Solutions for Chapter 2 Problems

1. Vectors in the Cartesian Coordinate System

P2.1: Given P(4,2,1) and $\mathbf{A}_{PQ}=2\mathbf{a}_x+4\mathbf{a}_y+6\mathbf{a}_z$, find the point Q.

$\mathbf{A}_{PQ} = 2\,\mathbf{a}_x + 4\,\mathbf{a}_y + 6\,\mathbf{a}_z = (Q_x-P_x)\mathbf{a}_x + (Q_y-P_y)\mathbf{a}_y+(Q_z-P_z)\mathbf{a}_z$
$Q_x-P_x=Q_x-4=2$; $Q_x=6$
$Q_y-P_y=Q_y-2=4$; $Q_y=6$
$Q_z-P_z=Q_z-1=6$; $Q_z=7$
Ans: Q(6,6,7)

P2.2: Given the points P(4,1,0)m and Q(1,3,0)m, fill in the table and make a sketch of the vectors found in (a) through (f).

	Vector	Mag	Unit Vector
a. Find the vector **A** from the origin to P	$\mathbf{A}_{OP} = 4\,\mathbf{a}_x + 1\,\mathbf{a}_y$	4.12	$\mathbf{A}_{OP} = 0.97\,\mathbf{a}_x + 0.24\,\mathbf{a}_y$
b. Find the vector **B** from the origin to Q	$\mathbf{B}_{OQ} = 1\,\mathbf{a}_x + 3\,\mathbf{a}_y$	3.16	$\mathbf{a}_{OQ} = 0.32\,\mathbf{a}_x + 0.95\,\mathbf{a}_y$
c. Find the vector **C** from P to Q	$\mathbf{C}_{PQ} = -3\,\mathbf{a}_x + 2\,\mathbf{a}_y$	3.61	$\mathbf{a}_{PQ} = -0.83\,\mathbf{a}_x + 0.55\,\mathbf{a}_y$
d. Find **A** + **B**	$\mathbf{A} + \mathbf{B} = 5\,\mathbf{a}_x + 4\,\mathbf{a}_y$	6.4	$\mathbf{a} = 0.78\,\mathbf{a}_x + 0.62\,\mathbf{a}_y$
e. Find **C** – **A**	$\mathbf{C} - \mathbf{A} = -7\,\mathbf{a}_x + 1\,\mathbf{a}_y$	7.07	$\mathbf{a} = -0.99\,\mathbf{a}_x + 0.14\,\mathbf{a}_y$
f. Find **B** - **A**	$\mathbf{B} - \mathbf{A} = -3\,\mathbf{a}_x + 2\,\mathbf{a}_y$	3.6	$\mathbf{a} = -0.83\,\mathbf{a}_x + 0.55\,\mathbf{a}_y$

a. $\mathbf{A}_{OP} = (4-0)\mathbf{a}_x + (1-0)\mathbf{a}_y + (0-0)\mathbf{a}_z = 4\,\mathbf{a}_x + 1\,\mathbf{a}_y$.

$$|\mathbf{A}_{OP}| = \sqrt{4^2+1^2} = \sqrt{17} = 4.12$$

$$\mathbf{a}_{OP} = \frac{4}{\sqrt{17}}\mathbf{a}_x + \frac{1}{\sqrt{17}}\mathbf{a}_y = 0.97\mathbf{a}_x + 0.24\mathbf{a}_y$$

(see Figure P2.2ab)

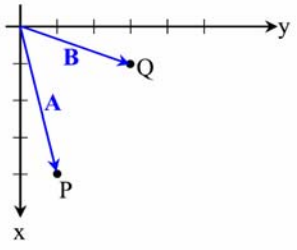


Fig. P2.2ab

b. $\mathbf{B}_{OQ} = (1-0)\mathbf{a}_x + (3-0)\mathbf{a}_y + (0-0)\mathbf{a}_z = 1\,\mathbf{a}_x + 3\,\mathbf{a}_y$.

$$|\mathbf{B}_{OQ}| = \sqrt{1^2+3^2} = \sqrt{10} = 3.16$$

$$\mathbf{a}_{OQ} = \frac{1}{\sqrt{10}}\mathbf{a}_x + \frac{3}{\sqrt{10}}\mathbf{a}_y = 0.32\mathbf{a}_x + 0.95\mathbf{a}_y$$

(see Figure P2.2ab)

c. $\mathbf{C}_{PQ} = (1-4)\mathbf{a}_x + (3-1)\mathbf{a}_y + (0-0)\mathbf{a}_z = -3\,\mathbf{a}_x + 2\,\mathbf{a}_y$.

$$|\mathbf{C}_{PQ}| = \sqrt{3^2+2^2} = \sqrt{13} = 3.61$$

$$\mathbf{a}_{PQ} = \frac{-3}{\sqrt{13}}\mathbf{a}_x + \frac{2}{\sqrt{13}}\mathbf{a}_y = -0.83\mathbf{a}_x + 0.55\mathbf{a}_y$$

(see Figure P2.2cd)

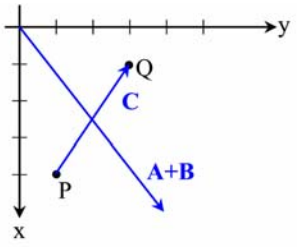


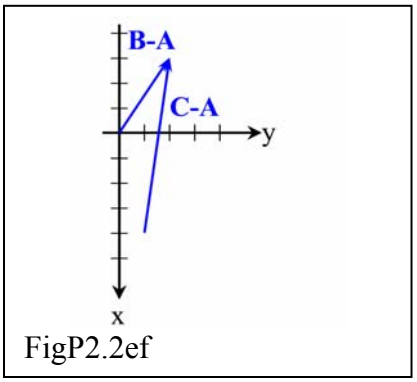
Fig. P2.2cd

d. $\mathbf{A} + \mathbf{B} = (4+1)\mathbf{a}_x + (1+3)\mathbf{a}_y + (0-0)\mathbf{a}_z = 5\ \mathbf{a}_x + 4\ \mathbf{a}_y$.

$$|\mathbf{A}+\mathbf{B}| = \sqrt{5^2+4^2} = \sqrt{41} = 6.4$$

$$\mathbf{a} = \frac{5}{\sqrt{41}}\mathbf{a}_x + \frac{4}{\sqrt{41}}\mathbf{a}_y = 0.78\mathbf{a}_x + 0.62\mathbf{a}_y$$

(see Figure P2.2cd)



FigP2.2ef

e. $\mathbf{C} - \mathbf{A} = (-3-4)\mathbf{a}_x + (2-1)\mathbf{a}_y + (0-0)\mathbf{a}_z = -7\ \mathbf{a}_x + 1\ \mathbf{a}_y$.

$$|\mathbf{C}-\mathbf{A}| = \sqrt{7^2+1^2} = \sqrt{50} = 7.07$$

$$\mathbf{a} = \frac{-7}{\sqrt{50}}\mathbf{a}_x + \frac{1}{\sqrt{50}}\mathbf{a}_y = -0.99\mathbf{a}_x + 0.14\mathbf{a}_y$$

(see Figure P2.2ef)

f. $\mathbf{B} - \mathbf{A} = (1-4)\mathbf{a}_x + (3-1)\mathbf{a}_y + (0-0)\mathbf{a}_z = -3\ \mathbf{a}_x + 2\ \mathbf{a}_y$.

$$|\mathbf{B}-\mathbf{A}| = \sqrt{3^2+2^2} = \sqrt{13} = 3.6$$

$$\mathbf{a} = \frac{-3}{\sqrt{13}}\mathbf{a}_x + \frac{2}{\sqrt{13}}\mathbf{a}_y = -0.83\mathbf{a}_x + 0.55\mathbf{a}_y$$

(see Figure P2.2ef)

P2.3: MATLAB: Write a program that will find the vector between a pair of arbitrary points in the Cartesian Coordinate System.

A program or function for this task is really overkill, as it is so easy to perform the task. Enter points P and Q (for example, P=[1 2 3]; Q=[6 5 4]). Then, the vector from P toQ is simply given by Q-P.

As a function we could have:

```
function PQ=vector(P,Q)
%   Given a pair of Cartesian points
%   P and Q, the program determines the
%   vector from P to Q.
PQ=Q-P;
```

Running this function we have:

```
>> P=[1 2 3];
>> Q=[6 5 4];
>> PQ=vector(P,Q)

PQ =
   5    3    1
```

Alternatively, we could simply perform the math in the command line window:

```
>> PQ=Q-P
PQ =
    5    3    1
>>
```

2. Coulomb's Law, Electric Field Intensity, and Field Lines

P2.4: Suppose Q_1(0.0, -3.0m, 0.0) = 4.0nC, Q_2(0.0, 3.0m, 0.0) = 4.0nC, and Q_3(4.0m, 0.0, 0.0) = 1.0nC. (a) Find the total force acting on the charge Q_3. (b) Repeat the problem after changing the charge of Q_2 to –4.0nC. (c) Find the electric field intensity for parts (a) and (b).

(a) $\mathbf{F}_{13} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_o R_{13}^2}\mathbf{a}_{13}$, where $\mathbf{R}_{13} = 4\,\mathbf{a}_x + 3\,\mathbf{a}_y =$, $R_{13} = 5m$, $\mathbf{a}_{13} = 0.8\,\mathbf{a}_x + 0.6\,\mathbf{a}_y$.

so

$$\mathbf{F}_{13} = \frac{\left(4x10^{-9}C\right)\left(1x10^{-9}C\right)\left(4\mathbf{a}_x + 3\mathbf{a}_y\right)/5}{4\pi\left(10^{-9}F/36\pi m\right)(5m)^2}\frac{FV}{C}\frac{NM}{VC}$$

$$= 1.15x10^{-9}\mathbf{a}_x + 0.86x10^{-9}\mathbf{a}_y\ N.$$

Similarly, $\mathbf{F}_{23} = 1.15x10^{-9}\mathbf{a}_x - 0.86x10^{-9}\mathbf{a}_y\ N$, so $\boxed{\mathbf{F}_{TOT} = 2.3\mathbf{a}_x\ nN}$

(b) with Q_2 = -4 nC, $\mathbf{F}_{13}$ is unchanged but $\mathbf{F}_{23} = -1.15x10^{-9}\mathbf{a}_x + 0.86x10^{-9}\mathbf{a}_y\ N$, so $\boxed{\mathbf{F}_{TOT} = 1.7\mathbf{a}_y\ nN}$.

(c) $\mathbf{E}_a = \dfrac{\mathbf{F}_{TOT}}{Q_3} = \dfrac{\left(2.3x10^{-9}\mathbf{a}_x N\right)}{\left(1x10^{-9}C\right)}\dfrac{VC}{Nm} = 2.3\mathbf{a}_x\dfrac{V}{m}$.

Likewise, $\mathbf{E}_b = 1.7\mathbf{a}_y\dfrac{V}{m}$.

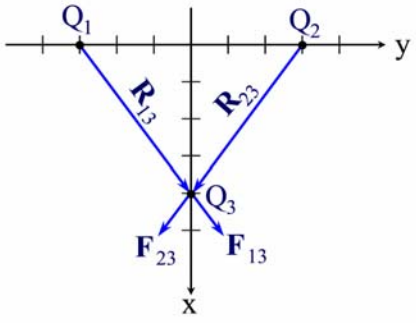


Fig. P2.4

P2.5: Find the force exerted by Q_1(3.0m, 3.0m, 3.0m) = 1.0 μC on Q_2(6.0m, 9.0m, 3.0m) = 10. nC.

$\mathbf{F}_{12} = \dfrac{Q_1 Q_2}{4\pi\varepsilon_o R_{12}^2}\mathbf{a}_{12}$, where

$\mathbf{R}_{12} = (6\text{-}3)\mathbf{a}_x + (9\text{-}3)\mathbf{a}_y + (3\text{-}3)\mathbf{a}_z = 3\,\mathbf{a}_x + 6\,\mathbf{a}_y$ m

$R_{12} = \sqrt{3^2 + 6^2} = \sqrt{45}m$, $\mathbf{a}_{12} = \dfrac{3\mathbf{a}_x + 6\mathbf{a}_y}{\sqrt{45}}$, and

$$\mathbf{F}_{12}=\frac{\left(1x10^{-6}C\right)\left(10x10^{-9}C\right)}{4\pi\left(10^{-9}F/36\pi m\right)\left(45m^2\right)}\frac{3\mathbf{a}_x+6\mathbf{a}_y}{\sqrt{45}}\frac{FV}{C}\frac{NM}{VC}\text{, so }\boxed{\mathbf{F}_{12}=0.89\mathbf{a}_x+1.8\mathbf{a}_y\ \mu N}.$$

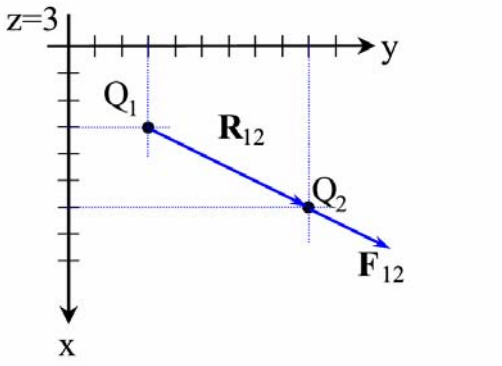


Fig. P2.5

P2.6: Suppose 10.0 nC point charges are located on the corners of a square of side 10.0 cm. Locating the square in the x-y plane (at z = 0.00) with one corner at the origin and one corner at P(10.0, 10.0, 0.00) cm, find the total force acting at point P.

We arbitrarily label the charges as shown in Figure P2.6. Then
$\mathbf{R}_{OP} = 0.1\ \mathbf{a}_x + 0.1\ \mathbf{a}_y$
$R_{OP} = 0.141$ m
$\mathbf{a}_{OP} = 0.707\ \mathbf{a}_x + 0.707\ \mathbf{a}_y$.

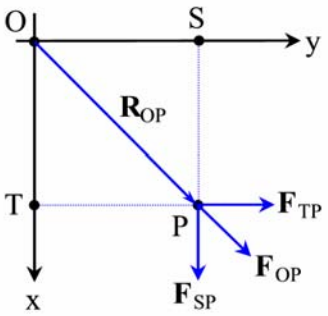


Fig. P2.6

$$\mathbf{F}_{OP}=\frac{(10nC)(10nC)(0.707)\left(\mathbf{a}_x+\mathbf{a}_y\right)}{(4\pi)\left(10^{-9}F/36\pi m\right)(0.141m)^2}$$

$$=32\left(\mathbf{a}_x+\mathbf{a}_y\right)\mu N$$

$$\mathbf{F}_{TP}=\frac{(10nC)(10nC)\mathbf{a}_y}{(4\pi)\left(10^{-9}F/36\pi m\right)(0.1m)^2}=90\mathbf{a}_y\mu N$$

$$\mathbf{F}_{SP}=\frac{(10nC)(10nC)\mathbf{a}_x}{(4\pi)\left(10^{-9}F/36\pi m\right)(0.1m)^2}=90\mathbf{a}_x\mu N$$

and then the total (adjusting to 2 significant digits) is:
$\mathbf{F}_{TOT}=120\left(\mathbf{a}_x+\mathbf{a}_y\right)\mu N.$

P2.7: 1.00 nC point charges are located at (0.00, -2.00, 0.00)m, (0.00, 2.00, 0.00)m, (0.00, 0.00, -2.00)m and (0.00, 0.00, +2.00)m. Find the total force acting on a 1.00 nC charge located at (2.00, 0.00, 0.00)m.

Figure P2.7a shows the situation, but we need only find the x-directed force from one of the charges on Q_t (Figure P2.7b) and multiply this result by 4. Because of the problem's symmetry, the rest of the components cancel.

$$\mathbf{F}_{1t} = \frac{QQ_t}{4\pi\varepsilon_o R}\boldsymbol{a}_R, \mathbf{R} = 2\boldsymbol{a}_x + 2\boldsymbol{a}_y, R = \sqrt{8}\ m, \boldsymbol{a}_R = \frac{2\boldsymbol{a}_x + 2\boldsymbol{a}_y}{\sqrt{8}},$$

$$\text{so } \mathbf{F}_{1t} = \frac{\left(1x10^{-9}C\right)\left(1x10^{-9}C\right)\left(2\mathbf{a}_x + 2\mathbf{a}_y \big/ \sqrt{8}\right)}{4\pi\left(10^{-9}F \big/ 36\pi m\right)\left(8m^2\right)} = 796x10^{-12}\left(\mathbf{a}_x + \mathbf{a}_y\right)N$$

The force from all charges is then

$$\mathbf{F}_{TOT} = (4)\left(796x10^{-12}\boldsymbol{a}_x\right)nN = 3.2\boldsymbol{a}_x nN.$$

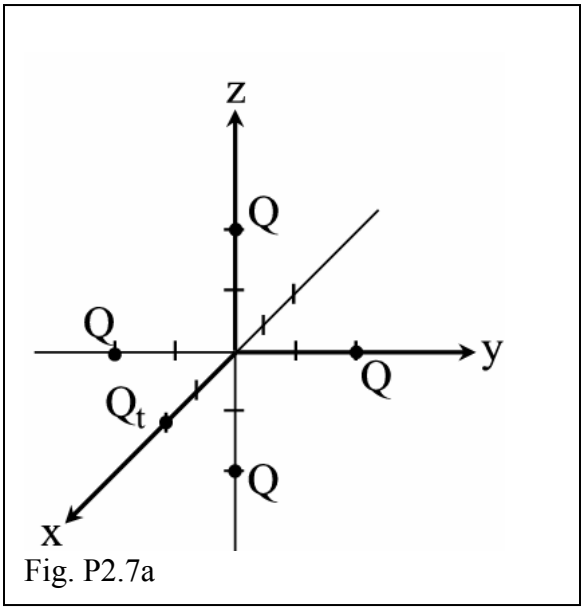


Fig. P2.7a

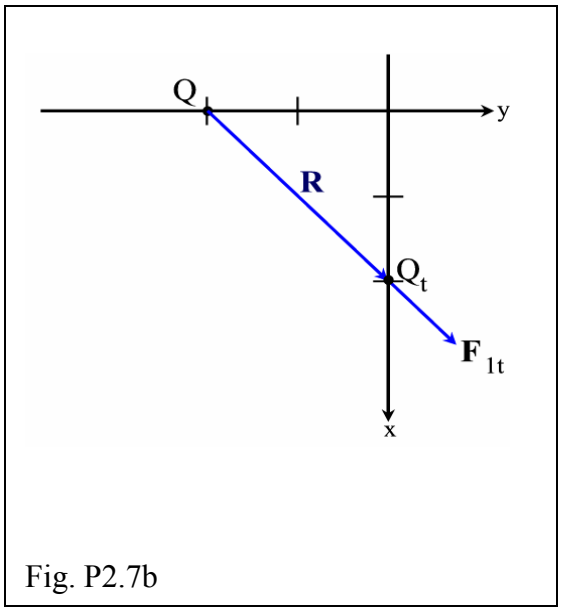


Fig. P2.7b

P2.8: A 20.0 nC point charge exists at P(0.00,0.00,-3.00m). Where must a 10.0 nC charge be located such that the total field is zero at the origin?

