Signals, Systems & Inference

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CHAPTER 2 SOLUTIONS

Note from the authors

These solutions represent a preliminary version of the Instructors' Solutions Manual (ISM). The book has a total of 350 problems, so it is possible and even likely that at this preliminary stage of preparing the ISM there are some omissions and errors in the draft solutions. It is also possible that an occasional problem in the book is now slightly different from an earlier version for which the solution here was generated. It is therefore important for an instructor to carefully review the solutions to problems of interest, and to modify them as needed. We will, from time to time, update these solutions with clarifications, elaborations, or corrections.

Many of these solutions have been prepared by the teaching assistants for the course in which this material has been taught at MIT, and their assistance is individually acknowledged in the book. For preparing solutions to the remaining problems in recent months, we are particularly grateful to Abubakar Abid (who also constructed the solution template), Leighton Barnes, Fisher Jepsen, Tarek Lahlou, Catherine Medlock, Lucas Nissenbaum, Ehimwenma Nosakhare, Juan Miguel Rodriguez, Andrew Song, and Guolong Su. We would also like to thank Laura von Bosau for her assistance in compiling the solutions.

Solution 2.1

(a) Recall that the phase delay and group delay of a frequency response $H(j\omega)$ are respectively defined by

$$\tau_p(\omega) = -\frac{\angle H(j\omega)}{\omega} \quad \text{and} \quad \tau_g(\omega) = -\frac{d}{d\omega}\angle_A H(j\omega)$$

where $\angle H(j\omega)$ may contain discontinuities of size $\pm\pi$ in order to compensate for the non-negativity of the magnitude function $|H(j\omega)|$ while the function $\angle_A H(j\omega)$ does not. For the frequency response $H(j\omega) = 3e^{-j3\omega}$ we have that $A(\omega) = |H(j\omega)| = 3$ for all ω hence $\angle_A H(j\omega) = \angle H(j\omega) = -3\omega$. Plugging this into the expressions above yields $\tau_p(\omega) = \tau_g(\omega) = 3$ for all ω.

TRUE **FALSE**

(b) This statement is true by the combined definition of eigenfunction and time-invariance properties.

TRUE **FALSE**

(c) Let $x(t)$ be a linear combination of $K > 1$ eigenfunctions, i.e. $x(t) = \sum_{k=1}^{K} \alpha_k e^{j\omega_k t}$, and observe that

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h(\tau) \sum_{k=1}^{K} \alpha_k e^{j\omega_k(t-\tau)} d\tau \\ &= \sum_{k=1}^{K} \alpha_k e^{j\omega_k t} \int_{-\infty}^{\infty} h(\tau) e^{-j\omega_k \tau} d\tau \\ &= \sum_{k=1}^{K} \alpha_k H(e^{j\omega_k}) e^{j\omega_k t} \end{aligned}$$

which is in general not equal to a single scalar times $x(t)$.

TRUE **FALSE**

For the remaining parts, we refer to the system characterized by:

$$h[n] = \left(\frac{2}{3}\right)^n u[n] - \delta[n] \qquad H(e^{j\Omega}) = \frac{\frac{2}{3}e^{-j\Omega}}{1 - \frac{2}{3}e^{-j\Omega}}.$$

(d) The system is causal since $h[n] = 0$ for all $n < 0$.

TRUE **FALSE**

(e) The system is BIBO stable since the impulse response is absolutely summable, i.e.

$$\sum_{n=0}^{\infty} |h[n]| < \infty.$$

This result can also be established by the fact that the ROC of the z-transform of $h[n]$ includes the unit circle. We may conclude this since we know that the Fourier transform $H(e^{j\Omega})$ exists.

TRUE **FALSE**

(f) The input $x[n] = (-1)^n$ for all n is an eigenfunction of a DT LTI system. This may be more visually apparent by rewriting $x[n]$ as $x[n] = e^{j\pi n}$ for all n. Therefore, from (b) we are guaranteed that the output is purely a (possibly complex) scalar times $x[n]$ and therefore does not take the form

$$y[n] = K_1(-1)^n + K_2\left(\frac{2}{3}\right)^{n-1} u[n-1]$$

since K_2 is expressly forbidden to be zero.

TRUE **FALSE**

Solution 2.2

(a) This part deals with an input signal $x[n] = \cos\left(\frac{\pi}{3}n\right) = \frac{1}{2}e^{j\frac{\pi}{3}n} + \frac{1}{2}e^{-j\frac{\pi}{3}n}$.

The output is given by (ii) $y[n] = 2x[n-1]$.

A short derivation reveals this:

$$
\begin{aligned}
y[n] &= \frac{1}{2}H\left(e^{j\frac{\pi}{3}}\right)e^{j\frac{\pi}{3}n} + \frac{1}{2}H\left(e^{-j\frac{\pi}{3}}\right)e^{-j\frac{\pi}{3}n} \\
&= \frac{1}{2}\left(2e^{-j\frac{\pi}{3}}\right)e^{j\frac{\pi}{3}n} + \frac{1}{2}\left(2e^{j\frac{\pi}{3}}\right)e^{-j\frac{\pi}{3}n} \\
&= e^{j\frac{\pi}{3}(n-1)} + e^{-j\frac{\pi}{3}(n-1)} \\
&= 2\cos\left(\frac{\pi}{3}(n-1)\right) \\
&= 2x[n-1]
\end{aligned}
$$

(b) This part deals with an input signal $x[n] = s[n]\cos\left(\frac{2\pi}{3}n\right) = \frac{1}{2}s[n]e^{j\frac{2\pi}{3}n} + \frac{1}{2}s[n]e^{-j\frac{2\pi}{3}n}$.

The output is approximately given by (ii) $y[n] = 2s[n]\cos\left(\frac{2\pi}{3}(n-1)\right)$.

Assume $s[n]$ is sufficiently bandlimited so that we may simply apply $H(e^{j\Omega})$ to the complex exponentials. We do not need to worry about applying a group delay shift to $s[n]$ since the phase curve has no slope at $\Omega = \pm\frac{2\pi}{3}$.

$$
\begin{aligned}
y[n] &\approx \frac{1}{2}s[n]H\left(e^{j\frac{2\pi}{3}}\right)e^{j\frac{2\pi}{3}n} + \frac{1}{2}s[n]H\left(e^{-j\frac{2\pi}{3}}\right)e^{-j\frac{2\pi}{3}n} \\
&= \frac{1}{2}s[n]\left(2e^{-j\frac{2\pi}{3}}\right)e^{j\frac{2\pi}{3}n} + \frac{1}{2}s[n]\left(2e^{j\frac{2\pi}{3}}\right)e^{-j\frac{2\pi}{3}n} \\
&= s[n]e^{j\frac{2\pi}{3}(n-1)} + s[n]e^{-j\frac{2\pi}{3}(n-1)} \\
&= 2s[n]\cos\left(\frac{2\pi}{3}(n-1)\right)
\end{aligned}
$$

Solution 2.3

We are given the squared magnitude frequency response of a filter $H(j\omega)$ to be

$$|H(j\omega)|^2 = \frac{\omega^2 + 1}{\omega^2 + 100}.$$

Making the substitution $\omega^2 \to -s^2$ gives

$$\begin{aligned} H(s)H(-s) &= \frac{-s^2 + 1}{-s^2 + 100} \\ &= \frac{(-s+1)(s+1)}{(-s+10)(s+10)}. \end{aligned}$$

We now factor the terms above in order to identify $H(s)$. Since the system is both causal and stable, as well as has a causal and stable inverse, then the filters poles and zeros must lie in the left half plane. Therefore,

$$H(s) = K\frac{s+1}{s+10}$$

where K may be either plus or minus one. Substituting $s = j\omega$ gives

$$H(j\omega) = \frac{j\omega + 1}{j\omega + 10}$$

where we have taken $K = 1$ so that $H(j\omega) > 0$ at $\omega = 0$.

