,x\right)$

21. $y'' = 0 \Rightarrow r^2 = 0 \Rightarrow r = 0$, repeated twice $\Rightarrow y = c_1 e^{0 \cdot x} + c_2 x\, e^{0 \cdot x} \Rightarrow y = c_1 + c_2 x$

22. $y'' + 8y' + 16y = 0 \Rightarrow r^2 + 8r + 16 = 0 \Rightarrow (r+4)^2 = 0 \Rightarrow r = -4$, repeated twice $\Rightarrow y = c_1 e^{-4x} + c_2 x\, e^{-4x}$

23. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0 \Rightarrow r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r = -2$, repeated twice $\Rightarrow y = c_1 e^{-2x} + c_2 x\, e^{-2x}$

24. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0 \Rightarrow r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow r = 3$, repeated twice $\Rightarrow y = c_1 e^{3x} + c_2 x\, e^{3x}$

25. $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0 \Rightarrow r^2 + 6r + 9 = 0 \Rightarrow (r+3)^2 = 0 \Rightarrow r = -3$, repeated twice $\Rightarrow y = c_1 e^{-3x} + c_2 x\, e^{-3x}$

26. $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 9y = 0 \Rightarrow 4r^2 - 12r + 9 = 0 \Rightarrow (2r-3)^2 = 0 \Rightarrow r = \frac{3}{2}$, repeated twice $\Rightarrow y = c_1 e^{\frac{3}{2}x} + c_2 x\, e^{\frac{3}{2}x}$

27. $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 0 \Rightarrow 4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)^2 = 0 \Rightarrow r = -\frac{1}{2}$, repeated twice $\Rightarrow y = c_1 e^{-\frac{1}{2}x} + c_2 x\, e^{-\frac{1}{2}x}$

28. $4\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0 \Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow (2r-1)^2 = 0 \Rightarrow r = \frac{1}{2}$, repeated twice $\Rightarrow y = c_1 e^{\frac{1}{2}x} + c_2 x\, e^{\frac{1}{2}x}$

29. $9\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + y = 0 \Rightarrow 9r^2 + 6r + 1 = 0 \Rightarrow (3r+1)^2 = 0 \Rightarrow r = -\frac{1}{3}$, repeated twice $\Rightarrow y = c_1 e^{-\frac{1}{3}x} + c_2 x\, e^{-\frac{1}{3}x}$

30. $9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0 \Rightarrow 9r^2 - 12r + 4 = 0 \Rightarrow (3r-2)^2 = 0 \Rightarrow r = \frac{2}{3}$, repeated twice $\Rightarrow y = c_1 e^{\frac{2}{3}x} + c_2 x\, e^{\frac{2}{3}x}$

31. $y'' + 6y' + 5y = 0,\ y(0) = 0,\ y'(0) = 3 \Rightarrow r^2 + 6r + 5 = 0 \Rightarrow (r+5)(r+1) = 0 \Rightarrow r = -5$ or $r = -1$
$\Rightarrow y = c_1 e^{-5x} + c_2 e^{-x} \Rightarrow y' = -5c_1 e^{-5x} - c_2 e^{-x}$; $y(0) = 0 \Rightarrow c_1 + c_2 = 0$, and $y'(0) = 3 \Rightarrow -5c_1 - c_2 = 3$
$\Rightarrow c_1 = -\frac{3}{4}$ and $c_2 = \frac{3}{4} \Rightarrow y = -\frac{3}{4}e^{-5x} + \frac{3}{4}e^{-x}$

32. $y'' + 16y = 0,\ y(0) = 2,\ y'(0) = -2 \Rightarrow r^2 + 16 = 0 \Rightarrow r = 0 \pm 4i \Rightarrow y = c_1 \cos 4x + c_2 \sin 4x$
$\Rightarrow y' = -4c_1 \sin 4x + 4c_2 \cos 4x$; $y(0) = 2 \Rightarrow c_1 = 2$, and $y'(0) = -2 \Rightarrow 4c_2 = -2 \Rightarrow c_1 = 2$ and $c_2 = -\frac{1}{2}$
$\Rightarrow y = 2\cos 4x - \frac{1}{2}\sin 4x$

33. $y'' + 12y = 0,\ y(0) = 0,\ y'(0) = 1 \Rightarrow r^2 + 12 = 0 \Rightarrow r = 0 \pm 2\sqrt{3}\,i \Rightarrow y = c_1 \cos 2\sqrt{3}\,x + c_2 \sin 2\sqrt{3}\,x$
$\Rightarrow y' = -2\sqrt{3}\,c_1 \sin 2\sqrt{3}\,x + 2\sqrt{3}\,c_2 \cos 2\sqrt{3}\,x$; $y(0) = 0 \Rightarrow c_1 = 0$, and $y'(0) = 1 \Rightarrow 2\sqrt{3}\,c_2 = 1$
$\Rightarrow c_1 = 0$ and $c_2 = \frac{1}{2\sqrt{3}} \Rightarrow y = \frac{1}{2\sqrt{3}}\sin 2\sqrt{3}\,x$

34. $12y'' + 5y' - 2y = 0,\ y(0) = 1,\ y'(0) = -1 \Rightarrow 12r^2 + 5r - 2 = 0 \Rightarrow (4r-1)(3r+2) = 0 \Rightarrow r = \frac{1}{4}$ or $r = -\frac{2}{3}$
$\Rightarrow y = c_1 e^{(1/4)x} + c_2 e^{-(2/3)x} \Rightarrow y' = \frac{1}{4}c_1 e^{(1/4)x} - \frac{2}{3}c_2 e^{-(2/3)x}$; $y(0) = 1 \Rightarrow c_1 + c_2 = 1$, and $y'(0) = -1$
$\Rightarrow \frac{1}{4}c_1 - \frac{2}{3}c_2 = -1 \Rightarrow c_1 = -\frac{4}{11}$ and $c_2 = \frac{15}{11} \Rightarrow y = -\frac{4}{11}e^{(1/4)x} + \frac{15}{11}e^{-(2/3)x}$

35. $y'' + 8y = 0,\ y(0) = -1,\ y'(0) = 2 \Rightarrow r^2 + 8 = 0 \Rightarrow r = 0 \pm 2\sqrt{2}\,i \Rightarrow y = c_1 \cos 2\sqrt{2}\,x + c_2 \sin 2\sqrt{2}\,x$
$\Rightarrow y' = -2\sqrt{2}\,c_1 \sin 2\sqrt{2}\,x + 2\sqrt{2}\,c_2 \cos 2\sqrt{2}\,x$; $y(0) = -1 \Rightarrow c_1 = -1$, and $y'(0) = 2 \Rightarrow 2\sqrt{2}\,c_2 = 2$
$\Rightarrow c_1 = -1$ and $c_2 = \frac{1}{\sqrt{2}} \Rightarrow y = -\cos 2\sqrt{2}\,x + \frac{1}{\sqrt{2}}\sin 2\sqrt{2}\,x$

36. $y'' + 4y' + 4y = 0,\ y(0) = 0,\ y'(0) = 1 \Rightarrow r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0 \Rightarrow r = -2$ repeated twice
$\Rightarrow y = c_1e^{-2x} + c_2x\,e^{-2x} \Rightarrow y' = -2c_1e^{-2x} + c_2e^{-2x} - 2c_2x\,e^{-2x};\ y(0) = 0 \Rightarrow c_1 = 0$, and
$y'(0) = 1 \Rightarrow -2c_1 + c_2 = 1 \Rightarrow c_1 = 0$ and $c_2 = 1 \Rightarrow y = x\,e^{-2x}$

37. $y'' - 4y' + 4y = 0,\ y(0) = 1,\ y'(0) = 0 \Rightarrow r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$ repeated twice
$\Rightarrow y = c_1e^{2x} + c_2x\,e^{2x} \Rightarrow y' = 2c_1e^{2x} + c_2e^{2x} + 2c_2x\,e^{2x};\ y(0) = 1 \Rightarrow c_1 = 1$, and $y'(0) = 0 \Rightarrow 2c_1 + c_2 = 0$
$\Rightarrow c_1 = 1$ and $c_2 = -2 \Rightarrow y = e^{2x} - 2x\,e^{2x}$

38. $4y'' - 4y' + y = 0,\ y(0) = 4,\ y'(0) = 4 \Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow (2r-1)^2 = 0 \Rightarrow r = \frac{1}{2}$ repeated twice
$\Rightarrow y = c_1e^{\frac{1}{2}x} + c_2x\,e^{\frac{1}{2}x} \Rightarrow y' = \frac{1}{2}c_1e^{\frac{1}{2}x} + c_2e^{\frac{1}{2}x} + \frac{1}{2}c_2x\,e^{\frac{1}{2}x};\ y(0) = 4 \Rightarrow c_1 = 4$,
and $y'(0) = 4 \Rightarrow \frac{1}{2}c_1 + c_2 = 4 \Rightarrow c_1 = 4$ and $c_2 = 2 \Rightarrow y = 4e^{\frac{1}{2}x} + 2x\,e^{\frac{1}{2}x}$

39. $4\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 9y = 0,\ y(0) = 2,\ \frac{dy}{dx}(0) = 1 \Rightarrow 4r^2 + 12r + 9 = 0 \Rightarrow (2r+3)^2 = 0 \Rightarrow r = -\frac{3}{2}$ repeated twice
$\Rightarrow y = c_1e^{-\frac{3}{2}x} + c_2x\,e^{-\frac{3}{2}x} \Rightarrow \frac{dy}{dx} = -\frac{3}{2}c_1e^{-\frac{3}{2}x} + c_2e^{-\frac{3}{2}x} - \frac{3}{2}c_2x\,e^{-\frac{3}{2}x};\ y(0) = 2 \Rightarrow c_1 = 2$, and $\frac{dy}{dx}(0) = 1$
$\Rightarrow -\frac{3}{2}c_1 + c_2 = 1 \Rightarrow c_1 = 2$ and $c_2 = 4 \Rightarrow y = 2e^{-\frac{3}{2}x} + 4x\,e^{-\frac{3}{2}x}$

40. $9\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0,\ y(0) = -1,\ \frac{dy}{dx}(0) = 1 \Rightarrow 9r^2 - 12r + 4 = 0 \Rightarrow (3r-2)^2 = 0 \Rightarrow r = \frac{2}{3}$ repeated twice
$\Rightarrow y = c_1e^{\frac{2}{3}x} + c_2x\,e^{\frac{2}{3}x} \Rightarrow \frac{dy}{dx} = \frac{2}{3}c_1e^{\frac{2}{3}x} + c_2e^{\frac{2}{3}x} + \frac{2}{3}c_2x\,e^{\frac{2}{3}x};\ y(0) = -1 \Rightarrow c_1 = -1$, and $\frac{dy}{dx}(0) = 1$
$\Rightarrow \frac{2}{3}c_1 + c_2 = 1 \Rightarrow c_1 = -1$ and $c_2 = \frac{5}{3} \Rightarrow y = -e^{\frac{2}{3}x} + \frac{5}{3}x\,e^{\frac{2}{3}x}$

41. $y'' - 2y' - 3y = 0 \Rightarrow r^2 - 2r - 3 = 0 \Rightarrow (r-3)(r+1) = 0 \Rightarrow r = 3$ or $r = -1 \Rightarrow y = c_1e^{3x} + c_2e^{-x}$

42. $6y'' - y' - y = 0 \Rightarrow 6r^2 - r - 1 = 0 \Rightarrow (2r-1)(3r+1) = 0 \Rightarrow r = \frac{1}{2}$ or $r = -\frac{1}{3} \Rightarrow y = c_1e^{\frac{1}{2}x} + c_2e^{-\frac{1}{3}x}$

43. $4y'' + 4y' + y = 0 \Rightarrow 4r^2 + 4r + 1 = 0 \Rightarrow (2r+1)^2 = 0 \Rightarrow r = -\frac{1}{2}$ repeated twice $\Rightarrow y = c_1e^{-\frac{1}{2}x} + c_2x\,e^{-\frac{1}{2}x}$

44. $9y'' + 12y' + 4y = 0 \Rightarrow 9r^2 + 12r + 4 = 0 \Rightarrow (3r+2)^2 = 0 \Rightarrow r = -\frac{2}{3}$ repeated twice $\Rightarrow y = c_1e^{-\frac{2}{3}x} + c_2x\,e^{-\frac{2}{3}x}$

45. $4y'' + 20y = 0 \Rightarrow 4r^2 + 20 = 0 \Rightarrow r = 0 \pm 5i \Rightarrow y = e^{0\cdot x}\left(c_1\cos\sqrt{5}\,x + c_2\sin\sqrt{5}\,x\right) \Rightarrow y = c_1\cos\sqrt{5}\,x + c_2\sin\sqrt{5}\,x$

46. $y'' + 2y' + 2y = 0 \Rightarrow r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2\pm\sqrt{(2)^2-4(1)(2)}}{2(1)} = -1 \pm i \Rightarrow y = e^{-x}(c_1\cos x + c_2\sin x)$

47. $25y'' + 10y' + y = 0 \Rightarrow 25r^2 + 10r + 1 = 0 \Rightarrow (5r+1)^2 = 0 \Rightarrow r = -\frac{1}{5}$ repeated twice $\Rightarrow y = c_1e^{-\frac{1}{5}x} + c_2x\,e^{-\frac{1}{5}x}$

48. $6y'' + 13y' - 5y = 0 \Rightarrow 6r^2 + 13r - 5 = 0 \Rightarrow (3r-1)(2r+5) = 0 \Rightarrow r = \frac{1}{3}$ or $r = -\frac{5}{2} \Rightarrow y = c_1e^{\frac{1}{3}x} + c_2e^{-\frac{5}{2}x}$

49. $4y'' + 4y' + 5y = 0 \Rightarrow 4r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4\pm\sqrt{(4)^2-4(4)(5)}}{2(4)} = -\frac{1}{2} \pm i \Rightarrow y = e^{-\frac{1}{2}x}(c_1\cos x + c_2\sin x)$

50. $y'' + 4y' + 6y = 0 \Rightarrow r^2 + 4r + 6 = 0 \Rightarrow r = \frac{-4\pm\sqrt{(4)^2-4(1)(6)}}{2(1)} = -2 \pm \sqrt{2}\,i \Rightarrow y = e^{-2x}\left(c_1\cos\sqrt{2}\,x + c_2\sin\sqrt{2}\,x\right)$

51. $16y'' - 24y' + 9y = 0 \Rightarrow 16r^2 - 24r + 9 = 0 \Rightarrow (4r-3)^2 = 0 \Rightarrow r = \frac{3}{4}$ repeated twice $\Rightarrow y = c_1e^{\frac{3}{4}x} + c_2x\,e^{\frac{3}{4}x}$

52. $6y'' - 5y' - 6y = 0 \Rightarrow 6r^2 - 5r - 6 = 0 \Rightarrow (2r-3)(3r+2) = 0 \Rightarrow r = \frac{3}{2}$ or $r = -\frac{2}{3} \Rightarrow y = c_1 e^{\frac{3}{2}x} + c_2 e^{-\frac{2}{3}x}$

53. $9y'' + 24y' + 16y = 0 \Rightarrow 9r^2 + 24r + 16 = 0 \Rightarrow (3r+4)^2 = 0 \Rightarrow r = -\frac{4}{3}$ repeated twice $\Rightarrow y = c_1 e^{-\frac{4}{3}x} + c_2 x\, e^{-\frac{4}{3}x}$

54. $4y'' + 16y' + 52y = 0 \Rightarrow 4r^2 + 16r + 52 = 0 \Rightarrow r = \frac{-16 \pm \sqrt{(16)^2 - 4(4)(52)}}{2(4)} = -2 \pm 3i \Rightarrow y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$

55. $6y'' - 5y' - 4y = 0 \Rightarrow 6r^2 - 5r - 4 = 0 \Rightarrow (3r-4)(2r+1) = 0 \Rightarrow r = \frac{4}{3}$ or $r = -\frac{1}{2} \Rightarrow y = c_1 e^{\frac{4}{3}x} + c_2 e^{-\frac{1}{2}x}$

56. $y'' - 2y' + 2y = 0,\ y(0) = 0,\ y'(0) = 2 \Rightarrow r^2 - 2r + 2 = 0 \Rightarrow r = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = 1 \pm i$
$\Rightarrow y = e^x(c_1 \cos x + c_2 \sin x) \Rightarrow y' = e^x(c_1 \cos x + c_2 \sin x) + e^x(-c_1 \sin x + c_2 \cos x)$; $y(0) = 0 \Rightarrow c_1 = 0$, and $y'(0) = 2 \Rightarrow c_1 + c_2 = 2 \Rightarrow c_1 = 0$ and $c_2 = 2 \Rightarrow y = 2e^x \sin x$

57. $y'' + 2y' + y = 0,\ y(0) = 1,\ y'(0) = 1 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$, repeated twice
$\Rightarrow y = c_1 e^{-x} + c_2 x\, e^{-x} \Rightarrow y' = -c_1 e^{-x} - c_2 x\, e^{-x} + c_2 e^{-x}$; $y(0) = 1 \Rightarrow c_1 = 1$, and $y'(0) = 1 \Rightarrow -c_1 + c_2 = 1$
$\Rightarrow c_1 = 1$ and $c_2 = 2 \Rightarrow y = e^{-x} + 2x\, e^{-x}$

58. $4y'' - 4y' + y = 0,\ y(0) = -1,\ y'(0) = 2 \Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow (2r-1)^2 = 0 \Rightarrow r = \frac{1}{2}$, repeated twice
$\Rightarrow y = c_1 e^{\frac{1}{2}x} + c_2 x\, e^{\frac{1}{2}x} \Rightarrow y' = \frac{1}{2}c_1 e^{\frac{1}{2}x} + \frac{1}{2}c_2 x\, e^{\frac{1}{2}x} + c_2 e^{\frac{1}{2}x}$; $y(0) = -1 \Rightarrow c_1 = -1$, and $y'(0) = 2$
$\Rightarrow \frac{1}{2}c_1 + c_2 = 2 \Rightarrow c_1 = -1$ and $c_2 = \frac{5}{2} \Rightarrow y = -e^{\frac{1}{2}x} + \frac{5}{2}x\, e^{\frac{1}{2}x}$

59. $3y'' + y' - 14y = 0,\ y(0) = 2,\ y'(0) = -1 \Rightarrow 3r^2 + r - 14 = 0 \Rightarrow (3r+7)(r-2) = 0 \Rightarrow r = -\frac{7}{3}$ or $r = 2$
$\Rightarrow y = c_1 e^{-\frac{7}{3}x} + c_2 e^{2x} \Rightarrow y' = -\frac{7}{3}c_1 e^{-\frac{7}{3}x} + 2c_2 e^{2x}$; $y(0) = 2 \Rightarrow c_1 + c_2 = 2$, and $y'(0) = -1 \Rightarrow -\frac{7}{3}c_1 + 2c_2 = -1$
$\Rightarrow c_1 = \frac{15}{13}$ and $c_2 = \frac{11}{13} \Rightarrow y = \frac{15}{13}e^{-(7/3)x} + \frac{11}{13}e^{2x}$

60. $4y'' + 4y' + 5y = 0,\ y(\pi) = 1,\ y'(\pi) = 0 \Rightarrow 4r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{(4)^2 - 4(4)(5)}}{2(4)} = -\frac{1}{2} \pm i$
$\Rightarrow y = e^{-\frac{1}{2}x}(c_1 \cos x + c_2 \sin x) \Rightarrow y' = -\frac{1}{2}e^{-\frac{1}{2}x}(c_1 \cos x + c_2 \sin x) + e^{-\frac{1}{2}x}(-c_1 \sin x + c_2 \cos x)$; $y(\pi) = 1$
$\Rightarrow -e^{-\frac{\pi}{2}}c_1 = 1$, and $y'(\pi) = 0 \Rightarrow -\frac{1}{2}e^{-\frac{\pi}{2}}c_1 + e^{-\frac{\pi}{2}}c_2 = 0 \Rightarrow c_1 = -e^{\frac{\pi}{2}}$ and $c_2 = -\frac{1}{2}e^{\frac{\pi}{2}}$
$\Rightarrow y = e^{-\frac{1}{2}x}\left(-e^{\frac{\pi}{2}}\cos x - \frac{1}{2}e^{\frac{\pi}{2}}\sin x\right)$

61. Let r_1 and r_2 be real roots with $r_1 \neq r_2$. If $e^{r_1 x}$ and $e^{r_2 x}$ are linearly independent, then $e^{r_1 x}$ is not a constant multiple of $e^{r_2 x}$ (and vice versa). Assume that $e^{r_1 x}$ is a constant multiple of $e^{r_2 x}$, then for some nonzero constant c, $e^{r_1 x} = c\, e^{r_2 x} \Rightarrow \frac{e^{r_1 x}}{e^{r_2 x}} = c$ $\Rightarrow e^{r_1 x - r_2 x} = c \Rightarrow e^{(r_1 - r_2)x} = c$. Since $r_1 \neq r_2$, c is not a constant, which is a contridiction. Thus $e^{r_1 x}$ and $e^{r_2 x}$ are linearly independent.

62. Let r be the only repeated real root. If e^{rx} and $x\, e^{rx}$ are linearly independent, then e^{rx} is not a constant multiple of $x\, e^{rx}$ (and vice versa). Assume that $x\, e^{rx}$ is a constant multiple of e^{rx}, then for some nonzero constant c, $x\, e^{rx} = c\, e^{rx}$ $\Rightarrow x\, e^{rx} - c\, e^{rx} = 0 \Rightarrow e^{rx}(x - c) = 0 \Rightarrow e^{rx} = 0$ or $x - c = 0$. Since $e^{rx} \neq 0 \Rightarrow x = c$, thus c is not a constant, which is a contridiction. Thus e^{rx} and $x\, e^{rx}$ are linearly independent.

63. Let $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ be complex roots. If $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are linearly independent, then $e^{\alpha x}\cos\beta x$ is not a constant multiple of $e^{\alpha x}\sin\beta x$ (and vice versa). Assume that $e^{\alpha x}\cos\beta x$ is a constant multiple of $e^{\alpha x}\sin\beta x$, then for some nonzero constant c, $e^{\alpha x}\cos\beta x = c\,e^{\alpha x}\sin\beta x \Rightarrow e^{\alpha x}\cos\beta x - c\,e^{\alpha x}\sin\beta x = 0 \Rightarrow e^{\alpha x}(\cos\beta x - c\sin\beta x) = 0$ $\Rightarrow e^{\alpha x} = 0$ or $\cos\beta x - c\sin\beta x = 0$. Since $e^{\alpha x} \neq 0 \Rightarrow c = \cot\beta x$, thus c is not a constant, which is a contridiction. Thus $e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$ are linearly independent.

64. Let y_1 and y_2 be linearly independent solutions of $P(x)\,y'' + Q(x)\,y' + R(x)\,y = 0$. Let $y_3 = y_1 + y_2 \Rightarrow y_3' = y_1' + y_2'$ $\Rightarrow y_3'' = y_1'' + y_2''$. Then $P(x)\,y_3'' + Q(x)\,y_3' + R(x)\,y_3 = P(x)\,(y_1'' + y_2'') + Q(x)(y_1' + y_2') + R(x)\,(y_1 + y_2)$
$= P(x)\,y_1'' + P(x)\,y_2'' + Q(x)y_1' + Q(x)\,y_2' + R(x)\,y_1 + R(x)\,y_2$
$= P(x)\,y_1'' + Q(x)y_1' + R(x)\,y_1 + P(x)\,y_2'' + Q(x)\,y_2' + R(x)\,y_2 = 0 + 0 = 0$. Let $y_4 = y_1 - y_2 \Rightarrow y_4' = y_1' - y_2'$
$\Rightarrow y_4'' = y_1'' - y_2''$. Then $P(x)\,y_4'' + Q(x)\,y_4' + R(x)\,y_4 = P(x)\,(y_1'' - y_2'') + Q(x)(y_1' - y_2') + R(x)\,(y_1 - y_2)$
$= P(x)\,y_1'' - P(x)\,y_2'' + Q(x)y_1' - Q(x)\,y_2' + R(x)\,y_1 - R(x)\,y_2$
$= P(x)\,y_1'' + Q(x)y_1' + R(x)\,y_1 - [P(x)\,y_2'' + Q(x)\,y_2' + R(x)\,y_2] = 0 - 0 = 0$. Thus y_3 and y_4 are both solutions.
Suppose that y_3 is a constant multiple of y_4, then there is a nonzero constant c such that $y_3 = c\,y_4$.
$\Rightarrow y_1 + y_2 = c(y_1 - y_2)$. If we solve this equation for y_1 we obtain $y_1 = -\frac{1+c}{1-c}y_2 \Rightarrow y_1$ is a constant multiple of y_2, which is a contradiction since y_1 and y_2 are linearly independent $\Rightarrow y_1 + y_2$ and $y_1 - y_2$ are linearly independent.