For zero field at the origin, we must cancel the $+\mathbf{a}_z$ directed field from Q_P by placing Q at the point Q(0,0,z) (see Figure P2.8). Then we have $\mathbf{E}_{tot} = \mathbf{E}_P + \mathbf{E}_Q = 0$.

$$\text{So, } \mathbf{E}_P = \frac{Q_P}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{\left(20x10^{-9}C\right)\mathbf{a}_z}{4\pi\left(10^{-9}F \big/ 36\pi m\right)(3m)^2}\frac{FV}{C} = 20\mathbf{a}_z\frac{V}{m}$$

and

$$\mathbf{E}_Q = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R$$

$$= \frac{\left(10x10^{-9}C\right)(-\mathbf{a}_z)}{4\pi\left(10^{-9}F/36\pi m\right)\left(z(m)\right)^2} = \frac{-90}{z^2}\mathbf{a}_z$$

So then

$$20\mathbf{a}_z - \frac{90}{z^2}\mathbf{a}_z = 0,$$

$$z^2 = \frac{90}{20}, \quad z = 2.12.$$

Thus, Q(0,0,2.12m).

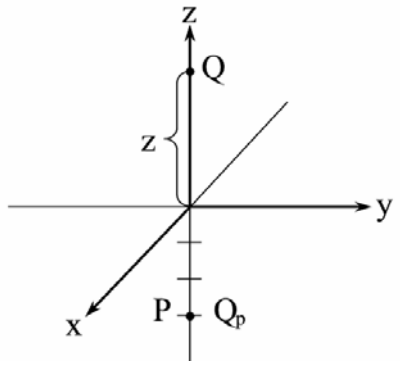


Fig. P2.8

3. The Spherical Coordinate System

P2.9: Convert the following points from Cartesian to Spherical coordinates:

a. P(6.0, 2.0, 6.0)
b. P(0.0, -4.0, 3.0)
c. P(-5.0,-1.0, -4.0)

(a) $r = \sqrt{6^2+2^2+6^2} = 8.7, \theta = \cos^{-1}\left(\frac{6}{8.7}\right) = 47^o, \phi = \tan^{-1}\left(\frac{2}{6}\right) = 18^o$

(b) $r = \sqrt{0^2+4^2+3^2} = 5, \theta = \cos^{-1}\left(\frac{3}{5}\right) = 53^o, \phi = \tan^{-1}\left(\frac{-4}{0}\right) = -90^o$

(c) $r = \sqrt{5^2+1^2+4^2} = 6.5, \theta = \cos^{-1}\left(\frac{-4.0}{6.5}\right) = 130^o, \phi = \tan^{-1}\left(\frac{-1}{-5}\right) = 190^o$

P2.10: Convert the following points from Spherical to Cartesian coordinates:

a. P(3.0, 30.°, 45.°)
b. P(5.0, π/4, 3π/2)
c. P(10., 135°, 180°)

(a)

$x = r\sin\theta\cos\phi = 3\sin 30^o \cos 45^o = 1.06$

$y = r\sin\theta\sin\phi = 3\sin 30^o \sin 45^o = 1.06$

$z = r\cos\theta = 3\cos 30^o = 2.6$

$so\ P(1.1, 1.1, 2.6).$

(b)

$x = r\sin\theta\cos\phi = 5\sin 45^o \cos 270^o = 0$

$y = r\sin\theta\sin\phi = 5\sin 45^o \sin 270^o = -3.5$

$z = r\cos\theta = 5\cos 45^o = 3.5$

so $P(0, -3.5, 3.5)$.

(c)

$x = r\sin\theta\cos\phi = 10\sin 135^o \cos 180^o = -7.1$

$y = r\sin\theta\sin\phi = 10\sin 135^o \sin 180^o = 0$

$z = r\cos\theta = 10\cos 135^o = -7.1$

so $P(-7.1, 0, -7.1)$.

P2.11: Given a volume defined by 1.0m ≤ r ≤ 3.0m, 0 ≤ θ ≤ 0°, 90° ≤ ϕ ≤ 90°, (a) sketch the volume, (b) perform the integration to find the volume, and (c) perform the necessary integrations to find the total surface area.

(a)

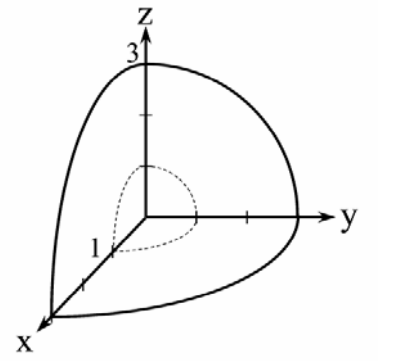


Fig. P2.11

(b)

$$V = \iiint r^2 \sin\theta dr d\theta d\phi = \int_1^3 r^2 dr \int_0^{90^o} \sin\theta d\theta \int_0^{\pi/2} d\phi = \frac{13\pi}{3} = 13.6m^3.$$

So volume $V = 14$ m^3.

(c) There are 5 surfaces: an inner, an outer, and 3 identical sides.

$$S_{side} = \iint r dr d\phi = \int_1^3 r dr \int_0^{\pi/2} d\phi = 2\pi m^2; \quad S_{sides} = 6\pi m^2$$

$$S_{outer} = \iint r^2 \sin\theta d\theta d\phi = 3^2 \int_0^{90^o} \sin\theta d\theta \int_0^{\pi/2} d\phi = \frac{9\pi}{2} m^2$$

$$S_{inner} = \frac{\pi}{2} m^2; S_{TOT} = 11\pi m^2 = 34.6m^2$$

So $S_{total} = 35$ m^2.

4. Line Charges and the Cylindrical Coordinate System

P2.12: Convert the following points from Cartesian to cylindrical coordinates:

a. P(0.0, 4.0, 3.0)
b. P(-2.0, 3.0, 2.0)
c. P(4.0, -3.0, -4.0)

(a) $\rho = \sqrt{0^2 + 4^2} = 4, \phi = \tan^{-1}\left(\frac{4}{0}\right) = 90^o, z = 3, \; so \; P(4.0, 90^o, 3.0)$

(b) $\rho = \sqrt{2^2 + 3^2} = 3.6, \phi = \tan^{-1}\left(\frac{3}{-2}\right) = 124^o, z = 2, \; so \; P(3.6, 120^o, 2.0)$

(c) $\rho = \sqrt{4^2 + 3^2} = 5, \phi = \tan^{-1}\left(\frac{-3}{4}\right) = -37^o, z = -4, \; so \; P(5.0, -37^o, -4.0)$

P2.13: Convert the following points from cylindrical to Cartesian coordinates:

a. P(2.83, 45.0°, 2.00)
b. P(6.00, 120.°, -3.00)
c. P(10.0, -90.0°, 6.00)

(a)

$x = \rho \cos\phi = 2.83 \cos 45^o = 2.00$

$y = \rho \sin\phi = 2.83 \sin 45^o = 2.00$

$z = z = 2.00$

$so \; P(2.00, 2.00, 2.00).$

(b)

$x = \rho \cos\phi = 6.00 \cos 120^o = -3.00$

$y = \rho \sin\phi = 6.00 \sin 120^o = 5.20$

$z = z = -3.00$

$so \; P(-3.00, 5.20, -3.00).$

(c)

$x = \rho \cos\phi = 10.0 \cos(-90.0^o) = 0$

$y = \rho \sin\phi = 10.0 \sin(-90.0^o) = -10.0$

$z = z = 6.00$

$so \; P(0, -10.0, 6.00).$

P2.14: A 20.0 cm long section of copper pipe has a 1.00 cm thick wall and outer diameter of 6.00 cm.

a. Sketch the pipe conveniently overlaying the cylindrical coordinate system, lining up the length direction with the z-axis
b. Determine the total surface area (this could actually be useful if, say, you needed to do an electroplating step on this piece of pipe)
c. Determine the weight of the pipe given the density of copper is 8.96 g/cm^3

(a) See Figure P2.14

(b) The top area, S_{top}, is equal to the bottom area. We must also find the inner area, S_{inner}, and the outer area, S_{outer}.

$$S_{top} = \iint \rho d\rho d\phi = \int_2^3 \rho d\rho \int_0^{2\pi} d\phi = 5\pi \ cm^2.$$

$$S_{bottom} = S_{top}.$$

$$S_{outer} = \iint \rho d\phi dz = 3\int_0^{2\pi} d\phi \int_0^{20} dz = 120\pi \ cm^2$$

$$S_{inner} == \iint \rho d\phi dz = 2\int_0^{2\pi} d\phi \int_0^{20} dz = 80\pi \ cm^2$$

The total area, then, is 210π cm^2, or $S_{tot} = 660$ cm^2.

(c) Determining the weight of the pipe requires the volume:

$$V = \iiint \rho d\rho d\phi dz$$

$$= \int_2^3 \rho d\rho \int_0^{2\pi} d\phi \int_0^{20} dz = 100\pi \ cm^3.$$

$$M_{pipe} = \left(8.96 \frac{g}{cm^3}\right)\left(100\pi cm^3\right)$$

$$= 2815g.$$

So $M_{pipe} = 2820$g.

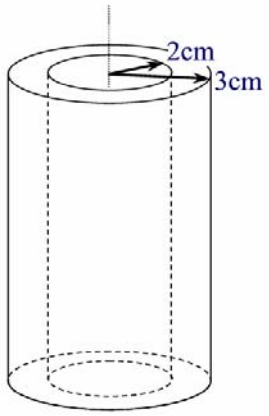


Fig. P2.14

P2.15: A line charge with charge density 2.00 nC/m exists at y = -2.00 m, x = 0.00. (a) A charge Q = 8.00 nC exists somewhere along the y-axis. Where must you locate Q so that the total electric field is zero at the origin? (b) Suppose instead of the 8.00 nC charge of part (a) that you locate a charge Q at (0.00, 6.00m, 0.00). What value of Q will result in a total electric field intensity of zero at the origin?

(a) The contributions to **E** from the line and point charge must cancel, or $\mathbf{E} = \mathbf{E}_L + \mathbf{E}_Q$.

For the line: $\mathbf{E}_L = \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho = \frac{\left(2nC/m\right)}{2\pi\left(10^{-9}F/36\pi m\right)(2m)}\mathbf{a}_y = 18\frac{V}{m}\mathbf{a}_y$

and for the point charge, where the point is located a distance y along the y-axis, we have: $\mathbf{E}_Q = \dfrac{Q}{4\pi\varepsilon_o y^2}\left(-\mathbf{a}_y\right) = \dfrac{(8nC)\left(-\mathbf{a}_y\right)}{4\pi\left(10^{-9}F/36\pi m\right)y^2} = \dfrac{72}{y^2}\left(-\mathbf{a}_y\right)$

Therefore:

$\dfrac{72}{y^2} = 18,$ or $y = \sqrt{\dfrac{72}{18}} = 2m.$

So $\boxed{\mathrm{Q}(0,2.0\mathrm{m},0)}$

(b)

$$\frac{Q}{4\pi\varepsilon_o(6)^2} = 18,$$

$$Q = \frac{(18)(36)}{9} = 72nC.$$

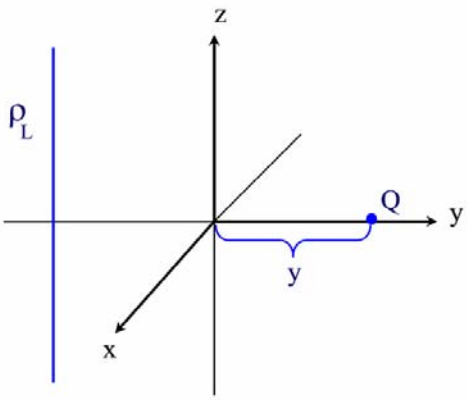


Fig. P2.15

P2.16: You are given two z-directed line charges of charge density +1 nC/m at x = 0, y = -1.0 m, and charge density –1.0 nC/m at x = 0, y = 1.0 m. Find **E** at P(1.0m,0,0).

The situation is represented by Figure P2.16a. A better 2-dimensional view in Figure P2.16b is useful for solving the problem.

$\mathbf{E}_1 = \dfrac{\rho_{L1}}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho$, and $\rho\mathbf{a}_\rho = \sqrt{2}\left(\dfrac{\mathbf{a}_x + \mathbf{a}_y}{\sqrt{2}}\right)$.

$\mathbf{E}_1 = \dfrac{\left(1x10^{-9}C\right)}{2\pi\left(10^{-9}F/36\pi m\right)\left(\sqrt{2}m\right)}\dfrac{\left(\mathbf{a}_x + \mathbf{a}_y\right)}{\sqrt{2}}\dfrac{FV}{C} = 9\left(\mathbf{a}_x + \mathbf{a}_y\right)\dfrac{V}{m}$, and $\mathbf{E}_2 = 9\left(\text{-}\mathbf{a}_x + \mathbf{a}_y\right)\dfrac{V}{m}$.

So $\mathbf{E}_{TOT} = 18\ \mathbf{a}_y$ V/m.

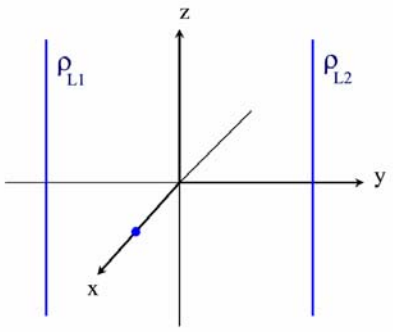


Fig. P2.16a

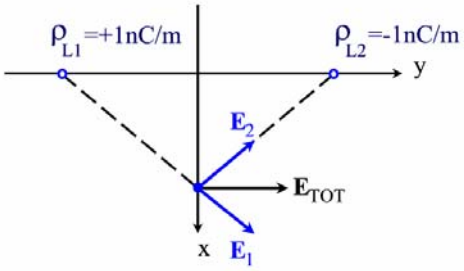


Fig. P2.16b

P2.17: MATLAB: Suppose you have a segment of line charge of length 2*L* centered on the z-axis and having a charge distribution ρ_L. Compare the electric field intensity at a point on the y-axis a distance *d* from the origin with the electric field at that point assuming the line charge is of infinite length. The ratio of *E* for the segment to *E* for the infinite line is to be plotted versus the ratio *L/d* using MATLAB.

This is similar to MATLAB 2.3. We have for the ideal case

$$\mathbf{E}_{ideal} = \frac{\rho_L}{2\pi\varepsilon_o \rho}\mathbf{a}_\rho = \frac{\rho_L}{2\pi\varepsilon_o d}\mathbf{a}_\rho.$$

For the actual *2L* case, we have an integration to perform (Equation (2.35) with different limits):

$$\mathbf{E}_{actual} = \frac{\rho_L \rho \mathbf{a}_\rho}{4\pi\varepsilon_o}\int_{-L}^{+L}\frac{dz}{\left(\rho^2+z^2\right)^{3/2}} = \frac{\rho_L d\mathbf{a}_y}{4\pi\varepsilon_o}\left[\frac{z}{d^2\sqrt{z^2+d^2}}\right]\Bigg|_{-L}^{+L}$$

$$\mathbf{E}_{actual} = \frac{\rho_L \mathbf{a}_y}{2\pi\varepsilon_o d}\left(\frac{L}{\sqrt{L^2+d^2}}\right).$$

Now we manipulate these expressions to get the following ratio:

$$\frac{\mathbf{E}_{actual}}{\mathbf{E}_{ideal}} = \frac{L/d}{\sqrt{1+\left(L/d\right)^2}}.$$

In the program, the actual to ideal field ratio is termed "Eratio" and the charged line half-length L ratioed to the distance d is termed "Lod".

```
%   M-File: MLP0217
%
%   This program is similar to ML0203.
%   It compares the E-field from a finite length
%   segment of charge (from -L to +L on the z-axis)
%   to the E-field from an infinite length line
%   of charge.  The ratio (E from segment to E from
%   infinite length line) is plotted versus the ratio
%   Lod=L/d, where d is the distance along the y axis.
%
%   Wentworth, 12/19/02
%
%   Variables:
%       Lod     the ratio L/d
%       Eratio  ratio of E from segment to E from line

clc     %clears the command window
clear   %clears variables

%   Initialize Lod array and calculate Eratio
Lod=0.1:0.01:100;
```

```
Eratio=Lod./(sqrt(1+Lod.^2));

%   Plot Eratio versus Lod
semilogx(Lod,Eratio)
grid on
xlabel('Lod=L/d')
ylabel('E ratio: segment to line')
```

Executing the program gives Figure P2.17.

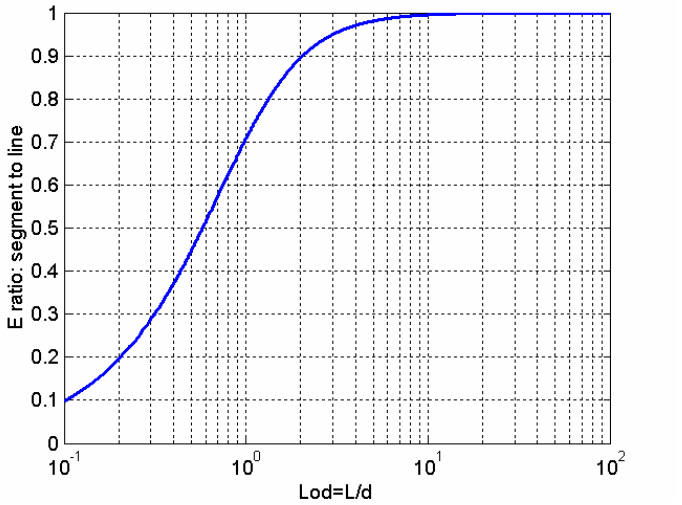


Fig. P2.17

So we see that the field from a line segment of charge appears equivalent to the field from an infinite length line if the test point is close to the line.

