CHAPTER 2: Analytic Functions

EXERCISES 2.1: Functions of a Complex Variable

1. a. $w = (3x^2 - 3y^2 + 5x + 1) + i(6xy + 5y + 1)$

   b. $w = \dfrac{x}{x^2+y^2} + i\left(-\dfrac{y}{x^2+y^2}\right)$

   c. $w = \dfrac{1}{z-i} = \dfrac{x}{x^2+(y-1)^2} + i\dfrac{-y+1}{x^2+(y-1)^2}$

   d. $w = \dfrac{2x^2-2y^2+3}{\sqrt{(x-1)^2+y^2}} + i\dfrac{4xy}{\sqrt{(x-1)^2+y^2}}$

   e. $w = e^{3x}\cos 3y + ie^{3x}\sin 3y$

   f. $w = (e^x + e^{-x})\cos y + i(e^x - e^{-x})\sin y$
   $= 2\cosh x \cos y + i2\sinh x \sin y$

2. a. $\mathbf{C}$

   b. $\mathbf{C} \setminus \{0\}$

   c. $\mathbf{C} \setminus \{i, -i\}$

   d. $\mathbf{C} \setminus \{1\}$

   e. $\mathbf{C}$

   f. $\mathbf{C}$

3. a. $\operatorname{Re} w > 5$

   b. $\operatorname{Im} w \geq 0$

   c. $|w| \geq 1$

   d. The intersection of $|w| < 2$ and $-\pi < \operatorname{Arg} w < \pi/2$

4. a. Taking θ from 0 to 2π, the points $z = re^{i\theta}$ traverse the circle $|z| = r$ exactly once in the counterclockwise direction. For the same values of θ the points $w = \dfrac{1}{re^{i\theta}} = \dfrac{1}{r}e^{-i\theta}$ traverse the circle $|w| = \dfrac{1}{r}$ exactly once in the clockwise direction, hence the mapping is onto.

   b. For $z = re^{i\theta_0}$ on the ray $\operatorname{Arg} z = \theta_0$, $w = \dfrac{1}{re^{i\theta_0}} = \dfrac{1}{r}e^{-i\theta_0}$ is on the ray $\operatorname{Arg} w = -\theta_0$. Taking values $0 < r < \infty$ shows that this mapping goes onto the ray $\operatorname{Arg} w = -\theta_0$.

4 (c) $|z-1| = 1$ $2\pi > \theta \geq 0 \Rightarrow z = 1 + e^{i\theta}$. $F(z) = 1/z. = 1/(1+e^{i\theta})$
$= (1+e^{-i\theta})/\{2(1+\cos\theta)\} = \frac{1}{2} - i(\frac{1}{2})\sin\theta/(1+\cos\theta)$
which is a vertical line at $x = \frac{1}{2}$.

5. a. domain: $\mathbf{C}$

   range: $\mathbf{C} \setminus \{0\}$

   b. $f(-z) = e^{-z} = \dfrac{1}{e^z} = \dfrac{1}{f(z)}$

   c. circle $|w| = e$

   d. ray $\operatorname{Arg} w = \pi/4$

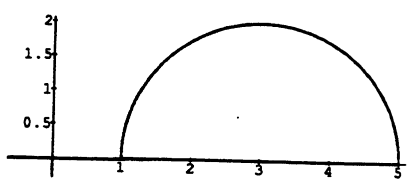
   e. infinite sector $0 \leq \operatorname{Arg} w \leq \pi/4$

6. a. $J\left(\dfrac{1}{z}\right) = \dfrac{1}{2}\left(\dfrac{1}{z} + \dfrac{1}{1/z}\right) = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right) = J(z)$

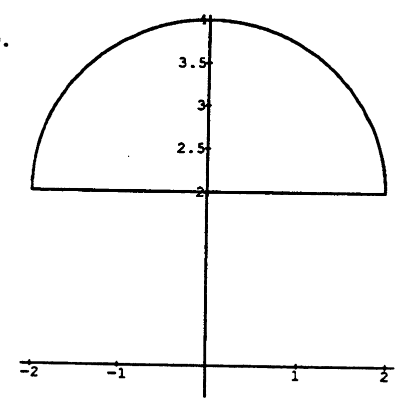
   b. For $z = e^{i\theta}$ on the unit circle $|z| = 1$, $J(z) = \dfrac{1}{2}\left(e^{i\theta} + \dfrac{1}{e^{i\theta}}\right) = \cos\theta$. For all values of θ, this ranges over the real interval $[-1, 1]$.

   c. For $z = re^{i\theta}$ on the circle $|z| = r$, $J(z) = \dfrac{1}{2}\left(re^{i\theta} + \dfrac{1}{re^{i\theta}}\right) = \dfrac{1}{2}\left(r + \dfrac{1}{r}\right)\cos\theta + i\dfrac{1}{2}\left(r - \dfrac{1}{r}\right)\sin\theta$. Setting u and v equal to the real and imaginary parts of this expression, respectively, one gets a pair of parametric equations that are equivalent to the ellipse $\dfrac{u^2}{[\frac{1}{2}(r+\frac{1}{r})]^2} + \dfrac{v^2}{[\frac{1}{2}(r-\frac{1}{r})]^2} = 1$, which has foci at ± 1.

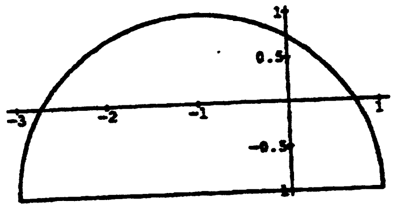
7. a.



b.



c.


1
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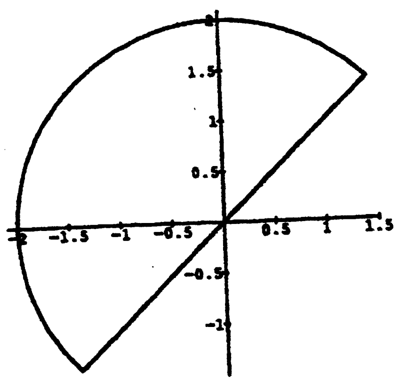
-3
-2
-1
1
-0.5

8. a.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1.5
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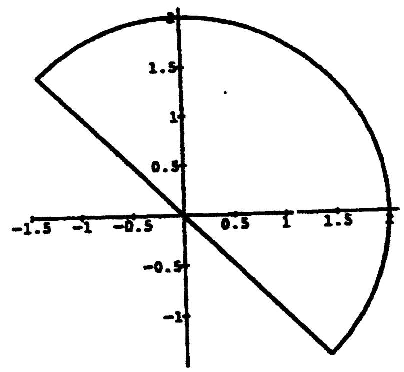
-1

b.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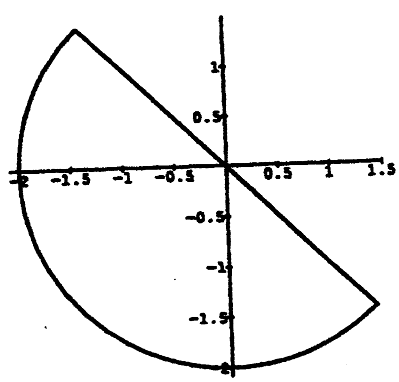
-1

c.


1
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-1.5
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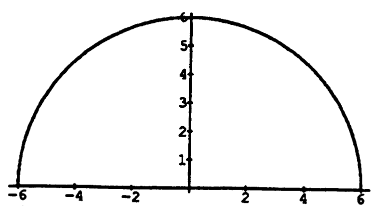
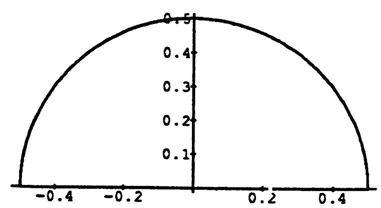
0.5
1
1.5
-0.5
-1
-1.5

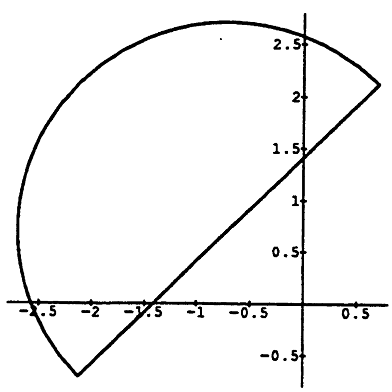
9. a.



