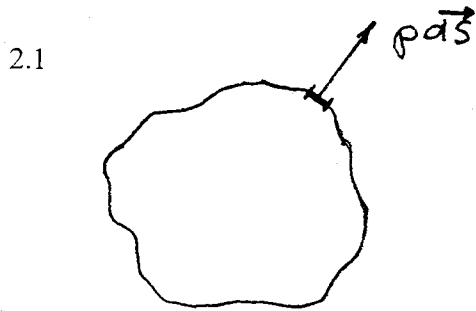


CHAPTER 2



$$\vec{F} = -\oiint p d\vec{S}$$

If  $p = \text{constant} = p_\infty$

$$\vec{F} = -p_\infty \oiint p d\vec{S} \quad (1)$$

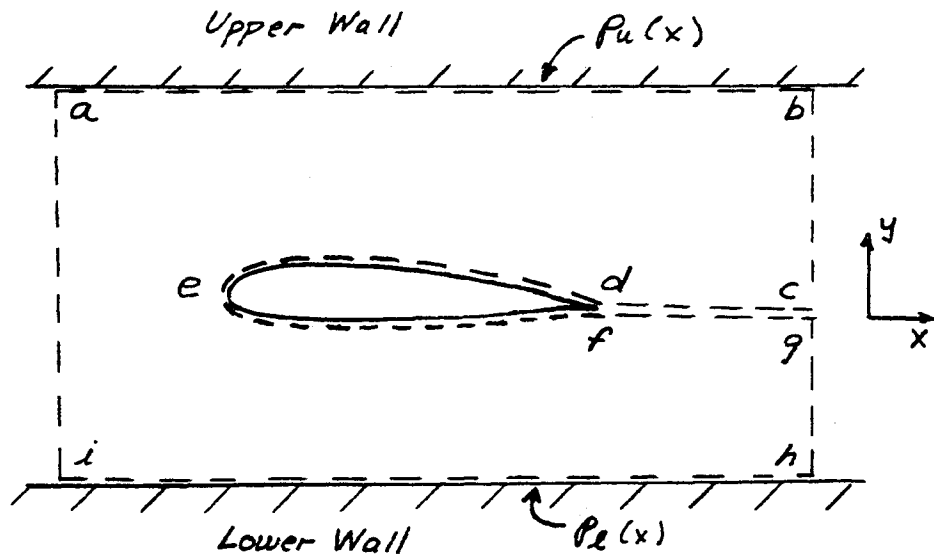
However, the integral of the surface vector over a closed surface is zero, i.e.,

$$\oiint d\vec{S} = 0$$

Hence, combining Eqs. (1) and (2), we have

$$\boxed{\vec{F} = 0}$$

2.2



Denote the pressure distributions on the upper and lower walls by  $p_u(x)$  and  $p_\ell(x)$  respectively. The walls are close enough to the model such that  $p_u$  and  $p_\ell$  are not necessarily equal to  $p_\infty$ . Assume that faces  $\underline{ai}$  and  $\underline{bh}$  are far enough upstream and downstream of the model such that

$$p = p_\infty \quad \text{and} \quad v = 0 \quad \text{and} \quad \underline{ai} \quad \text{and} \quad \underline{bh}.$$

Take the y-component of Eq. (2.66)

$$L = - \oint_S (\rho \vec{V} \cdot \vec{dS}) v - \iint_{abhi} (p \vec{dS})_y$$

The first integral = 0 over all surfaces, either because  $\vec{V} \cdot \vec{dS} = 0$  or because  $v = 0$ . Hence

$$L' = - \iint_{abhi} (p \vec{dS})_y = - \left[ \int_a^b p_u dx - \int_i^h p_\ell dx \right]$$

Minus sign because y-component is in downward Direction.