P2.18: A segment of line charge ρ_L =10 nC/m exists on the y-axis from the origin to y = +3.0 m. Determine **E** at the point (3.0, 0, 0)m.

It is clear from a sketch of the problem in Figure P2.18a that the resultant field will be directed in the x-y plane. The situation is redrawn in a temporary coordinate system in Figure P2.18b.

We have from Eqn (2.34) $\mathbf{E} = \int \frac{\rho_L dz}{4\pi\varepsilon_o} \frac{\rho \mathbf{a}_\rho - z\mathbf{a}_z}{\left(\rho^2 + z^2\right)^{3/2}} = E_\rho \mathbf{a}_\rho + E_z \mathbf{a}_z.$

For E_ρ we have:

$$E_\rho = \frac{\rho_L \rho}{4\pi\varepsilon_o} \int \frac{dz}{\left(\rho^2 + z^2\right)^{3/2}} = \frac{\rho_L \rho}{4\pi\varepsilon_o} \left[\frac{z}{\rho^2 \sqrt{z^2 + \rho^2}} \right]_0^3$$

With $\rho = 3$, we then have $E_\rho = 21.2$ V/m.
For E_z:

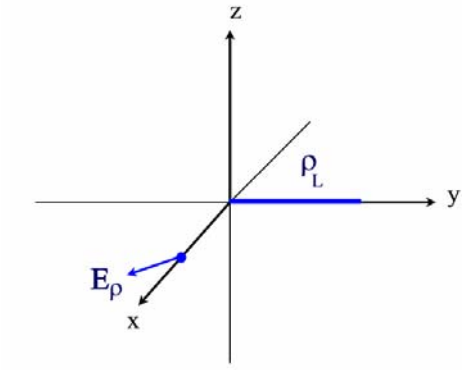


Fig. P2.18a

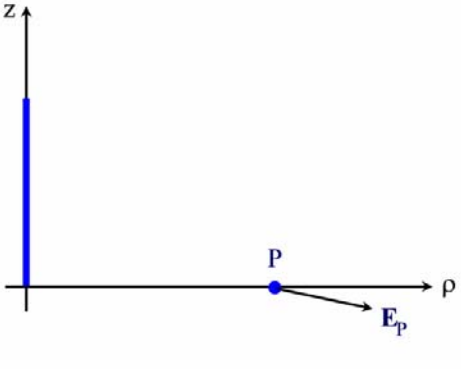


Fig. P2.18b

$$E_z = \frac{-\rho_L}{4\pi\varepsilon_o}\int \frac{z\,dz}{\left(\rho^2+z^2\right)^{3/2}} = \frac{\rho_L}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{\rho^2+z^2}}\right)\Bigg|_{z=0}^{\infty} = -8.79\frac{V}{m}$$

Thus we have $\mathbf{E}_{TOT} = 21\ \mathbf{a}_\rho - 8.8\ \mathbf{a}_z$ V/m.
Converting back to the original coordinates, we have $\mathbf{E}_{TOT} = 21\ \mathbf{a}_x - 8.8\ \mathbf{a}_y$ V/m.

5. Surface and Volume Charge

P2.19: In free space, there is a point charge Q = 8.0 nC at (-2.0,0,0)m, a line charge ρ_L = 10 nC/m at y = -9.0m, x = 0m, and a sheet charge ρ_s = 12. nC/m^2 at z = -2.0m. Determine **E** at the origin.

The situation is represented by Figure P2.19, and the total field is $\mathbf{E}_{TOT} = \mathbf{E}_Q + \mathbf{E}_L + \mathbf{E}_S$.

$$\mathbf{E}_Q = \frac{Q}{4\pi\varepsilon_o R^2}\mathbf{a}_R = \frac{\left(8x10^{-9}C\right)\mathbf{a}_x}{4\pi\left(10^{-9}F/36\pi m\right)(2m)^2}$$

$$= 18\mathbf{a}_x\frac{V}{m}$$

$$\mathbf{E}_L = \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho = \frac{\left(10x10^{-9}\,C/m\right)\mathbf{a}_y}{2\pi\left(10^{-9}F/36\pi m\right)(9m)}$$

$$= 20\mathbf{a}_y\frac{V}{m}$$

$$\mathbf{E}_s = \frac{\rho_s}{2\varepsilon_o}\mathbf{a}_N = \frac{\left(12x10^{-9}\,C/m^2\right)}{2\left(10^{-9}F/36\pi m\right)}\mathbf{a}_z$$

$$= 679\mathbf{a}_z\frac{V}{m}$$

So: $\mathbf{E}_{tot} = 18\ \mathbf{a}_x + 20\ \mathbf{a}_y + 680\ \mathbf{a}_z$ V/m.

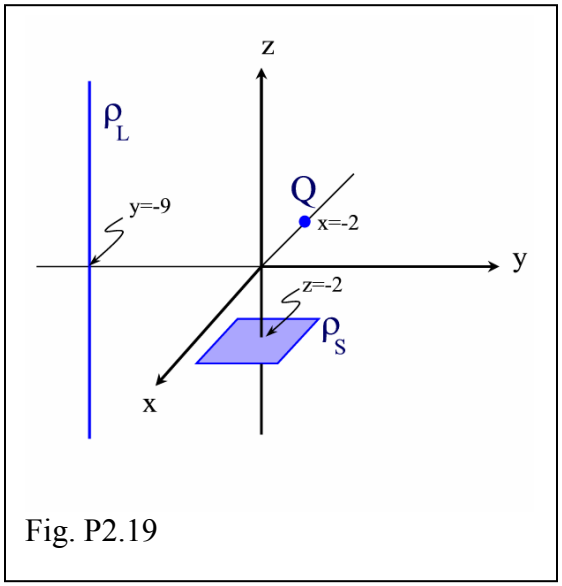


Fig. P2.19

P2.20: An infinitely long line charge (ρ_L = 21π nC/m) lies along the z-axis. An infinite area sheet charge (ρ_s = 3 nC/m^2) lies in the x-z plane at y = 10 m. Find a point on the y-axis where the electric field intensity is zero.

We have $\mathbf{E}_{TOT} = \mathbf{E}_L + \mathbf{E}_S$.

$$\mathbf{E}_L = \frac{\rho_L \mathbf{a}_\rho}{2\pi\varepsilon_o \rho} = \frac{\rho_L \mathbf{a}_y}{2\pi\varepsilon_o y}$$

$$= \frac{\left(21\pi x10^{-9} C/m\right)}{2\pi\left(10^{-9} F/36\pi m\right) y}\mathbf{a}_y = \frac{378\pi}{y}\mathbf{a}_y$$

$$\mathbf{E}_s = \frac{\rho_s \mathbf{a}_N}{2\varepsilon_o} = \frac{\left(3x10^{-9} C/m^2\right)\left(-\mathbf{a}_y\right)}{2\left(10^{-9} F/36\pi m\right)}$$

$$= -54\pi \mathbf{a}_y$$

so

$$\frac{378\pi}{y} - 54\pi = 0, \text{ or } y = 7.$$

Therefore, P(0, 7m, 0).

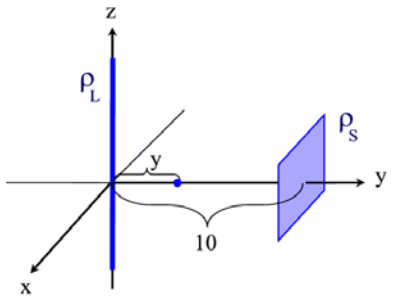


Fig. P2.20

P2.21: Sketch the following surfaces and find the total charge on each surface given a surface charge density of ρ_s = 1nC/m^2. Units (other than degrees) are meters.

(a) $-3 \le x \le 3, 0 \le y \le 4, z = 0$

(b) $1 \le r \le 4, 180° \le \phi \le 360°, \theta = \pi/2$

(c) $1 \le \rho \le 4, 180° \le \phi \le 360°, z = 0$

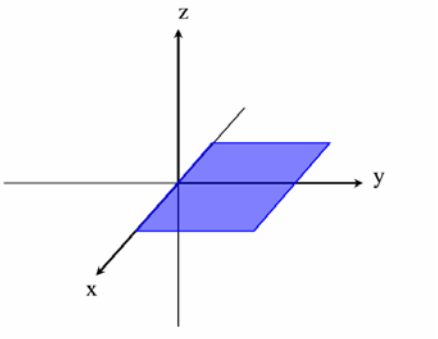


Fig. P2.21a

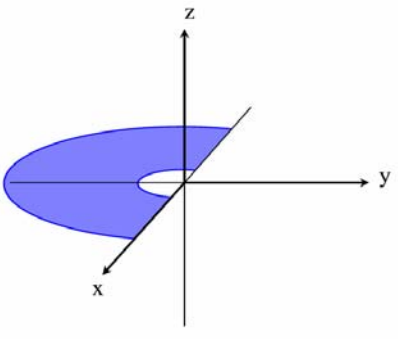


Fig. P2.21b&c

(a) $Q = \int \rho_s dS = \rho_s \int_{-3}^{3} dx \int_{0}^{4} dy = 24\rho_s = 24nC$

(b) $Q = \rho_s \int_1^4 r dr \int_\pi^{2\pi} d\phi = \frac{15\pi}{2}\rho_s = 24nC$

(c) $Q = \rho_s \int_1^4 \rho d\rho \int_\pi^{2\pi} d\phi = 24nC$

P2.22: Consider a circular disk in the x-y plane of radius 5.0 cm. Suppose the charge density is a function of radius such that $\rho_s = 12\rho$ nC/cm^2 (when ρ is in cm). Find the electric field intensity a point 20.0 cm above the origin on the z-axis.

From section 4 for a ring of charge of radius $a, \mathbf{E} = \frac{\rho_L a h \mathbf{a}_z}{2\varepsilon_o \left(a^2 + h^2\right)^{3/2}}$. Now we have

$\rho_L = \rho_s d\rho$ and $d\mathbf{E} = \frac{A\rho\, d\rho\, \rho h \mathbf{a}_z}{2\varepsilon_o \left(\rho^2 + h^2\right)^{3/2}}$, where $\rho_s = A\rho$ nC/cm^2. Now the total field is given by the integral:

$$\mathbf{E} = \frac{Ah\mathbf{a}_z}{2\varepsilon_o}\int \frac{\rho^2 d\rho}{\left(\rho^2 + h^2\right)^{3/2}}.$$

This can be solved using integration by parts, where $u = \rho$, du = dρ, $v = \frac{-1}{\sqrt{\rho^2 + h^2}}$, and $dv = \frac{\rho d\rho}{\sqrt{\rho^2 + h^2}}$. This leads to

$$\mathbf{E} = \frac{Ah}{2\varepsilon_o}\left[\frac{-a}{\sqrt{a^2 + h^2}} + \ln\left(\frac{a + \sqrt{a^2 + h^2}}{h}\right)\right]\mathbf{a}_z.$$

Plugging in the appropriate values we arrive at $\mathbf{E}$ = 6.7 kV/cm $\mathbf{a}_z$.

P2.23: Suppose a ribbon of charge with density ρ_s exists in the y-z plane of infinite length in the z direction and extending from $-a$ to $+a$ in the y direction. Find a general expression for the electric field intensity at a point d along the x-axis.

The problem is represented by Figure P2.23a. A better representation for solving the problem is shown in Figure P2.23b.

We have $d\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o \rho}\mathbf{a}_\rho$, where $\rho_L = \rho_s dy$. Then, since

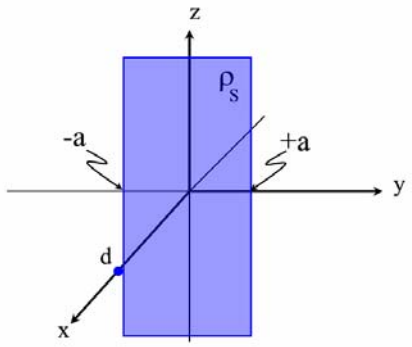
$$\rho\mathbf{a}_\rho = \frac{d\mathbf{a}_x - y\mathbf{a}_y}{\sqrt{d^2 + y^2}},$$

the integral becomes

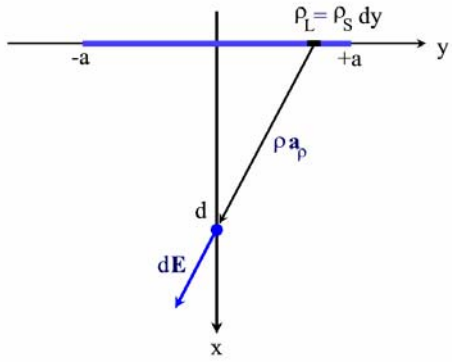
$$\mathbf{E} = \int \frac{\rho_s dy}{2\pi\varepsilon_o \sqrt{d^2 + y^2}} \frac{d\mathbf{a}_x - y\mathbf{a}_y}{\sqrt{d^2 + y^2}}.$$

It may be noted that the $\mathbf{a}_y$ component will cancel by symmetry. The $\mathbf{a}_x$ integral is found from the appendix and we have

$$\mathbf{E} = \frac{\rho_s}{\pi\varepsilon_o}\tan^{-1}\left(\frac{a}{d}\right)\mathbf{a}_x.$$



FigP2.23a



FigP2.23b

P2.24: Sketch the following volumes and find the total charge for each given a volume charge density of $\rho_v = 1\text{nC/m}^3$. Units (other than degrees) are meters.

(a) $0 \le x \le 4, 0 \le y \le 5, 0 \le z \le 6$

(b) $1 \le r \le 5, 0 \le \theta \le 60°$

(c) $1 \le \rho \le 5, 0° \le \phi \le 90°, 0 \le z \le 5$

(a) $$Q = \int \rho_v dv = \rho_v \int_0^4 dx \int_0^5 dy \int_0^6 dz = 120nC$$

(b)

$$Q = \int \rho_v dv$$

$$= \rho_v \int_1^5 r^2 dr \int_0^{60°} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 130nC$$

(c)

$$Q = \int \rho_v dv$$

$$= \rho_v \int_1^5 \rho \, d\rho \int_0^{\pi/2} d\phi \int_0^5 dz = 94nC$$

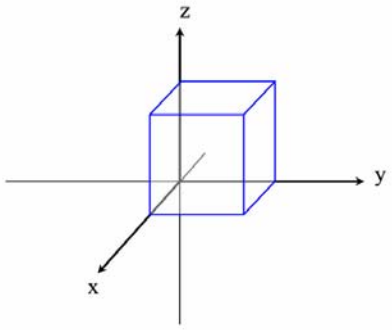


Fig. P2.24a

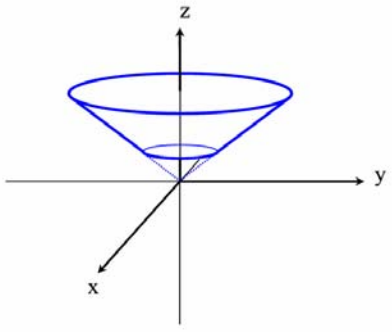


Fig. P2.24b

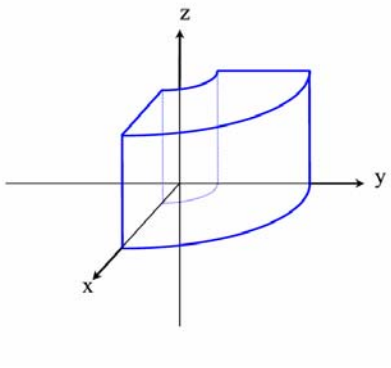


Fig. P2.24c

P2.25: You have a cylinder of 4.00 inch diameter and 5.00 inch length (imagine a can of tomatoes) that has a charge distribution that varies with radius as $\rho_v = (6\ \rho)$ nC/in^3 where ρ is in inches. (It may help you with the units to think of this as ρ_v (nC/in^3)= 6 (nC/in^4) ρ(in)). Find the total charge contained in this cylinder.

$$Q = \int \rho_v dv = \iiint (6\rho)\,\rho d\rho d\phi dz = 6\int_0^2 \rho^2 d\rho \int_0^{2\pi} d\phi \int_0^5 dz = 160\pi nC = 503nC$$

P2.26: MATLAB: Consider a rectangular volume with $0.00 \le x \le 4.00$ m, $0.00 \le y \le 5.00$ m and -6.00 m $\le z \le 0.00$ with charge density $\rho_v = 40.0$ nC/m^3. Find the electric field intensity at the point P(0.00,0.00,20.0m).

```
%   MLP0226
%   calculate E from a rectangular volume of charge

%   variables
%   xstart,xstop    limits on x for vol charge (m)
%   ystart,ystop
%   zstart,zstop
%   xt,yt,zt        test point (m)
%   rhov            vol charge density, nC/m^3
%   Nx,Ny,Nz        discretization points
%   dx,dy,dz        differential lengths
%   dQ              differential charge, nC
%   eo              free space permittivity (F/m)
%   dEi             differential field vector
%   dEix,dEiy,dEiz  x,y and z components of dEi
%   dEjx,dEjy,dEjz                      of dEj
%   dEkx,dEky,dEkz                      of dEk
%   Etot            total field vector, V/m
```

```
clc
clear

%   initialize variables
xstart=0;xstop=4;
ystart=0;ystop=5;
zstart=-6;zstop=0;
xt=0;yt=0;zt=20;
rhov=40e-9;
Nx=10;Ny=10;Nz=10;
eo=8.854e-12;

dx=(xstop-xstart)/Nx;
dy=(ystop-ystart)/Ny;
dz=(zstop-zstart)/Nz;
dQ=rhov*dx*dy*dz;

for k=1:Nz
    for j=1:Ny
        for i=1:Nx
            xv=xstart+(i-0.5)*dx;
            yv=ystart+(j-0.5)*dy;
            zv=zstart+(k-0.5)*dz;
            R=[xt-xv yt-yv zt-zv];
            magR=magvector(R);
            uvR=unitvector(R);
            dEi=(dQ/(4*pi*eo*magR^2))*uvR;
            dEix(i)=dEi(1);
            dEiy(i)=dEi(2);
            dEiz(i)=dEi(3);
        end
        dEjx(j)=sum(dEix);
        dEjy(j)=sum(dEiy);
        dEjz(j)=sum(dEiz);
    end
    dEkx(k)=sum(dEjx);
    dEky(k)=sum(dEjy);
    dEkz(k)=sum(dEjz);
end
Etotx=sum(dEkx);
Etoty=sum(dEky);
Etotz=sum(dEkz);
Etot=[Etotx Etoty Etotz]
```