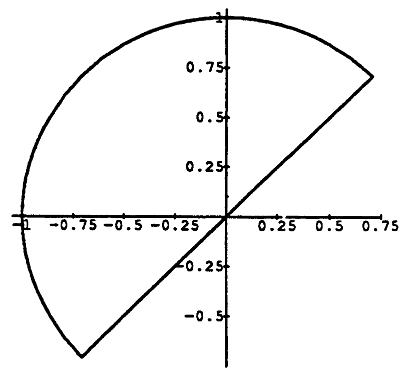
b.



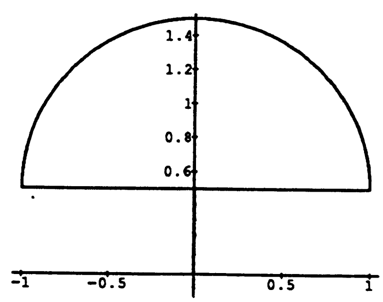
10. a. translate by i, rotate $\pi/4$



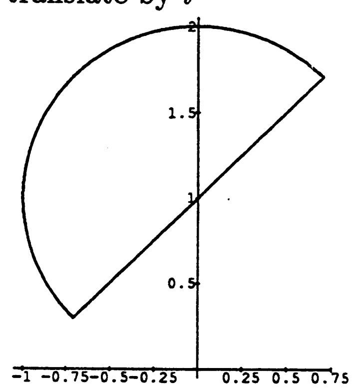
b. reduce by 1/2, rotate $\pi/4$



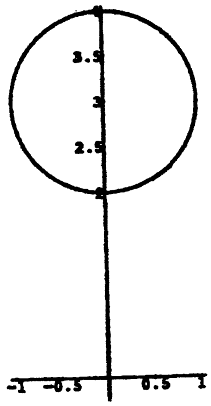
c. translate by i, reduce by 1/2



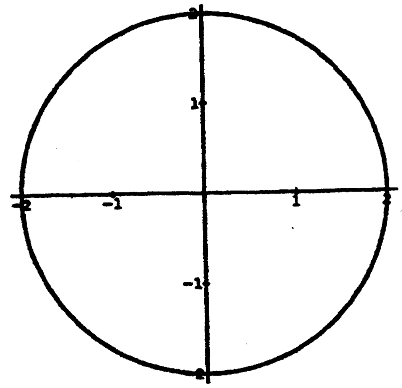
d. reduce by 1/2, rotate $\pi/4$, translate by i



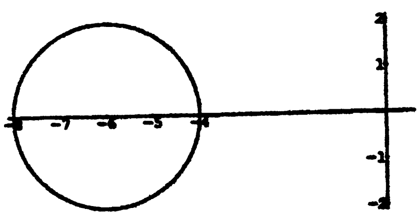
11. **a. translate by −3, rotate $-\pi/2$**



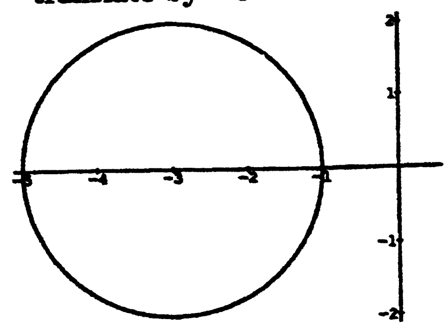
b. magnify by 2, rotate $-\pi/2$



c. translate by −3, magnify by 2



d. magnify by 2, rotate $-\pi/2$, translate by −3



12. Let $a = \rho e^{i\phi}$, $F(z) = \rho z$, $G(z) = e^{i\phi} z$, and $H(z) = z + b$. Then $H(G(F(z))) = az + b$.

13. (a) $w = u + iv = z^2 = (1 + iy)^2 = 1 - y^2 + i2y$
$u = 1 - y^2$, $v = 2y \Rightarrow y = v/2 \Rightarrow u = 1 - v^2/4$ a parabola in the w-plane.
(b) $w = u + iv = z^2 = (x + iy)^2 = (x + i/x)^2 = x^2 - 1/x^2 + 2i$
$u = x^2 - 1/x^2$, $v = 2$ a straight line in the w-plane.
(c) $w = u + iv = z^2 = (1 + e^{i\theta})^2 = (1 + 2e^{i\theta} + e^{i2\theta}) = (e^{-i\theta} + 2 + e^{i\theta})e^{i\theta}$
$= (2 + 2\cos\theta)e^{i\theta} = 2(1 + \cos\theta)e^{i\theta}$ a cardioid in the w-plane.

14. (a) $x_1 = 2x/(|z|^2 + 1)$, $x_2 = 2y/(|z|^2 + 1)$, $x_3 = (|z|^2 - 1)/(|z|^2 + 1)$
$w = e^{i\varphi}z = x\cos\varphi - y\sin\varphi + i(x\sin\varphi + y\cos\varphi)$, $|w| = |z|$
$\underline{x}_1 = (x\cos\varphi - y\sin\varphi)/(|z|^2 + 1)$, $\underline{x}_2 = (x\sin\varphi + y\cos\varphi)/(|z|^2 + 1)$, $\underline{x}_3 = x_3$
$\underline{x}_1 = (x_1\cos\varphi - x_2\sin\varphi)$, $\underline{x}_2 = (x_1\sin\varphi + x_2\cos\varphi)$, $\underline{x}_3 = x_3$ which corresponds to a rotation of an angle φ about the x_3 axis.
(b) $w = -1/z$. $|w| = 1/|z|$. $w = -1/(x+iy) = -x/|z| + iy/|z|$
$\underline{x}_1 = -x_1$, $\underline{x}_2 = x_2$, $\underline{x}_3 = -x_3$ so that $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ is obtained from (x_1, x_2, x_3) by a 180° rotation about the x_2 axis.

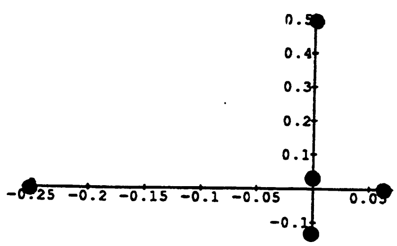
15. $w = (1+z)/(1 - z) = (1 + x + iy)/(1 - x - iy) = (1 - |z|^2 + i2y)/(1 - 2x + |z|^2)$
$|w|^2 = (1 + 2x + |z|^2)/(1 - 2x + |z|^2)$.
$(\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-x_3, x_2, x_1)$ so that $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ is obtained by a 90° counterclockwise rotation about the x_2 axis.

16. $w = (1 - iz)/(1 + iz) = (1 - ix + y)/(1 + ix - y) = (1 - |z|^2 + i2x)/(1 - 2y + |z|^2)$
$|w|^2 = (1 + 2y + |z|^2)/(1 - 2y + |z|^2)$.
$(\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-x_3, -x_1, x_2)$ so that $(\underline{x}_1, \underline{x}_2, \underline{x}_3)$ is obtained as a 90° counterclockwise rotation about the x_2 axis followed by a 90° counterclockwise rotation about the $x_3$axis.

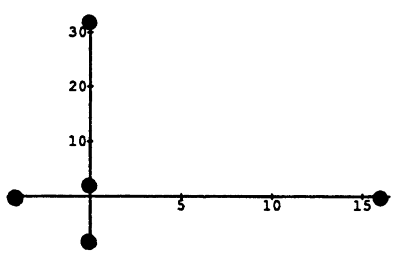
17. Any circle or line in the z-plane corresponds to a line or circle on the stenographic projection onto he Riemann sphere. The function $w=1/z$ rotates the Riemann sphere 180° about the x_1 axis. Lines and circles on the rotated sphere project to lines and circles in the w-plane. As a result lines and circles in the z-plane map to lines and circles in the w-plane.

