Chapter 2

Free Vibration of Single Degree of Freedom Systems

2.1

$$\delta_{st} = 5 \times 10^{-3} \ m$$

$$\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} = \left(\frac{9.81}{5 \times 10^{-3}}\right)^{1/2} = 44.2945 \ rad/sec = 7.0497 \ Hz$$

2.2

$$\tau_n = 0.21 \ sec = 2\pi\sqrt{\frac{m}{k}} \ , \quad \sqrt{m} = 0.21\sqrt{k}/2\pi$$

$$(i) \ (\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{1.5\,k}} = \frac{2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)}{\sqrt{1.5\,k}} = 0.1715 \ sec.$$

$$(ii) \ (\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{0.5\,k}} = 2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)\frac{1}{\sqrt{0.5\,k}} = 0.2970 \ sec.$$

2.3

$$\omega_n = 62.832 \ rad/sec = \sqrt{\frac{k}{m}} \ , \quad \sqrt{m} = \sqrt{k}/62.832$$

When spring constant is reduced, ω_n decreases.

$$(\omega_n)_{new} = 0.55\,\omega_n = 34.5576 \ rad/sec = \sqrt{\frac{k_{new}}{m_{new}}} = \sqrt{\frac{k-800}{m}}$$

$$\sqrt{\frac{k-800}{k}} \times 62.836 = 34.5576 \ , \quad \sqrt{\frac{k-800}{k}} = 0.55$$

$$\frac{k-800}{k} = (0.55)^2 = 0.3025$$

$$k = 1146.9534 \ N/m$$

$$\sqrt{m} = \sqrt{k}/62.832 \ ; \quad m = k/62.832^2 = \frac{1146.9534}{3947.8602}$$

$$m = 0.2905 \ kg$$

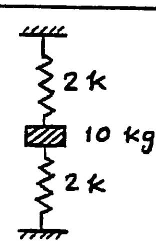
2.4

$$k = 100/\left(\frac{10}{1000}\right) = 10000 \ N/m$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4\,k}{m}} = \left(\frac{4 \times 10^4}{10}\right)^{1/2}$$

$$= 63.2456 \ rad/sec$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{6.2832}{63.2456} = 0.0993 \ sec$$



(2.5) $m = \dfrac{2000}{386.4}$.

Let $\omega_n = 7.5$ rad/sec.

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$k_{eq} = m\,\omega_n^2 = \left(\frac{2000}{386.4}\right)(7.5)^2 = 291.1491 \text{ lb/in} = 4\text{ k}$$

where k is the stiffness of the air spring.

Thus $k = \dfrac{291.1491}{4} = 72.7873$ lb/in.

(2.6) $x = A\cos(\omega_n t - \phi_o)$, $\dot{x} = -\omega_n A \sin(\omega_n t - \phi_o)$,

$\ddot{x} = -\omega_n^2 A \cos(\omega_n t - \phi_o)$

(a) $\omega_n A = 0.1$ m/sec ; $\tau_n = \dfrac{2\pi}{\omega_n} = 2$ sec, $\omega_n = 3.1416$ rad/sec

$A = 0.1/\omega_n = 0.03183$ m

(d) $x_0 = x(t=0) = A\cos(-\phi_o) = 0.02$ m

$\cos(-\phi_o) = \dfrac{0.02}{A} = 0.6283$

$\phi_o = 51.0724°$

(b) $\dot{x}_0 = \dot{x}(t=0) = -\omega_n A \sin(-\phi_o) = -0.1\sin(-51.0724°)$

$= 0.07779$ m/sec

(c) $\ddot{x}\big|_{max} = \omega_n^2 A = (3.1416)^2(0.03183) = 0.314151$ m/sec²

(2.7) For small angular rotation of bar PQ about P,

$$\frac{1}{2}(k_{12})_{eq}(\theta l_3)^2 = \frac{1}{2}k_1(\theta l_1)^2 + \frac{1}{2}k_2(\theta l_2)^2$$

i.e., $(k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2)/l_3^2$

Let k_{eq} = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

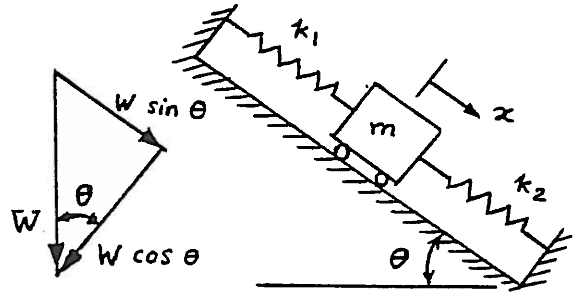
$$k_{eq} = \frac{(k_{12})_{eq}\,k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2\right\}k_3}{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m\,(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

2.8 $m = 2000$ kg , $\delta_{st} = 0.02$ m.

$$\omega_n = (g/\delta_{st})^{1/2} = \left(\frac{9.81}{0.02}\right)^{1/2} = 22.1472 \text{ rad/sec}$$

2.9 Let x be measured from the position of mass at which the springs are unstretched.



Equation of motion is

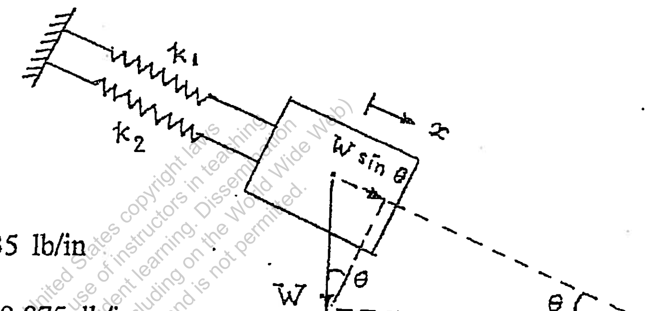
$$m\ddot{x} = -k_1(x+\delta_{st}) - k_2(x+\delta_{st}) + W \sin\theta \quad \text{---- } (E_1)$$

where $\delta_{st}(k_1+k_2) = W\sin\theta$.

Thus Eq. (E_1) becomes $m\ddot{x} + (k_1+k_2)x = 0 \Rightarrow \omega_n = \sqrt{\frac{k_1+k_2}{m}}$.

2.10

$$k_1 = \frac{A_1 E_1}{\ell_1} = \frac{\frac{\pi}{4}(0.05)^2\,(30\,(10^6))}{30\,(12)} = 163.6250 \text{ lb/in}$$



$$k_2 = \frac{A_2 E_2}{\ell_2} = \frac{163.6250\,(30)}{25} = 196.35 \text{ lb/in}$$

$$k_{eq} = k_1 + k_2 = 163.6250 + 196.35 = 359.975 \text{ lb/in}$$

Let x be measured from the unstretched length of the springs. The equation of motion is:

$$m\ddot{x} = -(k_1+k_2)(x+\delta_{st}) + W\sin\theta$$

where $(k_1+k_2)\,\delta_{st} = W\sin\theta$

$$\text{i.e.,} \quad m\ddot{x} + (k_1+k_2)\,x = 0$$

Thus the natural frequency of vibration of the cart is given by

$$\omega_n = \sqrt{\frac{k_1+k_2}{m}} = \sqrt{\frac{359.975\,(386.4)}{5000}} = 5.2743 \text{ rad/sec}$$

2.11 **Weight of electronic chassis = 500 N. To be able to use the unit in a vibratory environment with a frequency range of 0 - 5 Hz, its natural frequency must be away from the frequency of the environment. Let the natural frequency be ω_n = 10 Hz = 62.832 rad/sec. Since**

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 62.832$$

we have

$$k_{eq} = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(62.832)^2 = 20.1857\,(10^4)\ \text{N/m} \equiv 4\,k$$

so that k = spring constant of each spring = 50,464.25 N/m. For a helical spring,

$$k = \frac{G\,d^4}{8\,n\,D^3}$$

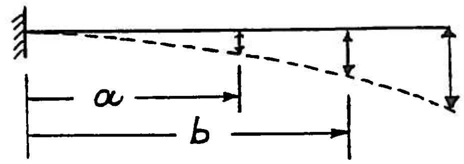
Assuming the material of springs as steel with $G = 80\,(10^9)$ Pa, $n = 5$ and $d = 0.005$ m, we find

$$k = 50,464.25 = \frac{80\,(10^9)\,(0.005)^4}{8\,(5)\,D^3}$$

This gives

$$D^3 = \frac{1250\,(10^{-3})}{50464.25} = 24,770.0\,(10^{-9}) \text{ or } D = 0.0291492 \text{ m} = 2.91492 \text{ cm}$$

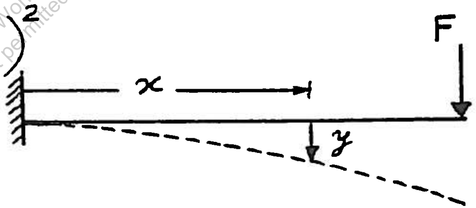
2.12 (i) with springs k_1 and k_2:



Let y_a, y_b, y_l be deflections of beam at distances a, b, l from fixed end.

$$\frac{1}{2}(k_{12})_{eq}\, y_l^2 = \frac{1}{2} k_1 y_a^2 + \frac{1}{2} k_2 y_b^2$$

i.e., $$(k_{12})_{eq} = k_1\left(\frac{y_a}{y_l}\right)^2 + k_2\left(\frac{y_b}{y_l}\right)^2$$



$$y = \frac{F x^2}{6EI}(3l - x)$$

@ $x = a$, $$y_a = \frac{F a^2}{6EI}(3l - a)$$

@ $x = b$, $$y_b = \frac{F b^2}{6EI}(3l - b)$$

@ $x = l$, $$y_l = \frac{F l^3}{3EI}$$

$$\omega_n = \left[\frac{k_1 k_3 \left(\frac{y_a}{y_l}\right)^2 + k_2 k_3 \left(\frac{y_b}{y_l}\right)^2}{m\left\{k_1\left(\frac{y_a}{y_l}\right)^2 + k_2\left(\frac{y_b}{y_l}\right)^2 + k_{beam}\right\}}\right]^{\frac{1}{2}} \quad \text{where } k_{beam} = \frac{3EI}{l^3}$$

$$= \left[\frac{k_1(3EI)\,a^4(3l-a)^2 + k_2(3EI)\,b^4(3l-b)^2}{m\,l^3\left\{k_1 a^4(3l-a)^2 + k_2 b^4 (3l-b)^2 + 12\,EI\,l^3\right\}}\right]^{\frac{1}{2}}$$

(ii) without springs k_1 and k_2:

$$\omega_n = \sqrt{\frac{k_{beam}}{m}} = \sqrt{\frac{3EI}{m\,l^3}}$$

(2.13) Let x_1, x_2 = displacements of pulleys 1, 2

$$x = 2x_1 + 2x_2 \qquad \text{----} \quad (E_1)$$

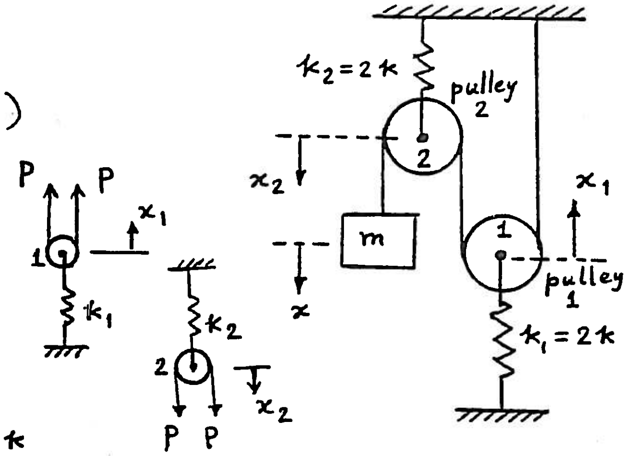
Let P = tension in rope.

For equilibrium of pulley 1,

$$2P = k_1 x_1 \qquad \text{----} \quad (E_2)$$

For equilibrium of pulley 2,

$$2P = k_2 x_2 \qquad \text{----} \quad (E_3)$$



where $\frac{1}{k_1} = \frac{1}{4k} + \frac{1}{4k} = \frac{1}{2k}$; $k_1 = 2k$

and $k_2 = k + k = 2k$

Combining Eqs. (E_1) to (E_3):

$$x = 2x_1 + 2x_2 = 2\left(\frac{2P}{k_1}\right) + 2\left(\frac{2P}{k_2}\right) = 4P\left(\frac{1}{2k} + \frac{1}{2k}\right) = \frac{4P}{k}$$

Let k_{eq} = equivalent spring constant of the system:

$$k_{eq} = \frac{P}{x} = \frac{k}{4}$$

Equation of motion of mass m: $m\ddot{x} + k_{eq}\, x = 0$

$$\therefore \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}}$$

(2.14) For a displacement of x of mass m, pulleys 1, 2 and 3 undergo displacements of 2x, 4x and 8x, respectively. The equation of motion of mass m can be written as

$$m\ddot{x} + F_0 = 0 \qquad (1)$$

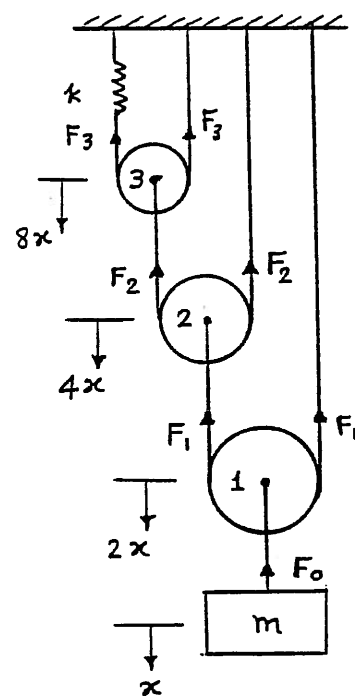
where $F_0 = 2F_1 = 4F_2 = 8F_3$ as shown in figure.

Since $F_3 = (8x)\,k$, Eq. (1) can be rewritten as

$$m\ddot{x} + 8F_3 = 8\,(8k) = 0 \qquad (2)$$

from which we can find

$$\omega_n = \sqrt{\frac{64\,k}{m}} = 8\sqrt{\frac{k}{m}} \qquad (3)$$



2.15 (a) $\omega_n = \sqrt{4k/M}$

(b) $\omega_n = \sqrt{4k/(M+m)}$

Initial conditions:

Velocity of falling mass $m = v = \sqrt{2gl}$ $\left(\because\ v^2 - u^2 = 2gl\right)$, $u \to 0$

$x = 0$ at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

Conservation of momentum:

$$(M+m)\,\dot{x}_0 = m\,v = m\sqrt{2gl}$$

$$\dot{x}_0 = \dot{x}(t=0) = \frac{m}{M+m}\sqrt{2gl}$$

Complete solution: $x(t) = A_0 \sin(\omega_n t + \phi_0)$

where $A_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{\frac{m^2 g^2}{16k^2} + \frac{m^2 g l}{2k(M+m)}}$

and $\phi_0 = \tan^{-1}\left(\frac{x_0\,\omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{-\sqrt{g}}{\sqrt{8lk(M+m)}}\right)$

2.16 (a) Velocity of anvil $= v = 50$ ft/sec $= 600$ in/sec. $x = 0$ at static equilibrium position.

$$x_0 = x(t{=}0) = -\frac{\text{weight}}{k_{eq}} = -\frac{m\,g}{4\,k}$$

Conservation of momentum:

$$(M+m)\,\dot{x}_0 = m\,v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t{=}0) = \frac{m\,v}{M+m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{4\,k}{M+m}}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n\,t + \phi_0)$$

where

$$A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = \left\{\frac{m^2\,g^2}{16\,k^2} + \frac{m^2\,v^2}{(M+m)\,4\,k}\right\}^{\frac{1}{2}}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0\,\omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(-\frac{m\,g}{4\,k}\sqrt{\frac{4\,k}{(M+m)}}\,\frac{(M+m)}{m\,v}\right) = \tan^{-1}\left(-\frac{g\sqrt{M+m}}{v\sqrt{4\,k}}\right)$$

Since v = 600, m = 12/386.4, M = 100/386.4, k = 100, we find

$$A_0 = \left\{ \left(\frac{12\ (386.4)}{4\ (100)\ (386.4)} \right)^2 + \left(\frac{12\ (600)}{386.4} \right)^2 \frac{386.4}{112\ (400)} \right\}^{\frac{1}{2}} = 1.7308 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(- \frac{386.4 \sqrt{112}}{\sqrt{386.4\ (600)} \sqrt{400}} \right) = \tan^{-1} (-0.01734) = -0.9934 \text{ deg}$$

(b) x = 0 at static equilibrium position: $x_0 = x(t=0) = 0$. Conservation of momentum gives:

$$M \dot{x}_0 = m\, v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t=0) = \frac{m\, v}{M}$$

Complete solution:

$$x(t) = A_0 \sin (\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{m^2\, v^2\, (M)}{M^2\, 4\, k} \right\}^{\frac{1}{2}} = \frac{m\, v}{\sqrt{4\, k\, M}} = \frac{12\ (600) \sqrt{386.4}}{386.4 \sqrt{4\ (100)\ (100)}} = 1.8314 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(\frac{x_0\, \omega_n}{\dot{x}_0} \right) = \tan^{-1} (0) = 0$$

2.17 $k_1 = \dfrac{3 E_1 I_1}{l_1^3}$ (at tip) ; $k_2 = \dfrac{48 E_2 I_2}{l_2^3}$ (at middle)

$k_{eq} = k_1 + k_2$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\left(\frac{3 E_1 I_1}{l_1^3} + \frac{48 E_2 I_2}{l_2^3} \right) \frac{g}{W}}$$

2.18 $k = \dfrac{AE}{l} = \dfrac{\left\{ \frac{\pi}{4} (0.01)^2 \right\} \left\{ 2.07 \times 10^{11} \right\}}{20} = 0.8129 \times 10^6$ N/m

m = 1000 kg

$$\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{0.8129 \times 10^6}{1000} \right)^{1/2} = 28.5114 \text{ rad/sec}$$

$\dot{x}_0 = 2$ m/s, $x_0 = 0$ (suddenly stopped while it has velocity)

Period of ensuing vibration $= \tau_n = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{28.5114} = 0.2204$ sec

Amplitude $= A = \dot{x}_0 / \omega_n = 2/28.5114 = 0.07015$ m

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2.19 $\omega_n = 2$ Hz $= 12.5664$ rad/sec $= \sqrt{\dfrac{k}{m}}$

$\sqrt{k} = 12.5664 \sqrt{m}$

$$\omega_n' = \sqrt{\frac{k'}{m'}} = \sqrt{\frac{k}{m+1}} = 6.2832 \text{ rad/sec}$$

$$\sqrt{k} = 6.2832\sqrt{m+1}$$
$$= 12.5664\sqrt{m}$$

$$\sqrt{m+1} = 2\sqrt{m} \quad , \quad m = \tfrac{1}{3}\ kg$$

$$k = (12.5664)^2\ m = 52.6381\ N/m$$

2.20 Cable stiffness $= k = \dfrac{AE}{\ell} = \dfrac{1}{4}\left(\dfrac{\pi}{4}(0.01)^2\right) 2.07\,(10^{11}) = 4.0644\,(10^6)$ N/m

$$\tau_n = 0.1 = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$$

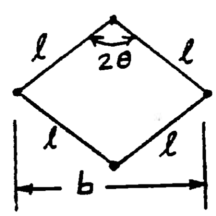
$$\omega_n = \frac{2\pi}{0.1} = 20\pi = \sqrt{\frac{k}{m}}$$

Hence

$$m = \frac{k}{\omega_n^2} = \frac{4.0644\,(10^6)}{(20\pi)^2} = 1029.53\ \text{kg}$$

2.21 $b = 2\ell \sin\theta$

Neglect masses of links.



(a) $k_{eq} = k\left(\dfrac{4\ell^2 - b^2}{b^2}\right) = k\left(\dfrac{4\ell^2 - 4\ell^2\sin^2\theta}{4\ell^2\sin^2\theta}\right)$

$$= k\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k\,g\,\text{cosec}^2\theta}{W}}$$ (from solution of problem 1.8)

(b) $\omega_n = \sqrt{\dfrac{kg}{W}}$ since $k_{eq} = k$.

2.22 $y = \sqrt{l^2 - (l\sin\theta - x)^2} - l\cos\theta = \sqrt{l^2(\cos^2\theta + \sin^2\theta) - (l\sin\theta - x)^2} - l\cos\theta$

$$= \sqrt{l^2\cos^2\theta - x^2 + 2\,l\,x\sin\theta} - l\cos\theta$$

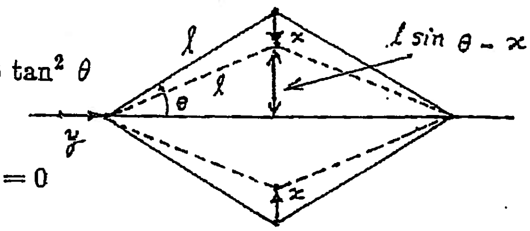
$$= \ell\cos\theta\sqrt{1 - \frac{x^2}{\ell^2\cos^2\theta} + \frac{2\,\ell\,x\sin\theta}{\ell^2\cos^2\theta}} - \ell\cos\theta$$

$$\frac{1}{2}k_{eq}\,x^2 = \frac{1}{2}k_1\,y^2 + \frac{1}{2}k_2\,y^2$$

with $\quad y \approx \ell\cos\theta\left(1 - \dfrac{1}{2}\dfrac{x^2}{\ell^2\cos^2\theta} + \dfrac{1}{2}\dfrac{2\,\ell\,x\sin\theta}{\ell^2\cos^2\theta}\right) - \ell\cos\theta$

$\approx \dfrac{x\sin\theta}{\cos\theta} = x\tan\theta \quad$ (since $x^2 << x$, it is neglected)

Thus k_{eq} can be expressed as



$$k_{eq} = (k_1 + k_2)\tan^2\theta$$

Equation of motion:

$$m\ddot{x} + k_{eq}\,x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{(k_1 + k_2)\,g}{W}}\,\tan\theta$$

2.23 (a) Neglect masses of rigid links. Let x = displacement of W. Springs are in series.

$$k_{eq} = \frac{k}{2}$$

Equation of motion:

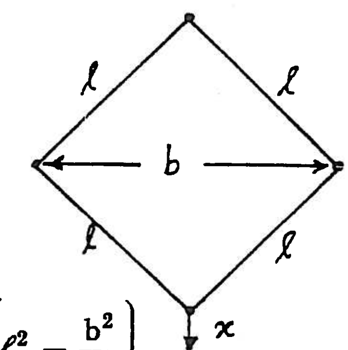
$$m\ddot{x} + k_{eq}\,x = 0$$

Natual frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2\,m}}$$

(b) Under a displacement of x of mass, each spring will be compressed by an an amount:

$$x_s = x\,\frac{2}{b}\sqrt{\ell^2 - \frac{b^2}{4}}$$



Equivalent spring constant:

$$\frac{1}{2}\,k_{eq}\,x^2 = 2\left(\frac{1}{2}\,k\,x_s^2\right)$$

$$\text{or}\quad k_{eq} = 2\,k\left(\frac{x_s}{x}\right)^2 = 2\,k\left(\frac{4}{b^2}\right)\left(\ell^2 - \frac{b^2}{4}\right) = \frac{8\,k}{b^2}\left(\ell^2 - \frac{b^2}{4}\right)$$

Equation of motion:

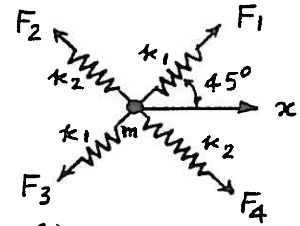
$$m\ddot{x} + k_{eq}\,x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8\,k}{b^2\,m}\left(\ell^2 - \frac{b^2}{4}\right)}$$

2.24 $F_1 = F_3 = k_1 \, x \cos 45^\circ$

$F_2 = F_4 = k_2 \, x \cos 135^\circ$



$$F = \text{force along } x = F_1 \cos 45^\circ + F_2 \cos 135^\circ + F_3 \cos 45^\circ + F_4 \cos 135^\circ$$

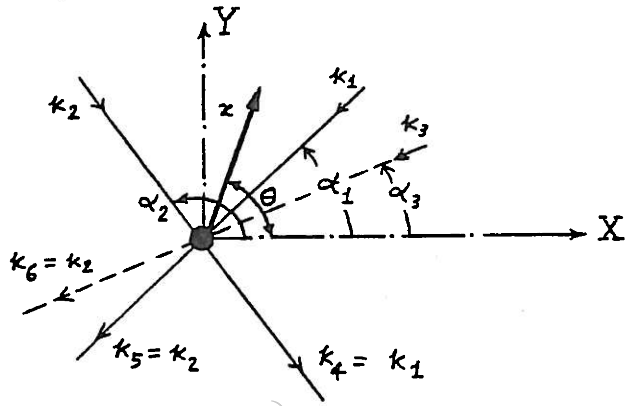
$$= 2x \left(k_1 \cos^2 45^\circ + k_2 \cos^2 135^\circ \right)$$

$$k_{eq} = \frac{F}{x} = 2\left(\frac{k_1}{2} + \frac{k_2}{2} \right) = k_1 + k_2$$

Equation of motion: $m\ddot{x} + (k_1 + k_2)\, x = 0$

2.25 Let α_i denote the angle made by i^{th} spring with respect to X axis.

Let x = displacement of mass along the direction defined by θ.



If k_{eq} = equivalent spring constant, the equivalence of potential energies gives

$$\frac{1}{2} k_{eq} \, x^2 = \frac{1}{2} \sum_{i=1}^{6} k_i \left\{ x \cos(\theta - \alpha_i) \right\}^2$$

$$k_{eq} = \sum_{i=1}^{6} k_i \cos^2(\theta - \alpha_i) = \sum_{i=1}^{6} k_i \left(\cos\theta \cos\alpha_i + \sin\theta \sin\alpha_i \right)^2$$

$$= \sum_{i=1}^{6} k_i \left(\cos^2\alpha_i \cos^2\theta + \sin^2\alpha_i \sin^2\theta \right) + 2 \sum_{i=1}^{6} \left(\cos\alpha_i \sin\alpha_i \cos\theta \sin\theta \right)$$

Natural frequency $= \omega_n = \sqrt{\dfrac{k_{eq}}{m}}$

For ω_n to be independent of θ, $\displaystyle\sum_{i=1}^{6} k_i \cos^2\alpha_i = \sum_{i=1}^{6} k_i \sin^2\alpha_i$ ---(E_1)

and $\displaystyle\sum_{i=1}^{6} k_i \cos\alpha_i \sin\alpha_i = 0$ ---(E_2)

(E_1) and (E_2) can be rewritten as

$$\sum_{i=1}^{6} k_i \left(\frac{1}{2} + \frac{1}{2} \cos 2\alpha_i \right) = \sum_{i=1}^{6} k_i \left(\frac{1}{2} - \frac{1}{2} \cos 2\alpha_i \right)$$

and $\displaystyle\frac{1}{2} \sum_{i=1}^{6} k_i \sin 2\alpha_i = 0$

i.e. $\displaystyle\sum_{i=1}^{6} k_i \cos 2\alpha_i = 0$ --- (E_3)

and $\displaystyle\sum_{i=1}^{6} k_i \sin 2\alpha_i = 0$ --- (E_4)

In the present example, (E_3) and (E_4) become

$$k_1 \cos 60° + k_2 \cos 240° + k_3 \cos 2\alpha_3 + k_1 \cos 420° + k_2 \cos 600° + k_3 \cos(360° + 2\alpha_3) = 0$$

$$k_1 \sin 60° + k_2 \sin 240° + k_3 \sin 2\alpha_3 + k_1 \sin 420° + k_2 \sin 600° + k_3 \sin(360° + 2\alpha_3) = 0$$

i.e., $\left.\begin{array}{l} k_1 - k_2 + 2 k_3 \cos 2\alpha_3 = 0 \\ \sqrt{3}\, k_1 - \sqrt{3}\, k_2 + 2 k_3 \sin 2\alpha_3 = 0 \end{array}\right\}$; $\begin{array}{l} 2 k_3 \cos 2\alpha_3 = k_2 - k_1 \quad \ldots (E_5) \\ 2 k_3 \sin 2\alpha_3 = \sqrt{3}\,(k_2 - k_1) \quad \ldots (E_6) \end{array}$

Squaring (E_5) and (E_6) and adding,

$$4 k_3^2 = (k_2 - k_1)^2 (1 + 3)$$

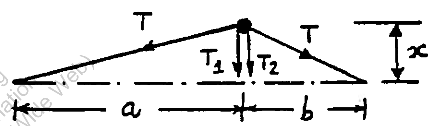
$$\therefore k_3 = \pm (k_2 - k_1) \Rightarrow k_3 = |k_2 - k_1|$$

Dividing (E_6) by (E_5),

$$\tan 2\alpha_3 = \sqrt{3}$$

$$\therefore \alpha_3 = \frac{1}{2} \tan^{-1}(\sqrt{3}) = 30°$$

2.26 $T_1 = \frac{x}{a} T, \quad T_2 = \frac{x}{b} T$



(a) $m \ddot{x} + (T_1 + T_2) = 0$

$$m \ddot{x} + \left(\frac{T}{a} + \frac{T}{b} \right) x = 0$$

(b) $\omega_n = \sqrt{\dfrac{\frac{T}{a} + \frac{T}{b}}{m}} = \sqrt{\dfrac{T}{m\,a\,b}(a+b)}$

2.27 $m = \dfrac{160}{386.4}\ \dfrac{\text{lb—sec}^2}{\text{inch}}$, $k = 10$ lb/inch.

Velocity of jumper as he falls through 200 ft:

$$m\,g\,h = \frac{1}{2}\, m\, v^2 \quad \text{or} \quad v = \sqrt{2\,g\,h} = \sqrt{2\,(386.4)\,(200\,(12))} = 1{,}361.8811 \text{ in/sec}$$

About static equilibrium position:

$$x_0 = x(t{=}0) = 0, \quad \dot{x}_0 = \dot{x}(t{=}0) = 1{,}361.8811 \text{ in/sec}$$

Response of jumper: $\quad x(t) = A_0 \sin(\omega_n\, t + \phi_0)$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \frac{\dot{x}_0}{\omega_n} = \frac{\dot{x}_0 \sqrt{m}}{\sqrt{k}} = \frac{1361.8811}{\sqrt{10}} \sqrt{\left(\frac{160}{386.4} \right)} = 277.1281 \text{ in}$$

and $\quad \phi_0 = \tan^{-1} \left(\dfrac{x_0\, \omega_n}{\dot{x}_0} \right) = 0$

2.28 The natural frequency of a vibrating rope is given by (see Problem 2.26):

$$\omega_n = \sqrt{\frac{T\,(a+b)}{m\,a\,b}}$$

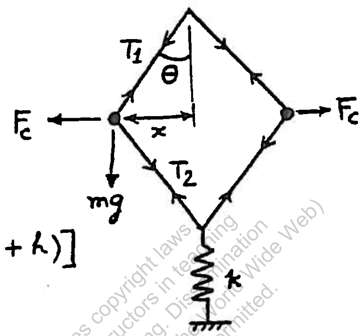
where T = tension in rope, m = mass, and a and b are lengths of the rope on both sides of the mass. For the given data:

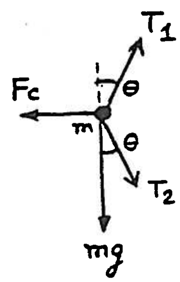
$$10 = \left\{ \frac{T\,(80+160)}{\left(\frac{120}{386.4}\right)(80)\,(160)} \right\}^{\frac{1}{2}} = \sqrt{T\,(0.060375)}$$

which yields

$$T = \frac{100}{0.060375} = 1{,}656.3147 \text{ lb}$$

2.29 When $\omega = 0$, total vertical height $= 2l + h$





When $\omega \neq 0$, total vertical height $= (2l\cos\theta + h)$

$$\text{spring force} = k[2l + h - (2l\cos\theta + h)] = 2kl\,(1-\cos\theta)$$

$$F_c = m\,x\,\omega^2$$

$$x = l\sin\theta$$

For vertical equilibrium of mass m,

$$mg + T_2\cos\theta = T_1\cos\theta \quad --- \; (E_1)$$

For horizontal equilibrium, $F_c = (T_1 + T_2)\sin\theta$

$$T_2 = (F_c - T_1\sin\theta)/\sin\theta \quad --- \; (E_2)$$

From (E_2), (E_1) can be expressed as

$$mg + \left(\frac{F_c - T_1\sin\theta}{\sin\theta}\right)\cos\theta = T_1\cos\theta$$

i.e. $$T_1 = \frac{mg + F_c\cot\theta}{2\cos\theta} = \frac{mg + m\omega^2 l\cos\theta}{2\cos\theta}$$

$$T_2 = \frac{F_c - T_1\sin\theta}{\sin\theta} = \frac{m\,x\,\omega^2 - \frac{mg}{2}\tan\theta - \frac{m\omega^2 l}{2}\sin\theta}{\sin\theta}$$

$$= \frac{ml\omega^2}{2} - \frac{mg}{2\cos\theta}$$

$$\text{spring force} = 2kl\,(1-\cos\theta) = 2T_2\cos\theta = ml\omega^2\cos\theta - mg$$

$$\cos\theta = \left(\frac{2kl + mg}{2kl + ml\omega^2}\right) \quad --- \; (E_3)$$

This equation defines the equilibrium position of mass m. For small oscillations about the equilibrium position, Newton's second law gives

$$2m\ddot{y} + k\,y = 0 \quad , \qquad \omega_n = \sqrt{\frac{2k}{m}}$$

2.30 (a) Let P = total spring force, F = centrifugal force acting on each ball. Equilibrium of moments about the pivot of bell crank lever (O) gives:

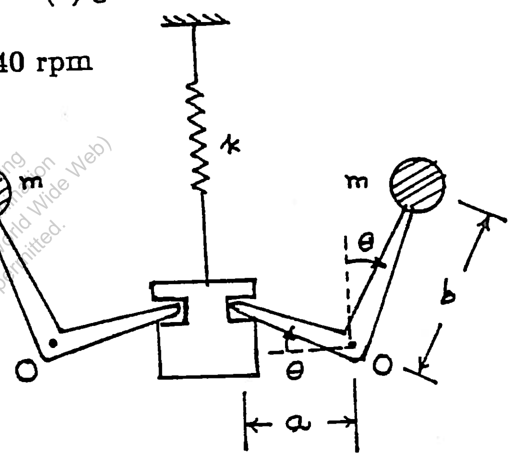
$$F\left(\frac{20}{100}\right) = \frac{P}{2}\left(\frac{12}{100}\right) \tag{1}$$

When $P = 10^4\left(\frac{1}{100}\right) = 100$ N, and

$$F = m\,r\,\omega^2 = m\,r\left(\frac{2\,\pi\,N}{60}\right)^2 = \frac{25}{9.81}\left(\frac{16}{100}\right)\left(\frac{2\,\pi\,N}{60}\right)^2 = 0.004471\ N^2$$

where N = speed of the governor in rpm. Equation (1) gives:

$$0.004471\ N^2\ (0.2) = \frac{100}{2}\ (0.12) \quad \text{or} \quad N = 81.9140 \text{ rpm}$$



(b) Consider a small displacement of the ball arm about the vertical position. Equilibrium about point O gives:

$$(m\,b^2)\,\ddot{\theta} + (k\ a\ \sin\theta)\ a\ \cos\theta = 0 \tag{2}$$

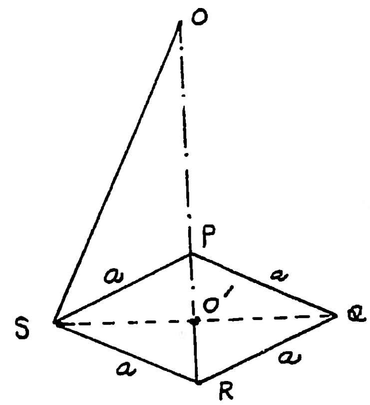
For small vallues of θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and hence Eq. (2) gives

$$m\,b^2\,\ddot{\theta} + k\,a^2\,\theta = 0$$

from which the natural frequency can be determined as

$$\omega_n = \left\{\frac{k\,a^2}{m\,b^2}\right\}^{\frac{1}{2}} = \left\{(10)^4\left(\frac{0.12}{0.20}\right)^2\ \frac{9.81}{25}\right\}^{\frac{1}{2}} = 37.5851 \text{ rad/sec}$$

2.31 $SO' = \frac{a}{\sqrt{2}}$, $OO' = h$, $OS = \sqrt{h^2 + \frac{a^2}{2}}$



When each wire stretches by x_s, let the resulting vertical displacement of the platform be x.

$$OS + x_s = \sqrt{(h+x)^2 + \frac{a^2}{2}}$$

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left\{ \sqrt{\frac{(h+x)^2 + \frac{a^2}{2}}{h^2 + \frac{a^2}{2}}} - 1 \right\}$$

$$= \sqrt{h^2 + \frac{a^2}{2}} \left[\sqrt{1 + \left\{ \frac{2hx + x^2}{\left(h^2 + \frac{a^2}{2}\right)} \right\}} - 1 \right]$$

For small x, x^2 is negligible compared to $2hx$ and $\sqrt{1+\theta} \simeq 1 + \frac{\theta}{2}$ and hence

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left[1 + \frac{hx}{\left(h^2 + \frac{a^2}{2}\right)} - 1 \right] = \frac{h}{\sqrt{h^2 + \frac{a^2}{2}}} \cdot x$$

Potential energy equivalence gives

$$\frac{1}{2} k_{eq} x^2 = 4 \left(\frac{1}{2} k\, x_s^2 \right)$$

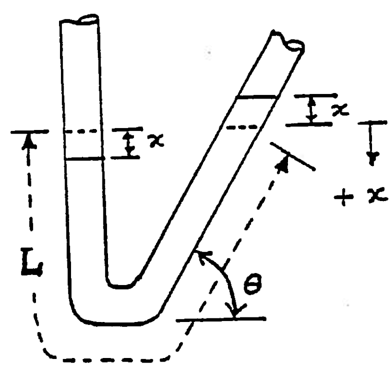
$$k_{eq} = 4k \left(\frac{x_s}{x} \right)^2 = \frac{4kh^2}{\left(h^2 + \frac{a^2}{2}\right)}$$

Equation of motion of M:

$$M\ddot{x} + k_{eq}\, x = 0$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{(k_{eq}/M)^{1/2}} = \frac{\pi\sqrt{M}}{h} \left(\frac{2h^2 + a^2}{2k} \right)^{1/2}$$

2.32 Equation of motion:



$$m\ddot{x} = \sum F_x$$

i.e., $(L A \rho)\ddot{x} = -2(A x \rho g)$

i.e., $\ddot{x} + \frac{2g}{L} x = 0$

where A = cross-sectional area of the tube and ρ = density of mercury. Thus the natural frequency is given by:

$$\omega_n = \sqrt{\frac{2g}{L}}$$

2.33 Assume same area of cross section for all segments of the cable. Speed of blades = 300 rpm = 5 Hz = 31.416 rad/sec.

$$\omega_n^2 = \frac{k_{eq}}{m} = (2\ (31.416))^2 = (62.832)^2$$

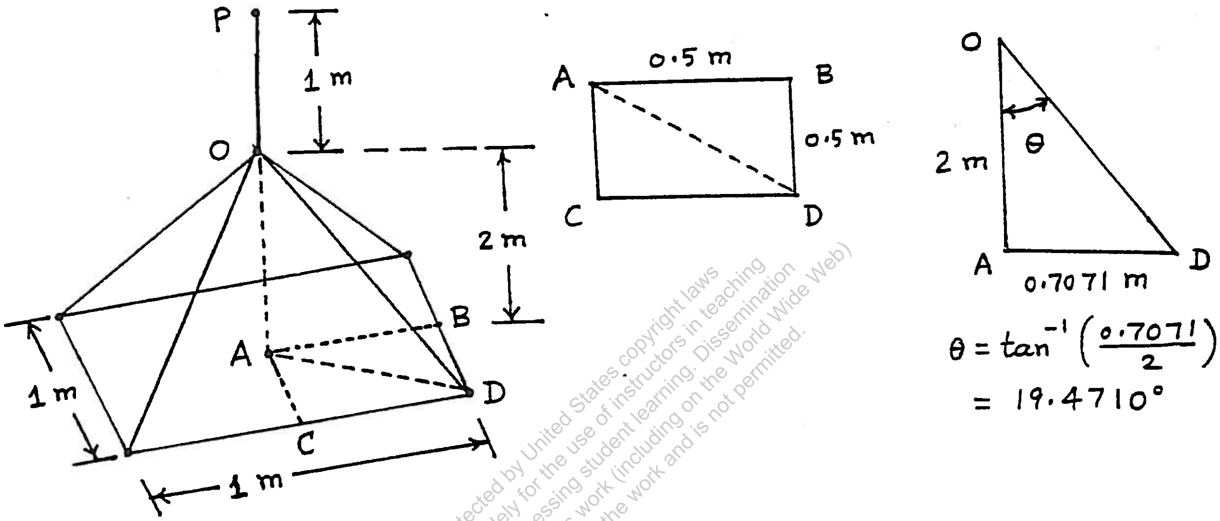
$$k_{eq} = m\ \omega_n^2 = 250\ (62.832)^2 = 98.6965\ (10^4)\ \text{N/m} \qquad (1)$$

$$AD = \sqrt{0.5^2 + 0.5^2} = 0.7071\ \text{m}\ ,\quad OD = \sqrt{2^2 + 0.7071^2} = 2.1213\ \text{m}$$

Stiffness of cable segments:

$$k_{PO} = \frac{A\,E}{\ell_{PO}} = \frac{A\ (207)\ (10^9)}{1} = 207\ (10^9)\ A\ \text{N/m}$$

$$K_{OD} = \frac{A\,E}{\ell_{OD}} = \frac{A\ (207)\ (10^9)}{2.1213} = 97.5817\ (10^9)\ A\ \text{N/m}$$



The total sttiffness of the four inclined cables (k_{ic}) is given by:

$$k_{ic} = 4\ k_{OD}\ \cos^2\theta$$
$$= 4\ (97.5817)\ (10^9)\ A\ \cos^2 19.4710° = 346.9581\ (10^9)\ A\ \text{N/m}$$

Equivalent stiffness of vertical and inclined cables is given by:

$$\frac{1}{k_{eq}} = \frac{1}{k_{PO}} + \frac{1}{k_{ic}}$$

i.e., $$k_{eq} = \frac{k_{PO}\ k_{ic}}{k_{PO} + k_{ic}}$$

$$= \frac{(207\ (10^9)\ A)\ (346.9581\ (10^9)\ A)}{(207\ (10^9)\ A) + (346.9581\ (10^9)\ A)} = 129.6494\ (10^9)\ A\ \text{N/m} \qquad (2)$$

Equating k_{eq} given by Eqs. (1) and (2), we obtain the area of cross section of cables as:

$$A = \frac{98.6965\ (10^4)}{129.6494\ (10^9)} = 7.6126\ (10^{-6})\ \text{m}^2$$

(2.34)

$$\frac{1}{2\pi}\left\{\frac{k_1}{m}\right\}^{\frac{1}{2}} = 5 \;;\; \frac{k_1}{m} = 4\,(\pi)^2\,(25) = 986.9651$$

$$\frac{1}{2\pi}\left\{\frac{k_1}{m+5000}\right\}^{\frac{1}{2}} = 4.0825 \;;\; \frac{k_1}{m+5000} = 4\,(\pi)^2\,(16.6668) = 657.9822$$

Using $k_1 = \dfrac{A\,E}{\ell_1}$ we obtain

$$\frac{k_1}{m} = \frac{A\,E}{\ell_1\,m} = \frac{A\,(207)\,(10^9)}{2\,m} = 986.9651$$

$$\text{i.e.,}\quad A = 9.5359\,(10^{-9})\,m \tag{1}$$

Also

$$\frac{k_1}{m+5000} = \frac{A\,E}{\ell_1\,(m+5000)} = 657.9822$$

$$\text{i.e.,}\quad \frac{A}{m+5000} = 6.3573\,(10^{-9}) \tag{2}$$

Using Eqs. (1) and (2), we obtain

$$A = 9.5359\,(10^{-9})\,m = 6.3573\,(10^{-9})\,m + 31.7865\,(10^{-6})$$

$$\text{i.e.,}\quad 3.1786\,(10^{-9})\,m = 31.7865\,(10^{-6})$$

$$\text{i.e.,}\quad m = 10000.1573\ \text{kg} \tag{3}$$

Equations (1) and (3) yield

$$A = 9.5359\,(10^{-9})\,m = 9.5359\,(10^{-9})\,(10000.1573) = 0.9536\,(10^{-4})\ \text{m}^2$$

(2.35) Longitudinal Vibration:

Let W_1 = part of weight W carried by length a of shaft

$W_2 = W - W_1$ = weight carried by length b

x = Elongation of length $a = \dfrac{W_1\,a}{AE}$

y = shortening of length $b = \dfrac{(W - W_1)(\ell - a)}{AE}$

E = Young's modulus

A = area of cross-section $= \pi d^2/4$

Since $x = y$, $\quad W_1 = \dfrac{W(\ell - a)}{\ell}$

x = elongation or static deflection of length $a = \dfrac{Wa\,(\ell - a)}{AE\,\ell}$

Considering the shaft of length a with end mass W_1/g as a spring-mass system,

$$\omega_n = \sqrt{\frac{g}{x}} = \left(\frac{g\,\ell\,AE}{Wa\,(\ell - a)}\right)^{1/2}$$

Transverse vibration:

spring constant of a fixed-fixed beam with off-center load

$$= k = \frac{3EI\,\ell^3}{a^3\,b^3} = \frac{3EI\,\ell^3}{a^3\,(\ell-a)^3}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left\{ \frac{3EI\,\ell^3\,g}{W\,a^3\,(\ell-a)^3} \right\}^{1/2} \quad \text{with} \quad I = \left(\pi d^4/64\right)$$

= moment of inertia

Torsional vibration:

If flywheel is given an angular deflection θ, resisting torques offered by lengths a and b are $\frac{GJ\theta}{a}$ and $\frac{GJ\theta}{b}$.