65. (a) $y'' + 4y = 0,\ y(0) = 0,\ y(\pi) = 1 \Rightarrow r^2 + 4 = 0 \Rightarrow r = 0 \pm 2i \Rightarrow y = e^{0\cdot x}(c_1\cos 2x + c_2\sin 2x)$
$\Rightarrow y = c_1\cos 2x + c_2\sin 2x;\ y(0) = 0 \Rightarrow c_1 = 0$, and $y(\pi) = 1 \Rightarrow c_1 = 1 \Rightarrow$ no solution
(b) $y'' + 4y = 0,\ y(0) = 0,\ y(\pi) = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r = 0 \pm 2i \Rightarrow y = e^{0\cdot x}(c_1\cos 2x + c_2\sin 2x)$
$\Rightarrow y = c_1\cos 2x + c_2\sin 2x;\ y(0) = 0 \Rightarrow c_1 = 0$, and $y(\pi) = 0 \Rightarrow c_1 = 0 \Rightarrow c_1 = 0$, c_2 can be any real number
$\Rightarrow y = c_2\sin 2x$

66. Let a, b, and c be positive constants, then $r = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$. There are three cases to consider.

Case I: Two distinct real solutions $\Rightarrow b^2 - 4ac > 0$. Since a, b, and c are positive $\Rightarrow 4ac > 0$ and $b > 0$
$\Rightarrow -4ac < 0$ and $-b < 0 \Rightarrow 0 < b^2 - 4ac < b^2 \Rightarrow 0 < \sqrt{b^2 - 4ac} < b \Rightarrow -b + \sqrt{b^2 - 4ac} < 0$
and $-b - \sqrt{b^2 - 4ac} < 0$. Since $2a > 0 \Rightarrow \frac{-b\pm\sqrt{b^2-4ac}}{2a} < 0$. Thus $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} < 0$ and
$r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} < 0$. The general solution to the differential equation is $y = c_1e^{r_1x} + c_2e^{r_2x}$.
$\lim_{x\to\infty}(c_1e^{r_1x} + c_2e^{r_2x}) = 0 + 0 = 0$.

Case II: One repeated real solution $\Rightarrow b^2 - 4ac = 0$. Since a, b, and c are positive $\Rightarrow 2a > 0$ and $-b < 0$
$\Rightarrow r = -\frac{b}{2a} < 0$. The general solution to the differential equation is $y = c_1e^{rx} + c_2x\,e^{rx}$. Since
$r < 0 \Rightarrow -r > 0$. $\lim_{x\to\infty}(c_1e^{rx} + c_2x\,e^{rx}) = \lim_{x\to\infty}\left(c_1e^{rx} + \frac{c_2x}{e^{-rx}}\right) = \lim_{x\to\infty}(c_1e^{rx}) + \lim_{x\to\infty}\left(\frac{c_2x}{e^{-rx}}\right)$
$= 0 + \lim_{x\to\infty}\left(\frac{c_2}{-r\,e^{-rx}}\right) = 0$, using L'hopital's rule to evaluate the limit of the second expression.

Case III: Two complex, nonreal solutions $\Rightarrow b^2 - 4ac < 0 \Rightarrow 4ac - b^2 > 0$. Since a, b, and c are positive $\Rightarrow 2a > 0$ and
$-b < 0 \Rightarrow -\frac{b}{2a} < 0$. The roots are $r_1 = -\frac{b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}i$ and $r_2 = -\frac{b}{2a} - \frac{\sqrt{4ac-b^2}}{2a}i$, thus $\alpha = -\frac{b}{2a} < 0$ and
$\beta = \frac{\sqrt{4ac-b^2}}{2a} > 0$. The general solution to the differential equation is $y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$. Since
$-1 \le \sin\beta x \le 1$ and $-1 \le \cos\beta x \le 1 \Rightarrow -|c_2| \le c_2\sin\beta x \le |c_2|$ and $-|c_1| \le c_1\cos\beta x \le |c_1|$
$\Rightarrow -|c_1| - |c_2| \le c_1\cos\beta x + c_2\sin\beta x \le |c_1| + |c_2|$
$\Rightarrow e^{\alpha x}(-|c_1| - |c_2|) \le e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) \le e^{\alpha x}(|c_1| + |c_2|)$
$\lim_{x\to\infty}[e^{\alpha x}(-|c_1| - |c_2|)] = (-|c_1| - |c_2|)\lim_{x\to\infty}e^{\alpha x} = 0$ and $\lim_{x\to\infty}[e^{\alpha x}(|c_1| + |c_2|)] = (|c_1| + |c_2|)\lim_{x\to\infty}e^{\alpha x} = 0$
thus by the Sandwich theorem, $\lim_{x\to\infty}[e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)] = 0$.

17.2 NONHOMOGENEOUS LINEAR EQUATIONS

1. $y'' - 3y' - 10y = -3 \Rightarrow r^2 - 3r - 10 = 0 \Rightarrow (r-5)(r+2) = 0 \Rightarrow r = 5$ or $r = -2 \Rightarrow y_c = c_1e^{5x} + c_2e^{-2x}$; $y_p = A$ $\Rightarrow y_p' = 0 \Rightarrow y_p'' = 0 \Rightarrow 0 - 3(0) - 10A = -3 \Rightarrow A = \frac{3}{10} \Rightarrow y = c_1e^{5x} + c_2e^{-2x} + \frac{3}{10}$

2. $y'' - 3y' - 10y = 2x - 3 \Rightarrow r^2 - 3r - 10 = 0 \Rightarrow (r-5)(r+2) = 0 \Rightarrow r = 5$ or $r = -2 \Rightarrow y_c = c_1e^{5x} + c_2e^{-2x}$; $y_p = Ax + B \Rightarrow y_p' = A \Rightarrow y_p'' = 0 \Rightarrow 0 - 3A - 10(Ax + B) = 2x - 3 \Rightarrow -10Ax + (-3A - 10B) = 2x - 3$ $\Rightarrow -10A = 2, -3A - 10B = -3 \Rightarrow A = -\frac{1}{5}, B = \frac{9}{25} \Rightarrow y = c_1e^{5x} + c_2e^{-2x} - \frac{1}{5}x + \frac{9}{25}$

3. $y'' - y' = \sin x \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0$ or $r = 1 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^x = c_1 + c_2e^x$; $y_p = A\sin x + B\cos x \Rightarrow y_p' = A\cos x - B\sin x \Rightarrow y_p'' = -A\sin x - B\cos x$ $\Rightarrow -A\sin x - B\cos x - (A\cos x - B\sin x) = \sin x \Rightarrow (-A + B)\sin x + (-A - B)\cos x = \sin x$ $\Rightarrow -A + B = 1, -A - B = 0 \Rightarrow A = -\frac{1}{2}, B = \frac{1}{2} \Rightarrow y = c_1 + c_2e^x - \frac{1}{2}\sin x + \frac{1}{2}\cos x$

4. $y'' + 2y' + y = x^2 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$ repeated twice $\Rightarrow y_c = c_1e^{-x} + c_2x\,e^{-x}$; $y_p = Ax^2 + Bx + C \Rightarrow y_p' = 2Ax + B \Rightarrow y_p'' = 2A \Rightarrow (Ax^2 + Bx + C) + 2(2Ax + B) + 2A = x^2$ $\Rightarrow Ax^2 + (4A + B)x + (2A + 2B + C) = x^2 \Rightarrow A = 1, 4A + B = 0, 2A + 2B + C = 0 \Rightarrow A = 1, B = -4,$ $C = 6 \Rightarrow y = c_1e^{-x} + c_2x\,e^{-x} + x^2 - 4x + 6$

5. $y'' + y = \cos 3x \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = e^{0\cdot x}(c_1\cos x + c_2\sin x) = c_1\cos x + c_2\sin x$; $y_p = A\sin 3x + B\cos 3x \Rightarrow y_p' = 3A\cos 3x - 3B\sin 3x \Rightarrow y_p'' = -9A\sin 3x - 9B\cos 3x$ $\Rightarrow -9A\sin 3x - 9B\cos 3x + (A\sin 3x + B\cos 3x) = \cos 3x \Rightarrow -8A\sin x - 8B\cos x = \cos 3x$ $\Rightarrow -8A = 0, -8B = 1 \Rightarrow A = 0, B = -\frac{1}{8} \Rightarrow y = c_1\cos x + c_2\sin x - \frac{1}{8}\cos 3x$

6. $y'' + y = e^{2x} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = e^{0\cdot x}(c_1\cos x + c_2\sin x) = c_1\cos x + c_2\sin x$; $y_p = A\,e^{2x} \Rightarrow y_p' = 2A\,e^{2x} \Rightarrow y_p'' = 4A\,e^{2x} \Rightarrow 4A\,e^{2x} + A\,e^{2x} = e^{2x} \Rightarrow 5A\,e^{2x} = e^{2x} \Rightarrow 5A = 1$ $\Rightarrow A = \frac{1}{5} \Rightarrow y = c_1\cos x + c_2\sin x + \frac{1}{5}e^{2x}$

7. $y'' - y' - 2y = 20\cos x \Rightarrow r^2 - r - 2 = 0 \Rightarrow (r-2)(r+1) = 0 \Rightarrow r = 2$ or $r = -1 \Rightarrow y_c = c_1e^{2x} + c_2e^{-x}$; $y_p = A\sin x + B\cos x \Rightarrow y_p' = A\cos x - B\sin x \Rightarrow y_p'' = -A\sin x - B\cos x$ $\Rightarrow -A\sin x - B\cos x - (A\cos x - B\sin x) - 2(A\sin x + B\cos x) = 20\cos x$ $\Rightarrow (-3A + B)\sin x + (-A - 3B)\cos x = 20\cos x \Rightarrow -3A + B = 0, -A - 3B = 20 \Rightarrow A = -2, B = -6$ $\Rightarrow y = c_1e^{2x} + c_2e^{-x} - 2\sin x - 6\cos x$

8. $y'' + y = 2x + 3e^x \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1\cos x + c_2\sin x$; $y_p = Ax + B + Ce^x \Rightarrow y_p' = A + Ce^x$ $\Rightarrow y_p'' = Ce^x \Rightarrow Ce^x + (Ax + B + Ce^x) = 2x + 3e^x \Rightarrow Ax + B + 2Ce^x = 2x + 3e^x \Rightarrow A = 2, B = 0, 2C = 3$ $\Rightarrow A = 2, B = 0, C = \frac{3}{2} \Rightarrow y = c_1\cos x + c_2\sin x + 2x + \frac{3}{2}e^x$

9. $y'' - y = e^x + x^2 \Rightarrow r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = 1$ or $r = -1 \Rightarrow y_c = c_1e^x + c_2e^{-x}$; $y_p = Axe^x + Bx^2 + Cx + D \Rightarrow y_p' = Ae^x + Axe^x + 2Bx + C \Rightarrow y_p'' = 2Ae^x + Axe^x + 2B$ $\Rightarrow (2Ae^x + Axe^x + 2B) - (Axe^x + Bx^2 + Cx + D) = e^x + x^2 \Rightarrow 2Ae^x - Bx^2 - Cx + (2B - D) = e^x + x^2$ $\Rightarrow 2A = 1, -B = 1, -C = 0, 2B - D = 0 \Rightarrow A = \frac{1}{2}, B = -1, C = 0, D = -2$ $\Rightarrow y = c_1e^x + c_2e^{-x} + \frac{1}{2}xe^x - x^2 - 2$

10. $y'' + 2y' + y = 6\sin 2x \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$, repeated twice $\Rightarrow y_c = c_1e^{-x} + c_2xe^{-x}$;
$y_p = A\sin 2x + B\cos 2x \Rightarrow y_p' = 2A\cos 2x - 2B\sin 2x \Rightarrow y_p'' = -4A\sin 2x - 4B\cos 2x$
$\Rightarrow -4A\sin 2x - 4B\cos 2x + 2(2A\cos 2x - 2B\sin 2x) + (A\sin 2x + B\cos 2x) = 6\sin 2x$
$\Rightarrow (-3A - 4B)\sin 2x + (4A - 3B)\cos 2x = 6\sin 2x \Rightarrow -3A - 4B = 6,\ 4A - 3B = 0 \Rightarrow A = -\frac{24}{25},\ B = -\frac{18}{25}$
$\Rightarrow y = c_1e^{-x} + c_2xe^{-x} - \frac{24}{25}\sin 2x - \frac{18}{25}\cos 2x$

11. $y'' - y' - 6y = e^{-x} - 7\cos x \Rightarrow r^2 - r - 6 = 0 \Rightarrow (r-3)(r+2) = 0 \Rightarrow r = 3$ or $r = -2 \Rightarrow y_c = c_1e^{3x} + c_2e^{-2x}$;
$y_p = Ae^{-x} + B\sin x + C\cos x \Rightarrow y_p' = -Ae^{-x} + B\cos x - C\sin x \Rightarrow y_p'' = Ae^{-x} - B\sin x - C\cos x$
$\Rightarrow Ae^{-x} - B\sin x - C\cos x - (-Ae^{-x} + B\cos x - C\sin x) - 6(Ae^{-x} + B\sin x + C\cos x) = e^{-x} - 7\cos x$
$\Rightarrow -4Ae^{-x} + (-7B + C)\sin x + (-B - 7C)\cos x = e^{-x} - 7\cos x \Rightarrow -4A = 1,\ -7B + C = 0,\ -B - 7C = -7$
$\Rightarrow A = -\frac{1}{4},\ B = \frac{7}{50},\ C = \frac{49}{50} \Rightarrow y = c_1e^{3x} + c_2e^{-2x} - \frac{1}{4}e^{-x} + \frac{7}{50}\sin x + \frac{49}{50}\cos x$

12. $y'' + 3y' + 2y = e^{-x} + e^{-2x} - x \Rightarrow r^2 + 3r + 2 = 0 \Rightarrow (r+1)(r+2) = 0 \Rightarrow r = -1$ or $r = -2 \Rightarrow y_c = c_1e^{-x} + c_2e^{-2x}$;
$y_p = Ax\,e^{-x} + Bx\,e^{-2x} + Cx + D \Rightarrow y_p' = Ae^{-x} - Ax\,e^{-x} + B\,e^{-2x} - 2Bx\,e^{-2x} + C$
$\Rightarrow y_p'' = -2Ae^{-x} + Ax\,e^{-x} - 4Be^{-2x} + 4Bx\,e^{-2x} \Rightarrow -2Ae^{-x} + Ax\,e^{-x} - 4Be^{-2x} + 4Bx\,e^{-2x}$
$+ 3(Ae^{-x} - Ax\,e^{-x} + B\,e^{-2x} - 2Bx\,e^{-2x} + C) + 2(Ax\,e^{-x} + Bx\,e^{-2x} + Cx + D) = e^{-x} + e^{-2x} - x$
$\Rightarrow Ax\,e^{-x} - B\,e^{-2x} + 2Cx + (3C + 2D) = e^{-x} + e^{-2x} - x \Rightarrow A = 1,\ -B = 1,\ 2C = -1,\ 3C + 2D = 0$
$\Rightarrow A = 1,\ B = -1,\ C = -\frac{1}{2},\ D = \frac{1}{4} \Rightarrow y = c_1e^{-x} + c_2e^{-2x} + x\,e^{-x} - x\,e^{-2x} - \frac{1}{2}x + \frac{1}{4}$

13. $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} = 15x^2 \Rightarrow r^2 + 5r = 0 \Rightarrow r(r+5) = 0 \Rightarrow r = 0$ or $r = -5 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^{-5x} = c_1 + c_2e^{-5x}$;
$y_p = Ax^3 + Bx^2 + Cx \Rightarrow y_p' = 3Ax^2 + 2Bx + C \Rightarrow y_p'' = 6Ax + 2B \Rightarrow 6Ax + 2B + 5(3Ax^2 + 2Bx + C) = 15x^2$
$\Rightarrow 15Ax^2 + (6A + 10B)x + (2B + 5C) = 15x^2 \Rightarrow 15A = 15,\ 6A + 10B = 0,\ 2B + 5C = 0 \Rightarrow A = 1,\ B = -\frac{3}{5},\ C = \frac{6}{25}$
$\Rightarrow y = c_1 + c_2e^{-5x} + x^3 - \frac{3}{5}x^2 + \frac{6}{25}x$

14. $\frac{d^2y}{dx^2} - \frac{dy}{dx} = -8x + 3 \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0$ or $r = 1 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^{1\cdot x} = c_1 + c_2e^x$;
$y_p = Ax^2 + Bx \Rightarrow y_p' = 2Ax + B \Rightarrow y_p'' = 2Ax \Rightarrow 2A - (2Ax + B) = -8x + 3 \Rightarrow -2Ax + (2A - B) = -8x + 3$
$\Rightarrow -2A = -8,\ 2A - B = 3 \Rightarrow A = 4,\ B = 5 \Rightarrow y = c_1 + c_2e^x + 4x^2 + 5x$

15. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = e^{3x} - 12x \Rightarrow r^2 - 3r = 0 \Rightarrow r(r-3) = 0 \Rightarrow r = 0$ or $r = 3 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^{3x} = c_1 + c_2e^{3x}$;
$y_p = Ax\,e^{3x} + Bx^2 + Cx \Rightarrow y_p' = Ae^{3x} + 3Ax\,e^{3x} + 2Bx + C \Rightarrow y_p'' = 6Ae^{3x} + 9Axe^{3x} + 2B$
$\Rightarrow 6Ae^{3x} + 9Axe^{3x} + 2B - 3(Ae^{3x} + 3Ax\,e^{3x} + 2Bx + C) = e^{3x} - 12x \Rightarrow 3Ae^{3x} - 6Bx + (2B - 3C) = e^{3x} - 12x$
$\Rightarrow 3A = 1,\ -6B = 12,\ 2B - 3C = 0 \Rightarrow A = \frac{1}{3},\ B = 2,\ C = \frac{4}{3} \Rightarrow y = c_1 + c_2e^{3x} + \frac{1}{3}x\,e^{3x} + 2x^2 + \frac{4}{3}x$

16. $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} = 42x^2 + 5x + 1 \Rightarrow r^2 + 7r = 0 \Rightarrow r(r+7) = 0 \Rightarrow r = 0$ or $r = -7 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^{-7x} = c_1 + c_2e^{-7x}$;
$y_p = Ax^3 + Bx^2 + Cx \Rightarrow y_p' = 3Ax^2 + 2Bx + C \Rightarrow y_p'' = 6Ax + 2B$
$\Rightarrow 6Ax + 2B + 7(3Ax^2 + 2Bx + C) = 42x^2 + 5x + 1 \Rightarrow 21Ax^2 + (6A + 14B)x + (2B + 7C) = 42x^2 + 5x + 1$
$\Rightarrow 21A = 42,\ 6A + 14B = 5,\ 2B + 7C = 1 \Rightarrow A = 2,\ B = -\frac{1}{2},\ C = \frac{2}{7} \Rightarrow y = c_1 + c_2e^{-7x} + 2x^3 - \frac{1}{1}x^2 + \frac{2}{7}x$

17. $y'' + y' = x, \Rightarrow r^2 + r = 0 \Rightarrow r(r+1) = 0 \Rightarrow r = 0$ or $r = -1 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^{-x} = c_1 + c_2e^{-x} \Rightarrow y_1 = 1,\ y_2 = e^{-x}$
$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix}} = \frac{-x\,e^{-x}}{-e^{-x}} = x$ and $v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & x \end{vmatrix}}{\begin{vmatrix} 1 & e^{-x} \\ 0 & -e^{-x} \end{vmatrix}} = \frac{x}{-e^{-x}} = -x\,e^x \Rightarrow v_1 = \int x\,dx = \frac{1}{2}x^2$ and
$v_2 = \int -x\,e^x\,dx = -x\,e^x + e^x \Rightarrow y_p = \frac{1}{2}x^2(1) + (-x\,e^x + e^x)e^{-x} = \frac{1}{2}x^2 - x + 1 \Rightarrow y = c_1 + c_2e^{-x} + \frac{1}{2}x^2 - x$

18. $y'' + y = \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1\cos x + c_2\sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \tan x}{1} = -\sin x \cdot \frac{\sin x}{\cos x} = -\frac{\sin^2 x}{\cos x}$ and $v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \tan x}{1} = \cos x \cdot \frac{\sin x}{\cos x} = \sin x$

$\Rightarrow v_1 = -\int \frac{\sin^2 x}{\cos x} dx = \int \frac{\cos^2 x - 1}{\cos x} dx = \int \left(\frac{\cos^2 x}{\cos x} - \frac{1}{\cos x}\right) dx = \int (\cos x - \sec x) dx = \sin x - \ln|\sec x + \tan x|$ and

$v_2 = \int \sin x \, dx = -\cos x \Rightarrow y_p = (\sin x - \ln|\sec x + \tan x|)(\cos x) + (-\cos x)(\sin x) = -(\cos x)\ln|\sec x + \tan x|$

$\Rightarrow y = c_1\cos x + c_2\sin x - (\cos x)\ln|\sec x + \tan x|$

19. $y'' + y = \sin x \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1\cos x + c_2\sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sin x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin^2 x}{1} = -\sin^2 x$ and $v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sin x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\sin x \cos x}{1} = \sin x \cos x$

$\Rightarrow v_1 = \int -\sin^2 x \, dx = \int \frac{\cos 2x - 1}{2} dx = \frac{1}{4}\sin 2x - \frac{1}{2}x$, and $v_2 = \int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x$

$\Rightarrow y_p = \left(\frac{1}{4}\sin 2x - \frac{1}{2}x\right)(\cos x) + \left(\frac{1}{2}\sin^2 x\right)(\sin x) = \frac{1}{2}\sin x \cos^2 x - \frac{1}{2}x\cos x + \frac{1}{2}\sin x - \frac{1}{2}\sin x \cos^2 x$

$= -\frac{1}{2}x\cos x + \frac{1}{2}\sin x \Rightarrow y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$

20. $y'' + 2y' + y = e^x \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$, repeated twice $\Rightarrow y_c = c_1e^{-x} + c_2x\,e^{-x}$

$\Rightarrow y_1 = e^{-x}, y_2 = x\,e^{-x}$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & x e^{-x} \\ e^x & e^{-x} - xe^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}} = \frac{-x}{e^{-2x}} = -x\,e^{2x}$ and $v_2' = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^x \end{vmatrix}}{\begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}} = \frac{1}{e^{-2x}} = e^{2x} \Rightarrow v_1 = -\int x\,e^{2x}\,dx = -\frac{1}{2}x\,e^{2x} + \frac{1}{4}e^{2x}$

and $v_2 = \int e^{2x}dx = \frac{1}{2}e^{2x} \Rightarrow y_p = \left(-\frac{1}{2}x\,e^{2x} + \frac{1}{4}e^{2x}\right)(e^{-x}) + \left(\frac{1}{2}e^{2x}\right)(x\,e^{-x}) = \frac{1}{4}e^x \Rightarrow y = c_1e^{-x} + c_2x\,e^{-x} + \frac{1}{4}e^x$

21. $y'' + 2y' + y = e^{-x} \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$, repeated twice $\Rightarrow y_c = c_1e^{-x} + c_2x\,e^{-x}$

$\Rightarrow y_1 = e^{-x}, y_2 = x\,e^{-x}$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & x e^{-x} \\ e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}} = \frac{-x\,e^{-2x}}{e^{-2x}} = -x$ and $v_2' = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - xe^{-x} \end{vmatrix}} = \frac{e^{-2x}}{e^{-2x}} = 1 \Rightarrow v_1 = -\int x\,dx = -\frac{1}{2}x^2$

and $v_2 = \int 1dx = x \Rightarrow y_p = \left(-\frac{1}{2}x^2\right)(e^{-x}) + (x)(x\,e^{-x}) = \frac{1}{2}x^2e^{-x} \Rightarrow y = c_1e^{-x} + c_2x\,e^{-x} + \frac{1}{2}x^2e^{-x}$