Now to run the program:

Etot =

```
  -6.9983   -8.7104   79.7668

>>
```

So $\mathbf{E} = -7.0\ \mathbf{a}_x - 8.7\ \mathbf{a}_y + 80.\ \mathbf{a}_z$ V/m

P2.27: MATLAB: Consider a sphere with charge density $\rho_v = 120$ nC/m^3 centered at the origin with a radius of 2.00 m. Now, remove the top half of the sphere, leaving a hemisphere below the x-y plane. Find the electric field intensity at the point P(8.00m,0.00,0.00). (*Hint: see MATLAB 2.4, and consider that your answer will now have two field components*.)

```
%   M-File: MLP0227
%
%   This program modifies ML0204 to find the field
%   at point P(8m,0,0) from a hemispherical
%   distribution of charge given by
%   rhov=120 nC/m^3 from 0 < r < 2m and
%   pi/2 < theta < pi.
%
%   Wentworth, 12/23/02
%
%   Variables:
%       d           y axis distance to test point (m)
%       a           sphere radius (m)
%       dV          differential charge volume where
%                   dV=delta_r*delta_theta*delta_phi
%       eo          free space permittivity (F/m)
%       r,theta,phi spherical coordinate location of
%                   center of a differential charge element
%       x,y,z       cartesian coord location of charge %
                    element
%       R           vector from charge element to P
%       Rmag        magnitude of R
%       aR          unit vector of R
%       dr,dtheta,dphi  differential spherical elements
%       dEi,dEj,dEk partial field values
%       Etot        total field at P resulting from charge

clc     %clears the command window
clear   %clears variables

%   Initialize variables
eo=8.854e-12;
d=8;a=2;
```

```
delta_r=40;delta_theta=72;delta_phi=144;

%   Perform calculation
for k=(1:delta_phi)
    for j=(1:delta_theta)
        for i=(1:delta_r)
            r=i*a/delta_r;
            theta=(pi/2)+j*pi/(2*delta_theta);
            phi=k*2*pi/delta_phi;
            x=r*sin(theta)*cos(phi);
            y=r*sin(theta)*sin(phi);
            z=r*cos(theta);
            R=[d-x,-y,-z];
            Rmag=magvector(R);
            aR=R/Rmag;
            dr=a/delta_r;
            dtheta=pi/delta_theta;
            dphi=2*pi/delta_phi;
            dV=r^2*sin(theta)*dr*dtheta*dphi;
            dQ=120e-9*dV;
            dEi=dQ*aR/(4*pi*eo*Rmag^2);
            dEix(i)=dEi(1);
            dEiy(i)=dEi(2);
            dEiz(i)=dEi(3);
        end
        dEjx(j)=sum(dEix);
        dEjy(j)=sum(dEiy);
        dEjz(j)=sum(dEiz);
    end
    dEkx(k)=sum(dEjx);
    dEky(k)=sum(dEjy);
    dEkz(k)=sum(dEjz);
end
Etotx=sum(dEkx);
Etoty=sum(dEky);
Etotz=sum(dEkz);

Etot=[Etotx Etoty Etotz]
```

Now to run the program:

```
Etot =
  579.4623    0.0000   56.5317
```

So $\mathbf{E} = 580\,\mathbf{a}_x + 57\,\mathbf{a}_z$ V/m.

6. Electric Flux Density

P2.28: Use the definition of dot product to find the three interior angles for the triangle bounded by the points P(-3.00, -4.00, 5.00), Q(2.00, 0.00, -4.00), and R(5.00, -1.00, 0.00).

Here we use $\mathbf{A}\bullet\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos\theta_{AB}$.

$\mathbf{PR} = (5--3)\mathbf{a}_x + (-1--4)\mathbf{a}_y + (0-5)\mathbf{a}_z$

$\mathbf{PR} = 8\mathbf{a}_x + 3\mathbf{a}_y - 5\mathbf{a}_z, |\mathbf{PR}| = 9.9$

$\mathbf{PQ} = 5\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z, |\mathbf{PQ}| = 11.0$

$\mathbf{PR}\bullet\mathbf{PQ} = (8)(5) + (3)(4) + (-5)(-9) = 97$

$= |9.9||11.0|\cos\theta_P$

$$\theta_p = \cos^{-1}\left(\frac{97}{(9.9)(11)}\right) = 27^\circ$$

$\mathbf{RQ} = -3\mathbf{a}_x + 1\mathbf{a}_y - 4\mathbf{a}_z, |\mathbf{RQ}| = 5.1$

$\mathbf{RP} = -8\mathbf{a}_x - 3\mathbf{a}_y + 5\mathbf{a}_z, |\mathbf{RP}| = 9.9$

$\mathbf{RQ}\bullet\mathbf{RP} = 1 = (5.1)(9.9)\cos\theta_R, \theta_R = 89^\circ$

$\theta_Q = 180^\circ - 27^\circ - 89^\circ = 64^\circ$

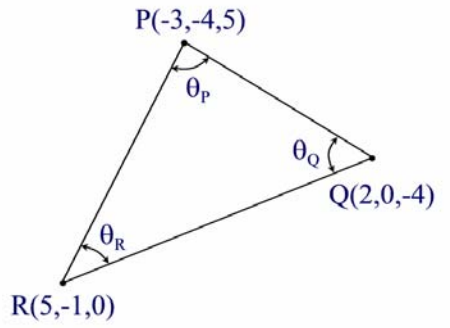


Fig. P2.28

P2.29: Given $\mathbf{D} = 2\rho\,\mathbf{a}_\rho + \sin\phi\,\mathbf{a}_z$ C/m^2, find the electric flux passing through the surface defined by $2.0 \le \rho \le 4.0$ m, $90.^\circ \le \phi \le 180^\circ$, and $z = 4.0$ m.

$\Psi = \int \mathbf{E}\bullet d\mathbf{S}, \quad d\mathbf{S} = \rho\, d\rho\, d\phi\,\mathbf{a}_z$

$$\Psi = \int \left(2\rho\mathbf{a}_\rho + \sin\phi\mathbf{a}_z\right)\bullet\rho\, d\rho\, d\phi\,\mathbf{a}_z = \int_2^4 \rho\, d\rho \int_{\pi/2}^{\pi} \sin\phi d\phi = 6C$$

P2.30: Suppose the electric flux density is given by $\mathbf{D} = 3r\,\mathbf{a}_r - \cos\phi\,\mathbf{a}_\theta + \sin^2\theta\,\mathbf{a}_\phi$ C/m^2. Find the electric flux through both surfaces of a hemisphere of radius 2.00 m and $0.00^\circ \le \theta \le 90.0^\circ$.

$\Psi_1 = \int \mathbf{D}\bullet d\mathbf{S}, \quad d\mathbf{S}_1 = r^2\sin\theta d\theta d\phi\,\mathbf{a}_r$

$\Psi_1 = \int \left(3r\mathbf{a}_r - \cos\phi\mathbf{a}_\theta + \sin^2\theta\mathbf{a}_\phi\right)\bullet\left(r^2\sin\theta d\theta d\phi\mathbf{a}_r\right)$

$$= 3r^3 \int_0^{\pi/2} \sin\theta d\theta \int_0^{2\pi} d\phi = 48\pi C$$

$d\mathbf{S}_2 = r dr d\phi\,\mathbf{a}_\theta$

$$\Psi_2 = \int -\cos\phi \mathbf{a}_\theta \bullet r dr d\phi \mathbf{a}_\theta$$

$$= -\left(\frac{r^2}{2}\Big|_0^2 \right) \sin\phi \Big|_0^{2\pi} = 0$$

$$\therefore \Psi = 48\pi C$$

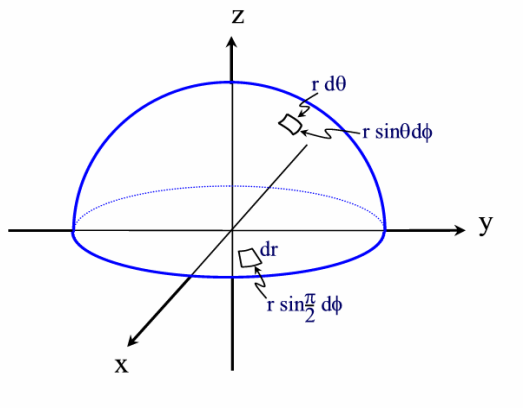


Fig. P2.30

7. Gauss's Law and Applications

P2.31: Given a 3.00 mm radius solid wire centered on the z-axis with an evenly distributed 2.00 coulombs of charge per meter length of wire, plot the electric flux density D_ρ versus radial distance from the z-axis over the range $0 \le \rho \le 9$ mm.

For a 1 m length, $\rho_v = \dfrac{2(C/m)}{\pi a^2} = 70.7x10^3 \dfrac{C}{m^3}$, $(a = 3mm = .003m)$

$Q_{enc} = \oint \mathbf{D} \bullet d\mathbf{S} = \int D_\rho \mathbf{a}_\rho \bullet \rho \, d\phi dz \mathbf{a}_\rho = 2\pi\rho L D_\rho$, where L is the length of the Gaussian surface. Note that this expression for Q_{enc} is valid for both Gaussian surfaces.

GS1 ($\rho < a$):

$$Q_{enc} = \int \rho_v dv = \rho_v \int_0^\rho \rho d\rho \int_0^{2\pi} d\phi \int_0^L dz = \rho_v \rho^2 \pi L$$

so $D_\rho = \dfrac{\rho_v \rho^2 \pi L}{2\pi\rho L} = \dfrac{\rho_v}{2}\rho$ for $\rho \le a$.

GS2 ($\rho > a$):

$$Q_{enc} = \rho_v a^2 \pi L, D_\rho = \frac{\rho_v a^2}{2}\frac{1}{\rho} \quad \text{for } \rho \ge a.$$

This is plotted with the following Matlab routine:

```
%   M-File: MLP0231
%
%   Gauss's Law Problem
%       solid cylinder with even charge
%
%   Variables
%   rhov    charge density (C/m^3)
%   a       radius of cylinder (m)
%   rho     radial distance from z-axis
%   rhomm   rho in mm
%   D       electric flux density (C/m^3)
```

```
%   N       number of data points
%   maxrad  max radius for plot (m)

clc;clear;

%   initialize variables
rhov=70.7e3;
a=0.003;
maxrad=.009;
N=100;
bndy=round(N*a/maxrad);

for i=1:bndy
    rho(i)=i*maxrad/N;
    rhomm(i)=rho(i)*1000;
    D(i)=rhov*rho(i)/2;
end

for i=bndy+1:N
    rho(i)=i*maxrad/N;
    rhomm(i)=rho(i)*1000;
    D(i)=(rhov*a^2)/(2*rho(i));
end
plot(rhomm,D)
xlabel('radial distance (mm)')
ylabel('elect. flux density (C/m^2)')
grid on
```

P2.32: Given a 2.00 cm radius solid wire centered on the z-axis with a charge density $\rho_v = 6\rho$ C/cm^3 (when ρ is in cm), plot the electric flux density D_ρ versus radial distance from the z-axis over the range $0 \leq \rho \leq 8$ cm.

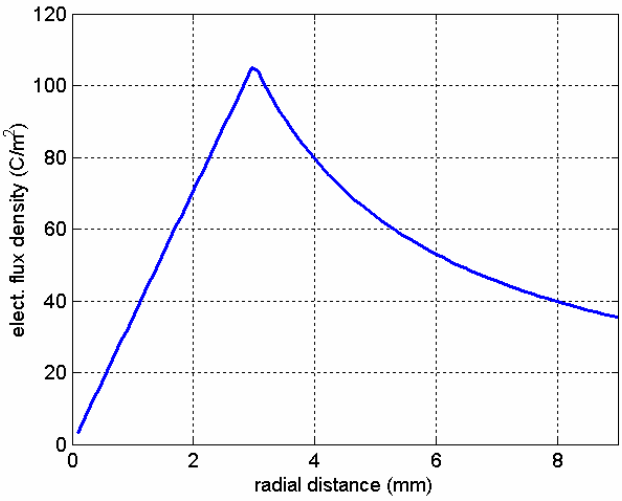


Fig. P2.31

Choose Gaussian surface length L, and as usual we have

$Q_{enc} = \oint \mathbf{D} \cdot d\mathbf{S} = \int D_\rho \mathbf{a}_\rho \cdot \rho d\phi dz \mathbf{a}_\rho = 2\pi L \rho D_\rho$, valid for both Gaussian surfaces.

In GS1 ($\rho < a$): $Q_{enc} = \int \rho_v dv = \int 6\rho^2 d\rho d\phi dz = 4\pi L \rho^3$,

so $D_\rho = \dfrac{4\pi L \rho^3}{2\pi L \rho} = 2\rho^2$ for $\rho \le a$.

For GS2 ($\rho > a$): $Q_{enc} = 4\pi L a^3$, $D_\rho = \dfrac{2a^3}{\rho}$ for $\rho \ge a$.

This is plotted for the problem values in the following Matlab routine.

```
%   M-File: MLP0232
%
%   Gauss's Law Problem
%       solid cylinder with radially-dependent charge
%
%   Variables
%   a       radius of cylinder (cm)
%   rho     radial distance from z-axis
%   D       electric flux density (C/cm^3)
%   N       number of data points
%   maxrad  max radius for plot (cm)

clc;clear;

%   initialize variables
a=2;
maxrad=8;
N=100;
bndy=round(N*a/maxrad);

for i=1:bndy
    rho(i)=i*maxrad/N;
    D(i)=2*rho(i)^2;
end

for i=bndy+1:N
    rho(i)=i*maxrad/N;
    D(i)=(2*a^3)/rho(i);
end
plot(rho,D)
xlabel('radial distance (cm)')
ylabel('elect. flux density (C/cm^2)')
grid on
```

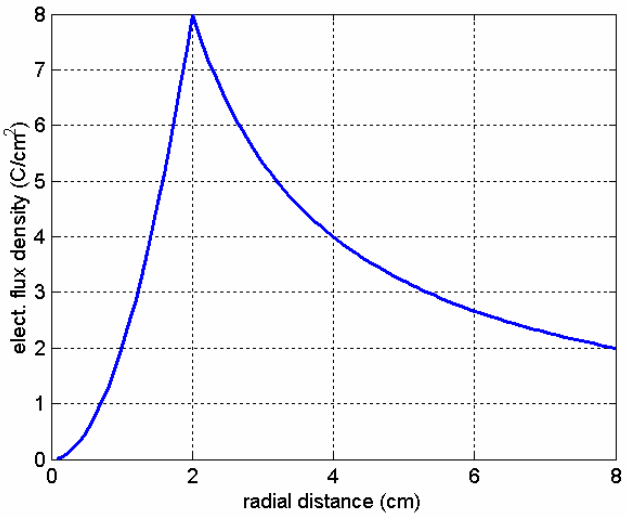


Fig. P2.32

P2.33: A cylindrical pipe with a 1.00 cm wall thickness and an inner radius of 4.00 cm is centered on the z-axis and has an evenly distributed 3.00 C of charge per meter length of pipe. Plot D_ρ as a function of radial distance from the z-axis over the range $0 \le \rho \le 10$ cm.

$Q_{enc} = \oint \mathbf{D} \cdot d\mathbf{S} = \int D_\rho \mathbf{a}_\rho \cdot \rho d\phi dz \mathbf{a}_\rho = 2\pi h \rho D_\rho$; this is true for all the Gaussian surfaces.

GS1 ($\rho < a$): since $Q_{enc} = 0$, $D_\rho = 0$.

GS2($a < \rho < b$):

$$\rho_v = \frac{3h}{\iiint \rho d\rho d\phi dz} = \frac{3}{\pi\left(b^2 - a^2\right)}$$

$$Q_{enc} = \int \rho_v dv$$

$$= \frac{3}{\pi\left(b^2 - a^2\right)} \int_a^\rho \rho d\rho \int_0^{2\pi} d\phi \int_0^h dz$$

$$= 3h \frac{\left(\rho^2 - a^2\right)}{\left(b^2 - a^2\right)}$$

So,

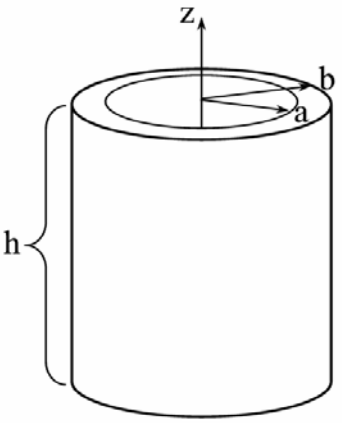


Fig. P2.33a

$$D_\rho = \frac{3h\left(\rho^2 - a^2\right)}{2\pi h \rho \left(b^2 - a^2\right)} = \frac{3}{2\pi\rho} \frac{\left(\rho^2 - a^2\right)}{\left(b^2 - a^2\right)}$$ for a $\le \rho \le b$.

GS3($\rho > b$):

$Q_{enc} = 3h,\ D_\rho = \dfrac{3}{2\pi\rho}$ for $\rho > b$.

