Introduction to
# Finite Elements in Engineering

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# Solutions Manual

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## *PREFACE*

*This solutions manual serves as an aid to professors in teaching from the book* **Introduction to Finite Elements in Engineering, 4th Edition**. *The problems in the book fall into the following categories:*

1. *Simple problems to understand the concepts*
2. *Derivations and direct solutions*
3. *Solutions requiring computer runs*
4. *Solutions requiring program modifications*

*Our basic philosophy in the development of this manual is to provide a complete guidance to the teacher in formulating, modeling, and solving the problems. Complete solutions are given for problems in all categories stated. For some larger problems such as those in three dimensional stress analysis, complete formulation and modeling aspects are discussed. The students should be able to proceed from the guidelines provided.*

*For problems involving distributed and other types of loading, the nodal loads are to be calculated for the input data. The programs do not generate the loads. This calculation and the boundary condition decisions enable the student to develop a physical sense for the problems. The students may be encouraged to modify the programs to calculate the loads automatically.*

*The students should be introduced to the programs in Chapter 12 right from the point of solving problems in Chapter 6. This will enable the students to solve larger problems with ease. The input data file for each program has been provided. Data for a problem should follow this format. The best strategy is to copy the example file and edit it for the problem under consideration. The data from program MESHGEN will need some editing to complete the information on boundary conditions, loads, and material properties.*

*We thank you for your enthusiastic response to our first three editions of the book. We look forward to receive your feedback of your experiences, comments, and suggestions for making improvements to the book and this manual.*

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# CHAPTER 1
# FUNDAMENTAL CONCEPTS

**1.1** We use the first three steps of Eq. 1.11

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = -\nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

Adding the above, we get

$$\varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

Adding and subtracting $\nu\frac{\sigma_x}{E}$ from the first equation,

$$\varepsilon_x = \frac{1+\nu}{E}\sigma_x - \frac{\nu}{E}\left(\sigma_x + \sigma_y + \sigma_z\right)$$

Similar expressions can be obtained for $\varepsilon_y$, and $\varepsilon_z$.

From the relationship for $\gamma_{yz}$ and Eq. 1.12,

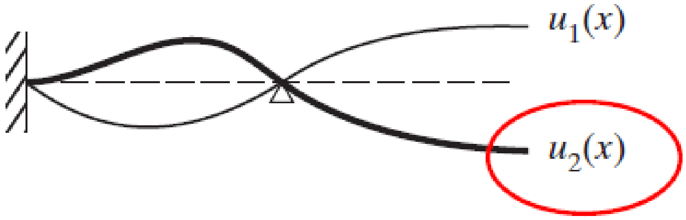
$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz} \qquad \text{etc.}$$

Above relations can be written in the form

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

where **D** is the material property matrix defined in Eq. 1.15. ■

**1.2** Note that $u_2(x)$ satisfies the zero slope boundary condition at the support.




■

**1.3** Plane strain condition implies that

$$\varepsilon_z = 0 = -\nu \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} + \frac{\sigma_z}{E}$$

which gives

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

We have, $\sigma_x = 20000\text{ psi} \quad \sigma_y = -10000\text{ psi} \quad E = 30\times10^6\text{ psi} \quad \nu = 0.3$.

On substituting the values,

$$\sigma_z = 3000\text{ psi} \qquad \blacksquare$$

**1.4** Displacement field

$$u = 10^{-4}\left(-x^2 + 2y^2 + 6xy\right)$$
$$v = 10^{-4}\left(3x + 6y - y^2\right)$$
$$\frac{\partial u}{\partial x} = 10^{-4}(-2x + 6y) \quad \frac{\partial u}{\partial y} = 10^{-4}(4y + 6x)$$
$$\frac{\partial v}{\partial x} = 3\times10^{-4} \qquad \frac{\partial v}{\partial y} = 10^{-4}(6 + 2y)$$
$$\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

at $x = 1, y = 0$

$$\varepsilon = 10^{-4}\begin{Bmatrix} -2 \\ 6 \\ 9 \end{Bmatrix}$$

$\blacksquare$

**1.5** On inspection, we note that the displacements $u$ and $v$ are given by

$$u = 0.1\,y + 4$$
$$v = 0$$

It is then easy to see that

$$\varepsilon_x = \frac{\partial u}{\partial x} = 0$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = 0$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.1$$