Solution 2.4

We are given a DT LTI system with impulse response $h[n] = \delta[n] - \delta[n-1]$.

a We write the frequency response of $h[n]$ as

$$\begin{aligned} H\left(e^{j\Omega}\right) &= 1 - e^{-j\Omega} \\ &= e^{-j\frac{\Omega}{2}}\left(e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right) \\ &= 2je^{-j\frac{\Omega}{2}}\sin\left(\frac{\Omega}{2}\right) \\ &= 2e^{-j\left(\frac{\Omega}{2}-\frac{\pi}{2}\right)}\sin\left(\frac{\Omega}{2}\right) \end{aligned}$$

where we use the identity $j = e^{j\frac{\pi}{2}}$. Therefore $\Theta(\Omega) = -\frac{\Omega}{2} + \frac{\pi}{2}$ for $-\pi \leq \Omega < \pi$. The group delay of this system is then given by

$$\tau_g(\Omega) = -\frac{d}{d\Omega}\left(-\frac{\Omega}{2} + \frac{\pi}{2}\right) = \frac{1}{2}$$

for $-\pi \leq \Omega < \pi$. The phase delay must take into account the fact that the amplitude is negative for $\pi < \Omega < 0$, hence

$$\tau_p(\Omega) = \begin{cases} \frac{1}{2} - \frac{\pi}{2\Omega}, & \Omega \in [0, \pi) \\ \frac{1}{2} + \frac{\pi}{2\Omega}, & \Omega \in [-\pi, 0) \end{cases} .$$

(b) We have that $H\left(e^{j\frac{\pi}{3}}\right) = 1 - e^{-j\frac{\pi}{3}} = e^{j\frac{\pi}{3}}$ and likewise that $H\left(e^{-j\frac{\pi}{3}}\right) = e^{-j\frac{\pi}{3}}$.

We can now write

$$\begin{aligned} y[n] &\approx \frac{1}{2}q[n]H\left(e^{j\frac{\pi}{3}}\right)e^{j\frac{\pi}{3}n} + \frac{1}{2}q[n]H\left(e^{-j\frac{\pi}{3}}\right)e^{-j\frac{\pi}{3}n} \\ &= q[n]\cos\left(\frac{\pi(n-\eta_0)}{3}\right) \end{aligned}$$

where $\eta_0 = \tau_p(\frac{\pi}{3}) = -1$. More generally, $\eta_0 = -1 + 6k$ for any integer k.

Loosely speaking, $q[n]$ is related to $p[n]$ by $q[n] = p$"$[n - \tau_g(\frac{\pi}{3})]$" where the quotes indicate further clarification is needed. The half-sample delay of the envelope $p[n]$ by the group delay is to be interpreted as if there was an underlying appropriately bandlimited CT signal $p(t)$ satisfying $p[n] = p(nT)$ for some arbitrary sampling interval T, say $T = 1$ without loss of generality, which then got shifted by $T/2 = \frac{1}{2}$ and then resampled, i.e., $q[n] = p(n - \frac{1}{2})$.

Solution 2.5

Consider the causal LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = x(t).$$

(a) Taking Laplace transforms of both sides gives

$$Y(s)(s+2) = X(s).$$

Solving for $H(s) = \frac{Y(s)}{X(s)}$ and substituting $s = j\omega$ yields

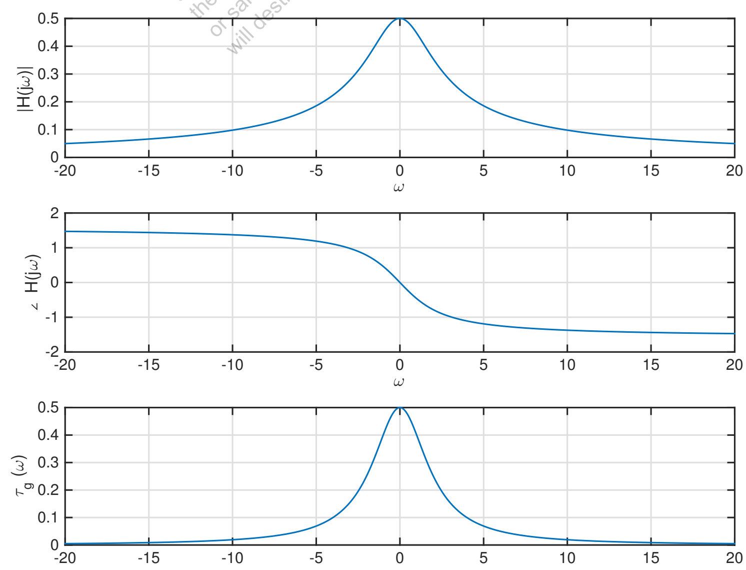
$$H(j\omega) = \frac{1}{j\omega + 2}.$$

The magnitude and phase responses of $H(j\omega)$ are depicted with the result of (b) below.

(b) Recall that the phase response of a system corresponds to the contributions of the zeros minus the contributions from the poles. Therefore, from (a) we conclude that $\angle_A H(j\omega) = -\tan^{-1}\left(\frac{\omega}{2}\right)$. The group delay is then computed by

$$\begin{aligned}\tau_g(\omega) &= -\frac{d}{d\omega}\left(-\tan^{-1}\left(\frac{\omega}{2}\right)\right)\\ &= \frac{2}{\omega^2+4}\end{aligned}$$

and is depicted in the figure below.



Solution 2.6

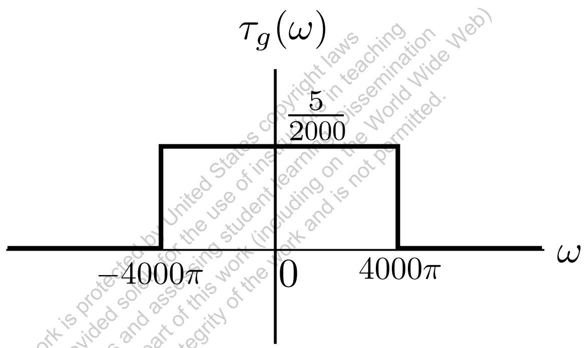
(a) First, we re-write the piecewise linear function $\angle H(j\omega)$ in terms of ω as

$$\angle H(j\omega) = \begin{cases} 10\pi - \frac{5}{2000}\omega, & \omega \in [0, 4000\pi] \\ -10\pi - \frac{5}{2000}\omega, & \omega \in [-4000\pi, 0) \\ 0, & \text{elsewhere} \end{cases} .$$

Computing the group-delay $\tau_g(\omega) = -\frac{d}{d\omega}\angle H(j\omega)$ then gives

$$\tau_g(\omega) = \begin{cases} \frac{5}{2000}, & \omega \in [-4000\pi, 4000\pi] \\ 0, & \text{elsewhere} \end{cases}$$

where we have chosen the value at DC to maintain continuity. A plot of the group delay is shown below.



(b) In obtaining an expression for $y(t)$ we make use of the fact the bandwidth of $p(t)$ is small as compared with w_1 and w_2 allowing the approximations discussed in Section 2.2.1 to hold. In particular, we make use of the following processing approximations:

$$\begin{aligned} h(t) * a_n p(t - nT)\cos(\omega_1 t) &\approx a_n p(t - nT - \tau_g(\omega_1))\cos(\omega_1(t - \tau_p(\omega_1))) \\ h(t) * b_n p(t - nT)\cos(\omega_2 t) &\approx b_n p(t - nT - \tau_g(\omega_2))\cos(\omega_2(t - \tau_p(\omega_2))) . \end{aligned}$$

From (a) we compute the following values: $\tau_g(\omega_1) = \frac{5}{2000}$, $\tau_g(\omega_2) = 0$, $\tau_p(\omega_1) = \frac{5}{2000}$, and $\tau_p(\omega_2) = 0$ Finally, we obtain an expression for the output using the linearity of the channel as:

$$\begin{aligned} y(t) &\approx \sum_{n=-\infty}^{\infty} a_n p(t - nT - \tau_g(\omega_1))\cos(\omega_1(t - \tau_p(\omega_1))) + \sum_{n=-\infty}^{\infty} b_n p(t - nT - \tau_g(\omega_2))\cos(\omega_2(t - \tau_p(\omega_2))) \\ &= \sum_{n=-\infty}^{\infty} a_n p\left(t - nT - \frac{1}{400}\right)\cos\left(\omega_1\left(t - \frac{1}{400}\right)\right) + \sum_{n=-\infty}^{\infty} b_n p(t - nT)\cos(\omega_2 t) \\ &= \sum_{n=-\infty}^{\infty} a_n p\left(t - nT - \frac{1}{400}\right)\cos\left(2\pi \times 1000\left(t - \frac{1}{400}\right)\right) + \sum_{n=-\infty}^{\infty} b_n p(t - nT)\cos((2\pi \times 3000)t) \end{aligned}$$

Solution 2.7

$\mathbf{y_2[n]}$

By examining the magnitude response of the filter, we see that the packet with the lowest frequency in the input is attenuated, while the intermediate frequency is only slightly attenuated relative to the highest frequency. The limits the choices to $y_2[n]$ and $y_4[n]$. The group delay of the filter shows that the packet with the highest frequency in the input is delayed by approximately 75 samples, and the intermediate-frequency packet is delayed by approximately 40 samples. These facts point to $y_2[n]$.