A plot with the appropriate values is generated by the following Matlab routine:

```
%   M-File: MLP0233
%   Gauss's Law Problem
%       cylindrical pipe with even charge distribution
%
%   Variables
%   a       inner radius of pipe (m)
%   b       outer radius of pipe (m)
%   rho     radial distance from z-axis (m)
%   rhocm   radial distance in cm
%   D       electric flux density (C/cm^3)
%   N       number of data points
%   maxrad  max radius for plot (m)
clc;clear;

%   initialize variables
a=.04;b=.05;maxrad=0.10;N=100;
bndya=round(N*a/maxrad);
bndyb=round(N*b/maxrad);

for i=1:bndya
    rho(i)=i*maxrad/N;
    rhocm(i)=rho(i)*100;
    D(i)=0;
end

for i=bndya+1:bndyb
    rho(i)=i*maxrad/N;
    rhocm(i)=rho(i)*100;
    D(i)=(3/(2*pi*rho(i)))*((rho(i)^2-a^2)/(b^2-a^2));
end

for i=bndyb+1:N
    rho(i)=i*maxrad/N;
    rhocm(i)=rho(i)*100;
    D(i)=3/(2*pi*rho(i));
end

plot(rhocm,D)
xlabel('radial distance (cm)')
ylabel('elect. flux density (C/m^2)')
grid on
```

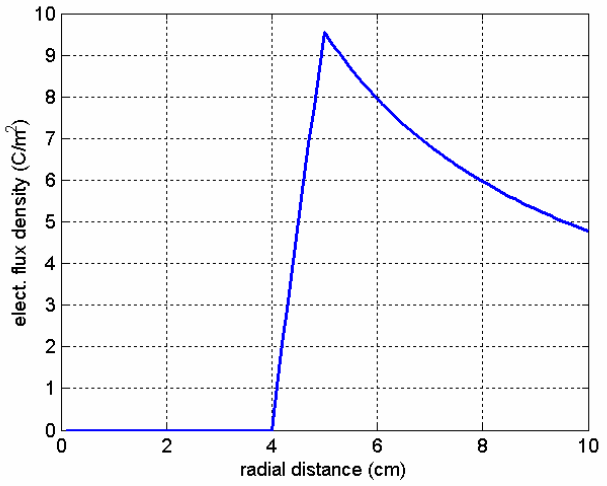


Fig. P2.33b

P2.34: An infinitesimally thin metallic cylindrical shell of radius 4.00 cm is centered on the z-axis and has an evenly distributed charge of 100. nC per meter length of shell. (a) Determine the value of the surface charge density on the conductive shell and (b) plot D_ρ as a function of radial distance from the z-axis over the range $0 \le \rho \le 12$ cm.

$$\rho_s = \frac{Q}{S} = \frac{100nC}{\int \rho d\phi dz} = \frac{100nC}{(.04m)(2\pi)(1m)} = 398\frac{nC}{m^2}.$$

For all Gaussian surfaces,
of height h and radius ρ, we have:

$Q_{enc} = \oint \mathbf{D} \bullet d\mathbf{S}$, where $d\mathbf{S} = \rho d\phi dz \mathbf{a}_\rho$,

$\therefore Q_{enc} = 2\pi\rho h D_\rho$.

GS1 ($\rho < a$): $Q_{enc} = 0$ so $D_\rho = 0$

GS2 ($\rho > a$):

$Q_{enc} = \int \rho_s dS = \rho_s \int \rho d\phi dz = 2\pi a h \rho_s$,

$D_\rho = \rho_s \frac{a}{\rho}$ for $\rho \ge a$.

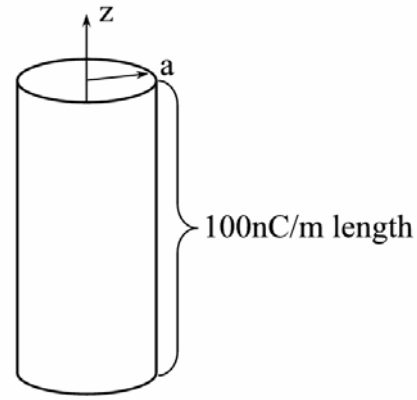


Fig. P2.34a

```
%   M-File: MLP0234
%
%   Gauss's Law Problem
%       cylindrical shell of charge
%
%   Variables
%   a       radius of cylinder (m)
%   Qs      surface charge density (nC/m^2)
%   rho     radial distance from z-axis (m)
```

```
%   rhocm   radial distance in cm
%   D       electric flux density (nC/cm^3)
%   N       number of data points
%   maxrad  max radius for plot (cm)

clc;clear;

%   initialize variables
a=.04;Qs=398;maxrad=0.12;N=100;
bndy=round(N*a/maxrad);

for i=1:bndy
    rho(i)=i*maxrad/N;
    rhocm(i)=rho(i)*100;
    D(i)=0;
end

for i=bndy+1:N
    rho(i)=i*maxrad/N;
    rhocm(i)=rho(i)*100;
    D(i)=Qs*a/rho(i);
end

plot(rhocm,D)
xlabel('radial distance (cm)')
ylabel('elect. flux density (nC/m^2)')
grid on
```

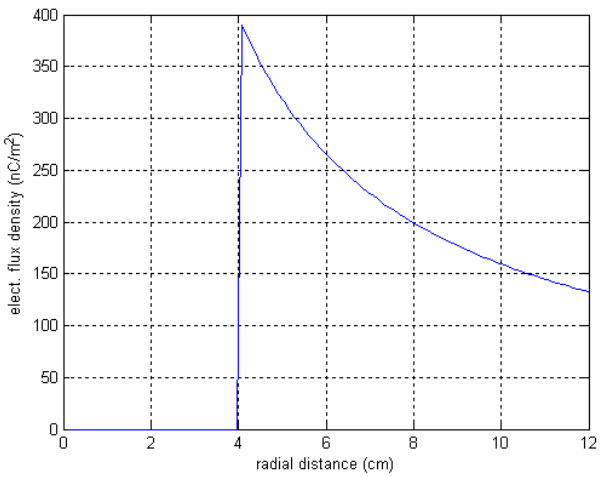


Fig. P2.34b

P2.35: A spherical charge density is given by $\rho_v = \rho_o\, r/a$ for $0 \le r \le a$, and $\rho_v = 0$ for $r > a$. Derive equations for the electric flux density for all r.

$Q_{enc} = \oint \mathbf{D}\bullet d\mathbf{S} = \int D_r \mathbf{a}_r \bullet r^2 \sin\theta d\theta d\phi \mathbf{a}_r = 4\pi r^2 D_r$. This is valid for each Gaussian surface.

GS1 ($r < a$): $Q_{enc} = \int \rho_v dv = \dfrac{\rho_o}{a}\int_0^r r^3 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \dfrac{\pi\rho_o r^4}{a}$.

So $D_r = \dfrac{\pi\rho_o r^4}{a 4\pi r^2} = \dfrac{\rho_o}{4a} r^2$ for $r \le a$.

GS2 ($r > a$): $Q_{enc} = \pi\rho_o a^3$, $D_r = \dfrac{\rho_o a^3}{4r^2}$ for $r \ge a$.

P2.36: A thick-walled spherical shell, with inner radius 2.00 cm and outer radius 4.00 cm, has an evenly distributed 12.0 nC charge. Plot D_r as a function of radial distance from the origin over the range $0 \le r \le 10$ cm.

Here we'll let a = inner radius and b = outer radius. Then
$Q_{enc} = \oint \mathbf{D}\bullet d\mathbf{S} = \int D_r \mathbf{a}_r \bullet r^2 \sin\theta d\theta d\phi \mathbf{a}_r = 4\pi r^2 D_r$; This is true for each Gaussian surface.
The volume containing charge is
$v = \int_a^b r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \dfrac{4}{3}\pi\left(b^3 - a^3\right)$.

So $\rho_v = \dfrac{Q}{v} = \dfrac{3Q}{4\pi\left(b^3 - a^3\right)}$.

Now we can evaluate Q_{enc} for each Gaussian surface.

GS1 ($r < a$): $Q_{enc} = 0$ so $D_r = 0$.

GS2 ($a < r < b$): $Q_{enc} = \int \rho_v dv = \rho_v \int_a^r r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \dfrac{\rho_v 4\pi}{3}\left(r^3 - a^3\right)$.

Inserting our value for ρ_v, we find

$D_r = \dfrac{Q}{4\pi r^2}\dfrac{\left(r^3 - a^3\right)}{\left(b^3 - a^3\right)}$ for $a \le r \le b$.

GS3 ($r > b$): $Q_{enc} = Q$, $D_r = \dfrac{Q}{4\pi r^2}$, for $r \ge b$.

This is plotted for appropriate values using the following Matlab routine:

```
%   M-File: MLP0236
%   Gauss's Law Problem
%       thick spherical shell with even charge
%
%   Variables
```

```
%   a       inner radius of sphere (m)
%   b       outer radius of sphere (m)
%   r       radial distance from origin (m)
%   rcm     radial distance in cm
%   D       electric flux density (nC/cm^3)
%   N       number of data points
%   maxr    max radius for plot (m)
%   Q       charge (nC)

clc;clear;

%   initialize variables
a=.02;b=.04;
Q=12;
maxrad=0.10;
N=100;
bndya=round(N*a/maxrad);
bndyb=round(N*b/maxrad);

for i=1:bndya
    r(i)=i*maxrad/N;
    rcm(i)=r(i)*100;
    D(i)=0;
end

for i=bndya+1:bndyb
    r(i)=i*maxrad/N;
    rcm(i)=r(i)*100;
    D(i)=(Q/(4*pi*r(i)^2))*(r(i)^3-a^3)/(b^3-a^3);
end

for i=bndyb+1:N
    r(i)=i*maxrad/N;
    rcm(i)=r(i)*100;
    D(i)=Q/(4*pi*r(i)^2);
end

plot(rcm,D)
xlabel('radial distance (cm)')
ylabel('elect. flux density (nC/m^2)')
grid on
```

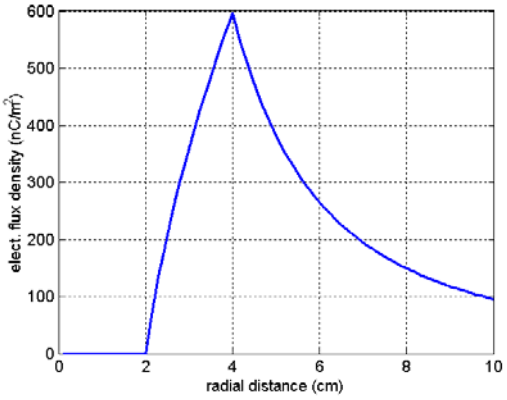


Fig. P2.36

P2.37: Given a coaxial cable with solid inner conductor of radius *a*, an outer conductor that goes from radius *b* to *c*, (so $c > b > a$), a charge $+Q$ that is evenly distributed

throughout a meter length of the inner conductor and a charge –*Q* that is evenly distributed throughout a meter length of the outer conductor, derive equations for the electric flux density for all ρ. You may orient the cable in any way you wish.

We conveniently center the cable on the z-axis. Then, for a Gaussian surface of length *L*, $Q_{enc} = \oint \mathbf{D} \bullet d\mathbf{S} = 2\pi\rho L D_\rho$; valid for all Gaussian surfaces.

GS1: ($\rho < a$): $\rho_v = \dfrac{Q}{(1m)(\pi a^2)}$;

$$Q_1 = \int \rho_v dv = \rho_v \int_0^\rho \rho d\rho \int_0^{2\pi} d\phi \int_0^L dz = \frac{QL}{a^2}\rho^2;$$

$$D_\rho = \frac{QL\rho^2}{a^2 2\pi\rho L} = \frac{Q}{2\pi a^2}\rho \quad \text{for } \rho \le a$$

GS2 ($a < \rho < b$): $Q_2 = QL$; $D_\rho = \dfrac{QL}{2\pi\rho L} = \dfrac{Q}{2\pi\rho}$ for $a \le \rho \le b$.

GS3 ($b < \rho < c$): $Q_3 = Q + \int \rho_{vo} dv$, where $\rho_{vo} = \dfrac{-Q}{(1m)\pi(c^2 - b^2)}$

$$Q_3 = Q + \frac{-Q}{\pi(c^2 - b^2)} \int_b^\rho \rho d\rho \int_0^{2\pi} d\phi \int_0^L dz = Q\frac{(c^2 - \rho^2)}{(c^2 - b^2)}$$

so $D_\rho = \dfrac{Q}{2\pi\rho}\dfrac{(c^2 - \rho^2)}{(c^2 - b^2)}$ for $b \le \rho \le c$.

GS4 ($\rho > c$): $Q_{enc} = 0$, $D_\rho = 0$.

8. Divergence and the Point Form of Gauss's Law

P2.38: Determine the charge density at the point P(3.0m,4.0m,0.0) if the electric flux density is given as $\mathbf{D} = xyz\,\mathbf{a}_z$ C/m^2.

$$\nabla \bullet \mathbf{D} = \frac{\partial D_z}{\partial z} = \frac{\partial(xyz)}{\partial z} = xy = \rho_v.$$

$\rho_v(3,4,0)=(3)(4)=12$ C/m^3.

P2.39: Given $\mathbf{D} = 3\mathbf{a}_x + 2xy\mathbf{a}_y + 8x^2y^3\mathbf{a}_z$ C/m^2, (a) determine the charge density at the point P(1,1,1). Find the total flux through the surface of a cube with $0.0 \le x \le 2.0$m, $0.0 \le y \le 2.0$m and $0.0 \le z \le 2.0$m by evaluating (b) the left side of the divergence theorem and (c) the right side of the divergence theorem.

(a) $\nabla\bullet\mathbf{D} = \dfrac{\partial}{\partial y}(2xy) = 2x, \quad \rho_v(1,1,1) = 2\dfrac{C}{m^3}.$

(b) $\oint \mathbf{D}\bullet d\mathbf{S} = \int \nabla\bullet\mathbf{D}dv = \int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{front}} + \int_{\text{back}}$

$\int_{top} = \int 8x^2y^3\mathbf{a}_z\bullet dxdy\mathbf{a}_z = 8\int_0^2 x^2dx\int_0^2 y^3dy = 85.3C$

$\int_{bottom} = \int 8x^2y^3\mathbf{a}_z\bullet(-dxdy\mathbf{a}_z) = -85.3C$

$\int_{left} = \int 2x\,y\big|_{\mathbf{y=0}}\,\mathbf{a}_y\bullet(-dxdz\mathbf{a}_y) = 0$

$\int_{right} = \int 2x\,y\big|_{y=2}\,\mathbf{a}_y\bullet dxdz\mathbf{a}_y = 16C$

$\int_{front} = \int 3\mathbf{a}_x\bullet dydz\mathbf{a}_x = 12C$

$\int_{back} = \int 3\mathbf{a}_x\bullet(-dydz\mathbf{a}_x) = -12C$

$\therefore Q_{enc} = \oint \mathbf{D}\bullet d\mathbf{S} = 16C.$

(c) $\nabla\bullet\mathbf{D} = \dfrac{\partial}{\partial y}(2xy) = 2x; \quad \int \nabla\bullet\mathbf{D}dv = 2\int_0^2 xdx\int_0^2 dy\int_0^2 dz = 16C.$

P2.40: Suppose $\mathbf{D} = 6\rho\cos\phi\,\mathbf{a}_\phi$ C/m^2. (a) Determine the charge density at the point (3m, 90°, -2m). Find the total flux through the surface of a quartered-cylinder defined by $0 \le \rho \le$ 4m, $0 \le \phi \le$ 90°, and -4m $\le z \le$ 0 by evaluating (b) the left side of the divergence theorem and (c) the right side of the divergence theorem.

(a) $(\nabla\bullet\mathbf{D})_{cylinder} = \dfrac{1}{\rho}\dfrac{\partial D_\phi}{\partial\phi} = \dfrac{1}{\rho}\dfrac{\partial(6\rho\cos\phi)}{\partial\phi} == -6\sin\phi.$

$\rho_v(3, 90^\circ, -2) = -6\dfrac{C}{m^3}.$

(b) $\oint \mathbf{D}\bullet d\mathbf{S} = \int_{\phi=0^\circ} + \int_{\phi=90^\circ} + \int_{top} + \int_{bottom} + \int_{outside}$,

note that the top, bottom and outside integrals yield zero since there is no component of D in the these d**S** directions.

$$\int_{\phi=0^\circ} = \int 6\rho\cos\phi\big|_{\phi=0^\circ}\mathbf{a}_\phi\bullet\left(-d\rho dz\mathbf{a}_\phi\right) = -192C$$

$$\int_{\phi=90^\circ} = \int 6\rho\cos\phi\big|_{\phi=90^\circ}\mathbf{a}_\phi\bullet\left(d\rho dz\mathbf{a}_\phi\right) = 0$$

So, $\oint \mathbf{D}\bullet d\mathbf{S} = -192C$.

(c)

$$\nabla\bullet\mathbf{D} = -6\sin\phi, \quad dv = \rho d\rho d\phi dz$$

$$\int \nabla\bullet\mathbf{D}dv = -6\int_0^{90^\circ}\sin\phi d\phi\int_0^4\rho d\rho\int_{-4}^0 dz = -192C.$$

P2.41: Suppose $\mathbf{D} = r^2\sin\theta\,\mathbf{a}_r + \sin\theta\cos\phi\,\mathbf{a}_\phi$ C/m^2. (a) Determine the charge density at the point (1.0m, 45°, 90°). Find the total flux through the surface of a volume defined by $0.0 \le r \le 2.0$ m, $0.0° \le \theta \le 90.°$, and $0.0 \le \phi \le 180°$ by evaluating (b) the left side of the divergence theorem and (c) the right side of the divergence theorem.

The volume is that of a quartered-sphere, as indicated in Figure P2.41.

(a)

$$\nabla\bullet\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi} = 4r\sin\theta - \frac{\sin\phi}{r} = \rho_v,$$

$$\rho_v\left(1, 45^\circ, 90^\circ\right) = 1.83\frac{C}{m^3}$$

(b) $\oint \mathbf{D}\bullet d\mathbf{S} = \int_{\phi=0^\circ} + \int_{\phi=180^\circ} + \int_{\theta=90^\circ} + \int_{r=2}$; note that $\int_{\theta=90^\circ} = 0$ since $D_\theta = 0$.

$$\int_{\phi=0^\circ} = \int \sin\theta\cos\phi\big|_{\phi=0^\circ}\mathbf{a}_\phi\bullet\left(-rdrd\theta\mathbf{a}_\phi\right) = -2C$$

$$\int_{\phi=180^\circ} = \int \sin\theta\cos\phi\big|_{\phi=180^\circ}\mathbf{a}_\phi\bullet rdrd\theta\mathbf{a}_\phi = -2C$$

$$\int_{r=2} = \int r^2\sin\theta\mathbf{a}_r\bullet r^2\sin\theta d\theta d\phi\mathbf{a}_r = r^4\int_0^{\pi/2}\sin^2\theta d\theta\int_0^\pi d\phi = 8\pi\int_0^{90^\circ}\left(1-\cos 2\theta\right)d\theta = 4\pi^2 C$$

Summing these terms we have $Q = 4(\pi^2 - 1)$C = 35.5C.

