

## CHAPTER 2

## 2.1 Limits of Sequences.

**2.1.0.** a) True. If  $x_n$  converges, then there is an  $M > 0$  such that  $|x_n| \leq M$ . Choose by Archimedes an  $N \in \mathbf{N}$  such that  $N > M/\varepsilon$ . Then  $n \geq N$  implies  $|x_n/n| \leq M/n \leq M/N < \varepsilon$ .

b) False.  $x_n = \sqrt{n}$  does not converge, but  $x_n/n = 1/\sqrt{n} \rightarrow 0$  as  $n \rightarrow \infty$ .

c) False.  $x_n = 1$  converges and  $y_n = (-1)^n$  is bounded, but  $x_n y_n = (-1)^n$  does not converge.

d) False.  $x_n = 1/n$  converges to 0 and  $y_n = n^2 > 0$ , but  $x_n y_n = n$  does not converge.

**2.1.1.** a) By the Archimedean Principle, given  $\varepsilon > 0$  there is an  $N \in \mathbf{N}$  such that  $N > 1/\varepsilon$ . Thus  $n \geq N$  implies

$$|(2 - 1/n) - 2| \equiv |1/n| \leq 1/N < \varepsilon.$$

b) By the Archimedean Principle, given  $\varepsilon > 0$  there is an  $N \in \mathbf{N}$  such that  $N > \pi^2/\varepsilon^2$ . Thus  $n \geq N$  implies

$$|1 + \pi/\sqrt{n} - 1| \equiv |\pi/\sqrt{n}| \leq \pi/\sqrt{N} < \varepsilon.$$

c) By the Archimedean Principle, given  $\varepsilon > 0$  there is an  $N \in \mathbf{N}$  such that  $N > 3/\varepsilon$ . Thus  $n \geq N$  implies

$$|3(1 + 1/n) - 3| \equiv |3/n| \leq 3/N < \varepsilon.$$

d) By the Archimedean Principle, given  $\varepsilon > 0$  there is an  $N \in \mathbf{N}$  such that  $N > 1/\sqrt{3\varepsilon}$ . Thus  $n \geq N$  implies

$$|(2n^2 + 1)/(3n^2) - 2/3| \equiv |1/(3n^2)| \leq 1/(3N^2) < \varepsilon.$$

**2.1.2.** a) By hypothesis, given  $\varepsilon > 0$  there is an  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $|x_n - 1| < \varepsilon/2$ . Thus  $n \geq N$  implies

$$|1 + 2x_n - 3| \equiv 2|x_n - 1| < \varepsilon.$$

b) By hypothesis, given  $\varepsilon > 0$  there is an  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $x_n > 1/2$  and  $|x_n - 1| < \varepsilon/4$ . In particular,  $1/x_n < 2$ . Thus  $n \geq N$  implies

$$|(\pi x_n - 2)/x_n - (\pi - 2)| \equiv 2|(x_n - 1)/x_n| < 4|x_n - 1| < \varepsilon.$$

c) By hypothesis, given  $\varepsilon > 0$  there is an  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $x_n > 1/2$  and  $|x_n - 1| < \varepsilon/(1 + 2e)$ . Thus  $n \geq N$  and the triangle inequality imply

$$|(x_n^2 - e)/x_n - (1 - e)| \equiv |x_n - 1| \left| 1 + \frac{e}{x_n} \right| \leq |x_n - 1| \left( 1 + \frac{e}{|x_n|} \right) < |x_n - 1|(1 + 2e) < \varepsilon.$$

**2.1.3.** a) If  $n_k = 2k$ , then  $3 - (-1)^{n_k} \equiv 2$  converges to 2; if  $n_k = 2k + 1$ , then  $3 - (-1)^{n_k} \equiv 4$  converges to 4.

b) If  $n_k = 2k$ , then  $(-1)^{3n_k} + 2 \equiv (-1)^{6k} + 2 = 1 + 2 = 3$  converges to 3; if  $n_k = 2k + 1$ , then  $(-1)^{3n_k} + 2 \equiv (-1)^{6k+3} + 2 = -1 + 2 = 1$  converges to 1.

c) If  $n_k = 2k$ , then  $(n_k - (-1)^{n_k} n_k - 1)/n_k \equiv -1/(2k)$  converges to 0; if  $n_k = 2k + 1$ , then  $(n_k - (-1)^{n_k} n_k - 1)/n_k \equiv (2n_k - 1)/n_k = (4k + 1)/(2k + 1)$  converges to 2.

**2.1.4.** Suppose  $x_n$  is bounded. By Definition 2.7, there are numbers  $M$  and  $m$  such that  $m \leq x_n \leq M$  for all  $n \in \mathbf{N}$ . Set  $C := \max\{1, |M|, |m|\}$ . Then  $C > 0$ ,  $M \leq C$ , and  $m \geq -C$ . Therefore,  $-C \leq x_n \leq C$ , i.e.,  $|x_n| < C$  for all  $n \in \mathbf{N}$ .

Conversely, if  $|x_n| < C$  for all  $n \in \mathbf{N}$ , then  $x_n$  is bounded above by  $C$  and below by  $-C$ .

**2.1.5.** If  $C = 0$ , there is nothing to prove. Otherwise, given  $\varepsilon > 0$  use Definition 2.1 to choose an  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $|b_n| \equiv b_n < \varepsilon/|C|$ . Hence by hypothesis,  $n \geq N$  implies

$$|x_n - a| \leq |C|b_n < \varepsilon.$$

By definition,  $x_n \rightarrow a$  as  $n \rightarrow \infty$ .

**2.1.6.** If  $x_n = a$  for all  $n$ , then  $|x_n - a| = 0$  is less than any positive  $\varepsilon$  for all  $n \in \mathbf{N}$ . Thus, by definition,  $x_n \rightarrow a$  as  $n \rightarrow \infty$ .

**2.1.7.** a) Let  $a$  be the common limit point. Given  $\varepsilon > 0$ , choose  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $|x_n - a|$  and  $|y_n - a|$  are both  $< \varepsilon/2$ . By the Triangle Inequality,  $n \geq N$  implies

$$|x_n - y_n| \leq |x_n - a| + |y_n - a| < \varepsilon.$$

By definition,  $x_n - y_n \rightarrow 0$  as  $n \rightarrow \infty$ .

b) If  $n$  converges to some  $a$ , then given  $\varepsilon = 1/2$ ,  $1 = |(n+1) - n| < |(n+1) - a| + |n - a| < 1$  for  $n$  sufficiently large, a contradiction.

c) Let  $x_n = n$  and  $y_n = n + 1/n$ . Then  $|x_n - y_n| = 1/n \rightarrow 0$  as  $n \rightarrow \infty$ , but neither  $x_n$  nor  $y_n$  converges.