Solution 2.8

$G(z)$ has a pole at $z = 0.7$ and a zero at infinity. This system does not have a causal and stable inverse, because the inverse system must have a pole at infinity, which means that the system cannot be both causal and stable.

Two decompositions of $G(z)$ into a minimum-phase and an all-pass system are:

$$G_{MP}(z) = \pm\frac{1}{1 - 0.7z^{-1}} \ , \quad |z| > 0.7$$

$$G_{AP}(z) = \pm z^{-1} \ , \quad |z| > 0 \ .$$

Solution 2.9

(a) We know

$$G(s)G(-s) = A(s)A(-s)M(s)M(-s) = M(s)M(-s) \ ,$$

so $M(s)$ must have the left-half-plane poles and zeros of

$$G(s)G(-s) = \frac{(s-2)(-s-2)}{(s+1)(-s+1)} = \frac{(s-2)(s+2)}{(s+1)(s-1)} \ .$$

Thus

$$M(s) = \pm\frac{s+2}{s+1} \ ,$$

and

$$A(s) = \frac{G(s)}{M(s)} = \pm\frac{s-2}{s+2} \ .$$

All-pass $A(s) = \pm\frac{s-2}{s+2}$. Minimum-phase $M(s) = \pm\frac{s+2}{s+1}$.

(b) Similarly,

$$H(z)H(z^{-1}) = (1-6z)(1-6z^{-1}) = B(z)B(z^{-1})N(z)N(z^{-1}) = N(z)N(z^{-1}) \ ,$$

so $N(z)$ must have the poles and zeros of this that are inside the unit circle. The zeros are at $\frac{1}{6}$ and 6, while the poles are at 0 and ∞. Hence

$$N(z) = K\,\frac{z-\frac{1}{6}}{z} = K(1-\frac{1}{6}z^{-1}) \ ,$$

and comparing with the preceding expression shows $K = \pm 6$. Then

$$B(z) = \frac{H(z)}{N(z)} = \mp z \ .$$

All-pass $B(z) = \mp z$, Minimum-phase $N(z) = \pm 6(1-\frac{1}{6}z^{-1})$.

Solution 2.10

If we represent the frequency responses using a magnitude and phase decomposition,

$$H_1(j\omega) = |H_1(j\omega)|e^{j\angle H_1(j\omega)} \text{ and } H_2(j\omega) = |H_2(j\omega)|e^{j\angle H_2(j\omega)}\ ,$$

then the cascade of the two systems has frequency response

$$H(j\omega) = H_1(j\omega)H_2(j\omega) = |H_1(j\omega)H_2(j\omega)|e^{j\angle H_1(j\omega)+j\angle H_2(j\omega)}.$$

Therefore, the group delay of the cascaded system will be

$$\tau_g = -\frac{d}{d\omega}\angle H(j\omega) = -\frac{d}{d\omega}(\angle H_1(j\omega) + \angle H_2(j\omega)) = \tau_{g1} + \tau_{g2}.$$

Solution 2.11

$\boxed{y_2[n]}$ is the most likely output signal for the given system.

The filter is a low-pass, so the high frequency components are removed, so it cannot be either $y_1[n]$ or $y_3[n]$ since they still contain the high frequency pulse.

There should be about a 40-sample delay of the low-frequency pulse and an 80-sample delay of the mid-frequency component. $y_4[n]$ has no delay for the mid-frequency component, but $y_2[n]$ has both of these delays.

Solution 2.12

Consider the DT causal LTI system with frequency response

$$H\left(e^{j\Omega}\right) = e^{-j4\Omega}\frac{1-\frac{1}{2}e^{j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}.$$

(a) The magnitude of the cascade of two systems is the product of the magnitudes and so

$$\begin{aligned}\left|H(e^{j\Omega})\right| &= \left|e^{-j4\Omega}\right| \cdot \left|\frac{1-\frac{1}{2}e^{j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}\right| \\ &= \left|\frac{1-\frac{1}{2}e^{j\Omega}}{1-\frac{1}{2}e^{-j\Omega}}\right|\end{aligned}$$

since $e^{-j4\Omega}$ is a linear phase factor and thus is an all-pass system with unity gain. Consistent with Eq. (2.28), the remaining term is an all-pass term since it corresponds to a system function $H(z)$ whose pole and zero are in a conjugate reciprocal location. This can be seen by replacing $e^{j\Omega}$ by z and observing the pole is at $z = \frac{1}{2}$ and the zero is at $z = 2$. Since the frequency response magnitude is constant, evaluating it for any value of Ω provides its value for all Ω. Substituting $\Omega = 0$ above gives

$$\begin{aligned}\left|H(e^{j0})\right| &= \left|\frac{1-\frac{1}{2}}{1-\frac{1}{2}}\right| \\ &= 1.\end{aligned}$$

(b) We are given that the group delay of the system

$$F\left(e^{j\Omega}\right) = e^{j3\Omega}H\left(e^{j\Omega}\right)$$

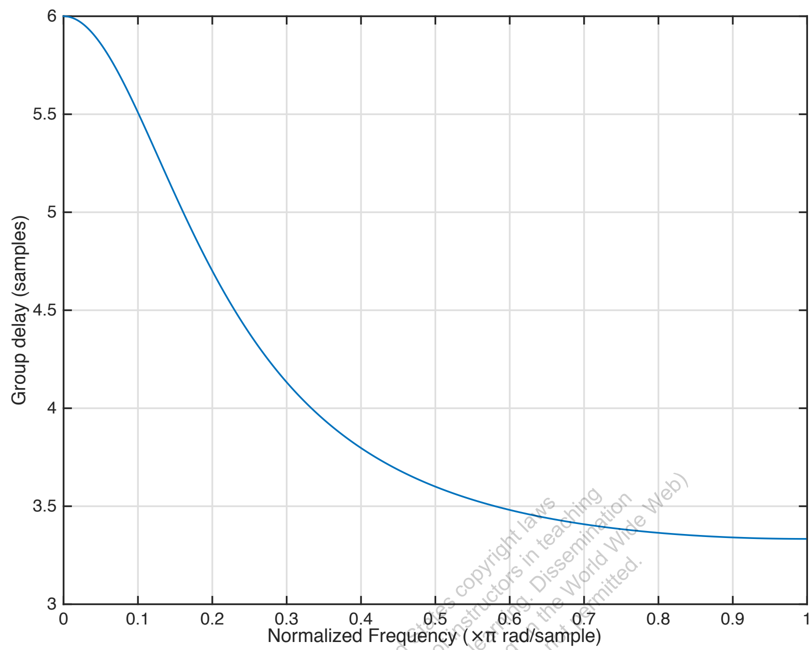
is given by

$$\tau_{g,F}(\Omega) = \frac{\frac{3}{4}}{\frac{5}{4}-\cos\Omega}.$$

Note that $F(e^{j\Omega})$ only differs from $H\left(e^{j\Omega}\right)$ by a linear phase factor. Moreover, the phases (and group delays) of a cascade of systems are the sum of the phases (and group delays), therefore it follows that the group delay of $H\left(e^{j\Omega}\right)$ is the group delay of $F(e^{j\Omega})$ raised by 3 for all Ω, i.e.

$$\begin{aligned}\tau_{g,H}(\Omega) &= \tau_{g,F}(\Omega)+3 \\ &= \frac{\frac{3}{4}}{\frac{5}{4}-\cos\Omega}+3.\end{aligned}$$

The group delay curve $\tau_{g,H}(\Omega)$ is plotted below for $\Omega \in [0,\pi]$.