22. $y'' - y = x \Rightarrow r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = 1$ or $r = -1 \Rightarrow y_c = c_1e^x + c_2e^{-x} \Rightarrow y_1 = e^x, y_2 = e^{-x}$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{-x\,e^{-x}}{-2} = \frac{1}{2}x\,e^{-x}$ and $v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & x \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{x\,e^x}{-2} = -\frac{1}{2}x\,e^x \Rightarrow v_1 = \frac{1}{2}\int x\,e^{-x}\,dx = -\frac{1}{2}x\,e^{-x} - \frac{1}{2}e^{-x}$

and $v_2 = -\frac{1}{2}\int x\,e^x\,dx = -\frac{1}{2}x\,e^x + \frac{1}{2}e^x \Rightarrow y_p = \left(-\frac{1}{2}x\,e^{-x} - \frac{1}{2}e^{-x}\right)e^x + \left(-\frac{1}{2}x\,e^x + \frac{1}{2}e^x\right)e^{-x} = -x$

$\Rightarrow y = c_1e^x + c_2e^{-x} - x$

23. $y'' - y = e^x \Rightarrow r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = 1$ or $r = -1 \Rightarrow y_c = c_1e^x + c_2e^{-x} \Rightarrow y_1 = e^x, y_2 = e^{-x}$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{-1}{-2} = \frac{1}{2}$ and $v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & e^x \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{e^{2x}}{-2} = -\frac{1}{2}e^{2x} \Rightarrow v_1 = \int \frac{1}{2}\,dx = \frac{1}{2}x$ and $v_2 = -\frac{1}{2}\int e^{2x}\,dx = -\frac{1}{4}e^{2x}$

$\Rightarrow y_p = \left(\frac{1}{2}x\right)e^x + \left(-\frac{1}{4}e^{2x}\right)e^{-x} = \frac{1}{2}x\,e^x - \frac{1}{4}e^x \Rightarrow y = c_1e^x + c_2e^{-x} + \frac{1}{2}x\,e^x$

24. $y'' - y = \sin x \Rightarrow r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = 1 \text{ or } r = -1 \Rightarrow y_c = c_1e^x + c_2e^{-x} \Rightarrow y_1 = e^x, y_2 = e^{-x}$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ \sin x & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{-e^{-x}\sin x}{-2} = \frac{1}{2}e^{-x}\sin x$ and $v_2' = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \sin x \end{vmatrix}}{\begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}} = \frac{e^x\sin x}{-2} = -\frac{1}{2}e^x\sin x$

$\Rightarrow v_1 = \frac{1}{2}\int e^{-x}\sin x\,dx = -\frac{1}{4}e^{-x}\cos x - \frac{1}{4}e^{-x}\sin x$ and $v_2 = -\frac{1}{2}\int e^x\sin x\,dx = \frac{1}{4}e^x\cos x - \frac{1}{4}e^x\sin x$

$\Rightarrow y_p = \left(-\frac{1}{4}e^{-x}\cos x - \frac{1}{4}e^{-x}\sin x\right)e^x + \left(\frac{1}{4}e^x\cos x - \frac{1}{4}e^x\sin x\right)e^{-x} = -\frac{1}{2}\sin x \Rightarrow y = c_1e^x + c_2e^{-x}e^x - \frac{1}{2}\sin x$

25. $y'' + 4y' + 5y = 10 \Rightarrow r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4\pm\sqrt{4^2-4(1)(5)}}{2(1)} = -2 \pm i \Rightarrow y = e^{-2x}(c_1\cos x + c_2\sin x)$

$\Rightarrow y_1 = e^{-2x}\cos x, y_2 = e^{-2x}\sin x \Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{-2x}\sin x \\ 10 & e^{-2x}\cos x - 2e^{-2x}\sin x \end{vmatrix}}{\begin{vmatrix} e^{-2x}\cos x & e^{-2x}\sin x \\ -e^{-2x}\sin x - 2e^{-2x}\cos x & e^{-2x}\cos x - 2e^{-2x}\sin x \end{vmatrix}} = \frac{-10e^{-2x}\sin x}{e^{-4x}} = -10e^{2x}\sin x$

and $v_2' = \frac{\begin{vmatrix} e^{-2x}\cos x & 0 \\ -e^{-2x}\sin x - 2e^{-2x}\cos x & 10 \end{vmatrix}}{\begin{vmatrix} e^{-2x}\cos x & e^{-2x}\sin x \\ -e^{-2x}\sin x - 2e^{-2x}\cos x & e^{-2x}\cos x - 2e^{-2x}\sin x \end{vmatrix}} = \frac{10e^{-2x}\cos x}{e^{-4x}} = 10e^{2x}\cos x$

$\Rightarrow v_1 = -\int 10e^{2x}\sin x\,dx = 2e^{2x}\cos x - 4e^{2x}\sin x$ and $v_2 = \int 10e^{2x}\cos x\,dx = 2e^{2x}\sin x + 4e^{2x}\cos x$

$\Rightarrow y_p = (2e^{2x}\cos x - 4e^{2x}\sin x)(e^{-2x}\cos x) + (2e^{2x}\sin x + 4e^{2x}\cos x)(e^{-2x}\sin x) = 2 \Rightarrow y = e^{-2x}(c_1\cos x + c_2\sin x) + 2$

26. $y'' - y' = 2^x \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0 \text{ or } r = 1 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^x = c_1 + c_2e^x \Rightarrow y_1 = 1, y_2 = e^x$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^x \\ 2^x & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{-2^x e^x}{e^x} = -2^x$ and $v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 2^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{2^x}{e^x} = \left(\frac{2}{e}\right)^x \Rightarrow v_1 = -\int 2^x\,dx = \frac{1}{\ln 2}2^x$ and

$v_2 = \int \left(\frac{2}{e}\right)^x dx = \frac{1}{\ln 2 - 1}\left(\frac{2}{e}\right)^x \Rightarrow y_p = \left(\frac{1}{\ln 2}2^x\right)(1) + \left(\frac{1}{\ln 2 - 1}\left(\frac{2}{e}\right)^x\right)e^x = \left(\frac{1}{\ln 2} + \frac{1}{\ln 2 - 1}\right)2^x = \frac{\ln 2 - 1}{\ln 2(\ln 2 - 1)}2^x$

$\Rightarrow y = c_1 + c_2e^x + \frac{\ln 2 - 1}{\ln 2(\ln 2 - 1)}2^x$

27. $\frac{d^2y}{dx^2} + y = \sec x, -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1\cos x + c_2\sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x\sec x}{1} = -\tan x$ and $v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{1}{1} = \sin x \Rightarrow v_1 = \int -\tan x\,dx$

$= -\int \frac{\sin x}{\cos x}dx = \ln|\cos x|$ and $v_2 = \int 1\,dx = x \Rightarrow y_p = (\ln|\cos x|)(\cos x) + (x)(\sin x) = \cos x\ln|\cos x| + x\sin x$

$\Rightarrow y = c_1\cos x + c_2\sin x + \cos x\ln|\cos x| + x\sin x$

28. $\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x\cos x \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0 \text{ or } r = 1 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^x = c_1 + c_2e^x$

$\Rightarrow y_1 = 1, y_2 = e^x \Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^x \\ e^x\cos x & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{-e^{2x}\cos x}{e^x} = -e^x\cos x$ and $v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & e^x\cos x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{e^x\cos x}{e^x} = \cos x$

$\Rightarrow v_1 = -\int e^x\cos x\,dx = -\frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x$ and and $v_2 = \int \cos x\,dx = \sin x$

$\Rightarrow y_p = \left(-\frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x\right)(1) + (\sin x)(e^x) = \frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x \Rightarrow y = c_1 + c_2e^x + \frac{1}{2}e^x\sin x - \frac{1}{2}e^x\cos x$

29. $y'' - 5y' = x\,e^{5x}, y_p = Ax^2e^{5x} + Bx\,e^{5x} \Rightarrow y_p' = 5Ax^2e^{5x} + 2Ax\,e^{5x} + 5Bx\,e^{5x} + B\,e^{5x}$

$\Rightarrow y_p'' = 25Ax^2e^{5x} + 20Ax\,e^{5x} + 2Ae^{5x} + 25Bx\,e^{5x} + 10B\,e^{5x}$

$\Rightarrow (25Ax^2e^{5x} + 20Ax\,e^{5x} + 2Ae^{5x} + 25Bx\,e^{5x} + 10B\,e^{5x}) - 5(5Ax^2e^{5x} + 2Ax\,e^{5x} + 5Bx\,e^{5x} + B\,e^{5x}) = x\,e^{5x}$

$\Rightarrow 10Ax\,e^{5x} + (2A + 5B)\,e^{5x} = x\,e^{5x} \Rightarrow 10A = 1, 2A + 5B = 0 \Rightarrow A = \frac{1}{10}, B = -\frac{1}{25} \Rightarrow y_p = \frac{1}{10}x^2e^{5x} - \frac{1}{25}x\,e^{5x};$

$r^2 - 5r = 0 \Rightarrow r(r-5) = 0 \Rightarrow r = 0 \text{ or } r = 5 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^{5x} = c_1 + c_2e^{5x}$

$\Rightarrow y = c_1 + c_2e^{5x} + \frac{1}{10}x^2e^{5x} - \frac{1}{25}x\,e^{5x}$

30. $y'' - y' = \cos x + \sin x$, $y_p = A\cos x + B\sin x \Rightarrow y_p' = -A\sin x + B\cos x \Rightarrow y_p'' = -A\cos x - B\sin x$
$\Rightarrow -A\cos x - B\sin x - (-A\sin x + B\cos x) = \cos x + \sin x \Rightarrow (A-B)\sin x + (-A-B)\cos x = \cos x + \sin x$
$\Rightarrow A - B = 1, -A - B = 1 \Rightarrow A = 0, B = -1 \Rightarrow y_p = -\sin x$; $r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0$ or $r = 1$
$\Rightarrow y_c = c_1 e^{0\cdot x} + c_2 e^x = c_1 + c_2 e^x \Rightarrow y = c_1 + c_2 e^{5x} - \sin x$

31. $y'' + y = 2\cos x + \sin x$, $y_p = Ax\sin x + Bx\cos x \Rightarrow y_p' = Ax\cos x + A\sin x - Bx\sin x + B\cos x$
$\Rightarrow y_p'' = -Ax\sin x + 2A\cos x - Bx\cos x - 2B\sin x$
$\Rightarrow (-Ax\sin x + 2A\cos x - Bx\cos x - 2B\sin x) + (Ax\sin x + Bx\cos x) = \cos x + \sin x$
$\Rightarrow -2B\sin x + 2A\cos x = 2\cos x + \sin x \Rightarrow -2B = 1, 2A = 2 \Rightarrow A = 1, B = -\frac{1}{2} \Rightarrow y_p = x\sin x - \frac{1}{2}x\cos x$;
$r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1\cos x + c_2\sin x \Rightarrow y = c_1\cos x + c_2\sin x + x\sin x - \frac{1}{2}x\cos x$

32. $y'' + y' - 2y = x e^x$, $y_p = Ax^2e^x + Bx e^x \Rightarrow y_p' = Ax^2e^x + 2Ax e^x + Bx e^x + B e^x$
$\Rightarrow y_p'' = Ax^2e^x + 4Ax e^x + 2Ae^x + Bx e^x + 2B e^x$
$\Rightarrow (Ax^2e^x + 4Ax e^x + 2Ae^x + Bx e^x + 2B e^x) + (Ax^2e^x + 2Ax e^x + Bx e^x + B e^x) - 2(Ax^2e^x + Bx e^x) = x e^x$
$\Rightarrow 6Ax e^x + (2A + 3B) e^x = x e^{5x} \Rightarrow 6A = 1, 2A + 3B = 0 \Rightarrow A = \frac{1}{6}, B = -\frac{1}{9} \Rightarrow y_p = \frac{1}{6}x^2e^x - \frac{1}{9}x e^x$;
$r^2 + r - 2 = 0 \Rightarrow (r+2)(r-1) = 0 \Rightarrow r = -2$ or $r = 1 \Rightarrow y_c = c_1e^{-2x} + c_2e^x \Rightarrow y = c_1e^{-2x} + c_2e^x + \frac{1}{6}x^2e^x - \frac{1}{9}x e^x$

33. $\frac{d^2y}{dx^2} - \frac{dy}{dx} = e^x + e^{-x} \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0$ or $r = 1 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^x = c_1 + c_2e^x$

(a) $y_1 = 1, y_2 = e^x \Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^x \\ e^x+e^{-x} & e^x \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{-e^{2x}-1}{e^x} = -e^x - e^{-x}$ and $v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & e^x+e^{-x} \end{vmatrix}}{\begin{vmatrix} 1 & e^x \\ 0 & e^x \end{vmatrix}} = \frac{e^x+e^{-x}}{e^x} = 1 + e^{-2x}$
$\Rightarrow v_1 = \int(-e^x - e^{-x})\,dx = -e^x + e^{-x}$ and $v_2 = \int(1 + e^{-2x})\,dx = x - \frac{1}{2}e^{-2x}$
$\Rightarrow y_p = (-e^x + e^{-x})(1) + \left(x - \frac{1}{2}e^{-2x}\right)(e^x) = -e^x + x e^x + \frac{1}{2}e^{-x} \Rightarrow y = c_1 + c_2e^x x e^x + \frac{1}{2}e^{-x}$

(b) $y_p = Ax e^x + B e^{-x} \Rightarrow y_p' = Ax e^x + A e^x - B e^{-x} \Rightarrow y_p'' = Ax e^x + 2A e^x + B e^{-x}$
$\Rightarrow (Ax e^x + 2A e^x + B e^{-x}) - (Ax e^x + A e^x - B e^{-x}) = e^x + e^{-x} \Rightarrow A e^x + 2B e^{-x} = e^x + e^{-x}$
$\Rightarrow A = 1, 2B = 1 \Rightarrow A = 1, B = \frac{1}{2} \Rightarrow y = c_1 + c_2e^x + x e^x + \frac{1}{2}e^{-x}$

34. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2e^{2x} \Rightarrow r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r = 2$, repeated twice $\Rightarrow y_c = c_1e^{2x} + c_2x e^{2x}$

(a) $y_1 = e^{2x}, y_2 = x e^{2x} \Rightarrow v_1' = \frac{\begin{vmatrix} 0 & x e^{2x} \\ 2e^{2x} & e^{2x}+2x e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x}+2x e^{2x} \end{vmatrix}} = \frac{-2x e^{4x}}{e^{4x}} = -2x$ and $v_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & x e^{2x} \\ 2e^{2x} & e^{2x}+2x e^{2x} \end{vmatrix}} = \frac{2e^{4x}}{e^{4x}} = 2$
$\Rightarrow v_1 = \int(-2x)\,dx = -x^2$ and $v_2 = \int 2\,dx = 2x \Rightarrow y_p = (-x^2)(e^{2x}) + (2x)(x e^{2x}) = x^2 e^x$
$\Rightarrow y = c_1e^{2x} + c_2x e^{2x} + x^2 e^x$

(b) $y_p = Ax^2 e^{2x} \Rightarrow y_p' = 2Ax e^{2x} + 2Ax^2 e^{2x} \Rightarrow y_p'' = 2A e^{2x} + 8Ax e^{2x} + 4Ax^2 e^{2x}$
$\Rightarrow (2A e^{2x} + 8Ax e^{2x} + 4Ax^2 e^{2x}) - 4(2Ax e^{2x} + 2Ax^2 e^{2x}) + 4Ax^2 e^{2x} = 2e^{2x} \Rightarrow 2A e^{2x} = 2e^{2x}$
$\Rightarrow 2A = 2 \Rightarrow A = 1 \Rightarrow y = c_1e^{2x} + c_2x e^{2x} + x^2 e^x$

35. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 5y = e^x + 4 \Rightarrow r^2 - 4r - 5 = 0 \Rightarrow (r-5)(r+1) = 0 \Rightarrow r = 5$ or $r = -1 \Rightarrow y_c = c_1e^{5x} + c_2\,e^{-x}$

(a) $y_1 = e^{5x}, y_2 = e^{-x} \Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{-x} \\ e^x+4 & -e^{-x} \end{vmatrix}}{\begin{vmatrix} e^{5x} & e^{-x} \\ 5e^{5x} & -e^{-x} \end{vmatrix}} = \frac{-1-4e^{-x}}{-6e^{4x}} = \frac{1}{6}e^{-4x} + \frac{2}{3}e^{-5x}$ and $v_2' = \frac{\begin{vmatrix} e^{5x} & 0 \\ 5e^{5x} & e^x+4 \end{vmatrix}}{\begin{vmatrix} e^{5x} & e^{-x} \\ 5e^{5x} & -e^{-x} \end{vmatrix}} = \frac{e^{6x}+4e^{5x}}{-6e^{4x}} = -\frac{1}{6}e^{2x} - \frac{2}{3}e^x$
$\Rightarrow v_1 = \int\left(\frac{1}{6}e^{-4x} + \frac{2}{3}e^{-5x}\right)dx = -\frac{1}{24}e^{-4x} - \frac{2}{15}e^{-5x}$ and $v_2 = \int\left(-\frac{1}{6}e^{2x} - \frac{2}{3}e^x\right)dx = -\frac{1}{12}e^{2x} - \frac{2}{3}e^x$
$\Rightarrow y_p = \left(-\frac{1}{24}e^{-4x} - \frac{2}{15}e^{-5x}\right)(e^{5x}) + \left(-\frac{1}{12}e^{2x} - \frac{2}{3}e^x\right)(e^{-x}) = -\frac{1}{8}e^x - \frac{4}{5} \Rightarrow y = c_1e^{5x} + c_2\,e^{-x} - \frac{1}{8}e^x - \frac{4}{5}$

(b) $y_p = Ae^x + B \Rightarrow y_p' = Ae^x \Rightarrow y_p'' = Ae^x \Rightarrow Ae^x - 4Ae^x + -5(Ae^x + B) = e^x + 4 \Rightarrow -8A e^x - 5B = e^x + 4$
$\Rightarrow -8A = 1, -5B = 4 \Rightarrow A = -\frac{1}{8}, B = -\frac{4}{5} \Rightarrow y = c_1e^{5x} + c_2\,e^{-x} - \frac{1}{8}e^x - \frac{4}{5}$

36. $\frac{d^2y}{dx^2} - 9\frac{dy}{dx} = 9e^{9x} \Rightarrow r^2 - 9r = 0 \Rightarrow r(r-9) = 0 \Rightarrow r = 0 \text{ or } r = 9 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^{9x} = c_1 + c_2e^{9x}$

(a) $y_1 = 1, y_2 = e^{9x} \Rightarrow v_1' = \frac{\begin{vmatrix} 0 & e^{9x} \\ 9e^{9x} & 9e^{9x} \end{vmatrix}}{\begin{vmatrix} 1 & e^{9x} \\ 0 & 9e^{9x} \end{vmatrix}} = \frac{-9e^{18x}}{9e^{9x}} = -e^{9x}$ and $v_2' = \frac{\begin{vmatrix} 1 & 0 \\ 0 & 9e^{9x} \end{vmatrix}}{\begin{vmatrix} 1 & e^{9x} \\ 0 & 9e^{9x} \end{vmatrix}} = \frac{9e^{9x}}{9e^{9x}} = 1$

$\Rightarrow v_1 = \int(-e^{9x})\,dx = -\frac{1}{9}e^{9x}$ and $v_2 = \int(1)\,dx = x \Rightarrow y_p = \left(-\frac{1}{9}e^{9x}\right)(1) + (x)(e^{9x}) = -\frac{1}{9}e^{9x} + x\,e^{9x}$

$\Rightarrow y = c_1 + c_2e^{9x} + x\,e^{9x}$

(b) $y_p = Ax\,e^{9x} \Rightarrow y_p' = 9Ax\,e^{9x} + A\,e^{9x} \Rightarrow y_p'' = 81Ax\,e^{9x} + 18A\,e^{9x}$

$\Rightarrow (81Ax\,e^{9x} + 18A\,e^{9x}) - 9(9Ax\,e^{9x} + A\,e^{9x}) = 9e^{9x} \Rightarrow 9A\,e^{9x} = 9e^{9x} \Rightarrow 9A = 9 \Rightarrow A = 1$

$\Rightarrow y = c_1 + c_2e^{9x} + x\,e^{9x}$

37. $y'' + y = \cot x,\ 0 < x < \pi \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1\cos x + c_2\sin x \Rightarrow y_1 = \cos x,\ y_2 = \sin x$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \cot x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x\cot x}{1} = -\sin x\,\frac{\cos x}{\sin x} = -\cos x$ and $v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \cot x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x\cot x}{1} = \cos x\cdot\frac{\cos x}{\sin x} = \frac{\cos^2 x}{\sin x}$

$\Rightarrow v_1 = -\int\cos x\,dx = \sin x$ and $v_2 = \int\frac{\cos^2 x}{\sin x}dx = \int\frac{1-\sin^2 x}{\sin x}dx = \int\left(\frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}\right)dx = \int\left(\csc x - \sin x\right)dx$

$= \ln|\csc x + \cot x| - \sin x \Rightarrow y_p = (\sin x)(\cos x) + (\ln|\csc x + \cot x| - \sin x)(\sin x) = -(\sin x)\ln|\csc x + \cot x|$

$\Rightarrow y = c_1\cos x + c_2\sin x - (\sin x)\ln|\csc x + \cot x|$

38. $y'' + y = \csc x,\ 0 < x < \pi \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1\cos x + c_2\sin x \Rightarrow y_1 = \cos x,\ y_2 = \sin x$

$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \csc x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x\csc x}{1} = -\sin x\,\frac{1}{\sin x} = -1$ and $v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \csc x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x\csc x}{1} = \cos x\cdot\frac{1}{\sin x} = \frac{\cos x}{\sin x}$

$\Rightarrow v_1 = \int(-1)\,dx = -x$ and $v_2 = \int\frac{\cos x}{\sin x}dx = \ln|\sin x| \Rightarrow y_p = (-x)(\cos x) + (\ln|\sin x|)(\sin x)$

$= -x\cos x + (\sin x)\ln|\sin x| \Rightarrow y = c_1\cos x + c_2\sin x - x\cos x + (\sin x)\ln|\sin x|$

39. $y'' - 8y' = e^{8x} \Rightarrow r^2 - 8r = 0 \Rightarrow r(r-8) = 0 \Rightarrow r = 0 \text{ or } r = 8 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^{8x} = c_1 + c_2e^{8x}$;

$y_p = Ax\,e^{8x} \Rightarrow y_p' = Ae^{8x} + 8Ax\,e^{8x} \Rightarrow y_p'' = 16Ae^{8x} + 64Ax\,e^{x} \Rightarrow (16Ae^{8x} + 64Ax\,e^{x}) - 8(Ae^{8x} + 8Ax\,e^{8x}) = e^{8x}$

$\Rightarrow 8Ae^{8x} = e^{8x} \Rightarrow 8A = 1 \Rightarrow A = \frac{1}{8} \Rightarrow y = c_1 + c_2e^{8x} + \frac{1}{8}x\,e^{8x}$

40. $y'' + 4y = \sin x \Rightarrow r^2 + 4 = 0 \Rightarrow r = 0 \pm 2i \Rightarrow y_c = e^{0\cdot x}(c_1\cos 2x + c_2\sin 2x) = c_1\cos 2x + c_2\sin 2x$;

$y_p = A\sin x + B\cos x \Rightarrow y_p' = A\cos x - B\sin x \Rightarrow y_p'' = -A\sin x - B\cos x$

$\Rightarrow (-A\sin x - B\cos x) + 4(A\sin x + B\cos x) = \sin x \Rightarrow 3A\sin x + 3B\cos x = \sin x \Rightarrow 3A = 1,\ 3B = 0$

$\Rightarrow A = \frac{1}{3},\ B = 0 \Rightarrow y = c_1\cos 2x + c_2\sin 2x + \frac{1}{3}\sin x$

41. $y'' - y' = x^3 \Rightarrow r^2 - r = 0 \Rightarrow r(r-1) = 0 \Rightarrow r = 0 \text{ or } r = 1 \Rightarrow y_c = c_1e^{0\cdot x} + c_2e^{x} = c_1 + c_2e^{x}$;