EXERCISES 2.2: Limits and Continuity

1. The first five terms are, respectively, $\frac{i}{2}, -\frac{1}{4}, -\frac{i}{8}, \frac{1}{16}$, and $\frac{i}{32}$. The sequence converges to 0 in a spiral-like fashion.



2. $2i, -4, -8i, 16, 32i$; divergent because terms grow in modulus without bound.



3. If $\lim_{n\to\infty} z_n = z_0$, then for any $\varepsilon>0$, there is an integer N such that $|z_n - z_0|<\varepsilon$ for all $n>N$. For the same integer N we have $|x_n - x_0|<=|z_n - z_0|<\varepsilon$ and $|y_n - y_0|<=|z_n - z_0|<\varepsilon$ for all $n>N$. Therefore, $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$.

If $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} y_n = y_0$, then for any $\varepsilon_1 >0$ and $\varepsilon_2 >0$ there are integers N_1 and N_2 such
$|x_n - x_0|<\varepsilon_1$ for all $n > N_1$ and $|y_n - y_0|<\varepsilon_2$ for all $n > N_2$. Given any $\varepsilon > 0$; let $\varepsilon_1 = \varepsilon/2$ and $\varepsilon_2 = \varepsilon/2$. Then
$|z_n - z_0|<=\varepsilon\ |x_n - x_0| + |y_n - y_0| < \varepsilon_1 + \varepsilon_2 = \varepsilon$ for all $n >$ maximum(N_1, N_2).
Thus $\lim_{n\to\infty} z_n = z_0$.

4. If $z_n = x_n + iy_n \to z_0 = x_0 + iy_0$, then $x_n \to x_0$ and $y_n \to y_0$ (see Problem 3).
$\underline{z}_n = x_n - iy_n \to x_0 - iy_0 = \underline{z}_0$.

If $\underline{z}_n = x_n - iy_n \to \underline{z}_0 = x_0 - iy_0$, then $x_n \to x_0$ and $y_n \to y_0$ (see Problem 3)..
$z_n = x_n + iy_n \to x_0 + iy_0 = z_0$. Thus $z_n \to z_0$ if and only if $\underline{z}_n \to \underline{z}_0$.

5. **$\lim_{n\to\infty} |z_n| = 0 \Longrightarrow$ There exists an integer N such that $||z_n| - 0| = |z_n| < \varepsilon$ whenever $n > N$. $\Longrightarrow |z_n - 0| < \varepsilon$ whenever $n > N$. $\Longrightarrow \lim_{n\to\infty} z_n = 0$, and conversely.**

6. **$z_0^n \to 0$ as $n \to \infty$ by problem 3, since the real-valued sequence $|z_0^n| \to 0$ as $n \to \infty$. On the other hand, if $|z_0| > 1$, then $|z_0^n| \to \infty$ as $n \to \infty$ so z_0^n diverges.**

7. **a. converges to 0**

   b. does not converge

   c. converges to π

   d. converges to $2+i$

   e. converges to 0

   f. does not converge

8. **Given $\varepsilon > 0$, choose $\delta = \varepsilon/6$. Then whenever $0 < |z-(1+i)| < \delta$,**

$$|6z - 4 - (2+6i)| = 6|z-(1+i)| < 6(\varepsilon/6) = \varepsilon$$

9. **Given $\varepsilon > 0$, choose $\delta = \dfrac{\varepsilon}{1+\varepsilon}$. Whenever $0 < |z-(-i)| < \delta$ notice that $|z| > 1-\delta$ and**

$$\left|\frac{1}{z} - i\right| = \left|\left(-\frac{i}{z}\right)(i+z)\right| = \frac{1}{|z|}|z-(-i)| < \left(\frac{1}{1-\delta}\right)\delta = \varepsilon$$

10. Given that f and g are continuous at z_0,

$$\lim_{z\to z_0} f(z) \pm g(z) = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z) = f(z_0) \pm g(z_0)$$

$\Longrightarrow f(z) \pm g(z)$ is continuous at z_0.

$$\lim_{z\to z_0} f(z)g(z) = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z) = f(z_0)g(z_0)$$

$\Longrightarrow f(z)g(z)$ is continuous at z_0.

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)} = \frac{f(z_0)}{g(z_0)}, \text{ provided } g(z_0) \neq 0$$

$\Longrightarrow \dfrac{f(z)}{g(z)}$ is continuous at z_0.

11. a. $-8i$
    b. $-\frac{7}{2}i$
    c. $6i$
    d. $-1/2$
    e. $2z_0$
    f. $4\sqrt{2}$

12. Clearly Arg z is discontinuous at $z = 0$. Let $a > 0$ be any real number and consider the sequence

$$z_n = -a - i/n \quad n = 1, 2, \ldots, \text{ which converges to } -a.$$

For each n, $-\pi < \operatorname{Arg} z_n < -\pi/2$, but $\operatorname{Arg}(-a) = \pi$.

13. $\lim_{z\to z_0} f(z)$ exists for all $z \neq -1$; f is continuous for all $z \neq 0, -1$; f has a removable discontinuity at $z = 0$.

14. Let z_0 be any complex number. Given $\varepsilon > 0$ choose $\delta = \varepsilon$. Then whenever $|z - z_0| < \delta$,

$$|g(z) - g(z_0)| = |\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0| < \varepsilon.$$

15. Given $\varepsilon > 0$ choose δ so that $|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$. Then, whenever $|z - z_0| < \delta$:

a. $|\overline{f(z)} - \overline{f(z_0)}| = |\overline{f(z) - f(z_0)}| = |f(z) - f(z_0)| < \varepsilon$

b. $|\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| = |\operatorname{Re}(f(z) - f(z_0))| \leq |f(z) - f(z_0)| < \varepsilon$

c. $|\operatorname{Im} f(z) - \operatorname{Im} f(z_0)| = |\operatorname{Im}(f(z) - f(z_0))| \leq |f(z) - f(z_0)| < \varepsilon$

d. $||f(z)| - |f(z_0)|| \leq |f(z) - f(z_0)| < \varepsilon$

16. **Given $\varepsilon > 0$, choose $\delta_0 > 0$ such that $|f(g(z)) - f(g(z_0))| < \varepsilon$ whenever $|g(z) - g(z_0)| < \delta_0$. Now choose $\delta > 0$ such that $|g(z) - g(z_0)| < \delta_0$ whenever $|z - z_0| < \delta$. Then $|f(g(z)) - f(g(z_0))| < \varepsilon$ whenever $|z - z_0| < \delta$; hence $f(g(z))$ is continuous at z_0.**

17. **No: Observe that although $\frac{1}{n} \to 0$ and $\frac{i}{n} \to 0$ as $n \to \infty$, $f\left(\frac{1}{n}\right) \to 1 + 2i$ and $f\left(\frac{i}{n}\right) \to 2i$; thus $\lim_{z\to 0} f(z)$ does not exist.**

18. If $\lim_{z\to z_0} f(z) = w_0$, then given $\varepsilon>0$ there exists $\delta>0$ such that $|f(z)-w_0|<\varepsilon$ for all $|z-z_0|<\delta$. Notice that
$|\underline{f(z)-w_0}| = |\underline{f(z)}-\underline{w_0}| = |f(z)-w_0|<\varepsilon$ for all $|z-z_0|<\delta$. So that $\lim_{z\to z_0}\underline{f(z)} = \underline{w_0}$.
$\lim_{x\to x_0, y\to y_0} \mu(x,y) = \lim_{z\to z_0} ((f(z)+\underline{f(z)})/2) = (w_0+\underline{w_0})/2 = \mu_0$.
$\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \lim_{z\to z_0} ((f(z)-\underline{f(z)})/2i) = (w_0-\underline{w_0})/2i = \upsilon_0$.
Thus, $\lim_{x\to x_0, y\to y_0} \mu(x,y) = \mu_0$ and $\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \upsilon_0$.