■

**1.6** The displacement field is given as

$$u = 1 + 3x + 4x^3 + 6xy^2$$
$$v = xy - 7x^2$$

**(a)** The strains are then given by

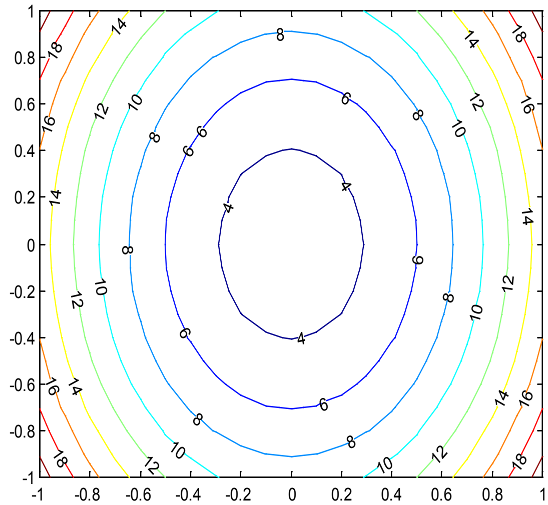
$$\varepsilon_x = \frac{\partial u}{\partial x} = 3 + 12x^2 + 6y^2$$
$$\varepsilon_y = \frac{\partial v}{\partial y} = x$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 12xy + y - 14x$$

**(b)** In order to draw the contours of the strain field using MATLAB, we need to create a script file, which may be edited as a text file and save with ".m" extension. The file for plotting $\varepsilon_x$ is given below

file "prob1p5b.m"

```
[X,Y] = meshgrid(-1:.1:1,-1:.1:1);
Z = 3.+12.*X.^2+6.*Y.^2;
[C,h] = contour(X,Y,Z);
clabel(C,h);
```

On running the program, the contour map is shown as follows:



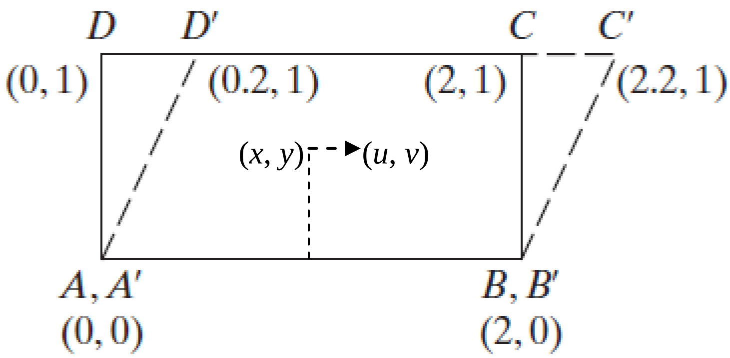
Contours of $\varepsilon_x$

Contours of $\varepsilon_y$ and $\gamma_{xy}$ are obtained by changing Z in the script file. The numbers on the contours show the function values.

**(c)** The maximum value of $\varepsilon_x$ is at any of the corners of the square region. The maximum value is 21.

■

**1.7**




a) $u = \frac{0.2}{1} y \Rightarrow u = 0.2y \qquad v = 0$

b) $\varepsilon_x = \frac{\partial u}{\partial x} = 0 \quad \varepsilon_y = \frac{\partial v}{\partial y} = 0 \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.2$

**1.8**

$$\sigma_x = 40\text{ MPa} \quad \sigma_y = 20\text{ MPa} \quad \sigma_z = 30\text{ MPa}$$
$$\tau_{yz} = -30\text{ MPa} \quad \tau_{xz} = 15\text{ MPa} \quad \tau_{xy} = 10\text{ MPa}$$
$$\mathbf{n} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}^{\mathrm{T}}$$