Total resisting torque $= M_t = GJ\left(\frac{1}{a} + \frac{1}{b}\right)\theta$

$$k_t = \frac{M_t}{\theta} = GJ\left(\frac{1}{a} + \frac{1}{b}\right) \quad \text{where} \quad J = \frac{\pi d^4}{32} = \text{polar moment of inertia}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left\{ \frac{GJ}{J_0}\left(\frac{1}{a} + \frac{1}{b}\right) \right\}^{1/2}$$

where J_0 = mass polar moment of inertia of the flywheel.

(2.36) $m_{eq_{end}}$ = equivalent mass of a uniform beam at the free end (see Problem 2.38) =

$$\frac{33}{140}\,m = \frac{33}{140}\left\{1\,(1)\,(150 \times 12)\,\frac{0.283}{386.4}\right\} = 0.3107$$

Stiffness of tower (beam) at free end:

$$k_b = \frac{3\,E\,I}{L^3} = \frac{3\,(30 \times 10^6)\,\left(\frac{1}{12}\,(1)\,(1^3)\right)}{(150 \times 12)^3} = 0.001286 \text{ lb/in}$$

Length of each cable:

$$OA = \sqrt{2} = 1.4142 \text{ ft} \;,\quad OB = \sqrt{2}\;15 = 21.2132 \text{ ft} \;,\quad AB = OB - OA = 19.7990 \text{ ft}$$

$$TB = \sqrt{TA^2 + AB^2} = \sqrt{100^2 + 19.7990^2} = 101.9412 \text{ ft}$$

$$\tan\theta = \frac{AT}{AB} = \frac{100}{19.7990} = 5.0508 \;,\quad \theta = 78.8008^\circ$$

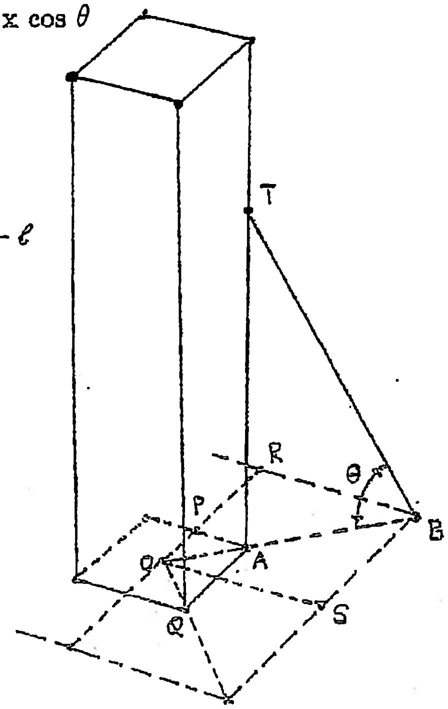
Axial stiffness of each cable:

$$k = \frac{A\,E}{\ell} = \frac{(0.5)\,(30 \times 10^6)}{(101.9412 \times 12)} = 12261.971 \text{ lb/in}$$

Axial extension of each cable (y_c) due to a horizontal displacement of x of tower:

$$\ell_1^2 = \ell^2 + x^2 - 2\,\ell\, x \cos(180^\circ - \theta) = \ell^2 + x^2 + 2\,\ell\, x \cos\theta$$

$$\text{or} \quad \ell_1 = \ell \left\{ 1 + \left(\frac{x}{\ell} \right)^2 + 2\,\frac{x}{\ell} \cos\theta \right\}^{\frac{1}{2}}$$



$$y_c = \ell_1 - \ell \approx \ell \left(1 + \frac{1}{2}\,\frac{x^2}{\ell^2} + \frac{1}{2}\,(2)\,\frac{x}{\ell} \cos\theta \right) - \ell$$

$$= \ell + x \cos\theta - \ell = x \cos\theta$$

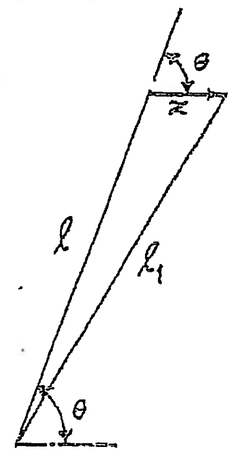
Equivalent stiffness of each cable, $k_{eq\,OB}$, in a horizontal direction, parallel to OAB, is given by

$$\frac{1}{2}\, k\, y_c^2 = \frac{1}{2}\, k_{eq\,OB}\, x^2 \quad \text{or} \quad k_{eq\,OB} = k \left(\frac{y_c}{x} \right)^2 = k \cos^2\theta$$

Equivalent stiffness of each cable, $k_{eq\,x}$, in a horizontal direction, parallel to the x-axis (along OS), can be found as

$$k_{eq\,x} = k_{eq\,OB} \cos^2 45^0 = \frac{1}{2}\, k_{eq\,OB} = \frac{1}{2}\, k \cos^2\theta$$

(since angle BOS is 45^0)



This gives

$$k_{eq\,x} = \frac{1}{2}\,(12261.971) \cos^2 78.8008^0 = 231.2709 \text{ lb/in}$$

In order to use the relation $k_{eq_{end}} = k_b + 4\, k_{eq_c} \left(\dfrac{y_{L1}}{y_L} \right)^2$, we find

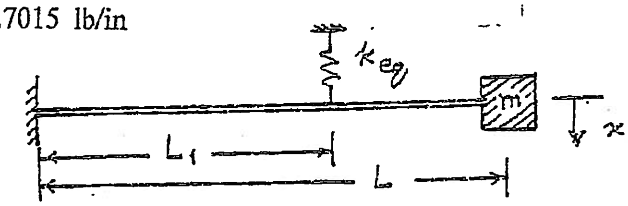
$$\frac{y_{L1}}{y_L} = \left(\frac{F\, L_1^2\,(3\,L - L_1)}{6\,E\,I} \; \frac{3\,E\,I}{F\,L^3} \right) = \frac{L_1^2\,(3\,L - L_1)}{2\,L^3}$$

$$= \frac{100^2\,(3\,(150) - 100)}{2\,(150)^3} = 0.5185 \text{ . Thus}$$

$$k_{eq_{end}} = k_b + 4\, k_{eq\,x}\,(0.5185)^2 = 0.001286 + 4(231.2709)(0.5185)^2$$

$$= 248.7015 \text{ lb/in}$$

Natural frequency:



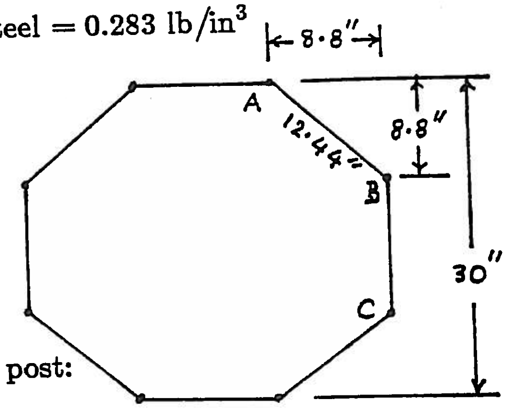
$$\omega_n = \left\{ \frac{k_{eq_{end}}}{m_{eq_{end}}} \right\}^{\frac{1}{2}} = \left(\frac{248.7015}{0.3107} \right)^{\frac{1}{2}} = 28.2923 \text{ rad/sec}$$

(2.37) Sides of the sign:

$$AB = \sqrt{8.8^2 + 8.8^2} = 12.44 \text{ in} \quad ; \quad BC = 30 - 8.8 - 8.8 = 12.4 \text{ in}$$

$$\text{Area} = 30\,(30) - 4\,(\frac{1}{2}\,(8.8)\,(8.8)) = 745.12 \text{ in}^2$$

$$\text{Thickness} = \frac{1}{8} \text{ in} \quad ; \quad \text{Weight density of steel} = 0.283 \text{ lb/in}^3$$



$$\text{Weight of sign} = (0.283)(\frac{1}{8})(745.12) = 26.64 \text{ lb}$$

$$\text{Weight of sign post} = (72)\,(2)\,(\frac{1}{4})\,(0.283) = 10.19 \text{ lb}$$

Stiffness of sign post (cantilever beam):

$$k = \frac{3\,E\,I}{\ell^3}$$

Area moments of inertia of the cross section of the sign post:

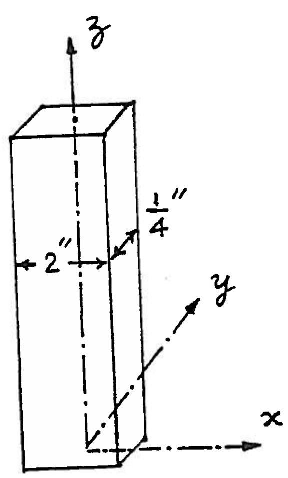
$$I_{xx} = \frac{1}{12}\,(2)\,(\frac{1}{4})^3 = \frac{1}{384} \text{ in}^4$$

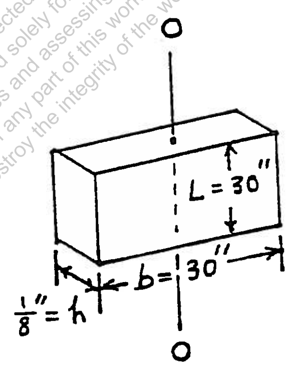
$$I_{yy} = \frac{1}{12}\,(\frac{1}{4})\,(2)^3 = \frac{1}{6} \text{ in}^4$$

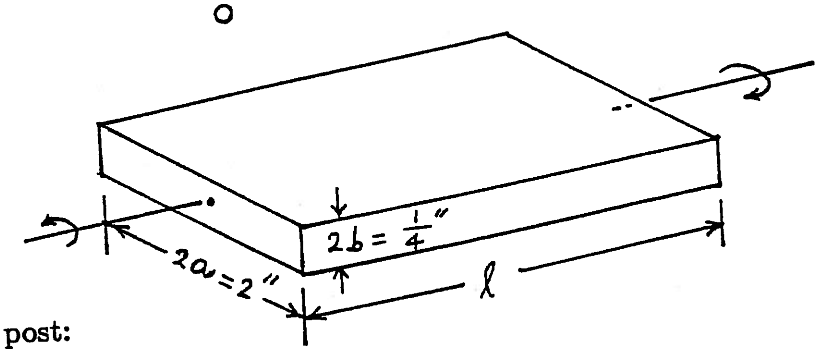
Bending stiffnesses of the sign post:

$$k_{xz} = \frac{3\,E\,I_{yy}}{\ell^3} = \frac{3\,(30\,(10^6))(\frac{1}{6})}{72^3} = 40.1877 \text{ lb/in}$$

$$k_{yz} = \frac{3\,E\,I_{xx}}{\ell^3} = \frac{3\,(30\,(10^6))\,(\frac{1}{384})}{72^3} = 0.6279 \text{ lb/in}$$







Torsional stiffness of the sign post:

$$k_t = 5.33 \; \frac{a\,b^3}{\ell} \; G \left\{ 1 - 0.63 \; \frac{b}{a} \left(1 - \frac{b^4}{12\,a^4} \right) \right\}$$

(See Ref: N. H. Cook, *Mechanics of Materials for Design*, McGraw-Hill, New York, 1984, p. 342).

Thus

$$k_t = 5.33 \left\{ \frac{(1)\,(\frac{1}{8})^3}{72} \right\} (11.5\,(10^6)) \left\{ 1 - (0.63)\,(\frac{1}{8}) \left(1 - \frac{(\frac{1}{8})^4}{12\,(1)^4} \right) \right\}$$

$$= 1531.7938 \text{ lb—in/rad}$$

Natural frequency for bending in xz plane:

$$\omega_{xz} = \left\{ \frac{k_{xz}}{m} \right\}^{\frac{1}{2}} = \left\{ \frac{40.1877}{\left(\frac{26.64}{386.4} \right)} \right\}^{\frac{1}{2}} = 24.1434 \text{ rad/sec}$$

Natural frequency for bending in yz plane:

$$\omega_{yz} = \left\{ \frac{k_{yz}}{m} \right\} = \left\{ \frac{0.6279}{\left(\frac{26.64}{386.4} \right)} \right\}^{\frac{1}{2}} = 3.0178 \text{ rad/sec}$$

By approximating the shape of the sign as a square of side 30 in (instead of an octagon), we can find its mass moment of inertia as:

$$I_{oo} = \frac{\gamma\,L}{3}\,(b^3\,h + h^3\,b) = \left(\frac{0.283}{386.4} \right) \left(\frac{30}{3} \right) \left(30^3\,(\frac{1}{8}) + (\frac{1}{8})^3\,(30) \right) = 24.7189$$

Natural torsional frequency:

$$\omega_t = \left\{ \frac{k_t}{I_{oo}} \right\}^{\frac{1}{2}} = \left\{ \frac{1531.7938}{24.7189} \right\}^{\frac{1}{2}} = 7.8720 \text{ rad/sec}$$

Thus the mode of vibration (close to resonance) is torsion in xy plane.

Let $l = h$.

2.38 (a) Pivoted:

$k_{eq} = 4\ k_{column} = 4\left(\frac{3EI}{l^3}\right) = \frac{12EI}{l^3}$

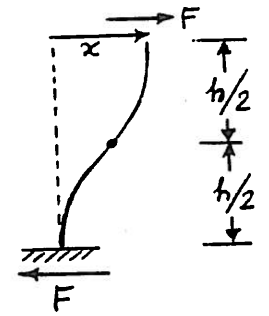
Let m_{eff1} = effective mass due to self weight of columns

Equation of motion: $\left(\frac{W}{g} + m_{eff1}\right)\ddot{x} + k_{eq}\,x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\dfrac{12EI}{l^3\left(\frac{W}{g} + m_{eff1}\right)}}$

(b) Fixed:

since the joint between column and floor does not permit rotation, each column will bend with inflection point at middle.



When force F is applied at ends,

$$x = 2 \; \frac{F(l/2)^3}{3EI} = \frac{F l^3}{12 EI}$$

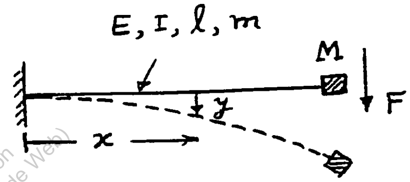
$$k_{column} = \frac{12EI}{l^3} \; ; \quad k_{eq} = 4\, k_{column} = \frac{48EI}{l^3}$$

Let m_{eff2} = effective mass of each column at top end

Equation of motion: $\left(\frac{W}{g} + m_{eff2}\right)\ddot{x} + k_{eq}\, x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\dfrac{48EI}{l^3\left(\frac{W}{g} + m_{eff2}\right)}}$

Effective mass (due to self weight):



(a) Let m_{eff1} = effective part of mass of beam (m) at end.

Thus vibrating inertia force at end is due to $(M + m_{eff1})$.

Assume deflection shape during vibration same as the static deflection shape with a tip load:

$$y(x,t) = Y(x)\cos(\omega_n t - \phi) \quad \text{where} \quad Y(x) = \frac{F x^2 (3l - x)}{6EI}.$$

$$Y(x) = \frac{Y_0}{2l^3}\, x^2 (3l - x) \quad \text{where} \quad Y_0 = \frac{F l^3}{3EI} = \text{max. tip deflection}$$

$$y(x,t) = \frac{Y_0}{2l^3}\left(3x^2 l - x^3\right)\cos(\omega_n t - \phi) \qquad (E_1)$$

Max. strain energy of beam = Max. work by force F

$$= \frac{1}{2} F Y_0 = \frac{3}{2}\,\frac{EI}{l^3}\, Y_0^2 \qquad (E_2)$$

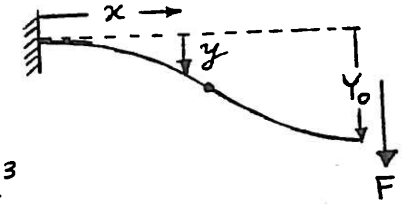
Max. kinetic energy due to distributed mass of beam

$$= \frac{1}{2}\,\frac{m}{l}\int_0^l \dot{y}^2(x,t)\Big|_{max} dx + \frac{1}{2}\left(\dot{y}_{max}\right)^2 M$$

$$= \frac{1}{2}\,\omega_n^2\, Y_0^2\left(\frac{33}{140}\, m\right) + \frac{1}{2}\,\omega_n^2\, Y_0^2\, M \qquad (E_3)$$

$$\therefore \; m_{eff1} = \frac{33}{140}\, m = 0.2357\, m$$

(b) Let $Y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$



$Y(0) = 0, \quad \frac{dY}{dx}(0) = 0, \quad Y(l) = Y_0, \quad \frac{dY}{dx}(l) = 0$

This leads to $\quad Y(x) = \frac{3 Y_0}{l^2} x^2 - \frac{2 Y_0}{l^3} x^3$

$$y(x,t) = Y_0 \left(3 \frac{x^2}{l^2} - 2 \frac{x^3}{l^3}\right) \cos(\omega_n t - \phi) \qquad (E_4)$$

$$\text{Maximum strain energy} = \frac{1}{2} EI \int_0^l \left(\frac{\partial^2 y}{\partial x^2}\right)^2 dx \Bigg|_{max} \qquad (E_5)$$

$$= \frac{6 EI Y_0^2}{l^3}$$

$$\text{Max. kinetic energy} = \frac{1}{2} M \omega_n^2 Y_0^2 + \frac{1}{2}\left(\frac{m}{l}\right) Y_0^2 \omega_n^2 \int_0^l \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3}\right)^2 dx \qquad (E_6)$$

$$= \frac{1}{2} \omega_n^2 Y_0^2 \left(M + \frac{13}{35} m\right)$$

$$\therefore \quad m_{eff\,2} = \frac{13}{35} m = 0.3714\, m$$

(2.39) Stiffnesses of segments:

$$A_1 = \frac{\pi}{4}(D_1^2 - d_1^2) = \frac{\pi}{4}(2^2 - 1.75^2) = 0.7363 \text{ in}^2$$

$$k_1 = \frac{A_1 E_1}{L_1} = \frac{(0.7363)(10^7)}{12} = 61.3583\,(10^4) \text{ lb/in}$$

$$A_2 = \frac{\pi}{4}(D_2^2 - d_2^2) = \frac{\pi}{4}(1.5^2 - 1.25^2) = 0.5400 \text{ in}^2$$

$$k_2 = \frac{A_2 E_2}{L_2} = \frac{(0.5400)(10^7)}{10} = 54.0\,(10^4) \text{ lb/in}$$

$$A_3 = \frac{\pi}{4}(D_3^2 - d_3^2) = \frac{\pi}{4}(1^2 - 0.75^2) = 0.3436 \text{ in}^2$$

$$k_3 = \frac{A_3 E_3}{L_3} = \frac{(0.3436)(10^7)}{8} = 42.9516\,(10^4) \text{ lb/in}$$

Equivalent stiffness (springs in series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

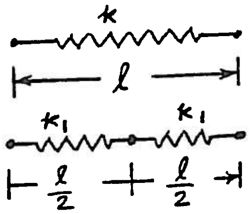
$$= 0.0162977\,(10^{-4}) + 0.0185185\,(10^{-4}) + 0.0232820\,(10^{-4}) = 0.0580982\,(10^{-4})$$

$$\text{or} \quad k_{eq} = 17.2122\,(10^4) \text{ lb/in}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq}\, g}{W}} = \sqrt{\frac{17.2122\,(10^4)\,(386.4)}{10}} = 2578.9157 \text{ rad/sec}$$

2.40



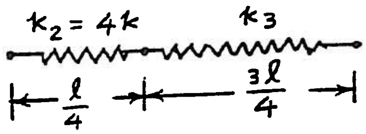
$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{total} = \frac{k_1}{2} \equiv k\ ;\ k_1 = 2k$$

$$\tau_n = 2\pi\sqrt{\frac{m}{k_{eq}}}$$

$$0.5 = 2\pi\sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$



$$\frac{1}{k_{total}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

$$k_3 = \frac{4}{3}k$$

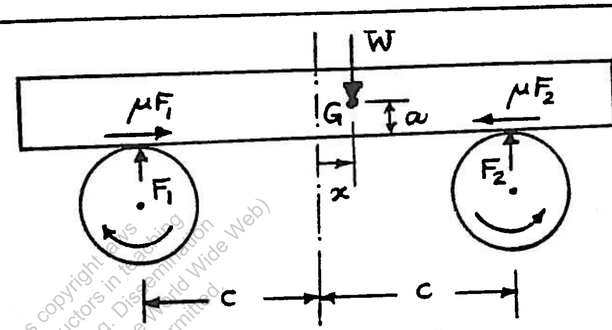
$$\tau_n = 2\pi\sqrt{\frac{m}{k_{eq}}} \quad \text{where} \quad k_{eq} = 4k + \frac{4}{3}k = \frac{16}{3}k$$

$$\therefore \tau_n = 2\pi\sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4}\cdot\sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4}\left(\frac{1}{2\pi}\right) = 0.4330 \text{ sec}$$

2.41 Let μ = coefficient of friction

x = displacement of C.G. of block

F_1, F_2 = net reactions between roller and block on left and right sides.



Reactions at left and right due to static load W are $W(c-x)/2c$ and $W(c+x)/2c$, respectively.

M = moment about G due to motion of block $= (\mu F_2 - \mu F_1)a$

Reactions at left and right to balance $M = \frac{M}{2c} = \frac{\mu a}{2c}(F_2 - F_1)$

$$F_1 = \frac{W(c-x)}{2c} - \frac{\mu a}{2c}(F_2 - F_1)\ ;\quad F_2 = \frac{W(c+x)}{2c} + \frac{\mu a}{2c}(F_2 - F_1)$$

Subtraction gives $\quad F_2 - F_1 = \frac{Wx}{c} + \frac{\mu a}{c}(F_2 - F_1)$

i.e., $\quad F_2 - F_1 = \frac{Wx}{c}\left(\frac{c}{c-\mu a}\right) = \frac{Wx}{c-\mu a}$

Restoring force $= \mu(F_2 - F_1) = \left(\frac{\mu W x}{c-\mu a}\right)$

Equation of motion : $\quad \frac{W}{g}\ddot{x} + \frac{\mu W}{(c-\mu a)}x = 0$

$$\omega_n = \omega = \sqrt{\frac{\mu W g}{W(c-\mu a)}} = \sqrt{\frac{\mu g}{c-\mu a}}$$

Solving this, we get $\quad \omega = \left[c\,\omega^2/(g + a\,\omega^2)\right]$

2.42 From problem 2.41,

Restoring force without springs $= \mu (F_2 - F_1) = \dfrac{\mu W x}{c - \mu a}$

spring restoring force $= 2kx$

Total restoring force $= \dfrac{\mu W x}{c-\mu a} + 2kx$

Equation of motion: $\dfrac{W}{g}\ddot{x} + \left(\dfrac{\mu W}{c-\mu a} + 2k\right)x = 0$

$$\omega_n = \omega = \left\{ \frac{[\mu W + 2k(c-\mu a)]\, g}{(c-\mu a)\, W} \right\}^{1/2}$$

Solution of this equation gives

$$\mu = \left(\frac{\omega^2 W c - 2 k g c}{W g + W \omega^2 a - 2 k g a} \right)$$

2.43 (a) Natural frequency of vibration of electromagnet (without the automobile):

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000.0\ (386.4)}{3000.0}} = 35.8887 \text{ rad/sec}$$

(b) When the automobile is dropped, the electromagnet moves up by a distance (x_0) from its static equilibrium position.

x_0 = static deflection (elongation of cable) under the weight of automobile

$$= \frac{W_{auto}}{k} = \frac{2000}{10000} = 0.2 \text{ in}$$

$\dot{x}_0 = 0$

Resultant motion of electromagnet (+x upwards):

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = x_0 = 0.2$$

and $$\phi_0 = \tan^{-1}\left(\frac{x_0\, \omega_n}{\dot{x}_0} \right) = \tan^{-1}(\infty) = 90^\circ$$

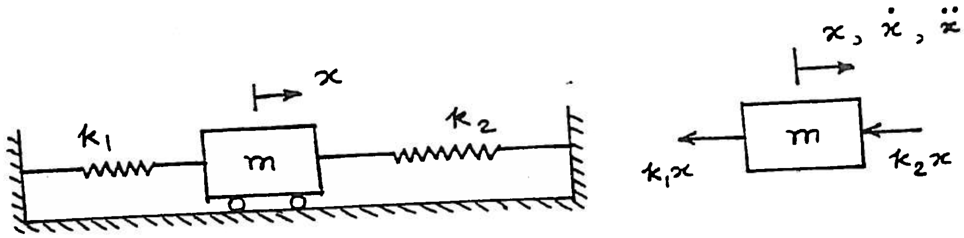
Hence $x(t) = 0.2 \sin(35.8887\, t + 90^\circ) = 0.2 \cos 35.8887\, t$

(c) Maximum x(t):

$$x(t)\big|_{max} = A_0 = 0.2 \text{ in}$$

Maximum tension in cable during motion $= k\, x(t)\big|_{max}$ + Weigh of electromagnet $= 10000\ (0.2) + 3000 = 5{,}000$ lb.

2.44



(a) Newton's second law of motion:

$$F(t) = -k_1\, x - k_2\, x = m\, \ddot{x} \quad \text{or} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

(b) D'Alembert's principle:

$$F(t) - m\, \ddot{x} = 0 \quad \text{or} \quad -k_1\, x - k_2\, x - m\, \ddot{x} = 0$$

$$\text{Thus} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

(c) Principle of virtual work:

When mass m is given a virtual displacement δx,
Virtual work done by the spring forces = $-(k_1 + k_2)\, x\, \delta x$
Virtual work done by the inertia force = $-(m\, \ddot{x})\, \delta x$
According to the principle of virtual work, the total virtual work done by all forces must be equal to zero:

$$-m\, \ddot{x}\, \delta x - (k_1 + k_2)\, x\, \delta x = 0 \quad \text{or} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

(d) Principle of conservation of energy:

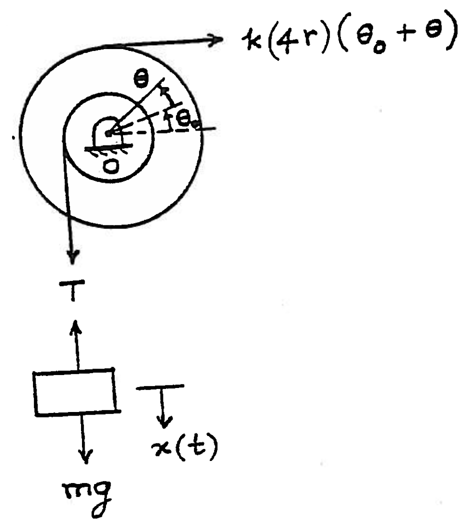
$$T = \text{kinetic energy} = \frac{1}{2}\, m\, \dot{x}^2$$

$$U = \text{strain energy} = \text{potential energy} = \frac{1}{2}\, k_1\, x^2 + \frac{1}{2}\, k_2\, x^2$$

$$T + U = \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, (k_1 + k_2)\, x^2 = c = \text{constant}$$

$$\frac{d}{dt}\,(T + U) = 0 \quad \text{or} \quad m\, \ddot{x} + (k_1 + k_2)\, x = 0$$

2.45



Equation of motion:

$$\text{Mass m:} \quad m\,g - T = m\,\ddot{x} \tag{1}$$

$$\text{Pulley } J_0: \quad J_0\,\ddot{\theta} = T\,r - k\,4\,r\,(\theta + \theta_0)\,4\,r \tag{2}$$

where θ_0 = angular deflection of the pulley under the weight, mg, given by:

$$m\,g\,r = k\,(4\,r\,\theta_0)\,4\,r \quad \text{or} \quad \theta_0 = \frac{m\,g}{16\,r\,k} \tag{3}$$

Substituting Eqs. (1) and (3) into (2), we obtain

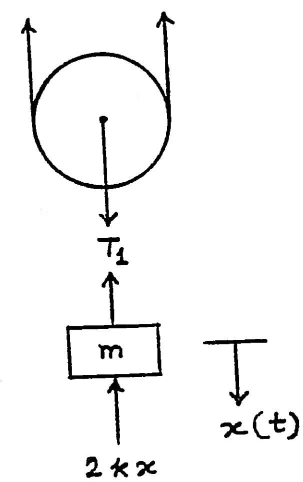
$$J_0\,\ddot{\theta} = (m\,g - m\,\ddot{x})\,r - k\,16\,r^2\,(\theta + \frac{m\,g}{16\,r\,k}) \tag{4}$$

Using $x = r\,\theta$ and $\ddot{x} = r\,\ddot{\theta}$, Eq. (4) becomes

$$(J_0 + m\,r^2)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.46 Consider the springs connected to the pulleys (by rope) to be in series. Then

$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{5k} \quad \text{or} \quad k_{eq} = \frac{5}{6}\,k$$



Let the displacement of mass m be x. Then the extension of the rope (springs connected to the pulleys) = 2 x. From the free body diagram, the equation of motion of mass m:

$$m\,\ddot{x} + 2\,k\,x + k_{eq}\,(2\,x) = 0$$

$$\text{or} \quad m\,\ddot{x} + \frac{11}{3}\,k\,x = 0$$

2.47 T = kinetic energy = $T_{mass} + T_{pulley}$

$$= \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 = \frac{1}{2}\,(m\,r^2 + J_0)\,\dot{\theta}^2$$

$$U = \text{potential energy} = \frac{1}{2}\,k\,x_s^2 = \frac{1}{2}\,k\,(4\,r\,\theta)^2 = \frac{1}{2}\,k\,(16\,r^2)\,\theta^2$$

Using $\frac{d}{dt}\,(T + U) = 0$ gives

$$(m\,r^2 + J_0)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.48 $T = \text{kinetic energy} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$

$U = \text{potential energy} = \frac{1}{2} k x_s^2$

where $\theta = \frac{x}{r}$, x_s = extension of spring $= 4 r \theta = 4 x$. Hence

$$T = \frac{1}{2} (m + \frac{J_0}{r^2}) \dot{x}^2 \quad ; \quad U = \frac{1}{2} (16 k) x^2$$

Using tthe relation $\frac{d}{dt} (T + U) = 0$, we obtain the equation of motion of the system as:

$$(m + \frac{J_0}{r^2}) \ddot{x} + 16 k x = 0$$

2.49 (a) stiffness of the cantilever beam of length l (k_b) at location of the mass:

$$k_b = \frac{3EI}{l^3} \tag{E1}$$

Since any transverse force F applied to the mass m is felt by each of the three springs k_1, k_2 and k_3, all the springs (k_1, k_2, k_3 and k_b) can be considered to be in series. The equivalent spring constant, k_{eq}, of the system is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} + \frac{1}{k_b}$$

$$= \frac{k_2 k_3 k_b + k_1 k_3 k_b + k_1 k_2 k_b + k_1 k_2 k_3}{k_1 k_2 k_3 k_4} \tag{E2}$$

or

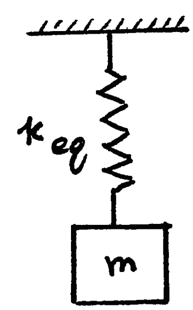
$$k_{eq} = \frac{k_1 k_2 k_3 k_b}{k_2 k_3 k_b + k_1 k_3 k_b + k_1 k_2 k_b + k_1 k_2 k_3} \tag{E3}$$

(b) Natural frequency of vibration of the system is given by

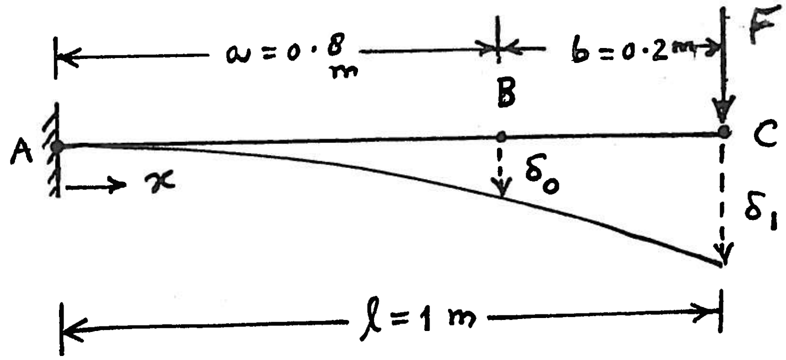
$$\omega_n = \sqrt{\frac{k_{eq}}{m}} \tag{E4}$$

where k_{eq} is given by Eq. (E3).

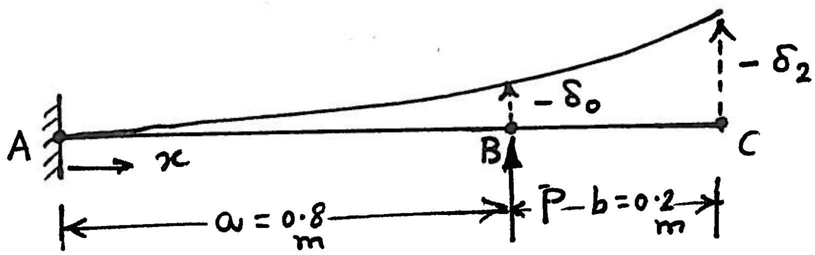
Equivalent system



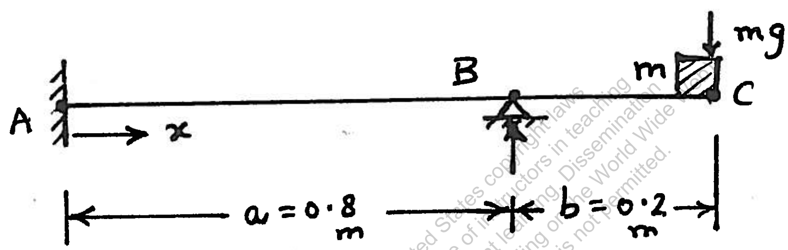
2.50



step (i)



step (ii)



Given system

To find the stiffness of the given system at point C (i.e. deflection produced at C by a force F applied at C), we use the following 6-step process:

step 1: Consider a cantilever beam with force applied at C (without support at B) and find deflections at B and C:

$$\delta(x) = \frac{F x^2}{6EI}(3l - x) \qquad (1)$$

$$\delta_B = (\text{with } x = a) = \frac{F a^2}{6EI}(3l - a) = \frac{F(0.8^2)(3 - 0.8)}{6EI}$$

$$= 0.2347\ F/EI \qquad (2)$$

$$\delta_C = (\text{with } x = l) = \frac{F l^2 (2l)}{6EI} = \frac{F(1^2)(2)}{6EI} = 0.3333\ F/EI \qquad (3)$$

$\hookrightarrow = \delta_1$

step 2: Consider a cantilever beam with force P applied at B (in upward direction) and find deflections at B and C:

$$\delta(x) = \frac{P x^2}{6EI}(3a - x) \qquad (4)$$

$$\delta_B = \frac{Pa^2}{6EI}(2a) = \frac{2P(0.8^3)}{6EI} = \frac{0.17067P}{EI} \qquad (5)$$

$$\delta_{BC}(x) = \frac{Pa^2}{6EI}(3x - a) \qquad (6)$$

$$\delta_C = (\text{at } x = l) = \frac{Pa^2}{6EI}(3l - a)$$

$$= \frac{P(0.8^2)(3 - 0.8)}{6EI} = \frac{0.2347\,P}{EI} \qquad (7)$$

step 3: Find the value of P needed to cause

δ_B (in Eq. (5)) = $-\delta_B$ (in Eq. (2))

i.e., $\frac{0.17067\,P}{EI} = -\frac{0.2347\,F}{EI}$

i.e., $P_{new} = -1.3749\,F$ (8)

step 4: Find the value of upward deflection caused by P_{new} at C (by using Eq. (8) in Eq. (7)):

$$\delta_C = \frac{0.2347}{EI}(-1.3749\,F) = -\frac{0.3227\,F}{EI} \qquad (9)$$

$\hookrightarrow = -\delta_2$

step 5: Superpose the deflections of step 2 with P_{new} (in place of P) to obtain zero deflection at B and $\left\{0.3333\frac{F}{EI} - 0.3227\frac{F}{EI}\right.$ (Eq. (3) + Eq. (9)) $= 0.0106\frac{F}{EI} \equiv \delta_1 - \delta_2 \Big\}$.

step 6: Thus we find net deflection of point C (δ_{cn}) as $\delta_{cn} = \delta_1 - \delta_2 = 0.0106\frac{F}{EI}$.

The stiffness of the beam (given system) due to force F applied at C is

$$k_c = \frac{F}{\delta_{cn}} = \frac{EI}{0.0106} = 94.3396\ EI$$

Here $E = 207 \times 10^9$ Pa and $I = \frac{1}{12}(0.05)(0.05)^3$

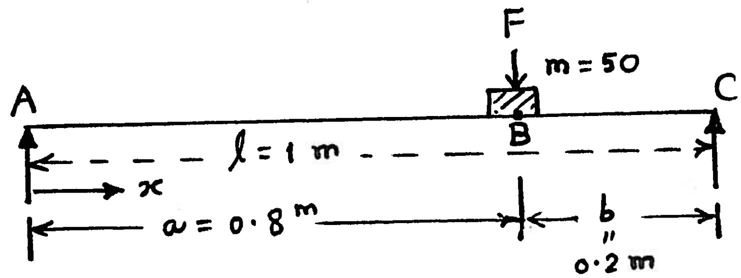
$= 52.1 \times 10^{-8}\ m^4$; $EI = 107,847$

Natural frequency of the system:

$$\omega_n = \sqrt{\frac{k_c}{m}} = \sqrt{\frac{94.3396\ (107,847)}{50}}$$

$$= 451.0930 \text{ rad/s}$$

2.51



Deflection δ due to F:

$$\delta_{AB}(x) = -\frac{F\,b\,x}{6EI\,l}\left(x^2 + b^2 - l^2\right)$$

At point B: $$\delta_B = -\frac{F(0.2)(0.8)\ (0.64 + 0.04 - 1.0)}{6EI(1.0)}$$

$$= \frac{0.008533\ F}{EI}$$

stiffness of beam at B $= \dfrac{F}{\delta_B} = \dfrac{EI}{0.008533\,F}$

i.e., $k_B = 117.1875\ EI$

Here $I = \frac{1}{12}(0.05)(0.05)^3 = 52.1 \times 10^{-8}\ m^4$

and $E = 207 \times 10^9$ Pa

$EI = (207 \times 10^9)(52.1 \times 10^{-8}) = 107{,}847.0\ N\text{-}m^2$

$\therefore\ k_B = 117.1875\ (107{,}847.0) = 12{,}638{,}820.3\ \frac{N}{m}$

Natural frequency of the system:

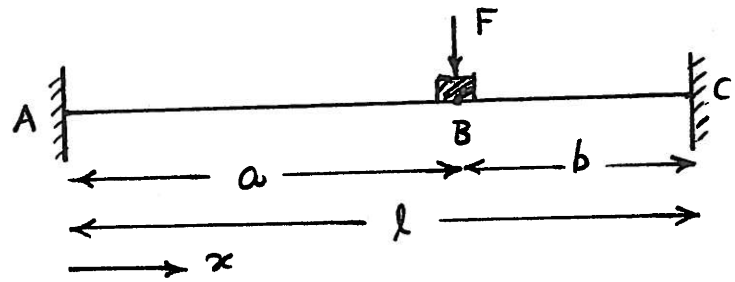
$$\omega_n = \sqrt{\frac{k_B}{m}} = \sqrt{\frac{12{,}638{,}820.3}{50}}$$

$$= 502.7588\ rad/s$$

2.52

$a = 0.8\ m$

$b = 0.2\ m$



$$y_{AB} = \frac{F b^2 x^2}{6EI\, l^3} \{ 3 a l - x(3a + b) \}$$

$$y_B = \frac{F(0.2^2)(0.8^2)}{6EI\,(1)^3} \{ 3(0.8)(1.0) - 0.8\,(3 \times 0.8 + 0.2) \}$$

$$= \frac{F(0.0256)(0.32)}{6EI} = \frac{0.001365\, F}{EI}$$

$$k_B = \frac{F}{y_B} = \frac{EI}{0.001365} = 732.4219\, EI$$

$m = 50\ kg$

$$EI = (207 \times 10^9)\left(\frac{1}{12}(0.05)(0.05)^3\right)$$

$$= 107{,}847.0\ N\text{-}m^2$$

$$k_B = 732.4219\,(107{,}847.0) = 78{,}989{,}504.65\ \frac{N}{m}$$

$$\omega_n = \sqrt{\frac{k_B}{m}} = \sqrt{\frac{78{,}989{,}504.65}{50}}$$

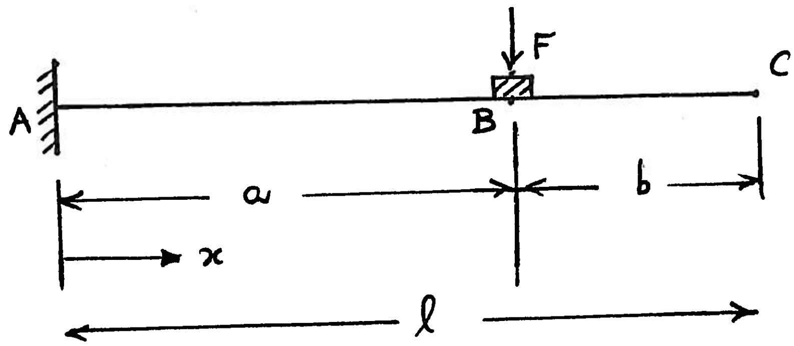
$$= 1256.8970\ rad/s$$

2.53

$a = 0.8\ m$

$b = 0.2\ m$

$l = 1.0\ m$



$$y_{AB} = \frac{F x^2}{6EI}(3a - x)$$

$$y_B = y_{AB}\Big|_{x=0.8} = \frac{F(0.8^2)}{6EI}(3 \times 0.8 - 0.8)$$

$$= \frac{0.17067\ F}{EI}$$

$$k_B = \frac{F}{y_B} = \frac{EI}{0.17067} = 5.85937\ EI$$

$m = 50\ kg$

$$EI = (207 \times 10^9)\left\{\frac{1}{12}(0.05)(0.05)^3\right\}$$

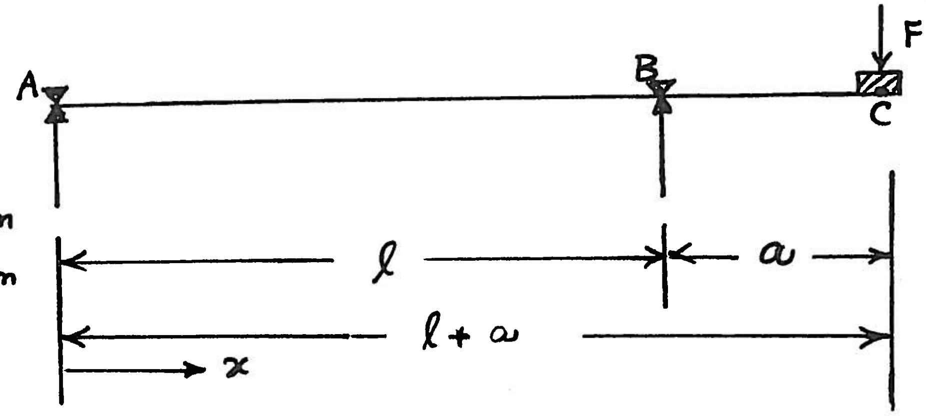
$$= 107{,}847.0$$

$$k_B = 5.85937\ (107{,}847.0) = 631{,}915.4764\ \frac{N}{m}$$

$$\omega_n = \sqrt{\frac{k_B}{m}} = \sqrt{\frac{631{,}915.4764}{50}}$$

$$= 112.4202\ rad/s$$

2.54



$l = 0.8\ m$
$a = 0.2\ m$

Beam on simple supports with overhang:

$$y_C = \frac{F a^2}{3EI}(l + a)$$

$$= \frac{F (0.2^2)(0.8 + 0.2)}{3EI} = \frac{0.04 F}{3EI}$$

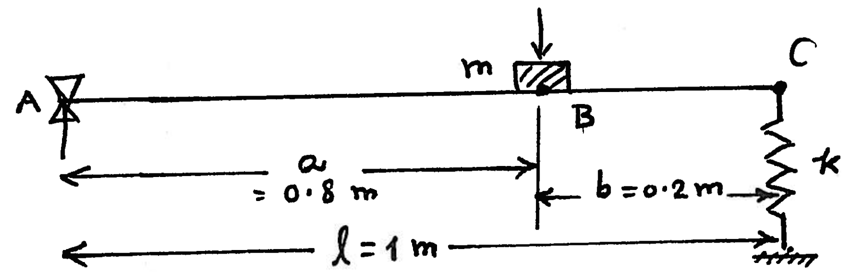
$$k_C = \frac{F}{y_C} = \frac{3EI}{0.04} = 75\ EI$$

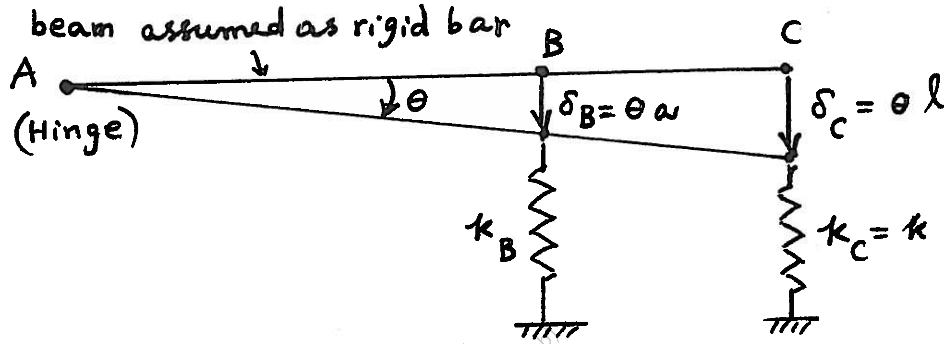
$$= 75\ (207 \times 10^9)\left(\frac{1}{12}(0.05)(0.05)^3\right)$$

$$= 75\ (107{,}847.0) = 808{,}852.5\ \frac{N}{m}$$

$$\omega_n = \sqrt{\frac{k_C}{m}} = \sqrt{\frac{808{,}8525.0}{50}} = 402.2070\ \text{rad/s}$$

2.55 Equivalent stiffness of spring k at location of mass m:





Assume the beam as a rigid bar ABC hinged at point A to find the equivalent stiffness of spring k at point B (k_B). Let the equivalent spring constant of k when located at B be k_B.