**2.1.8.** By Theorem 2.6, if  $x_n \rightarrow a$  then  $x_{n_k} \rightarrow a$ . Conversely, if  $x_{n_k} \rightarrow a$  for every subsequence, then it converges for the “subsequence”  $x_n$ .

## 2.2 Limit Theorems.

**2.2.0.** a) False. Let  $x_n = n^2$  and  $y_n = -n$  and note by Exercise 2.2.2a that  $x_n + y_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

b) True. Let  $\varepsilon > 0$ . If  $x_n \rightarrow -\infty$  as  $n \rightarrow \infty$ , then choose  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $x_n < -1/\varepsilon$ . Then  $x_n < 0$  so  $|x_n| = -x_n > 0$ . Multiply  $x_n < -1/\varepsilon$  by  $\varepsilon/(-x_n)$  which is positive. We obtain  $-\varepsilon < 1/x_n$ , i.e.,  $|1/x_n| = -1/x_n < \varepsilon$ .

c) False. Let  $x_n = (-1)^n/n$ . Then  $1/x_n = (-1)^n n$  has no limit as  $n \rightarrow \infty$ .

d) True. Since  $(2^x - x)' = 2^x \log 2 - 1 > 1$  for all  $x \geq 2$ , i.e.,  $2^x - x$  is increasing on  $[2, \infty)$ . In particular,  $2^x - x \geq 2^2 - 2 > 0$ , i.e.,  $2^x > x$  for  $x \geq 2$ . Thus, since  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ , we have  $2^{x_n} > x_n$  for  $n$  large, hence

$$2^{-x_n} < \frac{1}{x_n} \rightarrow 0$$

as  $n \rightarrow \infty$ .

**2.2.1.** a)  $|x_n| \leq 1/n \rightarrow 0$  as  $n \rightarrow \infty$  and we can apply the Squeeze Theorem.

b)  $2n/(n^2 + \pi) = (2/n)/(1 + \pi/n^2) \rightarrow 0/(1 + 0) = 0$  by Theorem 2.12.

c)  $(\sqrt{2n} + 1)/(n + \sqrt{2}) = ((\sqrt{2}/\sqrt{n}) + (1/n))/(1 + (\sqrt{2}/n)) \rightarrow 0/(1 + 0) = 0$  by Exercise 2.2.5 and Theorem 2.12.

d) An easy induction argument shows that  $2n + 1 < 2^n$  for  $n = 3, 4, \dots$ . We will use this to prove that  $n^2 \leq 2^n$  for  $n = 4, 5, \dots$ . It's surely true for  $n = 4$ . If it's true for some  $n \geq 4$ , then the inductive hypothesis and the fact that  $2n + 1 < 2^n$  imply

$$(n+1)^2 = n^2 + 2n + 1 \leq 2^n + 2n + 1 < 2^n + 2^n = 2^{n+1}$$

so the second inequality has been proved.

Now the second inequality implies  $n/2^n < 1/n$  for  $n \geq 4$ . Hence by the Squeeze Theorem,  $n/2^n \rightarrow 0$  as  $n \rightarrow \infty$ .

**2.2.2.** a) Let  $M \in \mathbf{R}$  and choose by Archimedes an  $N \in \mathbf{N}$  such that  $N > \max\{M, 2\}$ . Then  $n \geq N$  implies  $n^2 - n = n(n-1) \geq N(N-1) > M(2-1) = M$ .

b) Let  $M \in \mathbf{R}$  and choose by Archimedes an  $N \in \mathbf{N}$  such that  $N > -M/2$ . Notice that  $n \geq 1$  implies  $-3n \leq -3$  so  $1 - 3n \leq -2$ . Thus  $n \geq N$  implies  $n - 3n^2 = n(1 - 3n) \leq -2n \leq -2N < M$ .

c) Let  $M \in \mathbf{R}$  and choose by Archimedes an  $N \in \mathbf{N}$  such that  $N > M$ . Then  $n \geq N$  implies  $(n^2 + 1)/n = n + 1/n > N + 0 > M$ .

d) Let  $M \in \mathbf{R}$  satisfy  $M \leq 0$ . Then  $2 + \sin \theta \geq 2 - 1 = 1$  implies  $n^2(2 + \sin(n^3 + n + 1)) \geq n^2 \cdot 1 > 0 \geq M$  for all  $n \in \mathbf{N}$ . On the other hand, if  $M > 0$ , then choose by Archimedes an  $N \in \mathbf{N}$  such that  $N > \sqrt{M}$ . Then  $n \geq N$  implies  $n^2(2 + \sin(n^3 + n + 1)) \geq n^2 \cdot 1 \geq N^2 > M$ .

**2.2.3.** a) Following Example 2.13,

$$\frac{2 + 3n - 4n^2}{1 - 2n + 3n^2} = \frac{(2/n^2) + (3/n) - 4}{(1/n^2) - (2/n) + 3} \rightarrow \frac{-4}{3}$$

as  $n \rightarrow \infty$ .

b) Following Example 2.13,

$$\frac{n^3 + n - 2}{2n^3 + n - 2} = \frac{1 + (1/n^2) - (2/n^3)}{2 + (1/n^2) - (2/n^3)} \rightarrow \frac{1}{2}$$

as  $n \rightarrow \infty$ .

c) Rationalizing the expression, we obtain

$$\sqrt{3n+2} - \sqrt{n} = \frac{(\sqrt{3n+2} - \sqrt{n})(\sqrt{3n+2} + \sqrt{n})}{\sqrt{3n+2} + \sqrt{n}} = \frac{2n+2}{\sqrt{3n+2} + \sqrt{n}} \rightarrow \infty$$

as  $n \rightarrow \infty$  by the method of Example 2.13. (Multiply top and bottom by  $1/\sqrt{n}$ .)

d) Multiply top and bottom by  $1/\sqrt{n}$  to obtain

$$\frac{\sqrt{4n+1} - \sqrt{n}}{\sqrt{9n+1} - \sqrt{n+2}} = \frac{\sqrt{4+1/n} - \sqrt{1-1/n}}{\sqrt{9+1/n} - \sqrt{1+2/n}} \rightarrow \frac{2-1}{3-1} = \frac{1}{2}.$$

**2.2.4.** a) Clearly,

$$\frac{x_n}{y_n} - \frac{x}{y} = \frac{x_n y - x y_n}{y y_n} = \frac{x_n y - x y + x y - x y_n}{y y_n}.$$

Thus

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| \leq \frac{1}{|y_n|} |x_n - x| + \frac{|x|}{|y y_n|} |y_n - y|.$$

Since  $y \neq 0$ ,  $|y_n| \geq |y|/2$  for large  $n$ . Thus

$$\left| \frac{x_n}{y_n} - \frac{x}{y} \right| \leq \frac{2}{|y|} |x_n - x| + \frac{2|x|}{|y|^2} |y_n - y| \rightarrow 0$$

as  $n \rightarrow \infty$  by Theorem 2.12i and ii. Hence by the Squeeze Theorem,  $x_n/y_n \rightarrow x/y$  as  $n \rightarrow \infty$ .

b) By symmetry, we may suppose that  $x = y = \infty$ . Since  $y_n \rightarrow \infty$  implies  $y_n > 0$  for  $n$  large, we can apply Theorem 2.15 directly to obtain the conclusions when  $\alpha > 0$ . For the case  $\alpha < 0$ ,  $x_n > M$  implies  $\alpha x_n < \alpha M$ . Since any  $M_0 \in \mathbf{R}$  can be written as  $\alpha M$  for some  $M \in \mathbf{R}$ , we see by definition that  $x_n \rightarrow -\infty$  as  $n \rightarrow \infty$ .