(c) We know the magnitude of the input signal $x[n] = \cos(0.1n)\cos\left(\frac{\pi}{3}n\right)$ is preserved since the system is an all-pass with unity gain. We make use of the fact that the frequency content of $\cos(0.1n)$ is concentrated (impulses in fact) and that $\Omega = 0.1$ is sufficiently small as compared to $\Omega = \frac{\pi}{3}$ so that the approximation

$$y[n] \approx \cos\left(0.1\left(n - \tau_{g,H}\left(\frac{\pi}{3}\right)\right)\right)\cos\left(\frac{\pi}{3}\left(n - \tau_{p,H}\left(\frac{\pi}{3}\right)\right)\right)$$

is reasonable. Using the expression obtained in (b) we have that $\tau_{g,H}\left(\frac{\pi}{3}\right) = 4$ and that $\tau_{p,H}\left(\frac{\pi}{3}\right) = 1$ therefore

$$y[n] \approx \cos(0.1(n-4))\cos\left(\frac{\pi}{3}(n-1)\right).$$

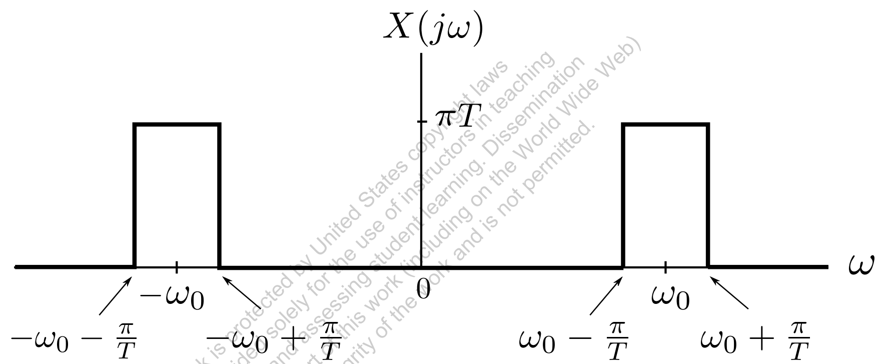
Solution 2.13

The input signal is $x(t) = m(t)\cos(\omega_0 t)$ where $m(t) = \frac{\sin(\pi t/T)}{\pi t/T}$.

(a) Sketch $X(j\omega)$. Recall the Fourier transform pairs

$$\begin{aligned}\cos(\omega_0 t) &\overset{FT}{\longleftrightarrow} \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]\\ \frac{\sin(\pi t/T)}{\pi t/T} &\overset{FT}{\longleftrightarrow} \begin{cases} T, & \omega \in [-\frac{\pi}{T}, \frac{\pi}{T}] \\ 0 & \text{elsewhere}\end{cases}.\end{aligned}$$

Utilizing the fact that multiplication of two signals in the time domain corresponds to the convolution of their Fourier transforms in the frequency domain, we obtain the following plot for $X(j\omega)$.



(b) Express $y(t)$ if $H(j\omega) = e^{-j\omega(4\cdot 10^{-6})}$.

We have that the corresponding impulse response is $h(t) = \delta(t - 4\cdot 10^{-6})$. Recall from the sifting property that $x(t) * \delta(t-T) = x(t-T)$. Therefore, $y(t)$ corresponds to a time-delay of the input, i.e.

$$\begin{aligned} y(t) &= x(t - 4\cdot 10^{-6}) \\ &= m(t - 4\cdot 10^{-6})\cos(\omega_0(t - 4\cdot 10^{-6})). \end{aligned}$$

This can equivalently be seen by using the approximation found in Section 2.2.1. In particular we can express the output as

$$y(t) \approx m\left(t - \tau_g\left(\omega_0\right)\right)\cos\left(\omega_0\left(t - \tau_p\left(\omega_0\right)\right)\right).$$

For an impulse response $h(t)$ corresponding to a simple delay the phase and group delays are the same. (See Problem 2.1(a) for an example)

(c) We shall make use of the fact that the bandwidth of the sinc function $m(t)$ is rather small as compared to the frequency of the cosine. Specifically, we are given that $\omega_0 = 2600\pi$ while $\frac{1}{T} = 75$ kHz. This validates the approximation model discussed in Section 2.2.1 where the effect of the filter on the envelope is shown to be approximately equal to a time delay by the group delay of the filter at ω_0 and the effect on the filter on the carrier sinusoid is a delay by the phase delay of the system. This is summarized as

$$y(t) \approx m\left(t - \tau_g\left(\omega_0\right)\right) \cos\left(\omega_0\left(t - \tau_p\left(\omega_0\right)\right)\right).$$

From Fig. P2.13-2 we find that $\tau_g(\omega_0) \approx 1\mu$s and $\tau_p(\omega_0) \approx \frac{4\pi}{\omega_0} = \frac{1}{650}$ seconds. Note that it is also sufficient to take $\tau_p(\omega) = 0$ since when multiplied by ω_0 the quantity $\frac{1}{650}$ becomes a multiple of 2π. Therefore, our final approximation is given by

$$y(t) \approx m\left(t - 1 \cdot 10^{-6}\right) \cos\left(\omega_0 t\right).$$

Note that if we were to use the same approximation in part (b) the solution to (b) would not change. This follows from the fact that the phase delay and group delay for a system with an impulse response $\delta(t - \Delta)$ are equal.

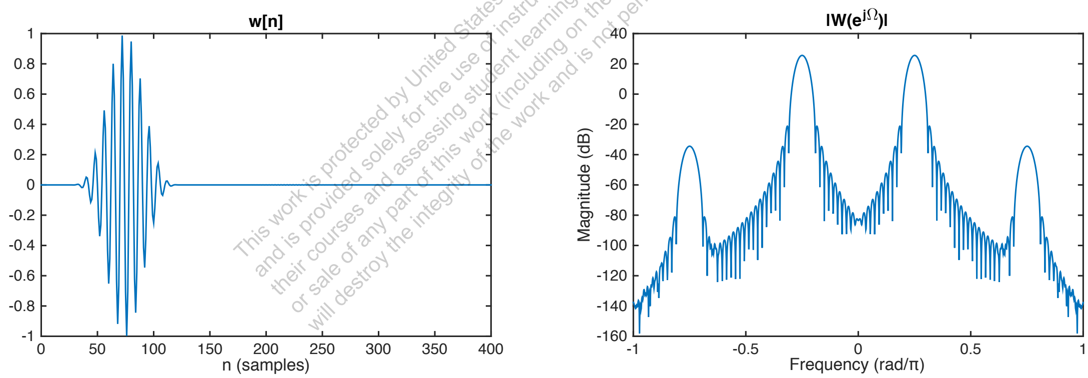
Solution 2.14

(a) Calculate the group delay of the filter $H(e^{j\Omega})$ and sketch $w[n]$ and $|W(e^{j\Omega})|$.

We are told that the first filter $h[n]$ and its associated frequency response $H(e^{j\Omega})$ are purely real. From this information we may conclude that the impulse response $h[n]$ is even symmetric about 0, i.e. $h[-n] = h[n]$ for all n, and consequently that the phase $\angle H(e^{j\Omega})$ is zero for all $\Omega \in [-\pi, \pi)$. Therefore,

$$\tau_g(\Omega) = 0, \quad -\pi \leq \Omega < \pi.$$

The magnitude of $H(e^{j\Omega})$ is such that the high frequency pulse associated with $x_2[n-150]$ will be attenuated by -60 dB or more while the low frequency pulse associated with $x_1[n]$ is passed with no magnitude change. The high frequency pulse has a strength of about 20 dB and thus will have a magnitude of approximately -40dB in the output. Therefore, the output $w[n]$ is approximately given by $w[n] \approx x_1[n] + 0.001x_2[n-150]$. A rough approximation to this is depicted in the figure below.

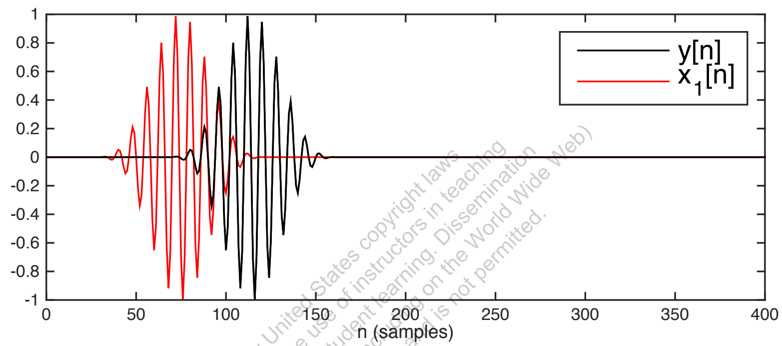


(b) Sketch what the output $y[n]$ will look like.