Then we equate the moments created at point A by the spring force due to k at C and the spring force due to k_B at B:

$$k_c\, \delta_c\, l = k_B\, \delta_B\, a$$

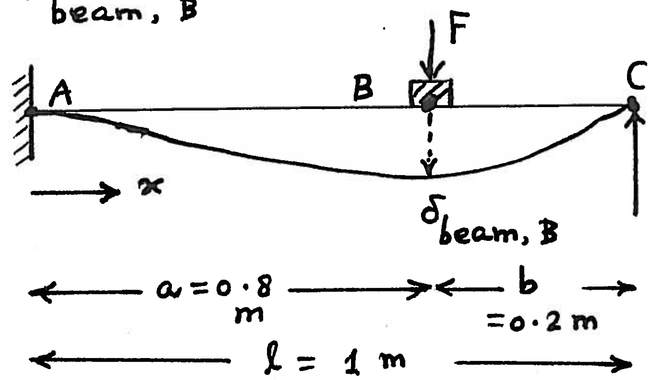
i.e., $$k_B = k_c \frac{\delta_c}{\delta_B} \frac{l}{a} = k \frac{\theta l}{a l} \cdot \frac{l}{a} = \frac{k l^2}{a^2}$$

$$= 10000 \left(\frac{1^2}{0.8^2} \right) = 15625 \text{ N/m}$$

Spring constant of the beam at location of mass m:

For simplicity, we assume that the spring at C acts as a simple support. This permits the computation of the equivalent spring constant of the beam ABC subjected to a force F at B.

$$k_{beam,B} = \frac{F}{\delta_{beam,B}}$$



$$\delta_{AB}(x) = \frac{F b^2 x^2}{6 E I l^3}\{3al - x(3a+b)\}$$

$$\delta_{beam,B} = \frac{F(0.2^2)(0.8^2)}{6EI(1^3)}\{3(0.8)(1.0) - 0.8(3 \times 0.8 + 0.2)\}$$

$$= 0.001365\ F/EI$$

$$\therefore\ k_{beam,B} = \frac{F}{\delta_{beam,B}} = \frac{EI}{0.001365} = 732.4219\,EI$$

$$EI = (207 \times 10^9)\frac{1}{12}(0.05)(0.05)^3 = (207 \times 10^9)(52.1 \times 10^{-8})$$

$$= 107{,}847.0$$

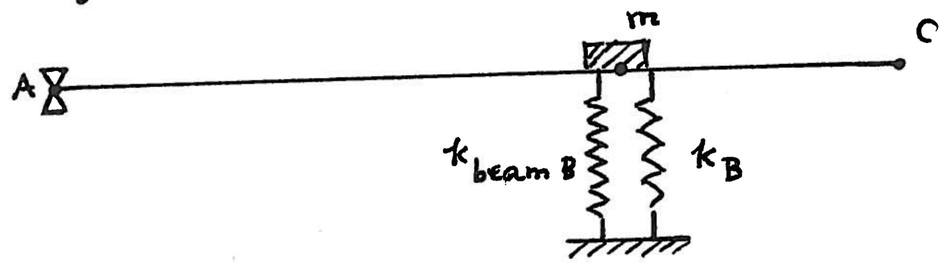
$$\therefore\ k_{beam,B} = 732.4219\,(107{,}847.0)$$

$$= 78.9895 \times 10^6\ N/m$$

Next, we consider the two springs k_B and $k_{beam,B}$ to be parallel so that the equivalent spring constant at B, $k_{eq\,B}$, is given by

$$k_{eq\,B} = k_B + k_{beam\,B} = 0.01562 \times 10^6 + 78.9895 \times 10^6$$

$\therefore \; k_{eq\,B} = 79.00512 \times 10^6 \;\; N/m$

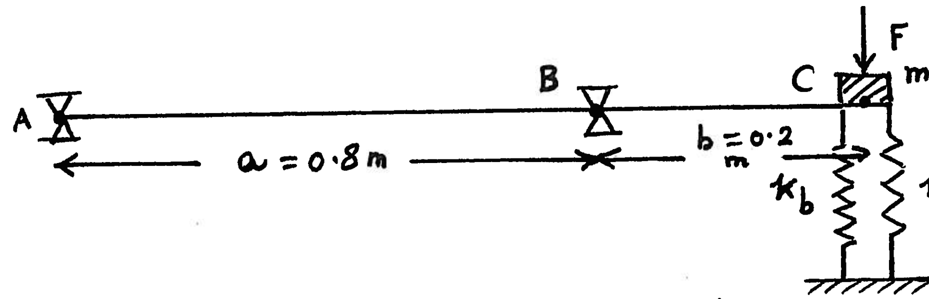


Natural frequency of vibration of the system:

$$\omega_n = \sqrt{\frac{k_{eq\,B}}{m}} = \sqrt{\frac{79.00512 \times 10^6}{50}}$$

$$= 1580.1024 \;\; rad/s$$

2.56



Simply supported beam with a overhang:

$$\delta_C = \frac{F\, b^2 (a+b)}{3EI} \quad ; \quad k_b = k_{\text{beam at } C} = \frac{F}{\delta_C}$$

$$= \frac{3EI}{b^2 (a+b)}$$

$$E = 207 \times 10^9 \text{ Pa}$$

$$I = \frac{1}{12} (0.05)(0.05)^3 = 52.1 \times 10^{-8} \text{ m}^4$$

$$k_b = \frac{3 (207 \times 10^9)(52.1 \times 10^{-8})}{(0.04)(1.0)}$$

$$= 8{,}088{,}525.0 \text{ N/m}$$

$$k_{eq\,C} = \text{equivalent stiffness constant at } C$$

$$= k_b + k = 8{,}088{,}525.0 + 10{,}000.0$$

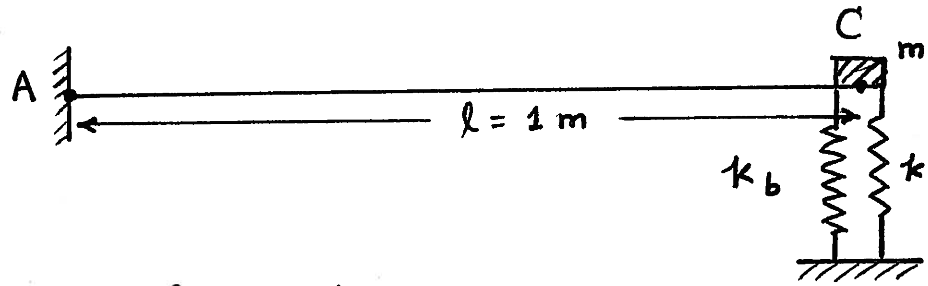
$$= 8{,}098{,}525.0 \text{ N/m}$$

Natural frequency of vibration of the system:

$$\omega_n = \sqrt{\frac{k_{eq\,C}}{m}} = \sqrt{\frac{8{,}098{,}525.0}{50}}$$

$$= 402.4556 \text{ rad/s}$$

(2.57)



For a cantilever beam,

$$k_b = k_{beam\ at\ C} = \frac{3EI}{l^3}$$

$$= \frac{3\ (207 \times 10^9)\ \frac{1}{12}\ (0.05)(0.05)^3}{1^3}$$

$$= 323{,}541.0 \text{ N/m}$$

$k_{eq\,C}$ = equivalent spring constant at C

$$= k_b + k = 323{,}541.0 + 10{,}000.0$$

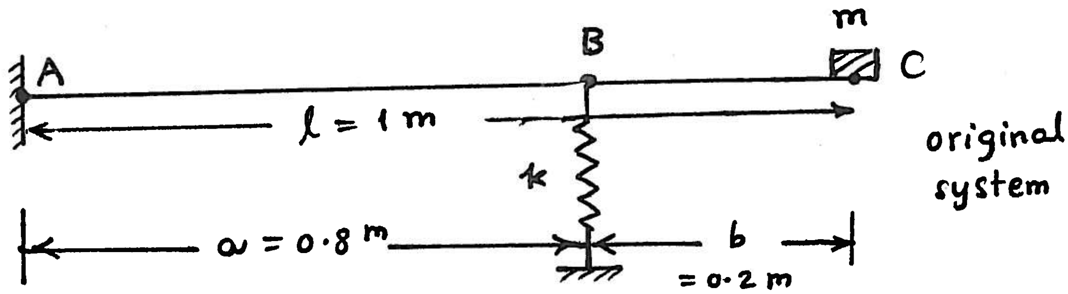
$$= 333{,}541.0 \text{ N/m}$$

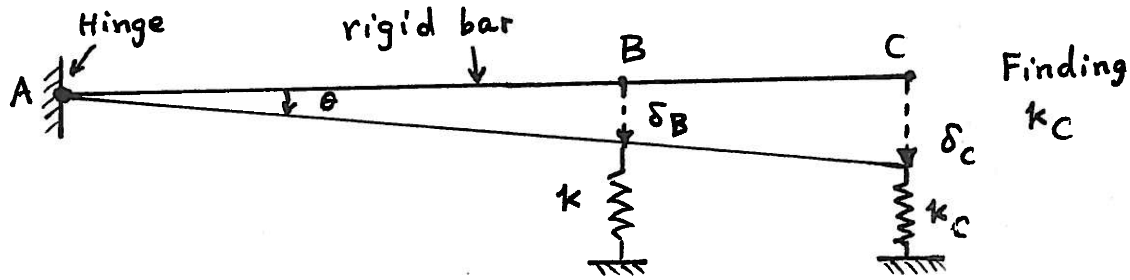
Natural frequency of vibration of the system:

$$\omega_n = \sqrt{\frac{k_{eq\,C}}{m}} = \sqrt{\frac{333{,}541.0}{50}}$$

$$= 81.6751 \text{ rad/s}$$

2.58





Assume the beam as a rigid bar ABC hinged at A to find the equivalent stiffness of spring k at point C (k_C). We equate the moments created at point A by the spring force due to k at B and the spring force due to k_C at C:

$$k_C \, \delta_c \, l = k \, \delta_B \, a$$

i.e., $$k_C = k \frac{\delta_B}{\delta_c} \cdot \frac{a}{l} = k \frac{\theta a}{\theta l} \frac{a}{l} = \frac{k a^2}{l^2}$$

$$= 10\,000 \frac{(0.64)}{(1^2)} = 6400 \text{ N/m}$$

$k_b = k_{beam}$ = stiffness constant of the beam at location of mass m

$$= \frac{3EI}{l^3} = \frac{3(207 \times 10^9)\left\{\frac{1}{12}(0.05)(0.05)^3\right\}}{(1)^3}$$

i.e.,

$$k_b = 323{,}541.0 \text{ N/m}$$

Equivalent spring constant at location of mass (m):

$$k_{eq} = k_b + k_C$$

$$= 323{,}541.0 + 6{,}400.0 = 329{,}941.0 \text{ N/m}$$

Natural frequency of vibration of the system:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{329{,}941.0}{50}}$$

$$= 81.2331 \text{ rad/s}$$

2.59

$$x(t) = A\cos(\omega_n t - \phi) \quad (1)$$

$k = 2000$ N/m, $m = 5$ kg

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{5}} = 20 \text{ rad/s}$$

$$A = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}}, \quad \phi = \tan^{-1}\left(\frac{\dot{x}_0}{x_0\omega_n}\right)$$

(a) $x_0 = 20$ mm, $\dot{x}_0 = 200$ mm/s

$$A = \left\{ (20)^2 + \left(\frac{200}{20}\right)^2 \right\}^{\frac{1}{2}} = 22.3607 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{200}{20(20)}\right) = \tan^{-1}(0.5)$$

$$= 26.5650^\circ \text{ or } 0.4636 \text{ rad}$$

Since both x_0 and $\dot{x}_0$ are positive, ϕ will lie in the first quadrant. Thus the response of the system is given by Eq. (1):

$$x(t) = 22.3607 \cos(20t - 0.4636) \text{ mm}$$

(b) $x_0 = -20$ mm, $\dot{x}_0 = 200$ mm/s

$$A = \left\{ (-20)^2 + \left(\frac{200}{20}\right)^2 \right\}^{\frac{1}{2}} = 22.3607 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{200}{(-20)(20)}\right) = \tan^{-1}(-0.5)$$

$$= -26.5650^\circ \text{ (or } -0.4636 \text{ rad) or}$$
$$153.4349^\circ \text{ (or } 2.6780 \text{ rad)}$$

Since x_0 is negative, ϕ lies in the second quadrant. Thus the response of the system is:

$$x(t) = 22.3607 \cos(20t - 2.6780) \text{ mm}$$

(c) $x_0 = 20$ mm, $\dot{x}_0 = -200$ mm/s

$$A = \left\{ (20)^2 + \left(-\frac{200}{20}\right)^2 \right\}^{\frac{1}{2}} = 22.3607 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{-200}{20(20)}\right) = \tan^{-1}(-0.5)$$

$$= -26.5650^\circ \text{ (or } -0.4636 \text{ rad) or}$$

$$333.4350^\circ \text{ (or } 5.8196 \text{ rad)}$$

Since $\dot{x}_0$ is negative, ϕ lies in the fourth quadrant. Thus the response of the system is given by

$$x(t) = 22.3607 \cos(20t + 0.4636) \text{ mm}$$

or $\qquad 22.3607 \cos(20t - 5.8196)$ mm

(d) $x_0 = -20$ mm, $\dot{x}_0 = -200$ mm/s

$$A = \left\{ (-20)^2 + \left(-\frac{200}{20}\right)^2 \right\}^{\frac{1}{2}} = 22.3607 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{-200}{(-20)(20)}\right) = \tan^{-1}(0.5)$$

$$= 26.5650^\circ \text{ (or } 0.4636 \text{ rad)}$$

$$\text{or } 206.5650^\circ \text{ (or } 3.5952 \text{ rad)}$$

Since both x_0 and $\dot{x}_0$ are negative, ϕ will be in the third quadrant. Hence the response of the system will be

$$x(t) = 22.3607 \cos(20t - 3.5952) \text{ mm}$$

2.60

$$x(t) = A \cos(\omega_n t - \phi) \qquad (1)$$

with

$$A = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}}, \quad \phi = \tan^{-1}\left(\frac{\dot{x}_0}{x_0 \omega_n}\right)$$

$m = 10$ kg, $k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \text{ rad/s}$$

(a) $x_0 = 10$ mm, $\dot{x}_0 = 100$ mm/s

$$A = \left\{ (10)^2 + \left(\frac{100}{10}\right)^2 \right\}^{\frac{1}{2}} = (100 + 100)^{\frac{1}{2}} = 14.1421 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{100}{10\,(10)}\right) = \tan^{-1}(1) = 45^\circ \text{ or } 0.7854 \text{ rad}$$

since both x_0 and $\dot{x}_0$ are positive, ϕ will be in the first quadrant. Hence the response of the system is given by Eq. (1):

$$x(t) = 14.1421 \cos(10t - 0.7854) \text{ mm}$$

(b) $x_0 = -10$ mm, $\dot{x}_0 = 100$ mm/s

$$A = \left\{ (-10)^2 + \left(\frac{100}{(10)}\right)^2 \right\}^{\frac{1}{2}} = 14.1421 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{100}{(-10)(10)}\right) = \tan^{-1}(-1) = -45^\circ \text{ or } 135^\circ$$

$$\text{or } (-0.7854 \text{ rad or } 2.3562 \text{ rad})$$

since x_0 is negative, ϕ lies in the second quadrant. Thus the response of the system is given by

$$x(t) = 14.1421 \cos(10t - 2.3562) \text{ mm}$$

(c) $x_0 = 10$ mm, $\dot{x}_0 = -100$ mm/s

$$A = \left\{ (10)^2 + \left(\frac{-100}{10} \right)^2 \right\}^{\frac{1}{2}} = 14.1421 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{-100}{10\,(10)} \right) = \tan^{-1}(-1)$$

$$= -45^\circ \text{ or } 315^\circ \ (\text{or } -0.7854 \text{ rad or } 5.4978 \text{ rad})$$

Since x_0 is positive and $\dot{x}_0$ is negative, ϕ lies in the fourth quadrant. Hence the response of the system is given by

$$x(t) = 14.1421 \cos(10t - 5.4978) \text{ mm}$$

(d) $x_0 = -10$ mm, $\dot{x}_0 = -100$ mm/s

$$A = \left\{ (-10)^2 + \left(\frac{-100}{(10)} \right)^2 \right\}^{\frac{1}{2}} = 14.1421 \text{ mm}$$

$$\phi = \tan^{-1}\left(\frac{-100}{-10\,(10)} \right) = \tan^{-1}(1) = 45^\circ \text{ or } 225^\circ$$

$$= (0.7854 \text{ rad or } 2.3562 \text{ rad})$$

Since both x_0 and $\dot{x}_0$ are negative, ϕ lies in the third quadrant. Thus the response of the system will be

$$x(t) = 14.1421 \cos(10t - 2.3562) \text{ mm}$$

2.61 Computation of phase angle ϕ_0 in Eq. (2.23):

case (i): x_0 and $\frac{\dot{x}_0}{\omega_n}$ are positive:

$\tan\phi_0$ = positive; hence ϕ_0 lies in first quadrant (as shown in Fig. A)

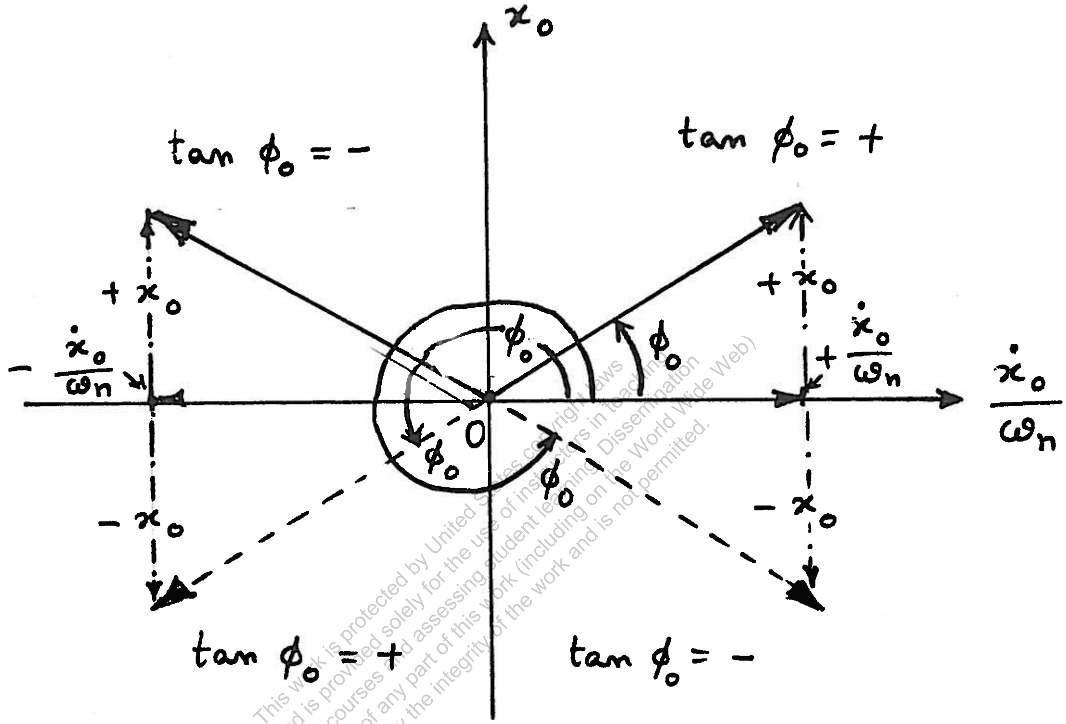


Figure A

case (ii): x_0 = positive, $\dot{x}_0$ (or $\frac{\dot{x}_0}{\omega_n}$) = negative

$\tan\phi_0$ = negative; ϕ_0 lies in second quadrant

case (iii): x_0 = negative, $\dot{x}_0$ (or $\frac{\dot{x}_0}{\omega_n}$) = negative

$\tan\phi_0$ = positive; ϕ_0 lies in third quadrant

case (iv): x_0 = negative, $\dot{x}_0$ (or $\frac{\dot{x}_0}{\omega_n}$) = positive

$\tan\phi_0$ = negative; ϕ_0 lies in fourth quadrant

2.62 $m = 5$ kg, $k = 2000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2000}{5}} = 20 \text{ rad/s}$$

(a) $x_0 = 20$ mm, $\dot{x}_0 = 200$ mm/s

Since x_0 and $\dot{x}_0$ are both positive, ϕ_0 lies in the first quadrant (From solution of Problem 2.61):

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{20\,(20)}{200}\right) = \tan^{-1}(2)$$

$$= 63.4349^\circ \text{ or } 1.1071 \text{ rad}$$

Response given by Eq. (2.23):

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

with $A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = \left\{(20)^2 + \left(\frac{200}{20}\right)^2\right\}^{\frac{1}{2}}$

$$= 22.3607 \text{ mm}$$

$$\therefore\ x(t) = 22.3607 \sin(20t + 1.1071) \text{ mm}$$

(b) $x_0 = -20$ mm, $\dot{x}_0 = 200$ mm/s

Since x_0 is negative and $\dot{x}_0$ is positive, ϕ_0 lies in the fourth quadrant (From Problem 2.61).

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{-20\,(20)}{200}\right)$$

$$= \tan^{-1}(2) = -63.4349^\circ\ (-1.1071 \text{ rad}) \text{ or}$$

$$296.5651^\circ\ (5.1.760 \text{ rad})$$

$$A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = \left\{(-20)^2 + \left(\frac{200}{20}\right)^2\right\}^{\frac{1}{2}}$$

$$= 22.3607 \text{ mm}$$

$$\therefore\ x(t) = 22.3607 \sin(2t + 5.1760) \text{ mm}$$

(c) $x_0 = 20$ mm, $\dot{x}_0 = -200$ mm/s

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{20(20)}{-200}\right) = \tan^{-1}(-2)$$

$$= -63.4349^\circ \text{ (or } -1.1071 \text{ rad) or}$$

$$116.5650^\circ \text{ (or } 2.0344 \text{ rad)}$$

Since x_0 is positive and $\dot{x}_0$ is negative, ϕ_0 lies in the second quadrant (From Problem 2.61).

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}} = \left\{ (20)^2 + \left(-\frac{200}{20}\right)^2 \right\}^{\frac{1}{2}}$$

$$= 22.3607 \text{ mm}$$

$$\therefore x(t) = 22.3607 \sin(20t + 2.0344) \text{ mm}$$

(d) $x_0 = -20$ mm, $\dot{x}_0 = -200$ mm/s

$$\phi_0 = \tan^{-1}\left(\frac{(-20)\,20}{-200}\right) = \tan^{-1}(2) = 63.4349^\circ$$

or 1.1071 rad (or 243.4349° or 4.2487 rad)

$$A_0 = \left\{ (-20)^2 + \left(-\frac{200}{20}\right)^2 \right\}^{\frac{1}{2}}$$

$$= 22.3607 \text{ mm}$$

$$\therefore x(t) = 22.3607 \sin(20t + 4.2487) \text{ mm}$$

(Since x_0 and $\dot{x}_0$ are both negative, ϕ_0 lies in the third quadrant, from solution of Problem 2.61).

2.63

$m = 10$ kg, $k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \text{ rad/s}$$

solution (response) of the system is given by

$$x(t) = A_0 \sin(\omega_n t + \phi_0) \text{ mm}$$

with

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}} \quad \text{and} \quad \phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right)$$

(a) $x_0 = 10$ mm, $\dot{x}_0 = 100$ mm/s

$$A_0 = \left\{(10)^2 + \left(\frac{100}{10}\right)^2\right\}^{\frac{1}{2}} = \sqrt{200} = 14.1421 \text{ mm}$$

$$\phi_0 = \tan^{-1}\left(\frac{10\,(10)}{100}\right) = \tan^{-1}(1) = 45^\circ \text{ or } 0.7854 \text{ rad}$$

Because x_0 and $\dot{x}_0$ are both positive, ϕ_0 lies in the first quadrant (from Problem 2.61).

$$\therefore\ x(t) = 14.1421 \sin(10t + 0.7854) \text{ mm}$$

(b) $x_0 = -10$ mm, $\dot{x}_0 = 100$ mm/s

$$A_0 = \left\{(-10)^2 + \left(\frac{100}{10}\right)^2\right\}^{\frac{1}{2}} = 14.1421 \text{ mm}$$

$$\phi_0 = \tan^{-1}\left(\frac{-10\,(10)}{100}\right) = \tan^{-1}(-1) = -45^\circ \text{ or}$$

-0.7854 rad (or 315° or 5.4978 rad)

Since x_0 is negative and $\dot{x}_0$ is positive, ϕ_0 lies in the fourth quadrant (from Problem 2.61).

$$\therefore\ x(t) = 14.1421 \sin(10t + 5.4978) \text{ mm}$$

(c) $x_0 = 10$ mm, $\dot{x}_0 = -100$ mm/s

$$A_0 = \left\{ (10)^2 + \left(-\frac{100}{10} \right)^2 \right\}^{\frac{1}{2}} = 14.1421 \text{ mm}$$

$$\phi_0 = \tan^{-1}\left(\frac{10\,(10)}{-100} \right) = \tan^{-1}(-1) = 135^\circ \text{ or } 2.3562 \text{ rad}$$

since x_0 is positive and $\dot{x}_0$ is negative, ϕ_0 lies in the second quadrant (from Problem 2.61).

$$\therefore x(t) = 14.1421 \sin(10t + 2.3562) \text{ mm}$$

(d) $x_0 = -10$ mm, $\dot{x}_0 = -100$ mm/s

$$A_0 = \left\{ (-10)^2 + \left(-\frac{100}{10} \right)^2 \right\}^{\frac{1}{2}} = 14.1421 \text{ mm}$$

$$\phi_0 = \tan^{-1}\left(\frac{-10\,(10)}{-100} \right) = \tan^{-1}(1) = 225^\circ \text{ or } 3.9270 \text{ rad}$$

since both x_0 and $\dot{x}_0$ are negative, ϕ_0 lies in the third quadrant (from Problem 2.61).

$$\therefore x(t) = 14.1421 \sin(10t + 3.9270) \text{ mm}$$

2.64

From Example 2.1, $m = 1$ kg, $k = 2500$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{1}} = 50 \text{ rad/s}$$

$x_0 = -2$ mm, $\dot{x}_0 = 100$ mm/s

Eq. (2.23) is: $x(t) = A_0 \sin(\omega_n t + \phi_0)$

with $A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}}$

and $\phi_0 = \tan^{-1} \left(\frac{x_0 \omega_n}{\dot{x}_0} \right)$

For the given data,

$$A_0 = \left\{ (-2)^2 + \left(\frac{100}{50} \right)^2 \right\}^{\frac{1}{2}} = 2.8284 \text{ mm}$$

$$\phi_0 = \tan^{-1} \left(\frac{(-2)(50)}{100} \right) = \tan^{-1}(-1)$$

$= -45°$ or -0.7854 rad

or

$315°$ or 5.4978 rad

Since x_0 is negative and $\dot{x}_0$ is positive, ϕ_0 lies in the fourth quadrant (from Problem 2.61).

$\therefore$ Response is given by

$$x(t) = 2.8284 \sin(50t + 5.4978) \text{ mm}$$

2.65 (a) The area moment of inertia of the solid circular cross-section of the tree (I) is given by

$$I = \frac{1}{64}\pi d^4 = \frac{1}{64}\pi (0.25)^4 = 0.000191748 \ m^4$$

The axial load acting on the top of the trunk is:

$$F = m_c g = 100\ (9.81) = 981 \ N$$

Assuming the trunk as a fixed-free column under axial load, the buckling load can be determined as

$$P_{cri} = \frac{1}{4}\frac{\pi^2 E I}{l^2} = \frac{\pi^2}{4}\frac{(1.2 \times 10^9)(191.748 \times 10^{-6})}{(10)^2}$$

$$= 5677.4573 \ N$$

since the axial force due to the mass of the crown (F) is smaller than the critical load, the tree trunk will not buckle.

(b) The spring constant of the trunk in sway (transverse) motion is given by (assuming the trunk as a fixed-free beam)

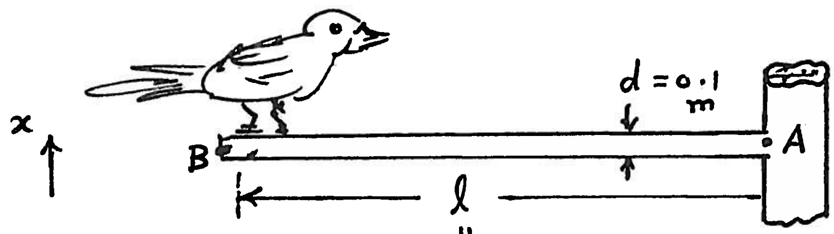
$$k = \frac{3EI}{l^3} = \frac{3\ (1.2 \times 10^9)(191.748 \times 10^{-6})}{(10)^3}$$

$$= 690.2928 \ N/m$$

Natural frequency of vibration of the tree is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{690.2928}{100}} = 2.6273 \ rad/s$$

2.66



(a) Mass of bird = m_b = 2 kg

Mass of beam (branch) = $m_{br} = \frac{\pi d^2}{4} l \rho$

$$m_{br} = \frac{\pi (0.1)^2}{4} (4)(700) = 21.9912 \text{ kg}$$

M = total mass at B = mass of bird + equivalent mass of beam (AB) at B

$$= 2 + 0.23 (21.9912) = 7.0580 \text{ kg}$$

(equivalent mass of a cantilever beam at its free end = 0.23 times its total mass)

k = stiffness of cantilever beam (branch) at end B

$$= \frac{3EI}{l^3} = \frac{3(10 \times 10^9) \frac{\pi}{64} (0.1)^4}{(4)^3}$$

$$= 2301.0937 \text{ N/m}$$

Thus the equation of motion of the bird, in free vibration, is given by

$$M \ddot{x} + kx = 0 \quad \text{(by assuming no damping)}$$

i.e.

$$7.0580 \ddot{x} + 2301.0937 x = 0$$

(b) Natural frequency of vibration of the bird:

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{2301.0937}{7.0580}} = 18.0562 \text{ rad/s}$$

2.67 Given: mass of bird (m) = 2 kg

height of branch (length of cantilever beam) $= h = 2$ m

density of branch $= \rho = 700\ kg/m^3$

Young's modulus of branch $= E = 10$ GPa

(a) Buckling load of a cantilever beam with axial force applied at free end is given by

$$P_{cri} = \frac{1}{4}\frac{\pi^2 E I}{h^2} \qquad (1)$$

Assuming the diameter of branch as d, the area moment of inertia (I) is given by

$$I = \frac{\pi d^4}{64} \qquad (2)$$

When critical load (P_{cri}) is set equal to the weight of bird,

$$P_{cri} = mg = 2(9.81) = 19.62\ N \qquad (3)$$

Equating Eq. (3) to Eq. (1), we obtain

$$19.62 = \frac{1}{4}\frac{\pi^2 (10 * 10^9)}{2^2}\left(\frac{\pi d^4}{64}\right)$$

$$= 0.3028\ d^4 * 10^9\ N$$

i.e.,
$$d^4 = \frac{19.62}{0.3028 \times 10^9} = 6.4735 * 10^{-8}$$

i.e.,
$$d = 1.5954 * 10^{-2} = 0.015954\ m$$

$\therefore$ Minimum diameter of the branch to avoid buckling under the weight of the bird (neglecting the weight of the branch) is $d = 1.595$ cm.

(b) Natural frequency of vibration of the system in bending ($\omega_{n,b}$) :

$$\omega_{n,b} = \sqrt{\frac{k}{m}}$$

where $m = 2$ kg (neglecting mass of branch), and

k = bending stiffness of cantilever beam of length, h

$$= \frac{3EI}{h^3} = \frac{3\,(10 * 10^9)\left\{\frac{\pi}{64}\,(0.01595)^4\right\}}{2^3}$$

$$= 11.9137 \text{ N/m}$$

Thus $\omega_{n,b} = \sqrt{\frac{11.9137}{2}} = 2.4407$ rad/s

Natural frequency of vibration of the system in axial motion ($\omega_{n,a}$):

$$\omega_{n,a} = \sqrt{\frac{k_a}{m}}$$

where $m = 2$ kg and

$$k_a = \frac{AE}{l} = \frac{\pi}{4}\,\frac{(0.01595)^2\,(10 * 10^9)}{(2)}$$

$$= 0.9990 * 10^6 \text{ N/m}$$

Thus $\omega_{n,a} = \sqrt{\frac{0.9990 \times 10^6}{2}}$

$$= 706.7531 \text{ rad/s}$$

2.68

$m = 2$ kg, $k = 500$ N/m, $x_0 = 0.1$ m, $\dot{x}_0 = 5$ m/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{500}{2}} = 15.8114 \text{ rad/s}$$

Displacement of mass (given by Eq. (2.21)):

$$x(t) = A \cos(\omega_n t - \phi)$$

where

$$A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right]^{\frac{1}{2}} = \left[0.1^2 + \left(\frac{5}{15.8114}\right)^2 \right]^{\frac{1}{2}} = \sqrt{0.11}$$

$$= 0.3317 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{\dot{x}_0}{\omega_n x_0}\right) = \tan^{-1}\left(\frac{5}{15.8114 * 0.1}\right)$$

$$= \tan^{-1}(3.1623) = 72.4516^\circ \text{ or } 1.2645 \text{ rad}$$

(ϕ will be in the first quadrant because both x_0 and $\dot{x}_0$ are positive)

$$x(t) = 0.3317 \cos(15.8114\, t - 1.2645) \text{ m}$$

$$\dot{x}(t) = -5.2446 \sin(15.8114\, t - 1.2645) \text{ m/s}$$

$$\ddot{x}(t) = -82.9251 \cos(15.8114\, t - 1.2645) \text{ m/s}^2$$

2.69

Data: $\omega_n = 10$ rad/s, $x_0 = 0.05$ m, $\dot{x}_0 = 1$ m/s

Response of undamped system:

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

$$= 0.05 \cos 10t + \left(\frac{1}{10}\right) \sin 10t$$

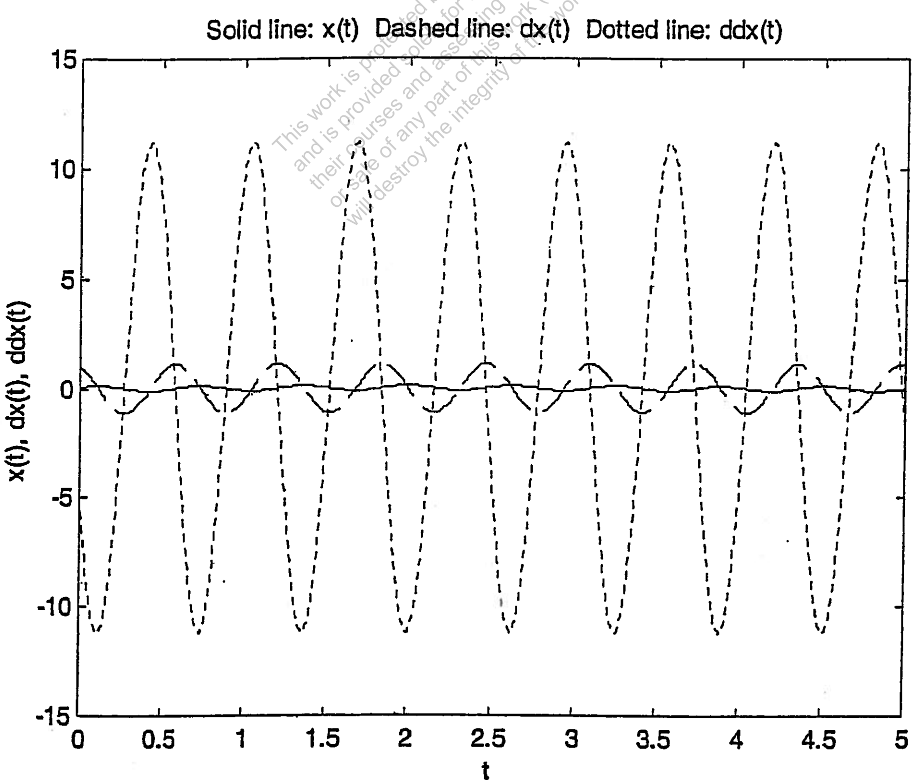
$$\therefore\ x(t) = 0.05 \cos 10t + 0.1 \sin 10t \quad \text{m} \qquad (E.1)$$

$$\dot{x}(t) = -0.5 \sin 10t + \cos 10t \quad \text{m/s} \qquad (E.2)$$

$$\ddot{x}(t) = -5 \cos 10t - 10 \sin 10t \quad \text{m/s}^2 \qquad (E.3)$$

Plotting of Eqs. (E.1) to (E.3):

```
% Ex2_52.m
for i = 1: 1001
   t(i) = (i-1)*5/1000;
   x(i) = 0.05 * cos(10*t(i)) + 0.1*sin(10*t(i));
   dx(i) = -0.5*sin(10*t(i)) + cos(10*t(i));
   ddx(i) = -5*cos(10*t(i)) - 10*sin(10*t(i));
end
plot(t, x);
hold on;
plot(t, dx, '--');
plot(t, ddx, ':');
xlabel('t');
ylabel('x(t), dx(t), ddx(t)');
title('Solid line: x(t)  Dashed line: dx(t)  Dotted line: ddx(t)');
```

2.70 Data: $\omega_d = 2$ rad/s, $\zeta = 0.1$, $X_0 = 0.01$ m, $\phi_0 = 1$ rad

Initial conditions ?

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n, \quad \omega_n = \omega_d / \sqrt{1-\zeta^2} = 2/\sqrt{1-0.01}$$

$$= 2.0101 \text{ rad/s} \qquad (E.1)$$

Eqs. (2.73), (2.75):

$$X_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{\frac{1}{2}} = 0.01 \qquad (E.2)$$

$$\phi_0 = \tan^{-1}\left(- \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d x_0} \right) = 1 \qquad (E.3)$$

Eqs. (E.2) and (E.3) lead to:

$$x_0^2 + \left(\frac{\dot{x}_0 + 0.20101\, x_0}{2} \right)^2 = 0.0001 \qquad (E.4)$$

$$- \left(\frac{\dot{x}_0 + 0.20101\, x_0}{2 x_0} \right) = \tan 1 = 0.7854$$

or

$$- (\dot{x}_0 + 0.20101\, x_0) = 1.5708\, x_0 \qquad (E.5)$$

Substitution of Eq. (E.5) into (E.4) yields

$$x_0 = 0.007864 \text{ m} \qquad (E.6)$$

Eqs. (E.6) and (E.5) give

$$\dot{x}_0 = -0.013933 \text{ m/s} \qquad (E.7)$$

2.71 Without passengers,

$$(\omega_n)_1 = \sqrt{\frac{k}{m}} = 20 \text{ rad/s} \Rightarrow k = 400\, m \qquad (E.1)$$

With passengers,

$$(\omega_n)_2 = \sqrt{\frac{k}{m+500}} = 17.32 \text{ rad/s} \qquad (E.2)$$

squaring Eq. (E.2), we get

$$\frac{k}{m+500} = (17.32)^2 = 299.9824 \qquad (E.3)$$

Using $k = 400\, m$ in (E.3) gives

$$m = 1499.6481 \text{ kg}$$

2.72 $\omega_n = \sqrt{k/m} = \sqrt{3200/2} = 40$ rad/s

$x_0 = 0$

$$X_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = 0.1$$

i.e. $\frac{\dot{x}_0}{\omega_n} = 0.1$ or $\dot{x}_0 = 0.1\,\omega_n = 4$ m/s

---

2.73 Data: $D = 0.5625''$, $G = 11.5 \times 10^6$ psi, $\rho = 0.282$ lb/in^3

$f = 193$ Hz, $k = 26.4$ lb/in

$$k = \text{spring rate} = \frac{d^4 G}{8 D^3 N} \Rightarrow \frac{d^4 (11.5 \times 10^6)}{8 (0.5625^3) N} = 26.4$$

or $$\frac{d^4}{N} = \frac{26.4 (8)(0.5625^3)}{11.5 \times 10^6} = 3.2686 \times 10^{-6} \quad (E.1)$$

$$f = \frac{1}{2}\sqrt{\frac{k g}{W}}$$

where $$W = \left(\frac{\pi d^2}{4}\right) \pi D N \rho = \frac{\pi^2}{4}(0.5625)(0.282)\, N d^2$$

$$= 0.391393\, N d^2$$

Hence $$f = \frac{1}{2}\sqrt{\frac{26.4 (386.4)}{0.391393\, N d^2}} = 193$$

or $$N d^2 = 0.174925 \quad (E.2)$$

Eqs. (E.1) and (E.2) yield

$$N = \frac{d^4}{3.2686 \times 10^{-6}} = \frac{0.174925}{d^2}$$

or $$d^6 = 0.571764 \times 10^{-6}$$

or $$d = 0.911037 \times 10^{-1} = 0.0911037 \text{ inch}$$

Hence $$N = \frac{0.174925}{d^2} = 21.075641$$

(2.74) Data: $D = 0.5625''$, $G = 4 \times 10^6$ psi, $\rho = 0.1$ lb/in^3

$f = 193$ Hz, $k = 26.4$ lb/in

k = spring rate $= \dfrac{d^4 G}{8 D^3 N} \Rightarrow \dfrac{d^4 (4 \times 10^6)}{8 (0.5625^3) N} = 26.4$

or $\dfrac{d^4}{N} = \dfrac{26.4 (8)(0.5625^3)}{4 \times 10^6} = 9.397266 \times 10^{-6}$ (E.1)

f = frequency $= \dfrac{1}{2} \sqrt{\dfrac{k g}{W}}$

where $W = \left(\dfrac{\pi d^2}{4}\right) \pi D N \rho = \dfrac{\pi^2}{4} (0.5625)(0.1)\, N d^2$

$= 0.138792\ N d^2$

Hence $f = \dfrac{1}{2} \sqrt{\dfrac{26.4\ (386.4)}{0.138792\ N d^2}} = 193$

or $N d^2 = 0.493290$ (E.2)

Eqs. (E.1) and (E.2) yield

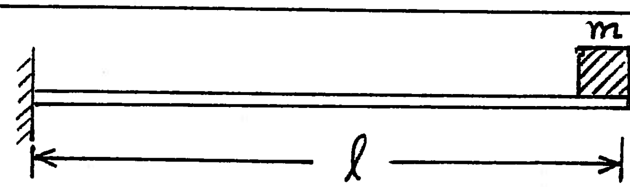
$N = \dfrac{d^4}{9.397266 \times 10^{-6}} = \dfrac{0.493290}{d^2}$

or $d^6 = 4.635575 \times 10^{-6}$

or $d = 0.129127$ inch

Hence $N = \dfrac{0.493290}{d^2} = 29.584728$

(2.75)



By neglecting the effect of self weight of the beam, and using a single degree of freedom model, the natural frequency of the system can be expressed as $\omega_n = \sqrt{\dfrac{k}{m}}$

where m = mass of the machine, and

k = stiffness of the cantilever beam :

$$k = \frac{3EI}{l^3}$$

where l = length, E = Young's modulus, and I = area moment of inertia of the beam section.

Assuming $E = 30 \times 10^6$ psi for steel and 10.5×10^6 psi for aluminum, we have

$$(\omega_n)_{steel} = \left\{ \frac{3\,(30 \times 10^6)\, I}{m\, l^3} \right\}^{\frac{1}{2}}$$

$$(\omega_n)_{aluminum} = \left\{ \frac{3\,(10.5 \times 10^6)\, I}{m\, l^3} \right\}^{\frac{1}{2}}$$

Ratio of natural frequencies:

$$\frac{(\omega_n)_{steel}}{(\omega_n)_{aluminum}} = \left(\frac{30}{10.5} \right)^{\frac{1}{2}} = 1.6903 = \frac{1}{0.59161}$$

Thus the natural frequency is reduced to 59.161% of its value if aluminum is used instead of steel.