(c)

$$\int \nabla \bullet \mathbf{D} dv = \int \left(4r \sin\theta - \frac{\sin\phi}{r} \right) r^2 \sin\theta dr d\theta d\phi$$

$$= 4\int_0^2 r^3 dr \int_0^{\pi/2} \sin^2\theta d\theta \int_0^{\pi} d\phi - \int_0^2 r dr \int_0^{\pi/2} \sin\theta d\theta \int_0^{\pi} \sin\phi d\phi = 4\pi^2 - 4 = 35.5C.$$

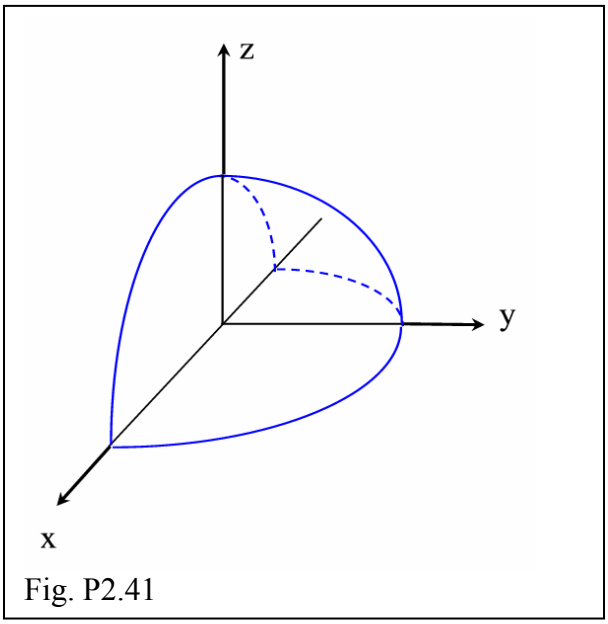


Fig. P2.41

9. Electric Potential

P2.42: A sheet of charge density ρ_s = 100 nC/m^2 occupies the x-z plane at y = 0. (a) Find the work required to move a 2.0 nC charge from P(-5.0m, 10.m, 2.0m) to *M*(2.0m, 3.0m, 0.0). (b)Find V_{MP}.

(a) $W = -Q\int_P^M \mathbf{E} \bullet d\mathbf{L}$; so we need **E** for the sheet charge.

$$\mathbf{E} = \frac{\rho_s}{2\varepsilon_o}\mathbf{a}_N = \frac{\left(100x10^{-9}C\right)}{2\left(8.854x10^{-12}\,F/m\right)}\frac{FV}{C}\mathbf{a}_y = 5.65x10^3\frac{V}{m}\mathbf{a}_y$$

Notice that we are only concerned with movement in the y-direction. We then have:

$$W = -2x10^{-9}C\int_{y=10}^{y=3}\left(5.65x10^3\frac{V}{m}\mathbf{a}_y\right)\bullet dy\mathbf{a}_y\left(\frac{J}{CV}\right) = 79\mu J$$

(b) $V_{MP} = \frac{W}{Q} = \frac{\left(79\mu J\right)}{\left(2x10^{-9}C\right)}\frac{CV}{J} = 39.5kV$; so $V_{MP} = 40kV$.

P2.43: A surface is defined by the function $2x + 4y^2 - \ln z = 12$. Use the gradient equation to find a unit vector normal to the plane at the point (3.00m,2.00m,1.00m).

Let $F = 2x + 4y^2 - \ln z = 12,$ then

$$\mathbf{a}_x = \frac{\nabla F}{|\nabla F|}; \quad \nabla F = 2\mathbf{a}_x + 8y\mathbf{a}_y - \frac{1}{z}\mathbf{a}_z,$$

At (3,2,1),

$$\nabla F = 2\mathbf{a}_x + 16\mathbf{a}_y - \mathbf{a}_z, |\nabla F| = \sqrt{2^2 + 16^2 + 1^2} = 16.16,$$

$$\mathbf{a}_N = 0.124\mathbf{a}_x + 0.990\mathbf{a}_y - 0.062\mathbf{a}_z$$

P2.44: For the following potential distributions, use the gradient equation to find **E**.

(a) $V = x+y^2z$ (V)

(b) $V = \rho^2 \sin\phi$(V)

(c) $V = r \sin\theta \cos\phi$ (V).

(a) $\mathbf{E} = -\nabla V = -\mathbf{a}_x - 2yz\mathbf{a}_y - y^2\mathbf{a}_z$

(b) $$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z\right) = -2\rho \sin\phi \mathbf{a}_\rho - \rho\cos\phi\mathbf{a}_\phi$$

(c)

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\mathbf{a}_\rho + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi\right) = -\sin\theta\cos\phi\mathbf{a}_\rho - \cos\theta\cos\phi\mathbf{a}_\theta + \sin\phi\mathbf{a}_\phi$$

P2.45: A 100 nC point charge is located at the origin. (a) Determine the potential difference V_{BA} between the point A(0.0,0.0,-6.0)m and point B(0.0,2.0,0.0)m. (b) How much work would be done to move a 1.0 nC charge from point A to point B against the electric field generated by the 100 nC point charge?

(a) $V_{BA} = -\int_A^A \mathbf{E} \bullet d\mathbf{L}.$

The potential difference is only a function of radial distance from the origin. Letting r_a = 6m and r_b = 2m, we then have

$$V_{BA} = -\int_{r_a}^{r_b} \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r \bullet dr\mathbf{a}_r = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{r_b} - \frac{1}{r_a}\right) = 300V.$$

(b) $W = Q_2 V_{BA} = \left(10^{-9}C\right)\left(300V\right)\frac{J}{CV} = 300nJ$

P2.46: MATLAB: Suppose you have a pair of charges Q_1(0.0, -5.0m, 0.0) = 1.0 nC and Q_2(0.0, 5.0m, 0.0) = 2.0 nC. Write a MATLAB routine to calculate the potential V_{RO} moving from the origin to the point R(5.0m, 0.0, 0.0). Your numerical integration will involve choosing a step size ΔL and finding the field at the center of the step. You should try several different step sizes to see how much this affects the solution.

```
%   M-File: MLP0246
%
%   Modify ML0207 to calculate the potential
%   difference going from the origin (O) to the point
%   R(5,0,0) given a pair of point charges
%   Q1(0,-5,0)=1nC and Q2(0,5,0)=2nC.
%
%   The approach will be to break up the distance
%   from O to R into k sections.  The total field E will
%   be found at the center of each section (located
%   at point P) and then dot(Ep,dLv) will give the
%   potential drop across the kth section.  Total
%   potential is found by summing the potential drops.
%
%   Wentworth, 1/7/03
%
%   Variables:
%       Q1,Q2   the point charges, in nC
%       k       number of numerical integration steps
%       dL      magnitude of one step
%       dLv     vector for a step
%       x(n)    x location at center of section at P
%       R1,R2   vector from Q1,Q2 to P
%       E1,E2   electric fields from Q1 & Q2 at P
%       Etot    total electric field at P
%       V(n)    portion of dot(Etot,dL) at P

clc     %clears the command window
clear   %clears variables

%   Initialize variables
k=64;
Q1=1;
Q2=2;
dL=5/k;
dLv=dL*[1 0 0];


%   Perform calculation
for n=1:k
    x(n)=(n-1)*dL+dL/2;
    R1=[x(n) 5 0];
    R2=[x(n) -5 0];
    Rmag1=magvector(R1);
    Rmag2=magvector(R2);
    E1=9*Q1*R1/Rmag1^3;
```

```
    E2=9*Q2*R2/Rmag2^3;
    Etot=E1+E2;
    V(n)=dot(Etot,dLv);
end

Vtot=sum(-V)

Now running the program:
Vtot =
   -1.5817
```

So V_{RO} = -1.6 V.

P2.47: For an infinite length line of charge density ρ_L = 20 nC/m on the z-axis, find the potential difference V_{BA} between point B(0, 2m, 0) and point A(0, 1m, 0).

$$V_{BA} = -\int_A^B \mathbf{E}\bullet d\mathbf{L}; \quad \mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho, d\mathbf{L} = d\rho\mathbf{a}_\rho,$$

$$\text{so } V_{BA} = -\int_A^B \frac{\rho_L}{2\pi\varepsilon_o\rho}\mathbf{a}_\rho \bullet d\rho\mathbf{a}_\rho = \frac{-\rho_L}{2\pi\varepsilon_o}\ln(2) = -250V$$

P2.48: Find the electric field at point P(0.0,0.0,8.0m) resulting from a surface charge density ρ_s = 5.0 nC/m^2 existing on the z = 0 plane from ρ = 2.0 m to ρ = 6.0 m. Assume V = 0 at a point an infinite distance from the origin.

(Method 1)
For a ring of charge it was previously found that

$$\mathbf{E} = \frac{\rho_L a h\mathbf{a}_z}{2\varepsilon_o\left(a^2+h^2\right)^{3/2}}.$$

We can then break up our disk into differential rings (see Figure P2.48), each contributing $d\mathbf{E}$ as:

$$d\mathbf{E} = \frac{\rho_S h}{2\varepsilon_o}\frac{\rho d\rho}{\left(\rho^2+h^2\right)^{3/2}}\mathbf{a}_z, \quad \text{where we've used } \rho_L = \rho_S d\rho.$$

So we then have

$$\mathbf{E} = \frac{\rho_S h\mathbf{a}_z}{2\varepsilon_o}\int\frac{\rho d\rho}{\left(\rho^2+h^2\right)^{3/2}}.$$

This is easy to integrate if we let $u = \rho^2 + h^2$, then $du = 2\,\rho\,d\rho$, and we have

$$\mathbf{E} = \frac{\rho_S h\mathbf{a}_z}{4\varepsilon_o}\int u^{-3/2}du = \frac{\rho_S h\mathbf{a}_z}{4\varepsilon_o}\frac{-2}{\sqrt{u}} = \frac{-\rho_S h\mathbf{a}_z}{2\varepsilon_o}\frac{1}{\sqrt{\rho^2+h^2}}\Bigg|_a^b$$

Solving, we arrive at

$$\mathbf{E} = \frac{-\rho_S h}{2\varepsilon_o}\left[\frac{1}{\sqrt{b^2+h^2}} - \frac{1}{\sqrt{a^2+h^2}}\right]\mathbf{a}_z.$$

Upon inserting the appropriate values we find **E** = 48 V/m $\mathbf{a}_z$.

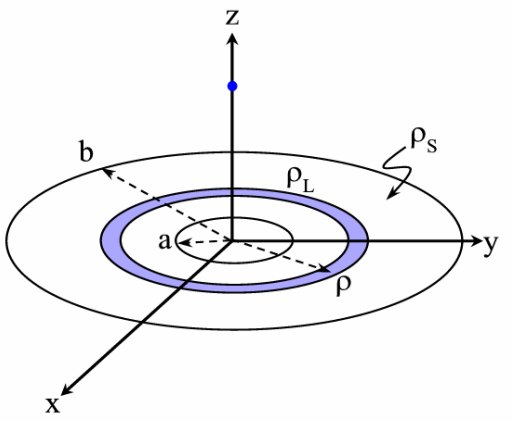


Fig. P2.48

(method 2)

Find an expression for potential and then evaluate the gradient at the point.

$$V = \int \frac{dQ}{4\pi\varepsilon_o R}, \quad R = \sqrt{\rho^2+h^2}, dQ = \rho_S \rho d\rho d\phi, \quad \text{so V}=\int \frac{\rho_S \rho d\rho d\phi}{4\pi\varepsilon_o\sqrt{\rho^2+h^2}}$$

$$\text{V}=\frac{\rho_S}{2\varepsilon_o}\int_a^b \frac{\rho d\rho}{\sqrt{\rho^2+h^2}} = \frac{\rho_S}{2\varepsilon_o}\sqrt{\rho^2+h^2}\Big|_a^b = \frac{\rho_S}{2\varepsilon_o}\left[\sqrt{b^2+h^2}-\sqrt{a^2+h^2}\right].$$

Now we let h = z and $\mathbf{E} = -\nabla V$;

$$\mathbf{E} = -\frac{\rho_S}{2\varepsilon_o}\left[\frac{\partial}{\partial z}\left(b^2+z^2\right)^{1/2} - \frac{\partial}{\partial z}\left(a^2+z^2\right)^{1/2}\right]\mathbf{a}_z$$

$$= -\frac{\rho_S}{2\varepsilon_o}\left[\frac{1}{2}\left(b^2+z^2\right)^{-1/2}2z - \frac{1}{2}\left(a^2+z^2\right)^{-1/2}2z\right]\mathbf{a}_z = -\frac{\rho_S}{2\varepsilon_o}\left[\frac{z}{\sqrt{b^2+z^2}} - \frac{z}{\sqrt{a^2+z^2}}\right]\mathbf{a}_z$$

Plugging in the values we find **E** = 48 V/m $\mathbf{a}_z$.

P2.49: Suppose a 6.0 m diameter ring with charge density 5.0 nC/m lies in the x-y plane with the origin at its center. Determine the potential difference V_{ho} between the point h(0.0,0.0,4.0)m and the origin. *(Hint: first find an expression for **E** on the z-axis as a general function of z.)*

For the ring of charge, replacing *h* with *z*, we have

$$\mathbf{E} = \frac{\rho_L a \mathbf{a}_z}{2\varepsilon_o}\frac{z}{\left(a^2+z^2\right)^{3/2}}$$

$$V_{ho} = -\int_0^h \mathbf{E}\cdot d\mathbf{L} = -\frac{\rho_L a}{2\varepsilon_o}\int \frac{z\,dz}{\left(a^2+z^2\right)^{3/2}}.$$

Letting $u = a^2 + z^2$, $du = 2z\,dz$, we have

$$V_{ho} = -\frac{\rho_L a}{4\varepsilon_o}\int u^{-3/2}du = \frac{\rho_L a}{2\varepsilon_o}\frac{1}{\sqrt{u}}.$$

Replacing u and evaluating from 0 to h,

$$V_{ho} = \frac{\rho_L a}{2\varepsilon_o}\left[\frac{1}{\sqrt{a^2+h^2}} - \frac{1}{a}\right]$$

$$= -36\pi V = -113V.$$

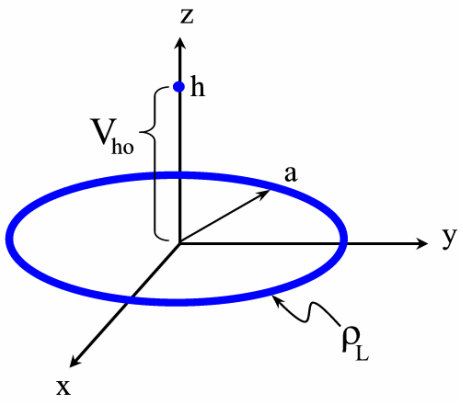


Fig. P2.49

10. Conductivity and Current

P2.50: A columnular beam of electrons from $0 \le \rho \le 1$ mm has a charge density ρ_v =-0.1 $\cos(\pi\rho/2)$ nC/mm^3 (where ρ is in mm) and a velocity of 6 x 10^6 m/sec in the $+\mathbf{a}_z$ direction. Find the current.

Let's let $\rho_v = \rho_o \cos\left(\frac{\pi\rho}{2}\right)$, where ρ_o = -0.1 nC/mm^3. Then we'll let $\mathbf{u} = u_o\mathbf{a}_z$, where u_o = 6x10^9 mm/s. Notice we convert the units to mm. Now,

$$\mathbf{J} = \rho_v\mathbf{u} = \rho_o u_o \cos\left(\frac{\pi\rho}{2}\right)\mathbf{a}_z,$$

and with $d\mathbf{S} = \rho\,d\rho\,d\phi\,\mathbf{a}_z$ we then have

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \rho_o u_o \int_0^a \rho\cos\left(\frac{\pi\rho}{2}\right)d\rho\int_0^{2\pi} d\phi.$$

This becomes

$$I = A\int_0^a \rho\cos\left(\frac{\pi\rho}{2}\right)d\rho, \text{ where } A = 2\pi\rho_o u_o.$$

Now we can integrate by parts, or $\int u\,dv = uv - \int v\,du$, where $u = A\rho$, $du = Ad\rho$, $v = \frac{2}{\pi}\sin\left(\frac{\pi\rho}{2}\right)$, and $dv = \cos\left(\frac{\pi\rho}{2}\right)d\rho$.

We then have

$$I = \frac{2Aa}{\pi}\sin\left(\frac{\pi a}{2}\right) + \frac{4A}{\pi^2}\left[\cos\left(\frac{\pi a}{2}\right) - 1\right].$$

To evaluate, we first find $A = 2\pi(-0.1\text{x}10^{-9})(6\text{x}10^9)$=3.77, and then
I = 2.40-1.53=0.87A.

I = 0.87A.

P2.51: Two spherical conductive shells of radius a and b ($b > a$) are separated by a material with conductivity σ. Find an expression for the resistance between the two spheres.

First find **E** for $a < r < b$, assuming $+Q$ at $r = a$ and $-Q$ at $r = b$. From Gauss's law:

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r$$

Now find V_{ab}:

$$V_{ab} = -\int_b^a \mathbf{E}\bullet d\mathbf{L} = -\int_b^a \frac{Q}{4\pi\varepsilon_o r^2}\mathbf{a}_r \bullet dr\mathbf{a}_r$$

$$= \frac{-Q}{4\pi\varepsilon_o}\int_b^a \frac{dr}{r^2} = \frac{Q}{4\pi\varepsilon_o}\frac{1}{r}\Big|_b^a = \frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{a}-\frac{1}{b}\right).$$

Now can find I:

$$I = \int \mathbf{J}\bullet d\mathbf{S} = \sigma\int \mathbf{E}\bullet d\mathbf{S} = \sigma\int \frac{Q}{4\pi\varepsilon_o}\frac{1}{r^2}\mathbf{a}_r \bullet r^2 \sin\theta d\theta d\phi \mathbf{a}_r$$

$$= \frac{\sigma Q}{4\pi\varepsilon_o}\int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi = \frac{\sigma Q}{\varepsilon_o}.$$

Finally, $R = \dfrac{V_{ab}}{I} = \dfrac{1}{4\pi\sigma}\left(\dfrac{1}{a}-\dfrac{1}{b}\right)$

P2.52: The typical length of each piece of jumper wire on a student's protoboard is 5.0 cm. Assuming AWG-20 (wire diameter 0.812 mm) copper wire, (a) determine the resistance for this length of wire. (b) Determine the power dissipated in the wire for 10. mA of current.

