n	x_n	y_n
1	0.1	-1.87392
2	0.2	-1.36127
3	0.3	-1.06476
4	0.4	-0.86734
5	0.5	-0.72143
6	0.6	-0.60353
7	0.7	-0.50028
8	0.8	-0.40303
9	0.9	-0.30541
10	1.0	-0.20195

Consequently the Runge-Kutta approximation to $y(1)$ is $y_{10} = -0.20195$. Comparing this to the corresponding Euler approximation from Problem 58 we have

$$|y_{\text{RK}} - y_{\text{E}}| = |0.20195 - 0.12355| = 0.07840.$$

63. Applying the Runge-Kutta method with $y' = \dfrac{3x}{y} + 2, x_0 = 1, y_0 = 2$, and $h = 0.05$ generates the sequence of approximants given in the table below.

n	x_n	y_n
1	1.05	2.17369
2	1.10	2.34506
3	1.15	2.51452
4	1.20	2.68235
5	1.25	2.84880
6	1.30	3.01404
7	1.35	3.17823
8	1.40	3.34151
9	1.45	3.50396
10	1.50	3.66568

Consequently the Runge-Kutta approximation to $y(1.5)$ is $y_{10} = 3.66568$. Comparing this to the corresponding Euler approximation from Problem 59 we have

$$|y_{\text{RK}} - y_{\text{E}}| = |3.66568 - 3.67185| = 0.00617.$$

Chapter 2 Solutions

Solutions to Section 2.1

True-False Review:

(a): TRUE. A diagonal matrix has no entries below the main diagonal, so it is upper triangular. Likewise, it has no entries above the main diagonal, so it is also lower triangular.

(b): FALSE. An $m \times n$ matrix has m row vectors and n column vectors.

(c): TRUE. This is a square matrix, and all entries off the main diagonal are zero, so it is a diagonal matrix (the entries *on* the diagonal also happen to be zero, but this is not required).

(d): FALSE. The main diagonal entries of a skew-symmetric matrix must be zero. In this case, $a_{11} = 4 \neq 0$, so this matrix is not skew-symmetric.

(e): FALSE. The form presented uses the same number along the entire main diagonal, but a symmetric matrix need not have identical entries on the main diagonal.

(f): TRUE. Since A is symmetric, $A = A^T$. Thus, $(A^T)^T = A = A^T$, so A^T is symmetric.

(g): FALSE. The trace of a matrix is the *sum* of the entries along the main diagonal.

(h): TRUE. If A is skew-symmetric, then $A^T = -A$. But A and A^T contain the same entries along the main diagonal, so for $A^T = -A$, both A and $-A$ must have the same main diagonal. This is only possible if all entries along the main diagonal are 0.

(i): TRUE. If A is both symmetric and skew-symmetric, then $A = A^T = -A$, and $A = -A$ is only possible if all entries of A are zero.

(j): TRUE. Both matrix functions are defined for values of t such that $t > 0$.

(k): FALSE. The $(3, 2)$-entry contains a function that is not defined for values of t with $t \leq 3$. So for example, this matrix functions is not defined for $t = 2$.

(l): TRUE. Each numerical entry of the matrix function is a constant function, which has domain $\mathbb{R}$.

(m): FALSE. For instance, the matrix function $A(t) = [t]$ and $B(t) = [t^2]$ satisfy $A(0) = B(0)$, but A and B are not the same matrix function.

Problems:

1(a). $a_{31} = 0, a_{24} = -1, a_{14} = 2, a_{32} = 2, a_{21} = 7, a_{34} = 4$.

1(b). $(1, 4)$ and $(3, 2)$.

2(a). $b_{12} = -1$, $b_{33} = 4$, $b_{41} = 0$, $b_{43} = 8$, $b_{51} = -1$, and $b_{52} = 9$.

2(b). $(1, 2)$, $(1, 3)$, $(2, 1)$, $(3, 2)$, and $(5, 1)$.

3. $\begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix}$; 2×2 matrix.

4. $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 4 & -2 \end{bmatrix}$; 2×3 matrix.

5. $\begin{bmatrix} -1 \\ 1 \\ 1 \\ -5 \end{bmatrix}$; 4×1 matrix.

6. $\begin{bmatrix} 1 & -3 & -2 \\ 3 & 6 & 0 \\ 2 & 7 & 4 \\ -4 & -1 & 5 \end{bmatrix}$; 4×3 matrix.

7. $\begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$; 3×3 matrix.

8. $\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \\ 3 & 2 & 1 & -0 \end{bmatrix}$; 4×4 matrix.

9. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$; 4×4 matrix.

10. $\text{tr}(A) = 1 + 3 = 4$.

11. $\text{tr}(A) = 1 + 2 + (-3) = 0$.

12. $\text{tr}(A) = 2 + 2 + (-5) = -1$.

13. Column vectors: $\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}$.
Row vectors: $[1 \ \ -1], [3 \ \ 5]$.

14. Column vectors: $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 7 \end{bmatrix}$.
Row vectors: $[1 \ \ 3 \ \ -4], [-1 \ \ -2 \ \ 5], [2 \ \ 6 \ \ 7]$.

15. Column vectors: $\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 10 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ 3 \end{bmatrix}$. Row vectors: $[2 \ \ 10 \ \ 6], [5 \ \ -1 \ \ 3]$.

16. $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 1 \end{bmatrix}$. Column vectors: $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$.

17. $A = \begin{bmatrix} -2 & 0 & 4 & -1 & -1 \\ 9 & -4 & -4 & 0 & 8 \end{bmatrix}$; column vectors: $\begin{bmatrix} -2 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \end{bmatrix}$.

18. $B = \begin{bmatrix} -2 & -4 \\ -6 & -6 \\ 3 & 0 \\ -1 & 0 \\ -2 & 1 \end{bmatrix}$; row vectors: $\begin{bmatrix} -2 & -4 \end{bmatrix}, \begin{bmatrix} -6 & -6 \end{bmatrix}, \begin{bmatrix} 3 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 1 \end{bmatrix}$.

19. $B = \begin{bmatrix} 2 & 5 & 0 & 1 \\ -1 & 7 & 0 & 2 \\ 4 & -6 & 0 & 3 \end{bmatrix}$. Row vectors: $[2 \ \ 5 \ \ 0 \ \ 1], [-1 \ \ 7 \ \ 0 \ \ 2], [4 \ \ -6 \ \ 0 \ \ 3]$.

20. $A = [\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p]$ has p columns and each column q-vector has q rows, so the resulting matrix has dimensions $q \times p$.

21. One example: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

22. One example: $\begin{bmatrix} 2 & 3 & 1 & 2 \\ 0 & 5 & 6 & 2 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

23. One example: $\begin{bmatrix} 1 & 3 & -1 & 2 \\ -3 & 0 & 4 & -3 \\ 1 & -4 & 0 & 1 \\ -2 & 3 & -1 & 0 \end{bmatrix}$.

24. One example: $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

25. The only possibility here is the zero matrix: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

26. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

27. One example: $\begin{bmatrix} t^2 - t & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

28. One example: $\begin{bmatrix} \frac{1}{\sqrt{3-t}} & \sqrt{t+2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

29. One example: $\begin{bmatrix} \frac{1}{t^2+1} \\ 0 \end{bmatrix}$.

30. One example: $\begin{bmatrix} t^2+1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

31. One example: Let A and B be 1×1 matrix functions given by

$$A(t) = [t] \qquad \text{and} \qquad B(t) = [t^2].$$

32. Let A be a symmetric upper triangular matrix. Then all elements below the main diagonal are zeros. Consequently, since A is symmetric, all elements above the main diagonal must also be zero. Hence, the only nonzero entries can occur along the main diagonal. That is, A is a diagonal matrix.

33. Since A is skew-symmetric, we know that $a_{ij} = -a_{ji}$ for all (i, j). But since A is symmetric, we know that $a_{ij} = a_{ji}$ for all (i, j). Thus, for all (i, j), we must have $-a_{ji} = a_{ji}$. That is, $a_{ji} = 0$ for all (i, j). That is, every element of A is zero.

Solutions to Section 2.2

True-False Review:

(a): FALSE. The correct statement is $(AB)C = A(BC)$, the associative law. A counterexample to the particular statement given in this review item can be found in Problem 5.

(b): TRUE. Multiplying from left to right, we note that AB is an $m \times p$ matrix, and right multiplying AB by the $p \times q$ matrix C, we see that ABC is an $m \times q$ matrix.

(c): TRUE. We have $(A + B)^T = A^T + B^T = A + B$, so $A + B$ is symmetric.

(d): FALSE. For example, let $A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix}$. Then A and B are skew-symmetric, but $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$ is not symmetric.

(e): FALSE. The correct equation is $(A+B)^2 = A^2+AB+BA+B^2$. The statement is false since $AB+BA$ does not necessarily equal $2AB$. For instance, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $(A+B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq (A+B)^2$.

(f): FALSE. For example, let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = 0$ even though $A \neq 0$ and $B \neq 0$.

(g): FALSE. For example, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then A is not upper triangular, despite the fact that AB is the zero matrix, hence automatically upper triangular.

(h): FALSE. For instance, the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is neither the zero matrix nor the identity matrix, and yet $A^2 = A$.

(i): TRUE. The derivative of each entry of the matrix is zero, since in each entry, we take the derivative of a constant, thus obtaining zero for each entry of the derivative of the matrix.

(j): FALSE. The correct statement is given in Problem 45. The problem with the statement as given is that the second term should be $\frac{dA}{dt}B$, not $B\frac{dA}{dt}$.

(k): FALSE. For instance, the matrix function $A = \begin{bmatrix} 2e^t & 0 \\ 0 & 3e^t \end{bmatrix}$ satisfies $A = \frac{dA}{dt}$, but A does not have the form $\begin{bmatrix} ce^t & 0 \\ 0 & ce^t \end{bmatrix}$.

(l): TRUE. This follows by exactly the same proof as given in the text for matrices of numbers (see part 3 of Theorem 2.2.23).

Problems:

1(a). $5A = \begin{bmatrix} -10 & 30 & 5 \\ -5 & 0 & -15 \end{bmatrix}$.

1(b). $-3B = \begin{bmatrix} -6 & -3 & 3 \\ 0 & -12 & 12 \end{bmatrix}$.

1(c). $iC = \begin{bmatrix} -1+i & -1+2i \\ -1+3i & -1+4i \\ -1+5i & -1+6i \end{bmatrix}$.

1(d). $2A - B = \begin{bmatrix} -6 & 11 & 3 \\ -2 & -4 & -2 \end{bmatrix}$.

1(e). $A + 3C^T = \begin{bmatrix} 1+3i & 15+3i & 16+3i \\ 5+3i & 12+3i & 15+3i \end{bmatrix}$.

1(f). $3D - 2E = \begin{bmatrix} 8 & 10 & 7 \\ 1 & 4 & 9 \\ 1 & 7 & 12 \end{bmatrix}$.

1(g). $D + E + F = \begin{bmatrix} 12 & -3-3i & -1+i \\ 3+i & 3-2i & 8 \\ 6 & 4+2i & 2 \end{bmatrix}$.

1(h). Solving for G and simplifying, we have that

$$G = -\frac{3}{2}A - B = \begin{bmatrix} 1 & -10 & -1/2 \\ 3/2 & -4 & 17/2 \end{bmatrix}.$$

1(i). Solving for H and simplifying, we have that $H = 4E - D - 2F =$

$$\begin{bmatrix} 8 & -20 & -8 \\ 4 & 4 & 12 \\ 16 & -8 & -12 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 1 \\ 1 & 2 & 5 \\ 3 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 12 & 4-6i & 2i \\ 2+2i & -4i & 0 \\ -2 & 10+4i & 6 \end{bmatrix} = \begin{bmatrix} -8 & -24+6i & -9-2i \\ 1-2i & 2+4i & 7 \\ 15 & -19-4i & -20 \end{bmatrix}.$$

1(j). We have $K^T = 2B - 3A$, so that $K = (2B - 3A)^T = 2B^T - 3A^T$. Thus,

$$K = 2\begin{bmatrix} 2 & 0 \\ 1 & 4 \\ -1 & -4 \end{bmatrix} - 3\begin{bmatrix} -2 & -1 \\ 6 & 0 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 10 & 3 \\ -16 & 8 \\ -5 & 1 \end{bmatrix}.$$

2(a). $-D = \begin{bmatrix} -4 & 0 & -1 \\ -1 & -2 & -5 \\ -3 & -1 & -2 \end{bmatrix}$.

2(b). $4B^T = 4\begin{bmatrix} 2 & 0 \\ 1 & 4 \\ -1 & -4 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 4 & 16 \\ -4 & -16 \end{bmatrix}$.

2(c). $-2A^T + C = -2\begin{bmatrix} -2 & -1 \\ 6 & 0 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} 1+i & 2+i \\ 3+i & 4+i \\ 5+i & 6+i \end{bmatrix} = \begin{bmatrix} 5+i & 4+i \\ -9+i & 4+i \\ 3+i & 12+i \end{bmatrix}$.

2(d). $5E + D = \begin{bmatrix} 10 & -25 & -10 \\ 5 & 5 & 15 \\ 20 & -10 & -15 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 1 \\ 1 & 2 & 5 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 14 & -25 & -9 \\ 6 & 7 & 20 \\ 23 & -9 & -13 \end{bmatrix}$.

2(e). We have

$$4A^T - 2B^T + iC = 4\begin{bmatrix} -2 & -1 \\ 6 & 0 \\ 1 & -3 \end{bmatrix} - 2\begin{bmatrix} 2 & 0 \\ 1 & 4 \\ -1 & -4 \end{bmatrix} + i\begin{bmatrix} 1+i & 2+i \\ 3+i & 4+i \\ 5+i & 6+i \end{bmatrix} = \begin{bmatrix} -13+i & -5+2i \\ 21+3i & -9+4i \\ 5+5i & -5+6i \end{bmatrix}.$$

2(f). We have

$$4E - 3D^T = \begin{bmatrix} 8 & -20 & -8 \\ 4 & 4 & 12 \\ 16 & -8 & -12 \end{bmatrix} - \begin{bmatrix} 12 & 3 & 9 \\ 0 & 6 & 3 \\ 3 & 15 & 6 \end{bmatrix} = \begin{bmatrix} -4 & -23 & -17 \\ 4 & -2 & 9 \\ 13 & -23 & -18 \end{bmatrix}.$$

2(g). We have $(1-6i)F+iD=$

$$\begin{bmatrix} 6-36i & -16-15i & 6+i \\ 7-5i & -12-2i & 0 \\ -1+6i & 17-28i & 3-18i \end{bmatrix} + \begin{bmatrix} 4i & 0 & i \\ i & 2i & 5i \\ 3i & i & 2i \end{bmatrix} = \begin{bmatrix} 6-32i & -16-15i & 6+2i \\ 7-4i & -12 & 5i \\ -1+9i & 17-27i & 3-16i \end{bmatrix}.$$

2(h). Solving for G, we have

$$\begin{aligned} G = A+(1-i)C^T &= \begin{bmatrix} -2 & 6 & 1 \\ -1 & 0 & -3 \end{bmatrix} + (1-i)\begin{bmatrix} 1+i & 3+i & 5+i \\ 2+i & 4+i & 6+i \end{bmatrix} \\ &= \begin{bmatrix} -2 & 6 & 1 \\ -1 & 0 & -3 \end{bmatrix} + \begin{bmatrix} 2 & 4-2i & 6-4i \\ 3-i & 5-3i & 7-5i \end{bmatrix} \\ &= \begin{bmatrix} 0 & 10-2i & 7-4i \\ 2-i & 5-3i & 4-5i \end{bmatrix}. \end{aligned}$$

2(i). Solve for H, we have

$$\begin{aligned} H &= \frac{3}{2}D - \frac{3}{2}E + 3I_3 \\ &= \begin{bmatrix} 6 & 0 & 3/2 \\ 3/2 & 3 & 15/2 \\ 9/2 & 3/2 & 3 \end{bmatrix} - \begin{bmatrix} 3 & -15/2 & -3 \\ 3/2 & 3/2 & 9/2 \\ 6 & -3 & -9/2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 15/2 & 9/2 \\ 0 & 9/2 & 3 \\ -3/2 & 9/2 & 21/2 \end{bmatrix}. \end{aligned}$$

2(j). We have $K^T = D^T + E^T - F^T = (D+E-F)^T$, so that

$$K = D+E-F = \begin{bmatrix} 0 & -7+3i & -1-i \\ 1-i & 3+2i & 8 \\ 8 & -6-2i & -4 \end{bmatrix}.$$

3(a).

$$AB = \begin{bmatrix} 5 & 10 & -3 \\ 27 & 22 & 3 \end{bmatrix}$$

3(b).

$$BC = \begin{bmatrix} 9 \\ 8 \\ -6 \end{bmatrix}$$

3(c). CA cannot be computed.

3(d).

$$A^T E = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} = \begin{bmatrix} 2-4i & 7+13i \\ -2 & 1+3i \\ 4-6i & 10+18i \end{bmatrix}$$

3(e).

$$CD = \begin{bmatrix} 2 & -2 & 3 \\ -2 & 2 & -3 \\ 4 & -4 & 6 \end{bmatrix}.$$

3(f).

$$C^T A^T = \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 6 & 10 \end{bmatrix}$$

3(g).

$$F^2 = \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix} \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix} = \begin{bmatrix} -1 & 10-10i \\ 0 & 15+8i \end{bmatrix}$$

3(h).

$$BD^T = \begin{bmatrix} 2 & -1 & 3 \\ 5 & 1 & 2 \\ 4 & 6 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 15 \\ 14 \\ -10 \end{bmatrix}$$

3(i).

$$A^T A = \begin{bmatrix} 1 & 3 \\ -1 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 2 & 14 \\ 2 & 2 & 2 \\ 14 & 2 & 20 \end{bmatrix}$$

3(j).

$$FE = \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix} \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} = \begin{bmatrix} -2+i & 13-i \\ 1-4i & 4+18i \end{bmatrix}$$

4(a).

$$AC = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

4(b).

$$DC = [10]$$

4(c).

$$DB = [6 \;\; 14 \;\; -4]$$

4(d). AD cannot be computed.

4(e). $EF = \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} \begin{bmatrix} i & 1-3i \\ 0 & 4+i \end{bmatrix} = \begin{bmatrix} 1+2i & 2-2i \\ 1 & 1+17i \end{bmatrix}.$

4(f). Since A^T is a 3×2 matrix and B is a 3×3 matrix, the product $A^T B$ cannot be constructed.

4(g). Since C is a 3×1 matrix, it is impossible to form the product $C \cdot C = C^2$.

4(h). $E^2 = \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} \begin{bmatrix} 2-i & 1+i \\ -i & 2+4i \end{bmatrix} = \begin{bmatrix} 4-5i & 1+7i \\ 3-4i & -11+15i \end{bmatrix}.$

4(i). $AD^T = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 16 \end{bmatrix}$.

4(j). $E^T A = \begin{bmatrix} 2-i & -i \\ 1+i & 2+4i \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 2-4i & -2 & 4-6i \\ 7+13i & 1+3i & 10+18i \end{bmatrix}$.

5. We have

$$\begin{aligned} ABC = (AB)C &= \left(\begin{bmatrix} -3 & 2 & 7 & -1 \\ 6 & 0 & -3 & -5 \end{bmatrix} \begin{bmatrix} -2 & 8 \\ 8 & -3 \\ -1 & -9 \\ 0 & 2 \end{bmatrix} \right) \begin{bmatrix} -6 & 1 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 15 & -95 \\ -9 & 65 \end{bmatrix} \begin{bmatrix} -6 & 1 \\ 1 & 5 \end{bmatrix} \\ &= \begin{bmatrix} -185 & -460 \\ 119 & 316 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} CAB = C(AB) &= \begin{bmatrix} -6 & 1 \\ 1 & 5 \end{bmatrix} \left(\begin{bmatrix} -3 & 2 & 7 & -1 \\ 6 & 0 & -3 & -5 \end{bmatrix} \begin{bmatrix} -2 & 8 \\ 8 & -3 \\ -1 & -9 \\ 0 & 2 \end{bmatrix} \right) \\ &= \begin{bmatrix} -6 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 15 & -95 \\ -9 & 65 \end{bmatrix} \\ &= \begin{bmatrix} -99 & 635 \\ -30 & 230 \end{bmatrix}. \end{aligned}$$

6.

$$A\mathbf{c} = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ -2 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ -5 \end{bmatrix} + (-2) \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ -38 \end{bmatrix}.$$

7.

$$A\mathbf{c} = \begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 5 \\ 7 & -6 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = 2 \begin{bmatrix} 3 \\ 2 \\ 7 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 1 \\ -6 \end{bmatrix} + (-4) \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -13 \\ -13 \\ -16 \end{bmatrix}.$$

8.

$$A\mathbf{c} = \begin{bmatrix} -1 & 2 \\ 4 & 7 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 4 \\ 5 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} -7 \\ 13 \\ 29 \end{bmatrix}.$$

9. We have

$$A\mathbf{c} = x \begin{bmatrix} a \\ e \end{bmatrix} + y \begin{bmatrix} b \\ f \end{bmatrix} + z \begin{bmatrix} c \\ g \end{bmatrix} + w \begin{bmatrix} d \\ h \end{bmatrix} = \begin{bmatrix} xa + yb + zc + wd \\ xe + yf + zg + wh \end{bmatrix}.$$

10(a). The dimensions of B should be $n \times r$ in order that ABC is defined.

10(b). The elements of the ith row of A are $a_{i1}, a_{i2}, \dots, a_{in}$ and the elements of the jth column of BC are

$$\sum_{m=1}^{r} b_{1m} c_{mj}, \quad \sum_{m=1}^{r} b_{2m} c_{mj}, \quad \dots \quad , \sum_{m=1}^{r} b_{nm} c_{mj},$$

so the element in the ith row and jth column of $ABC = A(BC)$ is

$$a_{i1}\sum_{m=1}^{r} b_{1m}c_{mj} + a_{i2}\sum_{m=1}^{r} b_{2m}c_{mj} + \cdots + a_{in}\sum_{m=1}^{r} b_{nm}c_{mj}$$
$$= \sum_{k=1}^{n} a_{ik}\left(\sum_{m=1}^{r} b_{km}c_{mj}\right) = \sum_{k=1}^{n}\left(\sum_{m=1}^{r} a_{ik}b_{km}\right)c_{mj}.$$

11(a).

$$A^2 = AA = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}.$$
$$A^3 = A^2A = \begin{bmatrix} -1 & -4 \\ 8 & 7 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}.$$
$$A^4 = A^3A = \begin{bmatrix} -9 & -11 \\ 22 & 13 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -31 & -24 \\ 48 & 17 \end{bmatrix}.$$

11(b).

$$A^2 = AA = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & -1 \end{bmatrix}.$$
$$A^3 = A^2A = \begin{bmatrix} -2 & 0 & 1 \\ 4 & -3 & 0 \\ 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -3 & 0 \\ 6 & 4 & -3 \\ -12 & 3 & 4 \end{bmatrix}.$$
$$A^4 = A^3A = \begin{bmatrix} 4 & -3 & 0 \\ 6 & 4 & -3 \\ -12 & 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ 4 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -3 \\ -20 & 9 & 4 \\ 10 & -16 & 3 \end{bmatrix}.$$

12(a). We apply the distributive property of matrix multiplication as follows:

$$(A+2B)^2 = (A+2B)(A+2B) = A(A+2B)+(2B)(A+2B) = (A^2+A(2B))+((2B)A+(2B)^2) = A^2+2AB+2BA+4B^2,$$

where scalar factors of 2 are moved in front of the terms since they commute with matrix multiplication.