Conversely, if $\lim_{x\to x_0, y\to y_0} \mu(x,y) = \mu_0$ and $\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \upsilon_0$, then (by Theorem 1.) $\mu_0 + i\upsilon_0 = \lim_{x\to x_0, y\to y_0} \mu(x,y) + i\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \lim_{z\to z_0} ((f(z)+\underline{f(z)})/2) + \lim_{z\to z_0} ((f(z)-\underline{f(z)})/2) = \lim_{z\to z_0} f(z) = w_0$.
Also $\mu_0 - i\upsilon_0 = \lim_{x\to x_0, y\to y_0} \mu(x,y) - i\lim_{x\to x_0, y\to y_0} \upsilon(x,y) = \lim_{z\to z_0} ((f(z)+\underline{f(z)})/2) - \lim_{z\to z_0} ((f(z)-\underline{f(z)})/2) = \lim_{z\to z_0} f(z) = \underline{w_0}$.

Thus, $\lim_{z\to z_0} f(z) = w_0$.

19. **$-\frac{1}{2} - i$, since $\lim_{\substack{x\to 1\\ y\to -1}} \frac{x}{x^2+3y} = -\frac{1}{2}$ and $\lim_{\substack{x\to 1\\ y\to -1}} xy = -1$.**

20. **For any z_0 in the complex plane,**

$$\lim_{z\to z_0} e^z = \lim_{\substack{x\to x_0\\ y\to y_0}} e^x \cos y + i \lim_{\substack{x\to x_0\\ y\to y_0}} e^x \sin y = e^{x_0}\cos y_0 + ie^{x_0}\sin y_0 = e^{z_0}.$$

21.
a. 1
b. 0
c. $-\pi/2 + i$
d. 1

22. By contradiction: Suppose $\lim_{z\to z_0} f(z) \neq w_0$. Then there is an $\varepsilon > 0$ for which there exists a sequence $\{z_n\}$ such that $|z_n - z_0| < \frac{1}{n}$ but $|f(z_n) - w_0| > \varepsilon$. For this sequence, $\lim_{n\to\infty} z_n = z_0$ but $\lim_{n\to\infty} f(z_n) \neq w_0$, contrary to hypothesis.

23. If $z_n \to \infty$, then for any M>0 there exist an integer N such $|z_n| > M$ for al $n > N$. Consider the chordal distance $\chi(z_n,\infty) = 2/\sqrt{(|z_n|^2 + 1)} < 2/\sqrt{(|z_n|^2)}$ $= 2/|z_n| < 2/M < \varepsilon$ for all $n > N$. Thus $z_n \to \infty$ as $n \to \infty$ is equivalent to $\chi(z_n,\infty) \to 0$ as $n \to \infty$.

24. If $\lim_{z\to z_0} f(z) = \infty$, then for any M>0 there exis $\delta > 0$ such that $|f(z)| > M$ for all $|z-z_0| < \delta$. Consider $\chi(f(z),\infty) = 2/\sqrt{(|f(z)|^2 + 1)} < 2/\sqrt{(|f(z)|^2)} = 2/|f(z)| <$ $2/M < \varepsilon$ for all $|z-z_0| < \delta$. Thus $\lim_{z\to z_0} f(z) = \infty$, is equivalent to $\lim_{z\to\infty} \chi(f(z),\infty) = 0$.

25. (a) ∞ (b) 3 (c) ∞ (d) ∞ (f) the limit does not exist.

EXERCISES 2.3: Analyticity

1. Let $\Delta z = z - z_0$ so that $\Delta z \to 0 \iff z \to z_0$. Then

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = L \iff$$

given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\left|\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} - L\right| < \varepsilon \text{ whenever} |\Delta z - 0| < \delta \iff$$

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - L\right| < \varepsilon \text{whenever} |z - z_0| < \delta \iff$$

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = L.$$

2. If $\lambda(z) = \dfrac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$, then $\lambda(z) \to 0$ as $z \to z_0$ and $f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0) = f(z)$.

3. $\lim_{z \to z_0} f(z) = \lim_{z \to z_0} [f(z_0) + f'(z_0)(z - z_0) + \lambda(z)(z - z_0)]$ $= f(z_0) + 0 + 0 = f(z_0)$.

4. a. $\lim_{\Delta z \to 0} \dfrac{\text{Re}\,(z+\Delta z) - \text{Re}\,(z)}{\Delta z} = \lim_{\Delta z \to 0} \dfrac{\text{Re}\,(\Delta z)}{\Delta z} = \begin{cases} 1, & \text{if } \Delta z = \Delta x \\ 0, & \text{if } \Delta z = i\Delta y \end{cases}$

b. $\lim_{\Delta z \to 0} \dfrac{\text{Im}\,(z+\Delta z) - \text{Im}(z)}{\Delta z} = \lim_{\Delta z \to 0} \dfrac{\text{Im}\,(\Delta z)}{\Delta z} = \begin{cases} 0, & \text{if } \Delta z = \Delta x \\ -i, & \text{if } \Delta z = i\Delta y \end{cases}$

c. Case 1, $z = 0$.

$$\lim_{\Delta z \to 0} \frac{|0+\Delta z| - |0|}{\Delta z} = \lim_{\Delta z \to 0} \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta x + i\Delta y} = \begin{cases} \pm 1, & \text{if } \Delta z = \Delta x \\ -i, & \text{if } \Delta z = \pm i\Delta y \end{cases}$$

Case 2, $z \neq 0$.

$$\begin{aligned}
&\lim_{\Delta z \to 0} \frac{|z+\Delta z| - |z|}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} - \sqrt{x^2+y^2}}{\Delta x + i\Delta y} \\
&= \lim_{\Delta z \to 0} \frac{(x+\Delta x)^2 + (y+\Delta y)^2 - (x^2+y^2)}{(\Delta x + i\Delta y)(\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} + \sqrt{x^2+y^2})} \\
&= \lim_{\Delta z \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2y\Delta y + (\Delta y)^2}{(\Delta x + i\Delta y)(\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} + \sqrt{x^2+y^2})} \\
&= \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}}, & \text{if } \Delta z = \Delta x,\ z \neq 0 \\ \dfrac{y}{i\sqrt{x^2+y^2}}, & \text{if } \Delta z = i\Delta y,\ z \neq 0 \end{cases}
\end{aligned}$$

5. Rule 5: $(f \pm g)'(z_0) = \lim_{\Delta z \to 0} \dfrac{(f \pm g)(z_0 + \Delta z) - (f \pm g)(z_0)}{\Delta z}$

$$\begin{aligned}
&= \lim_{\Delta z \to 0} \left[\frac{f(z_0+\Delta z) - f(z_0)}{\Delta z} \pm \frac{g(z_0+\Delta z) - g(z_0)}{\Delta z} \right] \\
&= f'(z_0) \pm g'(z_0)
\end{aligned}$$

Rule 7: $(fg)'(z_0) = \lim_{\Delta z \to 0} \dfrac{fg(z_0+\Delta z) - fg(z_0)}{\Delta z}$

$$= \lim_{\Delta z \to 0} \left\{ \frac{f(z_0 + \Delta z)g(z_0 + \Delta z) - f(z_0 + \Delta z)g(z_0)}{\Delta z} \right.$$

$$\left. + \frac{f(z_0 + \Delta z)g(z_0) - f(z_0)g(z_0)}{\Delta z} \right\}$$

$$= \lim_{\Delta z \to 0} \left\{ f(z_0 + \Delta z) \frac{[g(z_0 + \Delta z) - g(z_0)]}{\Delta z} \right.$$

$$\left. + g(z_0) \frac{[f(z_0 + \Delta z) - f(z_0)]}{\Delta z} \right\}$$

$$= f(z_0)g'(z_0) + g(z_0)f'(z_0)$$

6. Let $n > 0$ be an integer.
Then $\frac{d}{dz} z^{-n} = \frac{d}{dz}\left(\frac{1}{z^n}\right) = \frac{-nz^{n-1}}{z^{2n}}$ (using Rule 8) $= -nz^{-n-1}$.