Our sketch of $y[n]$ will be focused on the effect of the all-pass filter on the envelope of the waveform $x_1[n]$ and not the phase shift of the higher frequency carrier. To this end, we approximate the group delay at $\Omega = 0.25\pi$ from the phase curve in Figure P2.14-5 as:

$$\tau_g(0.25\pi) \quad \approx \quad -\frac{-5\pi - (-1\pi)}{0.3\pi - 0.2\pi} = 40 \text{ samples.}$$

Therefore, we conclude that $y[n] \approx x_1[n-40]$ as is depicted in the figure below. The signal $x_1[n]$ is also depicted for comparison.



Solution 2.15

The output of filter A for the depicted input $x[n]$ is $y_6[n]$.

The sharp peaks in the Fourier transform magnitude of the input signal $x[n]$ corresponding to high frequency content, i.e. the peaks closest to $\Omega = \pi$, are attenuated by approximately -100 dB or more and thus the output $y[n]$ of the filter will essentially not contain signal content at these frequencies. The frequency response magnitude of the filter is also such that the frequency content associated with the slow and medium varying pulses pass essentially untouched in magnitude. From this analysis we can eliminate the possible outputs $y_i[n]$ for $i = 1, 3, 4, 5$, and 7 since they each contain the high frequency pulse.

The group delay experienced by the low frequency pulse is approximately 220 samples. If we approximate the center of this pulse at $n = 100$ then the pulse should appear in the output centered at $n = 320$. The group delay experienced by the medium frequency pulse is approximately 60 samples. If we approximate the center of this pulse at $n = 170$ then the pulse should appear in the output centered at $n = 230$. The only filter output $y_i[n]$ of those remaining consistent with these observations is $y_6[n]$.

Solution 2.16

The impulse responses $h_1[n]$ and $h_2[h]$ are both zero before time $n = 20$ which manifests itself in the group delay curves as an additive offset of 20 samples. We conclude from this observation that we must select between A and C. As compared with $h_2[n]$, the impulse response $h1[n]$ appears to contain lower frequency content that emerges later, at a time that corresponds to the associated group delay of 40 in plot C. On the other hand, $h_2[n]$ has higher-frequency content that dominates around the associated group delay of 60 in plot A.

(1) Impulse response $h_1[n]$ corresponds to group delay plot C.

(2) Impulse response $h_2[n]$ corresponds to group delay plot A.

The remaining impulse responses $h_3[n]$ and $h_4[h]$ must correspond to choices B and D. The impulse response $h_3[n]$ ends with a low-frequency component that dominates around the associated group delay of 10 in plot D, whereas $h_4[n]$ had a higher-frequency component dominating at around the corresponding group delay of 10 in plot B.

(3) Impulse response $h_3[n]$ corresponds to group delay plot D.

(4) Impulse response $h_4[n]$ corresponds to group delay plot B.

Solution 2.17

(i) The impulse response $h_3(t)$ corresponds to group delay plot B.

A feature of the impulse response $h_3(t)$ that clearly distinguishes it from the others is that it is zero for all t approximately less than 0.01, which manifests itself in the group delay curve as an additive offset of 0.01 seconds.

(ii) The impulse response $h_4(t)$ corresponds to group delay plot A.

The impulse response $h_4(t)$ exhibits stronger high-frequency oscillations that persist to times around 0.06, and these are unaccompanied by low-frequency oscillations. Contrast this with $h_1(t)$, for which the low-frequency and high-frequency oscillations have equal group delays and are both visible out to time around 0.07.

(iii) The impulse response $h_2(t)$ corresponds to group delay plot C.

Note that $h_2(t)$, in contrast to $h_4(t)$, exhibits stronger low-frequency oscillations that persist to times around 0.08, and these are unaccompanied by high-frequency oscillations.

(iv) The impulse response $h_1(t)$ corresponds to group delay plot D.

The impulse response $h_1(t)$ appears to have an equal mix of low and high frequencies in its impulse response, and they both settle by times around 0.02.

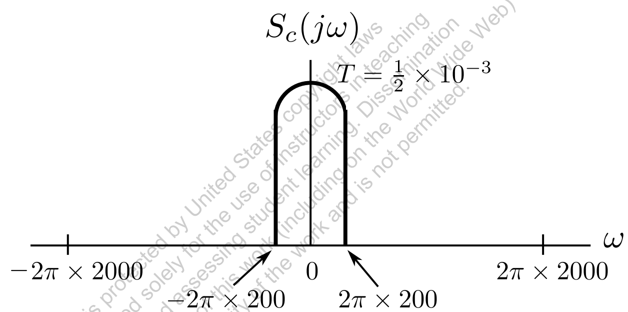
Solution 2.18

(a) Express $S_c(j\omega)$ in terms of $S_d(e^{j\Omega})$ and give a fully labeled sketch of $S_c(j\omega)$ in the interval $|\omega| < 2\pi \times 2000$ rad/sec.

Recall from the discussion of periodic sampling and reconstruction in Chapter 1 that

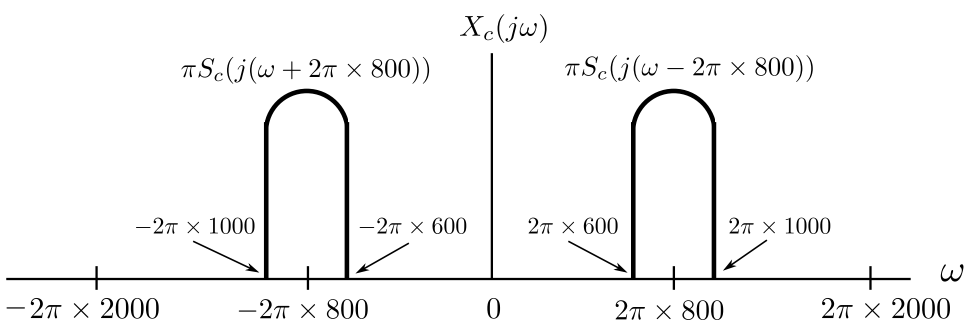
$$S_c(j\omega) = \begin{cases} TS_d(e^{j\Omega})|_{\Omega=\omega T} & |\omega| \leq \frac{\pi}{T} \\ 0, & \text{elsewhere} \end{cases} .$$

Using this equation we assemble the sketch of $S_c(j\omega)$ depicted below. Notice the periodic replications do not persist in the ω domain due to the interpolating LPF in the reconstruction process. This is relevant since we are asked to produce the plot for $|\omega| < 2\pi \times 2000$ and $\frac{\pi}{T} = 2\pi \times 1000$.



(b) Draw a detailed sketch of $X_c(j\omega)$ for $|\omega| < 2\pi \times 2000$ rad/sec.

The effect of the sinusoid translates two duplicates of the spectrum of $S_c(j\omega)$ scaled by π to the CT equivalent of $\pm\frac{4\pi}{5}$. This is depicted in the figure below.



(c) Give a time-domain expression for $x_c(t)$ in terms of $s_c(t)$.

We are given $x[n] = s_d[n] \cos\left(\frac{4\pi}{5}n\right)$. Going through the D/C converter with $T = 0.5$ms translates the DT frequency $\Omega = \frac{4\pi}{5}$ to the CT frequency $\omega = 2\pi \times 800$. The output of the D/C due to $s_d[n]$ will be denoted $s_c(t)$. Therefore, $x_c(t) = s_c(t) \cos(1600\pi t)$.

(d) Determine an approximate time-domain expression for $y_c(t)$ and $y_d[n]$ in terms of $s_c(t)$.