12(b). We apply the distributive property of matrix multiplication as follows:

$$\begin{aligned}(A+B+C)^2 = (A+B+C)(A+B+C) &= A(A+B+C) + B(A+B+C) + C(A+B+C) \\ &= A^2 + AB + AC + BA + B^2 + BC + CA + CB + C^2 \\ &= A^2 + B^2 + C^2 + AB + BA + AC + CA + BC + CB,\end{aligned}$$

as required.

12(c). We can use the formula for $(A+B)^3$ found in Example 2.2.20 and substitute $-B$ for B throughout the expression:

$$\begin{aligned}(A-B)^3 &= A^3 + A(-B)A + (-B)A^2 + (-B)^2A + A^2(-B) + A(-B)^2 + (-B)A(-B) + (-B)^3 \\ &= A^3 - ABA - BA^2 + B^2A - A^2B + AB^2 + BAB - B^3,\end{aligned}$$

as needed.

13. We have

$$A^2 = \begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ 6 & -6 \end{bmatrix} = \begin{bmatrix} -26 & 20 \\ -24 & 6 \end{bmatrix},$$

so that

$$A^2 + 4A + 18I_2 = \begin{bmatrix} -26 & 20 \\ -24 & 6 \end{bmatrix} + \begin{bmatrix} 8 & -20 \\ 24 & -24 \end{bmatrix} + \begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

14. We have

$$A^2 = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -7 & 12 & -4 \\ -4 & 7 & 6 \\ 5 & 3 & -2 \end{bmatrix}$$

and

$$A^3 = \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} -7 & 12 & -4 \\ -4 & 7 & 6 \\ 5 & 3 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 27 & 0 & -4 \\ -1 & 25 & -2 \\ 2 & -3 & 26 \end{bmatrix}.$$

Therefore, we have

$$A^3 + A - 26I_3 = \begin{bmatrix} 27 & 0 & -4 \\ -1 & 25 & -2 \\ 2 & -3 & 26 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 4 \\ 1 & 1 & 2 \\ -2 & 3 & 0 \end{bmatrix} - \begin{bmatrix} 26 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 26 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

15.

$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Substituting $A = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}$ for A, we have

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

that is,

$$\begin{bmatrix} 1 & 2x & 2z + xy \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since corresponding elements of equal matrices are equal, we obtain the following implications:

$$\begin{aligned} 2y = 1 &\Longrightarrow y = 1/2, \\ 2x = 1 &\Longrightarrow x = 1/2, \\ 2z + xy = 0 &\Longrightarrow 2z + (1/2)(1/2) = 0 \Longrightarrow z = -1/8. \end{aligned}$$

Thus, $A = \begin{bmatrix} 1 & 1/2 & -1/8 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$.

16. In order that $A^2 = A$, we require $\begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}\begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix} = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}$, that is, $\begin{bmatrix} x^2-2 & x+y \\ -2x-2y & -2+y^2 \end{bmatrix} = \begin{bmatrix} x & 1 \\ -2 & y \end{bmatrix}$, or equivalently, $\begin{bmatrix} x^2-x-2 & x+y-1 \\ -2x-2y+2 & y^2-y-2 \end{bmatrix} = 0_2$. Since corresponding elements of equal matrices are equal, it follows that

$$\begin{aligned} &x^2 - x - 2 = 0 \Longrightarrow x = -1 \text{ or } x = 2, \text{ and} \\ &y^2 - y - 2 = 0 \Longrightarrow y = -1 \text{ or } y = 2. \end{aligned}$$

Two cases arise from $x + y - 1 = 0$:
(a): If $x = -1$, then $y = 2$.
(b): If $x = 2$, then $y = -1$. Thus,

$$A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}.$$

17.

$$\sigma_1\sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\sigma_3.$$

$$\sigma_2\sigma_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = i\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = i\sigma_1.$$

$$\sigma_3\sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = i\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = i\sigma_2.$$

18.

$$\begin{aligned} [A, B] &= AB - BA \\ &= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 \\ 10 & 4 \end{bmatrix} - \begin{bmatrix} 5 & -2 \\ 8 & -2 \end{bmatrix} \\ &= \begin{bmatrix} -6 & 1 \\ 2 & 6 \end{bmatrix} \neq 0_2. \end{aligned}$$

19.

$$\begin{aligned} [A_1, A_2] &= A_1A_2 - A_2A_1 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0_2, \quad \text{thus } A_1 \text{ and } A_2 \text{ commute.} \end{aligned}$$

$$\begin{aligned} [A_1, A_3] &= A_1A_3 - A_3A_1 \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0_2, \quad \text{thus } A_1 \text{ and } A_3 \text{ commute.} \end{aligned}$$

$$
\begin{aligned}
[A_2, A_3] &= A_2A_3 - A_3A_2 \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \neq 0_2.
\end{aligned}
$$

Then $[A_3, A_2] = -[A_2, A_3] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \neq 0_2$. Thus, A_2 and A_3 do not commute.

20.

$$
\begin{aligned}
[A_1, A_2] &= A_1A_2 - A_2A_1 \\
&= \frac{1}{4}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \frac{1}{4}\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = A_3.
\end{aligned}
$$

$$
\begin{aligned}
[A_2, A_3] &= A_2A_3 - A_3A_2 \\
&= \frac{1}{4}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \frac{1}{4}\begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = A_1.
\end{aligned}
$$

$$
\begin{aligned}
[A_3, A_1] &= A_3A_1 - A_1A_3 \\
&= \frac{1}{4}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A_2.
\end{aligned}
$$

21.

$$
\begin{aligned}
&[A, [B, C]] + [B, [C, A]] + [C, [A, B]] \\
&= [A, BC - CB] + [B, CA - AC] + [C, AB - BA] \\
&= A(BC - CB) - (BC - CB)A + B(CA - AC) - (CA - AC)B + C(AB - BA) - (AB - BA)C \\
&= ABC - ACB - BCA + CBA + BCA - BAC - CAB + ACB + CAB - CBA - ABC + BAC \quad = 0.
\end{aligned}
$$

22.

Proof that $A(BC) = (AB)C$: Let $A = [a_{ij}]$ be of size $m \times n$, $B = [b_{jk}]$ be of size $n \times p$, and $C = [c_{kl}]$ be of size $p \times q$. Consider the (i, j)-element of $(AB)C$:

$$
[(AB)C]_{ij} = \sum_{k=1}^{p}\left(\sum_{h=1}^{n} a_{ih}b_{hk}\right)c_{kj} = \sum_{h=1}^{n} a_{ih}\left(\sum_{k=1}^{p} b_{hk}c_{kj}\right) = [A(BC)]_{ij}.
$$

Proof that $A(B+C) = AB + AC$**:** We have

$$\begin{aligned}[A(B+C)]_{ij} &= \sum_{k=1}^{n} a_{ik}(b_{kj}+c_{kj}) \\ &= \sum_{k=1}^{n}(a_{ij}b_{kj}+a_{ik}c_{kj}) \\ &= \sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} \\ &= [AB+AC]_{ij}.\end{aligned}$$

23.
Proof that $(A^T)^T = A$**:** Let $A = [a_{ij}]$. Then $A^T = [a_{ji}]$, so $(A^T)^T = [a_{ji}]^T = a_{ij} = A$, as needed.
Proof that $(A+C)^T = A^T + C^T$**:** Let $A = [a_{ij}]$ and $C = [c_{ij}]$. Then $[(A+C)^T]_{ij} = [A+C]_{ji} = [A]_{ji} + [C]_{ji} = a_{ji} + c_{ji} = [A^T]_{ij} + [C^T]_{ij} = [A^T + C^T]_{ij}$. Hence, $(A+C)^T = A^T + C^T$.

24. We have

$$(IA)_{ij} = \sum_{k=1}^{m} \delta_{ik}a_{kj} = \delta_{ii}a_{ij} = a_{ij},$$

for $1 \le i \le m$ and $1 \le j \le p$. Thus, $I_m A_{m\times p} = A_{m\times p}$.

25. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n\times n$ matrices. Then

$$\mathrm{tr}(AB) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} a_{ki}b_{ik}\right) = \sum_{k=1}^{n}\left(\sum_{i=1}^{n} b_{ik}a_{ki}\right) = \sum_{i=1}^{n}\left(\sum_{k=1}^{n} b_{ik}a_{ki}\right) = \mathrm{tr}(BA).$$

26(a). $B^TA^T = \begin{bmatrix} 0 & -7 & -1 \\ -4 & 1 & -3 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}.$

26(b). $C^TB^T = \begin{bmatrix} -9 & 1 \\ 0 & 1 \\ 3 & 5 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} 0 & -7 & -1 \\ -4 & 1 & -3 \end{bmatrix} = \begin{bmatrix} -4 & 64 & 6 \\ -4 & 1 & -3 \\ -20 & -16 & -18 \\ 8 & 12 & 8 \end{bmatrix}.$

26(c). Since D^T is a 3×3 matrix and A is a 1×3 matrix, it is not possible to compute the expression D^TA.

27(a). $AD^T = \begin{bmatrix} -3 & -1 & 6 \end{bmatrix}\begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -2 \\ 5 & 7 & -1 \end{bmatrix} = \begin{bmatrix} 35 & 42 & -7 \end{bmatrix}.$

27(b). First note that $C^TC = \begin{bmatrix} -9 & 1 \\ 0 & 1 \\ 3 & 5 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} -9 & 0 & 3 & -2 \\ 1 & 1 & 5 & -2 \end{bmatrix} = \begin{bmatrix} 82 & 1 & -22 & 16 \\ 1 & 1 & 5 & -2 \\ -22 & 5 & 34 & -16 \\ 16 & -2 & -16 & 8 \end{bmatrix}$. Therefore,

$$(C^TC)^2 = \begin{bmatrix} 82 & 1 & -22 & 16 \\ 1 & 1 & 5 & -2 \\ -22 & 5 & 34 & -16 \\ 16 & -2 & -16 & 8 \end{bmatrix}\begin{bmatrix} 82 & 1 & -22 & 16 \\ 1 & 1 & 5 & -2 \\ -22 & 5 & 34 & -16 \\ 16 & -2 & -16 & 8 \end{bmatrix} = \begin{bmatrix} 7465 & -59 & -2803 & 1790 \\ -59 & 31 & 185 & -82 \\ -2803 & 185 & 1921 & -1034 \\ 1790 & -82 & -1034 & 580 \end{bmatrix}.$$

27(c). $D^T B = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 0 & -2 \\ 5 & 7 & -1 \end{bmatrix} \begin{bmatrix} 0 & -4 \\ -7 & 1 \\ -1 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 2 & 2 \\ -48 & -10 \end{bmatrix}$.

28(a). We have

$$S = [\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3] = \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix},$$

so

$$AS = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -x & -y & z \\ 0 & y & 2z \\ x & -y & z \end{bmatrix} = \begin{bmatrix} -x & -y & 7z \\ 0 & y & 14z \\ x & -y & 7z \end{bmatrix} = [\mathbf{s}_1, \mathbf{s}_2, 7\mathbf{s}_3].$$

28(b).

$$S^T AS = S^T(AS) = \begin{bmatrix} -x & 0 & x \\ -y & y & -y \\ z & 2z & z \end{bmatrix} \begin{bmatrix} -x & -y & 7z \\ 0 & y & 14z \\ x & -y & 7z \end{bmatrix} = \begin{bmatrix} 2x^2 & 0 & 0 \\ 0 & 3y^2 & 0 \\ 0 & 0 & 42z^2 \end{bmatrix},$$

but $S^T AS = \text{diag}(1, 1, 7)$, so we have the following

$$2x^2 = 1 \Longrightarrow x = \pm\frac{\sqrt{2}}{2}$$
$$3y^2 = 1 \Longrightarrow y = \pm\frac{\sqrt{3}}{3}$$
$$6z^2 = 1 \Longrightarrow z = \pm\frac{\sqrt{6}}{6}.$$

29(a). We have

$$\begin{aligned} AS &= \begin{bmatrix} 1 & -4 & 0 \\ -4 & 7 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 2x & y \\ 0 & x & -2y \\ z & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -2x & 9y \\ 0 & -x & -18y \\ 5z & 0 & 0 \end{bmatrix} \\ &= [5\mathbf{s}_1, -\mathbf{s}_2, 9\mathbf{s}_3]. \end{aligned}$$

29(b). We have

$$S^T AS = \begin{bmatrix} 0 & 0 & z \\ 2x & x & 0 \\ y & -2y & 0 \end{bmatrix} \begin{bmatrix} 0 & -2x & 9y \\ 0 & -x & -18y \\ 5z & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5z^2 & 0 & 0 \\ 0 & -5x^2 & 0 \\ 0 & 0 & 45y^2 \end{bmatrix},$$

so in order for this to be equal to $\text{diag}(5, -1, 9)$, we must have

$$5z^2 = 5, \qquad -5x^2 = -1, \qquad 45y^2 = 9.$$

Thus, we must have $z^2 = 1$, $x^2 = \frac{1}{5}$, and $y^2 = \frac{1}{5}$. Therefore, the values of x, y, and z that we are looking for are $x = \pm\sqrt{\frac{1}{5}}$, $y = \pm\sqrt{\frac{1}{5}}$, and $z = \pm 1$.

30(a). $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

30(b). $\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$.

31. Suppose A is an $n \times n$ scalar matrix with trace k. If $A = aI_n$, then $\text{tr}(A) = na = k$, so we conclude that $a = k/n$. So $A = \frac{k}{n} I_n$, a uniquely determined matrix.

32. We have

$$B^T = \left[\frac{1}{2}(A + A^T)\right]^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A^T + A) = B$$

and

$$C^T = \left[\frac{1}{2}(A - A^T)\right]^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -C.$$

Thus, B is symmetric and C is skew-symmetric.

33. We have

$$B + C = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T) = \frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T = A.$$

34. We have

$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 & 0 \\ 9 & -2 & 3 \\ 2 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 4 & 9 & 2 \\ -1 & -2 & 5 \\ 0 & 3 & 5 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 8 & 8 & 2 \\ 8 & -4 & 8 \\ 2 & 8 & 10 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 1 \\ 4 & -2 & 4 \\ 1 & 4 & 5 \end{bmatrix}$$

and

$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}\left(\begin{bmatrix} 4 & -1 & 0 \\ 9 & -2 & 3 \\ 2 & 5 & 5 \end{bmatrix} - \begin{bmatrix} 4 & 9 & 2 \\ -1 & -2 & 5 \\ 0 & 3 & 5 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & -10 & -2 \\ 10 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -5 & -1 \\ 5 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}.$$

35.

$$B = \frac{1}{2}\left(\begin{bmatrix} 1 & -5 & 3 \\ 3 & 2 & 4 \\ 7 & -2 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 7 \\ -5 & 2 & -2 \\ 3 & 4 & 6 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & -2 & 10 \\ -2 & 4 & 2 \\ 10 & 2 & 12 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 6 \end{bmatrix}.$$

$$C = \frac{1}{2}\left(\begin{bmatrix} 1 & -5 & 3 \\ 3 & 2 & 4 \\ 7 & -2 & 6 \end{bmatrix} - \begin{bmatrix} 1 & 3 & 7 \\ -5 & 2 & -2 \\ 3 & 4 & 6 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 0 & -8 & -4 \\ 8 & 0 & 6 \\ 4 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 \\ 4 & 0 & 3 \\ 2 & -3 & 0 \end{bmatrix}.$$

36(a). If A is symmetric, then $A^T = A$, so that

$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}(A + A) = \frac{1}{2}(2A) = A$$

and

$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}(A - A) = \frac{1}{2}(0_n) = 0_n.$$

36(b). If A is skew-symmetric, then $A^T = -A$, so that

$$B = \frac{1}{2}(A + A^T) = \frac{1}{2}(A + (-A)) = \frac{1}{2}(0_n) = 0_n$$

and

$$C = \frac{1}{2}(A - A^T) = \frac{1}{2}(A - (-A)) = \frac{1}{2}(2A) = A.$$

37. If $A = [a_{ij}]$ and $D = \text{diag}(d_1, d_2, \dots, d_n)$, then we must show that the (i, j)-entry of DA is $d_i a_{ij}$. In index notation, we have

$$(DA)_{ij} = \sum_{k=1}^{n} d_i \delta_{ik} a_{kj} = d_i \delta_{ii} a_{ij} = d_i a_{ij}.$$

Hence, DA is the matrix obtained by multiplying the ith row vector of A by d_i, where $1 \leq i \leq n$.

38. If $A = [a_{ij}]$ and $D = \text{diag}(d_1, d_2, \dots, d_n)$, then we must show that the (i, j)-entry of AD is $d_j a_{ij}$. In index notation, we have

$$(AD)_{ij} = \sum_{k=1}^{n} a_{ik} d_j \delta_{kj} = a_{ij} d_j \delta_{jj} = a_{ij} d_j.$$

Hence, AD is the matrix obtained by multiplying the jth column vector of A by d_j, where $1 \leq j \leq n$.

39. Since A and B are symmetric, we have that $A^T = A$ and $B^T = B$. Using properties of the transpose operation, we therefore have

$$(AB)^T = B^T A^T = BA = AB,$$

and this shows that AB is symmetric.

40(a). We have $(AA^T)^T = (A^T)^T A^T = AA^T$, so that AA^T is symmetric.

40(b). We have $(ABC)^T = [(AB)C]^T = C^T (AB)^T = C^T (B^T A^T) = C^T B^T A^T$, as needed.

41. $A'(t) = \begin{bmatrix} 1 & \cos t \\ -\sin t & 4 \end{bmatrix}$.

42. $A'(t) = \begin{bmatrix} -2e^{-2t} \\ \cos t \end{bmatrix}$.

43. $A'(t) = \begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 1 \\ 0 & 3 & 0 \end{bmatrix}$.

44. $A'(t) = \begin{bmatrix} e^t & 2e^{2t} & 2t \\ 2e^t & 8e^{2t} & 10t \end{bmatrix}$.

45. We show that the (i, j)-entry of both sides of the equation agree. First, recall that the (i, j)-entry of AB is $\sum_{k=1}^{n} a_{ik} b_{kj}$, and therefore, the (i, j)-entry of $\frac{d}{dt}(AB)$ is (by the product rule)

$$\sum_{k=1}^{n} a'_{ik} b_{kj} + a_{ik} b'_{kj} = \sum_{k=1}^{n} a'_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} b'_{kj}.$$

The former term is precise the (i, j)-entry of the matrix $\frac{dA}{dt} B$, while the latter term is precise the (i, j)-entry of the matrix $A \frac{dB}{dt}$. Thus, the (i, j)-entry of $\frac{d}{dt}(AB)$ is precisely the sum of the (i, j)-entry of $\frac{dA}{dt} B$ and the (i, j)-entry of $A \frac{dB}{dt}$. Thus, the equation we are proving follows immediately.

46. We have

$$\int_0^1 \begin{bmatrix} e^t & e^{-t} \\ 2e^t & 5e^{-t} \end{bmatrix} dt = \begin{bmatrix} e^t & -e^{-t} \\ 2e^t & -5e^{-t} \end{bmatrix} \Big|_0^1 = \begin{bmatrix} e & -1/e \\ 2e & -5/e \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} e-1 & 1-1/e \\ 2e-2 & 5-5/e \end{bmatrix}.$$

47. We have

$$\int_0^{\pi/2} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} dt = \begin{bmatrix} \sin t \\ -\cos t \end{bmatrix} \Big|_0^{\pi/2} = \begin{bmatrix} \sin(\pi/2) \\ -\cos(\pi/2) \end{bmatrix} - \begin{bmatrix} \sin 0 \\ -\cos 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

48. We have

$$\int_0^1 \begin{bmatrix} e^t & e^{2t} & t^2 \\ 2e^t & 4e^{2t} & 5t^2 \end{bmatrix} dt = = \begin{bmatrix} e^t & \frac{1}{2}e^{2t} & \frac{t^3}{3} \\ 2e^t & 2e^{2t} & \frac{5}{3}t^3 \end{bmatrix} \Big|_0^1$$

$$= \begin{bmatrix} e & e^2/2 & 1/3 \\ 2e & 2e^2 & 5/3 \end{bmatrix} - \begin{bmatrix} 1 & 1/2 & 0 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} e-1 & \frac{e^2-1}{2} & 1/3 \\ 2e-2 & 2e^2-2 & 5/3 \end{bmatrix}.$$

49. We have

$$\int_0^1 \begin{bmatrix} e^{2t} & \sin 2t \\ t^2-5 & te^t \\ \sec^2 t & 3t-\sin t \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} & -\frac{1}{2}\cos 2t \\ \frac{t^3}{3}-5t & te^t-e^t \\ \tan t & \frac{3}{2}t^2+\cos t \end{bmatrix} \Big|_0^1$$

$$= \begin{bmatrix} \frac{e^2}{2} & -\frac{\cos 2}{2} \\ -14/3 & 0 \\ \tan 1 & \frac{3}{2}+\cos 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{e^2-1}{2} & \frac{1-\cos 2}{2} \\ -14/3 & 1 \\ \tan 1 & \frac{1}{2}+\cos 1 \end{bmatrix}.$$

50. $\int A(t)dt = \begin{bmatrix} \int -5dt & \int \frac{1}{t^2+1}dt & \int e^{3t}dt \end{bmatrix} = \begin{bmatrix} -5t & \tan^{-1}(t) & \frac{1}{3}e^{3t} \end{bmatrix}$.

51. $\int \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix} dt = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$.

52. $\int \begin{bmatrix} \sin t & \cos t & 0 \\ -\cos t & \sin t & t \\ 0 & 3t & 1 \end{bmatrix} dt = \begin{bmatrix} -\cos t & \sin t & 0 \\ -\sin t & -\cos t & t^2/2 \\ 0 & 3t^2/2 & t \end{bmatrix}$.