7. a. $18z^2 + 16z + i$
   b. $-12z(z^2 - 3i)^{-7}$
   c. $\frac{-iz^4 + (2 + 27i)z^2 + 2\pi z + 18}{(iz^3 + 2z + \pi)^2}$
   d. $\frac{-(z+2)^2(5z^2 + (16+i)z - 3 + 8i)}{(z^2 + iz + 1)^5}$
   e. $24i(z^3 - 1)^3(z^2 + iz)^{99}(53z^4 + 28iz^3 - 50z - 25i)$

8. Let $z = z_0 + \Delta z$. Then

$$\lim_{z \to z_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} = \left| \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right| = |f'(z_0)|.$$

$$\begin{aligned} \lim_{z \to z_0} \arg[f(z) - f(z_0)] - \arg(z - z_0) &= \lim_{z \to z_0} \arg\left[\frac{f(z) - f(z_0)}{z - z_0}\right] \\ \arg\left[\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}\right] &= \arg[f'(z_0)] \end{aligned}$$

9. a. $2-3i$

b. $\pm i$

c. $\dfrac{-1 \pm i\sqrt{15}}{2}$

d. $\dfrac{1}{2}, 1$

10. $\lim\limits_{\Delta z \to 0} \dfrac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}$

$$= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)(\overline{z}_0 + \overline{\Delta z}) - z_0 \overline{z}_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\overline{z}_0 + \frac{\overline{\Delta z}}{\Delta z} z_0 + \overline{\Delta z} \right) = \begin{cases} \overline{z_0} + z_0 & \text{if } \Delta z = \Delta x \\ \overline{z_0} - z_0 & \text{if } \Delta z = i\Delta y \end{cases}$$

If $z_0 = 0$, then the difference quotient is

$$\lim_{\Delta z \to 0} (0 + 0 + \overline{\Delta z}) = 0.$$

11. a. nowhere analytic

b. nowhere analytic

c. analytic except at $z = 5$

d. everywhere analytic

e. nowhere analytic

f. analytic except at $z = 0$

g. nowhere analytic

h. nowhere analytic

12. The case when $n = 1$ is trivial. Assume that the result holds for all positive integers less than or equal to n and define $Q(z) = P(z)(z - z_{n+1})$. Since $Q'(z) = P'(z)(z - z_{n+1}) + P(z)$, it follows that

$$\frac{Q'(z)}{Q(z)} = \frac{P'(z)}{P(z)} + \frac{1}{z - z_{n+1}} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_{n+1}}$$

13. a, b, d, f, and g are always true

14. $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{[f(z)-f(z_0)]/(z-z_0)}{[g(z)-g(z_0)]/(z-z_0)} = \frac{f'(z_0)}{g'(z_0)}$

15. $\frac{3}{5}$

16. Any point on the line through z_1 and z_2 has the form $z = -\frac{1}{2} + i\sqrt{3}\left(\frac{1}{2} - t\right)$, t real (see Section 1.3, Exercise 18). However, $f(z_2) - f(z_1) = 0$ but $f'(w) = 3w^2 \neq 0$ on the line in question.

17. $F'(z_0) = f(z_0)(gh)'(z_0) + f'(z_0)gh(z_0)$
$= f(z_0)[g(z_0)h'(z_0) + g'(z_0)h(z_0)] + f'(z_0)g(z_0)h(z_0)$
$= f'(z_0)g(z_0)h(z_0) + f(z_0)g'(z_0)h(z_0) + f(z_0)g(z_0)h'(z_0)$

EXERCISES 2.4: The Cauchy-Riemann Equations

1. a. $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$

   b. $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = 0$

   c. $\frac{\partial u}{\partial y} = 2 \neq -\frac{\partial v}{\partial x} = 1$

2. $\frac{\partial u}{\partial x} = 3x^2 + 3y^2 - 3 = \frac{\partial v}{\partial y}$, but $\frac{\partial u}{\partial y} = 6xy = \frac{\partial v}{\partial x}$. Therefore $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ only when $x = 0$ or $y = 0$. This means h is differentiable on the axes but h is nowhere analytic since lines are not open sets in the complex plane.

3. $\frac{\partial u}{\partial x} = 6x + 2 = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -6y = -\frac{\partial v}{\partial x}$. Since these partial derivatives exist and are continuous for all x and y, g is analytic. g can be written as $g(z) = 3z^2 + 2z - 1$.

4. $$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{u(\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$
$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{u(0, \Delta y) - u(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0}{\Delta y} = 0.$$
Similarly $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$.
However, when $\Delta z \to 0$ through real values ($\Delta z = \Delta x$)

$$\lim_{\Delta x \to 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = 0,$$

while along the real line $y = x$ ($\Delta z = \Delta x + i\Delta x$)

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f(0 + \Delta z) - f(0)}{\Delta z} &= \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^{4/3}(\Delta x)^{5/3} + i(\Delta x)^{5/3}(\Delta x)^{4/3}}{2(\Delta x)^2}}{\Delta x(1+i)} \\ &= \frac{1}{2}. \end{aligned}$$

Therefore f is not differentiable at $z = 0$.

5. $$\frac{\partial u}{\partial x} = 2e^{x^2-y^2}[x\cos(2xy) - y\sin(2xy)] = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -2e^{x^2-y^2}[y\cos(2xy) + x\sin(2xy)] = -\frac{\partial v}{\partial x}$$
f is entire because these first partials exist and are continuous for all x and y.

$$\begin{aligned} f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} &= 2e^{x^2-y^2}(x+iy)[\cos(2xy) + i\sin(2xy)] \\ &= 2e^{(x^2-y^2)}e^{i2xy}(x+iy) \\ &= 2ze^{z^2} \end{aligned}$$

(This derivative could have been obtained directly, since $f(z) = e^{z^2}$.)

6. $z = re^{i\theta} \Longrightarrow x = r\cos\theta$ and $y = r\sin\theta$ and

$$f(z) = u(x(r,\theta), y(r,\theta)) + iv(x(r,\theta), y(r,\theta))$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta$$

Similar applications of the chain rule yield

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y} r\cos\theta$$
$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y} r\cos\theta$$

Replace the partial derivatives on the right sides of the equations for $\frac{\partial u}{\partial r}$ and $\frac{\partial v}{\partial r}$ by their Cauchy-Riemann counterparts to obtain:

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial y}\cos\theta - \frac{\partial v}{\partial x}\sin\theta = \frac{1}{r}\frac{\partial v}{\partial \theta}$$
$$\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

7. Let $h(z) = f(z) - g(z)$. Then h is analytic in D and $h'(z) = 0$ so h is a constant function.

$$h(z) = c = f(z) - g(z) \Longrightarrow f(z) = g(z) + c$$

8. $u(x, y) = c$ in $D \Longrightarrow \frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$. Hence $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0$ so f is constant in D.

9. By contradiction. If f is analytic in a domain D then $v(x, y) = 0$ (a constant) $\Rightarrow f$ is constant (by condition 8) $\Rightarrow u$ is constant. (However, there is no open set in which $u(x, y) = |z^2 - z|$ is constant).

10. $\operatorname{Im} f(z) = 0$ in $D \Longrightarrow \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \Longrightarrow \frac{\partial u}{\partial x} = 0$ $\Longrightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0 \Longrightarrow f$ is constant in D.

11. $\operatorname{Re} f(z) = \frac{1}{2}[f(z) + \overline{f(z)}]$ is real valued and analytic if both f and $\overline{f}$ are analytic. Hence $\operatorname{Re} f(z)$ is constant by Exercise 10. It follows that $f(z)$ is constant by Exercise 8.