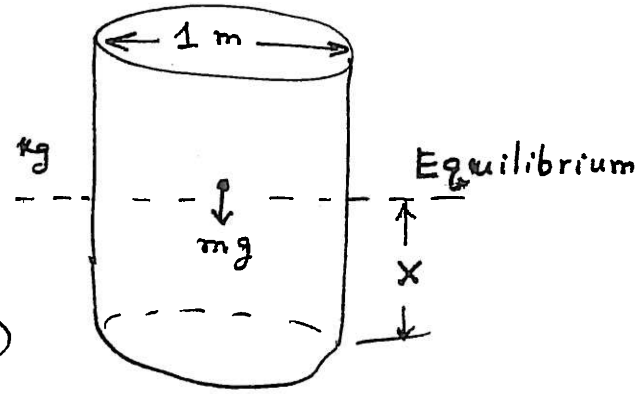
2.76

At equilibrium position,

M = Mass of drum = 500 kg

$= (\pi r^2)(x)(1050)$

(mass of salt water displaced at equilibrium)



$= \pi (0.5)^2 X (1050)$

$$\therefore X = \frac{500}{\pi (0.25)\ (1050)} = 0.6063 \text{ m}$$

Let the drum be displaced by a vertical distance x from the equilibrium position. Then the equation of motion can be expressed as

$$M \ddot{x} + \left(\begin{array}{l}\text{reaction force due to} \\ \text{the weight of salt} \\ \text{water displaced due} \\ \text{to } x\end{array}\right) = 0$$

or

$$M \ddot{x} + (\pi r^2)\, x \cdot (1050 * g) = 0$$

or

$$500 \ddot{x} + \pi (0.5)^2 x\, (1050 \times 9.81) = 0$$

or

$$\ddot{x} + \frac{0.25 \pi (1050 \times 9.81)}{500} x = 0$$

or

$$\ddot{x} + 16.18\, x = 0$$

from which the natural frequency of vibration can be determined as

$$\omega_n = \sqrt{16.18} = 4.0224 \text{ rad/s}$$

2.77

From the equation of motion, we note

$m = 500\ kg$ and spring force $= F = \dfrac{1000}{(0.025)^3}\, x^3\ N$

(a) By equating the weight of the mass and the spring force,

$$500\,(9.81) = \frac{1000}{(0.025)^3}\, x^3 \qquad (1)$$

we find the static equilibrium position of the system as

$$x_{st}^3 = \frac{500\,(9.81)\,(0.025^3)}{1000} = 76.641 \times 10^{-6}$$

or

$$x_{st} = 4.2477 \times 10^{-2} = 0.04248\ m$$

(b) The linearized spring constant, $\bar{k}$, about the static equilibrium position (x_{st}) is given by

$$\bar{k} = \left.\frac{dF}{dx}\right|_{x = x_{st}} = \left.\frac{3000}{(0.025^3)}\, x^2\right|_{x = x_{st}}$$

$$= \frac{3000}{(0.025)^3}\,(4.2477 \times 10^{-2})^2$$

$$= \frac{(3000)(4.2477)^2\, 10^{-4}}{15.625 \times 10^{-6}}$$

$$= \frac{(3 \times 10^3)(18.0429)\, 10^{-4}}{15.625 \times 10^{-6}} = 3.4642 \times 10^5\ N/m$$

(c) Natural frequency of vibration for small displacements:

$$\omega_n = \sqrt{\frac{\bar{k}}{m}} = \left(\frac{3.4642 \times 10^5}{500}\right)^{\frac{1}{2}} = 26.3218 \text{ rad/s}$$

(d) Natural frequency of vibration for small displacements when $m = 600$ kg :

In this case, the static equilibrium position is given by

$$\bar{x}_{st}^3 = \frac{600\,(9.81)\,(0.025^3)}{1000} = 5.886 \times (0.025)^3$$

$$\bar{x}_{st} = 1.8055 \times 0.025 = 0.04514 \text{ m}$$

The linearized spring constant, $\tilde{k}$, about the static equilibrium position $(\bar{x}_{st})$ is given by

$$\tilde{k} = \left.\frac{dF}{dx}\right|_{x=\bar{x}_{st}} = \frac{3000}{(0.025)^3}\left(\bar{x}_{st}\right)^2$$

$$= \frac{3000}{(0.025)^3}\left(4.514 \times 10^{-2}\right)^2$$

$$= \frac{3000\,(20.3748 \times 10^{-4})}{15.625 \times 10^{-6}} = 3.9120 \times 10^5 \text{ N/m}$$

Hence the natural frequency of vibration for small displacements:

$$\bar{\omega}_n = \sqrt{\frac{\tilde{k}}{m}} = \left(\frac{3.9120 \times 10^5}{600}\right)^{\frac{1}{2}} = 25.5342 \text{ rad/s}$$

2.78

$$\text{acceleration} = a = -10 \ m/s^2 = \ddot{x} = \frac{d^2x}{dt^2} \qquad (1)$$

Integration of Eq. (1) w.r.t. time gives

$$\dot{x} = \frac{dx}{dt} = -10\,t + c_1 \qquad (2)$$

At the brakes are applied, $t = 0$ and $\dot{x} = u = 100$ km/hour

$$u = \dot{x}\,(t=0) = \frac{100 \times 10^3}{60 \times 60} \ \frac{m}{s} = 27.7778 \ \frac{m}{s} = c_1$$

$$\therefore \ \frac{dx}{dt}(t) = -10\,t + 27.7778$$

$\frac{dx}{dt} = 0$ when the vehicle stops and hence the time taken before the vehicle stops, t_0, is given by

$$0 = -10\,t_0 + 27.7778$$

$$\text{or} \quad t_0 = 2.7778 \ s$$

The distance traveled before it stops is given by

$$s = u t_0 + \frac{1}{2} a t_0^2$$

$$= 27.7778\,(2.7778) + \frac{1}{2}(-10)\,(2.7778)^2$$

$$= 77.1612 - 38.5808$$

$$= 38.5803 \ m$$

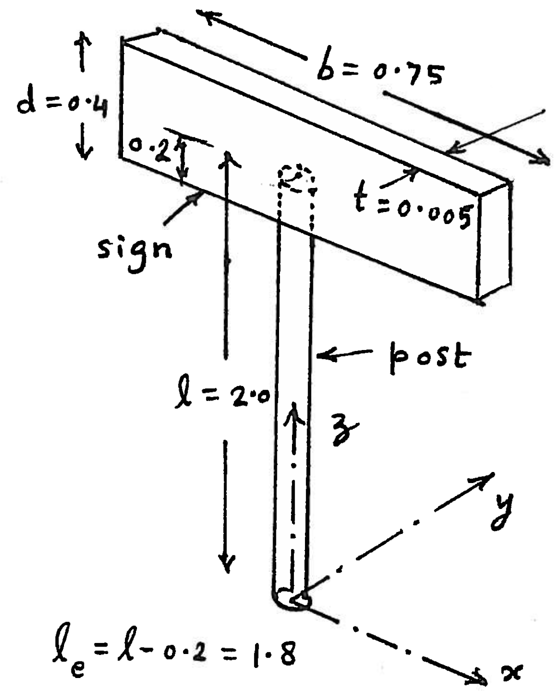
2.79

For hollow circular post,

$$I_{xx} = I_{yy} = \frac{\pi}{4}\left(r_o^4 - r_i^4\right)$$

$$= \frac{\pi}{4}\left(0.05^4 - 0.045^4\right)$$

$$= 1.6878 \times 10^{-6} \ m^4$$



Effective length of post (for bending stiffness) is

$$l_e = 2.0 - 0.2 = 1.8 \ m$$

Bending stiffness of the post in xz-plane:

$$k_{xz} = \frac{3 E I_{yy}}{l_e^3} = \frac{3\left(207 \times 10^9\right)\left(1.6878 \times 10^{-6}\right)}{(1.8)^3}$$

$$= 179.7194 \times 10^3 \ N/m$$

Mass of the post $= m = \pi\left(r_o^2 - r_i^2\right) l \rho$

$$= m = \pi\left(0.05^2 - 0.045^2\right)(2)\left(\frac{76500}{9.81}\right) = 23.2738 \ kg$$

Mass of traffic sign $= M = b d t \rho$

$$= M = 0.75(0.4)(0.005)\left(\frac{76500}{9.81}\right) = 11.6972 \ Kg$$

Equivalent mass of a cantilever beam of mass m with an end mass M (from back of front cover):

$$m_{eq} = M + 0.23 \ m = 11.6972 + 0.23\,(23.2738)$$

$$= 17.0502 \ kg$$

Natural frequency for vibration in xz plane:

$$\omega_n = \left(\frac{k_{xz}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{179.7194 \times 10^3}{17.0502}\right)^{\frac{1}{2}}$$

$$= 102.6674 \text{ rad/s}$$

Bending stiffness of the post in yz-plane:

$$k_{yz} = \frac{3 E I_{xx}}{l_e^3} = \frac{3(207 \times 10^9)(1.6878 \times 10^{-6})}{(1.8)^3}$$

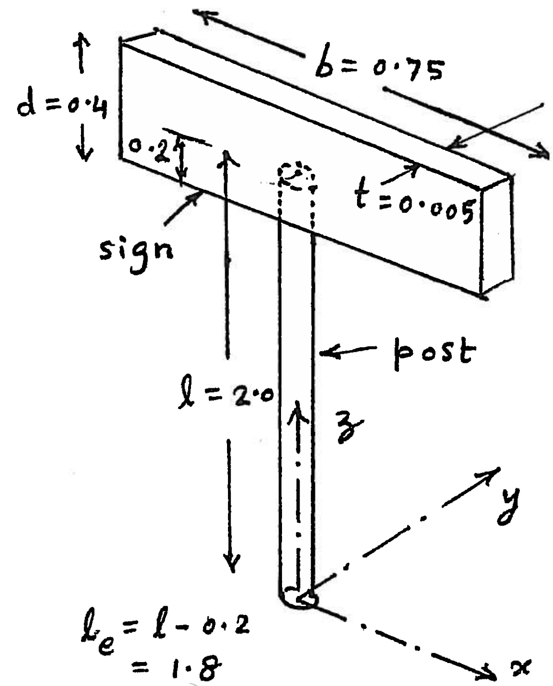
$$= 179.7194 \times 10^3 \text{ N/m}$$

Natural frequency for vibration in yz-plane:

$$\omega_n = \left(\frac{k_{yz}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{179.7194 \times 10^3}{17.0502}\right)^{\frac{1}{2}}$$

$$= 102.6674 \text{ rad/s}$$

2.80



For hollow circular post,

$$I_{xx} = I_{yy} = \frac{\pi}{4}\left(r_0^4 - r_i^4\right)$$

$$= \frac{\pi}{4}\left(0.05^4 - 0.045^4\right)$$

$$= 1.6878 \times 10^{-6} \ m^4$$

Effective length of post (for bending stiffness) is

$$l_e = 2.0 - 0.2 = 1.8 \ m$$

Bending stiffness of the post in xz-plane:

$$k_{xz} = \frac{3 E I_{yy}}{l_e^3} = \frac{3\,(111 \times 10^9)\,(1.6878 \times 10^{-6})}{(1.8)^3}$$

$$= 96.3727 \times 10^3 \ N/m$$

Mass of the post $= m = \pi\left(r_0^2 - r_i^2\right) l \rho$

$$= m = \pi\left(0.05^2 - 0.045^2\right)(2)\left(\frac{80100}{9.81}\right) = 24.3690 \ kg$$

Mass of traffic sign $= M = b d t \rho$

$$= M = 0.75\,(0.4)\,(0.005)\left(\frac{80100}{9.81}\right) = 12.2476 \ kg$$

Equivalent mass of a cantilever beam of mass m with an end mass M (from back of front cover):

$$m_{eq} = M + 0.23\, m = 12.2476 + 0.23\,(24.3690)$$

$$= 17.8525 \ kg$$

Natural frequency for vibration in xz plane:

$$\omega_n = \left(\frac{k_{xz}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{96.3727 \times 10^3}{17.8525}\right)^{\frac{1}{2}}$$

$$= 73.4729 \text{ rad/s}$$

Bending stiffness of the post in yz-plane:

$$k_{yz} = \frac{3EI_{xx}}{l_e^3} = \frac{3(111 \times 10^9)(1.6878 \times 10^{-6})}{(1.8)^3}$$

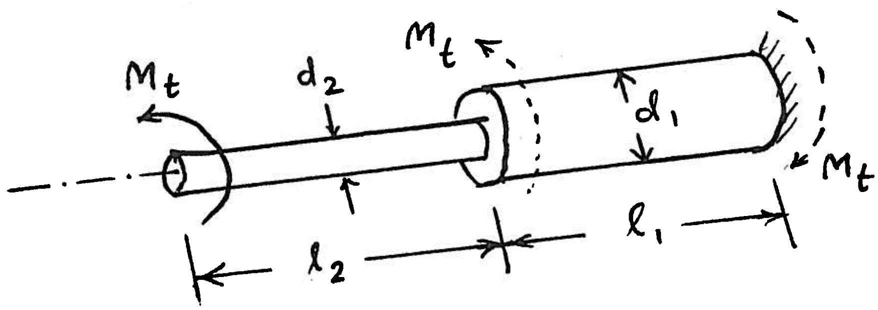
$$= 96.3727 \times 10^3 \text{ N/m}$$

Natural frequency for vibration in yz-plane:

$$\omega_n = \left(\frac{k_{yz}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{96.3727 \times 10^3}{17.8525}\right)^{\frac{1}{2}}$$

$$= 73.4729 \text{ rad/s}$$

2.81



Any applied moment M_t at the disk will be felt by every point along the stepped shaft. As such, the two steps of diameters d_1 and d_2 (with lengths l_1 and l_2) act as series torsional springs. Torsional spring constants of steps 1 and 2 are given by

(1) $k_{t1} = \dfrac{G\, I_{01}}{l_1}$; I_{01} = polar moment of inertia of shaft 1

$$= \frac{\pi d_1^4}{32}$$

(2) $k_{t2} = \dfrac{G\, I_{02}}{l_2}$; I_{02} = polar moment of inertia of shaft 2

$$= \frac{\pi d_2^4}{32}$$

Equivalent torsional spring constant, k_{teq}, is given by

$$\frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}}$$

or $$k_{teq} = \frac{k_{t1}\, k_{t2}}{k_{t1} + k_{t2}}$$

Natural frequency of heavy disk, of mass moment of inertia J, can be found as

$$\omega_n = \sqrt{\frac{k_{t\,eq}}{J}} = \sqrt{\frac{k_{t1}\, k_{t2}}{J(k_{t1} + k_{t2})}}$$

where k_{t1} and k_{t2} are given by Eqs. (1) and (2).

2.82 (a) Equation of motion of simple pendulum for small angular motions is given by

$$\ddot{\theta} + \frac{g_{mars}}{l}\,\theta = 0 \qquad (1)$$

and hence the natural frequency of vibration is

$$\omega_n = \sqrt{\frac{g_{mars}}{l}} = \sqrt{\frac{0.376\,(9.81)}{1}} = 1.9206 \text{ rad/s}$$

(b) Solution of Eq. (1) can be expressed, similar to Eq. (2.23), as

$$\theta(t) = A_0 \sin(\omega_n t + \phi_0) \qquad (2)$$

with $A_0 = \left\{ \theta_0^2 + \left(\frac{\dot{\theta}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}} = \sqrt{(0.08727)^2 + 0^2}$

$= 0.08727$ rad

since $\theta_0 = 5^\circ = 0.08727$ rad and $\dot{\theta}_0 = 0$.

and $\phi_0 = \tan^{-1}\left(\frac{\theta_0\,\omega_n}{\dot{\theta}_0}\right) = \tan^{-1}\left(\frac{0.08727 * 1.9206}{0}\right)$

$= \tan^{-1}(\infty) = 90^\circ$ or 1.5708 rad

$\therefore\ \theta(t) = 0.08727 \sin(1.9206\,t + 1.5708)$ rad

$\dot{\theta}(t) = 0.08727\,(1.9206) \cos(1.9206\,t + 1.5708)$

$= 0.1676 \cos(1.9206\,t + 1.5708)$ rad/s

Maximum velocity $= \dot{\theta}_{max} = 0.1676$ rad/s

(c) $\ddot{\theta}(t) = -0.1676\,(1.9206) \sin(1.9206\,t + 1.5708)$

$= -0.3219 \sin(1.9206\,t + 1.5708)$ rad/s^2

Maximum acceleration $= \ddot{\theta}_{max} = 0.3219$ rad/s^2

2.83 (a) Equation of motion of simple pendulum for small angular motions is

$$\ddot{\theta} + \frac{g_{moon}}{l}\,\theta = 0 \qquad (1)$$

Natural frequency of vibration is

$$\omega_n = \sqrt{\frac{g_{moon}}{l}} = \sqrt{\frac{1.6263}{1}} = 1.2753 \text{ rad/s}$$

(b) Solution of Eq. (1) can be written as (similar to Eq. (2.23)):

$$\theta(t) = A_0 \sin(\omega_n t + \phi_0) \qquad (2)$$

where $A_0 = \left\{ \theta_0^2 + \left(\frac{\dot{\theta}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}} = \left\{ (0.08727)^2 + 0 \right\}^{\frac{1}{2}}$

$= 0.08727$ rad

and $\phi_0 = \tan^{-1}\left(\frac{\theta_0\,\omega_n}{\dot{\theta}_0}\right) = \tan^{-1}(\infty) = 90^\circ$ or 1.5708 rad

$$\therefore\ \theta(t) = 0.08727 \sin(1.2753\,t + 1.5708) \text{ rad}$$

$$\dot{\theta}(t) = 0.08727(1.2753)\cos(1.2753t + 1.5708)$$

$$= 0.1113 \cos(1.2753\,t + 1.5708) \text{ rad/s}$$

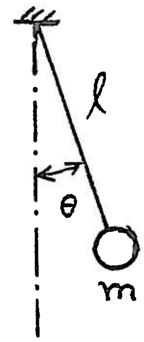
$$\dot{\theta}_{max} = 0.1113 \text{ rad/s}$$

(c) $\ddot{\theta}(t) = -0.1113(1.2753)\sin(1.2753\,t + 1.5708)$

$$= -0.1419 \sin(1.2753\,t + 1.5708) \text{ rad/s}^2$$

$$\therefore\ \ddot{\theta}\big|_{max} = 0.1419 \text{ rad/s}^2$$

2.84 For free vibration, apply Newton's second law of motion:



$$m l \ddot{\theta} + m g \sin\theta = 0 \qquad (E.1)$$

For small angular displacements, Eq. (E.1) reduces to

$$m l \ddot{\theta} + m g \theta = 0 \qquad (E.2)$$

or $$\ddot{\theta} + \omega_n^2 \theta = 0 \qquad (E.3)$$

where $$\omega_n = \sqrt{\frac{g}{l}} \qquad (E.4)$$

Solution of Eq. (E.3) is:

$$\theta(t) = \theta_0 \cos\omega_n t + \frac{\dot{\theta}_0}{\omega_n} \sin\omega_n t \qquad (E.5)$$

where θ_0 and $\dot{\theta}_0$ denote the angular displacement and angular velocity at $t = 0$. The amplitude of motion is given by

$$\Theta = \left\{ \theta_0^2 + \left(\frac{\dot{\theta}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} \qquad (E.6)$$

Using $\Theta = 0.5$ rad, $\theta_0 = 0$ and $\dot{\theta}_0 = 1$ rad/s, Eq. (E.6) gives

$$0.5 = \frac{\dot{\theta}_0}{\omega_n} = \frac{1}{\omega_n} \quad \text{or} \quad \omega_n = 2 \text{ rad/s}$$

2.85 The system of Fig. (A) can be drawn in equivalent form as shown in Fig. (B) where both pulleys have the same radius r_1. We notice in Fig. (B) that vibration can take place in only one way with one pulley moving clockwise and the other moving counter clockwise.†

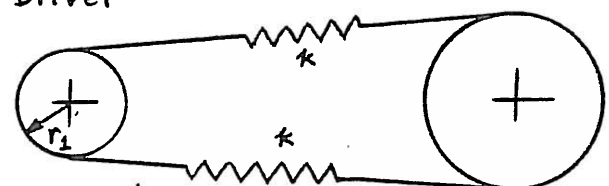


Fig. A

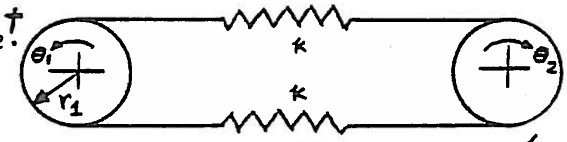


Fig. B

When pulleys rotate in opposite directions, $\dfrac{\theta_1}{\theta_2} = \dfrac{J_2}{J_1}$.

The spring force, which has the same value on either pulley is $-k_t(\theta_1+\theta_2)$ where k_t = torsional spring constant of the system. Equation of motion is

$$J_1\ddot{\theta}_1 + k_t(\theta_1+\theta_2) = 0 \quad \& \quad J_2\ddot{\theta}_2 + k_t(\theta_1+\theta_2) = 0$$

i.e. $J_1\ddot{\theta}_1 + k_t\left(1+\dfrac{J_1}{J_2}\right)\theta_1 = 0 \quad \& \quad J_2\ddot{\theta}_2 + k_t\left(\dfrac{J_2}{J_1}+1\right)\theta_2 = 0$

$$k_t = \frac{\Delta m_t}{\Delta\theta} = \left(\begin{matrix}\text{force in springs}\\ \text{due to } \Delta\theta\end{matrix}\right)\frac{r_1}{\Delta\theta}$$

$$= (2k\, r_1 \Delta\theta)\frac{r_1}{\Delta\theta} = 2k\, r_1^2$$

$$= 2k\left(\frac{125}{1000}\right)^2 = k/32 \quad \text{N-m/rad}$$

$\therefore$ Eq. (E_1) gives, for $\omega = 12\pi$ rad,

$$k = 454.7935 \text{ N/m}.$$

Either of these equations gives

$$\omega = \left\{k_t\left(\frac{J_1+J_2}{J_1 J_2}\right)\right\}^{1/2} \quad \text{---} (E_1)$$

Here $J_1 = 0.2/4 = 0.05$ kg-m^2,

$J_2' = J_2(\text{speed ratio})^2 = 0.2\left(\frac{1}{4}\right)^2 = 0.0125$ kg-m^2

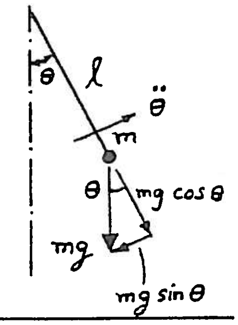
† The other possible motion is rotation of the two pulleys as a whole (as rigid body) in same direction. This will have a natural frequency of zero. See section 5.7.

2.86 $m\ell\,\ddot{\theta} + mg\sin\theta = 0$

For small θ, $\quad m\ell\,\ddot{\theta} + mg\,\theta = 0$

$$\omega_n = \sqrt{\frac{g}{\ell}}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{9.81}{0.5}}} = 1.4185 \text{ sec}$$



2.87 (a) $\omega_n = \sqrt{\dfrac{g}{\ell}}$

(b) $m\ell^2\ddot{\theta} + ka^2\sin\theta + mg\ell\sin\theta = 0\,;\quad m\ell^2\ddot{\theta} + (ka^2 + mg\ell)\,\theta = 0$

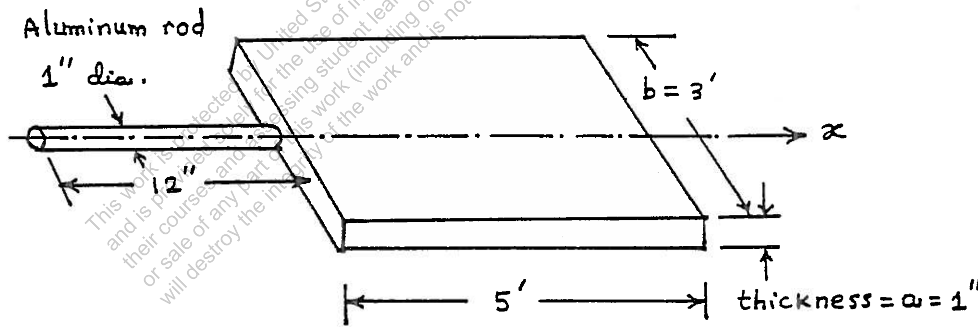
$$\omega_n = \sqrt{\frac{ka^2 + mg\ell}{m\ell^2}}$$

(c) $m\ell^2\ddot{\theta} + ka^2\sin\theta - mg\ell\sin\theta = 0\,;\quad m\ell^2\ddot{\theta} + (ka^2 - mg\ell)\,\theta = 0$

$$\omega_n = \sqrt{\frac{ka^2 - mg\ell}{m\ell^2}}$$

configuration (b) has the highest natural frequency.

2.88



$$m = \text{mass of a panel} = (5 \times 12)\,(3 \times 12)\,(1)\left(\frac{0.283}{386.4}\right) = 1.5820$$

$$J_0 = \text{mass moment of inertia of panel about x-axis} = \frac{m}{12}\,(a^2 + b^2)$$

$$= \frac{1.5820}{12}\,(1^2 + 36^2) = 170.9878$$

$$I_0 = \text{polar moment of inertia of rod} = \frac{\pi}{32}\,d^4 = \frac{\pi}{32}\,(1)^4 = 0.098175 \text{ in}^4$$

$$k_t = \frac{G\, I_0}{\ell} = \frac{(3.8\,(10^6))\,(0.098175)}{12} = 3.1089\,(10^4) \text{ lb—in/rad}$$

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{3.1089\,(10^4)}{170.9878}\right\}^{\frac{1}{2}} = 13.4841 \text{ rad/sec}$$

2.89

I_0 = polar moment of inertia of cross section of shaft AB

$$= \frac{\pi}{32}\, d^4 = \frac{\pi}{32}\,(1)^4 = 0.098175 \text{ in}^4$$

k_t = torsional stiffness of shaft AB $= \frac{G\, I_0}{\ell}$

$$= \frac{(12\,(10^6))\,(0.098175)}{6} = 19.635\,(10^4) \text{ lb—in/rad}$$

J_0 = mass moment of inertia of the three blades about y—axis

$$= 3\, J_0 \mid_{PQ} = 3 \left(\frac{1}{3}\, m\, \ell^2\right) = m\, \ell^2 = \left(\frac{2}{386.4}\right) (12)^2 = 0.7453$$

Torsional natural frequency:

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{19.635\,(10^4)}{0.7453}\right\}^{\frac{1}{2}} = 513.2747 \text{ rad/sec}$$

2.90

J_0 = mass moment of inertia of the ring = 1.0 kg—m^2.

I_{os} = polar moment of inertia of the cross section of steel shaft

$$= \frac{\pi}{32}\,(d_{os}^4 - d_{is}^4) = \frac{\pi}{4}\,(0.05^4 - 0.04^4) = 36.2266\,(10^{-8}) \text{ m}^4$$

I_{ob} = polar moment of inertia of cross section of brass shaft

$$= \frac{\pi}{32}\,(d_{ob}^4 - d_{ib}^4) = \frac{\pi}{32}\,(0.04^4 - 0.03^4) = 17.1806\,(10^{-8}) \text{ m}^4$$

k_{ts} = torsional stiffness of steel shaft

$$= \frac{G_s\, I_{os}}{\ell} = \frac{(80\,(10^9))\,(36.2266\,(10^{-8}))}{2} = 14490.64 \text{ N—m/rad}$$

k_{tb} = torsional stiffness of brass shaft

$$= \frac{G_b\, I_{ob}}{\ell} = \frac{(40\,(10^9))\,(17.1806\,(10^{-8}))}{2} = 3436.12 \text{ N—m/rad}$$

$$k_{t_{eq}} = k_{ts} + k_{tb} = 17{,}926.76 \text{ N—m/rad}$$

Torsional natural frequency:

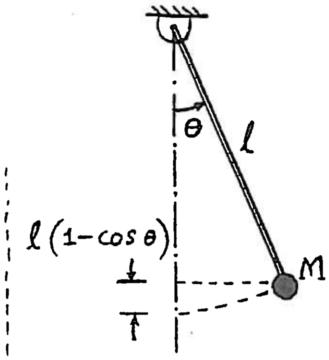
$$\omega_n = \sqrt{\frac{k_{t_{eq}}}{J_0}} = \sqrt{\frac{17926.76}{1}} = 133.8908 \text{ rad/sec}$$

Natural time period:

$$\tau_n = \frac{2\,\pi}{\omega_n} = \frac{2\,\pi}{133.8908} = 0.04693 \text{ sec}$$

2.91 Kinetic energy of system is

$$T = T_{rod} + T_{bob} = \frac{1}{2}\left(\frac{1}{3} m \ell^2\right) \dot{\theta}^2 + \frac{1}{2} M \ell^2 \dot{\theta}^2$$



Potential energy of system is (since mass of the rod acts through its center)

$$U = U_{rod} + U_{bob} = \frac{1}{2} m g \ell (1-\cos\theta) + \frac{1}{2} M g \ell (1-\cos\theta)$$

Equation of motion:

$$\frac{d}{dt}(T+U) = 0$$

i.e. $$\left(M+\frac{m}{3}\right)\ell^2 \ddot{\theta} + \left(M + \frac{m}{2}\right) g \ell \sin\theta = 0$$

For small angles, $$\ddot{\theta} + \frac{\left(M+\frac{m}{2}\right)}{\left(M+\frac{m}{3}\right)} \frac{g}{\ell} \cdot \theta = 0$$

$$\omega_n = \sqrt{\frac{\left(M+\frac{m}{2}\right) g}{\left(M+\frac{m}{3}\right)\ell}}$$

2.92 For the shaft, $$J = \frac{\pi d^4}{32} = \frac{\pi\,(0.05)^4}{32} = 61.3594 \times 10^{-8}\ m^4$$

$$k_t = \frac{G J}{\ell} = \frac{(0.793 \times 10^{11})(61.3594 \times 10^{-8})}{2} = 24329.002\ \text{N-m/rad}$$

For the disc,

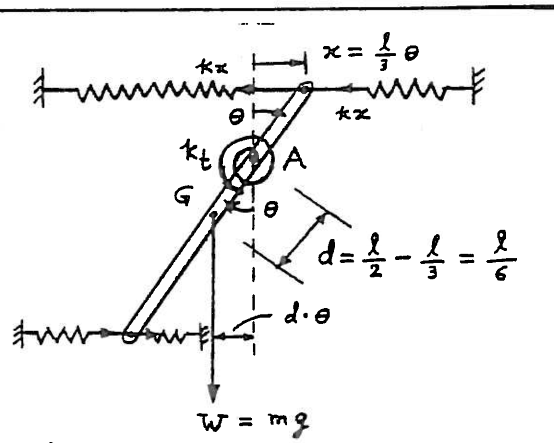
$$J_0 = \frac{M D^2}{8} = \left(\rho \frac{\pi D^2}{4} h\right)\frac{D^2}{8} = \frac{\rho \pi D^4 h}{32}$$

$$= \frac{(7.83 \times 10^3)\,\pi\,(1)^4 (0.1)}{32} = 76.8710\ \text{kg-m}^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{24329.002}{76.8710}\right)^{1/2} = 17.7902\ \text{rad/sec}$$

2.93 Equation of motion

$$J_A \ddot{\theta} = -W d\theta - 2k\left(\frac{\ell}{3}\theta\right)\frac{\ell}{3} - 2k\left(\frac{2\ell}{3}\theta\right)\frac{2\ell}{3} - k_t \theta$$



where

$$J_A = J_G + m d^2 = \frac{1}{12} m \ell^2 + m \frac{\ell^2}{36}$$

$$= \frac{1}{9} m \ell^2$$

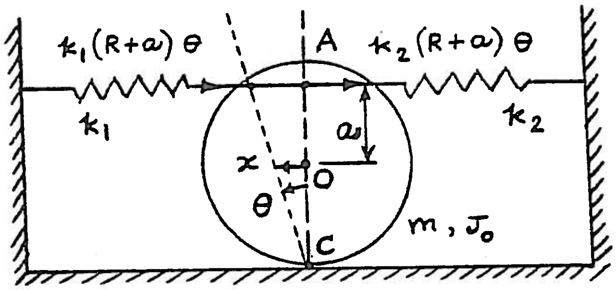
$$\therefore\ \frac{m\ell^2}{9}\ddot{\theta} + \left(m g d + 2k\frac{\ell^2}{9} + \frac{8k\ell^2}{9} + k_t\right)\theta = 0$$

$$\omega_n = \sqrt{\frac{\left(m g d + \frac{2}{9} k \ell^2 + \frac{8}{9} k \ell^2 + k_t\right) 9}{m \ell^2}} = \sqrt{\frac{9 m g d + 10 k \ell^2 + 9 k_t}{m \ell^2}}$$

For given data,

$$\omega_n = \sqrt{\frac{9(10)(9.81)(5/6) + 10\,(2000)\,(5)^2 + 9\,(1000)}{10\,(5)^2}} = 45.1547 \ \frac{\text{rad}}{\text{sec}}$$

2.94 $J_0 = \frac{1}{2} m R^2$, $J_C = \frac{1}{2} m R^2 + m R^2$



Let angular displacement $= \theta$

Equation of motion:

$$J_C\,\ddot{\theta} + k_1 (R+a)^2\theta + k_2 (R+a)^2 \theta = 0$$

$$\omega_n = \sqrt{\frac{(k_1 + k_2)\,(R+a)^2}{J_C}} = \sqrt{\frac{(k_1+k_2)(R+a)^2}{1.5\ m R^2}} \qquad (E_1)$$

Equation (E_1) shows that ω_n increases with the value of a.

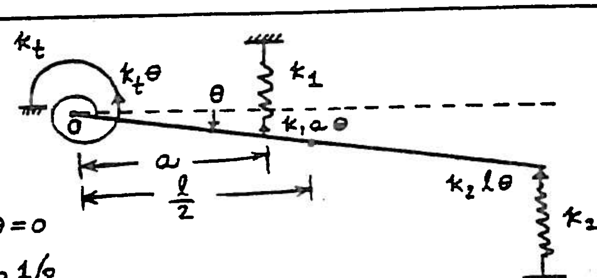
$\therefore$ ω_n will be maximum when $a = R$.

2.95 Net g acting on the pendulum $= 9.81 - 5 = 4.81\ \text{m/sec}^2 = g_n$

$$\omega_n = \sqrt{\frac{g_n}{l}} = \sqrt{\frac{4.81}{5}} = 3.1016\ \text{rad/sec}$$

$$\tau_n = 2\pi/\omega_n = 2.0258\ \text{sec}$$

2.96 Equation of motion:



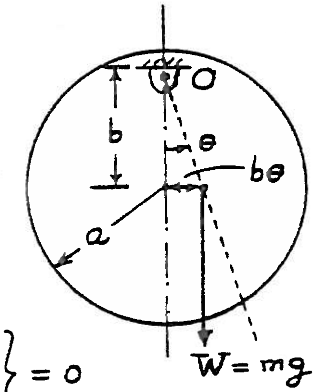
$$J_0\,\ddot{\theta} = -k_t\theta - (k_1 a\,\theta)a - (k_2 l\theta)\,l$$

where $J_0 = \frac{1}{12} m l^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$

$$\therefore \quad \frac{1}{3} m l^2\,\ddot{\theta} + (k_t + k_1 a^2 + k_2 l^2)\,\theta = 0$$

$$\omega_n = \left\{\frac{3\,(k_t + k_1 a^2 + k_2 l^2)}{m l^2}\right\}^{1/2}$$

2.97 $J_0 = J_G + m b^2 = \frac{1}{2} m a^2 + m b^2$



Equation of motion:

$$J_0\,\ddot{\theta} + m g b\,\theta = 0$$

$$\omega_n = \sqrt{\frac{m g b}{J_0}} = \sqrt{\frac{2 g b}{a^2 + 2 b^2}}$$

$$\frac{\partial \omega_n}{\partial b} = \frac{1}{2}\left(\frac{2 g b}{a^2 + 2 b^2}\right)^{-\frac{1}{2}} \left\{\frac{(a^2 + 2b^2)(2g) - 2 g b\,(4b)}{(a^2 + 2 b^2)^2}\right\} = 0$$

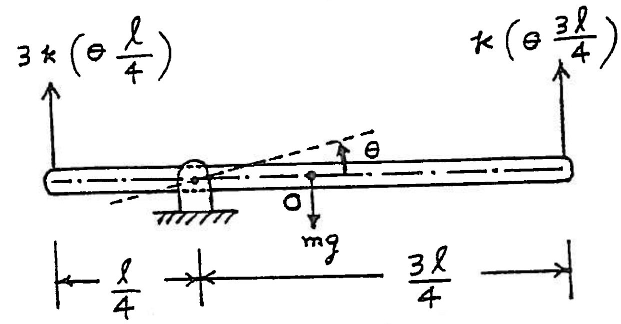
i.e., $b = \pm \frac{a}{\sqrt{2}}$

$$\omega_n\Big|_{b=+a/\sqrt{2}} = \sqrt{\frac{2g\,\frac{a}{\sqrt{2}}}{a^2 + 2(a^2/2)}} = \sqrt{\frac{g}{\sqrt{2}\,a}}$$

$b = -a/\sqrt{2}$ gives imaginary value for ω_n.

Since $\omega_n = 0$ when $b = 0$, we have $\omega_n\big|_{max}$ at $b = \frac{a}{\sqrt{2}}$.

2.98



Let θ be measured from static equilibrium position so that gravity force need not be considered.

(a) Newton's second law of motion:

$$J_0\,\ddot{\theta} = -3\,k\,(\theta\,\frac{\ell}{4})\,\frac{\ell}{4} - k\,(\theta\,\frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}) \quad \text{or} \quad J_0\,\ddot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

(b) D'Alembert's principle:

$$M(t) - J_0\,\ddot{\theta} = 0 \quad \text{or} \quad -3\,k\,(\theta\,\frac{\ell}{4})\,(\frac{\ell}{4}) - k\,(\theta\,\frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}) - J_0\,\ddot{\theta} = 0$$

$$\text{or} \quad J_0\,\ddot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

(c) Principle of virtual work:

Virtual work done by spring force:

$$\delta W_s = -3\,k\,(\theta\,\frac{\ell}{4})\,(\frac{\ell}{4}\,\delta\theta) - k\,(\theta\,\frac{3\,\ell}{4})\,(\frac{3\,\ell}{4}\,\delta\theta)$$

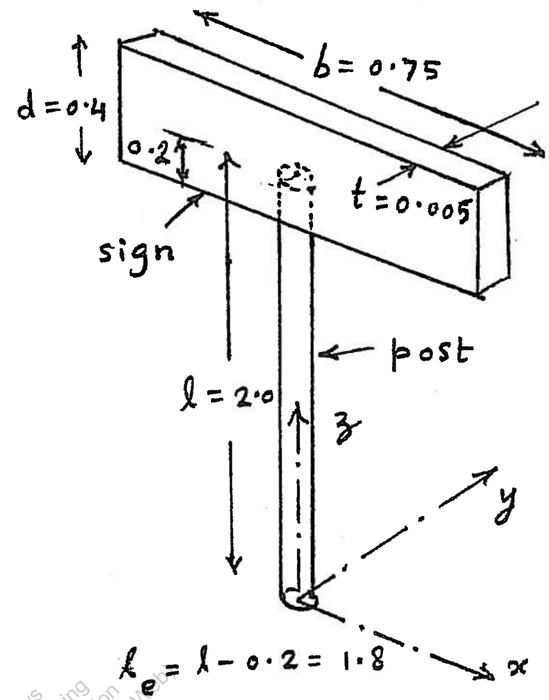
Virtual work done by inertia moment = - $(J_0\,\ddot{\theta})\,\delta\theta$

Setting total virtual work done by all forces/moments equal to zero, we obtain

$$J_0\,\ddot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

2.99

Torsional stiffness of the post (about z-axis):



$$k_t = \frac{\pi G}{2\, l_e}\left(r_o^4 - r_i^4\right)$$

$$= \frac{\pi\,(79.3\times 10^9)\left(0.05^4 - 0.045^4\right)}{2\,(1.8)}$$

$$= 148.7161 \times 10^3 \text{ N-m}$$

Mass moment of inertia of the sign about the z-axis:

$$J_{sign} = \frac{M}{12}\left(d^2 + b^2\right)$$

with

Mass of traffic sign $= M = b\,d\,t\,\rho$

$$= M = 0.75\,(0.4)\,(0.005)\left(\frac{76500}{9.81}\right) = 11.6972 \text{ kg}$$

Hence $J_{sign} = \dfrac{11.6972}{12}\left(0.40^2 + 0.75^2\right) = 0.7043 \text{ kg-m}^2$

Mass moment of inertia of the post about the z-axis:

$$J_{post} = \frac{m}{8}\left(d_o^2 + d_i^2\right)$$

with $d_o = 2\,r_o = 0.10$ m, $d_i = 2\,r_i = 2\,(0.045) = 0.09$ m

and

Mass of the post $= m = \pi\left(r_o^2 - r_i^2\right) l\, \rho$

$$= m = \pi\left(0.05^2 - 0.045^2\right)(2)\left(\frac{76500}{9.81}\right) = 23.2738 \text{ kg}$$

Hence

$$J_{post} = \frac{23.2738}{8}\left(0.10^2 + 0.09^2\right) = 0.052657 \text{ kg-m}^2$$

Equivalent mass moment of inertia of the post (J_{eff}) about the location of the sign:

$$J_{eff} = \frac{J_{post}}{3} = \frac{0.052657}{3} = 0.017552 \text{ kg-m}^2$$

(Derivation given below)

Natural frequency of torsional vibration of the traffic sign about the z-axis:

$$\omega_n = \left(\frac{k_t}{J_{sign} + J_{eff}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{148.7161 \times 10^3}{0.7043 + 0.017552}\right)^{\frac{1}{2}}$$

$$= 453.8945 \text{ rad/s}$$

Derivation:

Effect of the mass moment of inertia of the post or shaft (J_{eff}) on the natural frequency of vibration of a shaft carrying end mass moment of inertia (J_{sign}):

Let $\dot{\theta}$ be the angular velocity of the end mass moment of inertia (J_{sign}) during vibration. Assume a linear variation of the angular velocity of the shaft (post) so that at a distance x from the fixed end, the angular velocity is given by $\frac{\dot{\theta}\, x}{l}$.

The total kinetic energy of the shaft (post) is given by

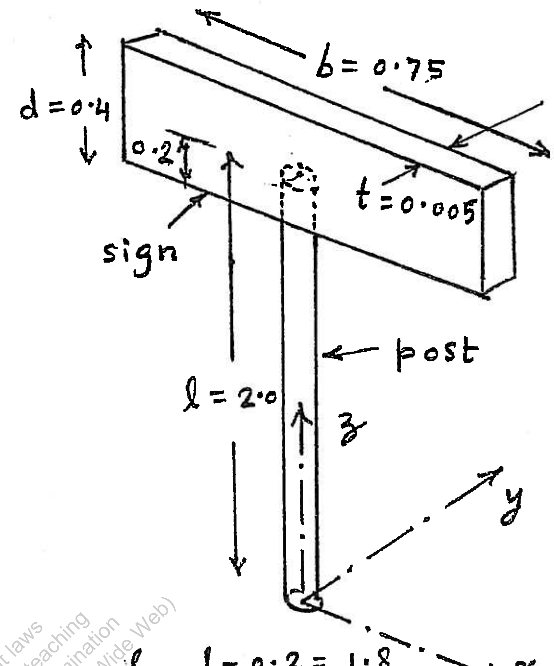
$$T_{post} = \frac{1}{2} \int_0^l \left(\frac{\dot{\theta}\, x}{l} \right) \left(\frac{J_{post}}{l} \right) dx$$

$$= \frac{1}{2} \frac{J_{post}}{3} \left(\dot{\theta} \right)^2$$

This shows that the effective mass moment of inertia of the shaft (post) at the end is $\frac{J_{post}}{3}$.