**2.2.5.** *Case 1.*  $x = 0$ . Let  $\epsilon > 0$  and choose  $N$  so large that  $n \geq N$  implies  $|x_n| < \epsilon^2$ . By (8) in 1.1,  $\sqrt{x_n} < \epsilon$  for  $n \geq N$ , i.e.,  $\sqrt{x_n} \rightarrow 0$  as  $n \rightarrow \infty$ .

*Case 2.*  $x > 0$ . Then

$$\sqrt{x_n} - \sqrt{x} = (\sqrt{x_n} - \sqrt{x}) \left( \frac{\sqrt{x_n} + \sqrt{x}}{\sqrt{x_n} + \sqrt{x}} \right) = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}.$$

Since  $\sqrt{x_n} \geq 0$ , it follows that

$$|\sqrt{x_n} - \sqrt{x}| \leq \frac{|x_n - x|}{\sqrt{x}}.$$

This last quotient converges to 0 by Theorem 2.12. Hence it follows from the Squeeze Theorem that  $\sqrt{x_n} \rightarrow \sqrt{x}$  as  $n \rightarrow \infty$ .

**2.2.6.** By the Density of Rationals, there is an  $r_n$  between  $x + 1/n$  and  $x$  for each  $n \in \mathbf{N}$ . Since  $|x - r_n| < 1/n$ , it follows from the Squeeze Theorem that  $r_n \rightarrow x$  as  $n \rightarrow \infty$ .

**2.2.7.** a) By Theorem 2.9 we may suppose that  $|x| = \infty$ . By symmetry, we may suppose that  $x = \infty$ . By definition, given  $M \in \mathbf{R}$ , there is an  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $x_n > M$ . Since  $w_n \geq x_n$ , it follows that  $w_n > M$  for all  $n \geq N$ . By definition, then,  $w_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

b) If  $x$  and  $y$  are finite, then the result follows from Theorem 2.17. If  $x = y = \pm\infty$  or  $-x = y = \infty$ , there is nothing to prove. It remains to consider the case  $x = \infty$  and  $y = -\infty$ . But by Definition 2.14 (with  $M = 0$ ),  $x_n > 0 > y_n$  for  $n$  sufficiently large, which contradicts the hypothesis  $x_n \leq y_n$ .

**2.2.8.** a) Take the limit of  $x_{n+1} = 1 - \sqrt{1 - x_n}$ , as  $n \rightarrow \infty$ . We obtain  $x = 1 - \sqrt{1 - x}$ , i.e.,  $x^2 - x = 0$ . Therefore,  $x = 0, 1$ .

b) Take the limit of  $x_{n+1} = 2 + \sqrt{x_n - 2}$  as  $n \rightarrow \infty$ . We obtain  $x = 2 + \sqrt{x - 2}$ , i.e.,  $x^2 - 5x + 6 = 0$ . Therefore,  $x = 2, 3$ . But  $x_1 > 3$  and induction shows that  $x_{n+1} = 2 + \sqrt{x_n - 2} > 2 + \sqrt{3 - 2} = 3$ , so the limit must be  $x = 3$ .

c) Take the limit of  $x_{n+1} = \sqrt{2 + x_n}$  as  $n \rightarrow \infty$ . We obtain  $x = \sqrt{2 + x}$ , i.e.,  $x^2 - x - 2 = 0$ . Therefore,  $x = 2, -1$ . But  $x_{n+1} = \sqrt{2 + x_n} \geq 0$  by definition (all square roots are nonnegative), so the limit must be  $x = 2$ .

This proof doesn't change if  $x_1 > -2$ , so the limit is again  $x = 2$ .

**2.2.9.** a) Let  $E = \{k \in \mathbf{Z} : k \geq 0 \text{ and } k \leq 10^{n+1}y\}$ . Since  $10^{n+1}y < 10$ ,  $E \subseteq \{0, 1, \dots, 9\}$ . Hence  $w := \sup E \in E$ . It follows that  $w \leq 10^{n+1}y$ , i.e.,  $w/10^{n+1} \leq y$ . On the other hand, since  $w + 1$  is not the supremum of  $E$ ,  $w + 1 > 10^{n+1}y$ . Therefore,  $y < w/10^{n+1} + 1/10^{n+1}$ .

b) Apply a) for  $n = 0$  to choose  $x_1 = w$  such that  $x_1/10 \leq x < x_1/10 + 1/10$ . Suppose

$$s_n := \sum_{k=1}^n \frac{x_k}{10^k} \leq x < \sum_{k=1}^n \frac{x_k}{10^k} + \frac{1}{10^n}.$$

Then  $0 < x - s_n < 1/10^n$ , so by a) choose  $x_{n+1}$  such that  $x_{n+1}/10^{n+1} \leq x - s_n < x_{n+1}/10^{n+1} + 1/10^{n+1}$ , i.e.,

$$\sum_{k=1}^{n+1} \frac{x_k}{10^k} \leq x < \sum_{k=1}^{n+1} \frac{x_k}{10^k} + \frac{1}{10^{n+1}}.$$

c) Combine b) with the Squeeze Theorem.

d) Since an easy induction proves that  $9^n > n$  for all  $n \in \mathbf{N}$ , we have  $9^{-n} < 1/n$ . Hence the Squeeze Theorem implies that  $9^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ . Hence, it follows from Exercise 1.4.4c and definition that

$$.4999 \dots = \frac{4}{10} + \lim_{n \rightarrow \infty} \sum_{k=2}^n \frac{9}{10^k} = \frac{4}{10} + \lim_{n \rightarrow \infty} \frac{1}{10} \left(1 - \frac{1}{9^n}\right) = \frac{4}{10} + \frac{1}{10} = 0.5.$$

Similarly,

$$.999 \dots = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{9}{10^k} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{9^n}\right) = 1.$$