$y_p = Ax^4 + Bx^3 + Cx^2 + Dx \Rightarrow y_p' = 4Ax^3 + 3Bx^2 + 2Cx + D \Rightarrow y_p'' = 12Ax^2 + 6Bx + 2C$

$\Rightarrow (12Ax^2 + 6Bx + 2C) - (4Ax^3 + 3Bx^2 + 2Cx + D) = x^3$

$\Rightarrow -4Ax^3 + (12A - 3B)x^2 + (6B - 2C)x + (2C - D) = x^3 \Rightarrow -4A = 1,\ 12A - 3B = 0,\ 6B - 2C = 0,\ 2C - D = 0$

$\Rightarrow A = -\frac{1}{4},\ B = -1,\ C = -3,\ D = -6 \Rightarrow y = c_1 + c_2e^{x} - \frac{1}{4}x^4 - x^3 - 3x^2 - 6x$

42. $y'' + 4y' + 5y = x + 2 \Rightarrow r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4\pm\sqrt{4^2-4(1)(5)}}{2(1)} = -2 \pm i \Rightarrow y_c = e^{-2x}(c_1\cos x + c_2\sin x)$

$y_p = Ax + B \Rightarrow y_p' = A \Rightarrow y_p'' = 0 \Rightarrow 0 + 4A + 5(Ax + B) = x + 2 \Rightarrow 5Ax + (4A + 5B) = x + 2 \Rightarrow 5A = 1,$

$4A + 5B = 2 \Rightarrow A = \frac{1}{5},\ B = \frac{6}{25} \Rightarrow y = e^{-2x}(c_1\cos x + c_2\sin x) + \frac{1}{5}x + \frac{6}{25}$

43. $y'' + 2y' = x^2 - e^x \Rightarrow r^2 + 2r = 0 \Rightarrow r(r+2) = 0 \Rightarrow r = 0$ or $r = -2 \Rightarrow y_c = c_1 e^{0 \cdot x} + c_2 e^{-2x} = c_1 + c_2 e^{-2x}$;
$y_p = Ax^3 + Bx^2 + Cx + De^x \Rightarrow y_p' = 3Ax^2 + 2Bx + C + De^x \Rightarrow y_p'' = 6Ax + 2B + De^x$
$\Rightarrow (6Ax + 2B + De^x) + 2(3Ax^2 + 2Bx + C + De^x) = x^2 - e^x \Rightarrow 6Ax^2 + (6A + 4B)x + (2B + 2C) + 3De^x = x^2 - e^x$
$\Rightarrow 6A = 1, 6A + 4B = 0, 2B + 2C = 0, 3D = -1 \Rightarrow A = \frac{1}{6}, B = -\frac{1}{4}, C = \frac{1}{8}, D = -\frac{1}{3}$
$\Rightarrow y = c_1 + c_2 e^{-2x} + \frac{1}{6}x^3 - \frac{1}{4}x^2 + \frac{1}{8}x - \frac{1}{3}e^x$

44. $y'' + 9y = 9x - \cos x \Rightarrow r^2 + 9 = 0 \Rightarrow r = 0 \pm 3i \Rightarrow y_c = e^{0 \cdot x}(c_1 \cos 3x + c_2 \sin 3x) = c_1 \cos 3x + c_2 \sin 3x$;
$y_p = Ax + B + C \sin x + D \cos x \Rightarrow y_p' = A + C \cos x - D \sin x \Rightarrow y_p'' = -C \sin x - D \cos x$
$\Rightarrow (-C \sin x - D \cos x) + 9(Ax + B + C \sin x + D \cos x) = 9x - \cos x \Rightarrow 9Ax + 9B + 8C \sin x + 8D \cos x = 9x - \cos x$
$\Rightarrow 9A = 9, 9B = 0, 8C = 0, 8D = -1 \Rightarrow A = 1, B = 0, C = 0, D = -\frac{1}{8} \Rightarrow y = c_1 \cos 3x + c_2 \sin 3x + x - \frac{1}{8}\cos x$

45. $\frac{d^2y}{dx^2} + y = \sec x \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y_1 = \cos x, y_2 = \sin x$
$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \sec x \tan x}{1} = -\tan^2 x$ and $v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \sec x \tan x}{1} = \tan x$
$\Rightarrow v_1 = \int -\tan^2 x \, dx = \int (1 - \sec^2 x) dx = x - \tan x$ and $v_2 = \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x|$
$\Rightarrow y_p = (x - \tan x)(\cos x) + (-\ln|\cos x|)(\sin x) = x \cos x - \sin x - (\sin x)\ln|\cos x|$
$\Rightarrow y = c_1 \cos x + c_2 \sin x + x \cos x - (\sin x)\ln|\cos x|$

46. $y'' - 3y' + 2y = e^x - 2e^{2x} \Rightarrow r^2 - 3r + 2 = 0 \Rightarrow (r-1)(r-2) = 0 \Rightarrow r = 1$ or $r = 2 \Rightarrow y_c = c_1 e^x + c_2 e^{2x}$;
$y_p = Ax e^{2x} + Bx e^x \Rightarrow y_p' = 2Ax e^{2x} + A e^{2x} + Bx e^x + Be^x \Rightarrow y_p'' = 4Ax e^{2x} + 4A e^{2x} + Bx e^x + 2Be^x$
$\Rightarrow (4Ax e^{2x} + 4A e^{2x} + Bx e^x + 2Be^x) - 3(2Ax e^{2x} + A e^{2x} + Bx e^x + Be^x) + 2(Ax e^{2x} + Bx e^x) = e^x - 2e^{2x}$
$\Rightarrow A e^{2x} - Be^x = e^x - 2e^{2x} \Rightarrow A = -2, -B = 1 \Rightarrow A = -2, B = -1 \Rightarrow y = c_1 e^x + c_2 e^{2x} - 2x e^{2x} - x e^x$

47. $y' - 3y = e^x \Rightarrow r - 3 = 0 \Rightarrow r = 3 \Rightarrow y_c = c_1 e^{3x}; y_p = Ae^x \Rightarrow y_p' = Ae^x \Rightarrow Ae^x - 3Ae^x = e^x \Rightarrow -2A e^x = e^x$
$\Rightarrow -2A = 1 \Rightarrow A = -\frac{1}{2} \Rightarrow y = c_1 e^{3x} - \frac{1}{2}e^x$

48. $y' + 4y = x \Rightarrow r + 4 = 0 \Rightarrow r = -4 \Rightarrow y_c = c_1 e^{-4x}; y_p = Ax + B \Rightarrow y_p' = A \Rightarrow A + 4(Ax + B) = x$
$\Rightarrow 4Ax + (A + 4B) = x \Rightarrow 4A = 1, A + 4B = 0 \Rightarrow A = \frac{1}{4}, B = -\frac{1}{16} \Rightarrow y = c_1 e^{-4x} + \frac{1}{4}x - \frac{1}{16}$

49. $y' - 3y = 5e^{3x} \Rightarrow r - 3 = 0 \Rightarrow r = 3 \Rightarrow y_c = c_1 e^{3x}; y_p = Ax e^{3x} \Rightarrow y_p' = 3Ax e^{3x} + A e^{3x}$
$\Rightarrow (3Ax e^{3x} + A e^{3x}) - 3Ax e^{3x} = 5e^{3x} \Rightarrow A e^{3x} = 5e^{3x} \Rightarrow A = 5 \Rightarrow y = c_1 e^{3x} + 5x e^{3x}$

50. $y' + y = \sin x \Rightarrow r + 1 = 0 \Rightarrow r = -1 \Rightarrow y_c = c_1 e^{-x}; y_p = A \sin x + B \cos x \Rightarrow y_p' = A \cos x - B \sin x$
$\Rightarrow (A \cos x - B \sin x) + (A \sin x + B \cos x) = \sin x \Rightarrow (A - B)\sin x + (A + B)\cos x = \sin x \Rightarrow A - B = 1, A + B = 0$
$\Rightarrow A = \frac{1}{2}, B = -\frac{1}{2} \Rightarrow y = c_1 e^{-x} + \frac{1}{2}\sin x - \frac{1}{2}\cos x$

51. $\frac{d^2y}{dx^2} + y = \sec^2 x, -\frac{\pi}{2} < x < \frac{\pi}{2}, y(0) = y'(0) = 1 \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x$
$\Rightarrow y_1 = \cos x, y_2 = \sin x$
$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec^2 x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{-\sin x \sec^2 x}{1} = -\sec x \tan x$ and $v_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec^2 x \end{vmatrix}}{\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}} = \frac{\cos x \sec^2 x}{1} = \sec x$
$\Rightarrow v_1 = \int -\sec x \tan x \, dx = -\sec x$ and $v_2 = \int \sec x \, dx = \ln|\sec x + \tan x|$
$\Rightarrow y_p = (-\sec x)(\cos x) + (\ln|\sec x + \tan x|)(\sin x) = -1 + (\sin x)\ln|\sec x + \tan x|$
$\Rightarrow y = c_1 \cos x + c_2 \sin x - 1 + (\sin x)\ln|\sec x + \tan x|; \; y(0) = 1 \Rightarrow 1 = c_1 - 1 \Rightarrow c_1 = 2$;

$\frac{dy}{dx} = -c_1 \sin x + c_2 \cos x + (\cos x)\ln|\sec x + \tan x| + (\sin x)\sec x,\ y'(0) = 1 \Rightarrow 1 = c_2$
$\Rightarrow y = 2\cos x + \sin x - 1 + (\sin x)\ln|\sec x + \tan x|$

52. $\frac{d^2y}{dx^2} + y = e^{2x},\ y(0) = 0,\ y'(0) = \frac{2}{5} \Rightarrow r^2 + 1 = 0 \Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x$
$y_p = A\,e^{2x} \Rightarrow y_p' = 2A\,e^{2x} \Rightarrow y_p'' = 4A\,e^{2x} \Rightarrow 4A\,e^{2x} + A\,e^{2x} = e^{2x} \Rightarrow 5A\,e^{2x} = e^{2x} \Rightarrow 5A = 1 \Rightarrow A = \frac{1}{5}$
$\Rightarrow y = c_1 \cos x + c_2 \sin x + \frac{1}{5}e^{2x} \Rightarrow y' = -c_1 \sin x + c_2 \cos x + \frac{2}{5}e^{2x};\ y(0) = 0 \Rightarrow c_1 + \frac{1}{5} = 0 \Rightarrow c_1 = -\frac{1}{5};$
$y'(0) = \frac{2}{5} \Rightarrow c_2 + \frac{2}{5} = \frac{2}{5} \Rightarrow c_2 = 0 \Rightarrow y = -\frac{1}{5}\cos x + \frac{1}{5}e^{2x}$

53. $y'' + y' = x,\ y_p = \frac{x^2}{2} - x,\ y(0) = 0,\ y'(0) = 0;\ y_p = \frac{x^2}{2} - x \Rightarrow y_p' = x - 1 \Rightarrow y_p'' = 1 \Rightarrow y'' + y' = 1 + x - 1 = x$
$\Rightarrow x = x \Rightarrow y_p$ satisfies the differential equation. $y'' + y' = 0 \Rightarrow r^2 + r = 0 \Rightarrow r(r+1) = 0 \Rightarrow r = 0$ or $r = -1$
$\Rightarrow y_c = c_1 e^{0 \cdot x} + c_2 e^{-x} = c_1 + c_2 e^{-x} \Rightarrow y = c_1 + c_2 e^{-x} + \frac{x^2}{2} - x \Rightarrow y' = -c_2 e^{-x} + x - 1;$
$y(0) = 0 \Rightarrow 0 = c_1 + c_2,\ y'(0) = 0 \Rightarrow -c_2 - 1 = 0 \Rightarrow c_1 = 1,\ c_2 = -1 \Rightarrow y = 1 - e^{-x} + \frac{x^2}{2} - x$

54. $y'' + y = x,\ y_p = 2\sin x + x,\ y(0) = 0,\ y'(0) = 0;\ y_p = 2\sin x + x \Rightarrow y_p' = 2\cos x + 1 \Rightarrow y_p'' = -2\sin x$
$\Rightarrow y'' + y = (-2\sin x) + (2\sin x + x) = x \Rightarrow x = x \Rightarrow y_p$ satisfies the differential equation. $y'' + y = 0 \Rightarrow r^2 + 1 = 0$
$\Rightarrow r = 0 \pm i \Rightarrow y_c = c_1 \cos x + c_2 \sin x \Rightarrow y = c_1 \cos x + c_2 \sin x + 2\sin x + x \Rightarrow y' = -c_1 \sin x + c_2 \cos x + 2\cos x + 1;$
$y(0) = 0 \Rightarrow 0 = c_1;\ y'(0) = 0 \Rightarrow 0 = c_2 + 3 \Rightarrow c_2 = -3 \Rightarrow y = -3\sin x + 2\sin x + x = -\sin x + x$

55. $\frac{1}{2}y'' + y' + y = 4e^x(\cos x - \sin x),\ y_p = 2e^x \cos x,\ y(0) = 0,\ y'(0) = 1;\ y_p = 2e^x \cos x \Rightarrow y_p' = 2e^x \cos x - 2e^x \sin x$
$\Rightarrow y_p'' = -4e^x \sin x \Rightarrow \frac{1}{2}y'' + y' + y = \frac{1}{2}(-4e^x \sin x) + (2e^x \cos x - 2e^x \sin x) + 2e^x \cos x = 4e^x \cos x - 4e^x \sin x$
$\Rightarrow 4e^x \cos x - 4e^x \sin x = 4e^x(\cos x - \sin x) \Rightarrow y_p$ satisfies the differential equation. $\frac{1}{2}y'' + y' + y = 0 \Rightarrow \frac{1}{2}r^2 + r + 1 = 0$
$\Rightarrow r = \frac{-1 \pm \sqrt{1^2 - 4\left(\frac{1}{2}\right)(1)}}{2\left(\frac{1}{2}\right)} = -1 \pm i \Rightarrow y_c = e^{-x}(c_1 \cos x + c_2 \sin x) \Rightarrow y = e^{-x}(c_1 \cos x + c_2 \sin x) + 2e^x \cos x$
$\Rightarrow y' = -e^{-x}(c_1 \cos x + c_2 \sin x) + e^{-x}(-c_1 \sin x + c_2 \cos x) + 2e^x \cos x - 2e^x \sin x;\ y(0) = 0 \Rightarrow c_1 + 2 = 0 \Rightarrow c_1 = -2;$
$y'(0) = 1 \Rightarrow -c_1 + c_2 + 2 = 1 \Rightarrow c_2 = -3 \Rightarrow y = e^{-x}(-2\cos x - 3\sin x) + 2e^x \cos x = 2(e^x - e^{-x})\cos x - 3e^{-x}\sin x$

56. $y'' - y' - 2y = 1 - 2x,\ y_p = x - 1,\ y(0) = 0,\ y'(0) = 1;\ y_p = x - 1 \Rightarrow y_p' = 1 \Rightarrow y_p'' = 0$
$\Rightarrow y'' - y' - 2y = 0 - 1 - 2(x - 1) = 1 - 2x \Rightarrow 1 - 2x = 1 - 2x \Rightarrow y_p$ satisfies the differential equation.
$y'' - y' - 2y = 0 \Rightarrow r^2 - r - 2 = 0 \Rightarrow (r - 2)(r + 1) = 0 \Rightarrow r = 2$ or $r = -1 \Rightarrow y_c = c_1 e^{2x} + c_2\, e^{-x}$
$\Rightarrow y = c_1 e^{2x} + c_2\, e^{-x} + x - 1 \Rightarrow y' = 2c_1 e^{2x} - c_2\, e^{-x} + 1;\ y(0) = 0 \Rightarrow c_1 + c_2 - 1 = 0 \Rightarrow c_1 + c_2 = 1,$
$y'(0) = 1 \Rightarrow 2c_1 - c_2 + 1 = 1 \Rightarrow 2c_1 - c_2 = 0.$ Thus $c_1 + c_2 = 1,\ 2c_1 - c_2 = 0 \Rightarrow c_1 = \frac{1}{3},\ c_2 = \frac{2}{3}$
$\Rightarrow y = \frac{1}{3}e^{2x} + \frac{2}{3}e^{-x} + x - 1$

57. $y'' - 2y' + y = 2e^x,\ y_p = x^2 e^x,\ y(0) = 1,\ y'(0) = 0;\ y_p = x^2 e^x \Rightarrow y_p' = x^2 e^x + 2x\,e^x \Rightarrow y_p'' = x^2 e^x + 4x\,e^x + 2\,e^x$
$\Rightarrow y'' - 2y' + y = (x^2 e^x + 4x\,e^x + 2\,e^x) - 2(x^2 e^x + 2x\,e^x) + x^2 e^x = x^2 e^x \Rightarrow x^2 e^x = x^2 e^x \Rightarrow y_p$ satisfies the differential equation. $y'' - 2y' + y = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r - 1)^2 = 0 \Rightarrow r = 1$, repeated twice $\Rightarrow y_c = c_1 e^x + c_2\, xe^x$
$\Rightarrow y = c_1 e^x + c_2\, xe^x + x^2 e^x \Rightarrow y' = c_1 e^x + c_2\, xe^x + c_2 e^x + x^2 e^x + 2x\,e^x;\ y(0) = 1 \Rightarrow c_1 = 1;\ y'(0) = 0 \Rightarrow c_1 + c_2 = 0$
$\Rightarrow c_2 = -1 \Rightarrow y = e^x - xe^x + x^2 e^x$

58. $y'' - 2y' + y = x^{-1}e^x,\ x > 0,\ y_p = x\,e^x \ln x,\ y(1) = e,\ y'(1) = 0;\ y_p = x\,e^x \ln x \Rightarrow y_p' = e^x + x\,e^x \ln x + e^x \ln x$
$\Rightarrow y_p'' = 2\,e^x + x\,e^x \ln x + e^x \ln x + \frac{e^x}{x}$
$\Rightarrow y'' - 2y' + y = \left(2\,e^x + x\,e^x \ln x + e^x \ln x + \frac{e^x}{x}\right) - 2(e^x + x\,e^x \ln x + e^x \ln x) + x\,e^x \ln x = \frac{e^x}{x} \Rightarrow \frac{e^x}{x} = x^{-1}e^x$
$\Rightarrow y_p$ satisfies the differential equation. $y'' - 2y' + y = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r - 1)^2 = 0 \Rightarrow r = 1$, repeated twice
$\Rightarrow y_c = c_1 e^x + c_2\, xe^x \Rightarrow y = c_1 e^x + c_2\, xe^x + x\,e^x \ln x \Rightarrow y' = c_1 e^x + c_2\, xe^x + c_2 e^x + e^x + x\,e^x \ln x + e^x \ln x;$

$y(1) = e \Rightarrow c_1 e + c_2 e = e$, $y'(1) = 0 \Rightarrow c_1 e + 2c_2 e + e = 0 \Rightarrow c_1 e + 2c_2 e = -e \Rightarrow c_1 + c_2 = 1$, $c_1 + 2c_2 = -1$ $\Rightarrow c_1 = 3, c_2 = -2 \Rightarrow y = 3e^x - 2xe^x + x e^x \ln x$

59. $x^2y'' + 2xy' - 2y = x^2$, $y_1 = x^{-2}$, $y_2 = x$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & x \\ \frac{x^2}{x^2} & 1 \end{vmatrix}}{\begin{vmatrix} x^{-2} & x \\ -2x^{-3} & 1 \end{vmatrix}} = \frac{-x}{3x^{-2}} = -\frac{1}{3}x^3 \text{ and } v_2' = \frac{\begin{vmatrix} x^{-2} & 0 \\ -2x^{-3} & \frac{x^2}{x^2} \end{vmatrix}}{\begin{vmatrix} x^{-2} & x \\ -2x^{-3} & 1 \end{vmatrix}} = \frac{x^{-2}}{3x^{-2}} = \frac{1}{3}$$

$$\Rightarrow v_1 = \int -\frac{1}{3}x^3\,dx = -\frac{1}{12}x^4 \text{ and } v_2 = \int \frac{1}{3}dx = \frac{1}{3}x \Rightarrow y_p = \left(-\frac{1}{12}x^4\right)\left(x^{-2}\right) + \left(\frac{1}{3}x\right)(x) = \frac{1}{4}x^2$$

60. $x^2y'' + xy' - y = x$, $y_1 = x^{-1}$, $y_2 = x$

$$\Rightarrow v_1' = \frac{\begin{vmatrix} 0 & x \\ \frac{x}{x^2} & 1 \end{vmatrix}}{\begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix}} = \frac{-1}{2x^{-1}} = -\frac{1}{2}x \text{ and } v_2' = \frac{\begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & \frac{x}{x^2} \end{vmatrix}}{\begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix}} = \frac{x^{-2}}{2x^{-1}} = \frac{1}{2}x^{-1}$$

$$\Rightarrow v_1 = \int -\frac{1}{2}x\,dx = -\frac{1}{4}x^2 \text{ and } v_2 = \int \frac{1}{2}x^{-1}dx = \frac{1}{2}\ln|x| \Rightarrow y_p = \left(-\frac{1}{4}x^2\right)\left(x^{-1}\right) + \left(\frac{1}{2}\ln|x|\right)(x) = -\frac{1}{4}x + \frac{1}{2}x\ln|x|$$

17.3 APPLICATIONS

1. $mg = 16 \Rightarrow m = \frac{16}{32}$; $k = 1$; $\delta = 1 \Rightarrow \frac{16}{32}\frac{d^2y}{dt^2} + 1\frac{dy}{dt} + 1y = 0 \Rightarrow \frac{1}{2}\frac{d^2y}{dt^2} + \frac{dy}{dt} + y = 0$, $y(0) = 2$, $y'(0) = 2$

2. $mg = 8$ lb $\Rightarrow m = \frac{8}{32}$; $8 = k \cdot 4 \Rightarrow k = 2$; $\delta = 1.5 \Rightarrow \frac{8}{32}\frac{d^2y}{dt^2} + 1.5\frac{dy}{dt} + 2y = 0 \Rightarrow \frac{1}{4}\frac{d^2y}{dt^2} + \frac{3}{2}\frac{dy}{dt} + 2y = 0$, $y(0) = -2$, $y'(0) = 3$

3. $20 = k \cdot \frac{1}{2} \Rightarrow k = 40$; $w = 25$ lb $\Rightarrow m = \frac{25}{32}$; $\delta = 0 \Rightarrow \frac{25}{32}\frac{d^2y}{dt^2} + 0 \cdot \frac{dy}{dt} + 40y = 0$. If $w = 25$ lb, it stretches the spring $25 = 40x \Rightarrow x = \frac{5}{8}$ ft $\Rightarrow$ spring is now stretched $\frac{6+5}{12} - \frac{5}{8} = \frac{7}{24}$ ft below equilibrium $\Rightarrow y(0) = \frac{7}{24}$; initial velocity is $v_0 \frac{\text{in}}{\text{sec}} = \frac{v_0}{12}\frac{\text{ft}}{\text{sec}} \Rightarrow y'(0) = \frac{v_0}{12}$. Thus we have $\frac{25}{32}\frac{d^2y}{dt^2} + 40y = 0$, $y(0) = \frac{7}{24}$, $y'(0) = \frac{v_0}{12}$

4. $w = 10$ lb $\Rightarrow m = \frac{10}{32}$; $10 = k \cdot \left(\frac{2}{12}\right) \Rightarrow k = 60\frac{\text{lb}}{\text{ft}}$; $\delta = \frac{20}{\sqrt{g}} = \frac{20}{\sqrt{32}} \Rightarrow \frac{10}{32}\frac{d^2y}{dt^2} + \frac{20}{\sqrt{32}}\frac{dy}{dt} + 60y = 0$ $\Rightarrow \frac{10}{32}\frac{d^2y}{dt^2} + \frac{5}{\sqrt{2}}\frac{dy}{dt} + 60y = 0$, $y(0) = \frac{1}{4}$, $y'(0) = 0$

5. $E(t) = 20\cos t$; $R\frac{dq}{dt} = 4\frac{dq}{dt}$; $\frac{1}{C}q = 10q$; $L\frac{di}{dt} = 2\frac{d^2q}{dt^2} \Rightarrow 2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 10q = 20\cos t$, $q(0) = 2$, $q'(0) = 3$

6. $L = 2$, $R = 12$, $C = \frac{1}{16}$, $E(t) = 300 \Rightarrow 2\frac{d^2q}{dt^2} + 12\frac{dq}{dt} + 16q = 300$, $q(0) = 0$, $q'(0) = 0$

7. $mg = 16 \Rightarrow m = \frac{16}{32}$; $k = 1$; resistance = velocity $\Rightarrow \delta = 1 \Rightarrow \frac{1}{2}\frac{d^2y}{dt^2} + 1\frac{dy}{dt} + 1y = 0$, $y(0) = 2$, $y'(0) = 2$ $\Rightarrow \frac{1}{2}r^2 + r + 1 = 0 \Rightarrow r^2 + 2r + 2 = 0 \Rightarrow r = \frac{-2\pm\sqrt{2^2-4(1)(2)}}{2(1)} = -1 \pm i \Rightarrow y = e^{-t}(c_1\cos t + c_2\sin t)$, $\Rightarrow y' = e^{-t}(-c_1\sin t + c_2\cos t) - e^{-t}(c_1\cos t + c_2\sin t)$; $y(0) = 2 \Rightarrow c_1 = 2$; $y'(0) = 2 \Rightarrow c_2 - c_1 = 2 \Rightarrow c_2 = 4$ $\Rightarrow y(t) = e^{-t}(2\cos t + 4\sin t)$. At $t = \pi$, $y = e^{-\pi}(2\cos\pi + 4\sin\pi) = -2e^{-\pi} \approx -0.0864 \Rightarrow$ 0.0864 ft above equilibrium.