In determining an expression for $y_c(t)$ we make the approximations discussed in Section 2.2.1. In particular, we approximate the output of the filter by

$$y_c(t) \approx s_c\left(t - \tau_g\left(2\pi \times 800\right)\right)\cos\left(2\pi \times 800\left(t - \tau_p\left(2\pi \times 800\right)\right)\right).$$

We justify these approximations from the sketch in part (b). In particular, the figure illustrates that the spectrum of $s_c(t)$ is sufficiently concentrated and small as compared with the carriers frequency of $2\pi \times 800$. From the phase and group delay curves in Figure P2.18-3 we find that $\tau_g(2\pi \times 800) = 0.25\text{ms}$ and $\tau_p(2\pi \times 800) = \frac{1}{2\pi \times 800}$. Finally, we obtain the expression

$$y_c(t) \approx s_c\left(t - 0.25 \times 10^{-3}\right)\cos\left(2\pi \times 800\left(t - \frac{1}{2\pi \times 800}\right)\right).$$

Using the relation $y_d[n] = y_c(nT)$ for $T = 0.5\text{ms}$ we obtain the expression

$$\begin{aligned} y_d[n] &= y_c(0.5 \times 10^{-3} n) \\ &= s_c\left(0.5 \times 10^{-3}\left(n - \frac{1}{2}\right)\right)\cos\left(\frac{4\pi}{5}n - 1\right). \end{aligned}$$

(e) The mapping from $x_d[n]$ to $y_d[n]$ does indeed correspond to an LTI system since no aliasing is introduced during either of the domain conversion systems. Subsequently, the frequency response $H(e^{j\Omega})$ is related to $H_c(j\omega)$ via

$$H\left(e^{j\Omega}\right) = H_c(j\omega)\big|_{\omega=\frac{\Omega}{T}}, \qquad -\pi \leq \Omega < \pi.$$

The group delay of $H(e^{j\Omega})$ at $\Omega = \frac{4\pi}{5}$ is $\frac{1}{2}$ sample, as is illustrated in the expression for $y_d[n]$ in part (d). We can develop a nice interpretation for what this means using the procedure developed in this problem. Namely, the action of a discrete time system corresponding to a half-sample delay is equivalently thought of as periodically sampling a continuous time signal every T seconds where the continuous time signal is a $\frac{T}{2}$ delayed version of the bandlimited interpolation (using parameter T) of the original discrete time signal. Notice that this interpretation does not depend on the value taken by T so long as the D/C and C/D both use the same value and that the time-shift is by $\frac{T}{2}$.

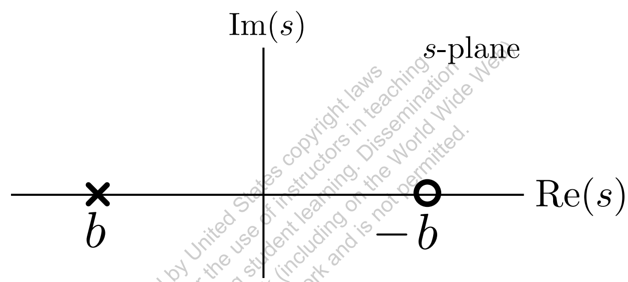
Solution 2.19

This problem pertains to a CT all-pass system of the form

$$H_{AP}(s) = B \prod_{k=1}^{M} \frac{s + b_k^*}{s - b_k}. \tag{1}$$

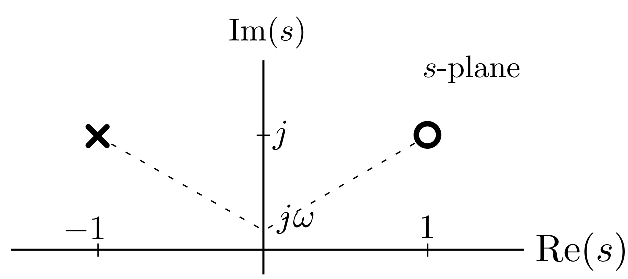
(a) Draw the pole-zero plot for $H_{AP}(s)$ with $M = 1$ and b_k real.

The pole-zero plot for $H_{AP}(s)$ is depicted below. Consistent with the convention established by Eq. (1), the pole is found in the s-plane at the value $s = b_k$ while the zero is found at $s = -b_k^*$. In our illustration we have selected $b < 0$ hence the pole is contained within the strict left half plane. Further, since b is chosen to be real we have that $b^* = b$.



(b) In this part we let b_k be complex valued. However, in order to ensure the system $H_{AP}(s)$ is both causal and stable, we require $\mathrm{Re}(b) < 0$, as is discussed in (a). We now argue the fact that the group delay of such a system is always positive at every frequency.

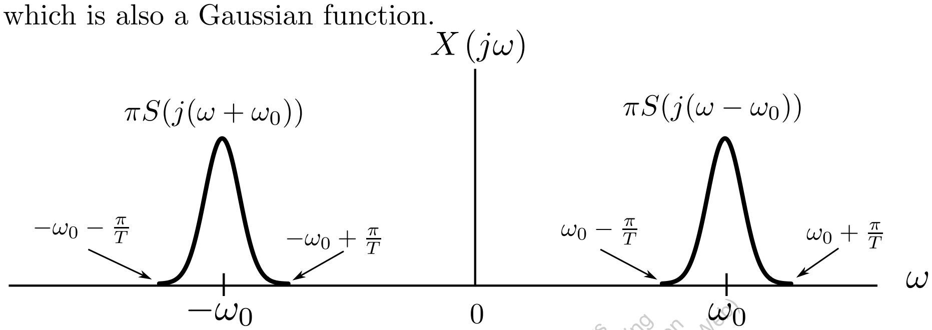
To see this geometrically, consider the example pole-zero plot in the figure below with $M = 1$ and $b = -1 + j$. Recall that the frequency response phase at any value ω is the angle of the vector from the zero to $s = j\omega$ (shown with a dotted line) minus the angle of the vector from the pole to $s = j\omega$ (also shown with a dotted line). The group delay corresponds to the negative rate of change of these vectors as we move ω. Since the pole is always in the left half plane and the zero is in the right half plane it is straightforward to see that the contribution of each is positive, hence the overall group delay is positive at all frequencies.



(c) For a general stable, causal CT all-pass system of the form of Eq. (1), we can conclude that the group delay is always positive for each frequency. From (b) we conclude that for $M = 1$ this is true. To extend our reasoning to $M > 1$ we rely upon the linearity of the phase response and the derivative required to compute the group delay. Recall that the phase response for $M > 0$ is the sum of the phase responses of each factor described in (b). Since the derivative is also a linear operator, the group delay is the sum of the group delays each factor in Eq. (1), each satisfying (b). Finally, since the sum of positive numbers is necessarily a positive number we conclude that the group delay of a causal, stable, all-pass CT system is positive everywhere.

Solution 2.20

We are given an input signal $x(t)$ of the form $x(t) = s(t)\cos(\omega_0 t)$ where $s(t) = e^{-\frac{t^2}{T^2}}$. The figure below sketches the Fourier transform $X(j\omega)$ of the signal $x(t)$. The width of $S(j\omega)$ is over exaggerated for clarity; the frequency $\omega_0 = 4\pi \times 10^{14}$ is about 10^8 times greater than $\frac{\pi}{T} = \pi \times 10^6$. We remark that the signal $s(t)$ is a Gaussian function which has a Fourier transform which is also a Gaussian function.



We now make use of the fact that the response of the optical fiber to the input signal $\cos(\omega t)$ is given by $10^{-\alpha(\omega)L}\cos(\omega t - \beta(\omega)L)$. Referring to the input signal $x(t)$, the bandwidth of envelope signal $s(t)$ is sufficiently small as compared to ω_0 so that the output of the optical fiber is approximated by

$$y(t) = s(t - \tau_g(\omega_0))10^{-\alpha(\omega_0)L}\cos(\omega_0 t - \beta(\omega_0)L)$$

where $\tau_g(\omega_0)$ is the group delay of the fiber at ω_0. We now justify this formula. The standard approximation formula when the frequency response is all-pass (Eq. (2.23a) or Eq. (2.23b)), is

$$y(t) = s(t - \tau_g(\omega_0))\cos(\omega_0(t - \tau_p(\omega_0))).$$

Comparing these two equations we see that $\beta(\omega_0)L$ plays the role of $\omega_0\tau_p(\omega_0)$. Using the fact that $\tau_p(\omega_0) = -\frac{\angle H_L(j\omega_0)}{\omega_0}$ we conclude that

$$\angle H_L(j\omega_0) = -\beta(\omega_0)L$$

Moreover, the function $\beta(\omega_0)$ is essentially linear over the frequency band of interest, as is depicted by the curve $\frac{d\beta(\omega)}{d\omega}$ in Figure P2.20-2. So, from the perspective of our input signal the optical fiber phase response is linear hence the fiber acts as a simple delay with magnitude $10^{-\alpha(\omega)L}$. With this approximation we compute the group delay as $\tau_g(\omega_0) = \tau_p(\omega_0)$ since linear phase systems have equal phase and group delays. (See problem 2.1(a) for an example of this). Therefore we have that $\tau_g(\omega_0) = \frac{\beta(\omega_0)L}{\omega_0}$. We also obtain the values $\alpha(\omega_0) = 0.1$ and $\beta(\omega_0) = 10^9$ from Figure P2.20-2 and are given that $L = 10$ and $T = 10^{-6}$. Finally, we write the output as:

$$\begin{aligned} y(t) &= s\left(t - \frac{1}{4\pi \times 10^4}\right)10^{-10\alpha\left(4\pi\times10^{14}\right)}\cos\left(4\pi\times10^{14}t - \beta\left(4\pi\times10^{14}\right)10\right) \\ &= 0.1e^{-10^{12}\left(t-\frac{1}{4\pi\times10^4}\right)^2}\cos\left(4\pi\times10^{14}t - 10^{10}\right). \end{aligned}$$

Note that the output is equivalently expressed as $y(t) = x(t) * 0.1\delta\left(t - \frac{1}{4\pi\times10^4}\right)$.