12. $|f(z)|$ constant in $D \Longrightarrow |f(z)|^2 = u^2 + v^2$ is constant in D. If $u = 0$ or $v = 0$ in D, then f is constant by Exercises 8 and 10. Otherwise,

$$\begin{aligned}\frac{\partial |f|^2}{\partial x} &= 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0\\ \frac{\partial |f|^2}{\partial y} &= 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = -2u\frac{\partial v}{\partial x} + 2v\frac{\partial u}{\partial x} = 0\end{aligned}$$

$$\begin{aligned}&\Longrightarrow \frac{1}{2}v\frac{\partial |f|^2}{\partial x} - \frac{1}{2}u\frac{\partial |f|^2}{\partial y} = 0 = (u^2+v^2)\frac{\partial v}{\partial x}\\ &\Longrightarrow \frac{\partial v}{\partial x} = 0 \Longrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial u}{\partial y} = 0\\ &\Longrightarrow f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0\\ &\Longrightarrow f \text{ is constant in } D.\end{aligned}$$

13. $|f(z)|$ is analytic and real-valued, so the result follows from Exercises 10 and 12.

14. If the line is vertical then $\operatorname{Re} f(z)$ is constant and this reduces to Problem 8. If the line is not vertical, then $v(x,y) = mu(x,y) + b$, and

$$\begin{aligned}\frac{\partial v}{\partial x} &= m\frac{\partial u}{\partial x} = m\frac{\partial v}{\partial y},\\ \frac{\partial v}{\partial y} &= m\frac{\partial u}{\partial y} = -m\frac{\partial v}{\partial x} = -m^2\frac{\partial v}{\partial y}.\end{aligned}$$

It follows that

$$\frac{\partial v}{\partial y} = 0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \text{ and } f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 0.$$

Hence $f(z)$ is constant.

15. $$\begin{aligned}J(x_0,y_0) &= \left.\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right|_{(x_0,y_0)}\\ &= \left[\frac{\partial u}{\partial x}(x_0,y_0)\right]^2 + \left[\frac{\partial v}{\partial x}(x_0,y_0)\right]^2\\ &= |f'(z_0)|^2 \quad \text{(using Equation (1))}\end{aligned}$$

16. a. $\frac{\partial \tilde{f}}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$

$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{1}{2i}$

$= \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

$\frac{\partial \tilde{f}}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$

$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(\frac{-1}{2i}\right)$

$= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

b. $\frac{\partial \tilde{f}}{\partial \eta} = 0 \Leftrightarrow 0 = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$ and $0 = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$$\Leftrightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

EXERCISES 2.5: Harmonic Functions

1. a. $u(x,y) = x^2 - y^2 + 2x + 1,\ \frac{\partial^2 u}{\partial x^2} = 2 = -\frac{\partial^2 u}{\partial y^2} \Longrightarrow \Delta u = 0$

   $v(x,y) = 2xy + 2y,\ \frac{\partial^2 v}{\partial x^2} = 0 = -\frac{\partial^2 v}{\partial y^2} \Longrightarrow \Delta v = 0$

   b. $u(x,y) = \frac{x}{x^2+y^2},\ \frac{\partial^2 u}{\partial x^2} = \frac{2x(x^2-3y^2)}{(x^2+y^2)^3} = -\frac{\partial^2 u}{\partial y^2} \Longrightarrow \Delta u = 0$

   $v(x,y) = -\frac{y}{x^2+y^2},\ \frac{\partial^2 v}{\partial x^2} = \frac{-2y(3x^2-y^2)}{(x^2+y^2)^3} = -\frac{\partial^2 v}{\partial y^2} \Longrightarrow \Delta v = 0$

   c. $u(x,y) = e^x \cos y,\ \frac{\partial^2 u}{\partial x^2} = e^x \cos y = -\frac{\partial^2 u}{\partial y^2} \Longrightarrow \Delta u = 0$

   $v(x,y) = e^x \sin y,\ \frac{\partial^2 v}{\partial x^2} = e^x \sin y = -\frac{\partial^2 v}{\partial y^2} \Longrightarrow \Delta v = 0$

2. $h(x,y) = ax^2 + bxy - ay^2$

3. a. $u = \text{Re}(-iz)$, $v = -x + a$, where a is a constant

b. $u = \text{Re}(-ie^z)$, $v = -e^x \cos y + a$

c. $u = \text{Re}\left(\frac{-i}{2}z^2 - iz - z\right)$, $v = -\frac{1}{2}(x^2 - y^2) - (x + y) + a$

d. It is straightforward to verify that $\Delta u = 0$.

$\frac{\partial u}{\partial x} = \cos x \cosh y = \frac{\partial v}{\partial y}$

$\Rightarrow v(x, y) = \int \cos x \cosh y dy = \cos x \sinh y + \psi(x)$

$\frac{\partial u}{\partial y} = \sin x \sinh y = -\frac{\partial v}{\partial x} = \sin x \sinh y + \psi'(x) \Rightarrow \psi(x) = a$

Thus, $v(x, y) = \cos x \sinh y + a$.

e. It is straightforward to verify that $\Delta u = 0$.

$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} = \frac{\partial v}{\partial y} \Rightarrow$

$v(x, y) = \int \frac{x}{x^2 + y^2} dy = \tan^{-1}\left(\frac{y}{x}\right) + \psi(x)$

$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2} = -\frac{\partial v}{\partial x} = \frac{y}{x^2 + y^2} - \psi'(x) \Rightarrow \psi(x) = a$

Thus, $v(x, y) = \tan^{-1}\left(\frac{y}{x}\right) + a$.

f. $u = \text{Re}\left(-ie^{z^2}\right)$, $v = -e^{x^2 - y^2}\cos(2xy) + a$.

4. Suppose v and w are both harmonic conjugates of u, and consider $\phi(x, y) = w(x, y) - v(x, y)$. Then (using the Cauchy-Riemann equations for v and w),

$$\frac{\partial \phi}{\partial x} = \frac{\partial w}{\partial x} - \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} - \left(-\frac{\partial u}{\partial y}\right) = 0$$

and similarly $\frac{\partial \phi}{\partial y} = 0$. Hence $\phi(x, y) = a$, from which it follows that

$$w(x, y) = v(x, y) + a.$$

5. If $f(z) = u(x, y) + iv(x, y)$ is analytic then $-if(z) = v(x, y) - iu(x, y)$ is analytic. Thus $-u$ is a harmonic conjugate of v.

6. Since $f(z) = u + iv$ is analytic, $\frac{1}{2}[f(z)]^2 = \frac{1}{2}(u^2 - v^2) + iuv$ is analytic. Thus $uv = \text{Im}\frac{1}{2}[f(z)]^2$ is harmonic.

7. $\phi(x, y) = x + 1$

8. a. Yes, because $\Delta(u + v) = \Delta u + \Delta v = 0$.

   b. No. Take $u = x, v = x^2 - y^2$ as an example.

   c. Yes, because $\Delta(u_x) = u_{xxx} + u_{xyy} = u_{xxx} + u_{yyx}$

$$= \frac{\partial}{\partial x}(\Delta u) = \frac{\partial}{\partial x}(0) = 0.$$

9. $\phi(x, y) = xy - 1$ $\left(\text{this is Im}\left(\frac{1}{2}z^2 - i\right)\right)$