8. $w = 8 \Rightarrow m = \frac{8}{32}$; $8 = k \cdot 4 \Rightarrow k = 2$; resistance = 1.5 v $\Rightarrow \delta = 1.5 \Rightarrow \frac{1}{4}\frac{d^2y}{dt^2} + 1.5\frac{dy}{dt} + 2y = 0$, $y(0) = -2$, $y'(0) = 3$ $\Rightarrow \frac{1}{4}r^2 + 1.5r + 2 = 0 \Rightarrow r^2 + 6r + 8 = 0 \Rightarrow (r+4)(r+2) \Rightarrow r = -4$ or $r = -2 \Rightarrow y = c_1e^{-4t} + c_2e^{-2t}$ $\Rightarrow y' = -4c_1e^{-4t} - 2c_2e^{-2t}$; $y(0) = -2 \Rightarrow c_1 + c_2 = -2$; $\Rightarrow y'(0) = 3 \Rightarrow -4c_1 - 2c_2 = 3 \Rightarrow c_1 = \frac{1}{2}$, $c_2 = -\frac{5}{2}$ $\Rightarrow y(t) = \frac{1}{2}e^{-4t} - \frac{5}{2}e^{-2t}$. At $t = 2$, $y = \frac{1}{2}e^{-8} - \frac{5}{2}e^{-2} \approx -0.0456 \Rightarrow 0.0456$ ft above equilibrium.

9. $20 = k \cdot \frac{1}{2} \Rightarrow k = 40$; $w = 25$ lb $\Rightarrow m = \frac{25}{32}$; $\delta = 0 \Rightarrow \frac{25}{32}\frac{d^2y}{dt^2} + 0 \cdot \frac{dy}{dt} + 40y = 0$. If $w = 25$ lb, it stretches the spring $25 = 40x \Rightarrow x = \frac{5}{8}$ ft $\Rightarrow$ spring is now stretched $\frac{6+5}{12} - \frac{5}{8} = \frac{7}{24}$ ft below equilibrium $\Rightarrow y(0) = \frac{7}{24}$; initial velocity is $v_0 \ \frac{\text{in}}{\text{sec}} = \frac{v_0}{12} \ \frac{\text{ft}}{\text{sec}} \Rightarrow y'(0) = \frac{v_0}{12}$. Thus we have $\frac{25}{32}\frac{d^2y}{dt^2} + 40y = 0$, $y(0) = \frac{7}{24}$, $y'(0) = \frac{v_0}{12} \Rightarrow \frac{25}{32}r^2 + 40 = 0 \Rightarrow r^2 + \frac{256}{5} = 0$
$\Rightarrow r = 0 \pm \frac{16}{\sqrt{5}}i \Rightarrow y = e^{0 \cdot t}\left(c_1\cos\left(\frac{16}{\sqrt{5}}t\right) + c_2\sin\left(\frac{16}{\sqrt{5}}t\right)\right) = c_1\cos\left(\frac{16}{\sqrt{5}}t\right) + c_2\sin\left(\frac{16}{\sqrt{5}}t\right)$, $y(0) = \frac{7}{24} \Rightarrow c_1 = \frac{7}{24}$;
$y' = -\frac{16}{\sqrt{5}}c_1\sin\left(\frac{16}{\sqrt{5}}t\right) + \frac{16}{\sqrt{5}}c_2\cos\left(\frac{16}{\sqrt{5}}t\right)$, $y'(0) = \frac{v_0}{12} \Rightarrow \frac{16}{\sqrt{5}}c_2 = \frac{v_0}{12} \Rightarrow c_2 = \frac{v_0\sqrt{5}}{192}$
$\Rightarrow y(t) = \frac{7}{24}\cos\left(\frac{16}{\sqrt{5}}t\right) + \frac{v_0\sqrt{5}}{192}\sin\left(\frac{16}{\sqrt{5}}t\right)$ (in feet) or $y(t) = \frac{7}{2}\cos\left(\frac{16}{\sqrt{5}}t\right) + \frac{v_0\sqrt{5}}{16}\sin\left(\frac{16}{\sqrt{5}}t\right)$ (in inches)

10. $m = 1, k = \frac{25}{4}, \delta = 3 \Rightarrow 1 \cdot \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + \frac{25}{4}y = 0, y(0) = -1, y'(0) = 3 \Rightarrow r^2 + 3r + \frac{25}{4} = 0$
$\Rightarrow 4r^2 + 12r + 25 = 0 \Rightarrow r = \frac{-12 \pm \sqrt{12^2 - 4(4)(25)}}{2(4)} = -\frac{3}{2} \pm 2i \Rightarrow y = e^{-\frac{3}{2}t}(c_1\cos 2t + c_2\sin 2t)$
$\Rightarrow y' = e^{-\frac{3}{2}t}\left(\left(-\frac{3}{2}c_1 + 2c_2\right)\cos 2t + \left(-2c_1 - \frac{3}{2}c_2\right)\sin 2t\right)$; $y(0) = -1 \Rightarrow c_1 = -1$, $y'(0) = 3 \Rightarrow -\frac{3}{2}c_1 + 2c_2 = 3$
$\Rightarrow c_1 = -1, c_2 = \frac{3}{4} \Rightarrow y(t) = e^{-\frac{3}{2}t}\left(-\cos 2t + \frac{3}{4}\sin 2t\right)$

11. $mg = 10 \Rightarrow m = \frac{5}{16}$; $10 = k \cdot \frac{1}{6} \Rightarrow k = 60$; $\delta = \frac{40}{\sqrt{32}} = 5\sqrt{2} \Rightarrow \frac{5}{16}\frac{d^2y}{dt^2} + 5\sqrt{2}\frac{dy}{dt} + 60y = 0, y(0) = \frac{1}{4}, y'(0) = 0$
$\Rightarrow \frac{5}{16}r^2 + 5\sqrt{2}\,r + 60 = 0 \Rightarrow r^2 + 16\sqrt{2}r + 192 = 0 \Rightarrow r = \frac{-16\sqrt{2} \pm \sqrt{\left(16\sqrt{2}\right)^2 - 4(1)(192)}}{2(1)} = -8\sqrt{2} \pm 8i$
$\Rightarrow y = e^{-8\sqrt{2}t}(c_1\cos 8t + c_2\sin 8t) \Rightarrow y' = e^{-8\sqrt{2}t}\left(\left(-8\sqrt{2}\,c_1 + 8c_2\right)\cos 8t + \left(-8c_1 - 8\sqrt{2}\,c_2\right)\sin 8t\right)$;
$y(0) = \frac{1}{4} \Rightarrow c_1 = \frac{1}{4}$, $y'(0) = 0 \Rightarrow -8\sqrt{2}\,c_1 + 8c_2 = 0 \Rightarrow c_1 = \frac{1}{4}, c_2 = \frac{\sqrt{2}}{4} \Rightarrow y = e^{-8\sqrt{2}t}\left(\frac{1}{4}\cos 8t + \frac{\sqrt{2}}{4}\sin 8t\right)$
Solve $y(t) = 0 \Rightarrow e^{-8\sqrt{2}t}\left(\frac{1}{4}\cos 8t + \frac{\sqrt{2}}{4}\sin 8t\right) = 0 \Rightarrow \frac{1}{4}\cos 8t + \frac{\sqrt{2}}{4}\sin 8t = 0 \Rightarrow \tan 8t = -\frac{1}{\sqrt{2}} \Rightarrow t \approx 0.3157$ sec

12. $w = mg$; $mg = k \cdot \frac{1}{2} \Rightarrow k = 2mg = 64m$; $\delta = 0 \Rightarrow m\frac{d^2y}{dt^2} + 0 \cdot \frac{dy}{dt} + 64m\,y = 0$, $y(0) = \frac{1}{6}$, $y'(0) = \frac{1}{6} \Rightarrow m\,r^2 + 64m = 0$,
$\Rightarrow r^2 + 64 = 0 \Rightarrow r = 0 \pm 8i \Rightarrow y = e^{0 \cdot t}(c_1\cos 8t + c_2\sin 8t) = c_1\cos 8t + c_2\sin 8t \Rightarrow y' = -8c_1\sin 8t + 8c_2\cos 8t$
$y(0) = \frac{1}{6} \Rightarrow c_1 = \frac{1}{6}$, $y'(0) = \frac{1}{6} \Rightarrow 8c_2 = \frac{1}{6} \Rightarrow c_1 = \frac{1}{6}, c_2 = \frac{1}{48} \Rightarrow y(t) = \frac{1}{6}\cos 8t + \frac{1}{48}\sin 8t$
(a) Solve $y(t) = 0 \Rightarrow \frac{1}{6}\cos 8t + \frac{1}{48}\sin 8t = 0 \Rightarrow \tan 8t = -8 \Rightarrow t \approx -0.1808$, first positive solution is $t = -0.1808 + \frac{\pi}{8}$ ≈ 0.2119 sec
(b) $y' = -\frac{4}{3}\sin 8t + \frac{1}{6}\cos 8t$, solve $y'(t) = 0 \Rightarrow -\frac{4}{3}\sin 8t + \frac{1}{6}\cos 8t = 0 \Rightarrow \tan 8t = \frac{1}{8} \Rightarrow t \approx 0.4082$ sec $\Rightarrow y(0.4082) \approx 0.1680$ ft
(c) $y'' = -\frac{32}{3}\cos 8t - \frac{4}{3}\sin 8t$, solve $y''(t) = 0 \Rightarrow -\frac{32}{3}\cos 8t - \frac{4}{3}\sin 8t = 0 \Rightarrow \tan 8t = -8 \Rightarrow t = \frac{\tan^{-1}(-8)}{8}$, first positive solution where maximum occurs is at $t = \frac{\tan^{-1}(-8)}{8} + \frac{\pi}{4} \Rightarrow y'\left(\frac{\tan^{-1}(-8)}{8} + \frac{\pi}{4}\right)$
$= -\frac{4}{3}\sin 8\left(\frac{\tan^{-1}(-8)}{8} + \frac{\pi}{4}\right) + \frac{1}{6}\cos 8\left(\frac{\tan^{-1}(-8)}{8} + \frac{\pi}{4}\right) = -\frac{4}{3}\sin\left(\tan^{-1}(-8) + 2\pi\right) + \frac{1}{6}\cos\left(\left(\tan^{-1}(-8) + 2\pi\right)\right)$
$= -\frac{4}{3}[\sin(\tan^{-1}(-8))\cos 2\pi + \cos(\tan^{-1}(-8))\sin 2\pi] + \frac{1}{6}[\cos(\tan^{-1}(-8))\cos 2\pi - \sin(\tan^{-1}(-8))\sin 2\pi]$
$= -\frac{4}{3}\left[-\frac{8}{\sqrt{65}} + 0\right] + \frac{1}{6}\left[\frac{1}{\sqrt{65}} - 0\right] = \frac{65}{6\sqrt{65}} = \frac{1}{6}\sqrt{65} \ \frac{\text{ft}}{\text{sec}} \Rightarrow 12\left(\frac{1}{6}\sqrt{65}\right) = 2\sqrt{65}\,\frac{\text{in}}{\text{sec}}$. Note that $2\sqrt{2g+1} = 2\sqrt{65}$ when $g = 32$.

13. First weight: $w = 10$ lb $\Rightarrow m = \frac{5}{16}$; $10 = k \cdot \left(\frac{5}{6}\right) \Rightarrow k = 12\frac{\text{lb}}{\text{ft}}$; $\delta = 0 \Rightarrow \frac{5}{16}\frac{d^2y}{dt^2} + 0 \cdot \frac{dy}{dt} + 12y = 0, y(0) = \frac{1}{6}, y'(0) = -\frac{1}{3}$
$\Rightarrow \frac{5}{16}r^2 + 12 = 0 \Rightarrow 5r^2 + 192 = 0 \Rightarrow r = 0 \pm \frac{8\sqrt{15}}{5}i \Rightarrow y = e^{0 \cdot t}\left(c_1\cos\frac{8\sqrt{15}}{5}t + c_2\sin\frac{8\sqrt{15}}{5}t\right)$
$= c_1\cos\frac{8\sqrt{15}}{5}t + c_2\sin\frac{8\sqrt{15}}{5}t \Rightarrow y' = -\frac{8\sqrt{15}}{5}c_1\sin\frac{8\sqrt{15}}{5}t + \frac{8\sqrt{15}}{5}c_2\cos\frac{8\sqrt{15}}{5}t$; $y(0) = \frac{1}{6} \Rightarrow c_1 = \frac{1}{6}$, $y'(0) = -\frac{1}{3}$
$\Rightarrow \frac{8\sqrt{15}}{5}c_2 = -\frac{1}{3} \Rightarrow c_1 = \frac{1}{6}, c_2 = -\frac{\sqrt{15}}{72} \Rightarrow y = \frac{1}{6}\cos\frac{8\sqrt{15}}{5}t - \frac{\sqrt{15}}{72}\sin\frac{8\sqrt{15}}{5}t$. The amplitude is $C = \sqrt{\left(\frac{1}{6}\right)^2 + \left(-\frac{\sqrt{15}}{72}\right)^2}$
$= \frac{\sqrt{159}}{72}$.

Second weight: $x = c_3\cos\omega t + c_4\sin\omega t$, $x(0) = \frac{\sqrt{159}}{72}$, $x'(0) = 2 \Rightarrow x' = -\omega c_3\sin\omega t + \omega c_4\cos\omega t$; $x(0) = \frac{\sqrt{159}}{72}$ $\Rightarrow c_3 = \frac{\sqrt{159}}{72}$, $x'(0) = 2 \Rightarrow \omega c_4 = 2 \Rightarrow c_3 = \frac{\sqrt{159}}{72}$, $c_4 = \frac{2}{\omega} \Rightarrow x = \frac{\sqrt{159}}{72}\cos\omega t + \frac{2}{\omega}\sin\omega t$. Since amplitude of second spring $= 2C \Rightarrow 2\left(\frac{\sqrt{159}}{72}\right) = \sqrt{\left(\frac{\sqrt{159}}{72}\right)^2 + \left(\frac{2}{\omega}\right)^2} \Rightarrow \omega = \frac{48}{\sqrt{53}} \Rightarrow \frac{48}{\sqrt{53}} = \sqrt{\frac{k}{m}} = \sqrt{\frac{12}{m}} \Rightarrow m = \frac{53}{192} \Rightarrow mg = \left(\frac{53}{192}\right)32$ $= 8.8333$ lbs

14. First spring: $m_1 g = k_1 \cdot \frac{1}{4} \Rightarrow k_1 = 128m_1$; $\delta_1 = 0 \Rightarrow m_1\frac{d^2y}{dt^2} + 0\cdot\frac{dy}{dt} + 128m_1 y = 0$, $y(0) = \frac{1}{12}$, $y'(0) = 0$ $\Rightarrow m_1 r^2 + 128\,m_1 = 0 \Rightarrow r^2 + 128 = 0 \Rightarrow r = 0 \pm 8\sqrt{2}i \Rightarrow y = e^{0\cdot t}\left(c_1\cos 8\sqrt{2}\,t + c_2\sin 8\sqrt{2}\,t\right)$ $= c_1\cos 8\sqrt{2}\,t + c_2\sin 8\sqrt{2}\,t \Rightarrow y' = -8\sqrt{2}c_1\sin 8\sqrt{2}\,t + 8\sqrt{2}c_2\cos 8\sqrt{2}\,t$; $y(0) = \frac{1}{12} \Rightarrow c_1 = \frac{1}{12}$, $y'(0) = 0$ $\Rightarrow 8\sqrt{2}c_2 = 0 \Rightarrow c_1 = \frac{1}{12}$, $c_2 = 0 \Rightarrow y(t) = \frac{1}{12}\cos 8\sqrt{2}t$
Second spring: $m_2 g = k_2 \cdot \frac{3}{4} \Rightarrow k_1 = \frac{128}{3}m_2$; $\delta_2 = 0 \Rightarrow m_2\frac{d^2x}{dt^2} + 0\cdot\frac{dx}{dt} + \frac{128}{3}m_2 x = 0$, $x(0) = \frac{1}{12}$, $x'(0) = 0$ $\Rightarrow \frac{d^2x}{dt^2} + \frac{128}{3}x = 0 \Rightarrow r^2 + \frac{128}{3} = 0 \Rightarrow r = 0 \pm 8\sqrt{\frac{2}{3}}i \Rightarrow x = e^{0\cdot t}\left(c_3\cos 8\sqrt{\frac{2}{3}}\,t + c_4\sin 8\sqrt{\frac{2}{3}}\,t\right)$ $= c_3\cos 8\sqrt{\frac{2}{3}}\,t + c_4\sin 8\sqrt{\frac{2}{3}}\,t \Rightarrow y' = -8\sqrt{\frac{2}{3}}c_3\sin 8\sqrt{\frac{2}{3}}\,t + 8\sqrt{\frac{2}{3}}c_4\cos 8\sqrt{\frac{2}{3}}\,t$; $x(0) = \frac{1}{12} \Rightarrow c_3 = \frac{1}{12}$, $x'(0) = 0$ $\Rightarrow 8\sqrt{\frac{2}{3}}c_4 = 0 \Rightarrow c_3 = \frac{1}{12}$, $c_4 = 0 \Rightarrow x(t) = \frac{1}{12}\cos 8\sqrt{\frac{2}{3}}\,t$
Equal velocities $\Rightarrow y' = x' \Rightarrow -\frac{2\sqrt{2}}{3}\sin 8\sqrt{2}t = -\frac{2}{3}\sqrt{\frac{2}{3}}\sin 8\sqrt{\frac{2}{3}}\,t \Rightarrow t \approx 0.2237$ sec

15. $mg = 16 \Rightarrow m = \frac{1}{2}$; $16 = k\cdot 4 \Rightarrow k = 4$; $\delta = 0 \Rightarrow \frac{1}{2}\frac{d^2y}{dt^2} + 0\cdot\frac{dy}{dt} + 4y = 0$, $y(0) = 5$, $y'(0) = 0 \Rightarrow \frac{1}{2}r^2 + 4 = 0$ $\Rightarrow r^2 + 8 = 0 \Rightarrow r = 0 \pm 2\sqrt{2}i \Rightarrow y = e^{0\cdot t}\left(c_1\cos 2\sqrt{2}\,t + c_2\sin 2\sqrt{2}\,t\right) = c_1\cos 2\sqrt{2}\,t + c_2\sin 2\sqrt{2}\,t$ $\Rightarrow y' = -2\sqrt{2}c_1\sin 2\sqrt{2}\,t + 2\sqrt{2}c_2\cos 2\sqrt{2}\,t$; $y(0) = 5 \Rightarrow c_1 = 5$, $y'(0) = 0 \Rightarrow 2\sqrt{2}c_2 = 0 \Rightarrow c_1 = 5$, $c_2 = 0$ $\Rightarrow y(t) = 5\cos 2\sqrt{2}\,t$. The amplitude is $C = \sqrt{5^2 + 0^2} = 5$
$y = c_3\cos 2\sqrt{2}\,t + c_4\sin 2\sqrt{2}\,t$, $y(0) = 5$, $y'(0) = v_0 \Rightarrow y' = -2\sqrt{2}c_3\sin 2\sqrt{2}\,t + 2\sqrt{2}c_4\cos 2\sqrt{2}\,t$; $y(0) = 5 \Rightarrow c_3 = 5$ $y'(0) = v_0 \Rightarrow 2\sqrt{2}c_4 = v_0 \Rightarrow c_3 = 5$, $c_4 = \frac{v_0}{2\sqrt{2}} \Rightarrow y(t) = 5\cos 2\sqrt{2}\,t + \frac{v_0}{2\sqrt{2}}\sin 2\sqrt{2}\,t$, and the new amplitude is $2\cdot 5$ $\Rightarrow 10 = \sqrt{5^2 + \left(\frac{v_0}{2\sqrt{2}}\right)^2} \Rightarrow v_0 = 10\sqrt{6} \approx 24.4949\ \frac{ft}{sec}$

16. $mg = 8 \Rightarrow m = \frac{1}{4}$; $8 = k\cdot\frac{3}{12} \Rightarrow k = 32$; $\delta = 2 \Rightarrow \frac{1}{4}\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 32y = 0$, $y(0) = 0$, $y'(0) = \frac{1}{3} \Rightarrow \frac{1}{4}r^2 + 2r + 32 = 0$ $\Rightarrow r^2 + 8r + 128 = 0 \Rightarrow r = \frac{-8\pm\sqrt{8^2-4(1)(128)}}{2(1)} = -4 \pm 4\sqrt{7}\,i \Rightarrow y = e^{-4t}\left(c_1\cos 4\sqrt{7}\,t + c_2\sin 4\sqrt{7}\,t\right)$ $\Rightarrow y' = e^{-4t}\left[\left(-4c_1 + 4\sqrt{7}c_2\right)\cos 4\sqrt{7}\,t + \left(-4\sqrt{7}c_1 - 4c_2\right)\sin 4\sqrt{7}\,t\right]$; $y(0) = 0 \Rightarrow c_1 = 0$, $y'(0) = \frac{1}{3}$ $\Rightarrow -4c_1 + 4\sqrt{7}c_2 = \frac{1}{3} \Rightarrow c_1 = 0$, $c_2 = \frac{1}{12\sqrt{7}} \Rightarrow y(t) = \frac{1}{12\sqrt{7}}e^{-4t}\sin 4\sqrt{7}\,t$. Solve $y(t) = 0 \Rightarrow \frac{1}{12\sqrt{7}}e^{-4t}\sin 4\sqrt{7}\,t = 0$ $\Rightarrow \sin 4\sqrt{7}\,t = 0 \Rightarrow t = \frac{\pi}{4\sqrt{7}} \approx 0.2969$ sec

17. δ decreases by 90% in 10 sec $\Rightarrow$ 10% remains $\Rightarrow e^{-10b} = \frac{1}{10} \Rightarrow b = -\frac{1}{10}\ln\left(\frac{1}{10}\right) = \frac{\ln 10}{10} \Rightarrow 2b = \frac{\delta}{m} \Rightarrow \delta = \frac{\ln 10}{5}m$ period $= 2$ sec $\Rightarrow 2 = \frac{2\pi}{\sqrt{\omega^2 - b^2}} \Rightarrow 4 = \frac{4\pi^2}{\omega^2 - b^2} \Rightarrow \omega^2 = \pi^2 + b^2 = \pi^2 + \left(\frac{\ln 10}{10}\right)^2 = \frac{100\pi^2 + (\ln 10)^2}{100} \Rightarrow \frac{k}{m} = \frac{100\pi^2 + (\ln 10)^2}{100}$ $k = \frac{100\pi^2 + (\ln 10)^2}{100}m \Rightarrow m\frac{d^2y}{dt^2} + \left(\frac{\ln 10}{5}m\right)\frac{dy}{dt} + \left(\frac{100\pi^2 + (\ln 10)^2}{100}m\right)y = 0$. When $y = \frac{1}{4}$ and $y' = -2$, then $m\frac{d^2y}{dt^2} + \left(\frac{\ln 10}{5}m\right)(-2) + \left(\frac{100\pi^2 + (\ln 10)^2}{100}m\right)\left(\frac{1}{4}\right) = 0 \Rightarrow \frac{d^2y}{dt^2} = \frac{2\ln 10}{5} - \frac{100\pi^2 + (\ln 10)^2}{400} \approx -1.5596\ \frac{ft}{sec^2}$