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6.011: Communication, Control and Signal Processing

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Problem Set 2 Solutions
State Space Models

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Problem 2.1

(i) (a) State variable choice:
A natural choice is $q_1(t) =$ position $y(t)$, and $q_2(t) =$ velocity $\dot{y}(t)$.
State-space model (state evolution equations and instantaneous output equation):

$$\begin{aligned} \dot{q}_1(t) &= q_2(t) \\ \dot{q}_2(t) &= -q_1^3(t) + x(t) \\ \\ y(t) &= q_1(t) \end{aligned}$$

(b) Equilibrium values of state variables when $x(t) \equiv 8$:
Set $q_1(t) \equiv \overline{q}_1$ and $q_2(t) \equiv \overline{q}_2$, and accordingly set $\dot{q}_1(t) \equiv 0$, $\dot{q}_2(t) \equiv 0$ in the preceding state-space model. This immediately yields

$$\overline{q}_1 = (8)^{1/3} = 2 , \qquad \overline{q}_2 = 0 .$$

(c) Linearized state-space model at the equilibrium above:
The key step here is to linearize the nonlinear term $y^3(t)$ or $q_1^3(t)$. Note that $d(y^3) = 3y^2\, dy$, so with

$$\widetilde{q}_1(t) = q_1(t) - \overline{q}_1$$

(and similar notation for all the other perturbations from equilibrium) we get, for small perturbations,

$$q_1^3(t) \approx \overline{q}_1^3 + (3\overline{q}_1^2)\, \widetilde{q}_1 .$$

The linearized model is then

$$\begin{aligned} \dot{\widetilde{q}}_1(t) &= \widetilde{q}_2(t) \\ \dot{\widetilde{q}}_2(t) &= -(3\overline{q}_1^2)\, \widetilde{q}_1 + \widetilde{x}(t) = -12\, \widetilde{q}_1 + \widetilde{x}(t) \\ \\ \widetilde{y}(t) &= \widetilde{q}_1(t) \end{aligned}$$

(ii) Now we are supposing that the given system is described by

$$\frac{d^2y(t)}{dt^2} = -y^3(t) + \frac{dx(t)}{dt} + x(t) \ .$$

(a) The obvious state variables chosen in (i) will not work as there is no way to account for the $\frac{dx(t)}{dt}$ term. Note that we cannot have differentials in the functions f_i because $\dot{\mathbf{q}}(t) = \mathbf{f}(\mathbf{q}(t), x(t))$ must fit the standard form for a system of first order ODE's.

The choice $q_1(t) = y(t)$ and $q_2(t) = \dot{y}(t) - x(t)$ gives

$$\begin{aligned} \dot{q}_1(t) &= q_2(t) + x(t) \\ \dot{q}_2(t) &= -q_1^3(t) + x(t) \\ \\ y(t) &= q_1(t) \ . \end{aligned}$$

(b) Setting $\dot{q}_1(t) = \dot{q}_2(t) = 0$ and $x(t) \equiv 8$ we get $\overline{q}_1 = 2$ and $\overline{q}_2 = -8$.

(c) Again, the only nonlinear term is $-q_1^3(t)$ and $\overline{q}_1$ has not changed. Thus we have

$$\begin{aligned} \dot{\widetilde{q}}_1(t) &= \widetilde{q}_2(t) + \widetilde{x}(t) \\ \dot{\widetilde{q}}_2(t) &= -(3\overline{q}_1^2)\,\widetilde{q}_1 + \widetilde{x}(t) = -12\,\widetilde{q}_1 + \widetilde{x}(t) \\ \\ \widetilde{y}(t) &= \widetilde{q}_1(t) \ . \end{aligned}$$

Problem 2.2

Using the given instantaneous output equation, we can write

$$x(t) = -\mathtt{d}^{-1}\mathbf{c}^T\mathbf{q}(t) + \mathtt{d}^{-1}y(t)$$

which will become the instantaneous output equation for the inverse system. Substituting the above equation in the given state evolution equation, we obtain

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{b}\Big(-\mathtt{d}^{-1}\mathbf{c}^T\mathbf{q}(t) + \mathtt{d}^{-1}y(t)\Big) = \left(\mathbf{A} - \mathbf{b}\mathtt{d}^{-1}\mathbf{c}^T\right)\mathbf{q}(t) + \mathbf{b}\mathtt{d}^{-1}y(t) \ ,$$

which is now a state evolution equation driven by $y(t)$ rather than $x(t)$, but with the same state vector $\mathbf{q}(t)$ as before. Thus,

$$\mathbf{A}_{in} = \mathbf{A} - \mathbf{b}\mathtt{d}^{-1}\mathbf{c}^T, \ \mathbf{b}_{in} = \mathbf{b}\mathtt{d}^{-1}, \ \mathbf{c}_{in}^T = -\mathtt{d}^{-1}\mathbf{c}^T, \ \mathtt{d}_{in} = \mathtt{d}^{-1} \ .$$

Problem 2.3

(a) Assuming temperature is held constant, the reaction rates k_f, k_r, and k_c are also constant. In this case the model is time-invariant. However, the model is nonlinear because the $q_1(t)q_2(t)$ terms ensure that some of the $\dot{q}_i(t)$ cannot be written as a linear combination of the state variables and the input.

(b) First we look at the rate of change $\frac{d}{dt}(q_2(t) + q_3(t)) = \dot{q}_2(t) + \dot{q}_3(t)$.

$$\begin{aligned} \dot{q}_2(t) + \dot{q}_3(t) &= -k_f q_1(t)q_2(t) + (k_r + k_c)q_3(t) \\ &\quad +k_f q_1(t)q_2(t) - (k_r + k_c)q_3(t) \\ &= 0\,. \end{aligned}$$

This means that $q_2(t) + q_3(t)$ will stay constant at its initial value $q_2(0) + q_3(0)$.

(c) Now we are assuming that $x(t) \equiv 0$ and we would like to find the equilibrium values $\bar{q}_i$ in terms of $\bar{q}_2 + \bar{q}_3 = E_0 > 0$. Setting $\dot{\mathbf{q}}(t) = 0$ the state evolution equations become

$$\begin{aligned} 0 &= -k_f\bar{q}_1\bar{q}_2 + k_r\bar{q}_3 \\ 0 &= -k_f\bar{q}_1\bar{q}_2 + (k_r + k_c)\bar{q}_3 \\ 0 &= k_f\bar{q}_1\bar{q}_2 - (k_r + k_c)\bar{q}_3 \\ 0 &= k_c\bar{q}_3\,. \end{aligned}$$

Immediately we get that

$$\bar{q}_3 = 0$$

and thus

$$\bar{q}_2 = E_0 > 0$$

and

$$k_f\bar{q}_1\bar{q}_2 = 0 \implies \bar{q}_1 = 0\,.$$

Finally note that the equilibrium value of $q_4(t)$ will depend on the dynamics of the reaction as it approaches equilibrium. Namely,

$$q_4(t) = q_4(0) + \int_0^t k_c q_3(\tau)d\tau$$

The linearized model around this equilibrium will have

$$A = \begin{bmatrix} -k_f\bar{q}_2 & -k_f\bar{q}_1 & k_r & 0 \\ -k_f\bar{q}_2 & -k_f\bar{q}_1 & k_r + k_c & 0 \\ k_f\bar{q}_2 & k_f\bar{q}_1 & -(k_r + k_c) & 0 \\ 0 & 0 & k_c & 0 \end{bmatrix} = \begin{bmatrix} -k_fE_0 & 0 & k_r & 0 \\ -k_fE_0 & 0 & k_r + k_c & 0 \\ k_fE_0 & 0 & -(k_r + k_c) & 0 \\ 0 & 0 & k_c & 0 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \ , \ \mathbf{c}^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \ , \ d = 0 \ .$$

(d) Suppose now we are looking for an equilibrium when $x(t) \equiv \bar{x} > 0$. This time the state evolution equations give

$$\begin{aligned} 0 &= -k_f\bar{q}_1\bar{q}_2 + k_r\bar{q}_3 + \bar{x} && (1) \\ 0 &= -k_f\bar{q}_1\bar{q}_2 + (k_r + k_c)\bar{q}_3 && (2) \\ 0 &= k_f\bar{q}_1\bar{q}_2 - (k_r + k_c)\bar{q}_3 && (3) \\ 0 &= k_c\bar{q}_3 \ . && (4) \end{aligned}$$

Again $\bar{q}_3 = 0$ and $\bar{q}_2 = E_0$. This time, however, (2) and (3) require that $\bar{q}_1 = 0$ while (1) requires $\bar{q}_1 = \bar{x}/(E_0 k_f) \neq 0$. This means that no full equilibrium is possible!