53. $\int \begin{bmatrix} e^t & e^{-t} \\ 2e^t & 5e^{-t} \end{bmatrix} dt = \begin{bmatrix} e^t & -e^{-t} \\ 2e^t & -5e^{-t} \end{bmatrix}$.

54. $\int \begin{bmatrix} e^{2t} & \sin 2t \\ t^2-5 & te^t \\ \sec^2 t & 3t-\sin t \end{bmatrix} dt = \begin{bmatrix} \frac{1}{2}e^{2t} & -\frac{1}{2}\cos 2t \\ \frac{t^3}{3}-5t & te^t-e^t \\ \tan t & \frac{3}{2}t^2+\cos t \end{bmatrix}$.

Solutions to Section 2.3

True-False Review:

(a): FALSE. The last column of the augmented matrix corresponds to the constants on the right-hand side of the linear system, so if the augmented matrix has n columns, there are only $n-1$ unknowns under consideration in the system.

(b): FALSE. Three distinct planes can intersect in a line (e.g. Figure 2.3.1, lower right picture). For instance, the xy-plane, the xz-plane, and the plane $y = z$ intersect in the x-axis.

(c): FALSE. The right-hand side vector must have m components, not n components.

(d): TRUE. If a linear system has two distinct solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, then any point on the line containing $\mathbf{x}_1$ and $\mathbf{x}_2$ is also a solution, giving us infinitely many solutions, not exactly two solutions.

(e): TRUE. The augmented matrix for a linear system has one additional column (containing the constants on the right-hand side of the equation) beyond the matrix of coefficients.

(f): FALSE. Because the vector $(x_1, x_2, x_3, 0, 0)$ has five entries, this vector belongs to $\mathbb{R}^5$. Vectors in $\mathbb{R}^3$ can only have three slots.

(g): FALSE. The two column vectors given have different numbers of components, so they are not the same vectors.

Problems:

1.

$$2 \cdot 1 - 3(-1) + 4 \cdot 2 = 13,$$
$$1 + (-1) - 2 = -2,$$
$$5 \cdot 1 + 4(-1) + 2 = 3.$$

2.

$$2 + (-3) - 2 \cdot 1 = -3,$$
$$3 \cdot 2 - (-3) - 7 \cdot 1 = 2,$$
$$2 + (-3) + 1 = 0,$$
$$2 \cdot 2 + 2(-3) - 4 \cdot 1 = -6.$$

3.

$$(1 - t) + (2 + 3t) + (3 - 2t) = 6,$$
$$(1 - t) - (2 + 3t) - 2(3 - 2t) = -7,$$
$$5(1 - t) + (2 + 3t) - (3 - 2t) = 4.$$

4.

$$s + (s - 2t) - (2s + 3t) + 5t = 0,$$
$$2(s - 2t) - (2s + 3t) + 7t = 0,$$
$$4s + 2(s - 2t) - 3(2s + 3t) + 13t = 0.$$

5. The two given lines are the same line. Therefore, since this line contains an infinite number of points, there must be an infinite number of solutions to this linear system.

6. These two lines are parallel and distinct, and therefore, there are no common points on these lines. In other words, there are no solutions to this linear system.

7. These two lines have different slopes, and therefore, they will intersect in exactly one point. Thus, this system of equations has exactly one solution.

8. The first and third equations describe lines that are parallel and distinct, and therefore, there are no common points on these lines. In other words, there are no solutions to this linear system.

9. $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -5 \\ 7 & 2 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $A^{\#} = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & -5 & 2 \\ 7 & 2 & -1 & 3 \end{array}\right]$.

10. $A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 2 & 4 & -3 & 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $A^{\#} = \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 3 \\ 2 & 4 & -3 & 7 & 2 \end{array}\right]$.

11. $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 5 & 6 & -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $A^{\#} = \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 3 & -2 & 0 \\ 5 & 6 & -5 & 0 \end{array}\right]$.

12. It is acceptable to use any variable names. We will use x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 - x_2 \quad +2x_3 + 3x_4 &= \ \ 1, \\ x_1 + x_2 \quad -2x_3 + 6x_4 &= -1, \\ 3x_1 + x_2 \quad +4x_3 + 2x_4 &= \ \ 2. \end{aligned}$$

13. It is acceptable to use any variable names. We will use x_1, x_2, x_3:

$$\begin{aligned} 2x_1 + \ \ x_2 \quad +3x_3 &= \ \ 3, \\ 4x_1 - \ \ x_2 \quad +2x_3 &= \ \ 1, \\ 7x_1 + 6x_2 \quad +3x_3 &= -5. \end{aligned}$$

14. The system of equations here only contains one equation: $4x_1 - 2x_2 - 2x_3 - 3x_5 = -9$.

15. This system of equations has three equations: $-3x_2 = -1, \quad 2x_1 - 7x_2 = 6, \quad 5x_1 + 5x_2 = 7$.

16. Given $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$, and an arbitrary constant c,

(a). we have

$$A\mathbf{z} = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0} + \mathbf{0} = \mathbf{0}$$

and

$$A\mathbf{w} = A(c\mathbf{x}) = c(A\mathbf{x}) = c\mathbf{0} = \mathbf{0}.$$

(b). No, because

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b},$$

and

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c\mathbf{b} \neq \mathbf{b}$$

in general.

17. $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4t \\ t^2 \end{bmatrix}$.

18. $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} t^2 & -t \\ -\sin t & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

19. $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & e^{2t} \\ -\sin t & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

20. $\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & -\sin t & 1 \\ -e^t & 0 & t^2 \\ -t & t^2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} t \\ t^3 \\ 1 \end{bmatrix}$.

21. We have

$$\mathbf{x}'(t) = \begin{bmatrix} 4e^{4t} \\ -2(4e^{4t}) \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix}$$

and

$$A\mathbf{x} + \mathbf{b} = \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} e^{4t} \\ -2e^{4t} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{4t} + (-1)(-2e^{4t}) + 0 \\ -2e^{4t} + 3(-2e^{4t}) + 0 \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ -8e^{4t} \end{bmatrix}.$$

22. We have

$$\mathbf{x}'(t) = \begin{bmatrix} 4(-2e^{-2t}) + 2\cos t \\ 3(-2e^{-2t}) + \sin t \end{bmatrix} = \begin{bmatrix} -8e^{-2t} + 2\cos t \\ -6e^{-2t} + \sin t \end{bmatrix}$$

and

$$\begin{aligned} A\mathbf{x} + \mathbf{b} &= \begin{bmatrix} 1 & -4 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4e^{-2t} + 2\sin t \\ 3e^{-2t} - \cos t \end{bmatrix} + \begin{bmatrix} -2(\cos t + \sin t) \\ 7\sin t + 2\cos t \end{bmatrix} \\ &= \begin{bmatrix} 4e^{-2t} + 2\sin t - 4(3e^{-2t} - \cos t) - 2(\cos t + \sin t) \\ -3(4e^{-2t} + 2\sin t) + 2(3e^{-2t} - \cos t) + 7\sin t + 2\cos t \end{bmatrix} = \begin{bmatrix} -8e^{-2t} + 2\cos t \\ -6e^{-2t} + \sin t \end{bmatrix}. \end{aligned}$$

23. We compute

$$\mathbf{x}' = \begin{bmatrix} 3e^t + 2te^t \\ e^t + 2te^t \end{bmatrix}$$

and

$$A\mathbf{x} + \mathbf{b} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2te^t + e^t \\ 2te^t - e^t \end{bmatrix} + \begin{bmatrix} 0 \\ 4e^t \end{bmatrix} = \begin{bmatrix} 2(2te^t + e^t) - (2te^t - e^t) + 0 \\ -(2te^t + e^t) + 2(2te^t - e^t) + 4e^t \end{bmatrix} = \begin{bmatrix} 2te^t + 3e^t \\ 2te^t + e^t \end{bmatrix}.$$

Therefore, we see from these calculations that $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$.

24. We compute

$$\mathbf{x}' = \begin{bmatrix} -te^t - e^t \\ -9e^{-t} \\ te^t + e^t - 6e^{-t} \end{bmatrix}$$

and

$$A\mathbf{x}+\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 2 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -te^t \\ 9e^{-t} \\ te^t + 6e^{-t} \end{bmatrix} + \begin{bmatrix} -e^t \\ 6e^{-t} \\ e^t \end{bmatrix} = \begin{bmatrix} -te^t \\ 2(-te^t) - 3(9e^{-t}) + 2(te^t + 6e^{-t}) \\ -te^t - 2(9e^{-t}) + 2(te^t + 6e^{-t}) \end{bmatrix} + \begin{bmatrix} -e^t \\ 6e^{-t} \\ e^t \end{bmatrix} = \begin{bmatrix} -te^t - e^t \\ -9e^{-t} \\ te^t + e^t - 6e^{-t} \end{bmatrix}$$

Therefore, we see from these calculations that $\mathbf{x}' = A\mathbf{x} + \mathbf{b}$.

Solutions to Section 2.4

True-False Review:

(a): TRUE. The precise row-echelon form obtained for a matrix depends on the particular elementary row operations (and their order). However, Theorem 2.4.15 states that there is a unique reduced row-echelon form for a matrix.

(b): FALSE. Upper triangular matrices could have pivot entries that are not 1. For instance, the following matrix is upper triangular, but not in row echelon form: $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

(c): TRUE. The pivots in a row-echelon form of an $n \times n$ matrix must move down and to the right as we look from one row to the next beneath it. Thus, the pivots must occur on or to the right of the main diagonal of the matrix, and thus all entries below the main diagonal of the matrix are zero.

(d): FALSE. This would not be true, for example, if A was a zero matrix with 5 rows and B was a nonzero matrix with 4 rows.

(e): FALSE. If A is a nonzero matrix and $B = -A$, then $A + B = 0$, so $\text{rank}(A + B) = 0$, but $\text{rank}(A)$, $\text{rank}(B) \geq 1$ so $\text{rank}(A) + \text{rank}(B) \geq 2$.

(f): FALSE. For example, if $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AB = 0$, so $\text{rank}(AB) = 0$, but $\text{rank}(A) + \text{rank}(B) = 1 + 1 = 2$.

(g): TRUE. A matrix of rank zero cannot have any pivots, hence no nonzero rows. It must be the zero matrix.

(h): TRUE. The matrices A and $2A$ have the same reduced row-echelon form, since we can move between the two matrices by multiplying the rows of one of them by 2 or 1/2, a matter of carrying out elementary row operations. If the two matrices have the same reduced row-echelon form, then they have the same rank.

(i): TRUE. The matrices A and $2A$ have the same reduced row-echelon form, since we can move between the two matrices by multiplying the rows of one of them by 2 or 1/2, a matter of carrying out elementary row operations.

Problems:

1. Neither.

2. Reduced row-echelon form.

3. Neither.

4. Row-echelon form.

5. Row-echelon form.

6. Reduced row-echelon form.

7. Reduced row-echelon form.

8. Reduced row-echelon form.

9.

$$\begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \text{Rank } (A) = 1.$$

$$\boxed{\textbf{1. } \mathrm{M}_1(\tfrac{1}{2}) \quad \textbf{2. } \mathrm{A}_{12}(4)}$$

10.

$$\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \stackrel{1}{\sim} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} \stackrel{2}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 7 \end{bmatrix} \stackrel{3}{\sim} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}, \text{Rank } (A) = 2.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(-2) \quad \textbf{3. } \mathrm{M}_2(\tfrac{1}{7})}$$

11.

$$\begin{bmatrix} 0 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 3 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. $A_{12}(-1)$, $A_{13}(-3)$ **2.** $A_{23}(-4)$

12.

$$\begin{bmatrix} 2 & 1 & 4 \\ 2 & -3 & 4 \\ 3 & -2 & 6 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & -2 & 6 \\ 2 & -3 & 4 \\ 2 & 1 & 4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 2 & -3 & 4 \\ 2 & 1 & 4 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -5 & 0 \\ 0 & -1 & 0 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & -5 & 0 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & -5 & 0 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. P_{13} **2.** $A_{21}(-1)$ **3.** $A_{12}(-2)$, $A_{13}(-3)$ **4.** P_{23} **5.** $M_2(-1)$ **6.** $A_{32}(5)$

13.

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & 1 & -2 \\ 2 & -1 & 3 \\ 2 & -2 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 2 & -1 & 3 \\ 0 & -1 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 13 \\ 0 & -1 & -2 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & -1 & -2 \\ 0 & -5 & 13 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & -5 & 13 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 23 \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \text{Rank }(A) = 3.$$

1. P_{12} **2.** $A_{21}(-1)$, $A_{23}(-1)$ **3.** $A_{12}(-2)$ **4.** P_{23} **5.** $M_2(-1)$ **6.** $A_{23}(5)$ **7.** $M_3(1/23)$.

14.

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \\ 2 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & 2 \\ 2 & -1 \\ 2 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 2 & 5 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & -7 \\ 0 & -1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 0 & -7 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. P_{12} **2.** $A_{21}(-1)$ **3.** $A_{12}(-2)$, $A_{13}(-2)$ **4.** P_{23} **5.** $M_2(-1)$, $A_{23}(7)$.

15.

$$\begin{bmatrix} 2 & -2 & -1 & 3 \\ 3 & -2 & 3 & 1 \\ 1 & -1 & 1 & 0 \\ 2 & -1 & 2 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 3 & -2 & 3 & 1 \\ 2 & -2 & -1 & 3 \\ 2 & -1 & 2 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 1 & 0 & 2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{Rank }(A) = 4.$$

$$\boxed{\textbf{1. } \mathrm{P}_{13} \quad \textbf{2. } \mathrm{A}_{12}(-3), \mathrm{A}_{13}(-2), \mathrm{A}_{14}(-2) \quad \textbf{3. } \mathrm{A}_{24}(-1) \quad \textbf{4. } \mathrm{M}_{3}(1/3)}$$

16.

$$\begin{bmatrix} 2 & -1 & 3 & 4 \\ 1 & -2 & 1 & 3 \\ 1 & -5 & 0 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 2 & -1 & 3 & 4 \\ 1 & -5 & 0 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 1 & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(-2), \mathrm{A}_{13}(-1) \quad \textbf{3. } \mathrm{M}_{2}(1/3)}$$

17.

$$\begin{bmatrix} 2 & 1 & 3 & 4 & 2 \\ 1 & 0 & 2 & 1 & 3 \\ 2 & 3 & 1 & 5 & 7 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 2 & 1 & 3 & 4 & 2 \\ 2 & 3 & 1 & 5 & 7 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 & -4 \\ 0 & 3 & -3 & 3 & 1 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 & -4 \\ 0 & 0 & 0 & -3 & 1 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & 0 & 2 & 1 & 3 \\ 0 & 1 & -1 & 2 & -4 \\ 0 & 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}, \text{Rank } (A) = 3.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(-2), \mathrm{A}_{13}(-2), \quad \textbf{3. } \mathrm{A}_{23}(-3) \quad \textbf{4. } \mathrm{M}_{3}(-\tfrac{1}{3})}$$

18.

$$\begin{bmatrix} 4 & 7 & 4 & 7 \\ 3 & 5 & 3 & 5 \\ 2 & -2 & 2 & -2 \\ 5 & -2 & 5 & -2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 5 & 3 & 5 \\ 2 & -2 & 2 & -2 \\ 5 & -2 & 5 & -2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & -1 & 0 & -1 \\ 0 & -6 & 0 & -6 \\ 0 & -12 & 0 & -12 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & -6 & 0 & -6 \\ 0 & -12 & 0 & -12 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Rank } (A) = 2.$$

$$\boxed{\textbf{1. } \mathrm{A}_{21}(-1) \quad \textbf{2. } \mathrm{A}_{12}(-3), \mathrm{A}_{13}(-2), \mathrm{A}_{14}(-5) \quad \textbf{3. } \mathrm{M}_{2}(-1) \quad \textbf{4. } \mathrm{A}_{23}(6), \mathrm{A}_{24}(12)}$$

19.

$$\begin{bmatrix} -4 & 2 \\ -6 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1/2 \\ -6 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}, \quad \text{Rank}(A) = 1.$$

$$\boxed{\textbf{1. } \mathrm{M}_{1}(-\tfrac{1}{4}) \quad \textbf{2. } \mathrm{A}_{12}(6)}$$

20.

$$\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \text{Rank } (A) = 2.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(-3) \quad \textbf{3. } \mathrm{M}_{2}(\tfrac{1}{5}) \quad \textbf{4. } \mathrm{A}_{21}(1)}$$

21.

$$\begin{bmatrix} 3 & 7 & 10 \\ 2 & 3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 3 & 7 & 10 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & 7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 7 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3, \text{Rank }(A) = 3.$$

1. P_{13} **2.** $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{13}(-3)$ **3.** $\mathrm{M}_2(-1)$ **4.** $\mathrm{A}_{21}(-2)$, $\mathrm{A}_{23}(-1)$ **5.** $\mathrm{M}_3(\frac{1}{4})$ **6.** $\mathrm{A}_{31}(5)$, $\mathrm{A}_{32}(-3)$

22.

$$\begin{bmatrix} 3 & -3 & 6 \\ 2 & -2 & 4 \\ 6 & -6 & 12 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank }(A) = 1.$$

1. $\mathrm{M}_1(\frac{1}{3})$, $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{13}(-6)$

23.

$$\begin{bmatrix} 3 & 5 & -12 \\ 2 & 3 & -7 \\ -2 & -1 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & -1 & 3 \\ 0 & 3 & -9 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & -3 \\ 0 & 3 & -9 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. $\mathrm{A}_{21}(-1)$, $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{13}(2)$ **2.** $\mathrm{M}_2(-1)$ **3.** $\mathrm{A}_{21}(-2)$, $\mathrm{A}_{23}(-3)$

24.

$$\begin{bmatrix} 1 & -1 & -1 & 2 \\ 3 & -2 & 0 & 7 \\ 2 & -1 & 2 & 4 \\ 4 & -2 & 3 & 8 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 4 & 0 \\ 0 & 2 & 7 & 0 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4, \text{Rank }(A) = 4.$$

1. $\mathrm{A}_{12}(-3)$, $\mathrm{A}_{13}(-2)$, $\mathrm{A}_{14}(-4)$ **2.** $\mathrm{A}_{21}(1)$, $\mathrm{A}_{23}(-1)$, $\mathrm{A}_{24}(-2)$ **3.** $\mathrm{A}_{31}(-2)$, $\mathrm{A}_{32}(-3)$, $\mathrm{A}_{34}(-1)$
4. $\mathrm{M}_4(-1)$ **5.** $\mathrm{A}_{41}(-5)$, $\mathrm{A}_{42}(-4)$, $\mathrm{A}_{43}(1)$

25.

$$\begin{bmatrix} 1 & -2 & 1 & 3 \\ 3 & -6 & 2 & 7 \\ 4 & -8 & 3 & 10 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \text{Rank }(A) = 2.$$

1. $\mathrm{A}_{12}(-3)$, $\mathrm{A}_{13}(-4)$ **2.** $\mathrm{M}_2(-1)$ **3.** $\mathrm{A}_{21}(-1)$, $\mathrm{A}_{23}(1)$

26.

$$\begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & -6 & -2 \\ 0 & 0 & -4 & -1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & -4 & -1 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{Rank } (A) = 3.$$

1. $A_{12}(-3)$, $A_{13}(-2)$ **2.** $M_2(-\frac{1}{6})$ **3.** $A_{21}(-2)$, $A_{23}(4)$ **4.** $M_3(3)$ **5.** $A_{32}(-\frac{1}{3})$, $A_{31}(-\frac{1}{3})$

Solutions to Section 2.5

True-False Review:

(a): FALSE. This process is known as Gaussian elimination. Gauss-Jordan elimination is the process by which a matrix is brought to *reduced* row echelon form via elementary row operations.

(b): TRUE. A homogeneous linear system always has the trivial solution $\mathbf{x} = \mathbf{0}$, hence it is consistent.

(c): TRUE. The columns of the row-echelon form that contain leading 1s correspond to leading variables, while columns of the row-echelon form that do not contain leading 1s correspond to free variables.

(d): TRUE. If the last column of the row-reduced augmented matrix for the system does not contain a pivot, then the system can be solved by back-substitution. On the other hand, if this column does contain a pivot, then that row of the row-reduced matrix containing the pivot in the last column corresponds to the impossible equation $0 = 1$.

(e): FALSE. The linear system $x = 0$, $y = 0$, $z = 0$ has a solution in $(0, 0, 0)$ even though none of the variables here is free.

(f): FALSE. The columns containing the leading 1s correspond to the *leading* variables, not the free variables.

Problems:

For the problems of this section, A will denote the coefficient matrix of the given system, and $A^{\#}$ will denote the augmented matrix of the given system.

1. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cc|c} 1 & -5 & 3 \\ 3 & -9 & 15 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 6 & 6 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & -5 & 3 \\ 0 & 1 & 1 \end{array}\right].$$

1. $A_{12}(-3)$ **2.** $M_2(\frac{1}{6})$

By back substitution, we find that $x_2 = 1$, and then $x_1 = 8$. Therefore, the solution is $(8, 1)$.

2. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cc|c} 4 & -1 & 8 \\ 2 & 1 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 2 \\ 2 & 1 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 2 \\ 0 & \frac{3}{2} & -3 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 2 \\ 0 & 1 & -2 \end{array}\right].$$

$$\boxed{\textbf{1.}\ \mathrm{M}_1(\tfrac{1}{4}) \quad \textbf{2.}\ \mathrm{A}_{12}(-2) \quad \textbf{3.}\ \mathrm{M}_2(\tfrac{2}{3})}$$

By back substitution, we find that $x_2 = -2$, and then $x_1 = \dfrac{3}{2}$. Therefore, the solution is $(\frac{3}{2}, -2)$.

3. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cc|c} 7 & -3 & 5 \\ 14 & -6 & 10 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{cc|c} 7 & -3 & 5 \\ 0 & 0 & 0 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{3}{7} & \frac{5}{7} \\ 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1.}\ \mathrm{A}_{12}(-2) \quad \textbf{2.}\ \mathrm{M}_2(\tfrac{1}{7})}$$

Observe that x_2 is a free variable, so we set $x_2 = t$. Then by back substitution, we have $x_1 = \frac{3}{7}t + \frac{5}{7}$. Therefore, the solution set to this system is

$$\left\{ \left(\frac{3}{7}t + \frac{5}{7}, t \right) : t \in \mathbb{R} \right\}.$$

4. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 2 & 5 & -1 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 5 & -1 \end{array}\right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

$$\boxed{\textbf{1.}\ \mathrm{A}_{12}(-3),\ \mathrm{A}_{13}(-2) \quad \textbf{2.}\ \mathrm{M}_2(-1) \quad \textbf{3.}\ \mathrm{A}_{23}(-2)}$$

The last augmented matrix results in the system:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1, \\ x_2 + 2x_3 &= 0, \\ x_3 &= -1. \end{aligned}$$

By back substitution we obtain the solution $(-2, 2, -1)$.