From Eq.1.8 we get

$$T_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$
$$= 35.607\text{ MPa}$$
$$T_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$
$$= -6.213\text{ MPa}$$
$$T_z = \tau_{xz} n_x + \tau_{yz} n_y + \sigma_z n_z$$
$$= 13.713\text{ MPa}$$
$$\sigma_n = T_x n_x + T_y n_y + T_z n_z$$
$$= 24.393\text{ MPa}$$

■

**1.9** From the derivation made in P1.1, we have

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu\varepsilon_y + \nu\varepsilon_z\right]$$

which can be written in the form

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-2\nu)\varepsilon_x + \nu\varepsilon_v\right]$$

and

$$\tau_{yz} = \frac{E}{2(1+\nu)}\gamma_{yz}$$

Lame's constants $\lambda$ and $\mu$ are defined in the expressions

$$\sigma_x = \lambda\varepsilon_v + 2\mu\varepsilon_x$$
$$\tau_{yz} = \mu\gamma_{yz}$$

On inspection,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$
$$\mu = \frac{E}{2(1+\nu)}$$

μ is same as the shear modulus *G*. ■

**1.10**

$$\varepsilon = 1.2 \times 10^{-5}$$
$$\Delta T = 30^0 C$$
$$E = 200\,\text{GPa}$$
$$\alpha = 12 \times 10^{-6} / ^0 C$$
$$\varepsilon_0 = \alpha \Delta T = 3.6 \times 10^{-4}$$
$$\sigma = E(\varepsilon - \varepsilon_0) = -69.6\,\text{MPa}$$

■

**1.11**

$$\varepsilon_x = \frac{du}{dx} = 1 + 2x^2$$
$$\delta = \int_0^L \frac{du}{dx} dx = \left( x + \frac{2}{3} x^3 \right) \Bigg|_0^L$$
$$= L\left(1 + \frac{2}{3} L^2\right)$$

■

**1.12** Following the steps of Example 1.1, we have

$$\begin{bmatrix} (80+40+50) & -80 \\ -80 & 80 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 60 \\ 50 \end{Bmatrix}$$

Above matrix form is same as the set of equations:

$$170\, q_1 - 80\, q_2 = 60$$

$$-80\, q_1 + 80\, q_2 = 50$$

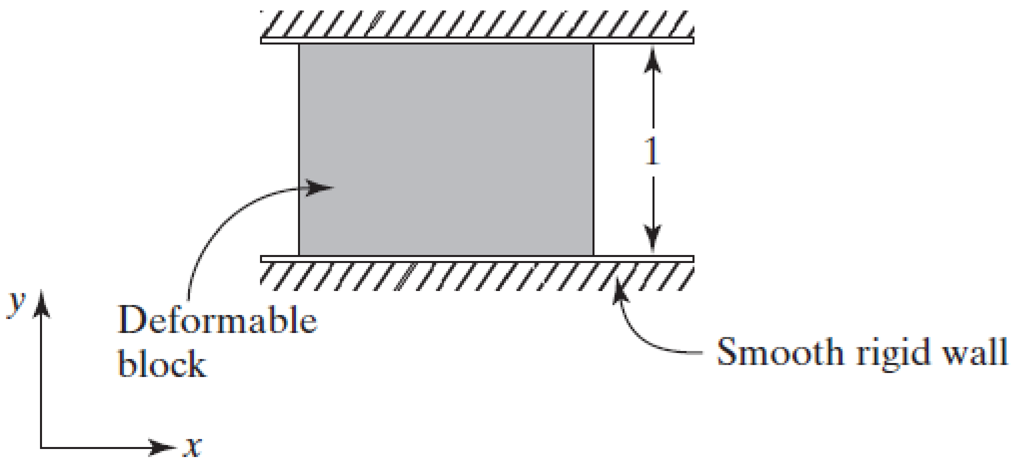
Solving for $q_1$ and $q_2$, we get

$$q_1 = 1.222 \text{ mm}$$

$$q_2 = 1.847 \text{ mm}$$

■

**1.13**




When the wall is smooth, $\sigma_x = 0$. $\Delta T$ is the temperature rise.