### 2.3 The Bolzano–Weierstrass Theorem.

**2.3.0.** a) False.  $x_n = 1/4 + 1/(n+4)$  is strictly decreasing and  $|x_n| \leq 1/4 + 1/5 < 1/2$ , but  $x_n \rightarrow 1/4$  as  $n \rightarrow \infty$ .

b) True. Since  $(n-1)/(2n-1) \rightarrow 1/2$  as  $n \rightarrow \infty$ , this factor is bounded. Since  $|\cos(n^2 + n + 1)| \leq 1$ , it follows that  $\{x_n\}$  is bounded. Hence it has a convergent subsequence by the Bolzano–Weierstrass Theorem.

c) False.  $x_n = 1/2 - 1/n$  is strictly increasing and  $|x_n| \leq 1/2 < 1 + 1/n$ , but  $x_n \rightarrow 1/2$  as  $n \rightarrow \infty$ .

d) False.  $x_n = (1 + (-1)^n)n$  satisfies  $x_n = 0$  for  $n$  odd and  $x_n = 2n$  for  $n$  even. Thus  $x_{2k+1} \rightarrow 0$  as  $k \rightarrow \infty$ , but  $x_n$  is NOT bounded.

**2.3.1.** Suppose that  $-1 < x_{n-1} < 0$  for some  $n \geq 0$ . Then  $0 < x_{n-1} + 1 < 1$  so  $0 < x_{n-1} + 1 < \sqrt{x_{n-1} + 1}$  and it follows that  $x_{n-1} < \sqrt{x_{n-1} + 1} - 1 = x_n$ . Moreover,  $\sqrt{x_{n-1} + 1} - 1 \leq 1 - 1 = 0$ . Hence by induction,  $x_n$  is increasing and bounded above by 0. It follows from the Monotone Convergence Theorem that  $x_n \rightarrow a$  as  $n \rightarrow \infty$ . Taking the limit of  $\sqrt{x_{n-1} + 1} - 1 = x_n$  we see that  $a^2 + a = 0$ , i.e.,  $a = -1, 0$ . Since  $x_n$  increases from  $x_0 > -1$ , the limit is 0. If  $x_0 = -1$ , then  $x_n = -1$  for all  $n$ . If  $x_0 = 0$ , then  $x_n = 0$  for all  $n$ .

Finally, it is easy to verify that if  $x_0 = \ell$  for  $\ell = -1$  or  $0$ , then  $x_n = \ell$  for all  $n$ , hence  $x_n \rightarrow \ell$  as  $n \rightarrow \infty$ .

**2.3.2.** If  $x_1 = 0$  then  $x_n = 0$  for all  $n$ , hence converges to 0. If  $0 < x_1 < 1$ , then by 1.4.1c,  $x_n$  is decreasing and bounded below. Thus the limit,  $a$ , exists by the Monotone Convergence Theorem. Taking the limit of  $x_{n+1} = 1 - \sqrt{1 - x_n}$ , as  $n \rightarrow \infty$ , we have  $a = 1 - \sqrt{1 - a}$ , i.e.,  $a = 0, 1$ . Since  $x_1 < 1$ , the limit must be zero.

Finally,

$$\frac{x_{n+1}}{x_n} = \frac{1 - \sqrt{1 - x_n}}{x_n} = \frac{1 - (1 - x_n)}{x_n(1 + \sqrt{1 - x_n})} \rightarrow \frac{1}{1 + 1} = \frac{1}{2}.$$

**2.3.3. Case 1.**  $x_0 = 2$ . Then  $x_n = 2$  for all  $n$ , so the limit is 2.

**Case 2.**  $2 < x_0 < 3$ . Suppose that  $2 < x_{n-1} \leq 3$  for some  $n \geq 1$ . Then  $0 < x_{n-1} - 2 \leq 1$  so  $\sqrt{x_{n-1} - 2} \geq x_{n-1} - 2$ , i.e.,  $x_n = 2 + \sqrt{x_{n-1} - 2} \geq x_{n-1}$ . Moreover,  $x_n = 2 + \sqrt{x_{n-1} - 2} \leq 2 + 1 = 3$ . Hence by induction,  $x_n$  is increasing and bounded above by 3. It follows from the Monotone Convergence Theorem that  $x_n \rightarrow a$  as  $n \rightarrow \infty$ . Taking the limit of  $2 + \sqrt{x_{n-1} - 2} = x_n$  we see that  $a^2 - 5a + 6 = 0$ , i.e.,  $a = 2, 3$ . Since  $x_n$  increases from  $x_0 > 2$ , the limit is 3.

**Case 3.**  $x_0 \geq 3$ . Suppose that  $x_{n-1} \geq 3$  for some  $n \geq 1$ . Then  $x_{n-1} - 2 \geq 1$  so  $\sqrt{x_{n-1} - 2} \leq x_{n-1} - 2$ , i.e.,  $x_n = 2 + \sqrt{x_{n-1} - 2} \leq x_{n-1}$ . Moreover,  $x_n = 2 + \sqrt{x_{n-1} - 2} \geq 2 + 1 = 3$ . Hence by induction,  $x_n$  is decreasing

and bounded above by 3. By repeating the steps in Case 2, we conclude that  $x_n$  decreases from  $x_0 \geq 3$  to the limit 3.

**2.3.4.** *Case 1.*  $x_0 < 1$ . Suppose  $x_{n-1} < 1$ . Then

$$x_{n-1} = \frac{2x_{n-1}}{2} < \frac{1+x_{n-1}}{2} = x_n < \frac{2}{2} = 1.$$

Thus  $\{x_n\}$  is increasing and bounded above, so  $x_n \rightarrow x$ . Taking the limit of  $x_n = (1+x_{n-1})/2$  as  $n \rightarrow \infty$ , we see that  $x = (1+x)/2$ , i.e.,  $x = 1$ .

*Case 2.*  $x_0 \geq 1$ . If  $x_{n-1} \geq 1$  then

$$1 = \frac{2}{2} \leq \frac{1+x_{n-1}}{2} = x_n \leq \frac{2x_{n-1}}{2} = x_{n-1}.$$

Thus  $\{x_n\}$  is decreasing and bounded below. Repeating the argument in Case 1, we conclude that  $x_n \rightarrow 1$  as  $n \rightarrow \infty$ .

**2.3.5.** The result is obvious when  $x = 0$ . If  $x > 0$  then by Example 2.2 and Theorem 2.6,

$$\lim_{n \rightarrow \infty} x^{1/(2n-1)} = \lim_{m \rightarrow \infty} x^{1/m} = 1.$$

If  $x < 0$  then since  $2n-1$  is odd, we have by the previous case that  $x^{1/(2n-1)} = -(-x)^{1/(2n-1)} \rightarrow -1$  as  $n \rightarrow \infty$ .