2.100

Torsional stiffness of the post (about z-axis):



$$k_t = \frac{\pi G}{2\, l_e}\left(r_o^4 - r_i^4\right)$$

$$= \frac{\pi\,(41.4\times 10^9)\left(0.05^4 - 0.045^4\right)}{2\,(1.8)}$$

$$= 77.6399 \times 10^3 \quad \text{N-m}$$

$l_e = l - 0.2 = 1.8$

Mass moment of inertia of the sign about the z-axis:

$$J_{sign} = \frac{M}{12}\left(d^2 + b^2\right)$$

with

Mass of traffic sign $= M = b\, d\, t\, \rho$

$$= M = 0.75\,(0.4)\,(0.005)\left(\frac{80\,100}{9.81}\right) = 12.2476 \quad \text{kg}$$

Hence $\quad J_{sign} = \dfrac{12.2476}{12}\left(0.40^2 + 0.75^2\right) = 0.7374 \quad \text{kg}-\text{m}^2$

Mass moment of inertia of the post about the z-axis:

$$J_{post} = \frac{m}{8}\left(d_o^2 + d_i^2\right)$$

with $d_o = 2\, r_o = 0.10$ m, $\quad d_i = 2\, r_i = 2\,(0.045) = 0.09$ m

and

Mass of the post $= m = \pi\,(r_o^2 - r_i^2)\, l\, \rho$

$$= m = \pi\left(0.05^2 - 0.045^2\right)(2)\left(\frac{76\,500}{9.81}\right) = 24.3690 \quad \text{kg}$$

Hence

$$J_{post} = \frac{24.3690}{8}\left(0.10^2 + 0.09^2\right) = 0.055135 \ \text{kg-m}^2$$

Equivalent mass moment of inertia of the post (J_{eff}) about the location of the sign:

$$J_{eff} = \frac{J_{post}}{3} = \frac{0.055135}{3} = 0.018378 \ \text{kg-m}^2$$

(Derivation given in the solution of Problem 2.79)

Natural frequency of torsional vibration of the traffic sign about the z-axis:

$$\omega_n = \left(\frac{k_t}{J_{sign} + J_{eff}}\right)^{\frac{1}{2}}$$

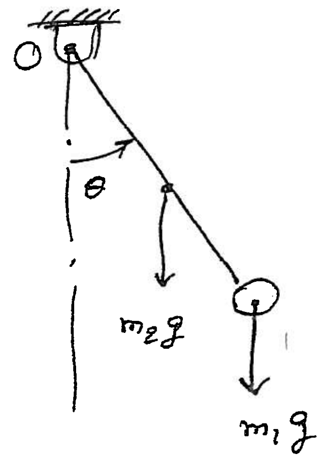
$$= \left(\frac{77.6399 \times 10^3}{0.7374 + 0.018378}\right)^{\frac{1}{2}}$$

$$= 320.5127 \ \text{rad/s}$$

(2.101)

Assume the end mass m_1 to be a point mass. Then the mass moment of inertia of m_1 about the pivot point is given by

$$I_1 = m_1 l^2 \quad (1)$$



For the uniform bar of length l and mass m_2, its mass moment of inertia about the pivot O is given by

$$I_2 = \frac{1}{12} m_2 l^2 + m_2 \left(\frac{l_2}{2}\right)^2 = \frac{1}{3} m_2 l^2 \quad (2)$$

Inertia moment about pivot point O is given by

$$I_0 \ddot{\theta} + m_2 g \cdot \frac{l}{2} \sin\theta + m_1 g \cdot l \sin\theta = 0 \quad (3)$$

where

$$I_0 = I_1 + I_2 = m_1 l^2 + \frac{1}{3} m_2 l^2, \quad (4)$$

For small angular displacement, $\sin\theta \simeq \theta$ and Eq. (3) can be expressed as

$$\left(m_1 l^2 + \frac{1}{3} m_2 l^2\right) \ddot{\theta} + \left(m_1 g l + \frac{m_2 g l}{2}\right) \theta = 0$$

or

$$\ddot{\theta} + \frac{3(2 m_1 g l + m_2 g l)}{2(3 m_1 l^2 + m_2 l^2)} \theta = 0$$

or $$\ddot{\theta} + \frac{g\ell\,(6m_1 + 3m_2)}{\ell^2\,(6m_1 + 2m_2)}\,\theta = 0$$

or $$\ddot{\theta} + \frac{g}{\ell}\left(\frac{6m_1 + 3m_2}{6m_1 + 2m_2}\right)\theta = 0 \qquad (5)$$

By expressing Eq. (5) as $\ddot{\theta} + \omega_n^2\,\theta = 0$, the natural frequency of vibration of the system can be expressed as

$$\omega_n = \sqrt{\frac{g}{\ell}\left(\frac{6m_1 + 3m_2}{6m_1 + 2m_2}\right)} \qquad (6)$$

2.102

Equation of motion for the angular motion of the forearm about the pivot point O:

$$I_0 \ddot{\theta}_t + m_2 g\, b \cos\theta_t + m_1 g \frac{b}{2} \cos\theta_t$$
$$- F_2 a_2 + F_1 a_1 = 0 \qquad (1)$$

where θ_t is the total angular displacement of the forearm, I_0 is the mass moment of inertia of the forearm and the mass carried:

$$I_0 = m_2 b^2 + \frac{1}{3} b^2 m_1 \qquad (2)$$

and the forces in the biceps and triceps muscles (F_2 and F_1) are given by

$$F_2 = - c_2 \theta_t \qquad (3)$$

$$F_1 = c_1 \dot{x} = c_1 a_1 \dot{\theta}_t \qquad (4)$$

where the linear velocity of the triceps can be expressed as

$$\dot{x} \simeq a_1 \dot{\theta}_t \qquad (5)$$

Using Eqs. (2) – (4), Eq. (1) can be rewritten as

$$I_0 \ddot{\theta}_t + \left(m_2 g b + \frac{1}{2} m_1 g b\right) \cos\theta_t$$
$$+ c_2 a_2 \theta_t + c_1 a_1^2 \dot{\theta}_t = 0 \qquad (6)$$

Let the forearm undergo small angular displacement (θ) about the static equilibrium position, $\bar{\theta}$, so that

$$\theta_t = \bar{\theta} + \theta \tag{7}$$

Using Taylor's series expansion of $\cos\theta_t$ about $\bar{\theta}$, the static equilibrium position, can be expressed as (for small values of θ):

$$\cos\theta_t = \cos(\bar{\theta}+\theta) \simeq \cos\bar{\theta} - \theta\sin\bar{\theta} \tag{8}$$

Using $\ddot{\theta}_t = \ddot{\theta}$ and $\dot{\theta}_t = \dot{\theta}$, Eq. (6) can be expressed as

$$I_0\ddot{\theta} + \left(m_2 g b + \tfrac{1}{2}m_1 g b\right)\left(\cos\bar{\theta} - \sin\bar{\theta}\,\theta\right) + c_2 a_2(\bar{\theta}+\theta) + c_1 a_1^2\dot{\theta} = 0$$

or

$$I_0\ddot{\theta} + \left(m_2 g b + \tfrac{1}{2}m_1 g b\right)\cos\bar{\theta} - \sin\bar{\theta}\left(m_2 g b + \tfrac{1}{2}m_1 g b\right)\theta + c_2 a_2\bar{\theta} + c_2 a_2\theta + c_1 a_1^2\dot{\theta} = 0 \tag{9}$$

Noting that the static equilibrium equation of the forearm at $\theta_t = \bar{\theta}$ is given by

$$\left(m_2 g b + \tfrac{1}{2}m_1 g b\right)\cos\bar{\theta} + c_2 a_2\bar{\theta} = 0 \tag{10}$$

In view of Eq. (10), Eq. (9) becomes

$$\left(m_2 b^2 + \tfrac{1}{3}b^2 m_1\right)\ddot{\theta} + c_1 a_1^2\dot{\theta} + \left\{c_2 a_2 - \sin\bar{\theta}\,g b\left(m_2 + \tfrac{1}{2}m_1\right)\right\}\theta = 0 \tag{11}$$

which denotes the equation of motion of the forearm.

The undamped natural frequency of the forearm can be expressed as

$$\omega_n = \sqrt{\frac{c_2 a_2 - \sin\bar{\theta} \;\; g b \left(m_2 + \frac{1}{2} m_1\right)}{b^2 \left(m_2 + \frac{1}{3} m_1\right)}} \qquad (12)$$

2.103

(a) $100\dot{v} + 20v = 0$

Using a solution similar to Eqs. (2.52) and (2.53), we find :

Free vibration response : $v(t) = v(0) \cdot e^{-\frac{20}{100}t}$

Time constant : $\tau = \frac{100}{20} = 5$ sec.

(b) $v(t) = v_h(t) + v_p(t)$

with $v_h(t) = A \cdot e^{-\frac{20}{100}t}$ where A = constant

and let $v_p(t) = C$ = constant

$\therefore$ Substitution in the Equation of motion gives

$$100(0) + 20C = 10 \quad \text{or} \quad c = \frac{1}{2}$$

$$\therefore \quad v(t) = A e^{-\frac{20}{100}t} + \frac{1}{2}$$

$$v(0) = A e^{-0} + \frac{1}{2} = 10 \quad \text{or} \quad A = \frac{19}{2}$$

Total response:

$$v(t) = \frac{19}{2} e^{-\frac{20}{100}t} + \frac{1}{2}$$

Free vibration response : $e^{-\frac{20}{100}t}$

Homogeneous solution: $\frac{19}{2} e^{-\frac{20}{100}t}$

Time constant: $\tau = \frac{100}{20} = 5$ sec

(c) Free vibration response :

$$v(t) = v(0)\, e^{\frac{20}{100} t}$$

This solution grows with time.

$\therefore$ No time constant can be found.

(d) Free vibration solution :

$$\omega(t) = 0.5\, e^{-\frac{50}{500} t} = 0.5\, e^{-0.1 t}$$

Time constant $= \tau = \frac{500}{50} = 10$ s.

2.104

Let $t=0$ when force is released.

Before the force is released, the system is at rest so that

$$F = k x \quad ; \quad t \le 0$$

$$\text{or} \quad x(0) = \frac{F}{k} \quad \text{or} \quad 0.1 = \frac{500}{k}$$

$$\therefore \quad k = 5000 \ \text{N/m}$$

The eqn of motion for $t > 0$ becomes

$$c\dot{x} + kx = 0 \qquad (E_1)$$

The solution of Eq. (E_1) is given by

$$x(t) = A \cdot e^{-\frac{k}{c}t} = A \cdot e^{-\frac{5000}{c}t}$$

At $t = 0$, $x(t) = 0.1$ and hence

$$0.1 = A \, e^{-0} \quad \text{or} \quad A = 0.1$$

$$\therefore \quad x(t) = 0.1 \, e^{-\frac{5000}{c}t} \quad ; \quad t > 0 \qquad (E_2)$$

Using $x(t = 10) = 0.01$ m in (E_2),

$$0.01 = 0.1 \, e^{-(5000/c)10} \quad \text{or} \quad e^{-(50{,}000/c)} = 0.1$$

$$\text{i.e.,} \quad -\frac{50000}{c} = \ln 0.1 = -2.3026$$

Hence $c = 21714.7$ N-s/m

2.105

$$m\dot{v} = F - D - mg$$

$$1000\,\dot{v} = 50\,000 - 2000\,v - 1000(9.81)$$

$$1{,}000\,\dot{v} + 2{,}000\,v = 40{,}190$$

or

$$0.5\,\dot{v} + v = 20.095 \qquad (E_1)$$

Solution of Eq. (E_1) with $v(0) = 0$ at $t = 0$:

$$v(t) = 20.095\left(1 - e^{-\frac{1}{0.5}t}\right)$$

or

$$\frac{dx}{dt}(t) = 20.095\left(1 - e^{-2t}\right) \qquad (E_2)$$

Integration of Eq. (E_2) gives

$$x(t) = 20.095\,t - 20.095\left(\frac{1}{-2}\cdot e^{-2t}\right) + c_1$$

$$= 20.095\,t + 10.0475\,e^{-2t} + c_1$$

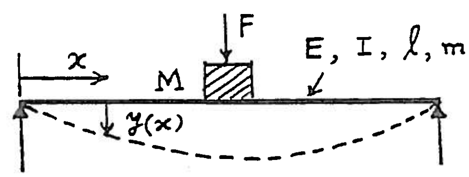
$$x(0) = 0$$

$$\Rightarrow 0 = 10.0475\,e^{0} + c_1$$

$$\text{or} \quad c_1 = -10.0475$$

$$\therefore x(t) = 20.095\,t + 10.0475\,e^{-2t} - 10.0475$$

2.106 Let m_{eff} = effective part of mass of beam (m) at middle. Thus vibratory inertia force at middle is due to $(M + m_{eff})$. Assume a deflection shape: $y(x,t) = Y(x) \cos(\omega_n t - \phi)$ where $Y(x)$ = static deflection shape due to load at middle given by:



$$Y(x) = Y_0 \left(3 \frac{x}{\ell} - 4 \frac{x^3}{\ell^3}\right) \; ; \; 0 \le x \le \frac{\ell}{2}$$

where Y_0 = maximum deflection of the beam at middle $= \dfrac{F \ell^3}{48 E I}$

Maximum strain energy of beam = maximum work done by force F $= \dfrac{1}{2} F Y_0$.

Maximum kinetic energy due to distributed mass of beam:

$$= 2\left\{\frac{1}{2} \frac{m}{\ell} \int_0^{\frac{\ell}{2}} \dot{y}^2(x,t)\,\Big|_{max}\, dx\right\} + \frac{1}{2}\left(\dot{y}_{max}\right)^2 M$$

$$= \frac{m\,\omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y^2(x)\, dx + \frac{1}{2}\omega_n^2 Y_{max}^2 M$$

$$= \frac{m\,\omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y_0^2 \left(\frac{9 x^2}{\ell^2} + 16 \frac{x^6}{\ell^6} - 24 \frac{x^4}{\ell^4}\right) dx + \frac{1}{2} Y_0^2 M \omega_n^2$$

$$= \frac{m\,\omega_n^2 Y_0^2}{\ell}\left[\frac{9}{\ell^2}\frac{x^3}{3} + \frac{16}{\ell^6}\frac{x^7}{7} - \frac{24}{\ell^4}\frac{x^5}{5}\right]\Bigg|_0^{\frac{\ell}{2}} + \frac{1}{2} Y_0^2 M \omega_n^2$$

$$= \frac{1}{2} Y_0^2 \omega_n^2 \left(\frac{17}{35} m + M\right)$$

This shows that $m_{eff} = \dfrac{17}{35} m = 0.4857\ m$

2.107 For small angular rotation of bar PQ about P,

$$\frac{1}{2}(k_{12})_{eq}(\theta \ell_3)^2 = \frac{1}{2} k_1 (\theta \ell_1)^2 + \frac{1}{2} k_2 (\theta \ell_2)^2$$

$$(k_{12})_{eq} = \frac{k_1 \ell_1^2 + k_2 \ell_2^2}{\ell_3^2}$$

Since $(k_{12})_{eq}$ and k_3 are in series,

$$k_{eq} = \frac{(k_{12})_{eq}\ k_3}{(k_{12})_{eq} + k_3} = \frac{k_1 k_3 \ell_1^2 + k_2 k_3 \ell_2^2}{k_1 \ell_1^2 + k_2 \ell_2^2 + k_3 \ell_3^2}$$

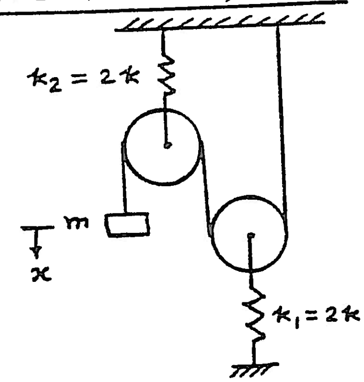
T = kinetic energy $= \frac{1}{2} m \dot{x}^2$, U = potential energy $= \frac{1}{2} k_{eq} x^2$

If $x = X \cos \omega_n t$,

$T_{max} = \frac{1}{2} m \omega_n^2 X^2$, $U_{max} = \frac{1}{2} k_{eq} X^2$

$T_{max} = U_{max}$ gives $\quad \omega_n = \sqrt{\dfrac{k_1 k_3 \ell_1^2 + k_2 k_3 \ell_2^2}{m(k_1 \ell_1^2 + k_2 \ell_2^2 + k_3 \ell_3^2)}}$

2.108 When mass m moves by x, spring k_1 deflects by $x/4$.



T = kinetic energy $= \frac{1}{2} m (\dot{x})^2$

U = potential energy $= 2\left\{\frac{1}{2}(2k)\left(\frac{x}{4}\right)^2\right\}$

$= \frac{1}{8} k x^2$

For harmonic motion,

$T_{max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{max} = \frac{1}{8} k X^2$

$T_{max} = U_{max}$ gives $\quad \omega_n = \sqrt{\dfrac{k}{4m}}$

2.109 Refer to the figure of solution of problem 2.24.

$T = \frac{1}{2} m \dot{x}^2 \quad , \quad U = \frac{1}{2}\left[2k_1 (x \cos 45°)^2 + 2k_2 (x \cos 135°)^2\right]$

$= \frac{1}{2}(k_1 + k_2) x^2$

For harmonic motion,

$T_{max} = \frac{1}{2} m \omega_n^2 X^2, \quad U_{max} = \frac{1}{2}(k_1 + k_2) X^2$

$T_{max} = U_{max}$ gives $\quad \omega_n = \sqrt{\dfrac{k_1 + k_2}{m}}$

2.110 kinetic energy (K.E.) $= \frac{1}{2} m \dot{x}^2$

Potential energy (P.E.) $= \frac{1}{2} T_1 x + \frac{1}{2} T_2 x$ = work done in displacing mass m by distance x against the total force (tension) of $T_1 + T_2$.

$T_1 = \frac{x}{a} T, \quad T_2 = \frac{x}{b} T \quad$ from solution of problem 2.26

Max. K.E. $= \frac{1}{2} m \omega_n^2 X^2, \quad$ Max. P.E. $= \frac{1}{2} T\left(\frac{1}{a} + \frac{1}{b}\right) X^2$

Max. K.E. = Max. P.E. gives $\quad \omega_n = \sqrt{\dfrac{T(a+b)}{m a b}} = \sqrt{\dfrac{T \ell}{m a (\ell - a)}}$

2.111 $T = K.E. = \frac{1}{2} J_A \dot{\theta}^2 = \frac{1}{2}(J_G + m d^2)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{12} m \ell^2 + m \frac{\ell^2}{36}\right)\dot{\theta}^2$

$= \frac{1}{2}\left(\frac{m \ell^2}{9}\right)\dot{\theta}^2$

$U = P.E. = m g d (1 - \cos\theta) + 2\left(\frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2\right) + \frac{1}{2} k_t \theta^2$

with $\cos\theta \simeq 1 - \frac{1}{2}\theta^2, \quad x_1 = \frac{\ell}{3}\theta$ and $x_2 = \frac{2\ell}{3}\theta$

$$U = mg\frac{l}{6}\frac{\theta^2}{2} + k\frac{l^2}{9}\theta^2 + k\frac{4l^2}{9}\theta^2 + \frac{1}{2}k_t\theta^2$$

$$T_{max} = \frac{1}{2}\left(\frac{ml^2}{9}\right)\Theta^2, \qquad U_{max} = \frac{1}{2}\frac{mgl}{6}\Theta^2 + \frac{1}{2}\left(\frac{10kl^2}{9}\right)\Theta^2 + \frac{1}{2}k_t\Theta^2$$

$T_{max} = U_{max}$ gives

$$\omega_n = \sqrt{\frac{mg\frac{l}{6} + \frac{10kl^2}{9} + k_t}{\frac{ml^2}{9}}} = 45.1547 \ \frac{rad}{sec} \text{ for given data}$$

2.112 Refer to the figure in the solution of problem 2.76

$$T = \frac{1}{2}J_0\dot{\theta}^2$$

$$U = \frac{1}{2}k_t\theta^2 + \frac{1}{2}k_1(\theta a)^2 + \frac{1}{2}k_2(\theta l)^2$$

For $\theta(t) = \Theta\cos\omega_n t$,

$$T_{max} = \frac{1}{2}J_0\omega_n^2\Theta^2, \qquad U_{max} = \frac{1}{2}\left(k_t + k_1a^2 + k_2l^2\right)\Theta^2$$

$T_{max} = U_{max}$ gives

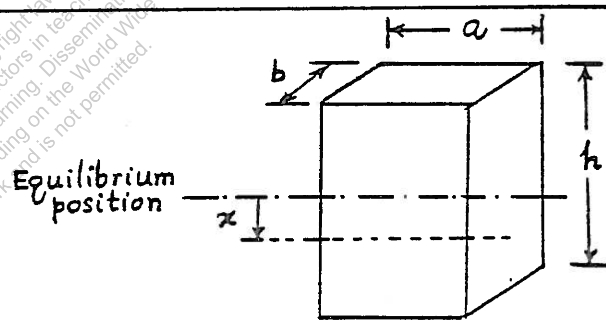
$$\omega_n = \sqrt{\frac{k_t + k_1a^2 + k_2l^2}{J_0}} = \sqrt{\frac{3(k_t + k_1a^2 + k_2l^2)}{ml^2}}$$

since $J_0 = ml^2/3$.

2.113 When prism is displaced by x from equilibrium position, the weight of oil displaced

$= \rho_o g\, ab\, x$ = restoring force

Mass of prism $= m = \rho_w abh$



Equation of motion:

$$m\ddot{x} + \text{restoring force} = 0$$

$$\rho_w abh\,\ddot{x} + \rho_o g ab x = 0$$

$$\omega_n = \sqrt{\frac{\rho_o g ab}{\rho_w abh}} = \sqrt{\frac{\rho_o g}{\rho_w h}} \qquad (E_1)$$

Since ω_n is independent of cross-section of the prism, ω_n remains same even for a circular wooden prism.

2.114

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 = \frac{1}{2}\left(mR^2 + \frac{1}{2}mR^2\right)\dot{\theta}^2$$

since $x = R\theta$ and $J_0 = \frac{1}{2}mR^2$.

$$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_1^2 = \frac{1}{2}(k_1 + k_2)(R + a)^2\theta^2$$

where $x_1 = (R + a)\,\theta$. Using $\frac{d}{dt}(T + U) = 0$, we obtain

$$\left(\frac{3}{2}\, m\, R^2\right) \ddot{\theta} + (k_1 + k_2)\,(R + a)^2\, \theta = 0$$

2.115 Let x(t) be measured from static equilibrium position of mass. T = kinetic energy of the system:

$$T = \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, J_0\, \dot{\theta}^2 = \frac{1}{2}\left(m + \frac{J_0}{r^2}\right) \dot{x}^2$$

since $\dot{\theta} = \frac{\dot{x}}{r}$ = angular velocity of pulley. U = potential energy of the system:

$$U = \frac{1}{2}\, k\, y^2 = \frac{1}{2}\, k\, (16\, x^2)$$

since $y = \theta\,(4\, r) = 4\, x$ = deflection of spring. $\frac{d}{dt}(T + U) = 0$ leads to:

$$m\, \ddot{x} + \frac{J_0}{r^2}\, \ddot{x} + 16\, k\, x = 0$$

This gives the natural frequency:

$$\omega_n = \sqrt{\frac{16\, k\, r^2}{m\, r^2 + J_0}}$$

(2.116)

Assume: No sliding of the cylinder.

Kinetic energy of the cylinder (T) = sum of translational and rotational kinetic energies

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J \dot{\theta}^2 \qquad (E_1)$$

Since the cylinder rolls without sliding,

$$x = \theta R \quad \text{or} \quad \theta = \frac{x}{R} \qquad (E_2)$$

Using Eq. (E_2), the kinetic energy can be expressed as

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J . \frac{\dot{x}^2}{R^2} = \frac{1}{2}\left(m + \frac{J}{R^2}\right) \dot{x}^2 \qquad (E_3)$$

$$= \frac{1}{2} m \dot{\theta}^2 R^2 + \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} \left(m R^2 + J\right) \dot{\theta}^2 \qquad (E_4)$$

The potential (or strain) energy, U, due to the deflection of the spring is given by

$$U = \frac{1}{2} k x^2 \qquad (E_5)$$

$$= \frac{1}{2} k R^2 \theta^2 \qquad (E_6)$$

Total energy is constant since the damping is absent.

$$\therefore \quad T + U = c = \text{constant} \qquad (E_7)$$

Using Eqs. (E_3) and (E_5) in Eq. (E_7), we obtain

$$\frac{1}{2}\left(m + \frac{J}{R^2}\right)\dot{x}^2 + \frac{1}{2} k x^2 = c \qquad (E_8)$$

Differentiating Eq. (E_8) w.r.t. time gives

$$\frac{1}{2}\left(m + \frac{J}{R^2}\right)(2\dot{x})\ddot{x} + \frac{1}{2} k (2 x \dot{x}) = 0$$

or

$$\left[\left(m + \frac{J}{R^2}\right)\ddot{x} + k x\right]\dot{x} = 0 \qquad (E_9)$$

Since $\dot{x} \neq 0$ for all t,

$$\left(m + \frac{J}{R^2}\right)\ddot{x} + x k = 0 \qquad (E_{10})$$

The natural frequency of vibration, from Eq. (E_{10}), is given by

$$\omega_n = \sqrt{\frac{k}{\left(m + \frac{J}{R^2}\right)}} \qquad (E_{11})$$

Since the mass moment of inertia of a cylinder can be expressed as

$$J = \frac{1}{2} m R^2 \qquad (E_{12})$$

Eqs. (E_{10}) and (E_{11}) become

$$\frac{3}{2} m \ddot{x} + k x = 0 \qquad (E_{13})$$

$$\omega_n = \sqrt{\frac{2k}{3m}} \qquad (E_{14})$$

Using Eqs. (E_4) and (E_6), the total energy of the system can be expressed as

$$\frac{1}{2}(mR^2 + J)\dot{\theta}^2 + \frac{1}{2}kR^2\theta^2 = c = \text{constant} \qquad (E_{15})$$

Differention of Eq. (E_{15}) with respect to time gives

$$\frac{1}{2}(mR^2+J)(2\dot{\theta}\ddot{\theta}) + \frac{1}{2}kR^2(2\theta\dot{\theta}) = 0 \qquad (E_{16})$$

$$\left[(mR^2+J)\ddot{\theta} + kR^2\theta\right]\dot{\theta} = 0 \qquad (E_{17})$$

Since $\dot{\theta} \neq 0$ for all t,

$$(mR^2+J)\ddot{\theta} + kR^2\theta = 0 \qquad (E_{18})$$

The natural frequency of vibraton, from Eq. (E_{18}), is given by

$$\omega_n = \sqrt{\frac{kR^2}{mR^2+J}} \qquad (E_{19})$$

Using Eq. (E_{12}), Eqs. (E_{18}) and (E_{19}) become

$$\frac{3}{2}mR^2\ddot{\theta} + kR^2\theta = 0 \qquad (E_{20})$$

$$\omega_n = \sqrt{\frac{k R^2}{\frac{3}{2} m R^2}} = \sqrt{\frac{2k}{3m}} \qquad (E_{21})$$

It can be seen that the two equations of motion, Eqs. (E_{10}) and (E_{18}), lead to the same natural frequency ω_n as shown in Eqs. (E_{14}) and (E_{21}).

2.117 Equation of motion: $m\ddot{x} + c\dot{x} + kx = 0$ (E.1)

(a) SI units (kg, N-s/m, N/m for m, c, k, respectively)

$m = 2$ kg, $c = 800$ N-s/m, $k = 4000$ N/m

Eq. (E.1) becomes

$$2\ddot{x} + 800\dot{x} + 4000x = 0 \qquad (E.2)$$

(b) British engineering units (slug, lb_f-s/ft, lb_f/ft for m, c, k)

m: 1 kg = 0.06852 slug

c: 1 N-s/m = 0.06852 lb_f-s/ft

(since 0.4 lb_f-s/ft = 5.837 N-s/m)

k: 1 N/m = 0.06852 lb_f/ft

Eq. (E.2) becomes

$$2(0.06852)\ddot{x} + 800(0.06852)\dot{x} + 4000(0.06852)x = 0 \qquad (E.3)$$

or $$2\ddot{x} + 800\dot{x} + 4000x = 0 \qquad (E.2)$$

(c) British absolute units (lb, poundal-s/ft, poundal/ft for m, c, k)

m: 1 kg = 2.2045 lb

c: $1 \frac{N-s}{m} = \frac{7.233 \text{ poundal-s}}{3.281 \text{ ft}} = 2.2045$ poundal-s/ft

k: $1 \frac{N}{m} = \frac{7.233 \text{ poundal}}{3.281 \text{ ft}} = 2.2045$ poundal/ft

Eq. (E.2) becomes

$$2(2.2045)\ddot{x} + 800(2.2045)\dot{x} + 4000(2.2045)x = 0 \qquad (E.4)$$

which can be seen to be same as Eq. (E.2).

(d) Metric engineering units (kg_f-s^2/m, kg_f-s/m, kg_f/m for m, c, k)

m: 1 kg = 0.10197 kg_f-s^2/m

$$c:\ 1\ \frac{N\text{-}s}{m} = \frac{\left(\frac{1}{9.807}\right) kg_f\text{-}s}{1\ m} = 0.10197\ kg_f\text{-}s/m$$

$$k:\ 1\ \frac{N}{m} = \frac{\left(\frac{1}{9.807}\right) kg_f}{1\ m} = 0.10197\ kg_f/m$$

Eq. (E.2) becomes

$$2(0.10197)\ddot{x} + 800(0.10197)\dot{x} + 4000(0.10197)x = 0 \qquad (E.5)$$

which can be seen to be same as Eq. (E.2).

(e) Metric absolute or cgs system (gram, dyne-s/cm, dyne/cm for m, c and k)

m: 1 kg = 1000 grams

$$c:\ 1\ \frac{N\text{-}s}{m} = \frac{10^5\ dyne\text{-}s}{10^2\ cm} = 1000\ dyne\text{-}s/cm$$

$$k:\ 1\ \frac{N}{m} = \frac{10^5\ dyne}{10^2\ cm} = 1000\ dyne/cm$$

Eq. (E.2) becomes

$$2(1000)\ddot{x} + 800(1000)\dot{x} + 4000(1000)x = 0 \qquad (E.6)$$

which can be seen to be same as Eq. (E.2).

(f) US Customory units (lb, lb_f-s/ft, lb_f/ft for m, c and k)

$$m:\ 1\ kg = 0.06852\ slug = 0.06252\ lb_f\text{-}s^2/ft$$
$$= 2.204\ lb_f/(32.2\ ft/s^2)$$

$$c:\ 1\ \frac{N\text{-}s}{m} = \frac{0.2248\ lb_f\text{-}s}{3.281\ ft} = 0.06852\ lb_f\text{-}s/ft$$

$$k:\ 1\ \frac{N}{m} = 0.2248\ lbf\text{-}s/3.281\ ft = 0.06852\ lb_f/ft$$

Eq. (E.2) becomes

$$2(0.06252)\ddot{x} + 800(0.06252)\dot{x} + 4000(0.06252)x = 0 \qquad (E.7)$$

which can be identified to be same as Eq. (E.2).

2.118

$m = 5$ kg, $c = 500$ N-s/m, $k = 5000$ N/m

Undamped natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5000}{5}} = 31.6228 \text{ rad/s}$$

critical damping constant: $c_c = 2\sqrt{km}$

$$= 2\sqrt{5000(5)}$$

$$= 316.2278 \text{ N-s/m}$$

Damping ratio:

$$\zeta = \frac{c}{c_c} = \frac{500}{316.2278} = 1.5811$$

Since it is overdamped, the system will not have damped frequency of vibration.

2.119

$m = 5$ kg, $c = 500$ N-s/m, $k = 50{,}000$ N/m

Undamped natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{50\,000}{5}} = 100 \text{ rad/s}$$

critical damping constant:

$$c_c = 2\sqrt{km} = 2(50000 * 5)^{\frac{1}{2}} = 1{,}000 \text{ N-s/m}$$

Damping ratio: $\zeta = \frac{c}{c_c} = \frac{500}{1000} = 0.5$

System is underdamped.

Damped natural frequency:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 100\sqrt{1-(0.5)^2}$$

$$= 86.6025 \text{ rad/s}$$

2.120 $m = 5$ kg, $c = 1000$ N-s/m, $k = 50\,000$ N/m

Undamped natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{50000}{5}} = 100 \text{ rad/s}$$

critical damping constant:

$$c_c = 2\sqrt{km} = 2\sqrt{50000(5)} = 1000 \text{ N-s/m}$$

Damping ratio:

$$\zeta = \frac{c}{c_c} = \frac{1000}{1000} = 1$$

system is critically damped.

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 100\sqrt{1-1^2} = 0$$

Damped natural frequency is zero.

2.121 Damped single d.o.f. system:

$m = 10$ kg, $k = 10\,000$ N/m, $\zeta = 0.1$ (underdamped)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10\,000}{10}} = 31.6228 \text{ rad/s}$$

Displacement of ~~mass~~ is given by Eq. (2.70 f):

$$x(t) = X\, e^{-\zeta \omega_n t} \cos(\omega_d t - \phi) \qquad (E.1)$$

where

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 31.6228\sqrt{1-0.01} = 31.4647 \text{ rad/s}$$

$$X = \left(x_0^2 \omega_n^2 + \dot{x}_0^2 + 2 x_0 \dot{x}_0 \zeta \omega_n\right)^{\frac{1}{2}} / \omega_d \qquad (2.73)$$

and

$$\phi = \tan^{-1}\left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{x_0 \omega_d}\right) \qquad (2.75)$$

(a) $x_0 = 0.2$ m, $\dot{x}_0 = 0$

$$X = \left\{(0.2)^2 (31.6228)^2\right\}^{\frac{1}{2}} / 31.4647 = 0.2010 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{0.1\,(31.6228)(0.2)}{0.2\,(31.4647)}\right) = \tan^{-1}(0.1005)$$

$$= 5.7391^\circ \text{ or } 0.1002 \text{ rad}$$

$$\therefore x(t) = 0.2010\, e^{-3.16228\, t} \cos(31.4647 t - 0.1002) \text{ m}$$

(b) $x_0 = -0.2$, $\dot{x}_0 = 0$

$$X = \left\{(-0.2)^2 (31.6228)^2\right\}^{\frac{1}{2}} / 31.4647 = 0.2010 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{0.1\,(31.6228)(-0.2)}{(-0.2)(31.4647)}\right) = \tan^{-1}(0.1005)$$

$$= 185.7391^\circ \text{ or } 3.2418 \text{ rad}$$

(since both numerator and denominator in Eq. (2.75) are negative, ϕ lies in third quadrant)

$$\therefore x(t) = 0.2010\, e^{-3.1623\, t} \cos(31.4647 t - 3.2418) \text{ m}$$

(c) $x_0 = 0, \; \dot{x}_0 = 0.2 \text{ m/s}$

$$X = \frac{\sqrt{(0.2)^2}}{31.4647} = 0.006356 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{0.2}{0}\right) = \tan(\infty) = 90^\circ \text{ or } 1.5708 \text{ rad}$$

$$\therefore x(t) = 0.006356 \, e^{-3.1623t} \cos(31.4647t - 1.5708) \text{ m}$$

2.122

Damped single d.o.f. system:

$m = 10$ kg, $k = 10,000$ N/m, $\zeta = 1.0$ (critically damped)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10,000}{10}} = 31.6228 \text{ rad/s}$$

Displacement of mass given by Eq. (2.80):

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)\, t\}\, e^{-\omega_n t}$$

(a) $x_0 = 0.2$ m, $\dot{x}_0 = 0$

$$x(t) = \{0.2 + 31.6228\,(0.2)\, t\}\, e^{-31.6228\, t}$$

$$= (0.2 + 6.32456\, t)\, e^{-31.6228\, t} \quad \text{m}$$

(b) $x_0 = -0.2$ m, $\dot{x}_0 = 0$

$$x(t) = \{-0.2 + 31.6228\,(-0.2)\, t\}\, e^{-31.6228\, t}$$

$$= -(0.2 + 6.32456\, t)\, e^{-31.6228\, t} \quad \text{m}$$

(c) $\dot{x}_0 = 0.2$ m/s, $x_0 = 0$

$$x(t) = \{0.2\, t\}\, e^{-31.6228\, t}$$

$$= 0.2\, t\, e^{-31.6228\, t} \quad \text{m}$$

(2.123) Single d.o.f. system:

$m = 10$ kg, $k = 10\,000$ N/m, $\zeta = 2.0$ (over damped)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10\,000}{10}} = 31.6228 \text{ rad/s}$$

Displacement of mass given by Eq. (2.81):

$$x(t) = C_1 \, e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} + C_2 \, e^{(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t}$$

where

$$C_1 = \frac{x_0 \,\omega_n \left(\zeta + \sqrt{\zeta^2 - 1}\right) + \dot{x}_0}{2\,\omega_n \sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0 \,\omega_n \left(\zeta - \sqrt{\zeta^2 - 1}\right) - \dot{x}_0}{2\,\omega_n \sqrt{\zeta^2 - 1}}$$

(a) $x_0 = 0.2$ m, $\dot{x}_0 = 0$

$$C_1 = \frac{0.2\,(31.6228)(2 + \sqrt{3})}{2\,(31.6228)\sqrt{3}} = 0.2155$$

$$C_2 = \frac{-0.2\,(31.6228)(2 - \sqrt{3})}{2\,(31.6228)\sqrt{3}} = -0.01547$$

$$x(t) = 0.2155 \, e^{(-2 + \sqrt{3})(31.6228)\,t} - 0.01547 \, e^{(-2 - \sqrt{3})(31.6228)\,t}$$

$$= 0.2155 \, e^{-8.4749\,t} - 0.01547 \, e^{-118.0163\,t} \text{ m}$$

(b) $x_0 = -0.2$ m, $\dot{x}_0 = 0$

$$C_1 = \frac{-0.2\,(31.6228)(2 + \sqrt{3})}{2\,(31.6228)\sqrt{3}} = -0.2155$$

$$C_2 = \frac{0.2\,(31.6228)(2 - \sqrt{3})}{2\,(31.6228)\sqrt{3}} = 0.01547$$

$$x(t) = -0.2155\, e^{(-2+\sqrt{3})(31.6228)\,t}$$

$$+ 0.01547\, e^{(-2-\sqrt{3})(31.6228)\,t}$$

$$= -0.2155\, e^{-8.4749\,t} + 0.01547\, e^{-18.0163\,t} \quad \text{m}$$

(c) $x_0 = 0,\ \dot{x}_0 = 0.2$ m/s

$$C_1 = \frac{0.2}{2\,(31.6228)\sqrt{3}} = 0.001826$$

$$C_2 = \frac{-0.2}{2\,(31.6228)\sqrt{3}} = -0.001826$$

$$x(t) = 0.001826 \left\{ e^{(-2+\sqrt{3})\,31.6228\,t} - e^{(-2-\sqrt{3})\,31.6228\,t} \right\}$$

$$= 0.001826 \left\{ e^{-8.4749\,t} - e^{-118.0163\,t} \right\} \quad \text{m}$$

(2.124) Torsional stiffness of the shaft of diameter d and length l is given by

$$k_t = \frac{G I_0}{l} = \frac{G}{l} \frac{\pi}{32} d^4 \qquad (1)$$

Since the shafts on the two sides of the disk act as parallel torsional springs (because the torque on the disk is shared by the two torsional springs), the resultant spring constant is given by

$$k_{t_{eq}} = k_{t_1} + k_{t_2} = \frac{G \pi d_1^4}{32 l_1} + \frac{G \pi d_2^4}{32 l_2}$$

$$= \frac{G \pi d^4}{32} \left(\frac{1}{l_1} + \frac{1}{l_2} \right)$$

$$= \frac{G \pi d^4}{32} \left(\frac{l_1 + l_2}{l_1 l_2} \right) \qquad (2)$$

Using $l_1 = l_2 = \frac{l}{2}$, Eq. (2) becomes

$$k_{t_{eq}} = \frac{G \pi d^4}{32} \frac{\left(\frac{l}{2} + \frac{l}{2}\right)}{\left(\frac{l^2}{4}\right)} = \frac{G \pi d^4}{8 l} \qquad (3)$$

Natural frequency of the disk in torsional vibration is given by

$$\omega_n = \sqrt{\frac{k_{t_{eq}}}{J}} = \sqrt{\frac{\pi G d^4}{8 l J}}$$

2.125 For pendulum, $\omega_n = \sqrt{g/l}$ in vaccum $= 0.5$ Hz $= \pi$ rad/sec

$$l = g/\pi^2 = 9.81/\pi^2 = 0.9940 \text{ m}$$

$\omega_d = \omega_n \sqrt{1-\zeta^2}$ in viscous medium $= 0.45$ Hz $= 0.9\pi$ rad/sec

$$\therefore \zeta^2 = \frac{\omega_n^2 - \omega_d^2}{\omega_n^2} = \pi^2\left(\frac{1-0.81}{\pi^2}\right) = 0.19$$

$\zeta = 0.4359$; System is under damped.

Equation of motion: $\quad m l^2 \ddot{\theta} + c_t \dot{\theta} + m g l \theta = 0$

$$c_{ct} = 2(m l^2)\omega_n = 2(1)(0.994)^2(\pi) = 6.2080$$

$$\zeta = \frac{c_t}{c_{ct}} = 0.4359$$

since $\omega_n = \sqrt{\frac{g}{l}} = \pi$, $\quad l = g/\omega_n^2 = 9.81/\pi^2 = 0.9939$ m

$$c_t = \zeta\, c_{ct} = \zeta(2)(m l^2)\,\omega_n = 0.4359(2)(1 * 0.9939^2)(\pi)$$

$$= 2.7061 \text{ N-m-s/rad}$$

2.126 From Eq. (2.85),

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \ln(18) \Rightarrow \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.8904$$

$$\zeta = \left\{\frac{(2.8904)^2}{(2.8904)^2 + 4\pi^2}\right\}^{\frac{1}{2}} = 0.4179$$

(a) If damping is doubled, $\zeta_{new} = 0.8358$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.8358)}{\sqrt{1-(0.8358)^2}} = 9.5656$$

$$\therefore \quad \frac{x_j}{x_{j+1}} = 14265.362$$

(b) If damping is halved, $\zeta = 0.2090$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.2090)}{\sqrt{1-(0.2090)^2}} = 1.3428$$

$$\therefore \quad \frac{x_j}{x_{j+1}} = 3.8296$$

2.127

$$x(t) = X\, e^{-\zeta\omega_n t} \sin\omega_d t \quad \text{where} \quad \omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

For maximum or minimum of $x(t)$,

$$\frac{dx}{dt} = X\, e^{-\zeta\omega_n t}\left(-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t\right) = 0$$

As $e^{-\zeta\omega_n t} \neq 0$ for finite t,

$$-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t = 0$$

i.e. $$\tan\omega_d t = \sqrt{1-\zeta^2}/\zeta$$

Using the relation

$$\sin\omega_d t = \pm\frac{\tan\omega_d t}{\sqrt{1+\tan^2\omega_d t}} = \pm\frac{\left(\sqrt{1-\zeta^2}/\zeta\right)}{\sqrt{1+\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)^2}} = \pm\sqrt{1-\zeta^2}$$

we obtain

$$\sin\omega_d t = \sqrt{1-\zeta^2}, \quad \cos\omega_d t = \zeta$$

and

$$\sin\omega_d t = -\sqrt{1-\zeta^2}, \quad \cos\omega_d t = -\zeta$$

$$\frac{d^2x}{dt^2} = X\, e^{-\zeta\omega_n t}\left[\zeta^2\omega_n^2 \sin\omega_d t - 2\zeta\omega_n\omega_d \cos\omega_d t - \omega_d^2 \sin\omega_d t\right]$$

when $\sin\omega_d t = \sqrt{1-\zeta^2}$ and $\cos\omega_d t = \zeta$,

$$\frac{d^2x}{dt^2} = -X\, e^{-\zeta\omega_n t}\,\omega_n^2\sqrt{1-\zeta^2} < 0$$

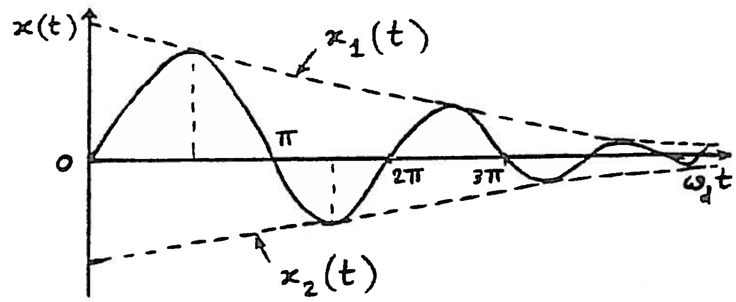
$\therefore$ $\sin\omega_d t = \sqrt{1-\zeta^2}$ corresponds to maximum of $x(t)$.

When $\sin\omega_d t = -\sqrt{1-\zeta^2}$ and $\cos\omega_d t = -\zeta$,

$$\frac{d^2x}{dt^2} = X\, e^{-\zeta\omega_n t}\, \omega_n^2 \sqrt{1-\zeta^2} > 0$$

$\therefore \sin \omega_d t = -\sqrt{1-\zeta^2}$ corresponds to minimum of $x(t)$.

Enveloping curves:



Let the curve passing through the maximum (or minimum) points be

$$x(t) = C\, e^{-\zeta\omega_n t}$$

For maximum points, $t_{max} = \dfrac{\sin^{-1}\left(\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

$$C\, e^{-\zeta\omega_n t_{max}} = X\, e^{-\zeta\omega_n t_{max}} \sin \omega_d t_{max}$$

i.e.

$$C = X\sqrt{1-\zeta^2}$$

$$\therefore x_1(t) = X\sqrt{1-\zeta^2}\; e^{-\zeta\omega_n t}$$

Similarly for minimum points, $t_{min} = \dfrac{\sin^{-1}\left(-\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

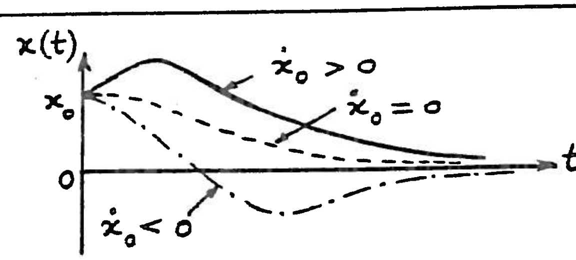
$$C\, e^{-\zeta\omega_n t_{min}} = X\, e^{-\zeta\omega_n t_{min}} \sin \omega_d t_{min}$$

i.e.

$$C = -X\sqrt{1-\zeta^2}$$

$$\therefore x_2(t) = -X\sqrt{1-\zeta^2}\; e^{-\zeta\omega_n t}$$

2.128 $x(t) = \left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right] e^{-\omega_n t}$ ---- (E_1)



For $x_0 > 0$, graph of Eq. (E_1) is shown for different $\dot{x}_0$. We assume $\dot{x}_0 > 0$ as it is the only case that gives a maximum.

For maximum of $x(t)$,

$$\frac{dx}{dt} = e^{-\omega_n t} \left\{ -(\dot{x}_0 + \omega_n x_0)\,\omega_n t + \dot{x}_0 \right\} = 0$$

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} \quad \text{---- } (E_2)$$

$$\frac{d^2x}{dt^2} = -e^{-\omega_n t}\left\{2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2(\dot{x}_0 + \omega_n x_0)t\right\} \quad \text{----}(E_3)$$

(E_2) and (E_3) give

$$\left.\frac{d^2x}{dt^2}\right|_{t=t_m} = -e^{-\omega_n t_m}\left\{2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2(\dot{x}_0 + \omega_n x_0)t_m\right\}$$

$$= -e^{-\omega_n\left(\frac{\dot{x}_0}{\omega_n(\dot{x}_0+\omega_n x_0)}\right)}\left\{\omega_n \dot{x}_0 + \omega_n^2 x_0\right\} \quad \text{----}(E_4)$$

For $x_0 > 0$ and $\dot{x}_0 > 0$, $\left.\frac{d^2x}{dt^2}\right|_{t_m} < 0$

Hence t_m given by Eq. (E_2) corresponds to a maximum of $x(t)$.

$$x\Big|_{t=t_m} = \left\{x_0 + (\dot{x}_0 + \omega_n x_0)\frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)}\right\}e^{-\omega_n t_m}$$

$$= \left(x_0 + \frac{\dot{x}_0}{\omega_n}\right)e^{-\left(\frac{\dot{x}_0}{\dot{x}_0 + \omega_n x_0}\right)} \quad \text{----}(E_5)$$

2.129 Equation (2.92) can be expressed as $\delta = \frac{1}{m}\ln\left(\frac{x_0}{x_m}\right)$

For half cycle, $m = \frac{1}{2}$ and hence $\delta = 2\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 2\ln\left(\frac{1}{0.15}\right)$

$= 3.7942$

Necessary damping ratio ζ_0 is

$$\zeta_0 = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{3.7942^2}{\sqrt{4\pi^2 + 3.7942^2}}$$

$= 0.5169$

(a) If $\zeta = \frac{3}{4}\zeta_0 = 0.3877$, the overshoot can be determined by finding δ from Eq. (2.85):

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.3877)}{\sqrt{1-0.3877^2}} = 2.6427 = 2\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$$\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 1.32135$$

$$x_{\frac{1}{2}} = x_0 / e^{1.32135} = 0.266775\ x_0$$

$\therefore$ overshoot is 26.6775%

(b) If $\zeta = \frac{5}{4}\zeta_0 = 0.6461$, δ is given by

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.6461)}{\sqrt{1-(0.6461)^2}} = 5.3189 = 2\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$\frac{x_0}{x_{\frac{1}{2}}} = 14.2888$, $x_{\frac{1}{2}} = 0.0700\ x_0$

$\therefore$ overshoot = 7%

2.130 (i) (a) Viscous damping, (b) Coulomb damping.