18. $mg = 10 \Rightarrow m = \frac{5}{16}$; $10 = k \cdot 2 \Rightarrow k = 5$; $\delta = \frac{10}{\sqrt{32}} = \frac{5\sqrt{2}}{4} \Rightarrow \frac{5}{16}\frac{d^2y}{dt^2} + \frac{5\sqrt{2}}{4}\frac{dy}{dt} + 5y = 0$, $y(0) = \frac{1}{2}$, $y'(0) = 0$

$\Rightarrow \frac{5}{16}r^2 + \frac{5\sqrt{2}}{4}r + 5 = 0 \Rightarrow r^2 + 4\sqrt{2}\,r + 16 = 0 \Rightarrow r = \frac{-4\sqrt{2} \pm \sqrt{\left(4\sqrt{2}\right)^2 - 4(1)(16)}}{2(1)} = -2\sqrt{2} \pm 2\sqrt{2}\,i$

$\Rightarrow y(t) = e^{-2\sqrt{2}t}\left(c_1\cos 2\sqrt{2}\,t + c_2\sin 2\sqrt{2}\,t\right)$

$\Rightarrow y' = e^{-2\sqrt{2}t}\left[\left(-2\sqrt{2}\,c_1 + 2\sqrt{2}\,c_2\right)\cos 2\sqrt{2}\,t + \left(-2\sqrt{2}\,c_1 - 2\sqrt{2}\,c_2\right)\sin 2\sqrt{2}\,t\right]$; $y(0) = \frac{1}{2} \Rightarrow c_1 = \frac{1}{2}$,

$y'(0) = 0 \Rightarrow -2\sqrt{2}\,c_1 + 2\sqrt{2}\,c_2 = 0 \Rightarrow c_1 = \frac{1}{2}, c_2 = \frac{1}{2} \Rightarrow y = e^{-2\sqrt{2}t}\left(\frac{1}{2}\cos 2\sqrt{2}\,t + \frac{1}{2}\sin 2\sqrt{2}\,t\right)$

To find maximum, solve $y'(t) = 0 \Rightarrow y'(t) = -2\sqrt{2}\,e^{-2\sqrt{2}t}\sin 2\sqrt{2}\,t \Rightarrow -2\sqrt{2}\,e^{-2\sqrt{2}t}\sin 2\sqrt{2}\,t = 0$

$\Rightarrow \sin 2\sqrt{2}\,t = 0 \Rightarrow t = 0, \frac{\pi}{2\sqrt{2}}, \frac{\pi}{\sqrt{2}}$, and at $t = \frac{\pi}{2\sqrt{2}}$, $y\left(\frac{\pi}{2\sqrt{2}}\right) \approx -0.02161$ ft ≈ -0.2593 in (above equilibrium)

19. $L = \frac{1}{5}, R = 1, C = \frac{5}{6}, E(t) = 0 \Rightarrow \frac{1}{5}\frac{d^2q}{dt^2} + 1 \cdot \frac{dq}{dt} + \frac{6}{5}q = 0$, $q(0) = 2$, $q'(0) = 4 \Rightarrow \frac{1}{5}r^2 + r + \frac{6}{5} = 0$

$\Rightarrow r^2 + 5r + 6 = 0 \Rightarrow (r+3)(r+2) = 0 \Rightarrow r = -3$ or $r = -2 \Rightarrow q(t) = c_1e^{-3t} + c_2\,e^{-2t} \Rightarrow q' = -3c_1e^{-3t} - 2c_2\,e^{-2t}$

$q(0) = 2 \Rightarrow c_1 + c_2 = 2$; $q'(0) = 4 \Rightarrow -3c_1 - 2c_2 = 4 \Rightarrow c_1 = -8, c_2 = 10 \Rightarrow q = -8e^{-3t} + 10e^{-2t}$

$\lim_{t\to\infty} q = \lim_{t\to\infty}(-8e^{-3t} + 10e^{-2t}) = 0$

20. $\frac{1}{C}q = 10q$; $R\frac{dq}{dt} = 4\frac{dq}{dt}$; $L\frac{di}{dt} = 2\frac{d^2q}{dt^2}$, $E(t) = 0 \Rightarrow 2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 10q = 0$, $q(0) = 2$, $q'(0) = 3 \Rightarrow 2r^2 + 4r + 10 = 0$

$\Rightarrow r^2 + 2r + 5 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = -1 \pm 2i \Rightarrow q(t) = e^{-t}(c_1\cos 2t + c_2\sin 2t)$

$\Rightarrow q' = e^{-t}((-c_1 + 2c_2)\cos 2t + (-2c_1 - c_2)\sin 2t)$; $q(0) = 2 \Rightarrow c_1 + c_2 = 2$; $q'(0) = 3 \Rightarrow -c_1 + 2c_2 = 3$

$\Rightarrow c_1 = \frac{1}{3}, c_2 = \frac{5}{3} \Rightarrow q(t) = e^{-t}\left(\frac{1}{3}\cos 2t + \frac{5}{3}\sin 2t\right)$

21. $mg = 16 \Rightarrow m = \frac{16}{32}$; $16 = k \cdot 4 \Rightarrow k = 4$; $\delta = 4.5$; $f(t) = 4 + e^{-2t} \Rightarrow \frac{1}{2}\frac{d^2y}{dt^2} + 4.5\frac{dy}{dt} + 4y = 4 + e^{-2t}$, $y(0) = 2$, $y'(0) = 4$

$\Rightarrow \frac{1}{2}r^2 + 4.5r + 4 = 0 \Rightarrow r^2 + 9r + 8 = 0 \Rightarrow (r+8)(r+1) = 0 \Rightarrow r = -8$ or $r = -1 \Rightarrow y_c = c_1e^{-8t} + c_2\,e^{-t}$;

$y_p = A + Be^{-2t} \Rightarrow y_p' = -2Be^{-2t} \Rightarrow y_p'' = 4Be^{-2t} \Rightarrow \frac{1}{2}(4Be^{-2t}) + 4.5(-2Be^{-2t}) + 4(A + Be^{-2t}) = 4 + e^{-2t}$

$\Rightarrow 4A - 3Be^{-2t} = 4 + e^{-2t} \Rightarrow 4A = 4, -3B = 1 \Rightarrow A = 1, B = -\frac{1}{3} \Rightarrow y(t) = c_1e^{-8t} + c_2\,e^{-t} + 1 - \frac{1}{3}e^{-2t}$

$\Rightarrow y' = -8c_1e^{-8t} - c_2\,e^{-t} + \frac{2}{3}e^{-2t}$; $y(0) = 2 \Rightarrow c_1 + c_2 + \frac{2}{3} = 2 \Rightarrow c_1 + c_2 = \frac{4}{3}$, $y'(0) = 4 \Rightarrow -8c_1 - c_2 + \frac{2}{3} = 4$

$\Rightarrow -8c_1 - c_2 = \frac{10}{3} \Rightarrow c_1 = -\frac{2}{3}, c_2 = 2 \Rightarrow y(t) = -\frac{2}{3}e^{-8t} + 2e^{-t} + 1 - \frac{1}{3}e^{-2t}$

22. $m = 10$; $k = 140$; $\delta = 90$; $f(t) = 5\sin t \Rightarrow 10\frac{d^2y}{dt^2} + 90\frac{dy}{dt} + 140y = 5\sin t$, $y(0) = 0$, $y'(0) = -1 \Rightarrow 10r^2 + 90r + 140 = 0$

$\Rightarrow r^2 + 9r + 14 = 0 \Rightarrow (r+2)(r+7) = 0 \Rightarrow r = -2$ or $r = -7 \Rightarrow y_c = c_1e^{-2t} + c_2\,e^{-7t}$; $y_p = A\sin t + B\cos t$

$\Rightarrow y_p' = A\cos t - B\sin t \Rightarrow y_p'' = -A\sin t - B\cos t$

$\Rightarrow 10(-A\sin t - B\cos t) + 90(A\cos t - B\sin t) + 140(A\sin t + B\cos t) = 5\sin t$

$\Rightarrow (130A - 90B)\sin t + (90A + 130B)\cos t = 5\sin t \Rightarrow A = \frac{13}{500}, B = -\frac{9}{500}$

$\Rightarrow y(t) = c_1e^{-2t} + c_2\,e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t \Rightarrow y' = -2c_1e^{-2t} - 7c_2\,e^{-7t} + \frac{13}{500}\cos t + \frac{9}{500}\sin t$;

$y(0) = 0 \Rightarrow c_1 + c_2 - \frac{9}{500} = 0$, $y'(0) = -1 \Rightarrow -2c_1 - 7c_2 + \frac{13}{500} = -1 \Rightarrow c_1 + c_2 = \frac{9}{500}, -2c_1 - 7c_2 = -\frac{513}{500}$

$\Rightarrow c_1 = -\frac{9}{50}, c_2 = \frac{99}{500} \Rightarrow y(t) = -\frac{9}{50}e^{-2t} + \frac{99}{500}\,e^{-7t} + \frac{13}{500}\sin t - \frac{9}{500}\cos t$

23. $m = 2 \Rightarrow mg = 2(9.8) = 19.6$; $19.6 = k \cdot 1.96 \Rightarrow k = 10$; $\delta = 4$; $f(t) = 20\cos t \Rightarrow 2\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 10y = 20\cos t$,

$y(0) = 2$, $y'(0) = 3 \Rightarrow 2r^2 + 4r + 10 = 0 \Rightarrow r = \frac{-4 \pm \sqrt{4^2 - 4(2)(10)}}{2(2)} = -1 \pm 2i \Rightarrow y_c = e^{-t}(c_1\cos 2t + c_2\sin 2t)$

$y_p = A\sin t + B\cos t \Rightarrow y_p' = A\cos t - B\sin t \Rightarrow y_p'' = -A\sin t - B\cos t$

$\Rightarrow 2(-A\sin t - B\cos t) + 4(A\cos t - B\sin t) + 10(A\sin t + B\cos t) = 20\cos t$

$\Rightarrow (8A - 4B)\sin t + (4A + 8B)\cos t = 20\cos t \Rightarrow 8A - 4B = 0, 4A + 8B = 20 \Rightarrow A = 1, B = 2$

$\Rightarrow y(t) = e^{-t}(c_1\cos 2t + c_2\sin 2t) + \sin t + 2\cos t \Rightarrow y' = e^{-t}((-c_1 + 2c_2)\cos 2t + (-2c_1 - c_2)\sin 2t) + \cos t - 2\sin t$;

$y(0) = 2 \Rightarrow c_1 + 2 = 2,\ y'(0) = 3 \Rightarrow -c_1 + 2c_2 + 1 = 3 \Rightarrow c_1 = 0,\ -c_1 + 2c_2 = 2 \Rightarrow c_1 = 0,\ c_2 = 1$
$\Rightarrow y(t) = e^{-t}\sin 2t + \sin t + 2\cos t;\ y(\pi) = -2 \Rightarrow$ 2 m above equilibrium

24. $mg = 8 \Rightarrow m = \frac{8}{32};\ 8 = k \cdot 4 \Rightarrow k = 2;\ \delta = 1.5;\ f(t) = 6 + e^{-t} \Rightarrow \frac{1}{4}\frac{d^2y}{dt^2} + 1.5\frac{dy}{dt} + 2y = 6 + e^{-t},\ y(0) = -2,\ y'(0) = 3$
$\Rightarrow \frac{1}{4}r^2 + 1.5r + 2 = 0 \Rightarrow r^2 + 6r + 8 = 0 \Rightarrow (r+2)(r+4) = 0 \Rightarrow r = -2 \text{ or } r = -4 \Rightarrow y_c = c_1e^{-2t} + c_2\,e^{-4t};$
$y_p = A + Be^{-t} \Rightarrow y_p' = -Be^{-t} \Rightarrow y_p'' = Be^{-t} \Rightarrow \frac{1}{4}(Be^{-t}) + 1.5(-Be^{-t}) + 2(A + Be^{-t}) = 6 + e^{-t}$
$\Rightarrow 2A + \frac{3}{4}Be^{-t} = 6 + e^{-t} \Rightarrow 2A = 6,\ \frac{3}{4}B = 1 \Rightarrow A = 3,\ B = \frac{4}{3} \Rightarrow y(t) = c_1e^{-2t} + c_2\,e^{-4t} + 3 + \frac{4}{3}e^{-t}$
$\Rightarrow y' = -2c_1e^{-2t} - 4c_2\,e^{-4t} - \frac{4}{3}e^{-t};\ y(0) = -2 \Rightarrow c_1 + c_2 + \frac{13}{3} = -2 \Rightarrow c_1 + c_2 = -\frac{19}{3},\ y'(0) = 3$
$\Rightarrow -2c_1 - 4c_2 - \frac{4}{3} = 3 \Rightarrow -2c_1 - 4c_2 = \frac{13}{3} \Rightarrow c_1 = -\frac{21}{2},\ c_2 = \frac{25}{6} \Rightarrow y(t) = -\frac{21}{2}e^{-2t} + \frac{25}{6}\,e^{-4t} + 3 + \frac{4}{3}e^{-t};$
$y(2) \approx 2.98953 \Rightarrow$ 2.99 ft below equilibrium

25. $L = 10,\ R = 10,\ C = \frac{1}{500},\ E(t) = 100 \Rightarrow 10\frac{d^2q}{dt^2} + 10\frac{dq}{dt} + 500q = 100,\ q(0) = 10,\ q'(0) = 0 \Rightarrow 10r^2 + 10r + 500 = 0$
$\Rightarrow r^2 + r + 50 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1^2 - 4(1)(50)}}{2(1)} = -\frac{1}{2} \pm \frac{\sqrt{199}}{2}i \Rightarrow q_c = e^{-\frac{1}{2}t}\left(c_1\cos\frac{\sqrt{199}}{2}t + c_2\sin\frac{\sqrt{199}}{2}t\right)$
$q_p = A \Rightarrow q_p' = 0 \Rightarrow q_p'' = 0 \Rightarrow 10(0) + 10(0) + 500A = 100 \Rightarrow 500A = 100 \Rightarrow A = \frac{1}{5}$
$\Rightarrow q(t) = e^{-\frac{1}{2}t}\left(c_1\cos\frac{\sqrt{199}}{2}t + c_2\sin\frac{\sqrt{199}}{2}t\right) + \frac{1}{5} \Rightarrow q' = e^{-\frac{1}{2}t}\left[\left(-\frac{1}{2}c_1 + \frac{\sqrt{199}}{2}c_2\right)\cos\frac{\sqrt{199}}{2}t + \left(-\frac{\sqrt{199}}{2}c_1 - \frac{1}{2}c_2\right)\sin\frac{\sqrt{199}}{2}t\right]$
$q(0) = 10 \Rightarrow c_1 + \frac{1}{5} = 10 \Rightarrow c_1 = \frac{49}{5},\ q'(0) = 0 \Rightarrow -\frac{1}{2}c_1 + \frac{\sqrt{199}}{2}c_2 = 0 \Rightarrow c_1 = \frac{49}{5},\ c_2 = \frac{49\sqrt{199}}{995}$
$\Rightarrow q(t) = e^{-\frac{1}{2}t}\left(\frac{49}{5}\cos\frac{\sqrt{199}}{2}t + \frac{49\sqrt{199}}{995}\sin\frac{\sqrt{199}}{2}t\right) + \frac{1}{5}$

26. $R\frac{dq}{dt} = 4\frac{dq}{dt};\ \frac{1}{C}q = 10q;\ L\frac{di}{dt} = 2\frac{d^2q}{dt^2},\ E(t) = 20\cos t \Rightarrow 2\frac{d^2q}{dt^2} + 4\frac{dq}{dt} + 10q = 0,\ q(0) = 2,\ q'(0) = 3 \Rightarrow 2r^2 + 4r + 10 = 0$
$\Rightarrow r^2 + 2r + 5 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{2^2 - 4(1)(5)}}{2(1)} = -1 \pm 2i \Rightarrow q_c = e^{-t}(c_1\cos 2t + c_2\sin 2t);\ q_p = A\sin t + B\cos t$
$\Rightarrow q_p' = A\cos t - B\sin t \Rightarrow q_p'' = -A\sin t - B\cos t$
$\Rightarrow 2(-A\sin t - B\cos t) + 4(A\cos t - B\sin t) + 10(A\sin t + B\cos t) = 20\cos t$
$\Rightarrow (8A - 4B)\sin t + (4A + 8B)\cos t = 20\cos t \Rightarrow 8A - 4B = 0,\ 4A + 8B = 20 \Rightarrow A = 1,\ B = 2$
$\Rightarrow q(t) = e^{-t}(c_1\cos 2t + c_2\sin 2t) + \sin t + 2\cos t \Rightarrow q' = e^{-t}((-c_1 + 2c_2)\cos 2t + (-2c_1 - c_2)\sin 2t) + \cos t - 2\sin t;$
$q(0) = 2 \Rightarrow c_1 + 2 = 2;\ q'(0) = 3 \Rightarrow -c_1 + 2c_2 + 1 = 3 \Rightarrow c_2 = 1 \Rightarrow q(t) = 2e^{-t}\sin 2t + \sin t + 2\cos t;\ y(10) \approx -2.222$
$\Rightarrow$ 2.222 ft above equilbrium

17.4 EULER EQUATIONS

1. $x^2y'' + 2x\,y' - 2y = 0 \Rightarrow r^2 + (2-1)r - 2 = 0 \Rightarrow r^2 + r - 2 = 0 \Rightarrow (r-1)(r+2) = 0 \Rightarrow r = 1 \text{ or } r = -2$
$\Rightarrow y = c_1e^z + c_2e^{-2z} = c_1e^{\ln x} + c_2e^{-2\ln x} \Rightarrow y = c_1x + \frac{c_2}{x^2}$

2. $x^2y'' + x\,y' - 4y = 0 \Rightarrow r^2 + (1-1)r - 4 = 0 \Rightarrow r^2 - 4 = 0 \Rightarrow (r-2)(r+2) = 0 \Rightarrow r = 2 \text{ or } r = -2$
$\Rightarrow y = c_1e^{2z} + c_2e^{-2z} = c_1e^{2\ln x} + c_2e^{-2\ln x} \Rightarrow y = c_1x^2 + \frac{c_2}{x^2}$

3. $x^2y'' - 6y = 0 \Rightarrow r^2 + (0-1)r - 6 = 0 \Rightarrow r^2 - r - 6 = 0 \Rightarrow (r-3)(r+2) = 0 \Rightarrow r = 3 \text{ or } r = -2$
$\Rightarrow y = c_1e^{3z} + c_2e^{-2z} = c_1e^{3\ln x} + c_2e^{-2\ln x} \Rightarrow y = c_1x^3 + \frac{c_2}{x^2}$

4. $x^2y'' + x\,y' - y = 0 \Rightarrow r^2 + (1-1)r - 1 = 0 \Rightarrow r^2 - 1 = 0 \Rightarrow (r-1)(r+1) = 0 \Rightarrow r = 1 \text{ or } r = -1$
$\Rightarrow y = c_1e^z + c_2e^{-z} = c_1e^{\ln x} + c_2e^{-\ln x} \Rightarrow y = c_1x + \frac{c_2}{x}$

5. $x^2y'' - 5x\,y' + 8y = 0 \Rightarrow r^2 + (-5-1)r + 8 = 0 \Rightarrow r^2 - 6r + 8 = 0 \Rightarrow (r-4)(r-2) = 0 \Rightarrow r = 4 \text{ or } r = 2$
$\Rightarrow y = c_1e^{4z} + c_2e^{2z} = c_1e^{4\ln x} + c_2e^{2\ln x} \Rightarrow y = c_1x^4 + c_2\,x^2$

6. $2x^2y'' + 7xy' + 2y = 0 \Rightarrow 2r^2 + (7-2)r + 2 = 0 \Rightarrow 2r^2 + 5r + 2 = 0 \Rightarrow (2r+1)(r+2) = 0 \Rightarrow r = -\frac{1}{2}$ or $r = -2$
$\Rightarrow y = c_1e^{-\frac{1}{2}z} + c_2e^{-2z} = c_1e^{-\frac{1}{2}\ln x} + c_2e^{-2\ln x} \Rightarrow y = \frac{c_1}{\sqrt{x}} + \frac{c_2}{x^2}$

7. $3x^2y'' + 4xy' = 0 \Rightarrow 3r^2 + (4-3)r = 0 \Rightarrow 3r^2 + r = 0 \Rightarrow r(3r+1) = 0 \Rightarrow r = 0$ or $r = -\frac{1}{3} \Rightarrow y = c_1e^{0\cdot z} + c_2e^{-\frac{1}{3}z}$
$= c_1e^{0\cdot\ln x} + c_2e^{-\frac{1}{3}\ln x} \Rightarrow y = c_1 + \frac{c_2}{\sqrt[3]{x}}$

8. $x^2y'' + 6xy' + 4y = 0 \Rightarrow r^2 + (6-1)r + 4 = 0 \Rightarrow r^2 + 5r + 4 = 0 \Rightarrow (r+1)(r+4) = 0 \Rightarrow r = -1$ or $r = -4$
$\Rightarrow y = c_1e^{-z} + c_2e^{-4z} = c_1e^{-\ln x} + c_2e^{-4\ln x} \Rightarrow y = \frac{c_1}{x} + \frac{c_2}{x^4}$

9. $x^2y'' - xy' + y = 0 \Rightarrow r^2 + (-1-1)r + 1 = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$, repeated twice
$\Rightarrow y = c_1e^z + c_2ze^z = c_1e^{\ln x} + c_2\ln x\, e^{\ln x} \Rightarrow y = c_1x + c_2x\ln x$

10. $x^2y'' - xy' + 2y = 0 \Rightarrow r^2 + (-1-1)r + 2 = 0 \Rightarrow r^2 - 2r + 2 = 0 \Rightarrow r = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)} = 1 \pm i$
$\Rightarrow y = e^z(c_1\cos z + c_2\sin z) = e^{\ln x}(c_1\cos(\ln x) + c_2\sin(\ln x)) \Rightarrow y = x\,(c_1\cos(\ln x) + c_2\sin(\ln x))$

11. $x^2y'' - xy' + 5y = 0 \Rightarrow r^2 + (-1-1)r + 5 = 0 \Rightarrow r^2 - 2r + 5 = 0 \Rightarrow r = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(5)}}{2(1)} = 1 \pm 2i$
$\Rightarrow y = e^z(c_1\cos 2z + c_2\sin 2z) = e^{\ln x}(c_1\cos(2\ln x) + c_2\sin(2\ln x)) \Rightarrow y = x\,(c_1\cos(2\ln x) + c_2\sin(2\ln x))$

12. $x^2y'' + 7xy' + 13y = 0 \Rightarrow r^2 + (7-1)r + 13 = 0 \Rightarrow r^2 + 6r + 13 = 0 \Rightarrow r = \frac{-6\pm\sqrt{(6)^2-4(1)(13)}}{2(1)} = -3 \pm 2i$
$\Rightarrow y = e^{-3z}(c_1\cos 2z + c_2\sin 2z) = e^{-3\ln x}(c_1\cos(2\ln x) + c_2\sin(2\ln x)) \Rightarrow y = \frac{1}{x^3}\,(c_1\cos(2\ln x) + c_2\sin(2\ln x))$

13. $x^2y'' + 3xy' + 10y = 0 \Rightarrow r^2 + (3-1)r + 10 = 0 \Rightarrow r^2 + 2r + 10 = 0 \Rightarrow r = \frac{-2\pm\sqrt{(2)^2-4(1)(10)}}{2(1)} = -1 \pm 3i$
$\Rightarrow y = e^{-z}(c_1\cos 3z + c_2\sin 3z) = e^{-\ln x}(c_1\cos(3\ln x) + c_2\sin(3\ln x)) \Rightarrow y = \frac{1}{x}\,(c_1\cos(3\ln x) + c_2\sin(3\ln x))$