(a) $R = \dfrac{1}{\sigma}\dfrac{L}{\pi a^2} = \dfrac{1}{5.8x10^7\,(S/m)}\dfrac{0.05m}{\pi\left(0.406x10^{-3}m\right)^2} = 1.67m\Omega$

so R = 1.7 mΩ

(b) $P = I^2R = \left(10x10^{-3}A\right)^2\left(1.7x10^{-3}\Omega\right) = 170nW$

P2.53: A densely wrapped coil of AWG-22 (0.644 mm diameter) copper magnet wire is 150 m long. The wire has a very thin insulative sheath. Determine the resistance for this length of wire.

$$R = \frac{1}{\sigma}\frac{L}{\pi a^2} = \frac{1}{5.8x10^7\ S/m}\frac{150m}{\pi\left(0.322x10^{-3}m\right)^2} = 7.94\Omega$$

so R = 7.9Ω

P2.54: Determine an expression for the power dissipated per unit length in coaxial cable of inner radius *a*, outer radius *b*, and conductivity between the conductors σ if a potential difference V_{ab} is applied.

From Eqn(2.84) we have $R = \frac{1}{2\pi\sigma L}\ln\left(\frac{b}{a}\right)$

Now for a given potential difference V_{ab} we have

$$P = \frac{V_{ab}^2}{R} = \frac{2\pi\sigma L V_{ab}^2}{\ln\left(b/a\right)}, \quad \text{so} \quad \frac{P}{L} = \frac{2\pi\sigma V_{ab}^2}{\ln\left(b/a\right)}.$$

P2.55: Find the resistance per unit length of a stainless steel pipe of inner radius 2.5 cm and outer radius 3.0 cm.

$$R = \frac{1}{\sigma}\frac{L}{\pi\left(b^2 - a^2\right)},$$

so we have $\frac{R}{L} = \frac{1}{\sigma}\frac{1}{\pi\left(b^2 - a^2\right)} = \left(\frac{1}{1.1x10^6\ S/m}\right)\left(\frac{1}{\pi\left(.030^2 - .025^2\right)m^2}\right) = 1.05\frac{m\Omega}{m}$

so R/L = 1.0 mΩ/m

P2.56: A nickel wire of diameter 5.0 mm is surrounded by a 0.50 mm thick layer of silver. What is the resistance per unit length for this wire? Assuming 1.0 m of this wire carries 1.0 A of current, determine the power dissipated in the nickel portion and in the silver portion of the wire.

We can treat this wire as two resistors in parallel. We have

$$\frac{R_{Ni}}{L} = \frac{1}{1.5x10^7}\frac{1}{\pi\left(2.5x10^{-3}\right)^2} = 3.4x10^{-3}\frac{\Omega}{m}$$

$$\frac{R_{Ag}}{L} = \frac{1}{6.2x10^7}\frac{1}{\pi\left[\left(3x10^{-3}\right)^2 - \left(2.5x10^{-3}\right)^2\right]} = 1.87x10^{-3}\frac{\Omega}{m}$$

$$\frac{R_{total}}{L} = \frac{R_{Ni}}{L}\left\|\frac{R_{Ag}}{L}\right. = 1.2\frac{m\Omega}{m}$$

To find the power dissipated, we first find the potential difference:

$V = IR_{total} = 1.2mV$

then

$$P_{Ni} = \frac{V^2}{R_{Ni}} = 0.42mW, P_{Ag} = \frac{V^2}{R_{Ag}} = 0.77mW$$

11. Dielectrics

P2.57: A material has 12.0 V/m $\mathbf{a}_x$ field intensity with permittivity 194.5 pF/m. Determine the electric flux density.

$$\mathbf{D} = \varepsilon\mathbf{E} = \left(194.5x10^{-12}\,F/m\right)\left(12\,V/m\right)\frac{C}{FV} = 2.3\frac{nC}{m^2}\mathbf{a}_x$$

P2.58: MATLAB: A 20 nC point charge at the origin is embedded in Teflon (ε_r = 2.1). Find and plot the magnitudes of the polarization vector, the electric field intensity and the electric flux density at a radial distance from 0.1 cm out to 10 cm.

We use the following equations:

$$E = \frac{Q}{4\pi\varepsilon_r\varepsilon_o r^2}, P = \chi_e\varepsilon_o E, D = \varepsilon_r\varepsilon_o E$$

```
%   M-File: MLP0258
%
%   Plot E, P and D vs distance r from a point
%   charge Q at the origin with a dielectric.
%
%   Variables
%   Q       charge (C)
%   eo      free space permittivity (F/m)
%   r       radial distance (m)
%   Chi     electric susceptibility
%   E       electric field intensity(V/m)
%   D       electric flux density (C/m^2)
%   P       polarization vector (C/m^2)

%   initialize variables
Q=20e-9;
er=2.1;
eo=8.854e-12;
Chi=er-1;

%   perform calculations
r=0.001:.001:0.100;
rcm=r.*100;
E=Q./(4*pi*r.^2);
P=Chi*eo*E;
D=er*eo*E;


%   plot data
```

```
subplot(2,1,1)
loglog(rcm,P,'--k',rcm,D,'-k')
legend('P','D')
ylabel('C/m^2')
grid on
subplot(2,1,2)
loglog(rcm,E)
ylabel('V/m')
xlabel('radial
distance (cm)')
grid on
```

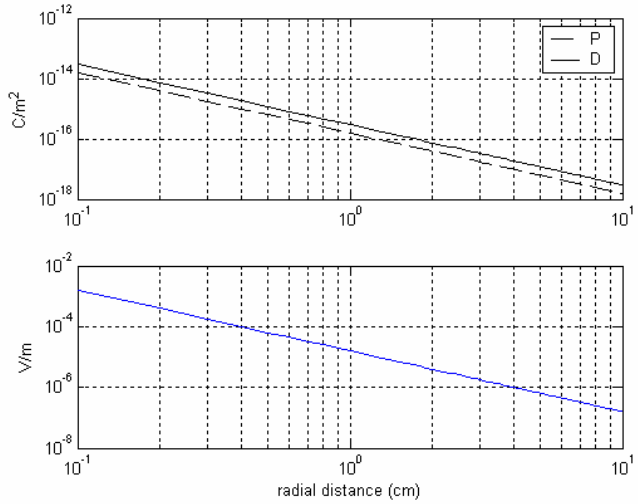


Fig. P2.58

P2.59: Suppose the force is very carefully measured between a pair of point charges separated by a dielectric material and is found to be 20 nN. The dielectric material is removed without changing the position of the point charges, and the force has increased to 100 nN. What is the relative permittivity of the dielectric?

$$F_1 = \frac{Q_1 Q_2}{4\pi\varepsilon_r\varepsilon_o R^2}, \quad F_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_o R^2}, \quad \frac{F_2}{F_1} = \varepsilon_r = \frac{100}{20} = 5$$

P2.60: The potential field in a material with $\varepsilon_r = 10.2$ is $V = 12\,xy^2$ (V). Find **E**, **P** and **D**.

$$\mathbf{E} = -\nabla V = -\frac{\partial\left(12xy^2\right)}{\partial x}\mathbf{a}_x - \frac{\partial\left(12xy^2\right)}{\partial y}\mathbf{a}_y = -12y^2\mathbf{a}_x - 24xy\mathbf{a}_y \ \ \frac{V}{m}$$

$$\mathbf{D} = \varepsilon_r\varepsilon_o\mathbf{E} = -1.1y^2\mathbf{a}_x - 2.2xy\mathbf{a}_y \ \ \frac{nC}{m^2}$$

$$\chi_e = \varepsilon_r - 1 = 9.2$$

$$\mathbf{P} = \chi_e\varepsilon_o\mathbf{E} = \left(9.2\right)\left(8.854x10^{-12}\right)\mathbf{E} = -9.8y^2\mathbf{a}_x - 2.00xy\mathbf{a}_y \ \ \frac{nC}{m^2}$$

P2.61: In a mineral oil dielectric, with breakdown voltage of 15 MV/m, the potential function is $V = x^3 - 6x^2 - 3.1x$ (MV). Is the dielectric likely to breakdown, and if so, where?

$$\mathbf{E} = -\nabla V = \left(-3x^2 + 12x + 3.1\right)\mathbf{a}_x \quad \frac{MV}{m}$$

$\frac{d\mathbf{E}}{dx} = -6x + 12, \quad \frac{d^2\mathbf{E}}{dx^2} = -6,$ so from 6x – 12 = 0 we find the maximum electric field occurs at x = 2m.

At x = 2m, we have **E** = -12+24+3.1 = 15.1 MV/m, exceeding the breakdown voltage.

12. Boundary Conditions

P2.62: For y < 0, $\varepsilon_{r1} = 4.0$ and $\mathbf{E}_1 = 3\mathbf{a}_x + 6\pi\mathbf{a}_y + 4\mathbf{a}_z$ V/m. At y = 0, $\rho_s = 0.25$ nC/m^2. If $\varepsilon_{r2} = 5.0$ for y > 0, find $\mathbf{E}_2$.

$\mathbf{E}_1 = 3\mathbf{a}_x + 6\pi\mathbf{a}_y + 4\mathbf{a}_z$ V/m

(a) $\mathbf{E}_{N1} = 6\pi\mathbf{a}_y$

(b) $\mathbf{E}_{T1} = 3\mathbf{a}_x + 4\mathbf{a}_z$

(d) $\mathbf{D}_{N1} = \varepsilon_{r1}\varepsilon_o\mathbf{E}_{N1} = 24\pi\varepsilon_o\,\mathbf{a}_y$

(g) $\mathbf{E}_2 = 3\mathbf{a}_x + 20.7\mathbf{a}_y + 4\mathbf{a}_z$ V/m

(f) $\mathbf{E}_{N2} = \mathbf{D}_{N2}/5\varepsilon_o = 20.7\mathbf{a}_y$

(c) $\mathbf{E}_{T2} = \mathbf{E}_{T1} = 3\mathbf{a}_x + 4\mathbf{a}_z$

(e) $\mathbf{D}_{N2} = 0.92\,\mathbf{a}_y$

(e) $\mathbf{a}_{21}\bullet\left(\mathbf{D}_1 - \mathbf{D}_2\right) = \rho_s, \quad -\mathbf{a}_y\bullet\left(D_{N1} - D_{N2}\right)\mathbf{a}_y = \rho_s, \quad D_{N2} - D_{N1} = \rho_s$

$$D_{N2} = \rho_s + D_{N1} = 0.25\frac{nC}{m^2} + 24\left(\frac{10^{-9}F}{36\pi m}\right)\pi\frac{nC}{m^2} = 0.92\frac{nC}{m^2}$$

P2.63: For $z \le 0$, $\varepsilon_{r1} = 9.0$ and for z > 0, $\varepsilon_{r2} = 4.0$. If $\mathbf{E}_1$ makes a 30° angle with a normal to the surface, what angle does $\mathbf{E}_2$ make with a normal to the surface?

Refer to Figure P2.63.

$E_{T1} = E_1 \sin\theta_1, \quad E_{T2} = E_2 \sin\theta_2,$ and $E_{T1} = E_{T2}$

also

$D_{N1} = \varepsilon_{r1}\varepsilon_o E_1 \cos\theta_1, \quad D_{N2} = \varepsilon_{r2}\varepsilon_o E_2 \cos\theta_2,$ and $D_{N1} = D_{N2}$ $\left(\text{since } \rho_s = 0\right)$

Therefore

$\frac{E_{T1}}{D_{N1}} = \frac{E_{T2}}{D_{N2}}$, and after routine math we find $\theta_2 = \tan^{-1}\left(\frac{\varepsilon_{r2}}{\varepsilon_{r1}}\tan\theta_1\right)$

Using this formula we obtain for this problem $\theta_2 = 14°$.

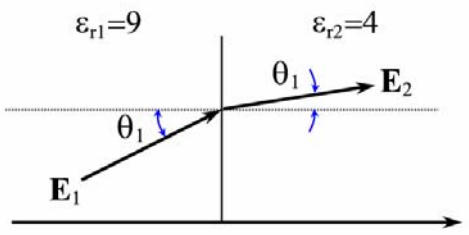


Fig. P2.63

P2.64: A plane defined by $3x + 2y + z = 6$ separates two dielectrics. The first dielectric, on the side of the plane containing the origin, has $\varepsilon_{r1} = 3.0$ and $\mathbf{E}_1 = 4.0\mathbf{a}_z$ V/m. The other dielectric has $\varepsilon_{r2} = 6.0$. Find $\mathbf{E}_2$.

We first use gradient to find a normal to the planar surface.
Let $F = 3x + 2y + z - 6 = 0$.

$$\nabla F = 3\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z, \quad \text{and } |\nabla F| = \sqrt{14},$$

$$\text{so } \mathbf{a}_N = \frac{\nabla F}{|\nabla F|} = 0.802\mathbf{a}_x + 0.534\mathbf{a}_y + 0.267\mathbf{a}_z.$$

Now we can work the boundary condition problem.

$$\mathbf{E}_1 = 4\mathbf{a}_z, \quad \mathbf{E}_{N1} = (\mathbf{E}_1 \bullet \mathbf{a}_N)\mathbf{a}_N = 0.857\mathbf{a}_x + 0.570\mathbf{a}_y + 0.285\mathbf{a}_z.$$

$$\mathbf{E}_{T1} = \mathbf{E}_1 - \mathbf{E}_{N1} = -0.857\mathbf{a}_x - 0.570\mathbf{a}_y + 3.715\mathbf{a}_z, \quad \mathbf{E}_{T2} = \mathbf{E}_{T1}$$

$$\mathbf{D}_{N1} = \varepsilon_{r1}\varepsilon_o\mathbf{E}_{N1} = \varepsilon_o\left[2.571\mathbf{a}_x + 1.710\mathbf{a}_y + 0.855\mathbf{a}_z\right], \quad \text{and } \mathbf{D}_{N2} = \mathbf{D}_{N1}$$

$$\mathbf{E}_{N2} = \frac{\mathbf{D}_{N2}}{\varepsilon_{r2}\varepsilon_o} = \frac{\mathbf{D}_{N2}}{6\varepsilon_o} = 0.429\mathbf{a}_x + 0.285\mathbf{a}_y + 0.143\mathbf{a}_z$$

Finally we have $\mathbf{E}_2 = \mathbf{E}_{T2} + \mathbf{E}_{N2} = -0.43\mathbf{a}_x - 0.29\mathbf{a}_y + 3.8\mathbf{a}_z \ \frac{\text{V}}{\text{m}}$.

P2.65: MATLAB: Consider a dielectric-dielectric charge free boundary at the plane $z = 0$. Construct a program that will allow the user to enter ε_{r1} (for $z < 0$), ε_{r2}, and $\mathbf{E}_1$, and will then calculate $\mathbf{E}_2$. (Just for fun, you may want to have the program calculate the angles that $\mathbf{E}_1$ and $\mathbf{E}_2$ make with a normal to the surface).

```
%   M-File: MLP0265
%
%   Given E1 at boundary between a pair of
%   dielectrics with no charge at boundary,
%   calculate E2.  Also calculates angles.
%
clc
clear
%   enter variables
disp('enter vector quantities in brackets,')
disp('for example: [1 2 3]')
er1=input('relative permittivity in material 1: ');
er2=input('relative permittivity in material 2: ');
a12=input('unit vector from mtrl 1 to mtrl 2: ');
E1=input('electric field intensity vector in mtrl 1: ');

%   perform calculations
```

```
En1=dot(E1,a12)*a12;
Et1=E1-En1;
Et2=Et1;
Dn1=er1*En1;    %ignores eo since it will factor out
Dn2=Dn1;
En2=Dn2/er2;
E2=Et2+En2

%   calculate the angles
th1=atan(magvector(Et1)/magvector(En1));
th2=atan(magvector(Et2)/magvector(En2));
th1r=th1*180/pi
th2r=th2*180/pi
```

Now run the program:

```
enter vector quantities in brackets,
for example: [1 2 3]
relative permittivity in material 1: 2
relative permittivity in material 2: 5
unit vector from mtrl 1 to mtrl 2: [0 0 1]
electric field intensity vector in mtrl 1: [3 4 5]

E2 =

   3    4    2


th1r =

   45


th2r =

  68.1986
```

P2.66: A 1.0 cm diameter conductor is sheathed with a 0.50 cm thickness of Teflon and then a 2.0 cm (inner) diameter outer conductor. (a) Use Laplace's equations to find an expression for the potential as a function of ρ in the dielectric. (b) Find **E** as a function of ρ. (c) What is the maximum potential difference that can be applied across this coaxial cable without breaking down the dielectric?

(a) Since V is only a function of ρ,

$$\nabla^2 V_{cyl} = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = 0$$

so $\rho\dfrac{\partial V}{\partial\rho} = A$

or $V = A\ln\rho + B$

where A and B are constants.

Now we apply boundary conditions.

BC1:

$0 = A\ln b + B, B = -A\ln b,$

$$\therefore V = A\ln\left(\frac{\rho}{b}\right)$$

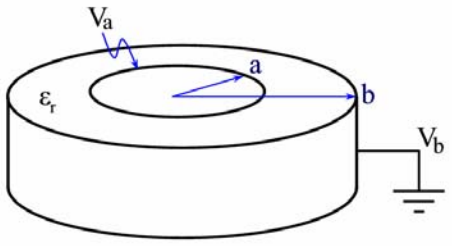


Fig. P2.66

BC2: $V_a = A\ln\left(\dfrac{a}{b}\right), \quad A = \dfrac{V_a}{\ln\left(a/b\right)}, \quad V = V_a\dfrac{\ln\left(\rho/b\right)}{\ln\left(a/b\right)}$

or

$V = -1.443V_a\ln\left(100\rho\right).$

(b) $\mathbf{E} = -\nabla V = -\dfrac{\partial V}{\partial\rho}\mathbf{a}_\rho = \dfrac{1.443V_a}{\rho}\mathbf{a}_\rho$

(c)

$$E_{\max} = \frac{1.443V_a}{.005} = 288.5V_a = E_{br} = 60x10^6,$$

so $V_a = \dfrac{60x10^6}{288.5} = 208kV, \quad \therefore \left(V_{ab}\right)_{\max} = 210kV$

P2.67: A 1.0 m long carbon pipe of inner diameter 3.0 cm and outer diameter 5.0 cm is cut in half lengthwise. Determine the resistance between the inner surface and the outer surface of one of the half sections of pipe.