**2.3.6.** a) Suppose that  $\{x_n\}$  is increasing. If  $\{x_n\}$  is bounded above, then there is an  $x \in \mathbf{R}$  such that  $x_n \rightarrow x$  (by the Monotone Convergence Theorem). Otherwise, given any  $M > 0$  there is an  $N \in \mathbf{N}$  such that  $x_N > M$ . Since  $\{x_n\}$  is increasing,  $n \geq N$  implies  $x_n \geq x_N > M$ . Hence  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

b) If  $\{x_n\}$  is decreasing, then  $-x_n$  is increasing, so part a) applies.

**2.3.7.** Choose by the Approximation Property an  $x_1 \in E$  such that  $\sup E - 1 < x_1 \leq \sup E$ . Since  $\sup E \notin E$ , we also have  $x_1 < \sup E$ . Suppose  $x_1 < x_2 < \dots < x_n$  in  $E$  have been chosen so that  $\sup E - 1/n < x_n < \sup E$ . Choose by the Approximation Property an  $x_{n+1} \in E$  such that  $\max\{x_n, \sup E - 1/(n+1)\} < x_{n+1} \leq \sup E$ . Then  $\sup E - 1/(n+1) < x_{n+1} < \sup E$  and  $x_n < x_{n+1}$ . Thus by induction,  $x_1 < x_2 < \dots$  and by the Squeeze Theorem,  $x_n \rightarrow \sup E$  as  $n \rightarrow \infty$ .

**2.3.8.** a) This follows immediately from Exercise 1.2.6.

b) By a),  $x_{n+1} = (x_n + y_n)/2 < 2x_n/2 = x_n$ . Thus  $y_{n+1} < x_{n+1} < \dots < x_1$ . Similarly,  $y_n = \sqrt{y_n^2} < \sqrt{x_n y_n} = y_{n+1}$  implies  $x_{n+1} > y_{n+1} > y_n \dots > y_1$ . Thus  $\{x_n\}$  is decreasing and bounded below by  $y_1$  and  $\{y_n\}$  is increasing and bounded above by  $x_1$ .

c) By b),

$$x_{n+1} - y_{n+1} = \frac{x_n + y_n}{2} - \sqrt{x_n y_n} < \frac{x_n + y_n}{2} - y_n = \frac{x_n - y_n}{2}.$$

Hence by induction and a),  $0 < x_{n+1} - y_{n+1} < (x_1 - y_1)/2^n$ .

d) By b), there exist  $x, y \in \mathbf{R}$  such that  $x_n \downarrow x$  and  $y_n \uparrow y$  as  $n \rightarrow \infty$ . By c),  $|x - y| \leq (x_1 - y_1) \cdot 0 = 0$ . Hence  $x = y$ .

**2.3.9.** Since  $x_0 = 1$  and  $y_0 = 0$ ,

$$\begin{aligned} x_{n+1}^2 - 2y_{n+1}^2 &= (x_n + 2y_n)^2 - 2(x_n + y_n)^2 \\ &= -x_n^2 + 2y_n^2 = \dots = (-1)^n(x_0 - 2y_0) = (-1)^n. \end{aligned}$$

Notice that  $x_1 = 1 = y_1$ . If  $y_{n-1} \geq n-1$  and  $x_{n-1} \geq 1$  then  $y_n = x_{n-1} + y_{n-1} \geq 1 + (n-1) = n$  and  $x_n = x_{n-1} + 2y_{n-1} \geq 1$ . Thus  $1/y_n \rightarrow 0$  as  $n \rightarrow \infty$  and  $x_n \geq 1$  for all  $n \in \mathbf{N}$ . Since

$$\left| \frac{x_n^2}{y_n^2} - 2 \right| = \left| \frac{x_n^2 - 2y_n^2}{y_n^2} \right| = \frac{1}{y_n^2} \rightarrow 0$$

as  $n \rightarrow \infty$ , it follows that  $x_n/y_n \rightarrow \pm\sqrt{2}$  as  $n \rightarrow \infty$ . Since  $x_n, y_n > 0$ , the limit must be  $\sqrt{2}$ .

**2.3.10.** a) Notice  $x_0 > y_0 > 1$ . If  $x_{n-1} > y_{n-1} > 1$  then  $y_{n-1}^2 - x_{n-1}y_{n-1} = y_{n-1}(y_{n-1} - x_{n-1}) > 0$  so  $y_{n-1}(y_{n-1} + x_{n-1}) < 2x_{n-1}y_{n-1}$ . In particular,

$$x_n = \frac{2x_{n-1}y_{n-1}}{x_{n-1} + y_{n-1}} > y_{n-1}.$$

It follows that  $\sqrt{x_n} > \sqrt{y_{n-1}} > 1$ , so  $x_n > \sqrt{x_n y_{n-1}} = y_n > 1 \cdot 1 = 1$ . Hence by induction,  $x_n > y_n > 1$  for all  $n \in \mathbf{N}$ .

Now  $y_n < x_n$  implies  $2y_n < x_n + y_n$ . Thus

$$x_{n+1} = \frac{2x_n y_n}{x_n + y_n} < x_n.$$

Hence,  $\{x_n\}$  is decreasing and bounded below (by 1). Thus by the Monotone Convergence Theorem,  $x_n \rightarrow x$  for some  $x \in \mathbf{R}$ .

On the other hand,  $y_{n+1}$  is the geometric mean of  $x_{n+1}$  and  $y_n$ , so by Exercise 1.2.6,  $y_{n+1} \geq y_n$ . Since  $y_n$  is bounded above (by  $x_0$ ), we conclude that  $y_n \rightarrow y$  as  $n \rightarrow \infty$  for some  $y \in \mathbf{R}$ .

b) Let  $n \rightarrow \infty$  in the identity  $y_{n+1} = \sqrt{x_{n+1}y_n}$ . We obtain, from part a),  $y = \sqrt{xy}$ , i.e.,  $x = y$ . A direct calculation yields  $y_6 > 3.141557494$  and  $x_7 < 3.14161012$ .