5. Converting the given system of equations to an augmented matrix and using Gaussian elimination, we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 3 & -1 & 0 & 1 \\ 2 & 1 & 5 & 4 \\ 7 & -5 & -8 & -3 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 2 & 1 & 5 & 4 \\ 7 & -5 & -8 & -3 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 0 & 5 & 15 & 10 \\ 0 & 9 & 27 & 18 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & -5 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 9 & 27 & 18 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1.}\ \mathrm{A}_{21}(-1) \quad \textbf{2.}\ \mathrm{A}_{12}(-2),\ \mathrm{A}_{13}(-7) \quad \textbf{3.}\ \mathrm{M}_2(\tfrac{1}{5}) \quad \textbf{4.}\ \mathrm{A}_{21}(2),\ \mathrm{A}_{23}(-9)}$$

The last augmented matrix results in the system:

$$\begin{aligned} x_1 \qquad + \;\; x_3 &= 1, \\ x_2 + 3x_3 &= 2. \end{aligned}$$

Let the free variable $x_3 = t$, a real number. By back substitution we find that the system has the solution set $\{(1-t, 2-3t, t) : \text{ for all real numbers } t\}$.

6. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrr|r} 3 & 5 & -1 & 14 \\ 1 & 2 & 1 & 3 \\ 2 & 5 & 6 & 2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 3 & 5 & -1 & 4 \\ 2 & 5 & 6 & 2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & -1 & -4 & -5 \\ 0 & 1 & 4 & -4 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 1 & 4 & -4 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & -9 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right].$$

1. P_{12} **2.** $\text{A}_{12}(-3)$, $\text{A}_{13}(-2)$ **3.** $\text{M}_2(-1)$ **4.** $\text{A}_{23}(-1)$ **5.** $\text{M}_4(-\frac{1}{9})$

This system of equations is inconsistent since $2 = \text{rank}(A) < \text{rank}(A^{\#}) = 3$.

7. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrr|r} 6 & -3 & 3 & 12 \\ 2 & -1 & 1 & 4 \\ -4 & 2 & -2 & -8 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{1}{2} & 2 \\ 2 & -1 & 1 & 4 \\ -4 & 2 & -2 & -8 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & \frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $\text{M}_1(\frac{1}{6})$ **2.** $\text{A}_{12}(-2)$, $\text{A}_{13}(4)$

Since x_2 and x_3 are free variables, let $x_2 = s$ and $x_3 = t$. The single equation obtained from the augmented matrix is given by $x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 = 2$. Thus, the solution set of our system is given by

$$\{(2 + \frac{s}{2} - \frac{t}{2}, s, t) : s, t \text{ any real numbers }\}.$$

8. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrr|r} 2 & -1 & 3 & 14 \\ 3 & 1 & -2 & -1 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 3 & 1 & -2 & -1 \\ 2 & -1 & 3 & 14 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & -5 & -15 \\ 2 & -1 & 3 & -14 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & -5 & -15 \\ 0 & -5 & 13 & 44 \\ 0 & -12 & 32 & 108 \\ 0 & -11 & 23 & 80 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & -5 & -15 \\ 0 & -12 & 32 & 108 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & -5 & -15 \\ 0 & -1 & 9 & 28 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & -5 & 13 & 44 \\ 0 & -11 & 23 & 80 \end{array}\right]$$

$$\overset{7}{\sim}\left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & 0 & -32 & -96 \\ 0 & 0 & -76 & -228 \end{array}\right] \overset{8}{\sim}\left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & 0 & 32 & 96 \\ 0 & 0 & -76 & -228 \end{array}\right] \overset{9}{\sim}\left[\begin{array}{ccc|c} 1 & 2 & -5 & -15 \\ 0 & 1 & -9 & -28 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{12} **2.** $A_{21}(-1)$ **3.** $A_{12}(-2)$, $A_{13}(-7)$, $A_{14}(-5)$ **4.** P_{23}
5. $A_{42}(-1)$ **6.** $M_2(-1)$ **7.** $A_{23}(5)$, $A_{24}(11)$ **8.** $M_3(-1)$ **9.** $M_3(\frac{1}{32})$, $A_{34}(76)$.

The last augmented matrix results in the system of equations:

$$\begin{aligned} x_1 - 2x_2 - 5x_3 &= -15, \\ x_2 - 9x_3 &= -28, \\ x_3 &= \quad 3. \end{aligned}$$

Thus, using back substitution, the solution set for our system is given by $\{(2, -1, 3)\}$.

9. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 2 & -1 & -4 & 5 \\ 3 & 2 & -5 & 8 \\ 5 & 6 & -6 & 20 \\ 1 & 1 & -3 & -3 \end{array}\right] \overset{1}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & -3 & -3 \\ 3 & 2 & -5 & 8 \\ 5 & 6 & -6 & 20 \\ 2 & -1 & -4 & -5 \end{array}\right] \overset{2}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & -3 & -3 \\ 0 & -1 & 4 & 17 \\ 0 & 1 & 9 & 35 \\ 0 & -3 & 2 & 11 \end{array}\right] \overset{3}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & -3 & -3 \\ 0 & 1 & -4 & -17 \\ 0 & 1 & 9 & 35 \\ 0 & -3 & 2 & 11 \end{array}\right]$$

$$\overset{4}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & -3 & -3 \\ 0 & 1 & -4 & -17 \\ 0 & 0 & 13 & 52 \\ 0 & 0 & -10 & -40 \end{array}\right] \overset{5}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & -3 & -3 \\ 0 & 1 & -4 & -17 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -10 & -40 \end{array}\right] \overset{6}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & -3 & -3 \\ 0 & 1 & -4 & -17 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{14} **2.** $A_{12}(-3)$, $A_{13}(-5)$, $A_{14}(-2)$ **3.** $M_2(-1)$ **4.** $A_{23}(-1)$, $A_{24}(3)$ **5.** $M_3(\frac{1}{13})$ **6.** $A_{34}(10)$

The last augmented matrix results in the system of equations:

$$\begin{aligned} x_1 + x_2 - 3x_3 &= -\ 3, \\ x_2 - 4x_3 &= -17, \\ x_3 &= \quad 4. \end{aligned}$$

By back substitution, we obtain the solution set $\{(10, -1, 4)\}$.

10. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 2 & 4 & -2 & 2 & 2 \\ 5 & 10 & -5 & 5 & 5 \end{array}\right] \overset{1}{\sim}\left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-2)$, $A_{13}(-5)$

The last augmented matrix results in the equation $x_1 + 2x_3 - x_3 + x_4 = 1$. Now x_2, x_3, and x_4 are free variables, so we let $x_2 = r$, $x_3 = s$, and $x_4 = t$. It follows that $x_1 = 1 - 2r + s - t$. Consequently, the solution set of the system is given by $\{(1 - 2r + s - t, r, s, t) : r, s, t \text{ and real numbers }\}$.

11. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 2 & -3 & 1 & -1 & 2 \\ 1 & -5 & 2 & -2 & 1 \\ 4 & 1 & -1 & 1 & 3 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & -7 & 3 & -3 & 0 \\ 0 & -7 & 3 & -3 & 0 \\ 0 & -7 & 3 & -3 & -1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & -7 & 3 & -3 & 0 \\ 0 & -7 & 3 & -3 & -1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrrr|r} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -\frac{3}{7} & \frac{3}{7} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-2)$, $A_{13}(-1)$, $A_{14}(-4)$ **2.** $M_2(-\frac{1}{7})$ **3.** $A_{23}(7)$, $A_{24}(7)$ **4.** P_{34} **5.** $M_3(-1)$

The given system of equations is inconsistent since $2 = \text{rank}(A) < \text{rank}(A^{\#}) = 3$.

12. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 4 & -3 & 2 \\ 2 & 4 & -1 & -10 & 5 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & -3 & -12 & 9 & -6 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrrrr|r} 1 & 2 & 1 & 1 & -2 & 3 \\ 0 & 0 & 1 & 4 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{13}(-2)$ **2.** $A_{23}(3)$

The last augmented matrix indicates that the first two equations of the initial system completely determine its solution. We see that x_4 and x_5 are free variables, so let $x_4 = s$ and $x_5 = t$. Then $x_3 = 2 - 4x_4 + 3x_5 = 2-4s+3t$. Moreover, x_2 is a free variable, say $x_2 = r$, so then $x_1 = 3-2r-(2-4s+3t)-s+2t = 1-2r+3s-t$. Hence, the solution set for the system is

$$\{(1 - 2r + 3s - t, r, 2 - 4s + 3t, s, t) : r, s, t \text{ any real numbers } \}.$$

13. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrr|r} 2 & -1 & -2 & 2 \\ 4 & 3 & -2 & -1 \\ 1 & 4 & 1 & 4 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 4 & 3 & -2 & -1 \\ 2 & -1 & -1 & 2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & -13 & -6 & -17 \\ 0 & -9 & -3 & -6 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & -9 & -3 & -6 \\ 0 & -13 & -6 & -17 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & 12 & 4 & 8 \\ 0 & -13 & -6 & -17 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & 12 & 4 & 8 \\ 0 & -1 & -2 & -9 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & -1 & -2 & -9 \\ 0 & 12 & 4 & 8 \end{array}\right] \overset{7}{\sim} \left[\begin{array}{rrr|r} 1 & 4 & 1 & 4 \\ 0 & 1 & 2 & 9 \\ 0 & 12 & 4 & 8 \end{array}\right]$$

$$\overset{8}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & -7 & -32 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & -20 & -100 \end{array}\right] \overset{9}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & -7 & -32 \\ 0 & 1 & 2 & 9 \\ 0 & 0 & 1 & 5 \end{array}\right] \overset{10}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right].$$

1. P_{13}	**2.** $A_{12}(-4)$, $A_{13}(-2)$	**3.** P_{23}	**4.** $M_2(-\frac{4}{3})$	**5.** $A_{23}(1)$
6. P_{23}	**7.** $M_2(-1)$	**8.** $A_{21}(-4)$, $A_{23}(-12)$	**9.** $M_3(-\frac{1}{20})$	**10.** $A_{31}(7)$, $A_{32}(-2)$

The last augmented matrix results in the solution $(3, -1, 5)$.

14. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 3 & 1 & 5 & 2 \\ 1 & 1 & -1 & 1 \\ 2 & 1 & 2 & 3 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 3 & 1 & 5 & 2 \\ 2 & 1 & 2 & 3 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -2 & 8 & -1 \\ 0 & -1 & 4 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & \frac{1}{2} \\ 0 & -1 & 4 & 1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -4 & 1/2 \\ 0 & 0 & 0 & 3/2 \end{array}\right].$$

We can stop here, since we see from this last augmented matrix that the system is inconsistent. In particular, $2 = \text{rank}(A) < \text{rank}(A^{\#}) = 3$.

1. P_{12}	**2.** $A_{12}(-3)$, $A_{13}(-2)$	**3.** $M_2(-\frac{1}{2})$	**4.** $A_{23}(1)$

15. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 3 & -2 & 4 & -9 \\ 1 & -4 & 2 & -3 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & -2 & 2 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & -4 & 4 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

.

1. $A_{12}(-3)$, $A_{13}(-1)$	**2.** $M_2(-\frac{1}{2})$	**3.** $A_{23}(4)$

The last augmented matrix results in the following system of equations:

$$x_1 - 2x_3 = -3 \quad \text{and} \quad x_2 - x_3 = 0.$$

Since x_3 is free, let $x_3 = t$. Thus, from the system we obtain the solutions $\{(2t-3, t, t) : t \text{ any real number }\}$.

16. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 2 & -1 & 3 & -1 & 3 \\ 3 & 2 & 1 & -5 & -6 \\ 1 & -2 & 3 & 1 & 6 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 3 & 2 & 1 & -5 & -6 \\ 2 & -1 & 3 & -1 & 3 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 0 & 8 & -8 & -8 & -24 \\ 0 & 3 & -3 & -3 & -9 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 3 & 1 & 6 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 3 & -3 & -3 & -9 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{13}	**2.** $A_{12}(-3)$, $A_{13}(-2)$	**3.** $M_2(\frac{1}{8})$	**4.** $A_{21}(2)$, $A_{23}(-3)$

The last augmented matrix results in the following system of equations:

$$x_1 + x_3 - x_4 = 0 \quad \text{and} \quad x_2 - x_3 - x_4 = -3.$$

Since x_3 and x_4 are free variables, we can let $x_3 = s$ and $x_4 = t$, where s and t are real numbers. It follows that the solution set of the system is given by $\{(t-s, s+t-3, s, t) : s, t \text{ any real numbers }\}$.

17. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 4 \\ 1 & -1 & -1 & -1 & 2 \\ 1 & 1 & -1 & 1 & -2 \\ 1 & -1 & 1 & 1 & -8 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 4 \\ 0 & -2 & -2 & 0 & -2 \\ 0 & 0 & -2 & 2 & -6 \\ 0 & -2 & 0 & 2 & -12 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 1 & 0 & -1 & 6 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & -1 & 5 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & -2 & 8 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right].$$

1. $\mathrm{A}_{12}(-1)$, $\mathrm{A}_{13}(-1)$, $\mathrm{A}_{14}(-1)$ **2.** $\mathrm{M}_2(-\frac{1}{2})$, $\mathrm{M}_3(-\frac{1}{2})$, $\mathrm{M}_4(-\frac{1}{2})$ **3.** $\mathrm{A}_{24}(-1)$
4. $\mathrm{A}_{32}(-1)$, $\mathrm{A}_{34}(1)$ **5.** $\mathrm{M}_4(-\frac{1}{2})$ **6.** $\mathrm{A}_{41}(1)$, $\mathrm{A}_{42}(-1)$, $\mathrm{A}_{43}(1)$

It follows from the last augmented matrix that the solution to the system is given by $(-1, 2, -1, -4)$.

18. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccccc|c} 2 & -1 & 3 & 1 & -1 & 11 \\ 1 & -3 & -2 & -1 & -2 & 2 \\ 3 & 1 & -2 & -1 & 1 & -2 \\ 1 & 2 & 1 & 2 & 3 & -3 \\ 5 & -3 & -3 & 1 & 2 & 2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccccc|c} 1 & -3 & -2 & -1 & -2 & 2 \\ 2 & -1 & 3 & 1 & -1 & 11 \\ 3 & 1 & -2 & -1 & 1 & -2 \\ 1 & 2 & 1 & 2 & 3 & -3 \\ 5 & -3 & -3 & 1 & 2 & 2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccccc|c} 1 & -3 & -2 & -1 & -2 & 2 \\ 0 & 5 & 7 & 3 & 3 & 7 \\ 0 & 10 & 4 & 2 & 7 & -8 \\ 0 & 5 & 3 & 3 & 5 & -5 \\ 0 & 12 & 7 & 6 & 12 & -8 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccccc|c} 1 & -3 & -2 & -1 & -2 & 2 \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 10 & 4 & 2 & 7 & -8 \\ 0 & 5 & 3 & 3 & 5 & -5 \\ 0 & 12 & 7 & 6 & 12 & -8 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{11}{5} & \frac{4}{5} & -\frac{1}{5} & \frac{31}{5} \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & -10 & -4 & 1 & -22 \\ 0 & 0 & -4 & 0 & 2 & -12 \\ 0 & 0 & -\frac{49}{5} & -\frac{6}{5} & \frac{24}{5} & -\frac{124}{5} \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & \frac{11}{5} & \frac{4}{5} & -\frac{1}{5} & \frac{31}{5} \\ 0 & 1 & \frac{7}{5} & \frac{3}{5} & \frac{3}{5} & \frac{7}{5} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{10} & \frac{11}{5} \\ 0 & 0 & -4 & 0 & 2 & -12 \\ 0 & 0 & -\frac{49}{5} & -\frac{6}{5} & \frac{24}{5} & -\frac{124}{5} \end{array}\right]$$

$$\overset{6}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{2}{25} & \frac{1}{50} & \frac{34}{25} \\ 0 & 1 & 0 & \frac{1}{25} & \frac{37}{50} & -\frac{42}{25} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{10} & \frac{11}{5} \\ 0 & 0 & 0 & \frac{8}{5} & \frac{8}{5} & -\frac{16}{5} \\ 0 & 0 & 0 & \frac{68}{25} & \frac{191}{50} & -\frac{81}{25} \end{array}\right] \overset{7}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{2}{25} & \frac{1}{50} & \frac{34}{25} \\ 0 & 1 & 0 & \frac{1}{25} & \frac{37}{50} & -\frac{42}{25} \\ 0 & 0 & 1 & \frac{2}{5} & -\frac{1}{10} & \frac{11}{5} \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & \frac{68}{25} & \frac{191}{50} & -\frac{81}{25} \end{array}\right] \overset{8}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{1}{10} & \frac{6}{5} \\ 0 & 1 & 0 & 0 & \frac{7}{10} & -\frac{8}{5} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & \frac{11}{10} & \frac{11}{5} \end{array}\right]$$

$$\overset{9}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & \frac{1}{10} & \frac{6}{5} \\ 0 & 1 & 0 & 0 & \frac{7}{10} & -\frac{8}{5} \\ 0 & 0 & 1 & 0 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \overset{10}{\sim} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

1. P_{12} **2.** $A_{12}(-2), A_{13}(-3), A_{14}(-1), A_{15}(-5)$ **3.** $M_2(\frac{1}{5})$ **4.** $A_{21}(3), A_{23}(-10), A_{24}(-5), A_{25}(-12)$ **5.** $M_3(-\frac{1}{10})$ **6.** $A_{31}(-\frac{11}{5}), A_{32}(-\frac{7}{5}), A_{34}(4), A_{35}(\frac{49}{5})$ **7.** $M_4(\frac{5}{8})$ **8.** $A_{41}(\frac{2}{25}), A_{42}(-\frac{1}{25}), A_{43}(-\frac{2}{5}), A_{45}(-\frac{68}{25})$ **9.** $M_5(\frac{10}{11})$ **10.** $A_{51}(-\frac{1}{10}), A_{52}(-\frac{7}{10}), A_{53}(\frac{1}{2}), A_{54}(-1)$

It follows from the last augmented matrix that the solution to the system is given by $(1, -3, 4, -4, 2)$.

19. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & -3 & 1 \\ 5 & -4 & 1 \\ 2 & 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 15 \\ -4 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 8 \\ 5 & -4 & 1 & 15 \\ 2 & 4 & -3 & -4 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 8 \\ 0 & 11 & -4 & -25 \\ 0 & 10 & -5 & -20 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 1 & 8 \\ 0 & 1 & 1 & -5 \\ 0 & 10 & -5 & -20 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -7 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & -15 & 30 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 4 & -7 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 1 & -2 \end{array}\right] \stackrel{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

1. $A_{12}(-5), A_{13}(-2)$ **2.** $A_{32}(-1)$ **3.** $A_{21}(3), A_{23}(-10)$ **4.** $M_3(-\frac{1}{15})$ **5.** $A_{31}(-4), A_{32}(-1)$

Thus, from the last augmented matrix, we see that $x_1 = 1$, $x_2 = -3$, and $x_3 = -2$.

20. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & 0 & 5 \\ 3 & -2 & 11 \\ 2 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 3 & -2 & 11 & 2 \\ 2 & -2 & 6 & 2 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & -2 & -4 & 2 \\ 0 & -2 & -4 & 2 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & -2 & -4 & 2 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{12}(-3), A_{13}(-2)$ **2.** $M_2(-1/2)$ **3.** $A_{23}(2)$

Hence, we have $x_1 + 5x_3 = 0$ and $x_2 + 2x_3 = -1$. Since x_3 is a free variable, we can let $x_3 = t$, where t is any real number. It follows that the solution set for the given system is given by $\{(-5t, -2t - 1, t) : t \in \mathbb{R}\}$.

21. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 0 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 5 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 0 & 1 & -1 & -2 \\ 0 & 5 & 1 & 8 \\ 0 & 2 & 1 & 5 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & -1 & -2 \\ 0 & 0 & 6 & 18 \\ 0 & 0 & 3 & 9 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{array}\right] \stackrel{3}{\sim} \left[\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-5),\ \mathrm{A}_{13}(-2) \quad \textbf{2. } \mathrm{M}_2(1/6) \quad \textbf{3. } \mathrm{A}_{21}(1),\ \mathrm{A}_{23}(-3)}$$

Consequently, from the last augmented matrix it follows that the solution set for the matrix equation is given by $\{(t, 1, 3) : t \in \mathbb{R}\}$.

22. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & -1 & 0 & -1 \\ 2 & 1 & 3 & 7 \\ 3 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 2 \\ 2 & 1 & 3 & 7 & 2 \\ 3 & -2 & 1 & 0 & 4 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 2 \\ 0 & 3 & 3 & 9 & -2 \\ 0 & 1 & 1 & 3 & -2 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 2 \\ 0 & 1 & 1 & 3 & -2 \\ 0 & 3 & 3 & 9 & -2 \end{array}\right] \stackrel{3}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-2),\ \mathrm{A}_{13}(-3) \quad \textbf{2. } \mathrm{P}_{23} \quad \textbf{3. } \mathrm{A}_{21}(1),\ \mathrm{A}_{23}(-3)}$$

From the last row of the last augmented matrix, it is clear that the given system is inconsistent.

23. The equation $A\mathbf{x} = \mathbf{b}$ reads

$$\begin{bmatrix} 1 & 1 & 0 & -1 \\ 3 & 1 & -2 & 3 \\ 2 & 3 & 1 & 1 \\ -2 & 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 3 \\ -9 \end{bmatrix}.$$

Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 3 & 1 & -2 & 3 & 8 \\ 2 & 3 & 1 & 2 & 3 \\ -2 & 3 & 5 & -2 & -9 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & -2 & -2 & 0 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 5 & 5 & 0 & -5 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & -2 & -2 & 0 & 2 \\ 0 & 5 & 5 & 0 & -5 \end{array}\right] \stackrel{3}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-3),\ \mathrm{A}_{13}(-2),\ \mathrm{A}_{14}(2) \quad \textbf{2. } \mathrm{P}_{23} \quad \textbf{3. } \mathrm{A}_{21}(-1),\ \mathrm{A}_{23}(2),\ \mathrm{A}_{24}(-5)}$$

From the last augmented matrix, we obtain the system of equations: $x_1 - x_3 + x_4 = 3$, $x_2 + x_3 = -1$. Since both x_3 and x_4 are free variables, we may let $x_3 = r$ and $x_4 = t$, where r and t are real numbers. The solution set for the system is given by $\{(3 + r - t, -r - 1, r, t) : r, t \in \mathbb{R}\}$.

24. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 5 & 1 & 7 \\ 1 & 1 & -k^2 & -k \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & -1 & 1 - k^2 & -3 - k \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 4 - k^2 & -2 - k \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-2),\ \mathrm{A}_{13}(-1) \quad \textbf{2. } \mathrm{A}_{23}(1)}$$

(a). If $k = 2$, then the last row of the last augmented matrix reveals an inconsistency; hence the system has no solutions in this case.

(b). If $k = -2$, then the last row of the last augmented matrix consists entirely of zeros, and hence we have only two pivots (first two columns) and a free variable x_3; hence the system has infinitely many solutions.

(c). If $k \neq \pm 2$, then the last augmented matrix above contains a pivot for each variable x_1, x_2, and x_3, and can be solved for a unique solution by back-substitution.

25. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 4 & 2 & -1 & 1 & 0 \\ 3 & -1 & 1 & k & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 4 & 2 & -1 & 1 & 0 \\ 3 & -1 & 1 & k & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -2 & -5 & 5 & 0 \\ 0 & -4 & -2 & k+3 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & -2 & -5 & 5 & 0 \\ 0 & -4 & -2 & k+3 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 10 & k-9 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & k+1 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(-2),\ \mathrm{A}_{13}(-4),\ \mathrm{A}_{14}(-3) \quad \textbf{3. } \mathrm{M}_2(-1) \quad \textbf{4. } \mathrm{A}_{23}(2),\ \mathrm{A}_{24}(4) \quad \textbf{5. } \mathrm{A}_{34}(-10)}$$

(a). Note that the trivial solution $(0, 0, 0, 0)$ exists under all circumstances, so there are no values of k for which there is no solution.

(b). From the last row of the last augmented matrix, we see that if $k = -1$, then the variable x_4 corresponds to an unpivoted column, and hence it is a free variable. In this case, therefore, we have infinitely solutions.

(c). Provided that $k \neq -1$, then each variable in the system corresponds to a pivoted column of the last augmented matrix above. Therefore, we can solve the system by back-substitution. The conclusion from this is that there is a unique solution, $(0, 0, 0, 0)$.

26. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 3 & 5 & -4 & 16 \\ 2 & 3 & -a & b \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & 4-a & b-8 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 4-a & b-8 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3-a & b-10 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-3),\ \mathrm{A}_{13}(-2) \quad \textbf{2. } \mathrm{M}_2(\tfrac{1}{2}) \quad \textbf{3. } \mathrm{A}_{21}(-1),\ \mathrm{A}_{23}(-1)}$$

(a). From the last row of the last augmented matrix above, we see that there is no solution if $a = 3$ and $b \neq 10$.

(b). From the last row of the augmented matrix above, we see that there are infinitely many solutions if $a = 3$ and $b = 10$, because in that case, there is no pivot in the column of the last augmented matrix corresponding to the third variable x_3.

(c). From the last row of the augmented matrix above, we see that if $a \neq 3$, then regardless of the value of b, there is a pivot corresponding to each variable x_1, x_2, and x_3. Therefore, we can uniquely solve the corresponding system by back-substitution.

27. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{cc|c} 1 & -a & 3 \\ 2 & 1 & 6 \\ -3 & a+b & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -a & 3 \\ 0 & 1+2a & 0 \\ 0 & b-2a & 10 \end{array}\right].$$

From the middle row, we see that if $a \neq -\frac{1}{2}$, then we must have $x_2 = 0$, but this leads to an inconsistency in solving for x_1 (the first equation would require $x_1 = 3$ while the last equation would require $x_1 = -\frac{1}{3}$. Now suppose that $a = -\frac{1}{2}$. Then the augmented matrix on the right reduces to $\left[\begin{array}{cc|c} 1 & -1/2 & 3 \\ 0 & b+1 & 10 \end{array}\right]$. If $b = -1$, then once more we have an inconsistency in the last row. However, if $b \neq -1$, then the row-echelon form obtained has full rank, and there is a unique solution. Therefore, we draw the following conclusions:

(a). There is no solution to the system if $a \neq -\frac{1}{2}$ or if $a = -\frac{1}{2}$ and $b = -1$.

(b). Under no circumstances are there an infinite number of solutions to the linear system.

(c). There is a unique solution if $a = -\frac{1}{2}$ and $b \neq -1$.

28. The corresponding augmented matrix for this linear system can be reduced to row-echelon form via

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 2 & 3 & 1 & y_2 \\ 3 & 5 & 1 & y_3 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 0 & 1 & -1 & y_2 - 2y_1 \\ 0 & 2 & -2 & y_3 - 3y_1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 1 & y_1 \\ 0 & 1 & -1 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_1 - 2y_2 + y_3 \end{array}\right].$$

1. $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{13}(-3)$ **2.** $\mathrm{A}_{23}(-2)$

For consistency, we must have $\text{rank}(A) = \text{rank}(A^{\#})$, which requires (y_1, y_2, y_3) to satisfy $y_1 - 2y_2 + y_3 = 0$. If this holds, then the system has an infinite number of solutions, because the column of the augmented matrix corresponding to y_3 will be unpivoted, indicating that y_3 is a free variable in the solution set.

29. Converting the given system of equations to an augmented matrix and using Gaussian elimination we obtain the following row-equivalent matrices. Since $a_{11} \neq 0$:

$$\left[\begin{array}{cc|c} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & \frac{a_{22}a_{11} - a_{21}a_{12}}{a_{11}} & \frac{a_{11}b_2 - a_{21}b_1}{a_{11}} \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & \frac{\Delta}{a_{11}} & \frac{\Delta_2}{a_{11}} \end{array}\right].$$

1. $\mathrm{M}_1(1/a_{11})$, $\mathrm{A}_{12}(-a_{21})$ **2.** Definition of Δ and Δ_2

(a). If $\Delta \neq 0$, then $\text{rank}(A) = \text{rank}(A^{\#}) = 2$, so the system has a unique solution (of course, we are assuming $a_{11} \neq 0$ here). Using the last augmented matrix above, $\left(\frac{\Delta}{a_{11}}\right) x_2 = \frac{\Delta_2}{a_{11}}$, so that $x_2 = \frac{\Delta_2}{\Delta}$. Using this, we can solve $x_1 + \frac{a_{12}}{a_{11}} x_2 = \frac{b_1}{a_{11}}$ for x_1 to obtain $x_1 = \frac{\Delta_1}{\Delta}$, where we have used the fact that $\Delta_1 = a_{22}b_1 - a_{12}b_2$.

(b). If $\Delta = 0$ and $a_{11} \neq 0$, then the augmented matrix of the system is $\left[\begin{array}{cc|c} 1 & \frac{a_{12}}{a_{11}} & \frac{b_1}{a_{11}} \\ 0 & 0 & \Delta_2 \end{array}\right]$, so it follows that the system has (i) no solution if $\Delta_2 \neq 0$, since $\text{rank}(A) < \text{rank}(A^{\#}) = 2$, and (ii) an infinite number of solutions if $\Delta_2 = 0$, since $\text{rank}(A^{\#}) < 2$.

(c). An infinite number of solutions would be represented as one line. No solution would be two parallel lines. A unique solution would be the intersection of two distinct lines at one point.

30. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 5 & 1 & 3 \\ 2 & 6 & 7 & 1 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 1 & 2 & 1 & 1 \\ 2 & 6 & 7 & 1 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 8/3 & 19/3 & -1 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 0 & 8/3 & 19/3 & -1 \\ 0 & 1/3 & 2/3 & 0 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 3 & 5 & 1 & 3 \\ 0 & 8/3 & 19/3 & -1 \\ 0 & 0 & -1/8 & 1/8 \end{array}\right].$$

$$\boxed{\textbf{1. } P_{12} \quad \textbf{2. } A_{12}(-1/3),\ A_{13}(-2/3) \quad \textbf{3. } P_{23} \quad \textbf{4. } A_{23}(-1/8)}$$

Using back substitution to solve the equivalent system yields the unique solution $(-2, 2, -1)$.

31. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & -1 & 3 & 14 \\ 3 & 1 & -2 & -1 \\ 7 & 2 & -3 & 3 \\ 5 & -1 & -2 & 5 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 3 & 1 & -2 & -1 \\ 2 & -1 & 3 & 14 \\ 5 & -1 & -2 & 5 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & 1/7 & -5/7 & -16/7 \\ 0 & -11/7 & 27/7 & 92/7 \\ 0 & -17/7 & 1/7 & 20/7 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & -17/7 & 1/7 & 20/7 \\ 0 & -11/7 & 27/7 & 92/7 \\ 0 & 1/7 & -5/7 & -16/7 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & -17/7 & 1/7 & 20/7 \\ 0 & 0 & 64/17 & 192/17 \\ 0 & 0 & -12/17 & -36/17 \end{array}\right]$$

$$\stackrel{5}{\sim} \left[\begin{array}{ccc|c} 7 & 2 & -3 & 3 \\ 0 & -17/7 & 1/7 & 20/7 \\ 0 & 0 & 64/17 & 192/17 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\begin{array}{c} \textbf{1. } P_{13} \quad \textbf{2. } A_{12}(-3/7),\ A_{13}(-2/7),\ A_{14}(-5/7) \quad \textbf{3. } P_{24} \\ \textbf{4. } A_{23}(-11/17),\ A_{24}(1/17) \quad \textbf{5. } A_{34}(3/16) \end{array}}$$

Using back substitution to solve the equivalent system yields the unique solution $(2, -1, 3)$.

32. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & -1 & -4 & 5 \\ 3 & 2 & -5 & 8 \\ 5 & 6 & -6 & 20 \\ 1 & 1 & -3 & -3 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & -20 \\ 3 & 2 & -5 & 8 \\ 2 & -1 & -4 & 5 \\ 1 & 1 & -3 & -3 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -8/5 & -7/5 & -4 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & -1/5 & -9/5 & -7 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & -8/5 & -7/5 & -4 \\ 0 & -1/5 & -9/5 & -7 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & 0 & -11/17 & -44/17 \\ 0 & 0 & -29/17 & -116/17 \end{array}\right]$$

$$\stackrel{5}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & 0 & -29/17 & -116/17 \\ 0 & 0 & -11/17 & -44/17 \end{array}\right] \stackrel{6}{\sim} \left[\begin{array}{ccc|c} 5 & 6 & -6 & 20 \\ 0 & -17/5 & -8/5 & -3 \\ 0 & 0 & -29/17 & -116/17 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{13} **2.** $A_{12}(-3/5)$, $A_{13}(-2/5)$, $A_{14}(-1/5)$ **3.** P_{23} **4.** $A_{23}(-8/17)$, $A_{24}(-1/17)$ **5.** P_{34} **6.** $A_{34}(-11/29)$

Using back substitution to solve the equivalent system yields the unique solution $(10, -1, 4)$.

33. We first use the partial pivoting algorithm to reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 2 & -1 & -1 & 2 \\ 4 & 3 & -2 & -1 \\ 1 & 4 & 1 & 4 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 4 & 3 & -2 & -1 \\ 2 & -1 & -1 & 2 \\ 1 & 4 & 1 & 4 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 0 & -5/2 & 0 & 5/2 \\ 0 & 13/4 & 3/2 & 17/4 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 0 & 13/4 & 3/2 & 17/4 \\ 0 & -5/2 & 0 & 5/2 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 4 & 3 & -2 & -1 \\ 0 & 13/4 & 3/2 & 17/4 \\ 0 & 0 & 15/13 & 75/13 \end{array}\right].$$

1. P_{12} **2.** $A_{12}(-1/2)$, $A_{13}(-1/4)$ **3.** P_{23} **4.** $A_{23}(10/13)$

Using back substitution to solve the equivalent system yields the unique solution $(3, -1, 5)$.

34.

(a). Let

$$A^{\#} = \left[\begin{array}{cccccc|c} a_{11} & 0 & 0 & \dots & 0 & & b_1 \\ a_{21} & a_{22} & 0 & \dots & 0 & & b_2 \\ a_{31} & a_{32} & a_{33} & \dots & 0 & & b_3 \\ \dots & \dots & \dots & \dots & \dots & & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & & b_n \end{array}\right]$$

represent the corresponding augmented matrix of the given system. Since $a_{11}x_1 = b_1$, we can solve for x_1 easily:

$$x_1 = \frac{b_1}{a_{11}}, \qquad (a_{11} \neq 0).$$

Now since $a_{21}x_1 + a_{22}x_2 = b_2$, by using the expression for x_1 we just obtained, we can solve for x_2:

$$x_2 = \frac{a_{11}b_2 - a_{21}b_1}{a_{11}a_{22}}.$$

In a similar manner, we can solve for $x_3, x_4, \dots, x_n$.

(b). We solve instantly for x_1 from the first equation: $x_1 = 2$. Substituting this into the middle equation, we obtain $2 \cdot 2 - 3 \cdot x_2 = 1$, from which it quickly follows that $x_2 = 1$. Substituting for x_1 and x_2 in the bottom equation yields $3 \cdot 2 + 1 - x_3 = 8$, from which it quickly follows that $x_3 = -1$. Consequently, the solution of the given system is $(2, 1, -1)$.

35. This system of equations is not linear in x_1, x_2, and x_3; however, the system *is* linear in x_1^3, x_2^2, and x_3, so we can first solve for x_1^3, x_2^2, and x_3. Converting the given system of equations to an augmented matrix and using Gauss-Jordan elimination we obtain the following equivalent matrices:

$$\left[\begin{array}{rrr|r} 4 & 2 & 3 & 12 \\ 1 & -1 & 1 & 2 \\ 3 & 1 & -1 & 2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 4 & 2 & 3 & 12 \\ 3 & 1 & -1 & 2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 0 & 6 & -1 & 4 \\ 0 & 4 & -4 & -4 \end{array}\right]$$

$$\overset{3}{\sim}\left[\begin{array}{rrr|r}1&-1&1&2\\0&4&-4&-4\\0&6&-1&4\end{array}\right]\overset{4}{\sim}\left[\begin{array}{rrr|r}1&-1&1&2\\0&1&-1&-1\\0&6&-1&4\end{array}\right]\overset{5}{\sim}\left[\begin{array}{rrr|r}1&0&0&1\\0&1&-1&-1\\0&0&5&10\end{array}\right]$$

$$\overset{6}{\sim}\left[\begin{array}{rrr|r}1&0&0&1\\0&1&-1&-1\\0&0&1&2\end{array}\right]\overset{7}{\sim}\left[\begin{array}{rrr|r}1&0&0&1\\0&1&0&1\\0&0&1&2\end{array}\right].$$

1. P_{12} **2.** $\mathrm{A}_{12}(-4)$, $\mathrm{A}_{13}(-3)$ **3.** P_{23} **4.** $\mathrm{M}_2(1/4)$
5. $\mathrm{A}_{21}(1)$, $\mathrm{A}_{23}(-6)$ **6.** $\mathrm{M}_2(1/5)$ **7.** $\mathrm{A}_{32}(1)$

Thus, taking only real solutions, we have $x_1^3 = 1$, $x_2^2 = 1$, and $x_3 = 2$. Therefore, $x_1 = 1$, $x_2 = \pm 1$, and $x_3 = 2$, leading to the *two* solutions $(1, 1, 2)$ and $(1, -1, 2)$ to the original system of equations. There is no contradiction of Theorem 2.5.9 here since, as mentioned above, this system is *not linear* in x_1, x_2, and x_3.

36. Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r}3&2&-1&0\\2&1&1&0\\5&-4&1&0\end{array}\right]\overset{1}{\sim}\left[\begin{array}{rrr|r}1&1&-2&0\\0&-1&5&0\\0&-9&11&0\end{array}\right]\overset{2}{\sim}\left[\begin{array}{rrr|r}1&1&-2&0\\0&1&-5&0\\0&-9&11&0\end{array}\right]\overset{3}{\sim}\left[\begin{array}{rrr|r}1&0&3&0\\0&1&-5&0\\0&0&-34&0\end{array}\right]$$

$$\overset{4}{\sim}\left[\begin{array}{rrr|r}1&0&3&0\\0&1&-5&0\\0&0&1&0\end{array}\right]\overset{5}{\sim}\left[\begin{array}{rrr|r}1&0&0&0\\0&1&0&0\\0&0&1&0\end{array}\right].$$

1. $\mathrm{A}_{21}(-1)$, $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{13}(-5)$ **2.** $\mathrm{M}_2(-1)$ **3.** $\mathrm{A}_{21}(-1)$, $\mathrm{A}_{23}(9)$
4. $\mathrm{M}_3(-1/34)$ **5.** $\mathrm{A}_{31}(-3)$, $\mathrm{A}_{32}(5)$

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

37. Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r}2&1&-1&0\\3&-1&2&0\\1&-1&-1&0\\5&2&-2&0\end{array}\right]\overset{1}{\sim}\left[\begin{array}{rrr|r}1&-1&-1&0\\3&-1&2&0\\2&1&-1&0\\5&2&-2&0\end{array}\right]\overset{2}{\sim}\left[\begin{array}{rrr|r}1&-1&-1&0\\0&2&5&0\\0&3&1&0\\0&7&3&0\end{array}\right]\overset{3}{\sim}\left[\begin{array}{rrr|r}1&-1&-1&0\\0&3&1&0\\0&2&5&0\\0&7&3&0\end{array}\right]$$

$$\overset{4}{\sim}\left[\begin{array}{rrr|r}1&-1&-1&0\\0&1&-4&0\\0&2&5&0\\0&7&3&0\end{array}\right]\overset{5}{\sim}\left[\begin{array}{rrr|r}1&0&-5&0\\0&1&-4&0\\0&0&13&0\\0&0&31&0\end{array}\right]\overset{6}{\sim}\left[\begin{array}{rrr|r}1&0&-5&0\\0&1&-4&0\\0&0&1&0\\0&0&31&0\end{array}\right]\overset{7}{\sim}\left[\begin{array}{rrr|r}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{array}\right].$$

1. P_{13} **2.** $\mathrm{A}_{12}(-3)$, $\mathrm{A}_{13}(-2)$, $\mathrm{A}_{14}(-5)$ **3.** P_{23} **4.** $\mathrm{A}_{32}(-1)$
5. $\mathrm{A}_{21}(1)$, $\mathrm{A}_{23}(-2)$, $\mathrm{A}_{24}(-7)$ **6.** $\mathrm{M}_3(1/13)$ **7.** $\mathrm{A}_{31}(5)$, $\mathrm{A}_{32}(4)$, $\mathrm{A}_{34}(-31)$

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

38. Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r}2&-1&-1&0\\5&-1&2&0\\1&1&4&0\end{array}\right]\overset{1}{\sim}\left[\begin{array}{rrr|r}1&1&4&0\\5&-1&2&0\\2&-1&-1&0\end{array}\right]\overset{2}{\sim}\left[\begin{array}{rrr|r}1&1&4&0\\0&-6&-18&0\\0&-3&-9&0\end{array}\right]$$

$$\overset{3}{\sim}\left[\begin{array}{ccc|c} 1 & 1 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -3 & -9 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{P}_{13} \quad \textbf{2. } \mathrm{A}_{12}(-5), \mathrm{A}_{13}(-2) \quad \textbf{3. } \mathrm{M}_2(-1/6) \quad \textbf{4. } \mathrm{A}_{21}(-1), \mathrm{A}_{23}(3)}$$

It follows that $x_1 + x_3 = 0$ and $x_2 + 3x_3 = 0$. Setting $x_3 = t$, where t is a free variable, we get $x_2 = -3t$ and $x_1 = -t$. Thus we have that the solution set of the system is $\{(-t, -3t, t) : t \in \mathbb{R}\}$.

39. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1+2i & 1-i & 1 & 0 \\ i & 1+i & -i & 0 \\ 2i & 1 & 1+3i & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} i & 1+i & -i & 0 \\ 1+2i & 1-i & 1 & 0 \\ 2i & 1 & 1+3i & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 1+2i & 1-i & 1 & 0 \\ 2i & 1 & 1+3i & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & -2-2i & 1+2i & 0 \\ 0 & -1-2i & 1+5i & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & -2-2i & 1+2i & 0 \\ 0 & 1 & 3i & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & 0 & -5+8i & 0 \\ 0 & 1 & 3i & 0 \end{array}\right]$$

$$\overset{6}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & 1 & 3i & 0 \\ 0 & 0 & -5+8i & 0 \end{array}\right] \overset{7}{\sim} \left[\begin{array}{ccc|c} 1 & 1-i & -1 & 0 \\ 0 & 1 & 3i & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{8}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

$$\boxed{\begin{array}{c} \textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{M}_1(-i) \quad \textbf{3. } \mathrm{A}_{12}(-1-2i), \mathrm{A}_{13}(-2i) \quad \textbf{4. } \mathrm{A}_{23}(-1) \quad \textbf{5. } \mathrm{A}_{32}(2+2i) \\ \textbf{6. } \mathrm{P}_{23} \quad \textbf{7. } \mathrm{M}_3(\frac{1}{-5+8i}) \quad \textbf{8. } \mathrm{A}_{21}(-1+i), \mathrm{A}_{31}(1), \mathrm{A}_{32}(-3i) \end{array}}$$

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

40. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 6 & -1 & 2 & 0 \\ 12 & 6 & 4 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 6 & -1 & 2 & 0 \\ 12 & 6 & 4 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{M}_1(1/3) \quad \textbf{2. } \mathrm{A}_{12}(-6), \mathrm{A}_{13}(-12) \quad \textbf{3. } \mathrm{M}_2(-1/5) \quad \textbf{4. } \mathrm{A}_{21}(-2/3), \mathrm{A}_{23}(2)}$$

From the last augmented matrix, we have $x_1 + \frac{1}{3}x_3 = 0$ and $x_2 = 0$. Since x_3 is a free variable, we let $x_3 = t$, where t is a real number. It follows that the solution set for the given system is given by $\{(t, 0, -3t) : t \in \mathbb{R}\}$.

41. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & 1 & -8 & 0 \\ 3 & -2 & -5 & 0 \\ 5 & -6 & -3 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 3 & -2 & -5 & 0 \\ 2 & 1 & -8 & 0 \\ 5 & -6 & -3 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 2 & 1 & -8 & 0 \\ 5 & -6 & -3 & 0 \\ 3 & -5 & 1 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 7 & -14 & 0 \\ 0 & 9 & -18 & 0 \\ 0 & 4 & -8 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & -3 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 9 & -18 & 0 \\ 0 & 4 & -8 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. P_{12} **2.** $\mathrm{A}_{21}(-1)$ **3.** $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{13}(-5)$, $\mathrm{A}_{14}(-3)$ **4.** $\mathrm{M}_2(1/7)$ **5.** $\mathrm{A}_{21}(3)$, $\mathrm{A}_{23}(-9)$, $\mathrm{A}_{24}(-4)$

From the last augmented matrix we have: $x_1 - 3x_3 = 0$ and $x_2 - 2x_3 = 0$. Since x_3 is a free variable, we let $x_3 = t$, where t is a real number. It follows that $x_2 = 2t$ and $x_1 = 3t$. Thus, the solution set for the given system is given by $\{(3t, 2t, t) : t \in \mathbb{R}\}$.

42. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ i & 1 & i & 0 \\ 1-2i & -1+i & 1-3i & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ 0 & 2-i & -1 & 0 \\ 0 & -4+2i & 2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ 0 & 2-i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 1+i & 1-i & 0 \\ 0 & 1 & \frac{-2-i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{6-2i}{5} & 0 \\ 0 & 1 & \frac{-2-i}{5} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $\mathrm{A}_{12}(-i)$, $\mathrm{A}_{13}(-1+2i)$ **2.** $\mathrm{A}_{23}(2)$ **3.** $\mathrm{M}_2(\frac{1}{2-i})$ **4.** $\mathrm{A}_{21}(-1-i)$

From the last augmented matrix we see that x_3 is a free variable. We set $x_3 = 5s$, where $s \in \mathbb{C}$. Then $x_1 = 2(i-3)s$ and $x_2 = (2+i)s$. Thus, the solution set of the system is $\{(2(i-3)s, (2+i)s, 5s) : s \in \mathbb{C}\}$.

43. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 3 & 0 & -1 & 0 \\ 5 & 1 & -1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 6 & -6 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & -10 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -10 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $\mathrm{A}_{13}(-3)$, $\mathrm{A}_{14}(-5)$ **2.** $\mathrm{M}_2(1/3)$ **3.** $\mathrm{A}_{21}(1)$, $\mathrm{A}_{23}(-3)$, $\mathrm{A}_{24}(-6)$
4. $\mathrm{M}_3(-1/6)$ **5.** $\mathrm{A}_{31}(-5/3)$, $\mathrm{A}_{32}(-2/3)$, $\mathrm{A}_{34}(10)$

Therefore, the unique solution to this system is $x_1 = x_2 = x_3 = 0$: $(0, 0, 0)$.

44. Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2 & -4 & 6 & 0 \\ 3 & -6 & 9 & 0 \\ 1 & -2 & 3 & 0 \\ 5 & -10 & 15 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 3 & -6 & 9 & 0 \\ 2 & -4 & 6 & 0 \\ 5 & -10 & 15 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

1. $\mathrm{M}_1(1/2)$ **2.** $\mathrm{A}_{12}(-3)$, $\mathrm{A}_{13}(-2)$, $\mathrm{A}_{14}(-5)$

From the last matrix we have that $x_1 - 2x_3 + 3x_3 = 0$. Since x_2 and x_3 are free variables, let $x_2 = s$ and let $x_3 = t$, where s and t are real numbers. The solution set of the given system is therefore $\{(2s - 3t, s, t) : s, t \in \mathbb{R}\}$.

45. Reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 4 & -2 & -1 & -1 & 0 \\ 3 & 1 & -2 & 3 & 0 \\ 5 & -1 & -2 & 1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 3 & 1 & -2 & 3 & 0 \\ 5 & -1 & -2 & 1 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 0 & 10 & -5 & 15 & 0 \\ 0 & 14 & -7 & 21 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 0 & 2 & -1 & 3 & 0 \\ 0 & 2 & -1 & 3 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 0 & 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{cccc|c} 1 & -3 & 1 & -4 & 0 \\ 0 & 1 & -1/2 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

1. $A_{21}(-1)$ **2.** $A_{12}(-3)$, $A_{13}(-5)$ **3.** $M_2(1/5)$, $M_3(1/7)$
4. $A_{23}(-1)$ **5.** $M_2(1/2)$

From the last augmented matrix above we have that $x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 = 0$ and $x_1 - 3x_2 + x_3 - 4x_4 = 0$. Since x_3 and x_4 are free variables, we can set $x_3 = 2s$ and $x_4 = 2t$, where s and t are real numbers. Then $x_2 = s - 3t$ and $x_1 = s - t$. It follows that the solution set of the given system is $\{(s-t, s-3t, 2s, 2t) : s, t \in \mathbb{R}\}$.

46. Reduce the augmented matrix of the system:

$$\left[\begin{array}{cccc|c} 2 & 1 & -1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 0 \\ 3 & -1 & 1 & -2 & 0 \\ 4 & 2 & -1 & 1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 2 & 1 & -1 & 1 & 0 \\ 3 & -1 & 1 & -2 & 0 \\ 4 & 2 & -1 & 1 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & -1 & -3 & 3 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & -2 & -5 & 5 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & -4 & -2 & 1 & 0 \\ 0 & -2 & -5 & 5 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 10 & -11 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & -3 & 3 & 0 \\ 0 & 0 & 10 & -11 & 0 \end{array}\right]$$

$$\overset{6}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 10 & -11 & 0 \end{array}\right] \overset{7}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right] \overset{8}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \overset{9}{\sim} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right].$$

1. P_{12} **2.** $A_{12}(-2)$, $A_{13}(-3)$, $A_{14}(-4)$ **3.** $M_2(-1)$ **4.** $A_{21}(-1)$, $A_{23}(4)$, $A_{24}(2)$
5. P_{34} **6.** $M_3(-1/3)$ **7.** $A_{31}(2)$, $A_{32}(-3)$, $A_{34}(-10)$ **8.** $M_4(-1)$ **9.** $A_{43}(1)$

From the last augmented matrix, it follows that the solution set to the system is given by $\{(0, 0, 0, 0)\}$.

47. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 3 & 4 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 3 & 4 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{11}{2} & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

1. $M_1(1/2)$ **2.** $A_{12}(-3)$ **3.** $M_2(2/11)$ **4.** $A_{21}(1/2)$

From the last augmented matrix, we see that $x_1 = x_2 = 0$. Hence, the solution set is $\{(0,0)\}$.

48. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1-i & 2i \\ 1+i & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 1-i & 2i & 0 \\ 1+i & -2 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 1+i & -2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & -1+i & 0 \\ 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{M}_1(\tfrac{1+i}{2}) \quad \textbf{2. } \mathrm{A}_{12}(-1-i)}$$

It follows that $x_1 + (-1+i)x_2 = 0$. Since x_2 is a free variable, we can let $x_2 = t$, where t is a complex number. The solution set to the system is then given by $\{(t(1-i), t) : t \in \mathbb{C}\}$.

49. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1+i & 1-2i \\ -1+i & 2+i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 1+i & 1-2i & 0 \\ -1+i & 2+i & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1+3i}{2} & 0 \\ -1+i & 2+i & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1+3i}{2} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{M}_1(\tfrac{1-i}{2}) \quad \textbf{2. } \mathrm{A}_{12}(1-i)}$$

It follows that $x_1 - \frac{1+3i}{2}x_2 = 0$. Since x_2 is a free variable, we can let $x_2 = r$, where r is any complex number. Thus, the solution set to the given system is $\{(\frac{1+3i}{2}r, r) : r \in \mathbb{C}\}$.

50. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -5 & -6 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -5 & -6 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -5 & -6 & 0 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-2), \mathrm{A}_{13}(-1) \quad \textbf{2. } \mathrm{P}_{23} \quad \textbf{3. } \mathrm{M}_2(-1) \quad \textbf{4. } \mathrm{A}_{21}(-2), \mathrm{A}_{23}(5) \quad \textbf{5. } \mathrm{M}_3(1/4) \quad \textbf{6. } \mathrm{A}_{31}(1), \mathrm{A}_{32}(-2)}$$

From the last augmented matrix, we see that the only solution to the given system is $x_1 = x_2 = x_3 = 0$: $\{(0,0,0)\}$.

51. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & 0 & -1 & 2 \\ 1 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 1 & 1 & 1 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 \\ 1 & 3 & 2 & 2 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrrr|r} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 3 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(1), \mathrm{A}_{13}(-1) \quad \textbf{2. } \mathrm{A}_{21}(-1), \mathrm{A}_{23}(-2) \quad \textbf{3. } \mathrm{A}_{31}(-1)}$$

From the last augmented matrix, we see that x_4 is a free variable. We set $x_4 = t$, where t is a real number. The last row of the reduced row echelon form above corresponds to the equation $x_3 + x_4 = 0$. Therefore, $x_3 = -t$. The second row corresponds to the equation $x_2 + x_4 = 0$, so we likewise find that $x_2 = -t$. Finally, from the first equation we have $x_1 - 3x_4 = 0$, so that $x_1 = 3t$. Consequently, the solution set of the original system is given by $\{(3t, -t, -t, t) : t \in \mathbb{R}\}$.

52. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2-3i & 1+i & i-1 \\ 3+2i & -1+i & -1-i \\ 5-i & 2i & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of this system:

$$\left[\begin{array}{ccc|c} 2-3i & 1+i & i-1 & 0 \\ 3+2i & -1+i & -1-i & 0 \\ 5-i & 2i & -2 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{-1+5i}{13} & \frac{-5-i}{13} & 0 \\ 3+2i & -1+i & -1-i & 0 \\ 5-i & 2i & -2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{-1+5i}{13} & \frac{-5-i}{13} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{M}_1\left(\tfrac{2+3i}{13}\right) \quad \textbf{2. } \mathrm{A}_{12}(-3-2i), \mathrm{A}_{13}(-5+i)}$$

From the last augmented matrix, we see that $x_1 + \frac{-1+5i}{13}x_2 + \frac{-5-i}{13}x_3 = 0$. Since x_2 and x_3 are free variables, we can let $x_2 = 13r$ and $x_3 = 13s$, where r and s are complex numbers. It follows that the solution set of the system is $\{(r(1-5i) + s(5+i), 13r, 13s) : r, s \in \mathbb{C}\}$.

53. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 3 & 0 \\ -2 & -3 & 0 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 1 & 3 & 0 & 0 \\ -2 & -3 & 0 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & 3 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(2), \mathrm{A}_{13}(-1) \quad \textbf{2. } \mathrm{P}_{23} \quad \textbf{3. } \mathrm{A}_{21}(-3), \mathrm{A}_{23}(-3)}$$

From the last augmented matrix we see that the solution set of the system is $\{(0, 0, t) : t \in \mathbb{R}\}$.

54. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & -1 & 7 \\ 2 & 1 & 8 \\ 1 & 1 & 5 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 3 & -1 & 7 & 0 \\ 2 & 1 & 8 & 0 \\ 1 & 1 & 5 & 0 \\ -1 & 1 & -1 & 0 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right] \stackrel{3}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-3), \mathrm{A}_{13}(-2), \mathrm{A}_{14}(-1), \mathrm{A}_{15}(1) \quad \textbf{2. } \mathrm{M}_2(-1) \quad \textbf{3. } \mathrm{A}_{23}(-1), \mathrm{A}_{24}(-1), \mathrm{A}_{25}(-1)}$$

From the last augmented matrix, we obtain the equations $x_1 + 3x_3 = 0$ and $x_2 + 2x_3 = 0$. Since x_3 is a free variable, we let $x_3 = t$, where t is a real number. The solution set for the given system is then given by $\{(-3t, -2t, t) : t \in \mathbb{R}\}$.

55. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 3 & -2 & 0 & 5 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 1 & -1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 5 & 0 \\ -1 & 2 & 0 & 1 & 0 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{rrrr|r} 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 & 0 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-3), \mathrm{A}_{13}(1) \quad \textbf{2. } \mathrm{A}_{21}(1), \mathrm{A}_{23}(-1)}$$

From the last augmented matrix we obtain the equations $x_1 + 3x_4 = 0$ and $x_2 + 2x_4 = 0$. Because x_3 and x_4 are free, we let $x_3 = t$ and $x_4 = s$, where s and t are real numbers. It follows that the solution set of the system is $\{(-3s, -2s, t, s) : s, t \in \mathbb{R}\}$.

56. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 0 & -9 & 0 \\ -2 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 1 & 0 & -3 & 0 & 0 \\ 3 & 0 & -9 & 0 & 0 \\ -2 & 0 & 6 & 0 & 0 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{rrrr|r} 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-3), \mathrm{A}_{13}(2)}$$

From the last augmented matrix we obtain $x_1 - 3x_3 = 0$. Therefore, x_2, x_3, and x_4 are free variables, so we let $x_2 = r$, $x_3 = s$, and $x_4 = t$, where r, s, t are real numbers. The solution set of the given system is therefore $\{(3s, r, s, t) : r, s, t \in \mathbb{R}\}$.

57. The equation $A\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 2+i & i & 3-2i \\ i & 1-i & 4+3i \\ 3-i & 1+i & 1+5i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 2+i & i & 3-2i & 0 \\ i & 1-i & 4+3i & 0 \\ 3-i & 1+i & 1+5i & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} i & 1-i & 4+3i & 0 \\ 2+i & i & 3-2i & 0 \\ 3-i & 1+i & 1+5i & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & -1-i & 3-3i & 0 \\ 2+i & i & 3-2i & 0 \\ 3-i & 1+i & 1+5i & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & -1-i & 3-4i & 0 \\ 0 & 1+4i & -7+3i & 0 \\ 0 & 5+3i & -4+20i & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|c} 1 & -1-i & 3-4i & 0 \\ 0 & 1 & \frac{5+31i}{17} & 0 \\ 0 & 5+3i & -4+20i & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{25-32i}{17} & 0 \\ 0 & 1 & \frac{5+31i}{17} & 0 \\ 0 & 0 & 10i & 0 \end{array}\right]$$

$$\overset{6}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & \frac{25-32i}{17} & 0 \\ 0 & 1 & \frac{5+31i}{17} & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \overset{7}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

1. P_{12} **2.** $\mathrm{M}_1(-i)$ **3.** $\mathrm{A}_{12}(-2-i)$, $\mathrm{A}_{13}(-3+i)$ **4.** $\mathrm{M}_2(\frac{1-4i}{17})$ **5.** $\mathrm{A}_{21}(1+i)$, $\mathrm{A}_{23}(-5-3i)$ **6.** $\mathrm{M}_3(-i/10)$ **7.** $\mathrm{A}_{31}(\frac{-25+32i}{17})$, $\mathrm{A}_{32}(\frac{-5-31i}{17})$

From the last augmented matrix above, we see that the only solution to this system is the trivial solution.

Solutions to Section 2.6

True-False Review:

(a): FALSE. An invertible matrix is also known as a *nonsingular* matrix.

(b): FALSE. For instance, the matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ does not contain a row of zeros, but fails to be invertible.

(c): TRUE. If A is invertible, then the unique solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$.

(d): FALSE. For instance, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $AB = I_2$, but A is not even a square matrix, hence certainly not invertible.

(e): FALSE. For instance, if $A = I_n$ and $B = -I_n$, then A and B are both invertible, but $A + B = 0_n$ is not invertible.

(f): TRUE. We have

$$(AB)B^{-1}A^{-1} = I_n \quad \text{and} \quad B^{-1}A^{-1}(AB) = I_n,$$

and therefore, AB is invertible, with inverse $B^{-1}A^{-1}$.

(g): TRUE. From $A^2 = A$, we subtract to obtain $A(A-I) = 0$. Left multiplying both sides of this equation by A^{-1} (since A is invertible, A^{-1} exists), we have $A - I = A^{-1}0 = 0$. Therefore, $A = I$, the identity matrix.

(h): TRUE. From $AB = AC$, we left-multiply both sides by A^{-1} (since A is invertible, A^{-1} exists) to obtain $A^{-1}AB = A^{-1}AC$. Since $A^{-1}A = I$, we obtain $IB = IC$, or $B = C$.

(i): TRUE. Any 5×5 invertible matrix must have rank 5, not rank 4 (Theorem 2.6.6).

(j): TRUE. Any 6×6 matrix of rank 6 is invertible (Theorem 2.6.6).

Problems:

1. We have

$$AA^{-1} = \begin{bmatrix} 4 & 9 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 7 & -9 \\ -3 & 4 \end{bmatrix} = \begin{bmatrix} (4)(7) + (9)(-3) & (4)(-9) + (9)(4) \\ (3)(7) + (7)(-3) & (3)(-9) + (7)(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

2. We have

$$AA^{-1} = \begin{bmatrix} 2 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} (2)(-1) + (-1)(-3) & (2)(1) + (-1)(2) \\ (3)(-1) + (-1)(-3) & (3)(1) + (-1)(2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

3. We have

$$\begin{aligned} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) &= \frac{1}{ad-bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2, \end{aligned}$$

and

$$\begin{aligned} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2. \end{aligned}$$

4. We have

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 3 & 5 & 1 \\ 1 & 2 & 1 \\ 2 & 6 & 7 \end{bmatrix} \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix} \\ &= \begin{bmatrix} (3)(8) + (5)(-5) + (1)(2) & (3)(-29) + (5)(19) + (1)(-8) & (3)(3) + (5)(-2) + (1)(1) \\ (1)(8) + (2)(-5) + (1)(2) & (1)(-29) + (2)(19) + (1)(-8) & (1)(3) + (2)(-2) + (1)(1) \\ (2)(8) + (6)(-5) + (7)(2) & (2)(-29) + (6)(19) + (7)(-8) & (2)(3) + (6)(-2) + (7)(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3. \end{aligned}$$

5. We have

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right] = [I_2|A^{-1}].$$

Therefore,

$$A^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-1) \quad \textbf{2. } \mathrm{A}_{21}(-2)}$$

6. We have

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 0 \\ 1-i & 1 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 0 \\ 0 & -1 & -1+i & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|cc} 1 & 1+i & 1 & 0 \\ 0 & 1 & 1-i & -1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 1+i \\ 0 & 1 & 1-i & -1 \end{array}\right] = [I_2|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} -1 & 1+i \\ 1-i & -1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-1+i) \quad \textbf{2. } \mathrm{M}_2(-1) \quad \textbf{3. } \mathrm{A}_{21}(-1-i)}$$

7. We have

$$[A|I_2] = \left[\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ i-1 & 2 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ 0 & 1-i & 1-i & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|cc} 1 & -i & 1 & 0 \\ 0 & 1 & 1 & \frac{1+i}{2} \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 1+i & \frac{-1+i}{2} \\ 0 & 1 & 1 & \frac{1+i}{2} \end{array}\right] = [I_2|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 1+i & \frac{-1+i}{2} \\ 1 & \frac{1+i}{2} \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(1-i) \quad \textbf{2. } \mathrm{M}_2(1/(1-i)) \quad \textbf{3. } \mathrm{A}_{21}(i)}$$

8. Note that $AB = 0_2$ for all 2×2 matrices B. Therefore, A is not invertible.

9. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 11 & 0 & 1 & 0 \\ 4 & -3 & 10 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 7 & -2 & 1 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 3 & 7 & -2 & 1 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -3 & 0 & 1 \\ 0 & 1 & 2 & -4 & 0 & 1 \\ 0 & 0 & 1 & 10 & 1 & -3 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -43 & -4 & 13 \\ 0 & 1 & 0 & -24 & -2 & 7 \\ 0 & 0 & 1 & 10 & 1 & -3 \end{array}\right] = [I_3|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} -43 & -4 & 13 \\ -24 & -2 & 7 \\ 10 & 1 & -3 \end{bmatrix}.$$

1. $A_{12}(-2), A_{13}(-4)$ **2.** P_{23} **3.** $A_{21}(1), A_{23}(-3)$ **4.** $A_{31}(-4), A_{32}(-2)$

10. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 3 & 5 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 3 & 5 & 1 & 1 & 0 & 0 \\ 2 & 6 & 7 & 0 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 & -3 & 0 \\ 0 & 2 & 5 & 0 & -2 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & 2 & 5 & 0 & -2 & 1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 2 & -5 & 0 \\ 0 & 1 & 2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -8 & 1 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -29 & 3 \\ 0 & 1 & 0 & -5 & 19 & -2 \\ 0 & 0 & 1 & 2 & -8 & 1 \end{array}\right] = [I_3|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 8 & -29 & 3 \\ -5 & 19 & -2 \\ 2 & -8 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-3), A_{13}(-2)$ **3.** $M_2(-1)$ **4.** $A_{21}(-2), A_{23}(-2)$ **5.** $A_{31}(3), A_{32}(-2)$

11. This matrix is not invertible, because the column of zeros guarantees that the rank of the matrix is less than three.

12. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 4 & 2 & -13 & 1 & 0 & 0 \\ 2 & 1 & -7 & 0 & 1 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 3 & 2 & 4 & 0 & 0 & 1 \\ 2 & 1 & -7 & 0 & 1 & 0 \\ 4 & 2 & -13 & 1 & 0 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 11 & 0 & -1 & 1 \\ 2 & 1 & -7 & 0 & 1 & 0 \\ 4 & 2 & -13 & 1 & 0 & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 11 & 0 & -1 & 1 \\ 0 & -1 & -29 & 0 & 3 & -2 \\ 0 & -2 & -57 & 1 & 4 & -4 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 1 & 11 & 0 & -1 & 1 \\ 0 & 1 & 29 & 0 & -3 & 2 \\ 0 & -2 & -57 & 1 & 4 & -4 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -18 & 0 & 2 & -1 \\ 0 & 1 & 29 & 0 & -3 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}\right]$$

$$\overset{6}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 18 & -34 & -1 & \\ 0 & 1 & 0 & -29 & 55 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{array}\right] = [I_3|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 18 & -34 & -1 \\ -29 & 55 & 2 \\ 1 & -2 & 0 \end{bmatrix}.$$

1. P_{13} **2.** $A_{21}(-1)$ **3.** $A_{12}(-2), A_{13}(-4)$ **4.** $M_2(-1)$
5. $A_{21}(-1), A_{23}(2)$ **6.** $A_{31}(18), A_{32}(-29)$

13. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 2 & 6 & -2 & 0 & 1 & 0 \\ -1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 2 & 4 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 2 & -3 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & \frac{1}{2} & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\stackrel{3}{\sim}\left[\begin{array}{rrr|rrr} 1 & 0 & -7 & 3 & -1 & 0 \\ 0 & 1 & 2 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & -5 & 4 & -\frac{3}{2} & 1 \end{array}\right] \stackrel{4}{\sim}\left[\begin{array}{rrr|rrr} 1 & 0 & -7 & 3 & -1 & 0 \\ 0 & 1 & 2 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{3}{10} & -\frac{1}{5} \end{array}\right]$$

$$\stackrel{5}{\sim}\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{13}{5} & \frac{11}{10} & -\frac{7}{5} \\ 0 & 1 & 0 & \frac{3}{5} & -\frac{1}{10} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{4}{5} & \frac{3}{10} & -\frac{1}{5} \end{array}\right] = [I_3|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} -\frac{13}{5} & \frac{11}{10} & -\frac{7}{5} \\ \frac{3}{5} & -\frac{1}{10} & \frac{2}{5} \\ -\frac{4}{5} & \frac{3}{10} & -\frac{1}{5} \end{bmatrix}.$$

1. $A_{12}(-2)$, $A_{13}(1)$ **2.** $M_2(\frac{1}{2})$ **3.** $A_{21}(-2)$, $A_{23}(-3)$ **4.** $M_3(-\frac{1}{5})$ **5.** $A_{31}(7)$, $A_{32}(-2)$

14. We have

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 1+i & -1 & 2i & 0 & 1 & 0 \\ 2 & 2i & 5 & 0 & 0 & 1 \end{array}\right] \stackrel{1}{\sim}\left[\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & -i & -2 & -1-i & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \stackrel{2}{\sim}\left[\begin{array}{ccc|ccc} 1 & i & 2 & 1 & 0 & 0 \\ 0 & 1 & -2i & 1-i & i & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\stackrel{3}{\sim}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -i & 1 & 0 \\ 0 & 1 & -2i & 1-i & i & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] \stackrel{4}{\sim}\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -i & 1 & 0 \\ 0 & 1 & 0 & 1-5i & i & 2i \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right] = [I_3|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} -i & 1 & 0 \\ 1-5i & i & 2i \\ -2 & 0 & 1 \end{bmatrix}.$$

1. $A_{12}(-1-i)$, $A_{13}(-2)$ **2.** $M_2(i)$ **3.** $A_{21}(-i)$ **4.** $A_{32}(2i)$

15. We have

$$[A|I_3] = \left[\begin{array}{rrr|rrr} 2 & 1 & 3 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \stackrel{1}{\sim}\left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 3 & 1 & 0 & 0 \\ 3 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \stackrel{2}{\sim}\left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 6 & -2 & 0 & -3 & 1 \end{array}\right]$$

$$\stackrel{3}{\sim}\left[\begin{array}{rrr|rrr} 1 & -1 & 2 & 0 & 1 & 0 \\ 0 & 3 & -1 & 1 & -2 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right]$$

Since $2 = \text{rank}(A) < \text{rank}(A^{\#}) = 3$, we know that A^{-1} does not exist (we have obtained a row of zeros in the block matrix on the left.