One approach is to consider the resistance for the half-section of pipe is twice the resistance for a complete cylindrical section, given by Eqn. (2.84). But we'll used the LaPlace equation approach instead.

Laplace: $\nabla^2 V_{cyl} = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial V}{\partial\rho}\right) = 0$; here we see V only depends on ρ

So: $\rho\dfrac{\partial V}{\partial\rho} = A; \quad V = A\ln\rho + B;$

where A and B are constants.

Now apply boundary conditions.

BC1:

$V_b = 0 = A\ln b + B; \quad B = -A\ln b;$

$V = A\ln\left(\frac{\rho}{b}\right)$

BC2:

$V_a = A\ln\left(\frac{a}{b}\right); \quad A = \frac{V_a}{\ln\left(a/b\right)};$

$V = V_a \frac{\ln\left(\rho/b\right)}{\ln\left(a/b\right)}$

$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho = -\frac{V_a}{\ln\left(a/b\right)}\frac{1}{\rho}\mathbf{a}_\rho$

J=σ**E**

$I = \int \mathbf{J}\bullet d\mathbf{S} = -\frac{\sigma V_a}{\ln\left(a/b\right)}\int_0^\pi \frac{1}{\rho}\rho d\phi \int_0^L dz = \frac{\pi L \sigma V_a}{\ln\left(b/a\right)}$

$R = \frac{V_a}{I} = \frac{\ln\left(b/a\right)}{\pi\sigma L} = 5.4\mu\Omega.$

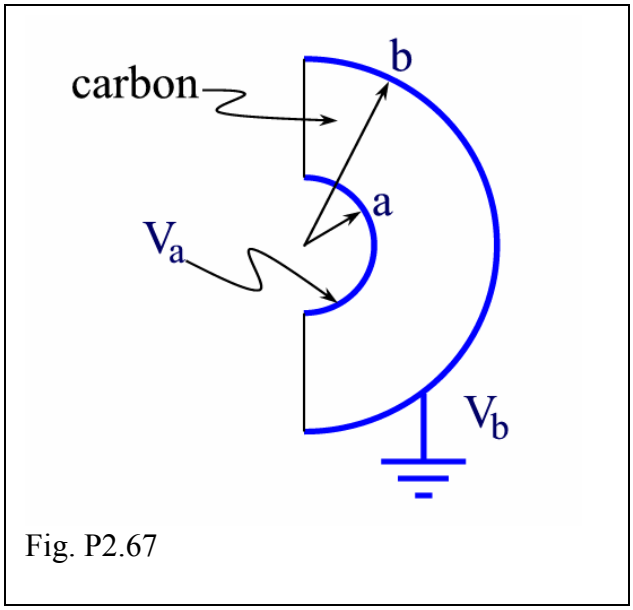


Fig. P2.67

P2.68: For a coaxial cable of inner conductor radius *a* and outer conductor radius *b* and a dielectric ε_r in-between, assume a charge density $\rho_v = \rho_o/\rho$ is added in the dielectric region. Use Poisson's equation to derive an expression for *V* and **E**. Calculate ρ_s on each plate.

$\nabla^2 V = -\frac{\rho_v}{\varepsilon} = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = \frac{-\rho_o}{\varepsilon\rho}$

so

$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) = \frac{\rho_o}{\varepsilon}; \quad \int d\left(\rho\frac{\partial V}{\partial\rho}\right) = \int\frac{\rho_o}{\varepsilon}d\rho; \quad \rho\frac{\partial V}{\partial\rho} = \frac{\rho_o}{\varepsilon}\rho + A$, where *A* is a constant.

$\frac{\partial V}{\partial\rho} = \frac{\rho_o}{\varepsilon} + \frac{A}{\rho}; \quad dV = \int\frac{\rho_o}{\varepsilon}d\rho + \int\frac{A}{\rho}d\rho; \quad V = \frac{\rho_o}{\varepsilon}\rho + A\ln\rho + B$, where *B* is a constant.

Now apply boundary conditions: $V = V_a$ at $\rho = a$ and $V = 0$ at $\rho = b$

Applying the second one gives us:

$V = \frac{\rho_o}{\varepsilon}(\rho - b) + A\ln\left(\rho/b\right).$

Applying the first one:

$$V_a = \frac{\rho_o}{\varepsilon}(a-b) + A\ln\left(a/b\right); \quad A = \frac{V_a + \frac{\rho_o}{\varepsilon}(b-a)}{\ln\left(a/b\right)}$$

Therefore,

$$\boxed{V = \frac{V_a + \frac{\rho_o}{\varepsilon}(b-a)}{\ln\left(a/b\right)}\ln\left(\frac{\rho}{b}\right) + \frac{\rho_o}{\varepsilon}(\rho - b).}$$

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial \rho}\mathbf{a}_\rho = -\frac{\partial}{\partial \rho}\left(K\ln\left(\frac{\rho}{b}\right) + \frac{\rho_o}{\varepsilon}\rho - \frac{\rho_o}{\varepsilon}b\right)\mathbf{a}_\rho$$

$$\boxed{\mathbf{E} = \left(-\frac{K}{\rho} - \frac{\rho_o}{\varepsilon}\right)\mathbf{a}_\rho}$$

where

$$K = \frac{V_a + \frac{\rho_o}{\varepsilon}(b-a)}{\ln\left(a/b\right)},$$

$$\text{so } \mathbf{E} = \left[\frac{-\left(V_a + \frac{\rho_o}{\varepsilon}(b-a)\right)}{\rho\ln\left(\frac{a}{b}\right)} - \frac{\rho_o}{\varepsilon}\right]\mathbf{a}_\rho.$$

$$D_N = \rho_s; \quad D_{Na} = \varepsilon\mathbf{E}\big|_{\rho=a} = \left[\frac{-\left(V_a + \frac{\rho_o}{\varepsilon}(b-a)\right)}{a\ln\left(\frac{a}{b}\right)} - \frac{\rho_o}{\varepsilon}\right] = \rho_{sa}$$

$$D_{Nb} = \varepsilon\mathbf{E}\big|_{\rho=b} = \left[\frac{-\left(V_a + \frac{\rho_o}{\varepsilon}(b-a)\right)}{b\ln\left(\frac{a}{b}\right)} - \frac{\rho_o}{\varepsilon}\right] = \rho_{sb}$$

P2.69: For the parallel plate capacitor given in Figure 2.51, suppose a charge density

$$\rho_v = \rho_o \sin\left(\frac{\pi z}{2d}\right)$$

is added between the plates. Use Poisson's equation to derive a new expression for V and $\mathbf{E}$. Calculate ρ_s on each plate.

$$\frac{\partial^2 V(z)}{\partial z^2} = \frac{-\rho_v}{\varepsilon} = \frac{-\rho_o \sin\left(\pi z/2d\right)}{\varepsilon}$$

$$\frac{\partial V(z)}{\partial z} = \frac{-\rho_o}{\varepsilon}\int \sin\left(\pi z/2d\right)dz = \frac{2\rho_o d}{\pi\varepsilon}\cos\left(\pi z/2d\right) + A$$

$$V(z) = \frac{2\rho_o d}{\pi\varepsilon}\int \cos\left(\pi z/2d\right)dz + A\int dz = \frac{2\rho_o d^2}{\pi^2\varepsilon}\sin\left(\pi z/2d\right) + Az + B$$

Now apply the boundary conditions:

$$V_a = 0 = B; \quad V_d = \frac{2\rho_o d^2}{\pi^2\varepsilon}\sin\left(\pi d/2d\right) + Ad; \quad A = \frac{V_d - \dfrac{2\rho_o d^2}{\pi^2\varepsilon}}{d}$$

$$\boxed{V(z) = \frac{2\rho_o d^2}{\pi^2\varepsilon}\sin\left(\pi z/2d\right) + \left(\frac{V_d}{d} - \frac{2\rho_o d}{\pi^2\varepsilon}\right)z}$$

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial z}\mathbf{a}_z = \left[-\frac{\partial}{\partial z}\left(\frac{2\rho_o d^2}{\pi^2\varepsilon}\sin\left(\pi z/2d\right)\right) + -\frac{\partial}{\partial z}\left(\left(\frac{V_d}{d} - \frac{2\rho_o d}{\pi^2\varepsilon}\right)z\right)\right]\mathbf{a}_z$$

$$\boxed{\mathbf{E} = \left(-\frac{\rho_o d}{\pi\varepsilon}\cos\left(\pi z/2d\right) - \frac{V_d}{d} + \frac{2\rho_o d}{\pi^2\varepsilon}\right)\mathbf{a}_z}$$

at z = 0, $D_N = \varepsilon E$, so $\rho_s\big|_{z=0} = -\dfrac{\rho_o d}{\pi} - \dfrac{\varepsilon V_d}{d} + \dfrac{2\rho_o d}{\pi^2} = \rho_s$.

at z = d, $D_N = \varepsilon E$, so $\rho_s\big|_{z=d} = -\dfrac{\varepsilon V_d}{d} + \dfrac{2\rho_o d}{\pi^2} = \rho_s$.

13. Capacitors

P2.70: A parallel plate capacitor is constructed such that the dielectric can be easily removed. With the dielectric in place, the capacitance is 48 nF. With the dielectric removed, the capacitance drops to 12 nF. Determine the relative permittivity of the dielectric.

$$C_1 = \frac{\varepsilon_r\varepsilon_o A}{d}; \quad C_2 = \frac{\varepsilon_o A}{d}; \quad \frac{C_1}{C_2} = \varepsilon_r = \frac{48}{12} = 4.0$$

P2.71: A parallel plate capacitor with a 1.0 m^2 surface area for each plate, a 2.0 mm plate separation, and a dielectric with relative permittivity of 1200 has a 12. V potential difference across the plates. (a) What is the minimum allowed dielectric strength for this capacitor? Calculate (b) the capacitance, and (c) the magnitude of the charge density on one of the plates.

(a) $E = \dfrac{12V}{0.002m} = 6\dfrac{kV}{m}; \quad (a)E_{br\min} = 6\dfrac{kV}{m}$

(b) $C = \dfrac{\varepsilon_r \varepsilon_o A}{d} = \dfrac{(1200)\left(8.854x10^{-12}\,F/m\right)\left(1m^2\right)}{0.002m} = 5.3\mu F$

(c) $C = \dfrac{Q}{V};\quad Q = CV = \left(5.3x10^{-6}\,F\right)(12V)\dfrac{C}{FV} = 64\mu C$

P2.72: A conical section of material extends from 2.0 cm ≤ r ≤ 9.0 cm for $0 \le \theta \le 30°$ with ε_r = 9.0 and σ = 0.020 S/m. Conductive plates are placed at each radial end of the section. Determine the resistance and capacitance of the section.

$\nabla^2 V = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial V}{\partial r}\right) = 0;\quad r^2 \dfrac{\partial V}{\partial r} = A;\quad V = -\dfrac{A}{r} + B$, where A and B are constants.

Boundary conditions: $r = a$, $V = 0$ and $r = b$, $V = V_b$

$$V = V_b \frac{\left(\frac{1}{a} - \frac{1}{r}\right)}{\left(\frac{1}{a} - \frac{1}{b}\right)}$$

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\mathbf{a}_r$$

$$\mathbf{E} = \frac{-V_b \mathbf{a}_r}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2},\quad \mathbf{D} = \frac{-\varepsilon_r \varepsilon_o V_b \mathbf{a}_r}{\left(\frac{1}{a} - \frac{1}{b}\right)r^2}$$

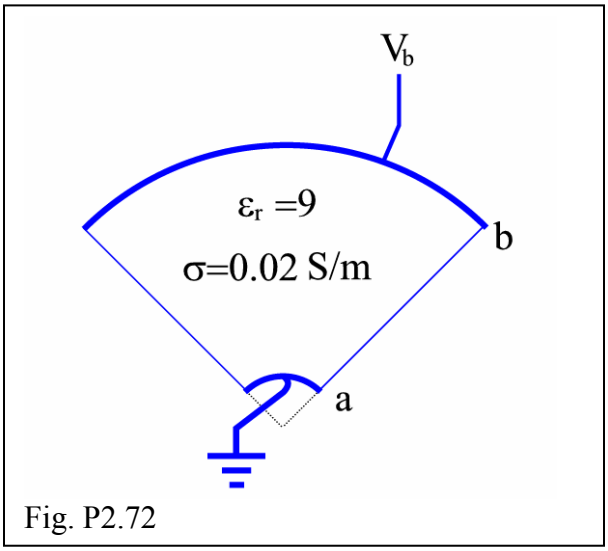


Fig. P2.72

$$\rho_{sb} = \frac{-\varepsilon_r \varepsilon_o V_b}{\left(\frac{1}{a} - \frac{1}{b}\right)b^2};$$

$Q = \int \rho_s dS;\quad dS = r^2 \sin\theta d\theta d\phi$

$$Q_b = \frac{\varepsilon_r \varepsilon_o V_b}{\left(\frac{1}{a}-\frac{1}{b}\right)b^2} b^2 \int_0^{30^o} \sin\theta d\theta \int_0^{2\pi} d\phi$$

$$= 1.73x10^{-12} V_b.$$

$$C = \frac{Q_b}{V_b} = 1.7\,pF; \quad RC = \frac{\varepsilon}{\sigma}; \quad R = \frac{1}{C}\frac{\varepsilon}{\sigma} = 2.3k\Omega$$

P2.73: An inhomogeneous dielectric fills a parallel plate capacitor of surface area 50. cm^2 and thickness 1.0 cm. You are given $\varepsilon_r = 3(1 + z)$, where z is measured from the bottom plate in cm. Determine the capacitance.

Place $+Q$ at $z = d$ and $-Q$ at $z = 0$.

$$\rho_s = \frac{Q}{S}, \quad \mathbf{D} = -\frac{Q}{S}\mathbf{a}_z, \quad \mathbf{E} = -\frac{Q}{\varepsilon_r \varepsilon_o S}\mathbf{a}_z$$

$$V_{do} = -\int_0^d \mathbf{E}\bullet d\mathbf{L} = -\int_0^d \frac{-Q}{\varepsilon_r \varepsilon_o S}\mathbf{a}_z \bullet dz\mathbf{a}_z = \frac{Q}{\varepsilon_o S}\int_0^d \frac{dz}{\varepsilon_r}$$

evaluating the integral:

$$\int_0^d \frac{dz}{\varepsilon_r} = \int_0^d \frac{dz}{3(1+z)} = \frac{1}{3}\ln(1+z)\Big|_0^1 = \frac{1}{3}\ln 2\ cm$$

$$C = \frac{Q}{V_{do}} = \frac{3\varepsilon_o S}{\ln 2} = \frac{3\left(8.854x10^{-12}\ F/m\right)\left(50cm^2\right)}{\left(\ln(2)cm\right)}\left(\frac{m}{100cm}\right)^2 = 19\,pF$$

P2.74: Given $\mathbf{E} = 5xy\mathbf{a}_x + 3z\mathbf{a}_z$ V/m, find the electrostatic potential energy stored in a volume defined by $0 \le x \le 2$ m, $0 \le y \le 1$ m, and $0 \le z \le 1$ m. Assume $\varepsilon = \varepsilon_o$.

$$W_E = \frac{1}{2}\varepsilon_o \int \mathbf{E}\bullet\mathbf{E}dv = \frac{1}{2}\varepsilon_o\left[\int 25x^2y^2dxdydz + \int 9z^2dxdydz\right]$$

$$W_E = \frac{1}{2}\varepsilon_o\left[25\int_0^2 x^2dx\int_0^1 y^2dy\int_0^1 dz + 9\int_0^2 dx\int_0^1 dy\int_0^1 z^2dz\right] = 125\,pJ$$

P2.75: Suppose a coaxial capacitor with inner radius 1.0 cm, outer radius 2.0 cm and length 1.0 m is constructed with 2 different dielectrics. When oriented along the z-axis, ε_r for $0° \le \phi \le 180°$ is 9.0, and for $180° \le \phi \le 360°$ is 4.0. (a) Calculate the capacitance. (b) If 9.0 V is applied across the conductors, determine the electrostatic potential energy stored in each dielectric for this capacitor.

(a) a coaxial line,

$$C = \frac{2\pi L\varepsilon_r\varepsilon_o}{\ln\left(b/a\right)}$$

But for only half the line,

$$C = \frac{\pi L\varepsilon_r\varepsilon_o}{\ln\left(b/a\right)}$$

So

$$C_1 = \frac{\pi L\varepsilon_{r1}\varepsilon_o}{\ln\left(b/a\right)} = 361pF$$

and

$$C_2 = \frac{\pi L\varepsilon_{r2}\varepsilon_o}{\ln\left(b/a\right)} = 161pF$$

So

$$C_{TOT} = C_1 + C_2 = 522\,pF$$

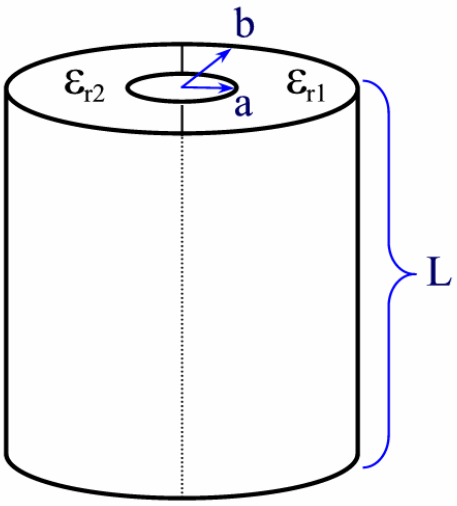


Fig. P2.75

(b) $W_{E1} = \frac{1}{2}C_1V^2 = 14.6nJ; \quad W_{E2} = \frac{1}{2}C_2V^2 = 6.5nJ$