## 2.4 Cauchy sequences.

**2.4.0.** a) False.  $a_n = 1$  is Cauchy and  $b_n = (-1)^n$  is bounded, but  $a_n b_n = (-1)^n$  does not converge, hence cannot be Cauchy by Theorem 2.29.

b) False.  $a_n = 1$  and  $b_n = 1/n$  are Cauchy, but  $a_n/b_n = n$  does not converge, hence cannot be Cauchy by Theorem 2.29.

c) True. If  $(a_n + b_n)^{-1}$  converged to 0, then given any  $M \in \mathbf{R}$ ,  $M \neq 0$ , there is an  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $|a_n + b_n|^{-1} < 1/|M|$ . It follows that  $n \geq N$  implies  $|a_n + b_n| > |M| > 0 > M$ . In particular,  $|a_n + b_n|$  diverges to  $\infty$ . But if  $a_n$  and  $b_n$  are Cauchy, then by Theorem 2.29,  $a_n + b_n \rightarrow x$  where  $x \in \mathbf{R}$ . Thus  $|a_n + b_n| \rightarrow |x|$ , NOT  $\infty$ .

d) False. If  $x_{2^k} = \log k$  and  $x_n = 0$  for  $n \neq 2^k$ , then  $x_{2^k} - x_{2^{k-1}} = \log(k/(k-1)) \rightarrow 0$  as  $k \rightarrow \infty$ , but  $x_k$  does not converge, hence cannot be Cauchy by Theorem 2.29.

**2.4.1.** Since  $(2n^2 + 3)/(n^3 + 5n^2 + 3n + 1) \rightarrow 0$  as  $n \rightarrow \infty$ , it follows from the Squeeze Theorem that  $x_n \rightarrow 0$  as  $n \rightarrow \infty$ . Hence by Theorem 2.29,  $x_n$  is Cauchy.

**2.4.2.** If  $x_n$  is Cauchy, then there is an  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $|x_n - x_N| < 1$ . Since  $x_n - x_N \in \mathbf{Z}$ , it follows that  $x_n = x_N$  for all  $n \geq N$ . Thus set  $a := x_N$ .

**2.4.3.** Suppose  $x_n$  and  $y_n$  are Cauchy and let  $\varepsilon > 0$ .

a) If  $\alpha = 0$ , then  $\alpha x_n = 0$  for all  $n \in \mathbf{N}$ , hence is Cauchy. If  $\alpha \neq 0$ , then there is an  $N \in \mathbf{N}$  such that  $n, m \geq N$  implies  $|x_n - x_m| < \varepsilon/|\alpha|$ . Hence

$$|\alpha x_n - \alpha x_m| \leq |\alpha| |x_n - x_m| < \varepsilon$$

for  $n, m \geq N$ .

b) There is an  $N \in \mathbf{N}$  such that  $n, m \geq N$  implies  $|x_n - x_m|$  and  $|y_n - y_m|$  are  $< \varepsilon/2$ . Hence

$$|x_n + y_n - (x_m + y_m)| \leq |x_n - x_m| + |y_n - y_m| < \varepsilon$$

for  $n, m \geq N$ .

c) By repeating the proof of Theorem 2.8, we can show that every Cauchy sequence is bounded. Thus choose  $M > 0$  such that  $|x_n|$  and  $|y_n|$  are both  $\leq M$  for all  $n \in \mathbf{N}$ . There is an  $N \in \mathbf{N}$  such that  $n, m \geq N$  implies  $|x_n - x_m|$  and  $|y_n - y_m|$  are both  $< \varepsilon/(2M)$ . Hence

$$|x_n y_n - (x_m y_m)| \leq |x_n - x_m| |y_m| + |x_n| |y_n - y_m| < \varepsilon$$

for  $n, m \geq N$ .

**2.4.4.** Let  $s_n = \sum_{k=1}^{n-1} x_k$  for  $n = 2, 3, \dots$ . If  $m > n$  then  $s_{m+1} - s_n = \sum_{k=n}^m x_k$ . Therefore,  $s_n$  is Cauchy by hypothesis. Hence  $s_n$  converges by Theorem 2.29.

**2.4.5.** Let  $x_n = \sum_{k=1}^n (-1)^k/k$  for  $n \in \mathbf{N}$ . Suppose  $n$  and  $m$  are even and  $m > n$ . Then

$$S := \sum_{k=n}^m \frac{(-1)^k}{k} \equiv \frac{1}{n} - \left( \frac{1}{n+1} - \frac{1}{n+2} \right) - \cdots - \left( \frac{1}{m-1} - \frac{1}{m} \right).$$

Each term in parentheses is positive, so the absolute value of  $S$  is dominated by  $1/n$ . Similar arguments prevail for all integers  $n$  and  $m$ . Since  $1/n \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that  $x_n$  satisfies the hypotheses of Exercise 2.4.4. Hence  $x_n$  must converge to a finite real number.

**2.4.6.** By Exercise 1.4.4c, if  $m \geq n$  then

$$|x_{m+1} - x_n| = \left| \sum_{k=n}^m (x_{k+1} - x_k) \right| \leq \sum_{k=n}^m \frac{1}{a^k} = \left( 1 - \frac{1}{a^m} - \left( 1 - \frac{1}{a^n} \right) \right) \frac{1}{a-1}.$$

Thus  $|x_{m+1} - x_n| \leq (1/a^n - 1/a^m)/(a-1) \rightarrow 0$  as  $n, m \rightarrow \infty$  since  $a > 1$ . Hence  $\{x_n\}$  is Cauchy and must converge by Theorem 2.29.

**2.4.7.** a) Suppose  $a$  is a cluster point for some set  $E$  and let  $r > 0$ . Since  $E \cap (a-r, a+r)$  contains infinitely many points, so does  $E \cap (a-r, a+r) \setminus \{a\}$ . Hence this set is nonempty. Conversely, if  $E \cap (a-s, a+s) \setminus \{a\}$  is always nonempty for all  $s > 0$  and  $r > 0$  is given, choose  $x_1 \in E \cap (a-r, a+r)$ . If distinct points  $x_1, \dots, x_k$  have been chosen so that  $x_k \in E \cap (a-r, a+r)$  and  $s := \min\{|x_1 - a|, \dots, |x_k - a|\}$ , then by hypothesis there is an  $x_{k+1} \in E \cap (a-s, a+s)$ . By construction,  $x_{k+1}$  does not equal any  $x_j$  for  $1 \leq j \leq k$ . Hence  $x_1, \dots, x_{k+1}$  are distinct points in  $E \cap (a-r, a+r)$ . By induction, there are infinitely many points in  $E \cap (a-r, a+r)$ .

b) If  $E$  is a bounded infinite set, then it contains distinct points  $x_1, x_2, \dots$ . Since  $\{x_n\} \subseteq E$ , it is bounded. It follows from the Bolzano–Weierstrass Theorem that  $x_n$  contains a convergent subsequence, i.e., there is an  $a \in \mathbf{R}$  such that given  $r > 0$  there is an  $N \in \mathbf{N}$  such that  $k \geq N$  implies  $|x_{n_k} - a| < r$ . Since there are infinitely many  $x_{n_k}$ 's and they all belong to  $E$ ,  $a$  is by definition a cluster point of  $E$ .