If we only require that the first three state variables are in equilibrium we instead get

$$\begin{aligned} 0 &= -k_f\bar{q}_1\bar{q}_2 + k_r\bar{q}_3 + \bar{x} \\ 0 &= -k_f\bar{q}_1\bar{q}_2 + (k_r + k_c)\bar{q}_3 \\ 0 &= k_f\bar{q}_1\bar{q}_2 - (k_r + k_c)\bar{q}_3 \ . \end{aligned}$$

Solving this system gives

$$\bar{q}_3 = \bar{x}/k_c$$

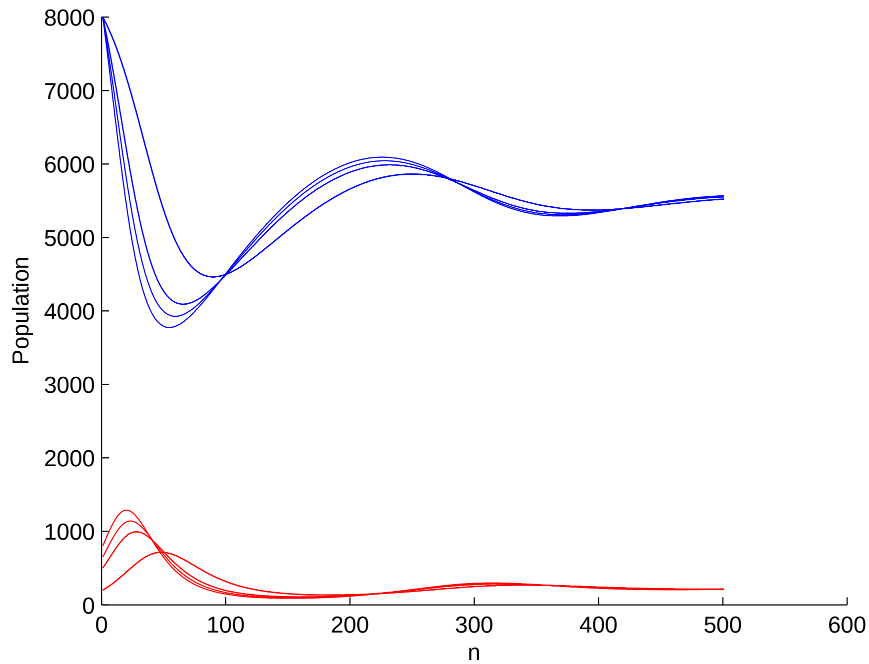
$$\bar{q}_2 = E_0 - \bar{x}/k_c$$

$$\bar{q}_1 = \frac{\bar{x} + k_r\bar{x}/k_c}{k_f(E_0 - \bar{x}/k_c)} = \frac{(k_c + k_r)\bar{x}}{k_c k_f E_0 - k_f\bar{x}} \ .$$

The rate of change of $y(t) = q_4(t)$ is $k_c q_3(t)$ which in this partial equilibirum is simply $k_c\bar{q}_3 = \bar{x}$.

Problem 2.4

(a) The following figure shows the response of the SIR model for a variety of initial conditions. In all cases the parameters used were $P = 10000$, $\beta = .01$, $\gamma = .2$, $\rho = .1$, and $x[n] = \bar{x} = .2$. The blue curves show $s[n]$ and the red curves show the corresponding $i[n]$.



(b) In the simulations from (a) we see that $i[n]$ settles to approximately 227.3 for large values of n. This is consistent with the computed equilibrium value i.e.

$$\bar{i} = \frac{\beta P}{\gamma} R_0 \left(1 - \frac{1}{R_0} - \bar{x}\right) = \frac{100}{.2}\frac{20}{11}\left(1 - \frac{11}{20} - .2\right) = \frac{25}{.11} \approx 227.27.$$

(c)

$$\mathbf{A}_{EE} = \begin{bmatrix} 1 - \beta R_0(1-\bar{x}) & -\gamma/R_0 \\ \beta(R_0(1-\bar{x}) - 1) & 1 \end{bmatrix} \approx \begin{bmatrix} .9855 & -.1100 \\ .0045 & 1 \end{bmatrix}$$

The matrix $\mathbf{A}_{EE}$ has eigenvalues

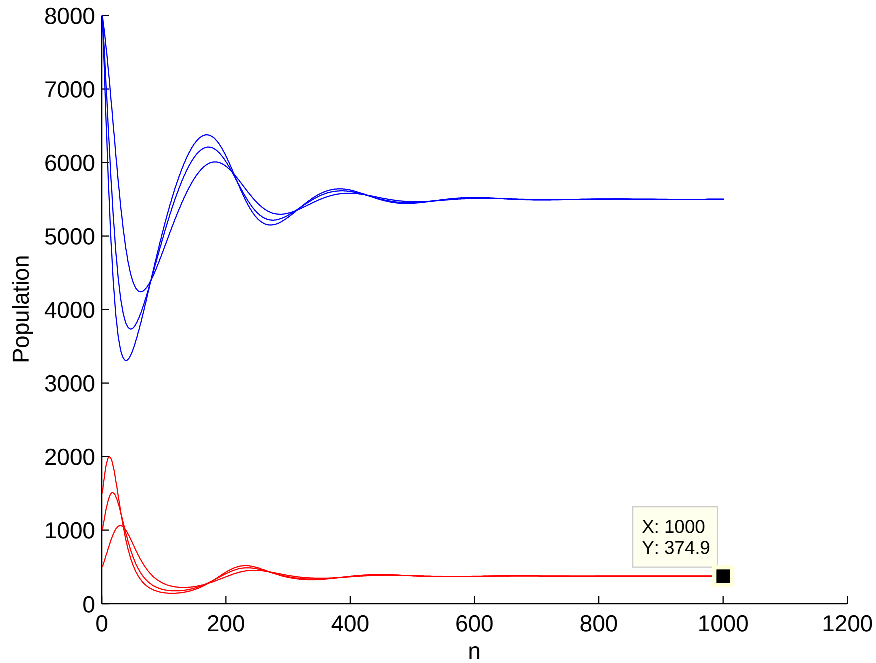
$$\begin{aligned} \lambda_1 &= 0.9927 + 0.0211j = .9930e^{.0213j} \\ \lambda_2 &= 0.9927 - 0.0211j = .9930e^{-.0213j} \end{aligned}$$

with associated eigenvectors

$$\begin{aligned} \mathbf{v}_1^T &= [0.9800, -0.0648 - 0.1884j] \\ \mathbf{v}_2^T &= [0.9800, -0.0648 + 0.1884j] \ . \end{aligned}$$

Note that since the eigenvalues have magnitude less than one, the linearized model is a stable system.

(d) If we use the feedback rule $x[n] = gi[n]/P$, then the endemic equilibrium value of i changes along with the gain g. See, for example, the plot with $g = 1$ and various initial conditions below.



We can find the new endemic equilibrium values by plugging $x[n] = gi[n]/P$ in to the state evolution equations:

$$\begin{aligned} 0 &= -\frac{\gamma i s}{P} + \beta(P - s) - \beta g i \\ 0 &= \frac{\gamma i s}{P} - \rho i - \beta i \,. \end{aligned}$$

The endemic (i.e. nonzero) solution for i is

$$i = \frac{P\beta}{g\beta + \rho + \beta}\left(1 - \frac{1}{R_0}\right)$$

which for our values gives $i = 375$ as in the plot. Note that as g increases, the value of i will decrease. However, there is no point at which a high enough gain will get rid of the endemic equilibrium entirely.