Note: In the above, the integrals over  $\underline{ia}$  and  $\underline{bh}$  cancel because  $p = p_\infty$  on both faces. Hence

$$L' = \int_i^h p_\ell dx - \int_a^b p_u dx$$

$$2.3 \quad \frac{dy}{dx} = \frac{v}{u} = \frac{cy / (x^2 + y^2)}{cx / (x^2 + y^2)} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln y = \ln x + c_1 = \ln (c_2 x)$$

$$y = c_2 x$$

The streamlines are straight lines emanating from the origin. (This is the velocity field and streamline pattern for a source, to be discussed in Chapter 3.)

$$2.4 \quad \frac{dy}{dx} = \frac{v}{u} = -\frac{x}{y}$$

$$y dy = -x dx$$

$$y^2 = -x^2 + \text{const}$$

$$x^2 + y^2 = \text{const.}$$

The streamlines are concentric with their centers at the origin. (This is the velocity field and streamline pattern for a vortex, to be discussed in Chapter 3.)

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2.5 From inspection, since there is no radial component of velocity, the streamlines must be circular, with centers at the origin. To show this more precisely,

$$u = -V_\theta \sin \theta = -cr \frac{y}{r} = -cy$$

$$v = V_\theta \cos \theta = cr \frac{x}{r} = cx$$

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{x}{y}$$

$$\boxed{y^2 + x^2 = \text{const.}}$$

This is the equation of a circle with the center at the origin. (This velocity field corresponds to solid body rotation.)

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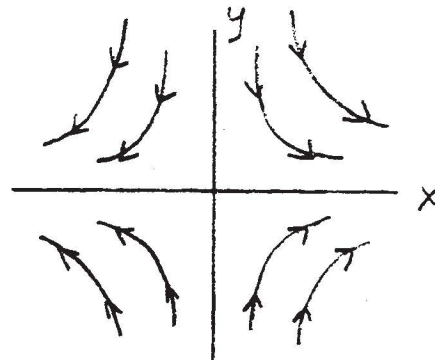
2.6  $\frac{dy}{dx} = \frac{v}{u} = -\frac{y}{x}$

$$\frac{dy}{y} = -\frac{dx}{x}$$

$$\ln y = -x \ln x + c_1$$

$$y = c_2/x$$

The streamlines are hyperbolas.



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2.7 (a)  $\frac{1}{\delta v} \frac{D(\delta v)}{Dt} = \nabla \cdot \vec{V}$

In polar coordinates:  $\nabla \cdot \vec{V} = \frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{1}{r} \frac{\partial V_\theta}{\partial \theta}$

Transformation:  $x = r \cos \theta$

$$y = r \sin \theta$$

$$V_r = u \cos \theta + v \sin \theta$$

$$V_\theta = -u \sin \theta + v \cos \theta$$

$$u = \frac{cx}{(x^2 + y^2)} = \frac{cr \cos\theta}{r^2} = \frac{c \cos\theta}{r}$$

$$v = \frac{cy}{(x^2 + y^2)} = \frac{cr \sin\theta}{r^2} = \frac{c \sin\theta}{r}$$

$$V_r = \frac{c}{r} \cos^2\theta + \frac{c}{r} \sin^2\theta = \frac{c}{r}$$

$$V_\theta = -\frac{c}{r} \cos\theta \sin\theta + \frac{c}{r} \cos\theta \sin\theta = 0$$

$$\nabla \cdot \vec{V} = \frac{1}{r} \frac{\partial}{\partial r} (c) + \frac{1}{r} \frac{\partial(0)}{\partial \theta} = 0$$

(b) From Eq. (2.23)

$$\nabla \times \vec{V} = e_z \left[ \frac{\partial V_\theta}{\partial r} + \frac{V_\theta}{r} - \frac{1}{r} \frac{\partial V_r}{\partial \theta} \right]$$

$$\nabla \times V = e_z [0 + 0 - 0] = \boxed{0}$$

The flowfield is irrotational.

$$2.8 \quad u = \frac{cy}{(x^2 + y^2)} = \frac{cr \sin\theta}{r^2} = \frac{c \sin\theta}{r}$$

$$v = \frac{-cx}{(x^2 + y^2)} = \frac{cr \cos\theta}{r^2} = -\frac{c \cos\theta}{r}$$

$$V_r = \frac{c}{r} \cos\theta \sin\theta - \frac{c}{r} \cos\theta \sin\theta = 0$$

$$V_\theta = -\frac{c}{r} \sin^2\theta - \frac{c}{r} \cos^2\theta = -\frac{c}{r}$$

$$(a) \quad \nabla \cdot \vec{V} = \frac{1}{r} \frac{\partial}{\partial r} (0) + \frac{1}{r} \frac{\partial(-c/r)}{\partial \theta} = 0 + 0 = \boxed{0}$$

$$\begin{aligned}
 \text{(b)} \quad \nabla \times \vec{V} &= e_z \left[ \frac{\partial(-c/r)}{\partial r} - \frac{c}{r^2} - \frac{1}{r} \frac{\partial(0)}{\partial \theta} \right] \\
 &= e_z \left[ \frac{c}{r^2} - \frac{c}{r^2} - 0 \right]
 \end{aligned}$$