**2.4.8.** a) To show  $E := [a, b]$  is sequentially compact, let  $x_n \in E$ . By the Bolzano–Weierstrass Theorem,  $x_n$  has a convergent subsequence, i.e., there is an  $x_0 \in \mathbf{R}$  and integers  $n_k$  such that  $x_{n_k} \rightarrow x_0$  as  $k \rightarrow \infty$ . Moreover, by the Comparison Theorem,  $x_n \in E$  implies  $x_0 \in E$ . Thus  $E$  is sequentially compact by definition.

b)  $(0, 1)$  is bounded and  $1/n \in (0, 1)$  has no convergent subsequence with limit in  $(0, 1)$ .

c)  $[0, \infty)$  is closed and  $n \in [0, \infty)$  is a sequence which has no convergent subsequence.

## 2.5 Limits supremum and infimum.

**2.5.1.** a) Since  $3 - (-1)^n = 2$  when  $n$  is even and 4 when  $n$  is odd,  $\limsup_{n \rightarrow \infty} x_n = 4$  and  $\liminf_{n \rightarrow \infty} x_n = 2$ .

b) Since  $\cos(n\pi/2) = 0$  if  $n$  is odd, 1 if  $n = 4m$  and  $-1$  if  $n = 4m + 2$ ,  $\limsup_{n \rightarrow \infty} x_n = 1$  and  $\liminf_{n \rightarrow \infty} x_n = -1$ .

c) Since  $(-1)^{n+1} + (-1)^n/n = -1 + 1/n$  when  $n$  is even and  $1 - 1/n$  when  $n$  is odd,  $\limsup_{n \rightarrow \infty} x_n = 1$  and  $\liminf_{n \rightarrow \infty} x_n = -1$ .

d) Since  $x_n \rightarrow 1/2$  as  $n \rightarrow \infty$ ,  $\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = 1/2$  by Theorem 2.36.

e) Since  $|y_n| \leq M$ ,  $|y_n/n| \leq M/n \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,  $\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = 0$  by Theorem 2.36.

f) Since  $n(1 + (-1)^n) + n^{-1}((-1)^n - 1) = 2n$  when  $n$  is even and  $-2/n$  when  $n$  is odd,  $\limsup_{n \rightarrow \infty} x_n = \infty$  and  $\liminf_{n \rightarrow \infty} x_n = 0$ .

g) Clearly  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Therefore,  $\limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = \infty$  by Theorem 2.36.

**2.5.2.** By Theorem 1.20,

$$\liminf_{n \rightarrow \infty} (-x_n) := \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} (-x_k) \right) = - \lim_{n \rightarrow \infty} \left( \sup_{k \geq n} x_k \right) = - \limsup_{n \rightarrow \infty} x_n.$$

A similar argument establishes the second identity.

**2.5.3.** a) Since  $\lim_{n \rightarrow \infty} (\sup_{k \geq n} x_k) < r$ , there is an  $N \in \mathbf{N}$  such that  $\sup_{k \geq N} x_k < r$ , i.e.,  $x_k < r$  for all  $k \geq N$ .

b) Since  $\lim_{n \rightarrow \infty} (\sup_{k \geq n} x_k) > r$ , there is an  $N \in \mathbf{N}$  such that  $\sup_{k \geq N} x_k > r$ , i.e., there is a  $k_1 \in \mathbf{N}$  such that  $x_{k_1} > r$ . Suppose  $k_\nu \in \mathbf{N}$  have been chosen so that  $k_1 < k_2 < \cdots < k_j$  and  $x_{k_\nu} > r$  for  $\nu = 1, 2, \dots, j$ . Choose  $N > k_j$  such that  $\sup_{k \geq N} x_k > r$ . Then there is a  $k_{j+1} > N > k_j$  such that  $x_{k_{j+1}} > r$ . Hence by induction, there are distinct natural numbers  $k_1, k_2, \dots$  such that  $x_{k_j} > r$  for all  $j \in \mathbf{N}$ .

**2.5.4.** a) Since  $\inf_{k \geq n} x_k + \inf_{k \geq n} y_k$  is a lower bound of  $x_j + y_j$  for any  $j \geq n$ , we have  $\inf_{k \geq n} x_k + \inf_{k \geq n} y_k \leq \inf_{j \geq n} (x_j + y_j)$ . Taking the limit of this inequality as  $n \rightarrow \infty$ , we obtain

$$\liminf_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n \leq \liminf_{n \rightarrow \infty} (x_n + y_n).$$

Note, we used Corollary 1.16 and the fact that the sum on the left is not of the form  $\infty - \infty$ . Similarly, for each  $j \geq n$ ,

$$\inf_{k \geq n} (x_k + y_k) \leq x_j + y_j \leq \sup_{k \geq n} x_k + y_j.$$

Taking the infimum of this inequality over all  $j \geq n$ , we obtain  $\inf_{k \geq n} (x_k + y_k) \leq \sup_{k \geq n} x_k + \inf_{j \geq n} y_j$ . Therefore,

$$\liminf_{n \rightarrow \infty} (x_n + y_n) \leq \limsup_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n.$$

The remaining two inequalities follow from Exercise 2.5.2. For example,

$$\begin{aligned} \limsup_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n &= -\liminf_{n \rightarrow \infty} (-x_n) - \limsup_{n \rightarrow \infty} (-y_n) \\ &\leq -\liminf_{n \rightarrow \infty} (-x_n - y_n) = \limsup_{n \rightarrow \infty} (x_n + y_n). \end{aligned}$$

b) It suffices to prove the first identity. By Theorem 2.36 and a),

$$\lim_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n \leq \liminf_{n \rightarrow \infty} (x_n + y_n).$$

To obtain the reverse inequality, notice by the Approximation Property that for each  $n \in \mathbf{N}$  there is a  $j_n > n$  such that  $\inf_{k \geq n} (x_k + y_k) > x_{j_n} - 1/n + y_{j_n}$ . Hence

$$\inf_{k \geq n} (x_k + y_k) > x_{j_n} - \frac{1}{n} + \inf_{k \geq n} y_k$$

for all  $n \in \mathbf{N}$ . Taking the limit of this inequality as  $n \rightarrow \infty$ , we obtain

$$\liminf_{n \rightarrow \infty} (x_n + y_n) \geq \lim_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n.$$

c) Let  $x_n = (-1)^n$  and  $y_n = (-1)^{n+1}$ . Then the limits infimum are both  $-1$ , the limits supremum are both  $1$ , but  $x_n + y_n = 0 \rightarrow 0$  as  $n \rightarrow \infty$ . If  $x_n = (-1)^n$  and  $y_n = 0$  then

$$\liminf_{n \rightarrow \infty} (x_n + y_n) = -1 < 1 = \limsup_{n \rightarrow \infty} x_n + \liminf_{n \rightarrow \infty} y_n.$$