1. P_{12} **2.** $A_{12}(-2)$, $A_{13}(-3)$ **3.** $A_{23}(-2)$

16. We have

$$[A|I_4] = \left[\begin{array}{rrrr|rrrr} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 2 & 0 & 3 & -4 & 0 & 1 & 0 & 0 \\ 3 & -1 & 7 & 8 & 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \end{array}\right] \stackrel{1}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & -10 & -2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & -3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\stackrel{2}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & -1 & 2 & 3 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & -1 & -3 & 0 & 1 & 0 \\ 0 & 2 & -1 & -10 & -2 & 1 & 0 & 0 \end{array}\right] \stackrel{3}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & -1 & 0 & 1 & -2 \\ 0 & 0 & -3 & -14 & 0 & 1 & 0 & -2 \end{array}\right]$$

$$\stackrel{4}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 3 & 5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 & 1 & 0 & -1 & 2 \\ 0 & 0 & -3 & -14 & 0 & 1 & 0 & -2 \end{array}\right] \stackrel{5}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & -10 & -3 & 0 & 3 & -5 \\ 0 & 1 & 0 & -3 & -2 & 0 & 1 & -1 \\ 0 & 0 & 1 & 5 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 3 & 1 & -3 & 4 \end{array}\right]$$

$$\stackrel{6}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 27 & 10 & -27 & 35 \\ 0 & 1 & 0 & 0 & 7 & 3 & -8 & 11 \\ 0 & 0 & 1 & 0 & -14 & -5 & 14 & -18 \\ 0 & 0 & 0 & 1 & 3 & 1 & -3 & 4 \end{array}\right] = [I_4|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 27 & 10 & -27 & 35 \\ 7 & 3 & -8 & 11 \\ -14 & -5 & 14 & -18 \\ 3 & 1 & -3 & 4 \end{bmatrix}.$$

1. $A_{12}(-2)$, $A_{13}(-3)$, $A_{14}(-1)$ **2.** P_{13} **3.** $A_{21}(1)$, $A_{23}(-2)$, $A_{24}(-2)$
4. $M_3(-1)$ **5.** $A_{31}(-3)$, $A_{32}(-1)$, $A_{34}(3)$ **6.** $A_{41}(10)$, $A_{42}(3)$, $A_{43}(5)$

17. We have

$$[A|I_4] = \left[\begin{array}{rrrr|rrrr} 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 3 & -1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \stackrel{1}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 3 & -1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\stackrel{2}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 4 & 2 & -3 & 0 & 1 & -2 & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & -6 & 0 & 0 & -3 & 1 \end{array}\right] \stackrel{3}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 & 0 & 0 \\ 0 & 5 & -2 & -6 & 0 & 0 & -3 & 1 \end{array}\right]$$

$$\stackrel{4}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{9}{2} & 1 & \frac{1}{2} & -1 & 0 \\ 0 & 0 & -\frac{9}{2} & -\frac{9}{4} & 0 & -\frac{5}{4} & -\frac{1}{2} & 1 \end{array}\right] \stackrel{5}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{9}{2} & -\frac{9}{4} & 0 & -\frac{5}{4} & -\frac{1}{2} & 1 \\ 0 & 0 & 0 & -\frac{9}{2} & 1 & \frac{1}{2} & -1 & 0 \end{array}\right]$$

$$\stackrel{6}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & -\frac{3}{4} & 0 & \frac{1}{4} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{5}{18} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 0 & -\frac{9}{2} & 1 & \frac{1}{2} & -1 & 0 \end{array}\right] \stackrel{7}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & -1 & 0 & \frac{1}{9} & -\frac{5}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{5}{18} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 0 & -\frac{9}{2} & 1 & \frac{1}{2} & -1 & 0 \end{array}\right]$$

$$\stackrel{8}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & -1 & 0 & \frac{1}{9} & -\frac{5}{9} & \frac{1}{9} \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{5}{18} & \frac{1}{9} & -\frac{2}{9} \\ 0 & 0 & 0 & 1 & -\frac{2}{9} & -\frac{1}{9} & \frac{2}{9} & 0 \end{array}\right] \stackrel{9}{\sim}\left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ 0 & 1 & 0 & 0 & -\frac{2}{9} & 0 & -\frac{1}{3} & \frac{1}{9} \\ 0 & 0 & 1 & 0 & \frac{1}{9} & \frac{1}{3} & 0 & -\frac{2}{9} \\ 0 & 0 & 0 & 1 & -\frac{2}{9} & -\frac{1}{9} & \frac{2}{9} & 0 \end{array}\right] = [I_4|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} 0 & \frac{2}{9} & -\frac{1}{9} & \frac{2}{9} \\ -\frac{2}{9} & 0 & -\frac{1}{3} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{3} & 0 & -\frac{2}{9} \\ -\frac{2}{9} & -\frac{1}{9} & \frac{2}{9} & 0 \end{bmatrix}.$$

1. P_{13} **2.** $A_{12}(-2)$, $A_{14}(-3)$ **3.** $M_2(\frac{1}{4})$ **4.** $A_{21}(2)$, $A_{23}(2)$, $A_{24}(-5)$
5. P_{34} **6.** $M_3(-\frac{2}{9})$ **7.** $A_{31}(-1)$, $A_{32}(-\frac{1}{2})$ **8.** $M_4(-\frac{2}{9})$ **9.** $A_{42}(1)$, $A_{43}(-\frac{1}{2})$

18. We have

$$= \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 7 & 8 & 0 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{6}{5} & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 7 & 8 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{2}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{6}{5} & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & -\frac{2}{5} & 0 & 0 & -\frac{7}{5} & 1 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -4 & 3 \\ 0 & 0 & 0 & -\frac{2}{5} & 0 & 0 & -\frac{7}{5} & 1 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -4 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & \frac{7}{2} & -\frac{5}{2} \end{array}\right] = [I_4|A^{-1}].$$

Thus,

$$A^{-1} = \begin{bmatrix} -2 & 1 & 0 & 0 \\ \frac{3}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -4 & 3 \\ 0 & 0 & \frac{7}{2} & -\frac{5}{2} \end{bmatrix}.$$

1. $A_{12}(-3)$, $M_3(\frac{1}{5})$ **2.** $A_{34}(-7)$ **3.** $A_{21}(1)$, $A_{13}(3)$ **4.** $M_2(-\frac{1}{2})$, $M_4(-\frac{5}{2})$

19. To determine the third column vector of A^{-1} without determining the whole inverse, we solve the system $\begin{bmatrix} -1 & -2 & 3 \\ -1 & 1 & 1 \\ -1 & -2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The corresponding augmented matrix $\left[\begin{array}{ccc|c} -1 & -2 & 3 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & -2 & -1 & 1 \end{array}\right]$ can be row-reduced to $\left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} \end{array}\right]$. Thus, back substitution yields $z = -\frac{1}{4}$, $y = -\frac{1}{6}$, and $x = -\frac{5}{12}$. Thus, the third column vector of A^{-1} is $\begin{bmatrix} -5/12 \\ -1/6 \\ -1/4 \end{bmatrix}$.

20. To determine the second column vector of A^{-1} without determining the whole inverse, we solve the linear system $\begin{bmatrix} 2 & -1 & 4 \\ 5 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The corresponding augmented matrix $\left[\begin{array}{ccc|c} 2 & -1 & 4 & 0 \\ 5 & 1 & 2 & 1 \\ 1 & -1 & 3 & 0 \end{array}\right]$ can be row-reduced to $\left[\begin{array}{rrr|r} 1 & -1 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$. Thus, back-substitution yields $z = -1$, $y = -2$, and $x = 1$. Thus, the second column vector of A^{-1} is $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$.

21. We have $A = \begin{bmatrix} 6 & 20 \\ 2 & 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -8 \\ 2 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} \frac{7}{2} & -10 \\ -1 & 3 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} \frac{7}{2} & -10 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -8 \\ 2 \end{bmatrix} = \begin{bmatrix} -48 \\ 14 \end{bmatrix}.$$

Hence, we have $x_1 = -48$ and $x_2 = 14$.

22. We have $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

So we have $x_1 = 4$ and $x_2 = -1$.

23. We have $A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 1 \\ 2 & 4 & -3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 7 & 5 & -3 \\ -2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

Hence, we have $x_1 = -2$, $x_2 = 2$, and $x_3 = 1$.

24. We have $A = \begin{bmatrix} 1 & -2i \\ 2-i & 4i \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ -i \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \frac{1}{2+8i} \begin{bmatrix} 4i & 2i \\ -2+i & 1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{2+8i} \begin{bmatrix} 4i & 2i \\ -2+i & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -i \end{bmatrix} = \frac{1}{2+8i} \begin{bmatrix} 2+8i \\ -4+i \end{bmatrix}.$$

Hence, we have $x_1 = 1$ and $x_2 = \frac{-4+i}{2+8i}$.

25. We have $A = \begin{bmatrix} 3 & 4 & 5 \\ 2 & 10 & 1 \\ 4 & 1 & 8 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \begin{bmatrix} -79 & 27 & 46 \\ 12 & -4 & -7 \\ 38 & -13 & -22 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -79 & 27 & 46 \\ 12 & -4 & -7 \\ 38 & -13 & -22 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}.$$

Hence, we have $x_1 = -6$, $x_2 = 1$, and $x_3 = 3$.

26. We have $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 12 \\ 24 \\ -36 \end{bmatrix}$, and the Gauss-Jordan method yields $A^{-1} = \frac{1}{12}\begin{bmatrix} -1 & 3 & 5 \\ 3 & 3 & -3 \\ 5 & -3 & -1 \end{bmatrix}$. Therefore, we have

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{12}\begin{bmatrix} -1 & 3 & 5 \\ 3 & 3 & -3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 12 \\ 24 \\ -36 \end{bmatrix} = \begin{bmatrix} -10 \\ 18 \\ 2 \end{bmatrix}.$$

Hence, $x_1 = -10$, $x_2 = 18$, and $x_3 = 2$.

27. We have

$$AA^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} (0)(0)+(1)(1) & (0)(-1)+(1)(0) \\ (-1)(0)+(0)(1) & (-1)(-1)+(0)(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2,$$

so $A^T = A^{-1}$.

28. We have

$$\begin{aligned} AA^T &= \begin{bmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \\ &= \begin{bmatrix} (\sqrt{3}/2)(\sqrt{3}/2)+(1/2)(1/2) & (\sqrt{3}/2)(-1/2)+(1/2)(\sqrt{3}/2) \\ (-1/2)(\sqrt{3}/2)+(\sqrt{3}/2)(1/2) & (-1/2)(-1/2)+(\sqrt{3}/2)(\sqrt{3}/2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

so $A^T = A^{-1}$.

29. We have

$$\begin{aligned} AA^T &= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \\ &= \begin{bmatrix} \cos^2\alpha+\sin^2\alpha & (\cos\alpha)(-\sin\alpha)+(\sin\alpha)(\cos\alpha) \\ (-\sin\alpha)(\cos\alpha)+(\cos\alpha)(\sin\alpha) & (-\sin\alpha)^2+\cos^2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \end{aligned}$$

so $A^T = A^{-1}$.

30. We have

$$\begin{aligned} AA^T &= \left(\frac{1}{1+2x^2}\right)\begin{bmatrix} 1 & -2x & 2x^2 \\ 2x & 1-2x^2 & -2x \\ 2x^2 & 2x & 1 \end{bmatrix}\left(\frac{1}{1+2x^2}\right)\begin{bmatrix} 1 & 2x & 2x^2 \\ -2x & 1-2x^2 & 2x \\ 2x^2 & -2x & 1 \end{bmatrix} \\ &= \left(\frac{1}{1+4x^2+4x^4}\right)\begin{bmatrix} 1+4x^2+4x^4 & 0 & 0 \\ 0 & 1+4x^2+4x^4 & 0 \\ 0 & 0 & 1+4x^2+4x^4 \end{bmatrix} = I_3, \end{aligned}$$

so $A^T = A^{-1}$.

31. For part 2, we have

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n,$$

and for part 3, we have

$$(A^{-1})^TA^T = (AA^{-1})^T = I_n^T = I_n.$$

32. We prove this by induction on k, with $k = 1$ trivial and $k = 2$ proven in part 2 of Theorem 2.6.10. Assuming the statement is true for a product involving $k-1$ matrices, we may proceed as follows:

$$\begin{aligned}(A_1A_2\cdots A_k)^{-1} = ((A_1A_2\cdots A_{k-1})A_k)^{-1} &= A_k^{-1}(A_1A_2\cdots A_{k-1})^{-1}\\ &= A_k^{-1}(A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}) = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}.\end{aligned}$$

In the second equality, we have applied part 2 of Theorem 2.6.10 to the two matrices $A_1A_2\cdots A_{k-1}$ and A_k, and in the third equality, we have assumed that the desired property is true for products of $k-1$ matrices.

33. Since A is skew-symmetric, we know that $A^T = -A$. We wish to show that $(A^{-1})^T = -A^{-1}$. We have

$$(A^{-1})^T = (A^T)^{-1} = (-A)^{-1} = -(A^{-1}),$$

which shows that A^{-1} is skew-symmetric. The first equality follows from part 3 of Theorem 2.6.10, and the second equality results from the assumption that A^{-1} is skew-symmetric.

34. Since A is symmetric, we know that $A^T = A$. We wish to show that $(A^{-1})^T = A^{-1}$. We have

$$(A^{-1})^T = (A^T)^{-1} = A^{-1},$$

which shows that A^{-1} is symmetric. The first equality follows from part 3 of Theorem 2.6.10, and the second equality results from the assumption that A is symmetric.

35. We have

$$\begin{aligned}(I_n - A^3)(I_n + A^3 + A^6 + A^9) &= I_n(I_n + A^3 + A^6 + A^9) - A^3(I_n + A^3 + A^6 + A^9)\\ &= I_n + A^3 + A^6 + A^9 - A^3 - A^6 - A^9 - A^{12} = I_n - A^{12} = I_n,\end{aligned}$$

where the last equality uses the assumption that $A^{12} = 0$. This calculation shows that $I_n - A^3$ and $I_n + A^3 + A^6 + A^9$ are inverses of one another.

36. We have

$$\begin{aligned}(I_n - A)(I_n + A + A^2 + A^3) &= I_n(I_n + A + A^2 + A^3) - A(I_n + A + A^2 + A^3)\\ &= I_n + A + A^2 + A^3 - A - A^2 - A^3 - A^4 = I_n - A^4 = I_n,\end{aligned}$$

where the last equality uses the assumption that $A^4 = 0$. This calculation shows that $I_n - A$ and $I_n + A + A^2 + A^3$ are inverses of one another.

37. We claim that the inverse of A^{15} is B^9. To verify this, use the fact that $A^5B^3 = I$ to observe that

$$A^{15}B^9 = A^5(A^5(A^5B^3)B^3)B^3 = A^5(A^5IB^3)B^3 = A^5(A^5B^3)B^3 = A^5IB^3 = A^5B^3 = I.$$

This calculation shows that the inverse of A^{15} is B^9.

38. We claim that the inverse of A^9 is B^{-3}. To verify this, use the fact that $A^3B^{-1} = I$ to observe that

$$A^9B^{-3} = A^3(A^3(A^3B^{-1})B^{-1})B^{-1} = A^3(A^3IB^{-1})B^{-1} = A^3(A^3B^{-1})B^{-1} = A^3IB^{-1} = A^3B^{-1} = I.$$

This calculation shows that the inverse of A^9 is B^{-3}.

39. We have

$$B = BI_n = B(AC) = (BA)C = I_nC = C.$$

40. YES. Since $BA = I_n$, we know that $A^{-1} = B$ (see Theorem 2.6.12). Likewise, since $CA = I_n$, $A^{-1} = C$. Since the inverse of A is unique, it must follow that $B = C$.

41. We can simply compute

$$\frac{1}{\Delta}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \frac{1}{\Delta}\begin{bmatrix} a_{22}a_{11} - a_{12}a_{21} & a_{22}a_{12} - a_{12}a_{22} \\ -a_{21}a_{11} + a_{11}a_{21} & -a_{21}a_{12} + a_{11}a_{22} \end{bmatrix}$$
$$= \frac{1}{\Delta}\begin{bmatrix} a_{11}a_{22} - a_{12}a_{21} & 0 \\ 0 & a_{11}a_{22} - a_{12}a_{21} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

Therefore,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\Delta}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

42. Assume that A is an invertible matrix and that $A\mathbf{x}_i = \mathbf{b}_i$ for $i = 1, 2, \dots, p$ (where each $\mathbf{b}_i$ is given). Use elementary row operations on the augmented matrix of the system to obtain the equivalence

$$[A|\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\ \dots\ \mathbf{b}_p] \sim [I_n|\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3\ \dots\ \mathbf{c}_p].$$

The solutions to the system can be read from the last matrix: $\mathbf{x}_i = \mathbf{c}_i$ for each $i = 1, 2, \dots, p$.

43. We have

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 2 \\ 2 & -1 & 4 & 1 & 2 & 3 \\ 1 & 1 & 6 & -1 & 5 & 2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & -1 & 2 \\ 0 & 1 & 2 & -1 & 4 & -1 \\ 0 & 2 & 5 & -2 & 6 & 0 \end{array}\right]$$
$$\overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 3 & 1 \\ 0 & 1 & 2 & -1 & 4 & -1 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 9 & -5 \\ 0 & 1 & 0 & -1 & 8 & -5 \\ 0 & 0 & 1 & 0 & -2 & 2 \end{array}\right].$$

Hence,

$$\mathbf{x}_1 = (0, -1, 0), \qquad \mathbf{x}_2 = (9, 8, -2), \qquad \mathbf{x}_3 = (-5, -5, 2).$$

1. $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{13}(-1)$ **2.** $\mathrm{A}_{21}(1)$, $\mathrm{A}_{23}(-2)$ **3.** $\mathrm{A}_{31}(-3)$, $\mathrm{A}_{32}(-2)$

44.

(a). Let $\mathbf{e}_i$ denote the ith column vector of the identity matrix I_m, and consider the m linear systems of equations

$$A\mathbf{x}_i = \mathbf{e}_i$$

for $i = 1, 2, \dots, m$. Since $\text{rank}(A) = m$ and each $\mathbf{e}_i$ is a column m-vector, it follows that

$$\text{rank}(A^{\#}) = m = \text{rank}(A)$$

and so each of the systems $A\mathbf{x}_i = \mathbf{e}_i$ above has a solution (Note that if $m < n$, then there will be an infinite number of solutions). If we let $B = [\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m]$, then

$$AB = A[\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m] = [A\mathbf{x}_1, A\mathbf{x}_2, \dots, A\mathbf{x}_m] = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m] = I_n.$$

(b). A right inverse for A in this case is a 3×2 matrix $\begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$ such that

$$\begin{bmatrix} a + 3b + c & d + 3e + f \\ 2a + 7b + 4c & 2d + 7e + 4f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus, we must have

$$a+3b+c=1, \quad d+3e+f=0, \quad 2a+7b+4c=0, \quad 2d+7e+4f=1.$$

The first and third equation comprise a linear system with augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 2 & 7 & 4 & 0 \end{array}\right]$ for a, b, and c. The row-echelon form of this augmented matrix is $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & -2 \end{array}\right]$. Setting $c=t$, we have $b=-2-2t$ and $a=7+5t$. Next, the second and fourth equation above comprise a linear system with augmented matrix $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 7 & 4 & 1 \end{array}\right]$ for d, e, and f. The row-echelon form of this augmented matrix is $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right]$. Setting $f=s$, we have $e=1-2s$ and $d=-3+5s$. Thus, right inverses of A are precisely the matrices of the form $\begin{bmatrix} 7+5t & -3+5s \\ -2-2t & 1-2s \\ t & s \end{bmatrix}$.

<u>Solutions to Section 2.7</u>

True-False Review:

(a): TRUE. Since every elementary matrix corresponds to a (reversible) elementary row operation, the reverse elementary row operation will correspond to an elementary matrix that is the inverse of the original elementary matrix.

(b): FALSE. For instance, the matrices $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ are both elementary matrices, but their product, $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, is not.

(c): FALSE. Every *invertible* matrix can be expressed as a product of elementary matrices. Since every elementary matrix is invertible and products of invertible matrices are invertible, any product of elementary matrices must be an invertible matrix.

(d): TRUE. Performing an elementary row operation on a matrix does not alter its rank, and the matrix EA is obtained from A by performing the elementary row operation associated with the elementary matrix E. Therefore, A and EA have the same rank.

(e): FALSE. If P_{ij} is a permutation matrix, then $P_{ij}^2 = I_n$, since permuting the ith and jth rows of I_n twice yields I_n. Alternatively, we can observe that $P_{ij}^2 = I_n$ from the fact that $P_{ij}^{-1} = P_{ij}$.