(iii) (a) $\tau_d = 0.2$ sec, $f_d = 5$ Hz, $\omega_d = 31.416$ rad/sec.
(b) $\tau_n = 0.2$ sec, $f_n = 5$ Hz, $\omega_n = 31.416$ rad/sec.

(ii) (a) $\frac{x_i}{x_{i+1}} = e^{\zeta \omega_n \tau_d}$

$$\ln\left(\frac{x_i}{x_{i+1}}\right) = \ln 2 = 0.6931 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

or $39.9590\ \zeta^2 = 0.4804$ or $\zeta = 0.1096$

Since $\omega_d = \omega_n \sqrt{1-\zeta^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(31.6065)^2 = 5.0916\ (10^4) \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

Hence $c = 2\,m\,\omega_n\,\zeta = 2\left(\frac{500}{9.81}\right)(31.6065)\,(0.1096) = 353.1164$ N–s/m

(b) From Eq. (2.135):

$$k = m\,\omega_n^2 = \frac{500}{9.81}(31.416)^2 = 5.0304\ (10^4) \text{ N/m}$$

Using N = W = 500 N,

$$\mu = \frac{0.002\ k}{4\ W} = \frac{(0.002)\,(5.0304\,(10^4))}{4\,(500)} = 0.0503$$

2.131 (a) $c_c = 2\sqrt{km} = 2\sqrt{5000 \times 50} = 1000$ N-s/m

(b) $c = c_c/2 = 500$ N-s/m

$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m}}\sqrt{1-\left(\frac{c}{c_c}\right)^2} = \sqrt{\frac{5000}{50}}\sqrt{1-\left(\frac{1}{2}\right)^2}$

$= 8.6603$ rad/sec

(c) From Eq. (2.85), $\delta = \frac{2\pi}{\omega_d}\left(\frac{c}{2m}\right) = \frac{2\pi}{8.6603}\left(\frac{500}{2 \times 50}\right)$

$= 3.6276$

2.132

To find the maximum of x(t), we set the derivative of x(t) with respect to time t equal to zero. Using Eq. (2.70),

$$x(t) = X\, e^{-\varsigma\,\omega_n t} \sin(\omega_d\, t - \phi)$$

$$\frac{dx(t)}{dt} = -X\,\varsigma\,\omega_n\, e^{-\varsigma\,\omega_n t} \sin(\omega_d\, t - \phi) + \omega_d\, X\, e^{-\varsigma\,\omega_n t} \cos(\omega_d\, t - \phi) = 0 \qquad \text{(E1)}$$

i.e.,

$$X\, e^{-\varsigma\,\omega_n t}\left[-\varsigma\,\omega_n \sin(\omega_d\, t - \phi) + \omega_d \cos(\omega_d\, t - \phi)\right] = 0 \qquad \text{(E2)}$$

Since $X\, e^{-\varsigma\,\omega_n t} \neq 0$,

we set the quantity inside the square brackets equal to zero. This yields

$$\tan(\omega_d\, t - \phi) = \frac{\omega_d}{\varsigma\,\omega_n} = \frac{\sqrt{1-\varsigma^2}\;\omega_n}{\varsigma\,\omega_n} = \frac{\sqrt{1-\varsigma^2}}{\varsigma} \qquad \text{(E3)}$$

or

$$\omega_d\, t - \phi = \tan^{-1}\left(\frac{\sqrt{1-\varsigma^2}}{\varsigma}\right) \qquad \text{(E4)}$$

(a)

In the present case, m = 2000 kg, $x_0 = 0$, $v = \dot{x}_0 = 10$ m/s, k = 80,000 N/m and c = 20,000 N-s/m and hence

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{80{,}000}{2000}} = 6.3245$ rad/s, $c_c = 2\sqrt{k\,m} = 2\sqrt{(80{,}000)(2000)} = 25{,}298.221$ N-s/m, $\varsigma = c/c_c = 0.7906$, $\omega_d = \omega_n \sqrt{1-\varsigma^2} = (6.3245)\sqrt{1-(0.7906)^2} = 3.8727$ rad/s,

$$\tan^{-1}\left(\frac{\sqrt{1-\varsigma^2}}{\varsigma}\right) = \tan^{-1}\left(\frac{\sqrt{1-0.7906^2}}{0.7906}\right) = \tan^{-1}(0.7745) = 0.6590 \text{ rad.}$$

For the given initial conditions, Eqs. (2.75) and (2.73) give

$$\phi = \tan^{-1}\left(\frac{10}{0}\right) = \tan^{-1}(\infty) = \frac{\pi}{2} = 1.5708 \text{ rad and } X = \frac{10}{3.8727} = 2.5822 \text{ m}$$

(b) Equation (E4) can be rewritten as

$$3.8727\, t = \phi + 0.6590 = 1.5708 + 0.6590 = 2.2298$$

which gives $t = t_{max}$ as $t_{max} = 0.5758\text{s}$.

(a) Using the value of t_{max} , Eq. (2.70) gives the maximum displacement of the car after engaging the springs and damper as

$$x(t_{max}) = x_{max} = 2.5822\, e^{-0.7906\,(6.3245)(0.5758)} \cos\,(3.8727*0.5758 - 1.5708)$$

$$= 2.5822\ (0.0562)\ \cos(0.6591) = 2.5822\ (0.0562)\ \cos(37.7635^0)$$

$$= 0.1147 \text{ m.}$$

Note: The condition used in Eq. (E1) is also valid for the minimum of x(t). As such, the sufficiency condition for the maximum of x(t) is to be verified. This implies that the second derivative, $\frac{d^2x(t)}{dt^2}$ at $t = t_{max}$, should be negative for maximum of x(t).

2.133 $\omega_n = 200$ cycles/min $= 20.944$ rad/sec , $\omega_d = 180$ cycles/min $= 18.8496\ \frac{\text{rad}}{\text{sec}}$

$J_0 = 0.2\ \text{kg-m}^2$

Since $\omega_d = \sqrt{1-\zeta^2}\ \omega_n$, $\zeta = \sqrt{1-\left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1-\left(\frac{18.8496}{20.944}\right)^2} = 0.4359$

$$= \frac{c_t}{(c_t)_{cri}} = \frac{c_t}{2\, J_0\, \omega_n}$$

$$c_t = 2\, J_0\, \omega_n\, \zeta = 2(0.2)(20.944)(0.4359)$$

$$= 3.6518\ \text{N-m-s/rad}$$

Eq. (2.72) can be used to obtain $\theta(t)$ for $\dot{\theta}_0 = 0$, $\theta_0 = 2^\circ = 0.03491$ rad and $t = \tau_d = \frac{2\pi}{\omega_d} = 0.3333$ sec ,

$$\theta(t) = e^{-\zeta\omega_n t}\ \theta_0 \left\{ \cos\omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin\omega_d t \right\}$$

$$= e^{-(0.4359)(20.944)(0.3333)}\ (0.03491) \left\{ \cos 18.8496 \times 0.3333 + \frac{0.4359 \times 20.944}{18.8496} \sin 18.8496 \times 0.3333 \right\}$$

$$= 0.001665 \text{ rad} = 0.09541^\circ$$

2.134 Assume that the bicycle and the boy fall as a rigid body by 5 cm at point A. Thus the mass (m_{eq}) will be subjected to an initial downward displacement of 5 cm (t = 0 assumed at point A):

$$x_0 = 0.05 \text{ m},\ \dot{x}_0 = 0$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{(50000)\ (9.81)}{800}} = 24.7614 \text{ rad/sec}$$

$$c_c = 2\, m\, \omega_n = 2 \left(\frac{800}{9.81} \right) (24.7614) = 4038.5566 \text{ N--s/m}$$

$$\zeta = \frac{c}{c_c} = \frac{1000.0}{4038.5566} = 0.2476 \text{ (underdamping)}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 24.7614 \sqrt{1 - 0.2476^2} = 23.9905 \text{ rad/sec}$$

Response of the system:

$$x(t) = X\, e^{-\zeta \omega_n t} \sin (\omega_d\, t + \phi)$$

where $X = \left\{ x_0^2 + \left(\dfrac{\dot{x}_0 + \zeta\, \omega_n\, x_0}{\omega_d} \right)^2 \right\}^{\frac{1}{2}}$

$$= \left\{ (0.05)^2 + \left(\frac{(0.2476)\,(24.7614)\,(0.05)}{23.9905} \right)^2 \right\}^{\frac{1}{2}} = 0.051607 \text{ m}$$

and $\phi = \tan^{-1} \left(\dfrac{x_0\, \omega_d}{\dot{x}_0 + \zeta\, \omega_n\, x_0} \right) = \tan^{-1} \left(\dfrac{0.05\,(23.9905)}{0.2476\,(24.7614)\,(0.05)} \right) = 75.6645^\circ$

Thus the displacement of the boy (positive downward) in vertical direction is given by

$$x(t) = 0.051607\, e^{-6.1309\, t} \sin (23.9905\, t + 75.6645^\circ) \text{ m}$$

2.135 Reduction in amplitude of viscously damped free vibration in one cycle = 0.5 in.

$$\frac{x_1}{x_2} = \frac{6.0}{5.5} = 1.0909; \quad \ln \frac{x_1}{x_2} = 0.08701 = \frac{2\,\pi\,\zeta}{\sqrt{1 - \zeta^2}}$$

i.e., $0.007571\,(1 - \zeta^2) = 39.478602\, \zeta^2$ or $\zeta = 0.013847$

2.136 $\tau_d = 0.2 \text{ sec} = \dfrac{2\pi}{\omega_d}$, $\omega_d = 31.416$ rad/sec

From Eq. (2.92) $\quad \delta = \dfrac{1}{50} \ln 10 = 0.04605$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.04605}{\sqrt{(2\pi)^2 + 0.04605^2}} = 0.007329$$

When damping is neglected,

$$\omega_n = \omega_d / \sqrt{1 - \zeta^2} = 31.417 \text{ rad/sec} \; ; \quad \tau_n = \frac{2\pi}{\omega_n} = 0.19999 \text{ sec}$$

Proportional decrease in period $= \left(\dfrac{0.2 - 0.19999}{0.2} \right) = 0.00005$

2.137 For critically damped system, Eq. (2.80) gives

$$x(t) = \{ x_0 + (\dot{x}_0 + \omega_n\, x_0)\, t \}\, e^{-\omega_n t} \qquad (E_1)$$

$$\dot{x}(t) = e^{-\omega_n t} \{ \dot{x}_0 - \dot{x}_0\, \omega_n t - \omega_n^2\, x_0\, t \} \qquad (E_2)$$

Let t_m = time at which $x = x_{max}$ and $\dot{x} = 0$ occur. Here $x_0 = 0$ and $\dot{x}_0$ = initial recoil velocity. By setting $\dot{x}(t) = 0$, Eq. (E_2) gives

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} = \frac{\dot{x}_0}{\omega_n \dot{x}_0} = \frac{1}{\omega_n} \qquad (E_3)$$

With Eq. (E_3) for t_m and $x_0 = 0$, (E_1) gives

$$x_{max} = \dot{x}_0 t_m e^{-\omega_n t_m} = \frac{\dot{x}_0 e^{-1}}{\omega_n} \qquad (E_4)$$

Using $x_{max} = 0.5$ m and $\dot{x}_0 = 10$ m/s, Eq. (E_4) gives

$\omega_n = \dot{x}_0 / (x_{max}\, e) = 10/(0.5 * 2.7183) = 7.3575$ rad/s

When mass of gun is 500 kg, stiffness of spring is

$k = \omega_n^2 m = (7.3575)^2 (500) = 27{,}066.403$ N/m

Note: Other values of $\dot{x}_0$ and m can also be used to find k. Finally, the stiffness corresponding to least cost can be chosen.

2.138 $k = 5000$ N/m, $\quad c_c = 0.2$ N-s/mm $= 200$ N-s/m

$= 2\sqrt{k m} = 2\sqrt{5000\, m}$

$m = 2$ kg

$\omega_n = \sqrt{k/m} = \sqrt{5000/2} = 50$ rad/sec

Logarithmic decrement $= \delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.0$

i.e., $\zeta = \dfrac{c}{c_c} = 0.3033$ and $c = 0.3033\,(0.2) = 60.66$ N-s/m

Assuming $x_0 = 0$ and $\dot{x}_0 = 1$ m/s,

$$x(t) = e^{-\zeta\omega_n t} \frac{\dot{x}_0}{\omega_n\sqrt{1-\zeta^2}} \sin\sqrt{1-\zeta^2}\,\omega_n t$$

For x_{max}, $\omega_n t \approx \pi/2$ and $\sin\sqrt{1-\zeta^2}\,\omega_n t \approx 1$

$$\therefore \quad x_{max} \simeq e^{-0.3033(\pi/2)} \frac{1}{50\sqrt{1-0.3033^2}}(1) = 0.01303 \text{ m}$$

2.139 For an overdamped system, Eq. (2.81) gives

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 e^{\omega_d t} + C_2 e^{-\omega_d t}\right) \qquad (E_1)$$

Using the relations $\quad e^{\pm x} = \cosh x \pm \sinh x \qquad (E_2)$

Eq. (E_1) can be rewritten as

$$x(t) = e^{-\zeta\omega_n t}\left(C_3 \cosh\omega_d t + C_4 \sinh\omega_d t\right) \qquad (E_3)$$

where $C_3 = C_1 + C_2$ and $C_4 = C_1 - C_2$.

Differentiating (E_3),

$$\dot{x}(t) = e^{-\zeta\omega_n t}\left[C_3\,\omega_d \sinh\omega_d t + C_4\,\omega_d \cosh\omega_d t\right]$$

$$-\zeta\omega_n\, e^{-\zeta\omega_n t}\left[C_3 \cosh\omega_d t + C_4 \sinh\omega_d t\right] \qquad (E_4)$$

Initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ with (E_3) and (E_4) give

$$C_3 = x_0\,, \quad C_4 = \left(\dot{x}_0 + \zeta\omega_n x_0\right)/\omega_d \qquad (E_5)$$

Thus (E_3) becomes

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sinh\omega_d t\right) + \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t}\sinh\omega_d t \qquad (E_6)$$

(i) When $\dot{x}_0 = 0$, Eq. (E_6) gives

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sinh\omega_d t\right) \qquad (E_7)$$

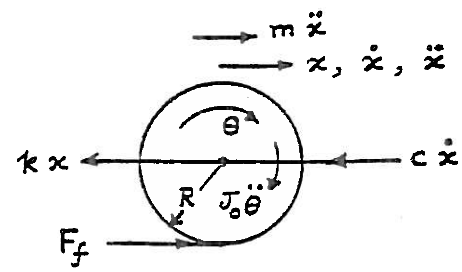
since $e^{-\zeta\omega_n t}$, $\cosh\omega_d t$, $\frac{\zeta\omega_n}{\omega_d}$ and $\sinh\omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

(ii) When $x_0 = 0$, Eq. (E_6) gives

$$x(t) = \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t}\sinh\omega_d t \qquad (E_8)$$

Here also, ω_d, $e^{-\zeta\omega_n t}$ and $\sinh\omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

2.140 Newton's second law of motion:



$$\sum F = m\ddot{x} = -kx - c\dot{x} + F_f \qquad (1)$$

$$\sum M = J_0\,\ddot{\theta} = -F_f\,R \qquad (2)$$

where F_f = friction force.

Using $J_0 = \frac{mR^2}{2}$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\frac{1}{2R}\left(mR^2\right)\frac{\ddot{x}}{R} = -\frac{1}{2}m\ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) yields:

$$\frac{3}{2}\,m\,\ddot{x}+c\,\dot{x}+k\,x=0 \qquad (4)$$

The undamped natural frequency is: $\omega_n=\sqrt{\frac{2\,k}{3\,m}}$ (5)

2.141 Newton's second law of motion: (measuring x from static equilibrium position of cylinder)

$$\sum F=m\,\ddot{x}=-k\,x-c\,\dot{x}-k\,x+F_f \qquad (1)$$

$$\sum M=J_0\,\ddot{\theta}=-F_f\,R \qquad (2)$$

where F_f = friction force. Using $J_0=\frac{1}{2}\,m\,R^2$ and $\ddot{\theta}=\frac{\ddot{x}}{R}$, Eq. (2) gives

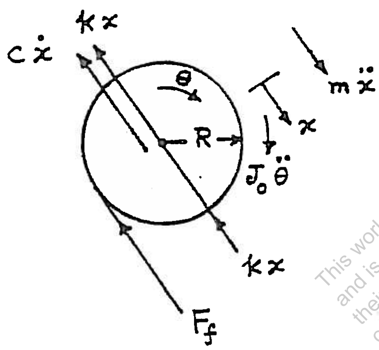
$$F_f=-\frac{1}{2}\,m\,\ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) gives

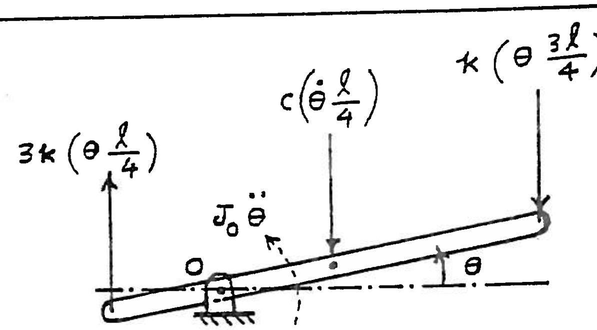
$$\frac{3}{2}\,m\,\ddot{x}+c\,\dot{x}+2\,k\,x=0 \qquad (4)$$

Undamped natural frequency of the system:

$$\omega_n=\sqrt{\frac{4\,k}{3\,m}} \qquad (4)$$



2.142



Consider a small angular displacement of the bar θ about its static equilibrium position. Newton's second law gives:

$$\sum M=J_0\,\ddot{\theta}=-k\left(\theta\,\frac{3\,\ell}{4}\right)\left(\frac{3\,\ell}{4}\right)-c\left(\dot{\theta}\,\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right)-3\,k\left(\theta\,\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right)$$

$$\text{i.e.,}\quad J_0\,\ddot{\theta}+\frac{c\,\ell^2}{16}\,\dot{\theta}+\frac{3}{4}\,k\,\ell^2\,\theta=0$$

where $J_0 = \frac{7}{48} m \ell^2$. The undamped natural frequency of torsional vibration is given by:

$$\omega_n = \sqrt{\frac{3 k \ell^2}{4 J_0}} = \sqrt{\frac{36 k}{7 m}}$$

2.143 Let δx = virtual displacement given to cylinder. Virtual work done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta})(\delta\theta) - (m \ddot{x}) \delta x = -(J_0 \ddot{\theta})(\frac{\delta x}{R}) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x$
Damping force: $\delta W_d = -(c \dot{x}) \delta x$
By setting the sum of virtual works equal to zero, we obtain:

$$-\frac{J_0}{R}\left(\frac{\ddot{x}}{R}\right) - m \ddot{x} - k x - c \dot{x} = 0 \quad \text{or} \quad \frac{3}{2} m \ddot{x} + c \dot{x} + k x = 0$$

2.144 Let δx = virtual displlacement given to cyllinder from its static equillibrium position. Virtualll works done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta - (m \ddot{x}) \delta\theta = -(J_0 \frac{\ddot{x}}{R})(\frac{\delta x}{R}) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x - (k x) \delta x = -2 k x \delta x$

Damping force: $\delta W_d = -(c \dot{x}) \delta x$
By setting the sum of virtual works equal to zero, we find

$$-\frac{J_0}{R} \frac{\ddot{x}}{R} - m \ddot{x} - 2 k x - c \dot{x} = 0 \tag{1}$$

Using $J_0 = \frac{1}{2} m R^2$, Eq. (1) can be rewritten as

$$\frac{3}{2} m \ddot{x} + c \dot{x} + 2 k x = 0 \tag{2}$$

2.145 See figure given in the solution of Problem 2.114. Let $\delta\theta$ be virtuall angular displacement given to the bar about its static equilibrium position. Virtual works done by various forces:
Inertia force: $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta$
Spring forces:

$$\delta W_s = -(k \theta \frac{3\ell}{4})(\frac{3\ell}{4} \delta\theta) - (3 k \theta \frac{\ell}{4})(\frac{\ell}{4} \delta\theta) = -\left(\frac{3}{4} k \ell^2 \theta\right) \delta\theta$$

Damping force: $\delta W_d = -(c \dot{\theta} \frac{\ell}{4})(\frac{\ell}{4} \delta\theta)$
By setting the sum of virtual works equal to zero, we get the equation of motion as:

$$J_0 \ddot{\theta} + c \frac{\ell^2}{16} \dot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

2.146 See solution of Problem 2.93. When wooden prism is given a displacement x, equation of motion becomes: $m\ddot{x}$ + restoring force $= 0$ where m = mass of prism = 40 kg and restoring force = weight of fluid displaced $= \rho_0\, g\, a\, b\, x = \rho_0\,(9.81)\,(0.4)\,(0.6)\,x = 2.3544\,\rho_0\,x$ where ρ_0 is the density of the fluid. Thus the equation of motion becomes:

$$40\,\ddot{x} + 2.3544\,\rho_0\,x = 0$$

$$\text{Natural frequency} = \omega_n = \sqrt{\frac{2.3544\,\rho_0}{40}}$$

$$\text{Since } \tau_n = \frac{2\,\pi}{\omega_n} = 0.5, \text{we find}$$

$$\omega_n = \frac{2\,\pi}{0.5} = 4\,\pi = \sqrt{\frac{2.3544\,\rho_0}{40}}$$

Hence $\rho_0 = 2682.8816\ \mathrm{kg/m^3}$.

2.147 Let x = displacement of mass and P = tension in rope on the left of mass. Equations of motion:

$$\sum F = m\,\ddot{x} = -k\,x - P \quad \text{or} \quad P = -m\,\ddot{x} - k\,x \tag{1}$$

$$\sum M = J_0\,\ddot{\theta} = P\,r_2 - c\,(\dot{\theta}\,r_1)\,r_1 \tag{2}$$

Using Eq. (1) in (2), we obtain

$$-(m\,\ddot{x} + k\,x)\,r_2 - c\,\dot{\theta}\,r_1^2 = J_0\,\ddot{\theta} \tag{3}$$

With $x = \theta\,r_2$, Eq. (3) can be written as:

$$(J_0 + m\,r_2^2)\,\ddot{\theta} + c\,r_1^2\,\dot{\theta} + k\,r_2^2\,\theta = 0 \tag{4}$$

For given data, Eq. (4) becomes

$$[5 + 10\,(0.25)^2]\,\ddot{\theta} + c\,(0.1^2)\,\dot{\theta} + k\,(0.25)^2\,\theta = 0$$

$$\text{or} \quad 5.625\,\ddot{\theta} + 0.01\,c\,\dot{\theta} + 0.0625\,k\,\theta = 0 \tag{5}$$

Since amplitude is reduced by 80% in 10 cycles,

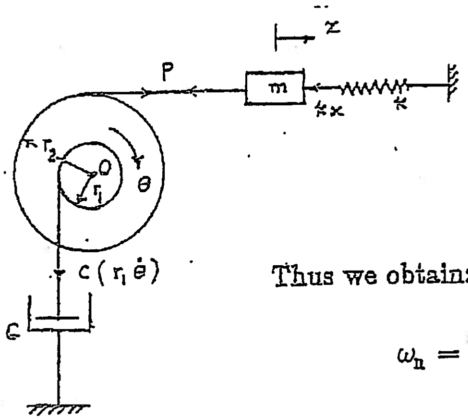
$$\frac{x_1}{x_{11}} = \frac{1.0}{0.2} = 5 = e^{10\,\zeta\,\omega_n\,\tau_d}$$

$$\ln\frac{x_1}{x_{11}} = \ln 5 = 1.6094 = 10\,\zeta\,\omega_n\,\tau_d \tag{6}$$

Since the natural frequency (assumed to be undamped torsional vibration frequency) is 5 Hz, $\omega_n = 2\,\pi\,(5) = 31.416$ rad/sec. Also

$$\tau_d = \frac{1}{f_d} = \frac{2\,\pi}{\omega_d} = \frac{2\,\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{0.2}{\sqrt{1-\zeta^2}} \tag{7}$$

Eq. (6) gives



$$1.6094 = 10\,\zeta\,(31.416)\left(\frac{0.2}{\sqrt{1-\zeta^2}}\right) = \frac{62.832\,\zeta}{\sqrt{1-\zeta^2}}$$

i.e., $\sqrt{1-\zeta^2} = \dfrac{62.832}{1.6094}\,\zeta = 39.0406\,\zeta$

i.e., $\zeta = 0.02561$

Thus we obtain:

$$\omega_n = \sqrt{\frac{0.0625\ k}{5.625}} = 31.416 \quad \text{or} \quad k = 8.8827\ (10^4)\ \text{N/m}$$

$$\zeta = 0.02561 = \frac{c_{eq}}{c_{eq\,cri}} = \frac{c_{eq}}{2\,m_{eq}\,\omega_n} = \frac{0.01\,c}{2\,(5.625)\,(31.4161)}$$

or $c = 905.1342$ N-s/m

2.148 Torque $= 2 \times 10^{-3}$ N-m

angle $= 50^\circ = 80$ divisions

For a torsional system, Eq. (2.84) gives

$$\frac{\theta_1}{\theta_2} = e^{\zeta\,\omega_n\,\tau_d} \qquad (E_1)$$

(b) For one cycle, $\tau_d = 2$ sec and (E_1) gives

$$\frac{80}{5} = e^{2\zeta\omega_n} \quad \text{or} \quad \zeta\omega_n = \frac{1}{2}\ln(16) = 1.3863 \qquad (E_2)$$

Since $\tau_d = \dfrac{2\pi}{\sqrt{\omega_n^2 - \zeta^2\omega_n^2}}$,

$$\omega_n^2 = \frac{(2\pi)^2}{\tau_d^2} + \zeta^2\omega_n^2 = \frac{4\pi^2}{4} + 1.3863^2 = 11.7915$$

i.e., $\omega_n = 3.4339$ rad/sec $\qquad (E_3)$

(d) Since angular displacement of rotor under applied torque

$= 50^\circ = 0.8727$ rad,

$$k_t = \text{torque/angular displacement} = 2\times10^{-3}/0.8727$$
$$= 2.2917 \times 10^{-3}\ \text{N-m/rad} \qquad (E_4)$$

(a) Mass moment of inertia of rotor is

$$J_0 = \frac{k_t}{\omega_n^2} = 2.2917\times10^{-3}/11.7915 = 1.9436\times10^{-4}\ \text{N-m-s}^2 \qquad (E_5)$$

(c) $c_t = 2\,J_0\,\zeta\,\omega_n \qquad (E_6)$

Eqs. (E_2) and (E_3) give $\zeta = \dfrac{\zeta\omega_n}{\omega_n} = \dfrac{1.3863}{3.4339} = 0.4037$

Eq. (E_6) gives $c_t = 5.3887 \times 10^{-4}$ N-m-s/rad.

2.149

(a) $m = 10$ kg
$c = 150$ N-s/m
$k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{150}{2(10)(10)} = 0.75$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 10\sqrt{1 - 0.75^2} = 6.61438 \text{ rad/s}$$

(under-damped)

(b) $m = 10$ kg
$c = 200$ N-s/m
$k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{200}{2(10)(10)} = 1.0$$

$$\omega_d = 10\sqrt{1 - 1.00^2} = 0$$

(critically-damped)

(c) $m = 10$ kg
$c = 250$ N-s/m
$k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{250}{2(10)(10)} = 1.25$$

ω_d = not applicable (over-damped)

2.150

(a) Underdamped system: Response: Eq. (2.70)

$$X_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{1/2} \qquad (E.1)$$

Using $x_0 = 0.1$, $\dot{x}_0 = 10$, $\zeta = 0.75$, $\omega_n = 10$, $\omega_d = 6.61438$, Eq. (E.1) gives $X_0 = 1.62832$ m.

$$\phi_0 = \tan^{-1}\left(- \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d x_0} \right)$$

$$= \tan^{-1}\left(- \frac{10 + 0.75(10)(0.1)}{6.61438(0.1)} \right) = -86.47908^\circ = -1.50935 \text{ rad}$$

Eq. (2.70) gives:

$$x(t) = 1.62832\, e^{-7.5 t} \cos(6.61438\, t + 1.50935) \text{ m}$$

(b) Critically damped system: Response: Eq. (2.80)

$$x(t) = \{ x_0 + (\dot{x}_0 + \omega_n x_0)\, t \}\, e^{-\omega_n t}$$

$$= \{0.1 + (10 + 10 * 0.1)\, t\}\, e^{-10t}$$

$$= (0.1 + 11\, t)\, e^{-10t} \quad m$$

(c) overdamped system: Response: Eq. (2.81)

Using $\sqrt{\zeta^2 - 1} = \sqrt{1.25^2 - 1} = 0.75$, we obtain

$$C_1 = \frac{x_0\, \omega_n \{\zeta + \sqrt{\zeta^2 - 1}\} + \dot{x}_0}{2\, \omega_n \sqrt{\zeta^2 - 1}} \qquad \text{Eq. (2.82)}$$

$$= \frac{0.1\,(10)\,\{1.25 + 0.75\} + 10}{2\,(10)\,(0.75)} = 0.8$$

$$C_2 = \frac{-x_0\, \omega_n \{\zeta - \sqrt{\zeta^2 - 1}\} - \dot{x}_0}{2\, \omega_n \sqrt{\zeta^2 - 1}} \qquad \text{Eq. (2.82)}$$

$$= \frac{-0.1\,(10)\,\{1.25 - 0.75\} - 10}{2\,(10)\,(0.75)} = -0.7$$

Eq. (2.81) gives

$$x(t) = C_1\, e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} + C_2\, e^{(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t}$$

$$= 0.8\, e^{(-1.25 + 0.75)(10)\,t} - 0.7\, e^{(-1.25 - 0.75)(10)\,t}$$

$$= 0.8\, e^{-5t} - 0.7\, e^{-20t} \quad m$$

2.151 Energy dissipated in a cycle of motion, $x(t) = X \sin \omega_d t$, is given by

$$\Delta W = \pi\, c\, \omega_d\, X^2 \qquad (E.1)$$

(a) $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{1000}{10}} = 10$ rad/s

$$\zeta = \frac{c}{2\, m\, \omega_n} = \frac{50}{2\,(10)\,(10)} = 0.25$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 10 \sqrt{1 - 0.25^2} = 9.682458 \text{ rad/s}$$

For $X = 0.2$ m, Eq. (E.1) gives

$$\Delta W = \pi (50)(9.682458)(0.2^2) = 60.83682 \text{ Joules}$$

(b) $\omega_n = \sqrt{\frac{k}{m}} = 10$ rad/s

$$\zeta = \frac{c}{2 m \omega_n} = \frac{150}{2(10)(10)} = 0.75$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 10 \sqrt{1-0.75^2} = 6.614378 \text{ rad/s}$$

For $X = 0.2$ m, Eq. (E.1) gives

$$\Delta W = \pi (150)(6.614378)(0.2^2) = 124.678385 \text{ Joules}$$

2.152

Equation of motion:

$$100\ddot{x} + 500\dot{x} + 10000x + 400x^3 = 0$$

(a) Static equilibrium position is given by $x = x_0$ so that, for the nonlinear spring,

$$10000 x_0 + 400 x_0^3 = mg = 100(9.81) = 981$$

The value of x_0 is given by the root of

$$400 x_0^3 + 10000 x_0 - 981 = 0$$

(Roots from MATLAB:

$x_0 = 0.0981$ m ; other roots: $-0.0490 \pm 5.0007i$)

(b) Linearized spring constant about the static equilibrium position of $x_0 = 0.0981$ m can be found as follows:

$$F(x) = 400x^3 + 10{,}000x$$

$$k_{linear} = \left.\frac{dF}{dx}\right|_{x = x_0} = 1200 x_0^2 + 10{,}000$$

$$= 1200(0.0981)^2 + 10000$$

$$= 10011.5483 \text{ N/m}$$

Linearized equation of motion:

$$100\ddot{x} + 500\dot{x} + 10011.5483x = 0$$

(c) Natural frequency of vibration for small displacements:

$$\omega_n = \left(\frac{10011.5483}{100}\right)^{\frac{1}{2}} = 10.0058 \text{ rad/s}$$

2.153

(a) static equilibrium position is given by $x = x_0$ such that

$$-400\,x_0^3 + 10000\,x_0 = mg = 100\,(9.81) = 981$$

or

$$-400\,x_0^3 + 10\,000\,x_0 - 981 = 0 \qquad (1)$$

Roots of Eq. (1) are: (from MATLAB)

$x_0 = 0.0981$; other roots: 4.9502 ; -5.0483

(b) Using the smallest positive root of Eq. (1) as the static equilibrium position, $x_0 = 0.0981$ m, the linearized spring constant about x_0 can be found as follows:

$$F(x) = -400\,x^3 + 10\,000\,x$$

$$k_{linear} = \left.\frac{dF}{dx}\right|_{x_0} = -1200\,x_0^2 + 10\,000$$

$$= 9988.4517 \text{ N/m}$$

Linearized equation of motion:

$$100\,\ddot{x} + 500\,\dot{x} + 9988.4517\,x = 0 \qquad (2)$$

(c) Natural frequency of vibration for small displacements:

$$\omega_n = \left(\frac{9988.4517}{100}\right)^{\frac{1}{2}} = 9.9942 \text{ rad/s}$$

2.154

Equation of motion:

$$J_o \ddot{\theta} + C_t \dot{\theta} + k_t \theta = 0$$

with $J_o = 25$ kg-m^2 and $k_t = 100$ N-m/rad.

For critical damping, Eq. (2.105) gives

$$c = C_c = 2\sqrt{J_o k_t} = 2\sqrt{25(100)}$$

$$= 100 \text{ N-m-s/rad.}$$

2.155

(a) $2\ddot{x} + 8\dot{x} + 16x = 0$

$m = 2, \; c = 8, \; k = 16$

$x(0) = 0, \; \dot{x}(0) = 1$

$c_c = 2\sqrt{km} = 2\sqrt{16(2)} = 11.3137$

since $c < c_c$, system is underdamped.

$$\zeta = \frac{c}{c_c} = \frac{8}{11.3137} = 0.7071$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{2}} = 2.8284 \text{ rad/s}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 2.8284\sqrt{1-0.7071^2} = 2.0 \text{ rad/s}$$

Eq. (2.72) gives the solution:

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-0.7071(2.8284)t}\left\{0 + \frac{1}{2}\sin 2t\right\}$$

$$= \frac{1}{2}e^{-2t}\sin 2t$$

(b) $3\ddot{x} + 12\dot{x} + 9x = 0$

$m = 3, \; c = 12, \; k = 9$

$x(0) = 0, \; \dot{x}(0) = 1$

$c_c = 2\sqrt{km} = 2\sqrt{9(3)} = 10.3923$

Since $c > c_c$, system is overdamped.

$$\zeta = \frac{c}{c_c} = \frac{12}{10.3922} = 1.1547$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{3}} = 1.7320$$

Solution is given by Eq. (2.81):

$$C_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}}$$

$$= \frac{1}{2\,(1.7320)\sqrt{(1.1547^2 - 1)}} = 0.5$$

$$C_2 = \frac{-x_0\,\omega_n\,(\zeta - \sqrt{\zeta^2-1}\,) - \dot{x}_0}{2\,\omega_n\sqrt{\zeta^2-1}} = -\frac{1}{2} = -0.5$$

Solution is:

$$x(t) = C_1\, e^{(-\zeta + \sqrt{\zeta^2-1}\,)\,\omega_n t} + C_2\, e^{(-\zeta - \sqrt{\zeta^2-1}\,)\,\omega_n t}$$

$$= 0.5\, e^{-t} - 0.5\, e^{-3t}$$

since

$$(-\zeta \pm \sqrt{\zeta^2-1}\,) = -1.1547 \pm \sqrt{1.1547^2 - 1}$$

$$= -1.1547 \pm 0.5773$$

$$= -1.732\,;\; -0.5774$$

(c) $2\ddot{x} + 8\dot{x} + 8x = 0$

$m = 2,\; c = 8,\; k = 8\;;\; x(0) = 0,\; \dot{x}(0) = 1$

$\zeta = c/c_c = c/(2\sqrt{k\,m}\,) = 8/(2\sqrt{8(2)}\,) = 1$

system is critically damped.

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2 \text{ rad/s}$

Solution is given by Eq. (2.80):

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)\, t\}\, e^{-\omega_n t}$$

$$= \{0 + (1 + 0)\, t\}\, e^{-2t}$$

$$= t\, e^{-2t}$$

2.156

(a) $2\ddot{x} + 8\dot{x} + 16x = 0$; $m = 2,\ c = 8,\ k = 16$

$x(0) = 1,\quad \dot{x}(0) = 0$

$c_c = 2\sqrt{km} = 2\sqrt{16(2)} = 11.3137$

Since $c < c_c$, system is underdamped

$\zeta = \dfrac{c}{c_c} = 0.7071,\quad \omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{16}{2}} = 2.8284$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 2.0$

solution is given by Eq. (2.72):

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-0.7071(2.8284)t}\left\{\cos 2t + \frac{0.7071\,(2.8284)(1)}{2}\sin 2t\right\}$$

$$= e^{-2t}\left(\cos 2t + \sin 2t\right)$$

(b) $3\ddot{x} + 12\dot{x} + 9x = 0$; $m = 3,\ c = 12,\ k = 9$

$x(0) = 1,\quad \dot{x}(0) = 0$

$c_c = 2\sqrt{km} = 2\sqrt{9(3)} = 10.3923$

Since $c > c_c$, system is overdamped.

$\zeta = \dfrac{c}{c_c} = \dfrac{12}{10.3923} = 1.1547$

$\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{9}{3}} = 1.7320$

$\sqrt{\zeta^2 - 1} = \sqrt{1.1547^2 - 1} = 0.5773$

$$C_1 = \frac{x_0 \omega_n (\zeta + \sqrt{\zeta^2 - 1})}{2 \omega_n \sqrt{\zeta^2-1}} = \frac{1(1.7320)(1.1547 + 0.5773)}{2(1.7320)(0.5773)}$$

$$= 1.5$$

$$C_2 = \frac{-x_0 \omega_n (\zeta - \sqrt{\zeta^2-1})}{2 \omega_n \sqrt{\zeta^2 - 1}}$$

$$= \frac{-1(1.7320)(1.1547 - 0.573)}{2(1.7320)(0.5773)} = -0.5$$

Solution is:

$$x(t) = 1.5\, e^{(-\zeta + \sqrt{\zeta^2-1})\omega_n t} - 0.5\, e^{(-\zeta - \sqrt{\zeta^2-1})\omega_n t}$$

$$= 1.5\, e^{-0.5774(1.732)t} - 0.5\, e^{-1.732(1.732)t}$$

$$= 1.5\, e^{-t} - 0.5\, e^{-3t}$$

(c) $2\ddot{x} + 8\dot{x} + 8x = 0$; $m = 2,\ c = 8,\ k = 8$

$x(0) = 1,\quad \dot{x}(0) = 0$

$$c_c = 2\sqrt{km} = 2\sqrt{8(2)} = 8$$

Since $c = c_c$, system is critically damped.

$$\zeta = \frac{c}{c_c} = \frac{8}{8} = 1$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2$$

Solution is given by Eq. (2.80):

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)t\}\, e^{-\omega_n t}$$

$$= \{1 + (0 + 2 \times 1)t\}\, e^{-2t}$$

$$= (1 + 2t)\, e^{-2t}$$

2.157

(a) $2\ddot{x} + 8\dot{x} + 16x = 0$

$m = 2,\ c = 8,\ k = 16;\ x(0) = 1,\ \dot{x}(0) = -1$

$c_c = 2\sqrt{km} = 2\sqrt{16(2)} = 11.3137$

Since $c < c_c$, system is underdamped.

$\zeta = \frac{c}{c_c} = \frac{8}{11.3137} = 0.7071$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{2}} = 2.8284$

$\omega_d = \sqrt{1-\zeta^2}\ \omega_n = 2.0$

Eq. (2.72) gives the solution as

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-(0.7071)(2.8284)t}\left\{\cos\omega_d t + \frac{-1 + 0.7071(2.8284)(1)}{2}\sin\omega_d t\right\}$$

$$= e^{-2t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

(b)

$3\ddot{x} + 12\dot{x} + 9x = 0$, $\quad m = 3,\ c = 12,\ k = 9$

$x(0) = 1,\ \dot{x}(0) = -1$

$c_c = 2\sqrt{km} = 2\sqrt{9(3)} = 2(5.1961) = 10.3923$

Since $c > c_c$, system is overdamped

$\zeta = \frac{c}{c_c} = \frac{12}{10.3923} = 1.1547$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{3}} = 1.732$

$$\sqrt{\zeta^2 - 1} = \sqrt{1.1547^2 - 1} = 0.5773$$

$$\zeta + \sqrt{\zeta^2 - 1} = 1.732$$

$$\zeta - \sqrt{\zeta^2 - 1} = 0.5774$$

$$C_1 = \frac{(1)\,\omega_n\,(1.732) \quad - 1}{2\,\omega_n\,(0.5773)} = \frac{2}{2} = 1$$

$$C_2 = \frac{-(1)\,\omega_n\,(0.5774) + 1}{2\,\omega_n\,(0.5773)} = \frac{-1+1}{2} = 0$$

Solution given by Eq. (2.81):

$$x(t) = C_1\, e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} + C_2\, e^{(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t}$$

$$= e^{-0.5774\,(1.732)\,t} = e^{-t}$$

(c) $2\ddot{x} + 8\dot{x} + 8x = 0$; $\quad m = 2,\ c = 8,\ k = 8$

$$x(0) = 1, \quad \dot{x}(0) = -1$$

$$c_c = 2\sqrt{km} = 2\sqrt{(8)(2)} = 8$$

$$\zeta = \frac{c}{c_c} = 1$$

Hence system is critically damped.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2$$

solution is given by Eq. (2.80):

$$x(t) = \left[x_0 + (\dot{x}_0 + \omega_n x_0)\, t \right] e^{-\omega_n t}$$

$$= \left[1 + \left(-1 + 2(1)\right) t \right] e^{-2t}$$

$$= (1 + t)\, e^{-2t}$$

2.158

Frequency in air = 120 cycles/min = $\frac{120}{60}(2\pi) = 4\pi$ rad/s

Frequency in liquid = 100 cycles/min = $\frac{100}{60}(2\pi)$

$= 3.3333\,\pi$ rad/s

Assuming damping to be negligible in air, we have

$$\omega_n = 4\pi = \sqrt{\frac{k}{m}} \Rightarrow k = (4\pi)^2 m = (4\pi)^2 (10)$$

$$= 1579.1441 \text{ N/m}$$

If damping ratio in liquid is ζ, and assuming underdamping, we have

$$\omega_d = 3.3333\,\pi = \omega_n \sqrt{1-\zeta^2}$$

or $$1-\zeta^2 = \left(\frac{3.3333\,\pi}{4\pi}\right)^2 = 0.6944$$

or $$\zeta = (1-0.6944)^{\frac{1}{2}} = 0.5528$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

or $$0.5528 = \frac{c}{2(10)(4\pi)}$$

or $$c = 0.5528\,(80\,\pi) = 138.9341 \text{ N-s/m}$$

2.159

(a) $\ddot{x} + 2\dot{x} + 9x = 0$

$m = 1,\ c = 2,\ k = 9;\quad c_c = 2\sqrt{km} = 2\sqrt{9(1)} = 6$

As $c < c_c$, system is underdamped.

$\zeta = \frac{c}{c_c} = \frac{2}{6} = 0.3333$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{1}} = 3$

$\sqrt{1-\zeta^2} = 0.9428;\quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 2.8284$

Solution is given by Eq. (2.70):

$$x(t) = X\, e^{-0.3333(3)t} \cos(0.9428 \times 3t - \phi)$$

$$= X\, e^{-t} \cos(2.8284\, t - \phi)$$

where X and ϕ depend on the initial conditions, as given by Eqs. (2.73) and (2.75), respectively.

Since the response (or solution) varies as e^{-t}, we can apply the concept of the time constant (τ) as the negative inverse of the exponential part. Hence the time constant is $\tau = 1$.

(b) $\ddot{x} + 8\dot{x} + 9x = 0;\quad m = 1,\ c = 8,\ k = 9$

$c_c = 2\sqrt{km} = 2\sqrt{9(1)} = 6;\quad \omega_n = \sqrt{\frac{k}{m}} = 3$

$\zeta = \frac{c}{c_c} = \frac{8}{6} = 1.3333$; Hence the system is overdamped.