**2.5.5.** a) For any  $j \geq n$ ,  $x_j \leq \sup_{k \geq n} x_k$  and  $y_j \leq \sup_{k \geq n} y_k$ . Multiplying these inequalities, we have  $x_j y_j \leq (\sup_{k \geq n} x_k)(\sup_{k \geq n} y_k)$ , i.e.,

$$\sup_{j \geq n} x_j y_j \leq (\sup_{k \geq n} x_k)(\sup_{k \geq n} y_k).$$

Taking the limit of this inequality as  $n \rightarrow \infty$  establishes a). The inequality can be strict because if

$$x_n = 1 - y_n = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$$

then  $\limsup_{n \rightarrow \infty} (x_n y_n) = 0 < 1 = (\limsup_{n \rightarrow \infty} x_n)(\limsup_{n \rightarrow \infty} y_n)$ .

b) By a),

$$\liminf_{n \rightarrow \infty} (x_n y_n) = -\limsup_{n \rightarrow \infty} (-x_n y_n) \geq -\limsup_{n \rightarrow \infty} (-x_n) \limsup_{n \rightarrow \infty} y_n = \liminf_{n \rightarrow \infty} x_n \limsup_{n \rightarrow \infty} y_n.$$

**2.5.6.** *Case 1.*  $x = \infty$ . By hypothesis,  $C := \limsup_{n \rightarrow \infty} y_n > 0$ . Let  $M > 0$  and choose  $N \in \mathbf{N}$  such that  $n \geq N$  implies  $x_n \geq 2M/C$  and  $\sup_{n \geq N} y_n > C/2$ . Then  $\sup_{k \geq N} (x_k y_k) \geq x_n y_n \geq (2M/C)y_n$  for any  $n \geq N$  and  $\sup_{k \geq N} (x_k y_k) \geq (2M/C) \sup_{n \geq N} y_n > M$ . Therefore,  $\limsup_{n \rightarrow \infty} (x_n y_n) = \infty$ .



*Case 2.*  $0 \leq x < \infty$ . By Exercise 2.5.6a and Theorem 2.36,

$$\limsup_{n \rightarrow \infty} (x_n y_n) \leq (\limsup_{n \rightarrow \infty} x_n)(\limsup_{n \rightarrow \infty} y_n) = x \limsup_{n \rightarrow \infty} y_n.$$

On the other hand, given  $\epsilon > 0$  choose  $n \in \mathbf{N}$  so that  $x_k > x - \epsilon$  for  $k \geq n$ . Then  $x_k y_k \geq (x - \epsilon)y_k$  for each  $k \geq n$ , i.e.,  $\sup_{k \geq n} (x_k y_k) \geq (x - \epsilon) \sup_{k \geq n} y_k$ . Taking the limit of this inequality as  $n \rightarrow \infty$  and as  $\epsilon \rightarrow 0$ , we obtain

$$\limsup_{n \rightarrow \infty} (x_n y_n) \geq x \limsup_{n \rightarrow \infty} y_n.$$

**2.5.7.** It suffices to prove the first identity. Let  $s = \inf_{n \in \mathbf{N}} (\sup_{k \geq n} x_k)$ .

*Case 1.*  $s = \infty$ . Then  $\sup_{k \geq n} x_k = \infty$  for all  $n \in \mathbf{N}$  so by definition,

$$\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} (\sup_{k \geq n} x_k) = \infty = s.$$

*Case 2.*  $s = -\infty$ . Let  $M > 0$  and choose  $N \in \mathbf{N}$  such that  $\sup_{k \geq N} x_k \leq -M$ . Then  $\sup_{k \geq n} x_k \leq \sup_{k \geq N} x_k \leq -M$  for all  $n \geq N$ , i.e.,  $\limsup_{n \rightarrow \infty} x_n = -\infty$ .

*Case 3.*  $-\infty < s < \infty$ . Let  $\epsilon > 0$  and use the Approximation Property to choose  $N \in \mathbf{N}$  such that  $\sup_{k \geq N} x_k < s + \epsilon$ . Since  $\sup_{k \geq n} x_k \leq \sup_{k \geq N} x_k < s + \epsilon$  for all  $n \geq N$ , it follows that

$$s - \epsilon < s \leq \sup_{k \geq n} x_k < s + \epsilon$$

for  $n \geq N$ , i.e.,  $\limsup_{n \rightarrow \infty} x_n = s$ .

**2.5.8.** It suffices to establish the first identity. Let  $s = \liminf_{n \rightarrow \infty} x_n$ .

*Case 1.*  $s = 0$ . Then by Theorem 2.35 there is a subsequence  $k_j$  such that  $x_{k_j} \rightarrow 0$ , i.e.,  $1/x_{k_j} \rightarrow \infty$  as  $j \rightarrow \infty$ . In particular,  $\sup_{k \geq n} (1/x_k) = \infty$  for all  $n \in \mathbf{N}$ , i.e.,  $\limsup_{n \rightarrow \infty} (1/x_n) = \infty = 1/s$ .

*Case 2.*  $s = \infty$ . Then  $x_k \rightarrow \infty$ , i.e.,  $1/x_k \rightarrow 0$ , as  $k \rightarrow \infty$ . Thus by Theorem 2.36,  $\limsup_{n \rightarrow \infty} (1/x_n) = 0 = 1/s$ .

*Case 3.*  $0 < s < \infty$ . Fix  $j \geq n$ . Since  $1/\inf_{k \geq n} x_k \geq 1/x_j$  implies  $1/\inf_{k \geq n} x_k \geq \sup_{j \geq n} (1/x_j)$ , it is clear that  $1/s \geq \limsup_{n \rightarrow \infty} (1/x_n)$ . On the other hand, given  $\epsilon > 0$  and  $n \in \mathbf{N}$ , choose  $j > N$  such that  $\inf_{k \geq n} x_k + \epsilon > x_j$ , i.e.,  $1/(\inf_{k \geq n} x_k + \epsilon) < 1/x_j \leq \sup_{k \geq n} (1/x_k)$ . Taking the limit of this inequality as  $n \rightarrow \infty$  and as  $\epsilon \rightarrow 0$ , we conclude that  $1/s \leq \limsup_{n \rightarrow \infty} (1/x_n)$ .

**2.5.9.** If  $x_n \rightarrow 0$ , then  $|x_n| \rightarrow 0$ . Thus by Theorem 2.36,  $\limsup_{n \rightarrow \infty} |x_n| = 0$ . Conversely, if  $\limsup_{n \rightarrow \infty} |x_n| \leq 0$ , then

$$0 \leq \liminf_{n \rightarrow \infty} |x_n| \leq \limsup_{n \rightarrow \infty} |x_n| \leq 0,$$

implies that the limits supremum and infimum of  $|x_n|$  are equal (to zero). Hence by Theorem 2.36, the limit exists and equals zero.