a) When the block is thin in the *z* direction, it corresponds to plane stress condition. The rigid walls in the *y* direction require $\varepsilon_y = 0$. The generalized Hooke's law yields the equations

$$\varepsilon_x = -\nu \frac{\sigma_y}{E} + \alpha \Delta T$$

$$\varepsilon_y = \frac{\sigma_y}{E} + \alpha \Delta T$$

From the second equation, setting $\varepsilon_y = 0$, we get $\sigma_y = -E\alpha\Delta T$. $\varepsilon_x$ is then calculated using the first equation as $(1-\nu)\alpha\Delta T$.

b) When the block is very thick in the *z* direction, plain strain condition prevails. Now we have $\varepsilon_z = 0$, in addition to $\varepsilon_y = 0$. $\sigma_z$ is not zero.

$$\varepsilon_x = -\nu \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} + \alpha \Delta T$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_z}{E} + \alpha \Delta T = 0$$

$$\varepsilon_z = -\nu \frac{\sigma_y}{E} + \frac{\sigma_z}{E} + \alpha \Delta T = 0$$

From the last two equations, we get

$$\sigma_y = \frac{-E\alpha\Delta T}{1+\nu}$$ ~~$\sigma_z = -\frac{1+2\nu}{1+\nu} E\alpha\Delta T$~~

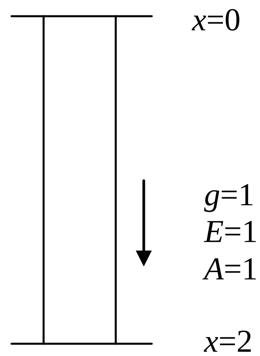
$\varepsilon_x$ is now obtained from the first equation. ■

and $\sigma_z$ from the third equation

**1.14** For thin block, it is plane stress condition. Treating the nominal size as 1, we may set the initial strain $\varepsilon_0 = \alpha \Delta T = \frac{0.1}{1}$ in part (a) of problem 1.13. Thus $\sigma_y = -0.1E$. ■

**1.15**

The potential energy $\Pi$ is given by




$$\Pi = \frac{1}{2}\int_0^2 EA\left(\frac{du}{dx}\right)^2 dx - \int_0^2 ugAdx$$

Consider the polynomial from Example 1.2,

$$u = a_3\left(-2x + x^2\right)$$

$$\frac{du}{dx} = \left(-2 + 2x\right)a_3 = 2\left(-1 + x\right)a_3$$

On substituting the above expressions and integrating, the first term of becomes

$$2a_3^{\ 2}\left(\frac{2}{3}\right)$$

and the second term

$$\int_0^2 ugAdx = \int_0^2 udx = a_3\left(-x^2 + \frac{x^3}{3}\right)\Big|_0^2$$

$$= -\frac{4}{3}a_3$$

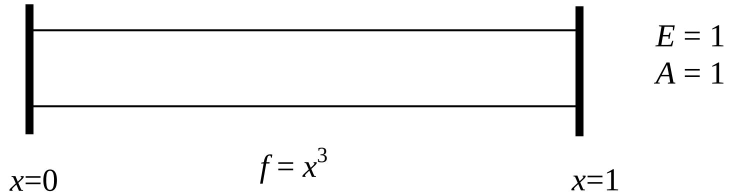
Thus

$$\Pi = \frac{4}{3}\left(a_3^{\ 2} + a_3\right)$$

$$\frac{\partial \Pi}{\partial a_3} = 0 \Rightarrow \quad a_3 = -\frac{1}{2}$$

this gives $u_{x=1} = -\frac{1}{2}\left(-2 + 1\right) = 0.5$ ■

**1.16**



We use the displacement field defined by $u = a_0 + a_1x + a_2x^2$.

$$u = 0 \text{ at } x = 0 \Rightarrow a_0 = 0$$

$$u = 0 \text{ at } x = 1 \Rightarrow a_1 + a_2 = 0 \Rightarrow a_2 = -a_1$$

We then have $u = a_1x(1-x)$, and $du/dx = a_1(1-x)$.