$\nabla \times \vec{V} = \vec{0}$  except at the origin, where  $r = 0$ . The flowfield is singular at the origin.

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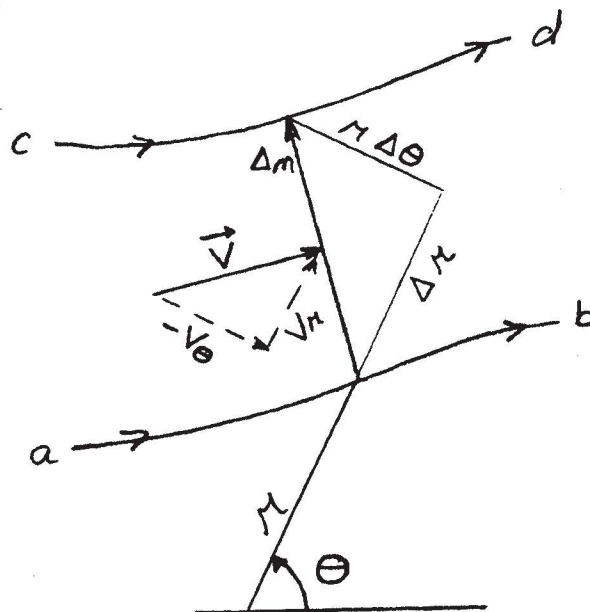
2.9  $V_r = 0, \quad V_\theta = cr$

$$\begin{aligned}
 \nabla \times \vec{V} &= e_z \left[ \frac{\partial(c/r)}{\partial r} + \frac{cr}{r} - \frac{1}{r} \frac{\partial(0)}{\partial \theta} \right] \\
 &= e_z (c + c - 0) = 2c e_z
 \end{aligned}$$

The vorticity is finite. The flow is not irrotational; it is rotational.

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2.10



Mass flow between streamlines =  $\Delta \bar{\psi}$

$$\Delta \bar{\psi} = \rho V \Delta n$$

$$\Delta \bar{\psi} = (-\rho V_\theta) \Delta r + \rho V_r (r\theta)$$

Let cd approach ab

$$d\bar{\psi} = -\rho V_\theta dr + \rho r V_r d\theta \quad (1)$$

Also, since  $\bar{\psi} = \bar{\psi}(r, \theta)$ , from calculus

$$d\bar{\psi} = \frac{\partial \bar{\psi}}{\partial r} dr + \frac{\partial \bar{\psi}}{\partial \theta} d\theta \quad (2)$$

Comparing Eqs. (1) and (2)

$$-\rho V_\theta = \frac{\partial \bar{\psi}}{\partial r}$$

and

$$\rho r V_r = \frac{\partial \bar{\psi}}{\partial \theta}$$

or:

$$\rho V_r = \frac{1}{r} \frac{\partial \bar{\psi}}{\partial \theta}$$

$$\rho V_\theta = -\frac{\partial \bar{\psi}}{\partial r}$$

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$$2.11 \quad u = cx = \frac{\partial \psi}{\partial y} : \psi = cxy + f(x) \quad (1)$$