$\sqrt{\zeta^2 - 1} = \sqrt{1.3333^2 - 1} = 0.8819$

$$-\zeta - \sqrt{\zeta^2 - 1} = -2.2152$$

$$-\zeta + \sqrt{\zeta^2 - 1} = -0.4514$$

Solution is given by Eq. (2.81):

$$x(t) = C_1 e^{-0.4514(3)t} + C_2 e^{-2.2152(3)t}$$

$$= C_1 e^{-1.3542\,t} + C_2 e^{-6.6456\,t}$$

Since the response is given by the sum of two exponentially decaying functions, two time constants can be associated with the two parts as

$$\tau_1 = \frac{1}{1.3512} = 0.7384 \;;\; \tau_2 = \frac{1}{6.6456} = 0.1505$$

(c) $\ddot{x} + 6\dot{x} + 9x = 0$; $m = 1,\ c = 6,\ k = 9$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{1}} = 3$$

$$c_c = 2\sqrt{km} = 2\sqrt{9(1)} = 6 \;;\; \zeta = \frac{c}{c_c} = 1$$

The system is critically damped. The solution is given by Eq. (2.80):

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)\,t\}\, e^{-\omega_n t}$$

$$= \{x_0 + (\dot{x}_0 + 3x_0)\,t\}\, e^{-3t}$$

Since the solution decreases exponentially, the concept of time constant (τ) can be applied to find $\tau = \frac{1}{3} = 0.3333$.

2.160

(a) Period of vibration $= \tau$

$$\omega_n = \sqrt{\frac{k_t}{J}}$$

$$\tau = \tau_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = 2\pi \cdot \sqrt{\frac{J}{k_t}}$$

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{J}{k_t}$$

$$\therefore \quad J = k_t \left(\frac{\tau}{2\pi}\right)^2$$

(b) $\tau = 0.5\ s$

$k_t = 5000\ N\text{-}m/rad$

$$J = 5000 \left(\frac{0.5}{2\pi}\right)^2 = 5000\ (0.006332)$$

$$= 31.6627\ N - m\text{-}s^2 = kg\text{-}m^2$$

2.161 Given: $m = 2$ kg, $c = 3$ N-s/m, $k = 40$ N/m

Natural frequency $= \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40}{2}} = 4.4721 \frac{rad}{s}$

c_c = critical damping $= 2\sqrt{km} = 2\sqrt{40 * 2}$

$= 17.8885$ N-s/m

ζ = damping ratio $= \frac{c}{c_c} = \frac{3}{17.8885} = 0.1677$

Type of response in free vibration: damped oscillations

For critical damping, we need to add 14.8885 N-s/m to the existing value of $c = 3$ N-s/m.

2.162 Response of the system:

$$x(t) = 0.05\, e^{-10t} + 10.5\, t\, e^{-10t} \text{ m}$$

This can be identified to correspond to critically damped system.

From the exponential terms, we find

$\omega_n = 10$ rad/s

From Eqs. (2.79), we find $C_1 = 0.05 = x_0$

and $C_2 = \dot{x}_0 + \omega_n x_0$ or $10.5 = \dot{x}_0 + 10\,(0.05)$

$\therefore\ x_0 = 0.05$ m, $\dot{x}_0 = 10.5 - 0.5 = 10$ m/s

Damping constant (c): $(\zeta = 1)$

$c = C_c = 2 m \omega_n = 2m(10) = 20 * \text{mass}.$

2.163

Characteristic Equations:

(a) $s_{1,2} = -4 \pm 5i$

$$(s+4+5i)(s+4-5i) = (s+4)^2 - (5i)^2$$

$$= s^2 + 8s + 16 + 25 = s^2 + 8s + 41 = 0$$

(b) $s_{1,2} = 4 \pm 5i$

$$(s-4-5i)(s-4+5i) = (s-4)^2 - (5i)^2$$

$$= s^2 + 16 - 8s + 25 = s^2 - 8s + 41 = 0$$

(c) $s_{1,2} = -4, -5$

$$(s+4)(s+5) = s^2 + 9s + 20$$

(d) $s_{1,2} = -4, -4$

$$(s+4)(s+4) = s^2 + 8s + 16 = 0$$

Undamped natural frequencies

(a) $m=1, \ c=8, \ k=41$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{41} = 6.4031$$

(b) $m=1, \ c=-8, \ k=41$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{41}{1}} = 6.4031$$

(c) $m=1, \ c=9, \ k=20$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{20} = 4.4721$$

(d) $m = 1, \; c = 8, \; k = 16$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{16} = 4.0$$

Damping ratios

$$m s^2 + c s + k = 0$$

$$\zeta = \frac{c}{2m} \cdot \frac{1}{\omega_n} = \frac{c}{2\sqrt{km}}$$

(a) $\zeta = \dfrac{8}{2\sqrt{41(1)}} = \dfrac{8}{2\sqrt{41}} = 0.6246$

(b) $\zeta = \dfrac{-8}{2\sqrt{41(1)}} = \dfrac{-8}{2\sqrt{41}} = -0.6246$

(c) $\zeta = \dfrac{9}{2\sqrt{20(1)}} = \dfrac{9}{8.9443} = 1.0062$

(d) $\zeta = \dfrac{8}{2\sqrt{16(1)}} = 1.0$

Damped frequencies

$$\omega_d = \sqrt{1-\zeta^2} \cdot \omega_n \quad \text{if} \quad \zeta < 1$$

(a) $\omega_d = \sqrt{1 - 0.6246^2} \cdot (6.4031) = 5.0004$

(b) ω_d : Not applicable

(c) ω_d : Not applicable

(d) $\omega_d = 0$

Time constants

$$\tau = \frac{1}{\zeta \omega_n} = \frac{2m}{c}$$

(a) $$\tau = \frac{1}{0.6246\ (6.4031)} = 0.2500 \text{ (Underdamped)}$$

(b) $$\tau = \frac{1}{-0.6246\ (6.4031)} = -0.2500$$

Not applicable ; negative damping.

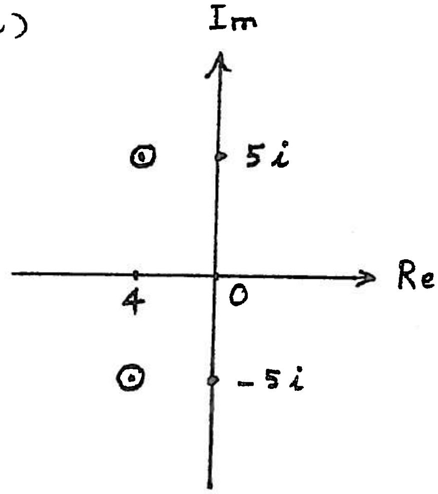
Response grows exponentially.

(c) $$\tau = \frac{1}{1.0062\ (4.4721)} = 0.2222 \text{ (overdamped)}$$

(d) $$\tau = \frac{1}{1.0\ (4.0)} = 0.25 \text{ (Undamped)}$$

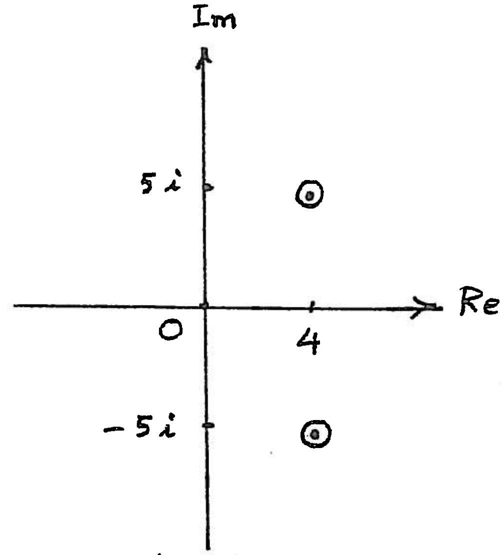
2.164

(a)



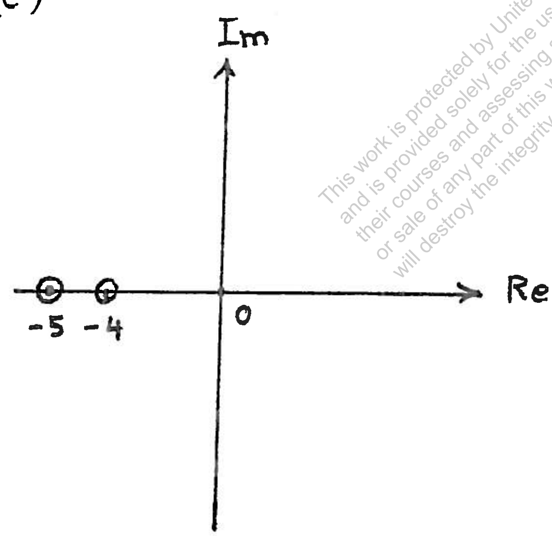
stable.
underdamped.
Oscillatory response.

(b)



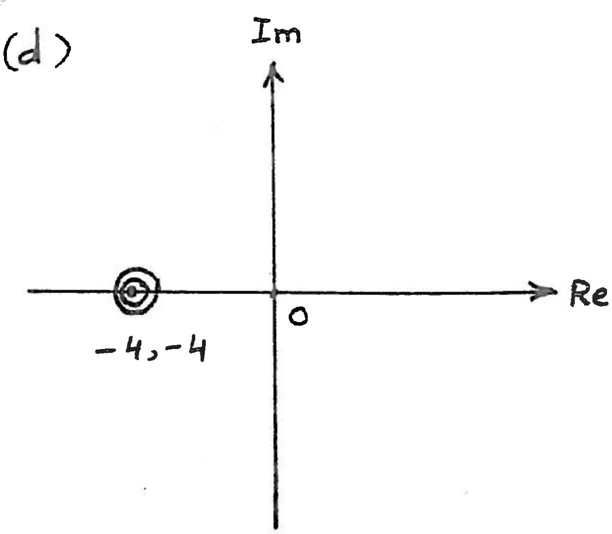
Unstable.
Response grows exponentially.

(c)



stable.
Response will not oscillate.

(d)



stable.
Response will not oscillate.

2.165

characteristic equation:

$$s^2 + a s + b = 0 \qquad (1)$$

where

$$a = c/m \qquad (2)$$

and

$$b = k/m \qquad (3)$$

Roots of Eq. (1):

$$s_{1,2} = \frac{-a \pm \quad a^2 - 4b}{2} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} \qquad (4)$$

and s_1 and s_2 are, in general, complex numbers.

Solution of Eq. (1) can be expressed as

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad (5)$$

where C_1 and C_2 are constants.

- When s_1 and s_2 are both real and negative, the solution in Eq. (5) approaches zero asymptotically.
- If s_1 and s_2 are complex, the nature of solution is governed by the real part of the roots. If real part is negative, the solution in Eq. (5) is oscillatory and approaches zero as $t \to \infty$.

The stability of the system in the s-plane is shown in Fig. a.

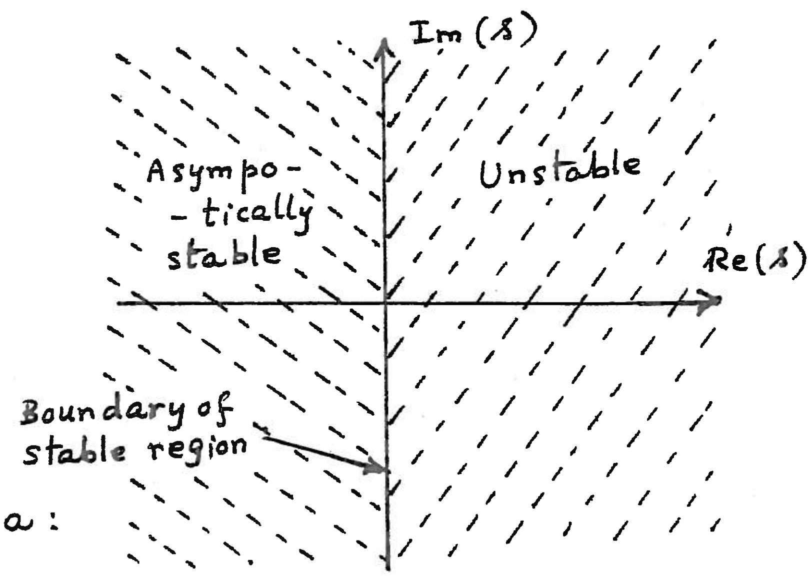


Figure a :

The stability of the system in the parameter space can be indicated as shown in Fig. b.

- When $a < 0$ and $b > 0$ (fourth quadrant), the curve $\left(\frac{a}{2}\right)^2 - b = 0$ separates the quadrant into two regions. In the top part (above the parabola), the roots s_1 and s_2 will be complex conjugate with positive real part. Hence the motion will be diverging oscillations.

  In the bottom part (below the parobola curve), both s_1 and s_2 will be real with at least one positive root. Hence the motion diverges without oscillation.

- When $a > 0$ and $b > 0$ (first quadrant in Fig. b):

  The curve given by $\left(\frac{a}{2}\right)^2 - b = 0$ (parabola) seperates the quadrant into two regions. In the top region, $\left(\frac{a^2}{4} > b\right)$, s_1 and s_2 will be real and negative. Hence the motion decays without oscillations (aperiodic decay).

  In the region $\frac{a^2}{4} < b$, s_1 and s_2 will be complex conjugates with negative real part. Hence the response is oscillatory and decays as time increases.

  Along the boundary curve $\left(\frac{a^2}{4} - b = 0\right)$, the roots s_1 and s_2 will be identical with $s_1 = s_2 = \frac{a}{2}$. Hence the motion decays with time t.

- When $a = 0$ and $b > 0$, the roots s_1 and s_2 will be pure imaginary complex conjugates. Hence the motion is oscillatory (harmonic) and stable.
- When $b < 0$ (second and third quadrants), s_1 and s_2 will be positive and hence the response diverges with no oscillations; thus the motion is unstable.

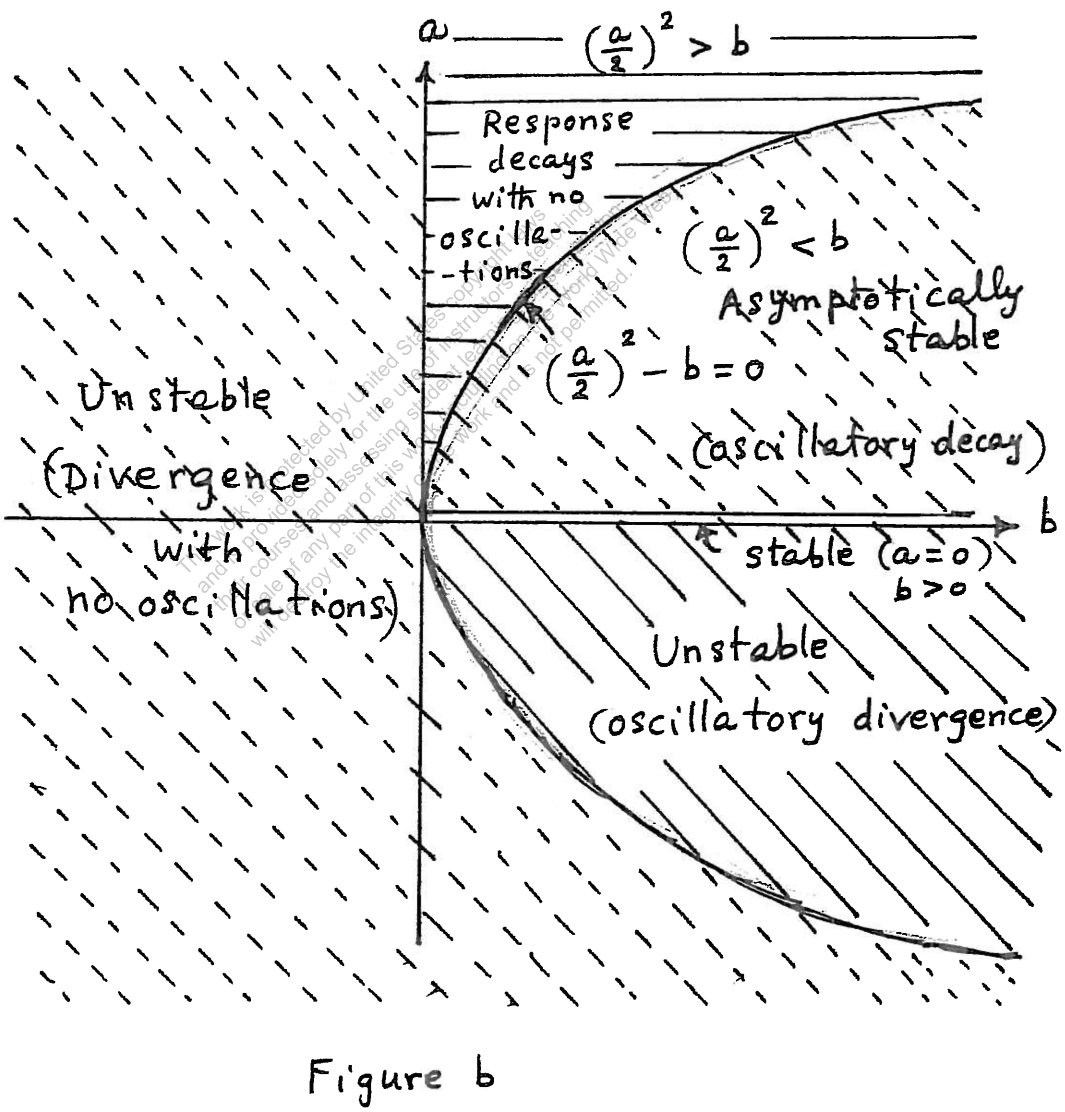


Figure b

2.166

characteristic equation:

$$2s^2 + cs + 18 = 0 \quad (1)$$

Roots of Eq. (1):

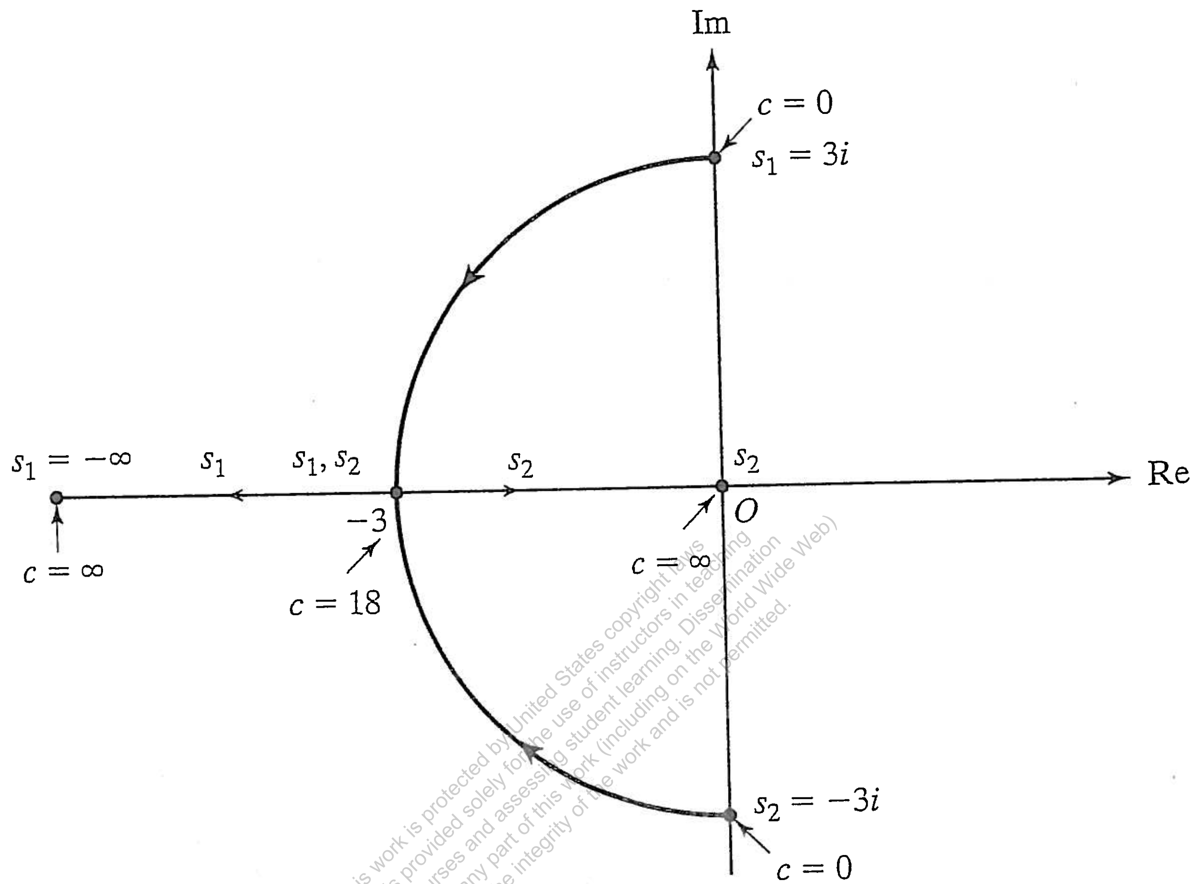
$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 144}}{4} \quad (2)$$

At $c = 0$, the roots are given by $s_{1,2} = \pm 3i$.

These roots are shown as dots in Fig. a. By increasing the value of c, the roots can be found as shown in the following Table.

c	s_2	s_1
0	$+3i$	$-3i$
2	$-0.5 + 2.96i$	$-0.5 - 2.96i$
4	$-1.0 + 2.83i$	$-1.0 - 2.83i$
8	$-2.0 + 2.24i$	$-2.0 - 2.24i$
11	$-2.75 + 1.20i$	$-2.75 - 1.20i$
12	-3.0	-3.0
14	$-3.5 + 1.80 = -1.70$	$-3.5 - 1.80 = -5.30$
20	$-5.0 + 4.0 = -1.0$	$-5.0 - 4.0 = -9.0$
100	$-25.0 + 24.82 = -0.18$	$-25.0 - 24.82 = -49.82$
1000	$-250 + 250 \simeq 0$	$-250 - 250 \cong -500$

Root locus is shown in Fig. a.



Problem 2.166 Root locus plot with variation of damping constant (c).

Fig. (a)

2.167

Characteristic equation:

$$2s^2 + 12s + k = 0 \tag{1}$$

Roots of Eq.(1):

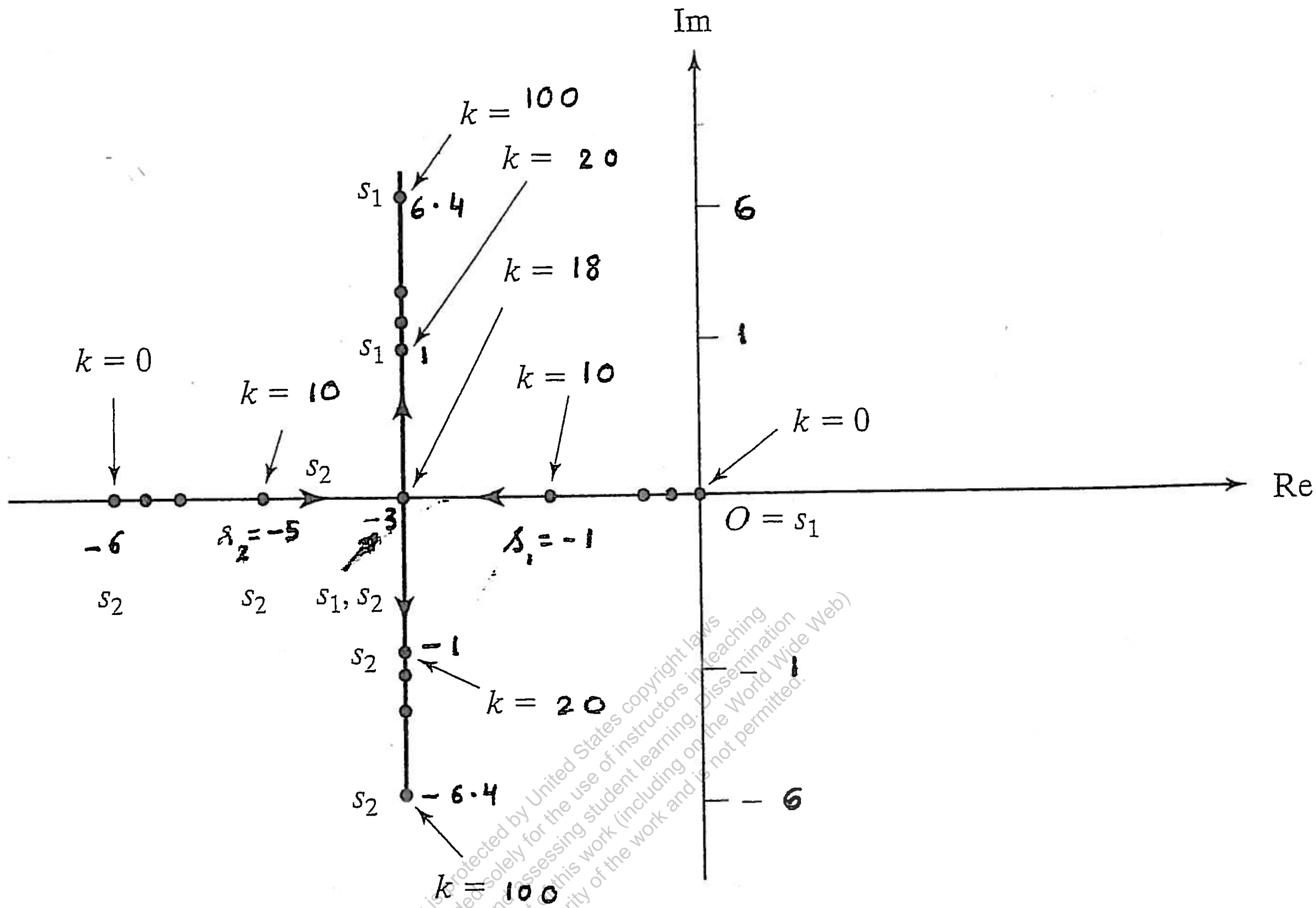
$$s_{1,2} = \frac{-12 \pm \sqrt{144 - 8k}}{4} \tag{2}$$

or

$$s_{1,2} = -3 \pm \sqrt{9 - \tfrac{1}{2}k} \tag{3}$$

Since k cannot be negative, we vary k from 0 to ∞. When $k = 18$, both s_1 and s_2 are real and equal to -3. In the range $0 < k < 18$, both s_1 and s_2 will be real and negative. When $k = 0$, $s_1 = 0$ and $s_2 = -6$. The variation of roots with increasing values of k is shown in the following Table and also in Fig. a.

k	s_1	s_2
0	0	-6.0
10	-1.0	-5.0
18	-3.0	-3.0
20	$-3 + i$	$-3 - i$
40	$-3 + 3.32i$	$-3 - 3.32i$
100	$-3 + 6.40i$	$-3 - 6.40i$
1000	$-3 + 22.16i$	$-3 - 22.16i$



Problem 2.167 Root locus plot with variation of spring constant (k).

Fig. (a)

2.168

Characteristic equation:

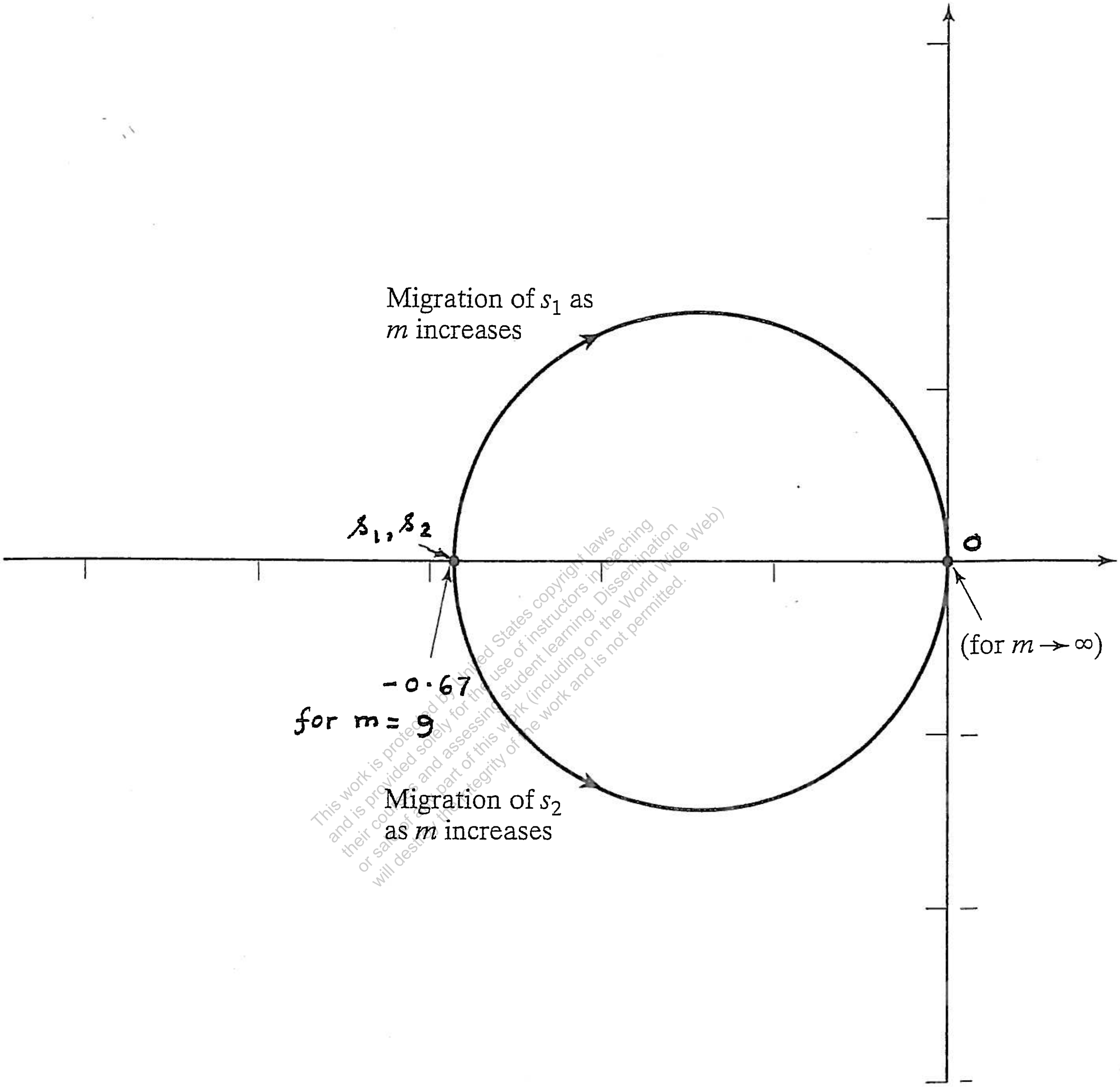
$$m s^2 + 12 s + 4 = 0 \tag{1}$$

Roots of Eq.(1):

$$s_{1,2} = \frac{-12 \pm \sqrt{144 - 16 m}}{2m} \tag{2}$$

Since negative and zero values of m are not possible, we vary m in the range $1 \leq m < \infty$. The roots given by Eq.(2) are shown in the following Table and also plotted in Fig. a.

m	s_1	s_2
1	-0.345	-11.655
4	-0.38	-2.62
8	-0.50	-1.00
9	-0.67	-0.67
10	$-0.6 + 0.2 i$	$-0.6 - 0.2 i$
20	$-0.3 + 0.33 i$	$-0.3 - 0.33 i$
100	$-0.06 + 0.19 i$	$-0.06 - 0.19 i$
500	$-0.012 + 0.089 i$	$-0.012 - 0.089 i$
1000	$-0.006 + 0.063 i$	$-0.006 - 0.063 i$



Problem 2.168

Root locus plot with variation of mass (m).

Fig. (a)

2.169 $m = 20$ kg, $k = 4000$ N/m

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{20}} = 14.1421$ rad/sec

Amplitudes of successive cycles : 50, 45, 40, 35 mm

Amplitudes of successive cycles diminish by $5\text{ mm} = 5 \times 10^{-3}$ m

System has Coulomb damping.

$$\frac{4\mu N}{k} = 5 * 10^{-3} \Rightarrow \mu N = \left\{ \frac{(5 \times 10^{-3})(4000)}{4} \right\} = 5 \text{ N}$$

$= \text{damping force}$

Frequency of damped vibration $= 14.1421$ rad/sec.

2.170 $m = 20$ kg, $k = 10000$ N/m, $\frac{4\mu N}{k} = \frac{150 - 100}{4}$ mm $= 12.5 \times 10^{-3}$ m

$$\mu = \frac{(12.5 \times 10^{-3})(10\,000)}{4(20 \times 9.81)} = 0.1593$$

Time elapsed $= 4\tau_n = 4 \times \frac{2\pi}{\omega_n} = 8\pi\sqrt{\frac{m}{k}} = 1.124$ sec

---

2.171 $m = 10$ kg, $k = 3000$ N/m, $\mu = 0.12$, $X = 100$ mm

$$\frac{4\mu N}{k} = \frac{4(0.12)(10 \times 9.81)}{3000} = 0.0157 \text{ m} = 15.7 \text{ mm}$$

As $6\left(\frac{4\mu N}{k}\right) = 94.2$ mm, mass comes to rest at $(100 - 94.2) = 5.8$ mm

---

2.172 $mg = 25$ N, $k = 1000$ N/m, damping force = constant

Mass released with $x_0 = 10$ cm and $\dot{x}_0 = 0$.

Static deflection of spring due to self weight of mass $= \frac{25}{1000}$

$= 0.025$ m

at $t = 0$: $x = 0.1$ m, $\dot{x} = 0$

$x_0 = 0.1$

$$x_1 = x_0 - 2\frac{\mu N}{k}, \quad x_2 = x_0 - \frac{4\mu N}{k}$$

$$x_3 = a_0 - \frac{6\mu N}{k}, \quad x_4 = x_0 - \frac{8\mu N}{k} = 0$$

i.e., $x_0 = \frac{8\mu N}{k} = 0.1$

Magnitude of damping force $= \mu N = \frac{x_0 k}{8} = \frac{(0.1)(1000)}{8}$
$= 12.5$ N

2.173 $m = 20$ kg, $k = 10{,}000$ N/m, $\mu N = 50$ N, $x_0 = 0.05$ m

(a) Number of half cycles elapsed before mass comes to rest (r) is given by:

$$r \geq \left\{ \frac{x_0 - \frac{\mu N}{k}}{2\frac{\mu N}{k}} \right\} = \frac{0.05 - \left(\frac{50}{10000}\right)}{2\left(\frac{50}{10\,000}\right)} = 4.5$$

$\therefore\ r = 5$

(b) Time elapsed before mass comes to rest:

$$t_p = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{20}{10000}} = 0.2810 \text{ sec}$$

Time taken $= (2.5 \text{ cycles})\ t_p = 0.7025$ sec

(c) Final extension of spring after 5 half-cycles:

$$x_5 = 0.05 - 5\left(\frac{2\mu N}{k}\right) = 0.05 - 5\left(2 * \frac{50}{10\,000}\right) = 0$$

(displacement from static equilibrium position = 0)

But static deflection $= \frac{mg}{k} = \frac{20 * 9.81}{10\,000} = 0.01962$ m

$\therefore$ Final extension of spring $= 1.9620$ cm.

2.174 (a) Equation of motion for angular oscillations of pendulum:

$$J_0\ddot{\theta} + mgl\sin\theta \pm mg\mu\frac{d}{2}\cos\theta = 0$$

For small angles, $\ddot{\theta} + \frac{mgl}{J_0}\left(\theta \pm \frac{\mu d}{2l}\right) = 0$

This shows that the angle of swing decreases by $\left(\frac{\mu d}{2l}\right)$ in each quarter cycle.

(b) For motion from right to left:

$$\theta(t) = A_1\cos\omega_n t + A_2\sin\omega_n t + \frac{\mu d}{2l}$$

where $\omega_n = \sqrt{\frac{mgl}{J_0}}$.

Let $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = 0$. Then $A_1 = \theta_0 - \frac{\mu d}{2l}$, $A_2 = 0$

$$\theta(t) = \left(\theta_0 - \frac{\mu d}{2\ell}\right)\cos\omega_n t + \frac{\mu d}{2\ell}$$

For motion from left to right:

$$\theta(t) = A_3 \cos\omega_n t + A_4 \sin\omega_n t - \frac{\mu d}{2\ell}$$

At $\omega_n t = \pi$, $\theta = -\theta_0 + \frac{2\mu d}{2\ell}$, $\dot{\theta} = 0$ from previous solution.

$$A_3 = \theta_0 - \frac{3\mu d}{2\ell}, \quad A_4 = 0$$

$$\theta(t) = \left(\theta_0 - \frac{3\mu d}{2\ell}\right)\cos\omega_n t - \frac{\mu d}{2\ell}$$

(c) The motion ceases when $\left(\theta_0 - n\,\frac{4\mu d}{2\ell}\right) < \frac{\mu d}{2\ell}$

where n denotes the number of cycles.

2.175 $x(t) = X \sin\omega t$ (under sinusoidal force $F_0 \sin\omega t$)

Damping force $= \mu N$

Total displacement per cycle $= 4X$

Energy dissipated per cycle $= \Delta W = 4\mu N X$ $\quad(E_1)$

If c_{eq} = equivalent viscous damping constant, energy dissipated per cycle is given by Eq. (2.98):

$$\Delta W = \pi\, c_{eq}\,\omega\, X^2 \qquad (E_2)$$

Equating (E_1) and (E_2) gives

$$c_{eq} = \frac{4\mu N X}{\pi\omega X^2} = \frac{4\mu N}{\pi\omega X} \qquad (E_3)$$

2.176 Due to viscous damping:

$$\delta = \ln\left(\frac{X_m}{X_{m+1}}\right) \simeq 2\pi\zeta$$

z_1 = percent decrease in amplitude per cycle at X_m

$$= 100\left(\frac{X_m - X_{m+1}}{X_m}\right) = 100\left(1 - \frac{X_{m+1}}{X_m}\right) = 100\left(1 - e^{-2\pi\zeta}\right)$$

Due to Coulomb damping:

z_2 = percent decrease in amplitude per cycle at X_m

$$= 100\left(\frac{X_m - X_{m+1}}{X_m}\right) = 100\left(\frac{4\mu N}{k X_m}\right)$$

When both types of damping are present:

$$z_1 + z_2\Big|_{X_m = 20\text{ mm}} = 2 \quad ; \quad z_1 + z_2\Big|_{X_m = 10\text{ mm}} = 3$$

i.e.,

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.02}\left(\frac{\mu N}{k}\right)=2$$

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.01}\left(\frac{\mu N}{k}\right)=3$$

The solution of these equations gives

$$50\left(1-e^{-2\pi\zeta}\right)=0.5 \quad \text{and} \quad \frac{\mu N}{k}=0.5\times 10^{-6}\ \text{m}$$

2.177 Coulomb damping.

(a) Natural frequency $=\omega_n=\dfrac{2\pi}{\tau_n}=\dfrac{2\pi}{1}=6.2832$ rad/sec. Reduction in amplitude in each cycle:

$$=\frac{4\mu N}{k}=4\mu g\frac{m}{k}=\frac{4\mu g}{\omega_n^2}=4\mu\left(\frac{9.81}{6.2832^2}\right)$$

$$=0.9940\,\mu=\frac{0.5}{100}=0.005\ \text{m}$$

Kinetic coefficient of friction $=\mu=0.00503$

(b) Number of half-cycles executed (r) is:

$$r\geq\frac{\left(x_0-\dfrac{\mu N}{k}\right)}{\left(\dfrac{2\mu N}{k}\right)}=\frac{\left(x_0-\dfrac{\mu g}{\omega_n^2}\right)}{\left(\dfrac{2\mu g}{\omega_n^2}\right)}$$

$$\geq\frac{\left(0.1-\dfrac{0.00503\,(9.81)}{6.2832^2}\right)}{\left(\dfrac{2\,(0.00503)\,(9.81)}{6.2832^2}\right)}$$

$$\geq 39.5032$$

$$\geq 40$$

Thus the block stops oscillating after 20 cycles.

2.178

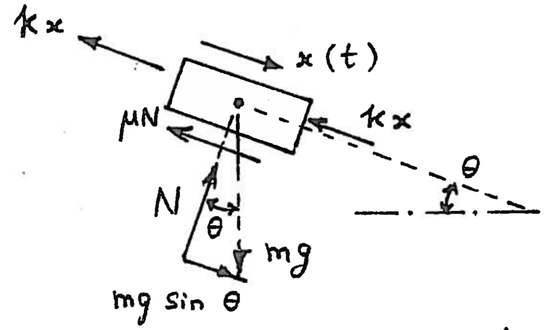
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000}{5}} = 44.721359 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{44.721359} = 0.140497 \text{ s}$$

Time taken to complete 10 cycles $= 10\,\tau_n$

$= 1.40497$ s

2.179



(a) $\theta = 30^\circ$

$N = mg\cos\theta$

case 1: When $x = +$ and $\dot{x} = +$ or $x = -$ and $\dot{x} = +$:

$$m\ddot{x} = -2kx - \mu N + mg\sin\theta$$

or $$m\ddot{x} + 2kx = -\mu mg\cos\theta + mg\sin\theta \qquad (E.1)$$

case 2: when $x = +$ and $\dot{x} = -$ or $x = -$ and $\dot{x} = -$:

$$m\ddot{x} = -2kx + \mu N + mg\sin\theta$$

or $$m\ddot{x} + 2kx = \mu mg\cos\theta + mg\sin\theta \qquad (E.2)$$

Eqs. (E.1) and (E.2) can be written as a single equation as:

$$m\ddot{x} + \mu mg\cos\theta \;\operatorname{sgn}(\dot{x}) + 2kx + mg\sin\theta = 0 \qquad (E.3)$$

(b) $x_0 = 0.1$ m, $\dot{x}_0 = 5$ m/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{20}} = 7.071068 \text{ rad/s}$$

Solution of Eq. (E.1):

$$x(t) = A_1\cos\omega_n t + A_2\sin\omega_n t - \frac{\mu mg\cos\theta}{k} + \frac{mg\sin\theta}{k} \qquad (E.4)$$

Solution of Eq. (E.2):

$$x(t) = A_3\cos\omega_n t + A_4\sin\omega_n t + \frac{\mu mg\cos\theta}{k} + \frac{mg\sin\theta}{k} \qquad (E.5)$$

Using the initial conditions in each half cycle, the constants A_1 and A_2 or A_3 and A_4 are to be found. For example, in the first half cycle, the motion starts from left toward right with $x_0 = 0.1$ and $\dot{x}_0 = 5$. These values can be used in Eq. (E.4) to find A_1 and A_2.

2.180 Friction force $= \mu N = 0.2\ (5) = 1$ N. $k = \dfrac{25}{0.10} = 250$ N/m. Reduction in amplitude in each cycle $= \dfrac{4\ \mu N}{k} = \dfrac{4\ (1)}{250} = 0.016$ m. Number of half-cycles executed before the motion ceases (r):

$$r \geq \left(\frac{x_0 - \frac{\mu N}{k}}{\frac{2\ \mu N}{k}} \right) = \frac{0.1 - 0.004}{0.008} \geq 12$$

Thus after 6 cycles, the mass stops at a distance of 0.1 - 6 (0.016) = 0.004 m from the unstressed position of the spring.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{250\ (9.81)}{5}} = 22.1472 \text{ rad/sec}$$

$$\tau_n = \frac{2\ \pi}{\omega_n} = 0.2837 \text{ sec}$$

Thus total time of vibration $= 6\ \tau_n = 1.7022$ sec.

2.181 Energy dissipated in each full load cycle is given by the area enclosed by the hysteresis loop.

The area can be found by counting the squares enclosed by the hysteresis loop. In Fig. 2.117, the number of squares is ≈ 33. Since each square $= \dfrac{100 \times 1}{1000} = 0.1$ N-m, the energy dissipated in a cycle is

$$\Delta W = 33 \times 0.1 = 3.3 \text{ N-m} = \pi k \beta X^2$$

Since the maximum deflection $= X = 4.3$ mm, and the slope of the force-deflection curve is

$$k \simeq \frac{1800 \text{ N}}{11 \text{ mm}} = 1.6364 \times 10^5 \text{ N/m},$$

the hysteresis damping constant β is given by

$$\beta = \frac{\Delta W}{\pi k X^2} = \frac{3.3}{\pi (1.6364 \times 10^5)(0.0043)^2} = 0.3472$$

$\delta = \pi \beta =$ logarithmic decrement $= \pi (0.3472) = 1.0908$

Equivalent viscous damping ratio $= \zeta_{eq} = \beta/2 = 0.1736.$

2.182 $\frac{x_j}{x_{j+1}} = \frac{2+\pi\beta}{2-\pi\beta} = 1.1$, $\beta = 0.03032$

$c_{eq} = \beta \sqrt{mk} = 0.03032 \sqrt{1 \times 2} = 0.04288$ N-s/m

$\Delta W = \pi k \beta X^2 = \pi (2)(0.03032)\left(\frac{10}{1000}\right)^2 = 19.05 \times 10^{-6}$ N-m

---

2.183 Logarithmic decrement $= \delta = \ln\left(\frac{x_j}{x_{j+1}}\right) \simeq \pi\beta$

For n cycles, $\delta = \frac{1}{n} \ln\left(\frac{x_0}{x_n}\right) \simeq \pi\beta$

$$\frac{1}{100} \ln\left(\frac{30}{20}\right) = 0.004055 = \pi\beta$$

$\beta = 0.001291$

2.184 $\delta = \frac{1}{n} \ln \frac{X_0}{X_n}$

$$= \frac{1}{100} \ln \frac{25}{10} = \frac{1}{100} \ln 2.5 = 0.0091629$$

$\delta = \pi \frac{h}{k}$

or $h = \frac{\delta k}{\pi} = \frac{(0.0091629)(200)}{\pi} = 0.583327$ N/m

---

2.185

(a) Equation of motion:

$$\ddot{\theta} + \frac{g}{\ell} \sin\theta = 0 \tag{1}$$

Linearization of $\sin\theta$ about an arbitrary value θ_0 using Taylor's series expansion (and retaining only upto the linear term):

$$\sin\theta = \sin\theta_0 + \cos\theta_0 \cdot (\theta - \theta_0) + \cdots \tag{2}$$

By defining $\underset{\sim}{\theta} = \theta - \theta_0$ so that $\theta = \underset{\sim}{\theta} + \theta_0$ with $\dot{\theta} = \dot{\underset{\sim}{\theta}}$ and $\ddot{\theta} = \ddot{\underset{\sim}{\theta}}$, we can express Eq. (1) as

$$\ddot{\underset{\sim}{\theta}} + \frac{g}{\ell}\left(\sin\theta_0 + \underset{\sim}{\theta} \cos\theta_0\right) = 0 \tag{3}$$

where g/ℓ, $\sin\theta_0$ and $\cos\theta_0$ are constants. Eq. (3) is the desired linear equation.