The potential energy is now written as

$$\Pi = \frac{1}{2}\int_0^1 \left(\frac{du}{dx}\right)^2 dx - \int_0^1 fudx$$
$$= \frac{1}{2}\int_0^1 a_1^2(1-2x)^2 dx - \int_0^1 x^3 a_1 x(1-x)dx$$
$$= \frac{1}{2}\int_0^1 a_1^2(1-4x+4x^2)dx - \int_0^1 a_1(x^4 - x^5)dx$$
$$= \frac{1}{2}a_1^2\left(1-\frac{4}{2}+\frac{4}{3}\right) - a_1\left(\frac{1}{5}-\frac{1}{6}\right)$$
$$= \frac{a_1^2}{6} - \frac{a_1}{30}$$

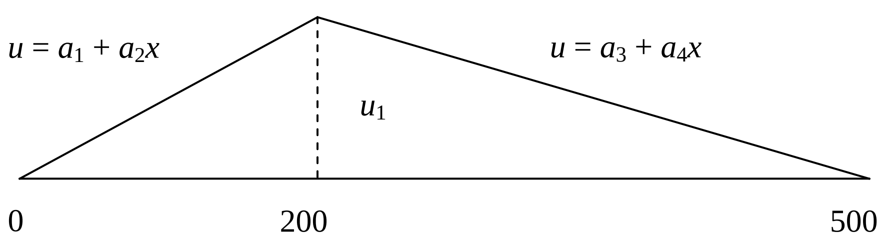
$$\frac{\partial \Pi}{\partial a_1} = 0 \quad \Rightarrow \quad \frac{a_1}{3} - \frac{1}{30} = 0$$

This yields, $a_1 = 0.1$

Displacemen $u = 0.1x(1-x)$

Stress $\sigma = E\, du/dx = 0.1(1-2x)$ ■

**1.17** Let $u_1$ be the displacement at $x = 200$ mm. Piecewise linear displacement that is continuous in the interval $0 \le x \le 500$ is represented as shown in the figure.



$0 \le x \le 200$

$u = 0$ at $x = 0 \Rightarrow a_1 = 0$

$u = u_1$ at $x = 200 \Rightarrow a_2 = u_1/200$

$\Rightarrow u = (u_1/200)x \qquad du/dx = u_1/200$

$200 \le x \le 500$

$u = 0$ at $x = 500 \Rightarrow a_3 + 500\, a_4 = 0$

$u = u_1$ at $x = 200 \Rightarrow a_3 + 200\, a_4 = u_1$

$\Rightarrow a_4 = -u_1/300 \qquad a_3 = (5/3)u_1$

$\Rightarrow u = (5/3)u_1 - (u_1/300)x \qquad du/dx = -u_1/200$

$$\Pi = \frac{1}{2}\int_0^{200} E_{al}A_1\left(\frac{du}{dx}\right)^2 dx + \frac{1}{2}\int_{200}^{500} E_{st}A_2\left(\frac{du}{dx}\right)^2 dx - 10000u_1$$

$$\Pi = \frac{1}{2}E_{al}A_1\left(\frac{u_1}{200}\right)^2 200 + \frac{1}{2}E_{st}A_2\left(-\frac{u_1}{300}\right)^2 300 - 10000u_1$$

$$= \frac{1}{2}\left(\frac{E_{al}A_1}{200} + \frac{E_{st}A_2}{300}\right)u_1^2 - 10000u_1$$

$$\frac{\partial \Pi}{\partial u_1} = 0 \Rightarrow \left(\frac{E_{al}A_1}{200} + \frac{E_{st}A_2}{300}\right)u_1 - 10000 = 0$$

Note that using the units MPa ($N/mm^2$) for modulus of elasticity and $mm^2$ for area and mm for length will result in displacement in mm, and stress in MPa.