14. $x^2y'' - 5xy' + 10y = 0 \Rightarrow r^2 + (-5-1)r + 10 = 0 \Rightarrow r^2 - 6r + 10 = 0 \Rightarrow r = \frac{-(-6)\pm\sqrt{(-6)^2-4(1)(10)}}{2(1)} = 3 \pm i$
$\Rightarrow y = e^{3z}(c_1\cos z + c_2\sin z) = e^{3\ln x}(c_1\cos(\ln x) + c_2\sin(\ln x)) \Rightarrow y = x^3(c_1\cos(\ln x) + c_2\sin(\ln x))$

15. $4x^2y'' + 8xy' + 5y = 0 \Rightarrow r^2 + (8-4)r + 5 = 0 \Rightarrow 4r^2 + 4r + 5 = 0 \Rightarrow r = \frac{-4\pm\sqrt{(4)^2-4(4)(5)}}{2(4)} = -\frac{1}{2} \pm i$
$\Rightarrow y = e^{-\frac{1}{2}z}(c_1\cos z + c_2\sin z) = e^{-\frac{1}{2}\ln x}(c_1\cos(\ln x) + c_2\sin(\ln x)) \Rightarrow y = \frac{1}{\sqrt{x}}\,(c_1\cos(\ln x) + c_2\sin(\ln x))$

16. $4x^2y'' - 4xy' + 5y = 0 \Rightarrow 4r^2 + (-4-4)r + 5 = 0 \Rightarrow 4r^2 - 8r + 5 = 0 \Rightarrow r = \frac{-(-8)\pm\sqrt{(-8)^2-4(4)(5)}}{2(4)} = 1 \pm \frac{1}{2}i$
$\Rightarrow y = e^z\left(c_1\cos\left(\frac{1}{2}z\right) + c_2\sin\left(\frac{1}{2}z\right)\right) = e^{\ln x}\left(c_1\cos\left(\frac{1}{2}\ln x\right) + c_2\sin\left(\frac{1}{2}\ln x\right)\right) \Rightarrow y = x\left(c_1\cos\left(\frac{1}{2}\ln x\right) + c_2\sin\left(\frac{1}{2}\ln x\right)\right)$

17. $x^2y'' + 3xy' + y = 0 \Rightarrow r^2 + (3-1)r + 1 = 0 \Rightarrow r^2 + 2r + 1 = 0 \Rightarrow (r+1)^2 = 0 \Rightarrow r = -1$, repeated twice
$\Rightarrow y = c_1e^{-z} + c_2ze^{-z} = c_1e^{-\ln x} + c_2\ln x\, e^{-\ln x} \Rightarrow y = \frac{c_1}{x} + \frac{c_2\ln x}{x}$

18. $x^2y'' - 3xy' + 9y = 0 \Rightarrow r^2 + (-3-1)r + 9 = 0 \Rightarrow r^2 - 4r + 9 = 0 \Rightarrow r = \frac{-(-4)\pm\sqrt{(-4)^2-4(1)9}}{2(1)} = 2 \pm \sqrt{5}\,i$
$\Rightarrow y = e^{2z}\left(c_1\cos\left(\sqrt{5}\,z\right) + c_2\sin\left(\sqrt{5}\,z\right)\right) = e^{2\ln x}\left(c_1\cos\left(\sqrt{5}\ln x\right) + c_2\sin\left(\sqrt{5}\ln x\right)\right)$
$\Rightarrow y = x^2\left(c_1\cos\left(\sqrt{5}\ln x\right) + c_2\sin\left(\sqrt{5}\ln x\right)\right)$

19. $x^2y'' + xy' = 0 \Rightarrow r^2 + (1-1)r = 0 \Rightarrow r^2 = 0 \Rightarrow r = 0$, repeated twice $\Rightarrow y = c_1e^{0\cdot z} + c_2z\,e^{0\cdot z} = c_1e^{0\cdot \ln x} + c_2 \ln x\, e^{0\cdot \ln x}$ $\Rightarrow y = c_1 + c_2 \ln x$

20. $4x^2y'' + y = 0 \Rightarrow 4r^2 + (0-4)r + 1 = 0 \Rightarrow 4r^2 - 4r + 1 = 0 \Rightarrow (2r-1)^2 = 0 \Rightarrow r = \frac{1}{2}$, repeated twice $\Rightarrow y = c_1e^{\frac{1}{2}z} + c_2z\,e^{\frac{1}{2}z} = c_1e^{\frac{1}{2}\ln x} + c_2 \ln x\, e^{\frac{1}{2}\ln x} \Rightarrow y = c_1\sqrt{x} + c_2\sqrt{x}\ln x$

21. $9x^2y'' + 15xy' + y = 0 \Rightarrow 9r^2 + (15-9)r + 1 = 0 \Rightarrow 9r^2 + 6r + 1 = 0 \Rightarrow (3r+1)^2 = 0 \Rightarrow r = -\frac{1}{3}$, repeated twice $\Rightarrow y = c_1e^{-\frac{1}{3}z} + c_2z\,e^{-\frac{1}{3}z} = c_1e^{-\frac{1}{3}\ln x} + c_2 \ln x\, e^{-\frac{1}{3}\ln x} \Rightarrow y = \frac{c_1}{\sqrt[3]{x}} + \frac{c_2 \ln x}{\sqrt[3]{x}}$

22. $16x^2y'' - 8xy' + 9y = 0 \Rightarrow 16r^2 + (-8-16)r + 9 = 0 \Rightarrow 16r^2 - 24r + 9 = 0 \Rightarrow (4r-3)^2 = 0 \Rightarrow r = \frac{3}{4}$, repeated twice $\Rightarrow y = c_1e^{\frac{3}{4}z} + c_2z\,e^{\frac{3}{4}z} = c_1e^{\frac{3}{4}\ln x} + c_2 \ln x\, e^{\frac{3}{4}\ln x} \Rightarrow y = c_1x^{3/4} + c_2\,x^{3/4}\ln x$

23. $16x^2y'' + 56xy' + 25y = 0 \Rightarrow 16r^2 + (56-16)r + 25 = 0 \Rightarrow 16r^2 + 40r + 25 = 0 \Rightarrow (4r+5)^2 = 0 \Rightarrow r = -\frac{5}{4}$, repeated twice $\Rightarrow y = c_1e^{-\frac{5}{4}z} + c_2z\,e^{-\frac{5}{4}z} = c_1e^{-\frac{5}{4}\ln x} + c_2 \ln x\, e^{-\frac{5}{4}\ln x} \Rightarrow y = \frac{c_1}{x^{5/4}} + \frac{c_2 \ln x}{x^{5/4}}$

24. $4x^2y'' - 16xy' + 25y = 0 \Rightarrow 4r^2 + (-16-4)r + 25 = 0 \Rightarrow 4r^2 - 20r + 25 = 0 \Rightarrow (2r-5)^2 = 0 \Rightarrow r = \frac{5}{2}$, repeated twice $\Rightarrow y = c_1e^{\frac{5}{2}z} + c_2z\,e^{\frac{5}{2}z} = c_1e^{\frac{5}{2}\ln x} + c_2 \ln x\, e^{\frac{5}{2}\ln x} \Rightarrow y = c_1x^{5/2} + c_2\,x^{5/2}\ln x$

25. $x^2y'' + 3xy' - 3y = 0,\ y(1) = 1,\ y'(1) = -1 \Rightarrow r^2 + (3-1)r - 3 = 0 \Rightarrow r^2 + 2r - 3 = 0 \Rightarrow (r-1)(r+3) = 0$ $\Rightarrow r = 1$ or $r = -3 \Rightarrow y = c_1e^z + c_2e^{-3z} = c_1e^{\ln x} + c_2e^{-3\ln x} \Rightarrow y = c_1x + \frac{c_2}{x^3} \Rightarrow y' = c_1 - \frac{3c_2}{x^4}$; $y(1) = 1 \Rightarrow c_1 + c_2 = 1;\ y'(1) = -1 \Rightarrow c_1 - 3c_2 = -1 \Rightarrow c_1 = \frac{1}{2},\ c_2 = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + \frac{1}{2x^3}$

26. $6x^2y'' + 7xy' - 2y = 0,\ y(1) = 0,\ y'(1) = 1 \Rightarrow 6r^2 + (7-6)r - 2 = 0 \Rightarrow 6r^2 + r - 2 = 0 \Rightarrow (2r-1)(3r+2) = 0$ $\Rightarrow r = \frac{1}{2}$ or $r = -\frac{2}{3} \Rightarrow y = c_1e^{\frac{1}{2}z} + c_2e^{-\frac{2}{3}z} = c_1e^{\frac{1}{2}\ln x} + c_2e^{-\frac{2}{3}\ln x} \Rightarrow y = c_1\sqrt{x} + \frac{c_2}{x^{2/3}} \Rightarrow y' = \frac{c_1}{2\sqrt{x}} - \frac{2c_2}{3x^{5/3}}$; $y(1) = 0 \Rightarrow c_1 + c_2 = 0;\ y'(1) = 1 \Rightarrow \frac{1}{2}c_1 - \frac{2}{3}c_2 = 1 \Rightarrow c_1 = \frac{6}{7},\ c_2 = -\frac{6}{7} \Rightarrow y = \frac{6}{7}\sqrt{x} - \frac{6}{7x^{2/3}}$

27. $x^2y'' - xy' + y = 0,\ y(1) = 1,\ y'(1) = 1 \Rightarrow r^2 + (-1-1)r + 1 = 0 \Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1$, repeated twice $\Rightarrow y = c_1e^z + c_2z\,e^z = c_1e^{\ln x} + c_2 \ln x\, e^{\ln x} \Rightarrow y = c_1x + c_2x\ln x \Rightarrow y' = c_1 + c_2\ln x + c_2$; $y(1) = 1 \Rightarrow c_1 = 1;\ y'(1) = 1 \Rightarrow c_1 + c_2 = 1 \Rightarrow c_1 = 1,\ c_2 = 0 \Rightarrow y = x$

28. $x^2y'' + 7xy' + 9y = 0,\ y(1) = 1,\ y'(1) = 0 \Rightarrow r^2 + (7-1)r + 9 = 0 \Rightarrow r^2 + 6r + 9 = 0 \Rightarrow (r+3)^2 = 0 \Rightarrow r = -3$, repeated twice $\Rightarrow y = c_1e^{-3z} + c_2z\,e^{-3z} = c_1e^{-3\ln x} + c_2 \ln x\, e^{-3\ln x} \Rightarrow y = \frac{c_1}{x^3} + \frac{c_2\ln x}{x^3} \Rightarrow y' = -\frac{3c_1}{x^4} + c_2\frac{1-3\ln x}{x^4}$; $y(1) = 1 \Rightarrow c_1 = 1;\ y'(1) = 0 \Rightarrow -3c_1 + c_2 = 0 \Rightarrow c_1 = 1,\ c_2 = 3 \Rightarrow y = \frac{1}{x^3} + \frac{3\ln x}{x^3}$

29. $x^2y'' - xy' + 2y = 0,\ y(1) = -1,\ y'(1) = 1 \Rightarrow r^2 + (-1-1)r + 2 = 0 \Rightarrow r^2 - 2r + 2 = 0 \Rightarrow r = \frac{-(-2)\pm\sqrt{(-2)^2-4(1)(2)}}{2(1)}$ $= 1 \pm i \Rightarrow y = e^z(c_1\cos z + c_2\sin z) = e^{\ln x}(c_1\cos(\ln x) + c_2\sin(\ln x)) \Rightarrow y = x\,(c_1\cos(\ln x) + c_2\sin(\ln x))$ $\Rightarrow y' = (c_1 + c_2)\cos(\ln x) + (c_2 - c_1)\sin(\ln x);\ y(1) = -1 \Rightarrow c_1 = -1;\ y'(1) = 1 \Rightarrow c_1 + c_2 = 1 \Rightarrow c_1 = -1,\ c_2 = 2$ $\Rightarrow y = x\,(-\cos(\ln x) + 2\sin(\ln x))$

30. $x^2y'' + 3xy' + 5y = 0,\ y(1) = 1,\ y'(1) = 0 \Rightarrow r^2 + (3-1)r + 5 = 0 \Rightarrow r^2 + 2r + 5 = 0 \Rightarrow r = \frac{-2\pm\sqrt{(2)^2-4(1)(5)}}{2(1)}$ $= -1 \pm 2i \Rightarrow y = e^{-z}(c_1\cos 2z + c_2\sin 2z) = e^{-\ln x}(c_1\cos(2\ln x) + c_2\sin(2\ln x)) \Rightarrow y = \frac{1}{x}\,(c_1\cos(2\ln x) + c_2\sin(2\ln x))$ $\Rightarrow y' = -\frac{1}{x^2}\,((c_1 - 2c_2)\cos(2\ln x) + (2c_1 + c_2)\sin(2\ln x));\ y(1) = 1 \Rightarrow c_1 = 1;\ y'(1) = 0 \Rightarrow -c_1 + 2c_2 = 0 \Rightarrow c_1 = 1,$ $c_2 = \frac{1}{2} \Rightarrow y = \frac{1}{x}\left(\cos(2\ln x) + \frac{1}{2}\sin(2\ln x)\right)$

17.5 POWER-SERIES SOLUTIONS

1. $y'' + 2y' = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2\sum_{n=1}^{\infty} n c_n x^{n-1} = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 2n c_n x^{n-1} = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 + 2(1)c_1 = 0$	$\Rightarrow c_2 = -c_1$
x^1	$3(2)c_3 + 2(2)c_2 = 0$	$\Rightarrow c_3 = -\frac{2}{3}c_2 = \frac{2}{3}c_1$
x^2	$4(3)c_4 + 2(3)c_3 = 0$	$\Rightarrow c_4 = -\frac{1}{2}c_3 = -\frac{1}{3}c_1$
x^3	$5(4)c_5 + 2(4)c_4 = 0$	$\Rightarrow c_5 = -\frac{2}{5}c_4 = \frac{2}{15}c_1$
x^4	$6(5)c_6 + 2(5)c_5 = 0$	$\Rightarrow c_6 = -\frac{1}{3}c_5 = -\frac{2}{45}c_1$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + 2(n+1)c_{n+1} = 0$	$\Rightarrow c_{n+2} = -\frac{2}{n+2}c_{n+1}$

or $c_n = -\frac{2}{n}c_{n-1} = \left(-\frac{2}{n}\right)\left(-\frac{2}{n-1}c_{n-2}\right) = \left(-\frac{2}{n}\right)\left(-\frac{2}{n-1}\right)\left(-\frac{2}{n-2}c_{n-3}\right) = \frac{(-2)^{n-1}}{n!}c_1$, $n \ge 2$. Thus

$y = c_0 + c_1 x - c_1 x^2 + \frac{2}{3}c_1 x^3 - \frac{1}{3}c_1 x^4 + \frac{2}{15}c_1 x^5 - \frac{2}{45}c_1 x^6 + \ldots = c_0 + c_1\left(x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{2}{15}x^5 - \frac{2}{45}x^6 + \ldots\right)$

or $y = c_0 + \frac{c_1}{2} - \frac{c_1}{2} + \frac{c_1}{2}(2x) - \frac{c_1}{2}\frac{(2x)^2}{2!} + \frac{c_1}{2}\frac{(2x)^3}{3!} - \frac{c_1}{2}\frac{(2x)^4}{4!} + \frac{c_1}{2}\frac{(2x)^5}{5!} - \frac{c_1}{2}\frac{(2x)^6}{6!} + \ldots$

$y = \left(c_0 + \frac{c_1}{2}\right) - \frac{c_1}{2}\left(1 - (2x) + \frac{(2x)^2}{2!} - \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} - \frac{(2x)^5}{5!} + \frac{(2x)^6}{6!} + \ldots\right) = \left(c_0 + \frac{c_1}{2}\right) - \frac{c_1}{2}\sum_{n=0}^{\infty}\frac{(-2x)^n}{n!}$

$= \left(c_0 + \frac{c_1}{2}\right) - \frac{c_1}{2}e^{-2x} = a + be^{-2x}$, where $a = c_0 + \frac{c_1}{2}$ and $b = -\frac{c_1}{2}$

2. $y'' + 2y' + y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 2\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 2n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 + 2(1)c_1 + c_0 = 0$	$\Rightarrow c_2 = -c_1 - \frac{1}{2}c_0$
x^1	$3(2)c_3 + 2(2)c_2 + c_1 = 0$	$\Rightarrow c_3 = -\frac{2}{3}c_2 - \frac{1}{6}c_1 = \frac{1}{2}c_1 + \frac{1}{3}c_0$
x^2	$4(3)c_4 + 2(3)c_3 + c_2 = 0$	$\Rightarrow c_4 = -\frac{1}{2}c_3 - \frac{1}{12}c_2 = -\frac{1}{6}c_1 - \frac{1}{8}c_0$
x^3	$5(4)c_5 + 2(4)c_4 + c_3 = 0$	$\Rightarrow c_5 = -\frac{2}{5}c_4 - \frac{1}{20}c_3 = \frac{1}{24}c_1 + \frac{1}{30}c_0$
x^4	$6(5)c_6 + 2(5)c_5 + c_4 = 0$	$\Rightarrow c_6 = -\frac{1}{3}c_5 - \frac{1}{30}c_4 = -\frac{1}{120}c_1 - \frac{1}{144}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + 2(n+1)c_{n+1} + c_n = 0$	$\Rightarrow c_{n+2} = -\frac{2}{n+2}c_{n+1} - \frac{1}{(n+2)(n+1)}c_n$

$y = c_0 + c_1 x + \left(-c_1 - \frac{1}{2}c_0\right)x^2 + \left(\frac{1}{2}c_1 + \frac{1}{3}c_0\right)x^3 + \left(-\frac{1}{6}c_1 - \frac{1}{8}c_0\right)x^4 + \left(\frac{1}{24}c_1 + \frac{1}{30}c_0\right)x^5 + \left(-\frac{1}{120}c_1 - \frac{1}{144}c_0\right)x^6 + \ldots$

$= c_0 - \frac{1}{2}c_0 x^2 + \frac{1}{3}c_0 x^3 - \frac{1}{8}c_0 x^4 + \frac{1}{30}c_0 x^5 - \frac{1}{144}c_0 x^6 + \ldots + c_1 x - c_1 x^2 + \frac{1}{2}c_1 x^3 - \frac{1}{6}c_1 x^4 + \frac{1}{24}c_1 x^5 - \frac{1}{120}c_1 x^6 + \ldots$

$= c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{30}x^5 - \frac{1}{144}x^6 + \ldots\right) + c_1\left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 + \frac{1}{24}x^5 - \frac{1}{120}x^6 + \ldots\right)$

Note that in each coefficient equation, the coefficient of c_1 can be given by $\frac{(-1)^{n-1}}{(n-1)!}$. To find the coefficient of c_0, note that:

$-\frac{1}{2} = -\left(1 - \frac{1}{2}\right) = -\left(\frac{1}{1!} - \frac{1}{2!}\right)$, $\frac{1}{3} = \frac{1}{2} - \frac{1}{6} = = \frac{1}{2!} - \frac{1}{3!}$, $-\frac{1}{8} = -\left(\frac{1}{6} - \frac{1}{24}\right) = -\left(\frac{1}{3!} - \frac{1}{4!}\right)$, $\frac{1}{30} = \frac{1}{24} - \frac{1}{120} = \frac{1}{4!} - \frac{1}{5!}$

so the coefficient of c_0 can be given by $(-1)^{n-1}\left[\frac{1}{(n-1)!} - \frac{1}{n!}\right] = \frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}$. Thus $c_n = \left[\frac{(-1)^{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}\right]c_0 + \frac{(-1)^{n-1}}{(n-1)!}c_1$

or $c_n = \frac{(-1)^n}{n!}c_0 + \frac{(-1)^{n-1}}{(n-1)!}(c_0 + c_1)$ for $n \ge 2$. Thus $y = c_0 + c_1 x + \sum_{n=2}^{\infty}\left(\frac{(-1)^n}{n!}c_0 + \frac{(-1)^{n-1}}{(n-1)!}(c_0 + c_1)\right)x^n$

$= c_0 + c_1 x + c_0\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}x^n + c_0\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}x^n + c_1\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}x^n$

$= c_0 - c_0 x + c_0\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}x^n + c_0 x + c_0\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}x^n + c_1 x + c_1\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}x^n$

$= c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n + c_0 x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}x^{n-1} + c_1 x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}x^{n-1} = c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n + (c_0 + c_1)\,x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(n-1)!}x^{n-1}$

$= c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}x^n + (c_0 + c_1)\,x\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}x^k = c_0 e^{-x} + (c_0 + c_1)\,x\,e^{-x} = a\,e^{-x} + b\,x\,e^{-x}$, where $a = c_0$ and $b = c_0 + c_1$

3. $y'' + 4y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4\sum_{n=0}^{\infty} c_n x^n = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} 4c_n x^n = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 + 4c_0 = 0$	$\Rightarrow c_2 = -2c_0$
x^1	$3(2)c_3 + 4c_1 = 0$	$\Rightarrow c_3 = -\frac{2}{3}c_1$
x^2	$4(3)c_4 + 4c_2 = 0$	$\Rightarrow c_4 = -\frac{1}{3}c_2 = \frac{2}{3}c_0$
x^3	$5(4)c_5 + 4c_3 = 0$	$\Rightarrow c_5 = -\frac{1}{5}c_3 = \frac{2}{15}c_1$
x^4	$6(5)c_6 + 4c_4 = 0$	$\Rightarrow c_6 = -\frac{2}{15}c_4 = -\frac{4}{45}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + 4c_n = 0$	$\Rightarrow c_{n+2} = -\frac{4}{(n+2)(n+1)}c_n$

$y = c_0 + c_1 x - 2c_0 x^2 - \frac{2}{3}c_1 x^3 + \frac{2}{3}c_0 x^4 + \frac{2}{15}c_1 x^5 - \frac{4}{45}c_0 x^6 + \ldots$

$= c_0 - 2c_0 x^2 + \frac{2}{3}c_0 x^4 - \frac{4}{45}c_0 x^6 + \ldots + c_1 x - \frac{2}{3}c_1 x^3 + \frac{2}{15}c_1 x^5 + \ldots$

$= c_0\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \ldots\right) + \frac{c_1}{2}\left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} + \ldots\right) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(2x)^{2n} + \frac{c_1}{2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2x)^{2n+1}$

$= c_0 \cos 2x + \frac{c_1}{2}\sin 2x = a\cos 2x + b\sin 2x$, where $a = c_0$ and $b = \frac{c_1}{2}$

4. $y'' - 3y' + 2y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 3\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 3n c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 - 3(1)c_1 + 2c_0 = 0$	$\Rightarrow c_2 = \frac{3}{2}c_1 - c_0$
x^1	$3(2)c_3 - 3(2)c_2 + 2c_1 = 0$	$\Rightarrow c_3 = c_2 - \frac{1}{3}c_1 = \frac{7}{6}c_1 - c_0$
x^2	$4(3)c_4 - 3(3)c_3 + 2c_2 = 0$	$\Rightarrow c_4 = \frac{3}{4}c_3 - \frac{1}{6}c_2 = \frac{5}{8}c_1 - \frac{7}{12}c_0$
x^3	$5(4)c_5 - 3(4)c_4 + 2c_3 = 0$	$\Rightarrow c_5 = \frac{3}{5}c_4 - \frac{1}{10}c_3 = \frac{31}{120}c_1 - \frac{1}{4}c_0$
x^4	$6(5)c_6 - 3(5)c_5 + 2c_4 = 0$	$\Rightarrow c_6 = \frac{1}{2}c_5 - \frac{1}{15}c_4 = \frac{7}{80}c_1 - \frac{31}{360}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - 3(n+1)c_{n+1} + 2c_n = 0$	$\Rightarrow c_{n+2} = \frac{3}{n+2}c_{n+1} - \frac{2}{(n+2)(n+1)}c_n$

$y = c_0 + c_1 x + \left(\frac{3}{2}c_1 - c_0\right)x^2 + \left(\frac{7}{6}c_1 - c_0\right)x^3 + \left(\frac{5}{8}c_1 - \frac{7}{12}c_0\right)x^4 + \left(\frac{31}{120}c_1 - \frac{1}{4}c_0\right)x^5 + \left(\frac{7}{80}c_1 - \frac{31}{360}c_0\right)x^6 + \ldots$