(b) At the equilibrium (reference) positions indicated by

$$\theta_e = n\pi; \quad n = 0, \pm\pi, \pm 2\pi, \ldots \tag{4}$$

$\sin\theta_e = \sin\theta_0 = 0$. Hence Eq. (3) takes the form

$$\ddot{\underset{\sim}{\theta}} + \frac{g}{\ell} \cos\theta_e \, \underset{\sim}{\theta} = 0 \tag{5}$$

The characteristic equation corresponding to Eq. (5) is

$$s^2 + \frac{g}{\ell}\cos\theta_e = 0 \tag{6}$$

The roots of Eq. (6) are

$$s = \pm\sqrt{-\frac{g\cos\theta_e}{\ell}} \tag{7}$$

For $\theta_e = 0$, $s = \pm i\sqrt{\frac{g}{l}}$ (8)

Both the values of s are imaginary. Hence the system is neutrally stable.

For $\theta_e = \pi$, $s = \pm\sqrt{\frac{g}{l}}$ (9)

Here one value of s is positive and the other value of s is negative (both are real). Hence the system is unstable.

ALTERNATIVE APPROACH:

The potential energy of the pendulum is given by

$$V(\theta) = V_0 - \frac{mg}{l}\cos\theta \tag{10}$$

where V_0 is a constant. The equilibrium states, $\theta = \theta_e$, of Eq. (10) are given by the stationary value of $V(\theta)$:

$$\frac{dV}{d\theta} = \frac{mg}{l}\sin\theta = 0 \tag{11}$$

Roots of Eq. (11) give the equilibrium states as

$$\theta_e = n\pi \quad ; \quad n = 0, \pm 1, \pm 2, \ldots \tag{12}$$

Second derivative of $V(\theta)$ is

$$\frac{d^2V}{d\theta^2} = \frac{mg}{l}\cos\theta \tag{13}$$

$$= \begin{cases} \text{positive for } \theta = 0, 2\pi, 4\pi, \ldots \\ \text{negative for } \theta = \pi, 3\pi, \ldots \end{cases}$$

Thus the potential energy is minimum at $\theta_e = 0, 2\pi, 4\pi, \ldots$ and maximum at $\theta_e = \pi, 3\pi, \ldots$

Hence the pendulum is stable at $\theta_e = 0$ and unstable at $\theta_e = \pi$.

2.186

(a) Equation of motion:

Mass moment of inertia of the circular disk about point O is $J + ML^2 = J_d$. (1)

Mass moment of inertia of the rod about point O is $J_r = \frac{1}{12} m l^2 + m \left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$ (2)

For small angular displacements (θ) of the rigid bar about the pivot point O, the free body diagram is shown in Fig. a.

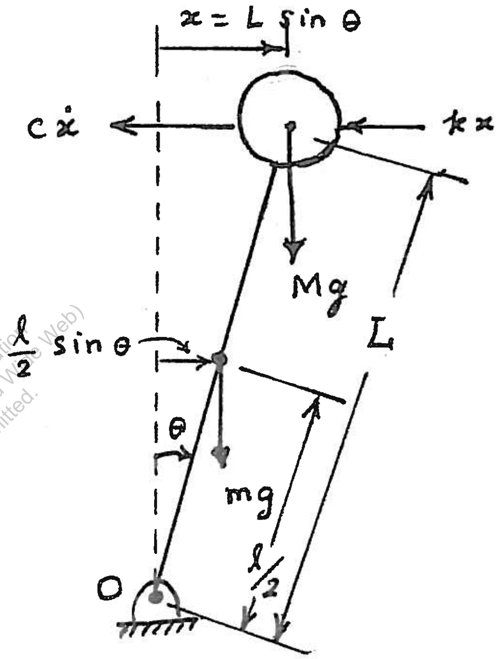


Figure a.

The equation of motion for the angular motion of the rigid bar, using Newton's second law of motion, is:

$$(J_r + J_d)\,\ddot{\theta} - mg\frac{l}{2}\sin\theta - MgL\sin\theta + c\dot{x}L\cos\theta + kxL\cos\theta = 0 \quad (3)$$

Since θ is small, $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$. Thus Eq. (3) can be expressed as

$$(J_r + J_d)\,\ddot{\theta} - \frac{mgl}{2}\theta - MgL\theta + cL^2 + kL^2 = 0 \quad (4)$$

Eq. (4) can be written as

$$J_o\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0 \quad (5)$$

where

$$J_0 = J_r + J_d \tag{6}$$

$$C_t = c L^2 \tag{7}$$

$$k_t = -\frac{mg\ell}{2} - MgL + kL^2 \tag{8}$$

(b) The characteristic equation for the differential equation (5) is given by

$$J_0 s^2 + C_t s + k_t = 0 \tag{9}$$

whose roots are given by

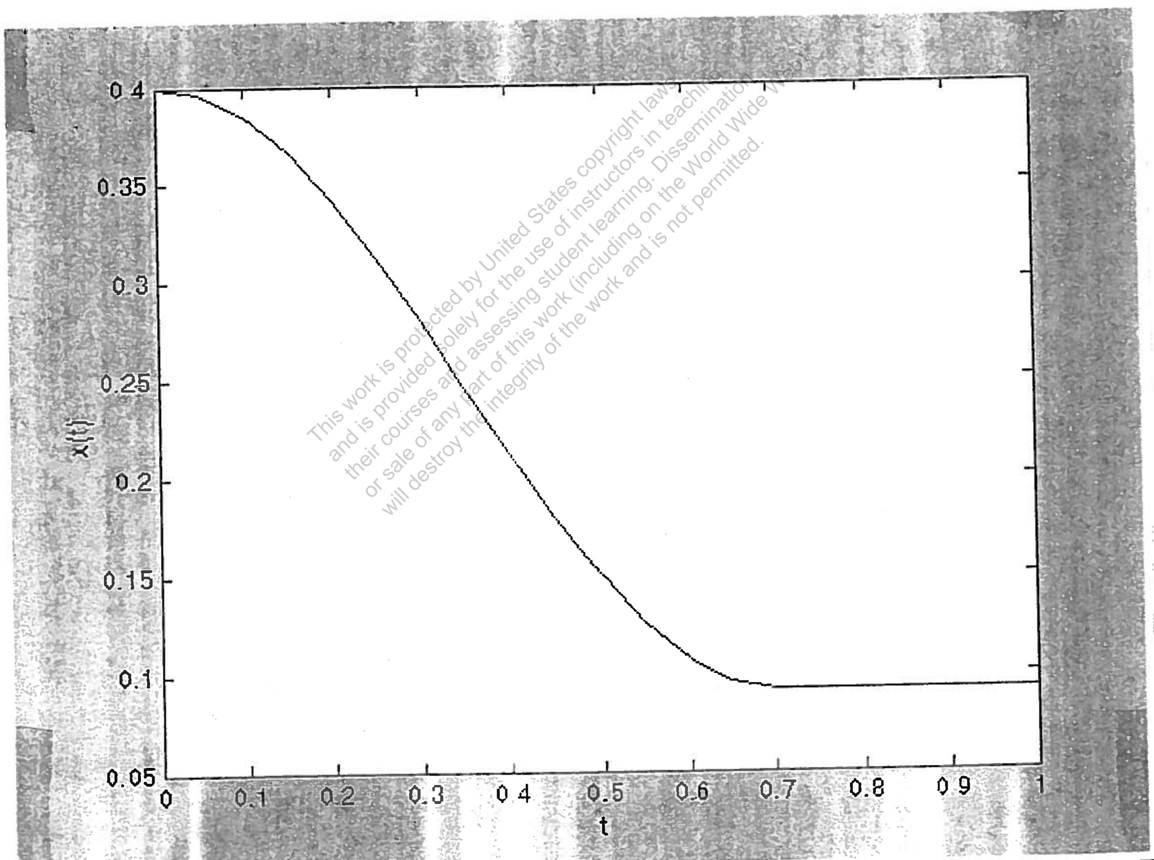
$$s_{1,2} = \frac{-C_t \pm \sqrt{C_t^2 - 4 J_0 k_t}}{2 J_0} \tag{10}$$

It can be shown (see Section 3.11.1) that the system will be stable if C_t and k_t are positive. In Eq. (9), $C_t > 0$ and $J_0 > 0$ while $k_t > 0$ only when $kL^2 > \frac{mg\ell}{2} + MgL$ (i.e., when the moment due to the restoring force of the spring is larger than the moment due to the gravity force).

2.187

```
% Ex2_187.m
% This program will use dfunc1.m
tspan = [0: 0.05: 8];
x0 = [0.4; 0.0];
[t, x] = ode23('dfunc1', tspan, x0);
plot(t, x(:, 1));
xlabel('t');
ylabel('x(t)');
```

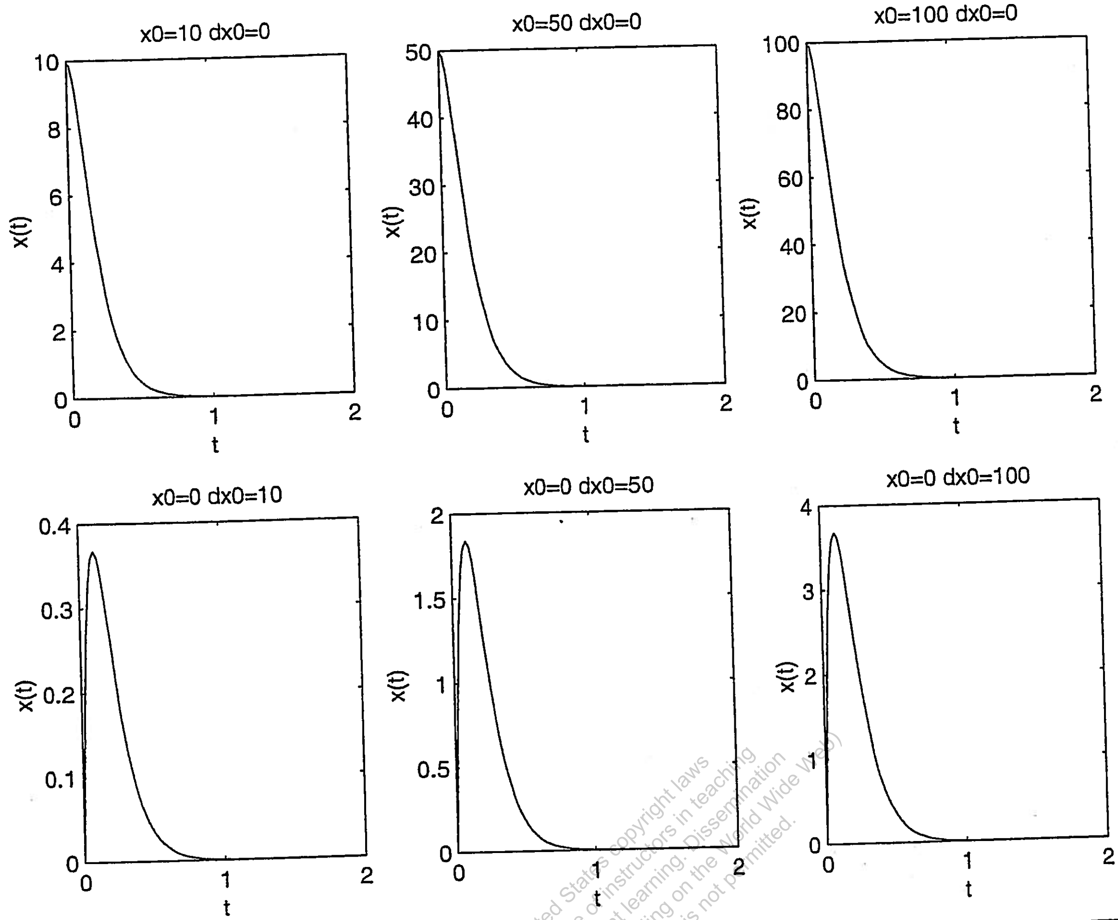
```
% dfunc1.m
function f = dfunc1(t, x)
u = 0.5;
k = 100;
m = 5;
f = zeros(2,1);
f(1) = x(2);
f(2) = -u * 9.81 * sign(x(2)) - k * x(1) / m;
```



2.188

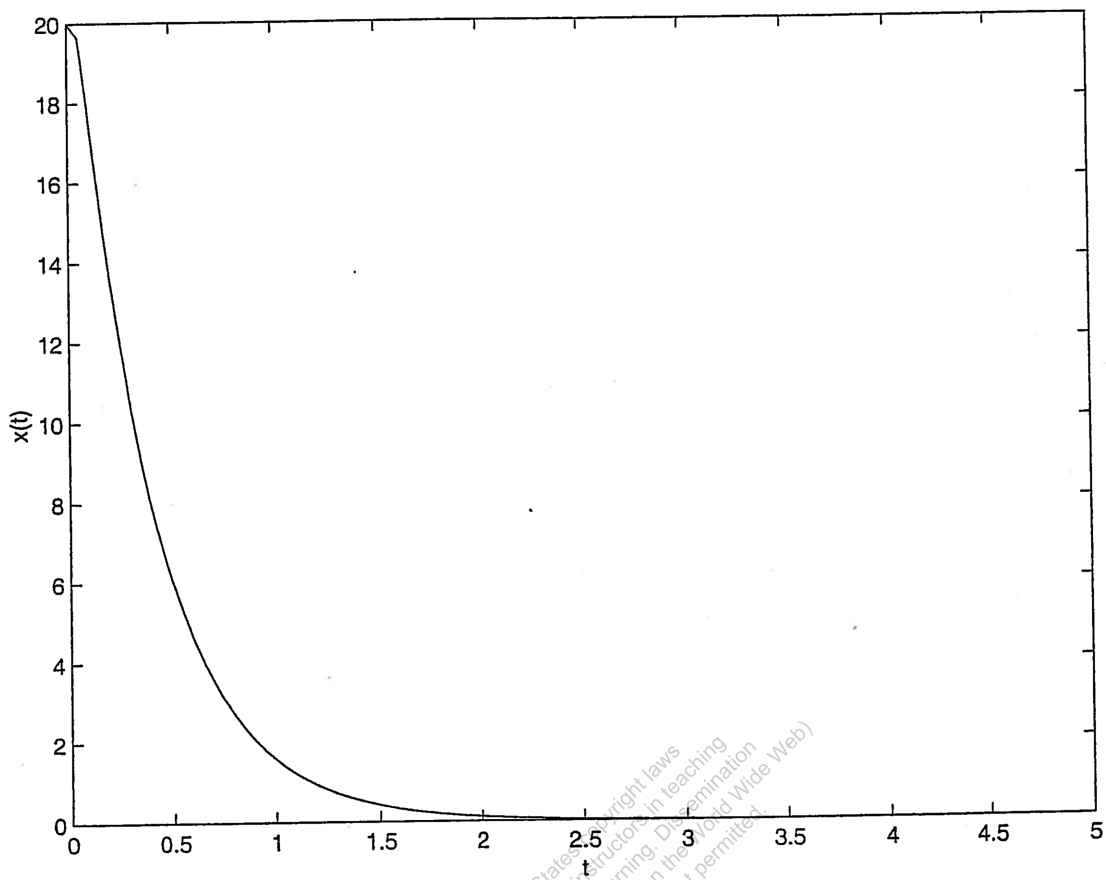
```
% Ex2_188.m
wn = 10;
dx0 = 0;
x0 = 10;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x1(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
```

```
x0 = 50;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x2(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
x0 = 100;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x3(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
x0 = 0;
dx0 = 10;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x4(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
dx0 = 50;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x5(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
dx0 = 100;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x6(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
subplot(231);
plot(t,x1);
title('x0=10 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(232);
plot(t,x2);
title('x0=50 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(233);
plot(t,x3);
title('x0=100 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(234);
plot(t,x4);
title('x0=0 dx0=10');
xlabel('t');
ylabel('x(t)');
subplot(235);
plot(t,x5);
title('x0=0 dx0=50');
xlabel('t');
ylabel('x(t)');
subplot(236);
plot(t,x6);
title('x0=0 dx0=100');
xlabel('t');
ylabel('x(t)');
```



2.189

```
% Ex2_189.m
wn = 10;
zeta = 2.0;
dx0 = 50;
x0 = 20;
c1 = ( x0*wn*( zeta + sqrt(zeta^2-1) ) + dx0 )/( 2*wn*sqrt(zeta^2-1) );
c2 = ( -x0*wn*( zeta - sqrt(zeta^2-1) ) - dx0 )/( 2*wn*sqrt(zeta^2-1) );
for i = 1:101
    t(i) = 5*(i-1)/100;
    x(i) = c1*exp( (-zeta + sqrt(zeta^2-1)) *wn*t(i) ) ...
        + c2*exp( (-zeta - sqrt(zeta^2-1)) *wn*t(i) );
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```

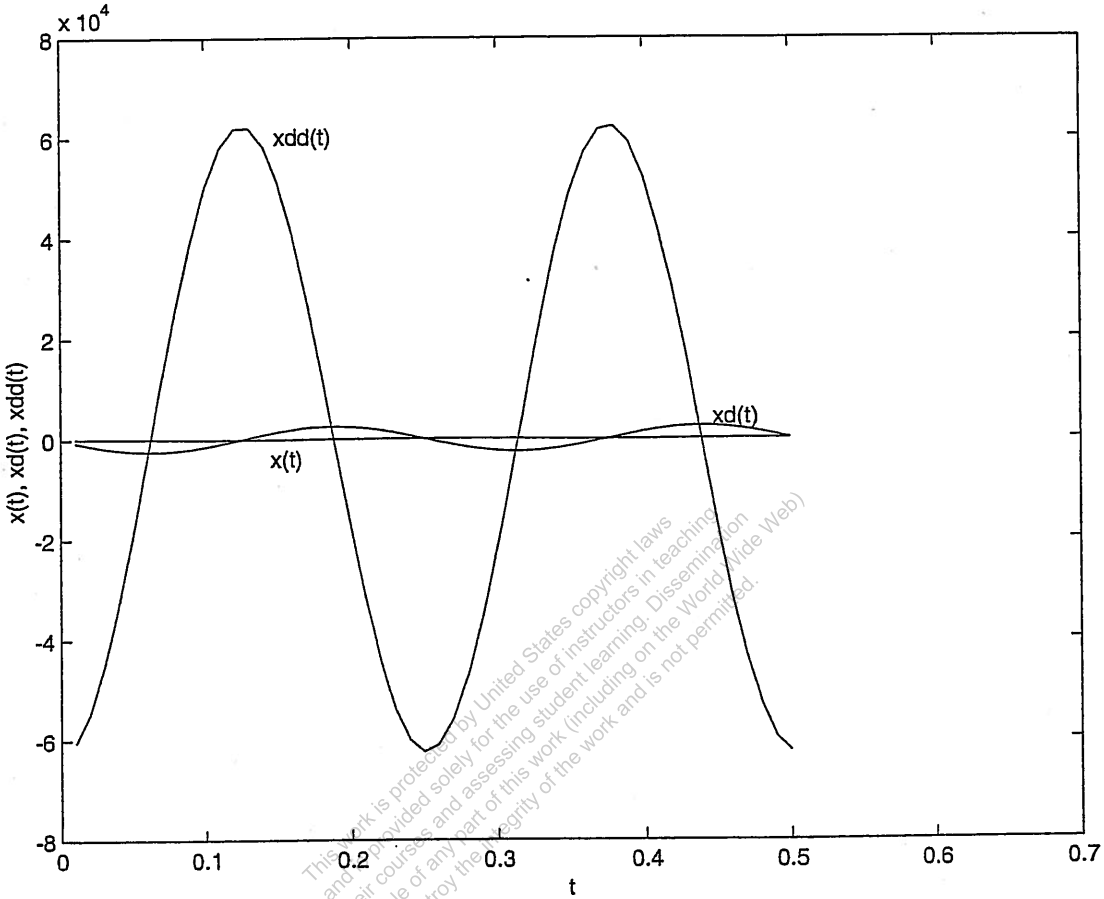


2.190

Results of Ex2_190.m

```
*****************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=      4.00000000e+000
k=      2.50000000e+003
c=      0.00000000e+000
x0=     1.00000000e+002
xd0=    -1.00000000e+001
n=      50
delt=   1.00000000e-002


system is undamped

Results:

 i    time(i)          x(i)            xd(i)            xdd(i)

 1  1.000000e-002   9.679228e+001   -6.282079e+002   -6.049518e+004
 2  2.000000e-002   8.756649e+001   -1.207348e+003   -5.472905e+004
 3  3.000000e-002   7.289623e+001   -1.711420e+003   -4.556014e+004
 4  4.000000e-002   5.369364e+001   -2.109085e+003   -3.355853e+004
 5  5.000000e-002   3.115264e+001   -2.375618e+003   -1.947040e+004
 6  6.000000e-002   6.674722e+000   -2.494445e+003   -4.171701e+003
 :
```

44	4.400000e-001	8.425659e-001	2.499931e+003	-5.266037e+002
45	4.500000e-001	2.555609e+001	2.417001e+003	-1.597256e+004
46	4.600000e-001	4.868066e+001	2.183793e+003	-3.042541e+004
47	4.700000e-001	6.877850e+001	1.814807e+003	-4.298656e+004
48	4.800000e-001	8.460003e+001	1.332986e+003	-5.287502e+004
49	4.900000e-001	9.516153e+001	7.682859e+002	-5.947596e+004
50	5.000000e-001	9.980636e+001	1.558176e+002	-6.237897e+004

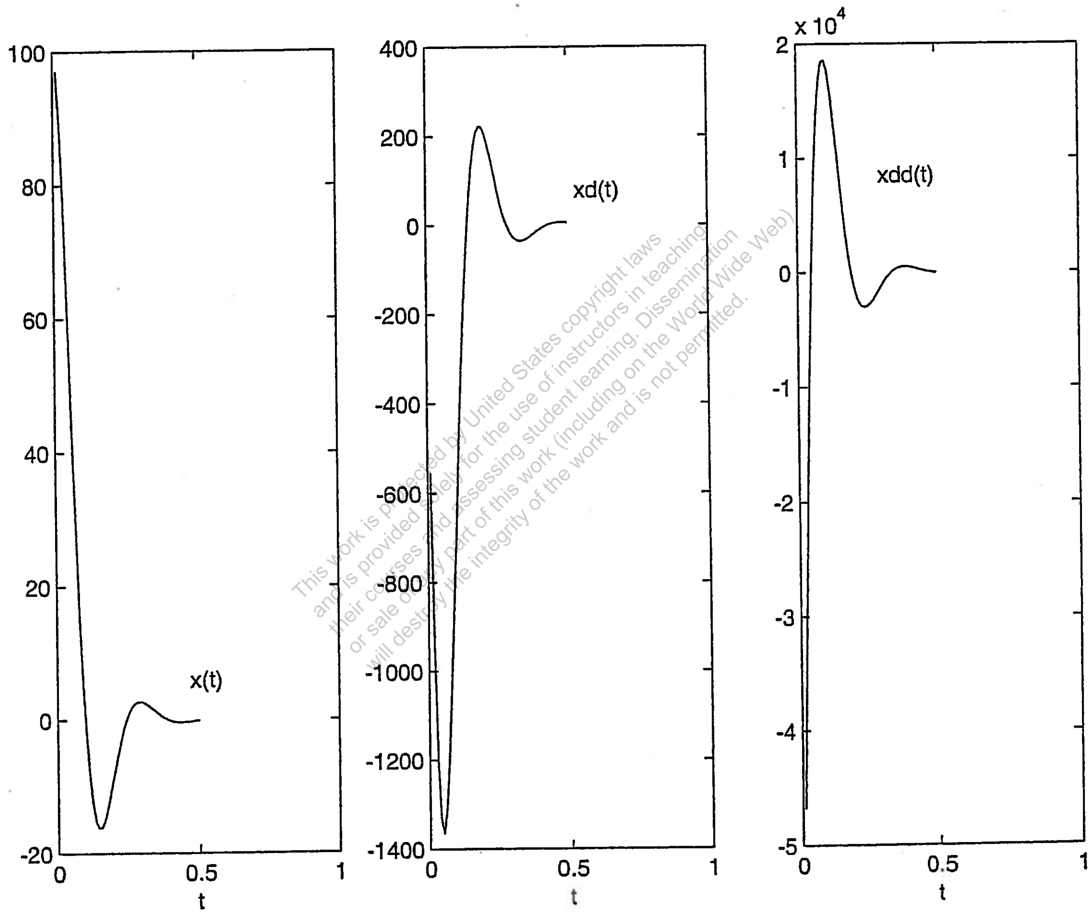


2.191

Results of Ex2_191.m

```
****************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=      4.00000000e+000
k=      2.50000000e+003
c=      1.00000000e+002
x0=     1.00000000e+002
xd0=    -1.00000000e+001
n=      50
delt=   1.00000000e-002


system is under damped

Results:
```

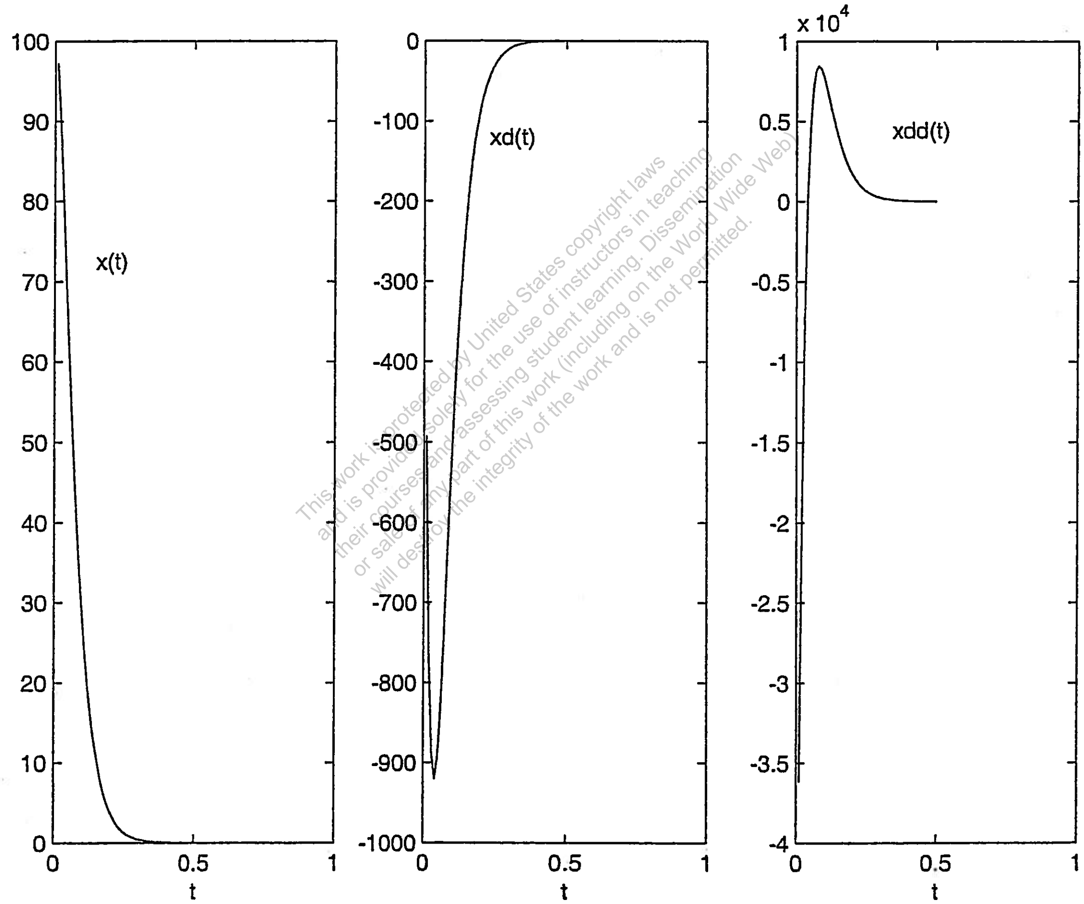
i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.704707e+001	-5.547860e+002	-4.678477e+004
2	2.000000e-002	8.940851e+001	-9.485455e+002	-3.216668e+004
3	3.000000e-002	7.854100e+001	-1.203024e+003	-1.901253e+004
4	4.000000e-002	6.575661e+001	-1.335030e+003	-7.722135e+003
5	5.000000e-002	5.218268e+001	-1.364393e+003	1.495649e+003
6	6.000000e-002	3.874058e+001	-1.312202e+003	8.592187e+003
⋮				
45	4.500000e-001	-4.071590e-001	3.283084e+000	1.723973e+002
46	4.600000e-001	-3.667451e-001	4.698554e+000	1.117518e+002
47	4.700000e-001	-3.150951e-001	5.542443e+000	5.837337e+001
48	4.800000e-001	-2.575358e-001	5.894760e+000	1.359090e+001
49	4.900000e-001	-1.985409e-001	5.844858e+000	-2.203340e+001
50	5.000000e-001	-1.416733e-001	5.484453e+000	-4.856551e+001



2.192

Results of Ex2_192.m

```
*****************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=       4.00000000e+000
k=       2.50000000e+003
c=       2.00000000e+002
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002


system is critically damped

Results:
```



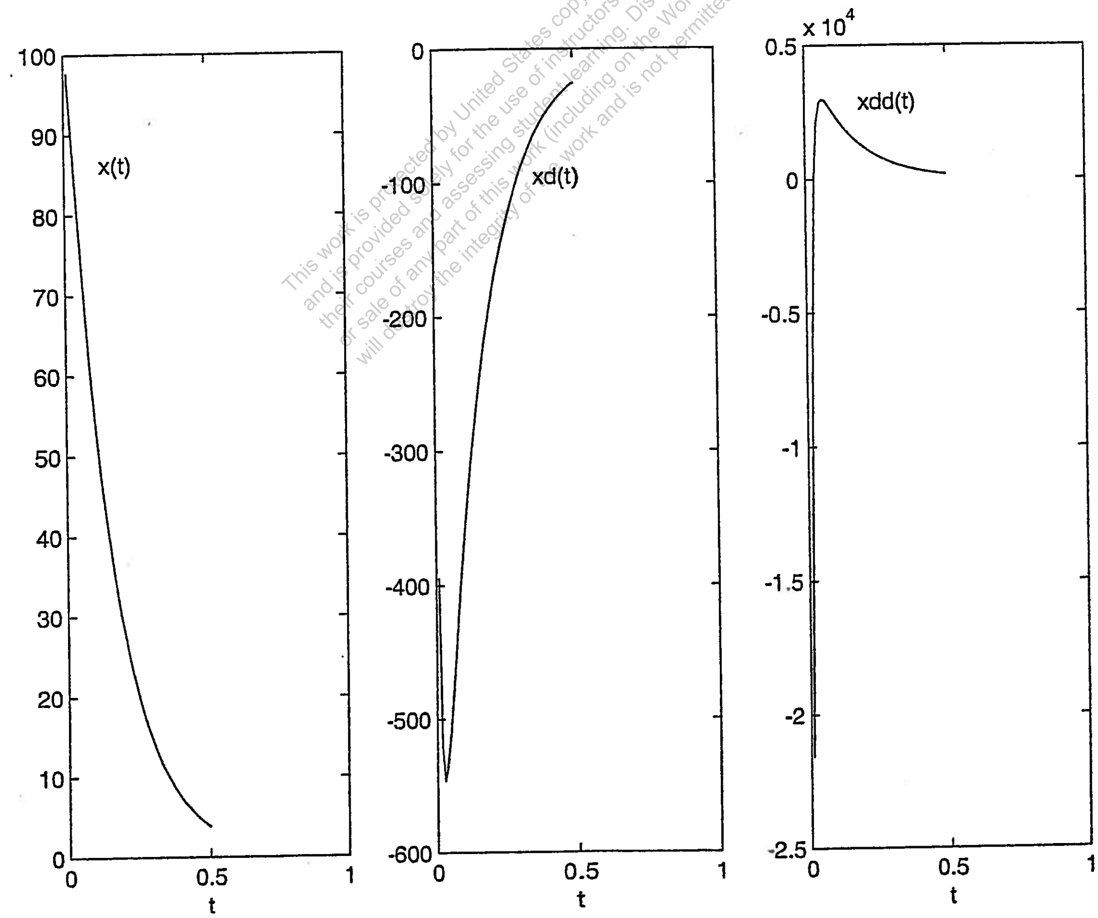
i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.727222e+001	-4.925915e+002	-3.616556e+004
2	2.000000e-002	9.085829e+001	-7.611960e+002	-1.872663e+004
3	3.000000e-002	8.252244e+001	-8.868682e+002	-7.233113e+003
4	4.000000e-002	7.342874e+001	-9.196986e+002	9.196986e+001
5	5.000000e-002	6.432033e+001	-8.946112e+002	4.530357e+003
⋮				

44	4.400000e-001	1.996855e-002	-4.576266e-001	1.040098e+001
45	4.500000e-001	1.587541e-002	-3.644970e-001	8.302721e+000
46	4.600000e-001	1.261602e-002	-2.901765e-001	6.623815e+000
47	4.700000e-001	1.002181e-002	-2.309008e-001	5.281410e+000
48	4.800000e-001	7.957984e-003	-1.836505e-001	4.208785e+000
49	4.900000e-001	6.316833e-003	-1.460059e-001	3.352274e+000
50	5.000000e-001	5.012349e-003	-1.160293e-001	2.668750e+000

2.193

Results of Ex2_193.m

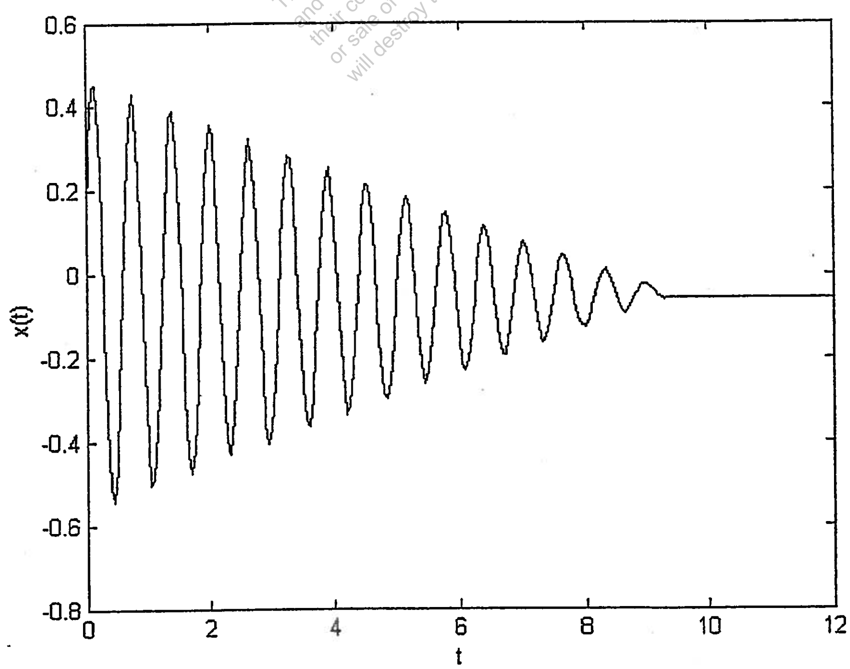
```
***************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=      4.00000000e+000
k=      2.50000000e+003
c=      4.00000000e+002
x0=     1.00000000e+002
xd0=    -1.00000000e+001
n=      50
delt=   1.00000000e-002

system is over damped

Results:
```



i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.764929e+001	-3.945541e+002	-2.157540e+004
2	2.000000e-002	9.294636e+001	-5.205155e+002	-6.039927e+003
3	3.000000e-002	8.756294e+001	-5.463949e+002	-8.734340e+001
4	4.000000e-002	8.214078e+001	-5.344391e+002	2.105923e+003
5	5.000000e-002	7.691749e+001	-5.090344e+002	2.830006e+003
⋮				
45	4.500000e-001	5.281309e+000	-3.537806e+001	2.369881e+002
46	4.600000e-001	4.939118e+000	-3.308581e+001	2.216329e+002
47	4.700000e-001	4.619098e+000	-3.094209e+001	2.072727e+002
48	4.800000e-001	4.319813e+000	-2.893726e+001	1.938429e+002
49	4.900000e-001	4.039920e+000	-2.706233e+001	1.812832e+002
50	5.000000e-001	3.778161e+000	-2.530888e+001	1.695374e+002

2.194

```
% Ex2_194.m
% This program will use dfunc2_194.m
tspan = [0: 0.05: 12];
x0 = [0.1; 5];
[t, x] = ode23 ('dfunc2_194', tspan, x0);
plot(t, x(:, 1));
xlabel('t');
ylabel('x(t)');

% dfunc2_194.m
function f = dfunc2_194(t, x)
u = 0.1;
k = 1000;
m = 20;
g = 9.81;
theta = 30 * pi/180;
f = zeros(2,1);
f(1) = x(2);
f(2) = -u*g*cos(theta)*sign(x(2)) - 2*k*x(1)/m - g*sin(theta);
```

2.195 The equations for the natural frequencies of vibration were derived in Problem 2.35.

Operating speed of turbine is:

$$\omega_o = (2400)\,\frac{2\pi}{60} = 251.328 \text{ rad/sec}$$

Thus we need to satisfy:

$$\omega_n\Big|_{axial} = \left\{\frac{g\,l\,A\,E}{W\,a\,(l-a)}\right\}^{1/2} \geq \omega_o \qquad (E_1)$$

$$\omega_n\Big|_{transverse} = \left\{\frac{3\,E\,I\,l^3\,g}{W\,a^3\,(l-a)^3}\right\}^{1/2} \geq \omega_o \qquad (E_2)$$

$$\omega_n\Big|_{circumferential} = \left\{\frac{G\,J}{J_o}\left(\frac{1}{a} + \frac{1}{l-a}\right)\right\}^{1/2} \geq \omega_o \qquad (E_3)$$

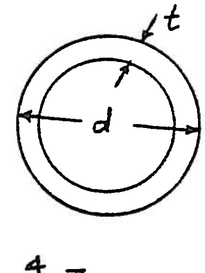
where $A = \frac{\pi d^2}{4}$, $W = 1000 \times 9.81 = 9810$ N,

$I = \frac{\pi d^4}{64}$, $J = \frac{\pi d^2}{32}$, $J_o = 500$ kg-m^2,

and $E = 207 \times 10^9$ N/m^2, $G = 79.3 \times 10^9$ N/m^2 (for steel).

The unknowns d, l and a can be determined to satisfy the inequalities (E_1), (E_2) and (E_3) using a trial and error procedure.

2.196 From solution of problem 2.38, the requirements can be stated as:



$$\omega_n\Big|_{\text{pivot ends}} = \sqrt{\frac{12\,EI}{l^3\left(\frac{W}{g} + m_{eff1}\right)}} \geq \omega_0 \qquad (E_1)$$

where $E = 30 \times 10^6$ psi and $I = \frac{\pi}{64}\left[d^4 - (d-2t)^4\right]$

$$\omega_n\Big|_{\text{fixed ends}} = \sqrt{\frac{48\,EI}{l^3\left(\frac{W}{g} + m_{eff2}\right)}} \geq \omega_0 \qquad (E_2)$$

with $m_{eff1} = (0.2357\ m)$, $m_{eff2} = (0.3714\ m)$,

m = mass of each column $= \frac{\pi}{4}\left[d^2 - (d-2t)^2\right]\frac{l\,\rho}{g}$,

$\rho = 0.283$ lb/in^3, $g = 386.4$ in/sec^2,

l = length of column = 96 in.,

W = weight of floor = 4000 lb,

$$\underset{\sim}{w} = \text{weight of columns} = 4\left\{\frac{\pi}{4}\left[d^2 - (d-2t)^2\right] l\,\rho\right\} \qquad (E_3)$$

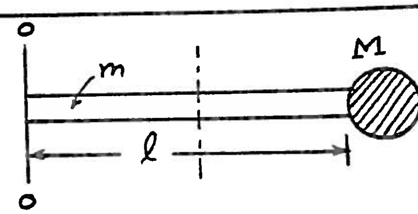
Frequency limit $= \omega_0 = 50 \times 2\pi = 314.16$ rad/sec.

Problem: Find d and t such that $\underset{\sim}{w}$ given by Eq. (E_3) is minimized while satisfying the inequalities (E_1) and (E_2).

This problem can be solved either by graphical optimization or by using a trial and error procedure.

2.197

$$J_0 = \frac{m l^2}{12} + \frac{m l^2}{4} + M l^2 = \frac{1}{3} m l^2 + M l^2 \qquad \text{--- } (E_1)$$



(i) Viscous damping:

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{k_t}{\frac{1}{3} m l^2 + M l^2}\right)^{\frac{1}{2}} \qquad \text{--- } (E_2)$$

$$(c_t)_{cri} = 2\,J_0\,\omega_n = 2\sqrt{J_0\,k_t} \qquad \text{--- } (E_3)$$

For critical damping, Eq. (2.80) gives

$$\theta(t) = \left\{\theta_0 + \left(\dot{\theta}_0 + \omega_n\,\theta_0\right)t\right\} e^{-\omega_n t} \qquad \text{--- } (E_4)$$

For $\theta_0 = 75^\circ = 1.309$ rad and $\dot{\theta}_0 = 0$,

$$\theta(t) = (1.309 + 1.309\,\omega_n t)\, e^{-\omega_n t} \qquad \text{--- } (E_5)$$

For $\theta = 5^\circ = 0.08727$ rad, Eq. (E_5) becomes

$$0.08727 = 1.309\,(1 + \omega_n t)\, e^{-\omega_n t} \qquad \text{--- } (E_6)$$

Let time to return = 2 sec. Then Eq. (E_6) gives

$$0.08727 = 1.309\,(1 + 2\omega_n)\, e^{-2\omega_n} \qquad \text{--- } (E_7)$$

Solve (E_7) by trial and error to find ω_n. Then choose the values of m, M and k_t to get the desired value of ω_n. Find the damping constant $(c_t)_{cri}$ using Eq. (E_3).

(ii) Coulomb damping:

(a) Follow the procedure of part(i) to find the value of ω_n.

(b) Derive expression for the equivalent torsional viscous damping constant $(c_t)_{eq}$ for Coulomb damping. This expression, for small amounts of damping, is

$$(c_t)_{eq} = \left\{ 4\, T_d \big/ \pi\, \omega_n\, \Theta \right\} \qquad \text{--- } (E_8)$$

where T_d = friction (damping) torque, and Θ = amplitude of angular oscillations.

(c) If $(c_t)_{eq}$ is to be equal to $(c_t)_{cri} = 2\sqrt{J_0\, k_t}$, we find

$$T_d = \frac{\pi\, \omega\, \Theta}{4} \left(2 \sqrt{J_0\, k_t} \right) \qquad \text{--- } (E_9)$$

2.198 Let x = vertical displacement of the mass (lunar excursion module), x_s = resulting deflection of each inclined leg (spring). From equivalence of potential energy, we find:
k_{eq_1} = stiffness of each leg in vertical direction = $k \cos^2 \alpha$
Hence for the four legs, the equivalent stiffness in vertical direction is:

$$k_{eq} = 4\, k \cos^2 \alpha$$

Similarly, the equivalent damping coefficient of the four legs in vertical direction is:

$$c_{eq} = 4\, c \cos^2 \alpha$$

where c = damping constant of each leg (in axial motion). Modeling the system as a single degree of freedom system, the equation of motion is:

$$m_{eq}\, \ddot{x} + c_{eq}\, \dot{x} + k_{eq}\, x = 0$$

and the damped period of vibration is:

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{2\pi}{\sqrt{\frac{k_{eq}}{m_{eq}}}\sqrt{1-\left(\frac{c_{eq}^2}{4\, k_{eq}\, m_{eq}}\right)}}$$

Using m_{eq} = 2000 kg, $k_{eq} = 4\, k \cos^2 \alpha$, $c_{eq} = 4\, c \cos^2 \alpha$, and $\alpha = 20°$, the values of k and c can be determined (by trial and error) so as to achieve a value of τ_d between 1 s and 2 s. Once k and c are known, the spring (helical) and damper (viscous) can be designed suitably.

2.199 Assume no damping. Neglect masses of telescoping boom and strut. Find stiffness of telescoping boom in vertical direction (see Example 2.5). Find the equivalent stiffness of telescoping boom together with the strut in vertical direction. Model the system as a single degree of freedom system with natural time period:

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m_{eq}}{k_{eq}}}$$

Using τ_n = 1 s and $m_{eq} = \left(\frac{W_c + W_f}{g}\right) = \frac{300}{386.4}$, determine the axial stiffness of the strut (k_s). Once k_s is known, the cross section of the strut (A_s) can be found from:

$$k_s = \frac{A_s\, E_s}{\ell_s}$$

with $E_s = 30\,(10^6)$ psi and ℓ_s = length of strut (known).