$$v = -cy = -\frac{\partial \psi}{\partial x} : \psi = cxy + f(y) \quad (2)$$

Comparing Eqs. (1) and (2),  $f(x)$  and  $f(y) = \text{constant}$

$$\boxed{\psi = c x y + \text{const.}} \quad (3)$$

$$u = cx = \frac{\partial \psi}{\partial x} : \phi = cx^2 + f(y) \quad (4)$$

$$v = -cy = \frac{\partial \psi}{\partial y} : \phi = -cy^2 + f(x) \quad (5)$$

Comparing Eqs. (4) and (5),  $f(y) = -cy^2$  and  $f(x) = cx^2$

$$\boxed{\phi = c(x^2 - y^2)} \quad (6)$$

Differentiating Eq. (3) with respect to  $x$ , holding  $\psi = \text{const.}$

$$0 = cx \frac{dy}{dx} + cy$$

or,

$$\left(\frac{dy}{dx}\right)_{\psi=\text{const}} = -y/x \quad (7)$$

Differentiating Eq. (6) with respect to  $x$ , holding  $\phi = \text{const.}$

$$0 = 2cx - 2cy \frac{dy}{dx}$$

or,

$$\left(\frac{dy}{dx}\right)_{\phi=\text{const}} = x/y \quad (8)$$

Comparing Eqs. (7) and (8), we see that

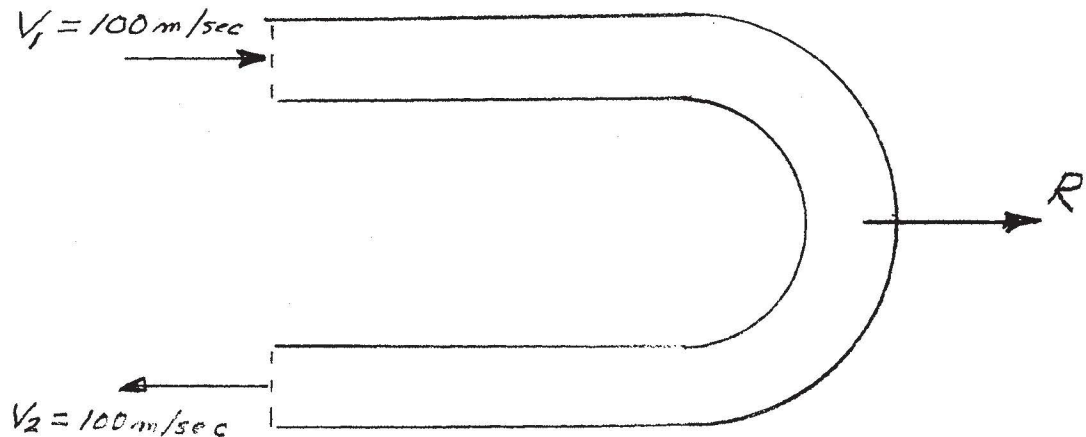
$$\left(\frac{dy}{dx}\right)_{\psi=\text{const}} = - \frac{1}{\left(\frac{dy}{dx}\right)_{\phi=\text{const}}}$$

Hence, lines of constant  $\psi$  are perpendicular to lines of constant  $\phi$ .

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2.12. The geometry of the pipe is shown below.



As the flow goes through the U-shape bend and is turned, it exerts a net force  $R$  on the internal surface of the pipe. From the symmetric geometry,  $R$  is in the horizontal direction, as shown, acting to the right. The equal and opposite force,  $-R$ , exerted by the pipe on the flow is the mechanism that reverses the flow velocity. The cross-sectional area of the pipe inlet is  $\pi d^2/4$  where  $d$  is the inside pipe diameter. Hence,  $A = \pi d^2/4 = \pi(0.5)^2/4 = 0.196\text{m}^2$ . The mass flow entering the pipe is

$$\dot{m} = \rho_1 A V_1 = (1.23)(0.196)(100) = 24.11 \text{ kg/sec.}$$

Applying the momentum equation, Eq. (2.64) to this geometry, we obtain a result similar to Eq. (2.75), namely

$$R = - \iint (\rho \mathbf{V} \cdot d\mathbf{S}) \mathbf{V} \quad (1)$$

Where the pressure term in Eq. (2.75) is zero because the pressure at the inlet and exit are the same values. In Eq. (1), the product  $(\rho \mathbf{V} \cdot d\mathbf{S})$  is negative at the inlet ( $\mathbf{V}$  and  $d\mathbf{S}$  are in opposite directions), and is positive at the exit ( $\mathbf{V}$  and  $d\mathbf{S}$  are in the same direction). The magnitude of  $\rho \mathbf{V} \cdot d\mathbf{S}$  is simply the mass flow,  $\dot{m}$ . Finally, at the inlet  $V_1$  is to the right, hence it is in the positive  $x$ -direction. At the exit,  $V_2$  is to the left, hence it is in the negative  $x$ -direction. Thus,  $V_2 = -V_1$ . With this, Eq. (1) is written as