$= c_0 - c_0 x^2 - c_0 x^3 - \frac{7}{12}c_0 x^4 - \frac{1}{4}c_0 x^5 - \frac{31}{360}c_0 x^6 + \ldots + c_1 x + \frac{3}{2}c_1 x^2 + \frac{7}{6}c_1 x^3 + \frac{5}{8}c_1 x^4 + \frac{31}{120}c_1 x^5 + \frac{7}{80}c_1 x^6 + \ldots$

$= c_0\left(1 - x^2 - \frac{1}{3}x^3 - \frac{7}{12}x^4 - \frac{1}{4}x^5 - \frac{31}{360}x^6 - \ldots\right) + c_1\left(x + \frac{3}{2}x^2 + \frac{7}{6}x^3 + \frac{5}{8}x^4 + \frac{31}{120}x^5 + \frac{7}{80}x^6 + \ldots\right)$

Note that if we use the techniques of Section 17.2, our solution is $y = a e^x + b e^{2x} = a\sum_{n=0}^{\infty} \frac{x^n}{n!} + b\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$. Thus

$(a+b) + (a+2b)x + \ldots = c_0 + c_1 x + \left(\frac{3}{2}c_1 - c_0\right)x^2 + \left(\frac{7}{6}c_1 - c_0\right)x^3 + \left(\frac{5}{8}c_1 - \frac{7}{12}c_0\right)x^4 + \left(\frac{31}{120}c_1 - \frac{1}{4}c_0\right)x^5 + \ldots$

If the series are equivalent, then take $c_0 = a + b$ and $c_1 = a + 2b$ and substitute in our first series for c_0 and c_1 to obtain

$y = (a+b) + (a+2b)x + \left(\frac{3}{2}(a+2b) - (a+b)\right)x^2 + \left(\frac{7}{6}(a+2b) - (a+b)\right)x^3 + \left(\frac{5}{8}(a+2b) - \frac{7}{12}(a+b)\right)x^4$
$\quad + \left(\frac{31}{120}(a+2b) - \frac{1}{4}(a+b)\right)x^5 + \left(\frac{7}{80}(a+2b) - \frac{31}{360}(a+b)\right)x^6 + \ldots$

$= (a+b) + (a+2b)x + \left(\frac{1}{2}a + 2b\right)x^2 + \left(\frac{1}{6}a + \frac{4}{3}b\right)x^3 + \left(\frac{1}{24}a + \frac{2}{3}b\right)x^4 + \left(\frac{1}{120}a + \frac{4}{15}b\right)x^5 + \left(\frac{1}{720}a + \frac{4}{45}b\right)x^6 + \ldots$

$= a + ax + \frac{1}{2}ax^2 + \frac{1}{6}ax^3 + \frac{1}{24}ax^4 + \frac{1}{120}ax^5 + \frac{1}{720}ax^6 + \ldots + b + 2bx + 2bx^2 + \frac{4}{3}bx^3 + \frac{2}{3}bx^4 + \frac{4}{15}bx^5 + \frac{4}{45}bx^6 + \ldots$

$= a\left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \ldots\right) + b\left(1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \frac{(2x)^5}{5!} + \frac{(2x)^6}{6!} + \ldots\right)$

$= a\sum_{n=0}^{\infty} \frac{x^n}{n!} + b\sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = a e^x + b e^{2x}$

5. $x^2y'' - 2xy' + 2y = 0 \Rightarrow x^2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} n\,c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 2n\,c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$

power of x	coefficient equation	
x^0	$2c_0 = 0$	$\Rightarrow c_0 = 0$
x^1	$-2(1)c_1 + 2c_1 = 0$	$\Rightarrow 0 = 0$
x^2	$2(1)c_2 - 2(2)c_2 + 2c_2 = 0$	$\Rightarrow 0 = 0$
x^3	$3(2)c_3 - 2(3)c_3 + 2c_3 = 0$	$\Rightarrow c_3 = 0$
x^4	$4(3)c_4 - 2(4)c_4 + 2c_4 = 0$	$\Rightarrow c_4 = 0$
$\vdots$	$\vdots$	$\vdots$
x^n	$n(n-1)c_n - 2n\,c_n + 2c_n = 0$	$\Rightarrow c_n = 0,\ n \ge 3$

$y = c_1 x + c_2 x^2$

6. $y'' - xy' + y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} n\,c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n\,c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 + c_0 = 0$	$\Rightarrow c_2 = -\frac{1}{2}c_0$
x^1	$3(2)c_3 - (1)c_1 + c_1 = 0$	$\Rightarrow c_3 = 0$
x^2	$4(3)c_4 - (2)c_2 + c_2 = 0$	$\Rightarrow c_4 = \frac{1}{12}c_2 = -\frac{1}{24}c_0$
x^3	$5(4)c_5 - (3)c_3 + c_3 = 0$	$\Rightarrow c_5 = \frac{1}{10}c_3 = 0$
x^4	$6(5)c_6 - (4)c_4 + c_4 = 0$	$\Rightarrow c_6 = \frac{2}{15}c_4 = -\frac{1}{180}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - n\,c_n + c_n = 0$	$\Rightarrow c_{n+2} = -\frac{n-1}{(n+2)(n+1)}c_n$

$y = c_0 + c_1 x - \frac{1}{2}c_0 x^2 - \frac{1}{24}c_0 x^4 - \frac{1}{180}c_0 x^6 + \ldots = c_0\left(1 - \frac{1}{2}x^2 - \frac{1}{24}x^4 - \frac{1}{180}x^6 + \ldots\right) + c_1 x$

7. $(1+x)y'' - y = 0 \Rightarrow (1+x)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 - c_0 = 0$	$\Rightarrow c_2 = \frac{1}{2}c_0$
x^1	$3(2)c_3 + 2(1)c_2 - c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{3}c_2 + \frac{1}{6}c_1 = \frac{1}{6}c_1 - \frac{1}{6}c_0$
x^2	$4(3)c_4 + 3(2)c_3 - c_2 = 0$	$\Rightarrow c_4 = -\frac{1}{2}c_3 + \frac{1}{12}c_2 = -\frac{1}{12}c_1 + \frac{1}{8}c_0$
x^3	$5(4)c_5 + 4(3)c_4 - c_3 = 0$	$\Rightarrow c_5 = -\frac{3}{5}c_4 + \frac{1}{12}c_3 = \frac{7}{120}c_1 - \frac{1}{12}c_0$
x^4	$6(5)c_6 + 5(4)c_5 - c_4 = 0$	$\Rightarrow c_6 = -\frac{2}{3}c_5 + \frac{1}{30}c_4 = -\frac{1}{24}c_1 + \frac{43}{720}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} + n(n+1)c_{n+1} - c_n = 0$	$\Rightarrow c_{n+2} = -\frac{n}{(n+2)}c_{n+1} + \frac{1}{(n+2)(n+1)}c_n$

$y = c_0 + c_1 x + \frac{1}{2}c_0 x^2 + \left(\frac{1}{6}c_1 - \frac{1}{6}c_0\right)x^3 + \left(-\frac{1}{12}c_1 + \frac{1}{8}c_0\right)x^4 + \left(\frac{7}{120}c_1 - \frac{1}{12}c_0\right)x^5 + \left(-\frac{1}{24}c_1 + \frac{43}{720}c_0\right)x^6 + \ldots$

$= c_0\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{8}x^4 - \frac{1}{12}x^5 + \frac{43}{720}x^6 + \ldots\right) + c_1\left(x + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{7}{120}x^5 - \frac{1}{24}x^6 + \ldots\right)$

8. $(1-x^2)y''-4xy'+6y=0 \Rightarrow (1-x^2)\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}-4x\sum_{n=1}^{\infty}n\,c_n x^{n-1}+6\sum_{n=0}^{\infty}c_n x^n=0$

$\Rightarrow \sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}-\sum_{n=2}^{\infty}n(n-1)c_n x^n-\sum_{n=1}^{\infty}4n\,c_n x^n+\sum_{n=0}^{\infty}6c_n x^n=0$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2+6c_0=0$	$\Rightarrow c_2=-3c_0$
x^1	$3(2)c_3-4(1)c_1+6c_1=0$	$\Rightarrow c_3=-\frac{1}{3}c_1$
x^2	$4(3)c_4-2(1)c_2-4(2)c_2+6c_2=0$	$\Rightarrow c_4=\frac{1}{3}c_2=-c_0$
x^3	$5(4)c_5-3(2)c_3-4(3)c_3+6c_3=0$	$\Rightarrow c_5=\frac{3}{5}c_3=-\frac{1}{5}c_1$
x^4	$6(5)c_6-4(3)c_4-4(4)c_4+6c_4=0$	$\Rightarrow c_6=\frac{11}{15}c_4=-\frac{11}{15}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2}-n(n-1)c_n-4nc_n+6c_n=0$	$\Rightarrow c_{n+2}=\frac{n^2+3n-6}{(n+2)(n+1)}c_n$

$y=c_0+c_1x-3c_0x^2-\frac{1}{3}c_1x^3-c_0x^4-\frac{1}{5}c_1x^5-\frac{11}{15}c_0x^6-\ldots$
$=c_0\left(1-3x^2-x^4-\frac{11}{15}x^6-\ldots\right)+c_1\left(x-\frac{1}{3}x^3-\frac{1}{5}x^5-\ldots\right)$

9. $(x^2-1)y''+2xy'-2y=0 \Rightarrow (x^2-1)\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}+2x\sum_{n=1}^{\infty}n\,c_n x^{n-1}-2\sum_{n=0}^{\infty}c_n x^n=0$

$\Rightarrow \sum_{n=2}^{\infty}n(n-1)c_n x^n-\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}+\sum_{n=1}^{\infty}2n\,c_n x^n-\sum_{n=0}^{\infty}2c_n x^n=0$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$-2(1)c_2-2c_0=0$	$\Rightarrow c_2=-c_0$
x^1	$-3(2)c_3+2(1)c_1-2c_1=0$	$\Rightarrow c_3=0$
x^2	$2(1)c_2-4(3)c_4+2(2)c_2-2c_2=0$	$\Rightarrow c_4=\frac{1}{3}c_2=-\frac{1}{3}c_0$
x^3	$3(2)c_3-5(4)c_5+2(3)c_3-2c_3=0$	$\Rightarrow c_5=\frac{1}{2}c_3=0$
x^4	$4(3)c_4-6(5)c_6+2(4)c_4-2c_4=0$	$\Rightarrow c_6=\frac{3}{5}c_4=-\frac{1}{5}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$n(n-1)c_n-(n+2)(n+1)c_{n+2}+2nc_n-2c_n=0$	$\Rightarrow c_{n+2}=\frac{n-1}{(n+1)}c_n$

$y=c_0+c_1x-c_0x^2-\frac{1}{3}c_0x^4-\frac{1}{5}c_0x^6-\ldots=c_0\left(1-x^2-\frac{1}{3}x^4-\frac{1}{5}x^6-\ldots\right)+c_1x$

10. $y''+y'-x^2y=0 \Rightarrow \sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}+\sum_{n=1}^{\infty}n\,c_n x^{n-1}-x^2\sum_{n=0}^{\infty}c_n x^n=0$

$\Rightarrow \sum_{n=2}^{\infty}n(n-1)c_n x^{n-2}+\sum_{n=1}^{\infty}n\,c_n x^{n-1}-\sum_{n=0}^{\infty}c_n x^{n+2}=0$

<u>power of x</u>	<u>coefficient equation</u>	
x^0	$2(1)c_2+(1)c_1=0$	$\Rightarrow c_2=-\frac{1}{2}c_1$
x^1	$3(2)c_3+(2)c_2=0$	$\Rightarrow c_3=-\frac{1}{3}c_2=\frac{1}{6}c_1$
x^2	$4(3)c_4+(3)c_3-c_0=0$	$\Rightarrow c_4=-\frac{1}{4}c_3+\frac{1}{12}c_0=-\frac{1}{24}c_1+\frac{1}{12}c_0$
x^3	$5(4)c_5+(4)c_4-c_1=0$	$\Rightarrow c_5=-\frac{1}{5}c_4+\frac{1}{20}c_1=\frac{1}{120}c_1+\frac{1}{30}c_0$
x^4	$6(5)c_6+(5)c_5-c_2=0$	$\Rightarrow c_6=-\frac{1}{6}c_5+\frac{1}{30}c_2=-\frac{13}{720}c_1-\frac{1}{180}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2}+(n+1)c_{n+1}-c_{n-2}=0$	$\Rightarrow c_{n+2}=-\frac{1}{(n+2)}c_{n+1}+\frac{1}{(n+2)(n+1)}c_{n-2}$

$y=c_0+c_1x-\frac{1}{2}c_0x^2+\frac{1}{6}c_1x^3+\left(-\frac{1}{24}c_1+\frac{1}{12}c_0\right)x^4+\left(\frac{1}{120}c_1+\frac{1}{30}c_0\right)x^5+\left(-\frac{13}{720}c_1-\frac{1}{180}c_0\right)x^6+\ldots$
$=c_0\left(1-\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{30}x^5-\frac{1}{180}x^6+\ldots\right)+c_1\left(x+\frac{1}{6}x^3-\frac{1}{24}x^4+\frac{1}{120}x^5-\frac{13}{720}x^6+\ldots\right)$

11. $(x^2-1)y''-6y=0 \Rightarrow (x^2-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 6\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} 6c_n x^n = 0$

power of x	coefficient equation	
x^0	$-2(1)c_2 - 6c_0 = 0$	$\Rightarrow c_2 = -3c_0$
x^1	$-3(2)c_3 - 6c_1 = 0$	$\Rightarrow c_3 = -c_1$
x^2	$2(1)c_2 - 4(3)c_4 - 6c_2 = 0$	$\Rightarrow c_4 = -\frac{1}{3}c_2 = c_0$
x^3	$3(2)c_3 - 5(4)c_5 - 6c_3 = 0$	$\Rightarrow c_5 = 0$
x^4	$4(3)c_4 - 6(5)c_6 - 6c_4 = 0$	$\Rightarrow c_6 = -\frac{1}{5}c_4 = -\frac{1}{5}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$n(n-1)c_n - (n+2)(n+1)c_{n+2} - 6c_n = 0$	$\Rightarrow c_{n+2} = -\frac{n-3}{(n+1)}c_n$

$y = c_0 + c_1 x - 3c_0 x^2 - c_1 x^3 + c_0 x^4 - \frac{1}{5}c_0 x^6 - \ldots = c_0\left(1 - 3x^2 + x^4 - \frac{1}{5}x^6 + \ldots\right) + c_1(x - x^3)$

12. $xy'' - (x+2)y' + 2y = 0 \Rightarrow x\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - (x+2)\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} - \sum_{n=1}^{\infty} n c_n x^n - \sum_{n=1}^{\infty} 2n c_n x^{n-1} + \sum_{n=0}^{\infty} 2c_n x^n = 0$

power of x	coefficient equation	
x^0	$-2(1)c_1 + 2c_0 = 0$	$\Rightarrow c_1 = c_0$
x^1	$2(1)c_2 - (1)c_1 - 2(2)c_2 + 2c_1 = 0$	$\Rightarrow c_2 = \frac{1}{2}c_1 = \frac{1}{2}c_0$
x^2	$3(2)c_3 - (2)c_2 - 2(3)c_3 + 2c_2 = 0$	$\Rightarrow 0 = 0$
x^3	$4(3)c_4 - (3)c_3 - 2(4)c_4 + 2c_3 = 0$	$\Rightarrow c_4 = \frac{1}{4}c_3$
x^4	$5(4)c_5 - (4)c_4 - 2(5)c_5 + 2c_4 = 0$	$\Rightarrow c_5 = \frac{1}{5}c_4 = \frac{1}{20}c_3$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+1)\,n\,c_{n+1} - n\,c_n - 2(n+1)c_{n+1} + 2c_n = 0$	$\Rightarrow c_{n+1} = \frac{1}{(n+1)}c_n$

$y = c_0 + c_0 x + \frac{1}{2}c_0 x^2 + c_3 x^3 + \frac{1}{4}c_3 x^4 + \frac{1}{20}c_3 x^5 + \ldots = c_0\left(1 + x + \frac{1}{2}x^2\right) + c_3\left(x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5 + \ldots\right)$

13. $(x^2-1)y'' + 4xy' + 2y = 0 \Rightarrow (x^2-1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 4x\sum_{n=1}^{\infty} n c_n x^{n-1} + 2\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$

power of x	coefficient equation	
x^0	$-2(1)c_2 + 2c_0 = 0$	$\Rightarrow c_2 = c_0$
x^1	$-3(2)c_3 + 4(1)c_1 + 2c_1 = 0$	$\Rightarrow c_3 = c_1$
x^2	$2(1)c_2 - 4(3)c_4 + 4(2)c_2 + 2c_2 = 0$	$\Rightarrow c_4 = c_2 = c_0$
x^3	$3(2)c_3 - 5(4)c_5 + 4(3)c_3 + 2c_3 = 0$	$\Rightarrow c_5 = c_3 = c_1$
x^4	$4(3)c_4 - 6(5)c_6 + 4(4)c_4 + 2c_4 = 0$	$\Rightarrow c_6 = c_4 = c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$n(n-1)c_n - (n+2)(n+1)c_{n+2} + 4nc_n + 2c_n = 0$	$\Rightarrow c_{n+2} = c_n$

$y = c_0 + c_1 x + c_0 x^2 + c_3 x^3 + c_1 x^4 + c_1 x^5 + c_0 x^6 - \ldots = c_0(1 + x^2 + x^4 + x^6 + \ldots) + c_1(x + x^3 + x^5 + \ldots)$

14. $y'' - 2xy' + 4y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} n c_n x^{n-1} + 4\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 4c_n x^n = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 + 4c_0 = 0$	$\Rightarrow c_2 = -2c_0$
x^1	$3(2)c_3 - 2(1)c_1 + 4c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{3}c_1$
x^2	$4(3)c_4 - 2(2)c_2 + 4c_2 = 0$	$\Rightarrow c_4 = 0$
x^3	$5(4)c_5 - 2(3)c_3 + 4c_3 = 0$	$\Rightarrow c_5 = \frac{1}{10}c_3 = -\frac{1}{30}c_1$
x^4	$6(5)c_6 - 2(4)c_4 + 4c_4 = 0$	$\Rightarrow c_6 = \frac{2}{15}c_4 = 0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - 2n c_n + 4c_n = 0$	$\Rightarrow c_{n+2} = \frac{2(n-1)}{(n+2)(n+1)}c_n$

$y = c_0 + c_1 x - 2c_0 x^2 - \frac{1}{3}c_1 x^3 - \frac{1}{30}c_1 x^5 - \ldots = c_0(1 - 2x^2) + c_1\left(x - \frac{1}{3}x^3 - \frac{1}{30}x^5 - \ldots\right)$

15. $y'' - 2xy' + 3y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 2n c_n x^n + \sum_{n=0}^{\infty} 3c_n x^n = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 + 3c_0 = 0$	$\Rightarrow c_2 = -\frac{3}{2}c_0$
x^1	$3(2)c_3 - 2(1)c_1 + 3c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{6}c_1$
x^2	$4(3)c_4 - 2(2)c_2 + 3c_2 = 0$	$\Rightarrow c_4 = \frac{1}{12}c_2 = -\frac{1}{8}c_0$
x^3	$5(4)c_5 - 2(3)c_3 + 3c_3 = 0$	$\Rightarrow c_5 = \frac{3}{20}c_3 = -\frac{1}{40}c_1$
x^4	$6(5)c_6 - 2(4)c_4 + 3c_4 = 0$	$\Rightarrow c_6 = \frac{1}{6}c_4 = -\frac{1}{48}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - 2n c_n + 3c_n = 0$	$\Rightarrow c_{n+2} = \frac{2n-3}{(n+2)(n+1)}c_n$

$y = c_0 + c_1 x - \frac{3}{2}c_0 x^2 - \frac{1}{6}c_1 x^3 - \frac{1}{8}c_0 x^4 - \frac{1}{40}c_1 x^5 - \frac{1}{48}c_0 x^6 - \ldots$
$= c_0\left(1 - \frac{3}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{48}x^6 - \ldots\right) + c_1\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - \ldots\right)$

16. $(1 - x^2)y'' - xy' + 4y = 0 \Rightarrow (1 - x^2)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} n c_n x^{n-1} + 4\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 4c_n x^n = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 + 4c_0 = 0$	$\Rightarrow c_2 = -2c_0$
x^1	$3(2)c_3 - (1)c_1 + 4c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{2}c_1$
x^2	$4(3)c_4 - 2(1)c_2 - (2)c_2 + 4c_2 = 0$	$\Rightarrow c_4 = 0$
x^3	$5(4)c_5 - 3(2)c_3 - (3)c_3 + 4c_3 = 0$	$\Rightarrow c_5 = \frac{1}{4}c_3 = -\frac{1}{8}c_1$
x^4	$6(5)c_6 - 4(3)c_4 - (4)c_4 + 4c_4 = 0$	$\Rightarrow c_6 = \frac{2}{5}c_4 = 0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - n(n-1)c_n - nc_n + 4c_n = 0$	$\Rightarrow c_{n+2} = \frac{n-2}{(n+1)}c_n$

$y = c_0 + c_1 x - 2c_0 x^2 - \frac{1}{2}c_1 x - \frac{1}{8}c_1 x^5 - \ldots = c_0(1 - x^2) + c_1\left(x - \frac{1}{2}x^3 - \frac{1}{8}x^5 - \ldots\right)$

17. $y'' - xy' + 3y = 0 \Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=1}^{\infty} n c_n x^{n-1} + 3\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 3c_n x^n = 0$

power of x	coefficient equation	
x^0	$2(1)c_2 + 3c_0 = 0$	$\Rightarrow c_2 = -\frac{3}{2}c_0$
x^1	$3(2)c_3 - (1)c_1 + 3c_1 = 0$	$\Rightarrow c_3 = -\frac{1}{3}c_1$
x^2	$4(3)c_4 - (2)c_2 + 3c_2 = 0$	$\Rightarrow c_4 = -\frac{1}{12}c_2 = \frac{1}{8}c_0$
x^3	$5(4)c_5 - (3)c_3 + 3c_3 = 0$	$\Rightarrow c_5 = 0$
x^4	$6(5)c_6 - (4)c_4 + 3c_4 = 0$	$\Rightarrow c_6 = \frac{1}{30}c_4 = \frac{1}{240}c_0$
$\vdots$	$\vdots$	$\vdots$
x^n	$(n+2)(n+1)c_{n+2} - n c_n + 3c_n = 0$	$\Rightarrow c_{n+2} = \frac{n-3}{(n+2)(n+1)}c_n$

$y = c_0 + c_1 x - \frac{3}{2}c_0 x^2 - \frac{1}{6}c_1 x^3 + \frac{1}{8}c_0 x^4 + \frac{1}{240}c_0 x^6 - \ldots = c_0\left(1 - \frac{3}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{240}x^6 + \ldots\right) + c_1\left(x - \frac{1}{3}x^3\right)$

18. $x^2y'' - 4xy' + 6y = 0 \Rightarrow x^2\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 4x\sum_{n=1}^{\infty} n c_n x^{n-1} + 6\sum_{n=0}^{\infty} c_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=1}^{\infty} 4n c_n x^n + \sum_{n=0}^{\infty} 6c_n x^n = 0$

power of x	coefficient equation	
x^0	$6c_0 = 0$	$\Rightarrow c_0 = 0$
x^1	$-4(1)c_1 + 6c_1 = 0$	$\Rightarrow c_1 = 0$
x^2	$2(1)c_2 - 4(2)c_2 + 6c_2 = 0$	$\Rightarrow 0 = 0$
x^3	$3(2)c_3 - 4(3)c_3 + 6c_3 = 0$	$\Rightarrow 0 = 0$
x^4	$4(3)c_4 - 4(4)c_4 + 6c_4 = 0$	$\Rightarrow c_4 = 0$
$\vdots$	$\vdots$	$\vdots$
x^n	$n(n-1)c_n - 4n c_n + 6c_n = 0$	$\Rightarrow c_n = 0$

$y = c_2 x^2 + c_3 x^3$