10. Let $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned}
\frac{\partial \phi}{\partial r} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial \phi}{\partial x}\cos\theta + \frac{\partial \phi}{\partial y}\sin\theta \\
\frac{\partial^2 \phi}{\partial r^2} &= \frac{\partial^2 \phi}{\partial x^2}\frac{\partial x}{\partial r}\cos\theta + \frac{\partial^2 \phi}{\partial y \partial x}\frac{\partial y}{\partial r}\cos\theta \\
&\quad + \frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial x}{\partial r}\sin\theta + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial y}{\partial r}\sin\theta \\
&= \frac{\partial^2 \phi}{\partial x^2}\cos^2\theta + \frac{\partial^2 \phi}{\partial y \partial x}2\sin\theta\cos\theta + \frac{\partial^2 \phi}{\partial y^2}\sin^2\theta \\
\frac{\partial \phi}{\partial \theta} &= \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial \phi}{\partial x}(-r\sin\theta) + \frac{\partial \phi}{\partial y}r\cos\theta \\
\frac{\partial^2 \phi}{\partial \theta^2} &= \frac{\partial^2 \phi}{\partial x^2}\frac{\partial x}{\partial \theta}(-r\sin\theta) + \frac{\partial^2 \phi}{\partial y \partial x}\frac{\partial y}{\partial \theta}(-r\sin\theta) + \frac{\partial \phi}{\partial x}(-r\cos\theta) \\
&\quad + \frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial x}{\partial \theta}(r\cos\theta) + \frac{\partial^2 \phi}{\partial y^2}\frac{\partial y}{\partial \theta}(r\cos\theta) + \frac{\partial \phi}{\partial y}(-r\sin\theta) \\
&= \frac{\partial^2 \phi}{\partial x^2}r^2\sin^2\theta + \frac{\partial^2 \phi}{\partial y \partial x}(-2r^2\sin\theta\cos\theta) + \frac{\partial^2 \phi}{\partial y^2}r^2\cos^2\theta \\
&\quad + \frac{\partial \phi}{\partial x}(-r\cos\theta) + \frac{\partial \phi}{\partial y}(-r\sin\theta).
\end{aligned}$$

Combining these partial derivatives, one gets

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0.$$

11. $\operatorname{Im} f(z)=y-\dfrac{y}{x^2+y^2}=0 \Longrightarrow yx^2+y^3-y=y(x^2+y^2-1)=0.$
The points satisfying $x^2+y^2-1=0$ lie on the circle $|z|=1$. The points (other than $z=0$) satisfying $y=0$ lie on the real axis.

12. $f(z)=z^n=r^n(\cos\theta+i\sin\theta)^n=r^n(\cos n\theta+i\sin n\theta)\Longrightarrow$
$\operatorname{Re} f(z)=r^n\cos n\theta$ and $\operatorname{Im} f(z)=r^n\sin n\theta$ are harmonic since f is analytic.

13. $\phi(x,y)=\operatorname{Im} z^4=r^4\sin 4\theta=-4xy^3+4x^3y$

14. Let $\phi(x,y)=\ln|f(z)|=\frac{1}{2}\ln(u^2+v^2)$

$$\frac{\partial\phi}{\partial x}=\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}=\frac{u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}}{u^2+v^2}$$

$$\frac{\partial^2\phi}{\partial x^2}=\frac{(v^2-u^2)\left[\left(\frac{\partial u}{\partial x}\right)^2-\left(\frac{\partial v}{\partial x}\right)^2\right]-4uv\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}}{(u^2+v^2)^2}+\frac{u\frac{\partial^2 u}{\partial x^2}+v\frac{\partial^2 v}{\partial x^2}}{u^2+v^2}$$

A similar calculation yields $\dfrac{\partial^2\phi}{\partial y^2}$. By applying Laplace's equation and the Cauchy-Riemann equations of u and v to $\dfrac{\partial^2\phi}{\partial x^2}+\dfrac{\partial^2\phi}{\partial y^2}$, the sum simplifies to reveal that $\Delta\phi=0$.

15. Consider $\varphi(z)=\operatorname{Re}(Az^n+Bz^{-n})+C$ which is harmonic for $1\le|z|\le 2$. Consider the polar form for z. $z=re^{i\theta}$ and select n=3 to agree with the cosine argument. $\varphi(re^{i\theta})=Ar^3\operatorname{Re}(e^{i3\theta})+Br^{-3}\operatorname{Re}(e^{-i3\theta})+C$.
$\varphi(re^{i\theta})=Ar^3\cos3\theta+Br^{-3}\cos3\theta+C=(Ar^3+Br^{-3})\cos3\theta+C$.

$r=1\Rightarrow(A+B)\cos3\theta+C=0\Rightarrow A+B=0,\ C=0.$
$r=2\Rightarrow(A*8+B/8)\cos3\theta=5\cos3\theta.\ A=40/63,\ B=-40/63$
$\varphi(re^{i\theta})=(40/63)(r^3-r^{-3})\cos3\theta=(40/63)\operatorname{Re}(z^3-z^{-3}).$

16. $\phi(x,y)=\dfrac{1}{\ln 3}\ln|z|-1$ or $\phi(x,y)=\ln\left|\dfrac{z}{3}\right|$ are two possibilities.

17. a. $\phi(x,y) = \text{Re}(z^2 + 5z + 1) = x^2 - y^2 + 5x + 1$

b. $\phi(x,y) = 2\text{Re}\left(\dfrac{z^2}{z+2i}\right) = \dfrac{2x(x^2+4y+y^2)}{x^2+y^2+4y+4}$

18. Let $u = \phi_x$, $v = -\phi_y$. Then

$$\frac{\partial u}{\partial x} = \phi_{xx} = -\phi_{yy} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = \phi_{xy} = -\frac{\partial v}{\partial x}$$

19. $\cos^2\theta = (\tfrac{1}{2})\cos 2\theta + \tfrac{1}{2} = \varphi(z) = A\text{Re}(r^{-2}e^{-i2\theta})n + B = Ar^{-2}\cos 2\theta + B$. In the limit as $r \to \infty$ $\varphi(z) = \tfrac{1}{2} \Rightarrow B = \tfrac{1}{2}$. On the circle $|z|=1$, $r = 1. \Rightarrow A = \tfrac{1}{2}$. $\varphi(z) = (\tfrac{1}{2})r^{-2}\cos 2\theta + \tfrac{1}{2} = \text{Re}\,[1/(2z^2)] + \tfrac{1}{2}$.

20. In order that $\dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x}$, let $v(x,y) = \int_0^y \dfrac{\partial u}{\partial x}(x,\eta)d\eta + \psi(x)$. Then

$$\begin{aligned}\frac{\partial v}{\partial x} &= \int_0^y \frac{\partial^2 u}{\partial x^2}(x,\eta)d\eta + \psi'(x)\\ &= -\int_0^y \frac{\partial^2 u}{\partial y^2}(x,\eta)d\eta + \psi'(x) \quad \text{(because } u \text{ is harmonic)}\\ &= -\frac{\partial u}{\partial y}(x,y) + \frac{\partial u}{\partial y}(x,0) + \psi'(x).\end{aligned}$$

In order that $\dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y}$, it must be true that $\psi'(x) = -\dfrac{\partial u}{\partial y}(x,0)$. Thus,

$$\psi(x) = -\int_0^x \frac{\partial u}{\partial y}(\zeta,0)d\zeta + a$$

and

$$v(x,y) = \int_0^y \frac{\partial u}{\partial x}(x,\eta)d\eta - \int_0^x \frac{\partial u}{\partial y}(\zeta,0)d\zeta + a.$$

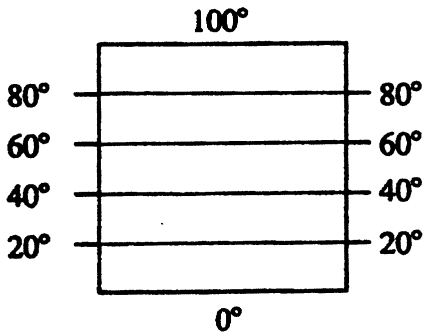
21. It is easily verified that $u = \ln|z|$ satisfies Laplace's equation on $\mathbf{C}\setminus\{0\}$ and that $u + iv = \ln|z| + i\text{Arg}(z)$ satisfies the Cauchy-Riemann equations on the domain $D = \mathbf{C}\setminus$ {nonpositive real axis}, so that

Arg (z) is a harmonic conjugate of u on D. By Problem 4, any harmonic conjugate of u has to be of the form Arg$(z)+a$ in D. It is impossible to have a harmonic conjugate of this form that is continuous on $\mathbf{C}\setminus\{0\}$.