Thus, $E_{al} = 70000$ MPa, $E_{st} = 200000$, and $A_1 = 900$ $mm^2$, $A_2 = 1200$ $mm^2$. On substituting these values into the above equation, we get

$$u_1 = 0.009 \text{ mm}$$

This is precisely the solution obtained from strength of materials approach ■

**1.18**

In the Galerkin method, we start from the equilibrium equation

$$\frac{d}{dx}EA\frac{du}{dx} + g = 0$$

Following the steps of Example 1.3, we get

$$\int_0^2 -EA\frac{du}{dx}\frac{d\phi}{dx}dx + \int_0^2 g\phi dx$$

Introducing

$$u = (2x - x^2)u_1, \text{ and}$$
$$\phi = (2x - x^2)\phi_1$$

where $u_1$ and $\phi_1$ are the values of $u$ and $\phi$ at $x$ = 1 respectively,

$$\phi_1\left[-u_1\int_0^2 (1-2x)^2\,dx + \int_0^2 (2x - x^2)dx\right] = 0$$

On integrating, we get

$$\phi_1\left(-\frac{8}{3}u_1 + \frac{4}{3}\right) = 0$$

This is to be satisfied for every $\phi_1$, which gives the solution

$$u_1 = 0.5 \qquad \blacksquare$$

**1.19** We use

$$u = a_1 + a_2 x + a_3 x^2 + a_4 x^3$$
$$u = 0 \text{ at } x = 0$$
$$u = 0 \text{ at } x = 2$$

This implies that

$$0 = a_1$$
$$0 = a_1 + 2a_2 + 4_3 + 8a_4$$

and

$$u = a_3(x^2 - 2x) + a_4(x^3 - 4x)$$
$$\frac{du}{dx} = 2a_3(x-1) + a_4(3x^2 - 4)$$

$a_3$ and $a_4$ are considered as independent variables in

$$\Pi = \frac{1}{2}\int_0^2 \left[2a_3(x-1) + a_4(3x^2-4)\right]^2 dx - 2(-a_3 - 3a_4)$$

on expanding and integrating the terms, we get

$$\Pi = 1.333a_3{}^2 + 12.8a_4{}^2 + 8a_3a_4 + 2a_3 + 6a_4$$

We differentiate with respect to the variables and equate to zero.

$$\frac{\partial \Pi}{\partial a_3} = 2.667a_3 + 8a_4 + 2 = 0$$

$$\frac{\partial \Pi}{\partial a_4} = 8a_3 + 25.6a_4 + 6 = 0$$

On solving, we get

$a_3 = -0.74856$ and $a_4 = -0.00045$.

On substituting in the expression for $u$, at $x = 1$,

$u_1 = 0.749$

This approximation is close to the value obtained in the example problem. ■

**1.20**

**(a)** $$\Pi = \frac{1}{2}\int_0^L \sigma^T \varepsilon A dx - \int_0^L T(x) u dx$$

$$\sigma = E\varepsilon \quad \text{and} \quad \varepsilon = \frac{du}{dx}$$

On substitution,

$$\Pi = \frac{1}{2}\int_0^{60} EA\left(\frac{du}{dx}\right)^2 dx - \int_0^{30} Tudx - \int_{30}^{60} Tudx$$

$$\Pi = \frac{1}{2}\left(60\times 10^6\right)\int_0^{60}\left(\frac{du}{dx}\right)^2 dx - \int_0^{30} 10xudx - \int_{30}^{60} 300udx$$

**(b)**

Since $u = 0$ at $x = 0$ and $x = 60$, and $u = a_0 + a_1x + a_2x^2$, we have

$$u = a_2 x(x - 60)$$

$$\frac{du}{dx} = a_2(2x - 60)$$

On substituting and integrating,

$$\Pi = 216\times 10^{10} a_2^{\ 2} + 8775000a_2$$

Setting $d\Pi/da_2 = 0$ gives

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