$$R = - [-\dot{m} V_1 + \dot{m} V_2] = \dot{m} (V_1 - V_2)$$

$$= \dot{m} [V_1 - (-V_1)] = \dot{m} (2V_1)$$

$$R = (24.11)(2)(100) = \boxed{4822 \text{ N}}$$

2.13 From Example 2.1, we have

$$u = V_{\infty} \left[ 1 + \frac{h}{\beta} \frac{2\pi}{\ell} \left( \cos \frac{2\pi x}{\ell} \right) e^{-2\pi\beta y/\ell} \right] \quad (2.35)$$

and

$$v = -V_{\infty} h \frac{2\pi}{\ell} \left( \sin \frac{2\pi x}{\ell} \right) e^{-2\pi\beta y/\ell} \quad (2.36)$$

Thus,

$$\frac{\partial \varphi}{\partial x} = u = V_{\infty} + \left( \frac{V_{\infty} h}{\beta} \right) \left( \frac{2\pi}{\ell} \right) \left( \cos \frac{2\pi x}{\ell} \right) e^{-2\pi\beta y/\ell} \quad (2.35a)$$

Integrating (2.35a) with respect to  $x$ , we have

$$\varphi = V_{\infty} x + \left( \frac{V_{\infty} h}{\beta} \right) \left( \frac{2\pi}{\ell} \right) \left( \sin \frac{2\pi x}{\ell} \right) \frac{1}{\left( \frac{2\pi}{\ell} \right)} e^{-2\pi\beta y/\ell} + f(y)$$

$$\varphi = V_{\infty} x + \frac{V_{\infty} h}{\beta} \left( \sin \frac{2\pi x}{\ell} \right) e^{-2\pi\beta y/\ell} + f(y) \quad (2.35b)$$

From (2.36)

$$\frac{\partial \varphi}{\partial y} = v = -V_{\infty} h \frac{2\pi}{\ell} \left( \sin \frac{2\pi x}{\ell} \right) e^{-2\pi\beta y/\ell} \quad (2.36a)$$

Integrating (2.36a) with respect to  $y$ , we have

$$\varphi = V_{\infty} h \left( \frac{2\pi}{\ell} \right) \left( \sin \frac{2\pi x}{\ell} \right) \left( e^{-2\pi\beta y/\ell} \right) \frac{1}{\left( \frac{2\pi\beta}{\ell} \right)} + f(x)$$

$$\varphi = \frac{V_{\infty} h}{\beta} \left( \sin \frac{2\pi x}{\ell} \right) \left( e^{-2\pi\beta y/\ell} \right) + f(x) \quad (2.36b)$$

Comparing (2.35b) and (2.36b), which represent the same function for  $\varphi$ , we see in (2.36b) that  $f(x) = V_{\infty} x$ . So the velocity potential for the compressible subsonic flow over a wavy well is:

$$\varphi = V_{\infty} x + \frac{V_{\infty} h}{\beta} \left( \sin \frac{2\pi x}{\ell} \right) e^{-2\pi\beta y/\ell}$$

2.14 The equation of a streamline can be found from Eq. (2.118)

$$\frac{dy}{dx} = \frac{v}{u}$$

For the flow over the wavy wall in Example 2.1,

$$\frac{dy}{dx} = \frac{-V_{\infty} h \frac{2\pi}{\ell} \left( \sin \frac{2\pi x}{\ell} \right) e^{-2\pi\beta y/\ell}}{V_{\infty} \left[ 1 + \frac{h}{\beta} \frac{2\pi}{\ell} \left( \cos \frac{2\pi x}{\ell} \right) e^{-2\pi\beta y/\ell} \right]}$$

As  $y \rightarrow \infty$ , then  $e^{-2\pi\beta y/\ell} \rightarrow 0$ . Thus,

$$\frac{dy}{dx} \rightarrow \frac{0}{V_{\infty} + 0} = 0$$

The slope is zero. Hence, the streamline at  $y \rightarrow \infty$  is straight.