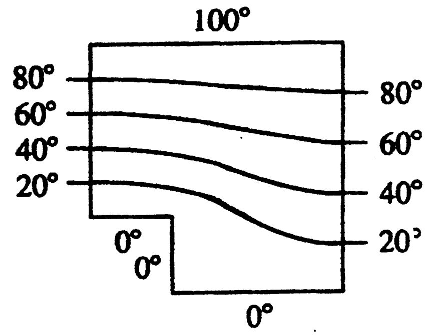
22. $$\begin{aligned}\frac{\partial u}{\partial x} &= \phi_{xx}\phi_y + \phi_x\phi_{yx} + \psi_{xx}\psi_y + \psi_x\psi_{yx}\\ &= -\phi_{yy}\phi_y + \phi_x\phi_{yx} - \psi_{yy}\psi_y + \psi_x\psi_{yx} = \frac{\partial v}{\partial y}\end{aligned}$$

EXERCISES 2.6: Steady-State Temperature as a Harmonic Function.

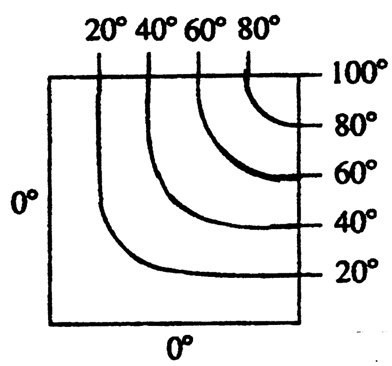
1. a.



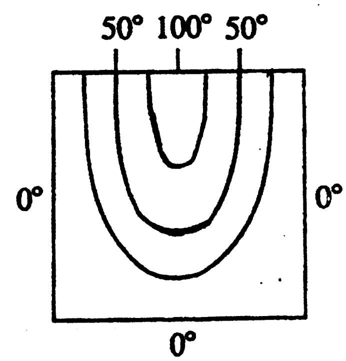
b.



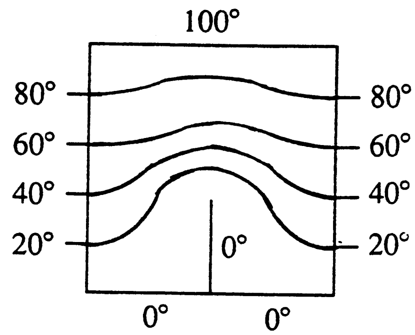
c.



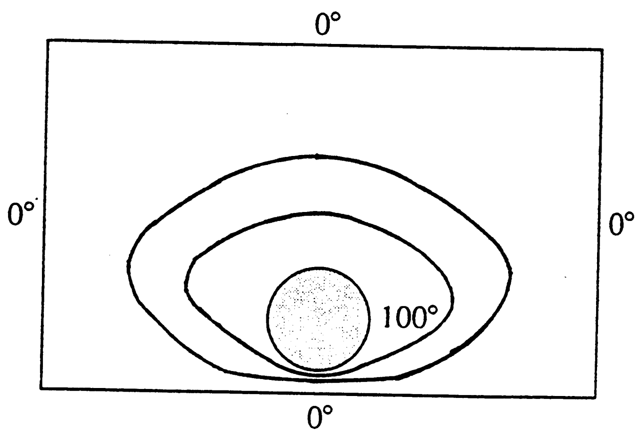
d.



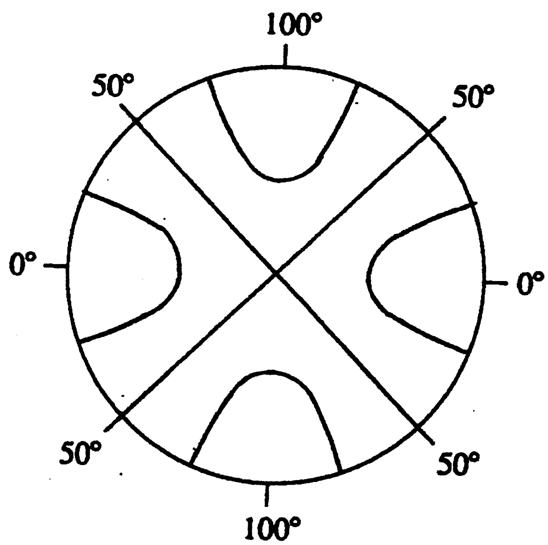
e.



2. This does not violate the maximum principle.



3. This does not violate the maximum principle.



Exercises 2.7

1. $f(z) = z^2 + c$ where c is a real constant.
   $\zeta_1 = (1 + \sqrt{(1-4c)})/2$, $\zeta_2 = (1 - \sqrt{(1-4c)})/2$
   Only ζ_2 is an attractor for $-3/4 < c < 1/4$.
2. $f(\varsigma) = \varsigma$ and $f'(\varsigma) > 1$ Therefore we can pick a real number ρ between 1 and $|f'(\varsigma)|$ such that $|f(z) - \varsigma| = \rho|z-\varsigma|$ for all z in a sufficiently small disk around ς. If any point z_0 in this disk is the seed for an orbit $z_1 = f(z_0)$, $z_2 = f(z_1)$, … $z_n = f(z_{n-1})$, then we have $|z_n-\varsigma| \geq \rho|z_{n-1}-\varsigma| \geq \ldots \geq \rho^n|z_{n-1}-\varsigma|$. Because $\rho>1$, the point z_n moves away from ς until the magnitude of the derivative becomes 1 or less. The orbit is out of the disk.
3. (a) Fixed points are $\zeta_1 = i$, $\zeta_2 = -i$. Both are repellors.
   (b) Fixed points are $\zeta_1 = 1/2$, $\zeta_2 = -1/2$, $\zeta_3 = -1$. Fixed points ζ_1 and ζ_3 are repellors, but fixed point ζ_2 is an attractor.
4. $z_0 = e^{i2\pi\alpha}$ with α an irrational real number. $z_n = e^{i2\pi\alpha 2^n}$. Because $|z_n| = 1$, the trajectory will follow the unit circle. If iterations p and q coincide, $2\pi\alpha 2^p - 2\pi\alpha 2^q = 2\pi\alpha(2^p - 2^q) = 2\pi k$ for some integer k. But because $(2^p - 2^q)$ is an integer that can be represented by m, the equation $2\pi\alpha m=2\pi k$ is satisfied only if $k=\alpha m$ or $\alpha = k/m$. Because α is irrational it cannot be represented by a rational number and no iterations repeat.
5. Fixed points are $\zeta_1 = -1/2 + i\sqrt{5}/2$ (an attractor) and $\zeta_2 = -1/2 - i\sqrt{5}/2$ (a repellor).
6. $f(z) = z^2$. The seed is z_0. $z_1 = z_0^2$, $z_2 = z_0^4$, … $z_n = z_0$^(2^n). To have an n cycle $z_n = z_0 = z_0$^(2^n). Or $z_n/z_0 = z_0$^$(2^n-1) = 1 = e^{i2\pi}$. Solving gives $z_0 = e$^$(i2\pi/(2^n-1))$.
7. The cycle is 4. $2^4(2\pi/p) = 2\pi \bmod p \Rightarrow 2^4 = 1 \bmod p$. p=3,5,15. 3 will give repeated cycles of length 2. 5 and 15 will give the desired cycles of length 4.
8. Student Matlab: n=100;c=.253; zo=0;y(1)=zo;
   for k=1:n-1,y(k+1)=y(k)^2+c;end
   plot(y)
9. If $|\alpha| \leq 1$ the whole complex plane is the filled Julia set. If $|\alpha| \geq 1$ the origin is the filled Julia set.
10. $f(z) = z - F(z)/F'(z)$. $f(\zeta) = \zeta - F(\zeta)/F'(\zeta) = \zeta \Rightarrow F(\zeta)/F'(\zeta) = 0 \Rightarrow F(\zeta)=0$ with the possible exception of the points where $F'(\zeta) = 0$.
    $f'(z) = 1 - F'(z)/F'(z) + F(z)F''(z)/(F'(z))^2 = F(z)F''(z)/(F'(z))^2$
    $f'(\zeta) = F(\zeta)F''(\zeta)/(F'(\zeta))^2 = 0$ where $F'(\zeta) \neq 0$ and every zero of F(z) is an attractor as long as $F'(\zeta) \neq 0$.