(f): FALSE. For example, consider the elementary matrices $E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then we have $E_1E_2 = \begin{bmatrix} 1 & 1 \\ 0 & 7 \end{bmatrix}$ and $E_2E_1 = \begin{bmatrix} 1 & 7 \\ 0 & 7 \end{bmatrix}$.

(g): FALSE. For example, consider the elementary matrices $E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Then we have $E_1E_2 = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2E_1 = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(h): FALSE. The only matrices we perform an LU factorization for are invertible matrices for which the reduction to upper triangular form can be accomplished without permuting rows.

(i): FALSE. The matrix U need not be a *unit* upper triangular matrix.

(j): FALSE. As can be seen in Example 2.7.8, a 4×4 matrix with LU factorization will have 6 multipliers, not 10 multipliers.

Problems:

1.

Permutation Matrices: $P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$

Scaling Matrices: $M_1(k) = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_2(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad M_3(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{bmatrix}.$

Row Combinations:

$$A_{12}(k) = \begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{13}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ k & 0 & 1 \end{bmatrix}, \quad A_{23}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & k & 1 \end{bmatrix},$$

$$A_{21}(k) = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{31}(k) = \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_{32}(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{bmatrix}.$$

2. We have

$$\begin{bmatrix} -4 & -1 \\ 0 & 3 \\ -3 & 7 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -1 & -8 \\ 0 & 3 \\ -3 & 7 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 8 \\ 0 & 3 \\ -3 & 7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 8 \\ 0 & 3 \\ 0 & 31 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 8 \\ 0 & 1 \\ 0 & 31 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 8 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{A}_{31}(-1) \quad \textbf{2. } \mathrm{M}_1(-1) \quad \textbf{3. } \mathrm{A}_{13}(3) \quad \textbf{4. } \mathrm{M}_2(\tfrac{1}{3}) \quad \textbf{5. } \mathrm{A}_{23}(-31)}$$

Elementary Matrices: $\mathrm{A}_{23}(31)$, $\mathrm{M}_2(\frac{1}{3})$, $\mathrm{A}_{13}(3)$, $\mathrm{M}_1(-1)$, $\mathrm{A}_{31}(-1)$.

3. We have

$$\begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 11 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(-3) \quad \textbf{3. } \mathrm{M}_2(\tfrac{1}{11})}$$

Elementary Matrices: $\mathrm{M}_2(\frac{1}{11})$, $A_{12}(-3)$, P_{12}.

4. We have

$$\begin{bmatrix} 5 & 8 & 2 \\ 1 & 3 & -1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 5 & 8 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & -7 & 7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 3 & -1 \\ 0 & 1 & -1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(-5) \quad \textbf{3. } \mathrm{M}_2(-\tfrac{1}{7})}$$

Elementary Matrices: $\mathrm{M}_2(-\frac{1}{7})$, $\mathrm{A}_{12}(-5)$, P_{12}.

5. We have

$$\begin{bmatrix} 3 & -1 & 4 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 3 \\ 3 & -1 & 4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & -10 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix}.$$

1. P_{13} **2.** $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{13}(-3)$ **3.** $\mathrm{A}_{23}(-2)$ **4.** $\mathrm{M}_2(-\frac{1}{5})$

Elementary Matrices: $\mathrm{M}_2(-\frac{1}{5})$, $\mathrm{A}_{23}(-2)$, $\mathrm{A}_{13}(-3)$, $\mathrm{A}_{12}(-2)$, P_{13}.

6. We have

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -2 & -3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -2 & -4 & -6 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

1. $\mathrm{A}_{12}(-2)$, $\mathrm{A}_{13}(-3)$ **2.** $\mathrm{M}_2(-1)$ **3.** $\mathrm{A}_{23}(2)$

Elementary Matrices: $\mathrm{A}_{23}(2)$, $\mathrm{M}_2(-1)$, $\mathrm{A}_{13}(-3)$, $\mathrm{A}_{12}(-2)$.

7. We reduce A to the identity matrix:

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. $\mathrm{A}_{12}(-1)$ **2.** $\mathrm{A}_{21}(-2)$

The elementary matrices corresponding to these row operations are $E_1 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$. We have $E_2E_1A = I_2$, so that

$$A = E_1^{-1}E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since E_1^{-1} and E_2^{-1} are elementary matrices.

8. We reduce A to the identity matrix:

$$\begin{bmatrix} -2 & -3 \\ 5 & 7 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 1 \\ -2 & -3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. $\mathrm{A}_{12}(2)$ **2.** P_{12} **3.** $\mathrm{A}_{12}(2)$ **4.** $\mathrm{A}_{21}(1)$ **5.** $\mathrm{M}_2(-1)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have $E_5E_4E_3E_2E_1A = I_2$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

9. We reduce A to the identity matrix:

$$\begin{bmatrix} 3 & -4 \\ -1 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -1 & 2 \\ 3 & -4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{M}_1(-1) \quad \textbf{3. } \mathrm{A}_{12}(-3) \quad \textbf{4. } \mathrm{M}_2(\tfrac{1}{2}) \quad \textbf{5. } \mathrm{A}_{21}(2)}$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}.$$

We have $E_5E_4E_3E_2E_1A = I_2$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

10. We reduce A to the identity matrix:

$$\begin{bmatrix} 4 & -5 \\ 1 & 4 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 \\ 4 & -5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & -21 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(-4) \quad \textbf{3. } \mathrm{M}_2(-\tfrac{1}{21}) \quad \textbf{4. } \mathrm{A}_{21}(-4)}$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{21} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}.$$

We have $E_4E_3E_2E_1A = I_2$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -21 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

11. We reduce A to the identity matrix:

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 3 & 1 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 4 & 3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-2) \quad \textbf{2. } \mathrm{A}_{13}(-3) \quad \textbf{3. } \mathrm{A}_{23}(-1) \quad \textbf{4. } \mathrm{M}_2(\tfrac{1}{4}) \quad \textbf{5. } \mathrm{A}_{32}(-\tfrac{1}{2}) \quad \textbf{6. } \mathrm{A}_{21}(1)}$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $E_6E_5E_4E_3E_2E_1A = I_3$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

12. We reduce A to the identity matrix:

$$\begin{bmatrix} 0 & -4 & -2 \\ 1 & -1 & 3 \\ -2 & 2 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ -2 & 2 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ 0 & 0 & 8 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & -1 & 3 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $\mathrm{A}_{13}(2)$ **3.** $\mathrm{M}_3(\frac{1}{8})$ **4.** $\mathrm{A}_{32}(2)$ **5.** $\mathrm{A}_{31}(-3)$ **6.** $\mathrm{M}_2(-\frac{1}{4})$ **7.** $\mathrm{A}_{21}(1)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{8} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_5 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_7 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $E_7E_6E_5E_4E_3E_2E_1A = I_3$, so

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

13. We reduce A to the identity matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 8 & 0 \\ 3 & 4 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & -4 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

1. $\mathrm{M}_2(\frac{1}{8})$ **2.** $\mathrm{A}_{13}(-3)$ **3.** $\mathrm{A}_{21}(-2)$ **4.** $\mathrm{A}_{23}(2)$ **5.** $\mathrm{M}_3(-\frac{1}{4})$ **6.** $\mathrm{A}_{31}(-3)$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have $E_6E_5E_4E_3E_2E_1A = I_3$, so

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which is the desired expression since each E_i^{-1} is an elementary matrix.

14. We reduce A to the identity matrix:

$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & -7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(-2) \quad \textbf{3. } \mathrm{M}_2(-\tfrac{1}{7}) \quad \textbf{4. } \mathrm{A}_{21}(-3)}$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

Direct multiplication verifies that $E_4E_3E_2E_1A = I_2$.

15. We have

$$\begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & -2 \\ 0 & \frac{13}{3} \end{bmatrix} = U.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(\tfrac{1}{3})}$$

Hence, $E_1 = \mathrm{A}_{12}(\frac{1}{3})$. Then Equation (2.7.3) reads $L = E_1^{-1} = \mathrm{A}_{12}(-\frac{1}{3}) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}$. Verifying Equation (2.7.2):

$$LU = \begin{bmatrix} 1 & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & \frac{13}{3} \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} = A.$$

16. We have

$$\begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & 3 \\ 0 & -\frac{13}{2} \end{bmatrix} = U \Longrightarrow m_{21} = \frac{5}{2} \Longrightarrow L = \begin{bmatrix} 1 & 0 \\ \frac{5}{2} & 1 \end{bmatrix}.$$

Then

$$LU = \begin{bmatrix} 1 & 0 \\ \frac{5}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 0 & -\frac{13}{2} \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix} = A.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-\tfrac{5}{2})}$$

17. We have

$$\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & 1 \\ 0 & \frac{1}{3} \end{bmatrix} = U \Longrightarrow m_{21} = \frac{5}{3} \Longrightarrow L = \begin{bmatrix} 1 & 0 \\ \frac{5}{3} & 1 \end{bmatrix}.$$

Then

$$LU = \begin{bmatrix} 1 & 0 \\ \frac{5}{3} & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} = A.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-\tfrac{5}{3})}$$

18. We have

$$\begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 4 & 4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 16 \end{bmatrix} = U \Longrightarrow m_{21} = 2, m_{31} = -1, m_{32} = 4.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 16 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 2 \\ 6 & -1 & 1 \\ -3 & 5 & 2 \end{bmatrix} = A.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-2),\ \mathrm{A}_{13}(1) \quad \textbf{2. } \mathrm{A}_{23}(-4)}$$

19. We have

$$\begin{bmatrix} 5 & 2 & 1 \\ -10 & -2 & 3 \\ 15 & 2 & -3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 5 \\ 0 & -4 & -6 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix} = U \Longrightarrow m_{21} = -2, m_{31} = 3, m_{32} = -2.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 2 & 1 \\ -10 & -2 & 3 \\ 15 & 2 & -3 \end{bmatrix} = A.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(2),\ \mathrm{A}_{13}(-3) \quad \textbf{2. } \mathrm{A}_{23}(2)}$$

20. We have

$$\begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & -4 \\ 3 & -1 & 7 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 2 & 1 & -1 \\ 0 & 4 & 2 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 4 & 22 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix} = U.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-2),\ \mathrm{A}_{13}(-3),\ \mathrm{A}_{14}(-1) \quad \textbf{2. } \mathrm{A}_{23}(-1),\ \mathrm{A}_{24}(-2) \quad \textbf{3. } \mathrm{A}_{34}(-2)}$$

Hence,

$$m_{21} = 2, \quad m_{31} = 3, m_{41} = 1, m_{32} = 1, m_{42} = 2, m_{43} = 2.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & 3 \\ 0 & 2 & -1 & -10 \\ 0 & 0 & 2 & 9 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & 3 \\ 2 & 0 & 3 & -4 \\ 3 & -1 & 7 & 8 \\ 1 & 3 & 4 & 5 \end{bmatrix} = A.$$

21. We have

$$\begin{bmatrix} 2 & -3 & 1 & 2 \\ 4 & -1 & 1 & 1 \\ -8 & 2 & 2 & -5 \\ 6 & 1 & 5 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & -10 & 6 & 3 \\ 0 & 10 & 2 & -4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 4 & 2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = U.$$

$\boxed{\textbf{1. } A_{12}(-2),\ A_{13}(4),\ A_{14}(-3) \quad \textbf{2. } A_{23}(2),\ A_{24}(-2) \quad \textbf{3. } A_{34}(-1)}$

Hence,

$$m_{21} = 2, \quad m_{31} = -4, \quad m_{41} = 3, \quad m_{32} = -2, \quad m_{42} = 2, \quad m_{43} = 1.$$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix} \text{ and } LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -4 & -2 & 1 & 0 \\ 3 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 & 2 \\ 0 & 5 & -1 & -3 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 & 2 \\ 4 & -1 & 1 & 1 \\ -8 & 2 & 2 & -5 \\ 6 & 1 & 5 & 2 \end{bmatrix} = A.$$

22. We have

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} = U \Longrightarrow m_{21} = 2 \Longrightarrow L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

$\boxed{\textbf{1. } A_{12}(-2)}$

We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$. Then $U\mathbf{x} = \mathbf{y}$ yields $\mathbf{x} = \begin{bmatrix} -11 \\ 7 \end{bmatrix}$.

23. We have

$$\begin{bmatrix} 1 & -3 & 5 \\ 3 & 2 & 2 \\ 2 & 5 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -13 \\ 0 & 11 & -8 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -3 & 5 \\ 0 & 11 & -13 \\ 0 & 0 & 5 \end{bmatrix} = U \Longrightarrow m_{21} = 3, m_{31} = 2, m_{32} = 1.$$

$\boxed{\textbf{1. } A_{12}(-3),\ A_{13}(-2) \quad \textbf{2. } A_{23}(-1)}$

Hence, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$. We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}$. Then $U\mathbf{x} = \mathbf{y}$ yields $\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$.

24. We have

$$\begin{bmatrix} 2 & 2 & 1 \\ 6 & 3 & -1 \\ -4 & 2 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & 2 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -4 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 2 & 2 & 1 \\ 0 & -3 & -4 \\ 0 & 0 & -4 \end{bmatrix} = U \Longrightarrow m_{21} = 3, m_{31} = -2, m_{32} = -2.$$

1. $\mathrm{A}_{12}(-3)$, $\mathrm{A}_{13}(2)$ **2.** $\mathrm{A}_{23}(2)$

Hence, $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix}$. We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$. Then $U\mathbf{x} = \mathbf{y}$ yields $\mathbf{x} = \begin{bmatrix} -1/12 \\ 1/3 \\ 1/2 \end{bmatrix}$.

25. We have

$$\begin{bmatrix} 4 & 3 & 0 & 0 \\ 8 & 1 & 2 & 0 \\ 0 & 5 & 3 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 4 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 5 & 3 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 4 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & -5 & 7 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 4 & 3 & 0 & 0 \\ 0 & -5 & 2 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 13 \end{bmatrix} = U.$$

1. $\mathrm{A}_{12}(-2)$ **2.** $\mathrm{A}_{23}(1)$ **3.** $\mathrm{A}_{34}(1)$

The only nonzero multipliers are $m_{21} = 2, m_{32} = -1$, and $m_{43} = -1$. Hence, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$. We now solve the triangular systems $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$. From $L\mathbf{y} = \mathbf{b}$, we obtain $\mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 4 \end{bmatrix}$. Then $U\mathbf{x} = \mathbf{y}$ yields $\mathbf{x} = \begin{bmatrix} 677/1300 \\ -9/325 \\ -37/65 \\ 4/13 \end{bmatrix}$.

26. We have

$$\begin{bmatrix} 2 & -1 \\ -8 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & -1 \\ 0 & -1 \end{bmatrix} = U \Longrightarrow m_{21} = -4 \Longrightarrow L = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}.$$

1. $\mathrm{A}_{12}(4)$

We now solve the triangular systems

$$L\mathbf{y}_i = \mathbf{b}_i, \qquad U\mathbf{x}_i = \mathbf{y}_i$$

for $i = 1, 2, 3$. We have

$$L\mathbf{y}_1 = \mathbf{b}_1 \Longrightarrow \mathbf{y}_1 = \begin{bmatrix} 3 \\ 11 \end{bmatrix}. \text{ Then } U\mathbf{x}_1 = \mathbf{y}_1 \Longrightarrow \mathbf{x}_1 = \begin{bmatrix} -4 \\ -11 \end{bmatrix};$$

$$L\mathbf{y}_2 = \mathbf{b}_2 \Longrightarrow \mathbf{y}_2 = \begin{bmatrix} 2 \\ 15 \end{bmatrix}. \text{ Then } U\mathbf{x}_2 = \mathbf{y}_2 \Longrightarrow \mathbf{x}_2 = \begin{bmatrix} -6.5 \\ -15 \end{bmatrix};$$

$$L\mathbf{y}_3 = \mathbf{b}_3 \Longrightarrow \mathbf{y}_3 = \begin{bmatrix} 5 \\ 11 \end{bmatrix}. \text{ Then } U\mathbf{x}_3 = \mathbf{y}_3 \Longrightarrow \mathbf{x}_3 = \begin{bmatrix} -3 \\ -11 \end{bmatrix}.$$

27. We have

$$\begin{bmatrix} -1 & 4 & 2 \\ 3 & 1 & 4 \\ 5 & -7 & 1 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -1 & 4 & 2 \\ 0 & 13 & 10 \\ 0 & 13 & 11 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} -1 & 4 & 2 \\ 0 & 13 & 10 \\ 0 & 0 & 1 \end{bmatrix} = U.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(3), \mathrm{A}_{13}(5) \quad \textbf{2. } \mathrm{A}_{23}(-1)}$$

Thus, $m_{21} = -3$, $m_{31} = -5$, and $m_{32} = 1$. We now solve the triangular systems

$$L\mathbf{y}_i = \mathbf{b}_i, \qquad U\mathbf{x}_i = \mathbf{y}_i$$

for $i = 1, 2, 3$. We have

$$L\mathbf{y}_1 = \mathbf{e}_1 \Longrightarrow \mathbf{y}_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}. \text{ Then } U\mathbf{x}_1 = \mathbf{y}_1 \Longrightarrow \mathbf{x}_1 = \begin{bmatrix} -29/13 \\ -17/13 \\ 2 \end{bmatrix};$$

$$L\mathbf{y}_2 = \mathbf{e}_2 \Longrightarrow \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}. \text{ Then } U\mathbf{x}_2 = \mathbf{y}_2 \Longrightarrow \mathbf{x}_2 = \begin{bmatrix} 18/13 \\ 11/13 \\ -1 \end{bmatrix};$$

$$L\mathbf{y}_3 = \mathbf{e}_3 \Longrightarrow \mathbf{y}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \text{ Then } U\mathbf{x}_3 = \mathbf{y}_3 \Longrightarrow \mathbf{x}_3 = \begin{bmatrix} -14/13 \\ -10/13 \\ 1 \end{bmatrix}.$$

28. Observe that if P_i is an elementary permutation matrix, then $P_i^{-1} = P_i = P_i^T$. Therefore, we have

$$P^{-1} = (P_1P_2 \dots P_k)^{-1} = P_k^{-1}P_{k-1}^{-1} \dots P_2^{-1}P_1^{-1} = P_k^T P_{k-1}^T \dots P_2^T \dots P_1^T = (P_1P_2 \dots P_k)^T = P^T.$$

29.

(a). Let A be an invertible upper triangular matrix with inverse B. Therefore, we have $AB = I_n$. Write $A = [a_{ij}]$ and $B = [b_{ij}]$. We will show that $b_{ij} = 0$ for all $i > j$, which shows that B is upper triangular. We have

$$\sum_{k=1}^{n} a_{ik}b_{kj} = \delta_{ij}.$$

Since A is upper triangular, $a_{ik} = 0$ whenever $i > k$. Therefore, we can reduce the above summation to

$$\sum_{k=i}^{n} a_{ik}b_{ij} = \delta_{ij}.$$

Let $i = n$. Then the above summation reduces to $a_{nn}b_{nj} = \delta_{nj}$. If $j = n$, we have $a_{nn}b_{nn} = 1$, so $a_{nn} \neq 0$. For $j < n$, we have $a_{nn}b_{nj} = 0$, and therefore $b_{nj} = 0$ for all $j < n$.

Next let $i = n-1$. Then we have

$$a_{n-1,n-1}b_{n-1,j} + a_{n-1,n}b_{nj} = \delta_{n-1,j}.$$

Setting $j = n-1$ and using the fact that $b_{n,n-1} = 0$ by the above calculation, we obtain $a_{n-1,n-1}b_{n-1,n-1} = 1$, so $a_{n-1,n-1} \neq 0$. For $j < n-1$, we have $a_{n-1,n-1}b_{n-1,j} = 0$ so that $b_{n-1,j} = 0$.

Next let $i = n-2$. Then we have $a_{n-2,n-2}b_{n-2,j} + a_{n-2,n-1}b_{n-1,j} + a_{n-2,n}b_{nj} = \delta_{n-2,j}$. Setting $j = n-2$ and using the fact that $b_{n-1,n-2} = 0$ and $b_{n,n-2} = 0$, we have $a_{n-2,n-2}b_{n-2,n-2} = 1$, so that $a_{n-2,n-2} \neq 0$. For $j < n-2$, we have $a_{n-2,n-2}b_{n-2,j} = 0$ so that $b_{n-2,j} = 0$.

Proceeding in this way, we eventually show that $b_{ij} = 0$ for all $i > j$.

For an invertible lower triangular matrix A with inverse B, we can either modify the preceding argument, or we can proceed more briefly as follows: Note that A^T is an invertible upper triangular matrix with inverse B^T. By the preceding argument, B^T is upper triangular. Therefore, B is lower triangular, as required.

(b). Let A be an invertible unit upper triangular matrix with inverse B. Use the notations from (a). By (a), we know that B is upper triangular. We simply must show that $b_{jj} = 0$ for all j. From $a_{nn}b_{nn} = 1$ (see proof of (a)), we see that if $a_{nn} = 1$, then $b_{nn} = 1$. Moreover, from $a_{n-1,n-1}b_{n-1,n-1} = 1$, the fact that $a_{n-1,n-1} = 1$ proves that $b_{n-1,n-1} = 1$. Likewise, the fact that $a_{n-2,n-2}b_{n-2,n-2} = 1$ implies that if $a_{n-2,n-2} = 1$, then $b_{n-2,n-2} = 1$. Continuing in this fashion, we prove that $b_{jj} = 1$ for all j.

For the last part, if A is an invertible unit lower triangular matrix with inverse B, then A^T is an invertible unit upper triangular matrix with inverse B^T, and by the preceding argument, B^T is a unit upper triangular matrix. This implies that B is a unit lower triangular matrix, as desired.

30.

(a). Since A is invertible, Corollary 2.6.13 implies that both L_2 and U_1 are invertible. Since $L_1U_1 = L_2U_2$, we can left-multiply by L_2^{-1} and right-multiply by U_1^{-1} to obtain $L_2^{-1}L_1 = U_2U_1^{-1}$.

(b). By Problem 29, we know that L_2^{-1} is a unit lower triangular matrix and U_1^{-1} is an upper triangular matrix. Therefore, $L_2^{-1}L_1$ is a unit lower triangular matrix and $U_2U_1^{-1}$ is an upper triangular matrix. Since these two matrices are equal, we must have $L_2^{-1}L_1 = I_n$ and $U_2U_1^{-1} = I_n$. Therefore, $L_1 = L_2$ and $U_1 = U_2$.

31. The system $A\mathbf{x} = \mathbf{b}$ can be written as $QR\mathbf{x} = \mathbf{b}$. If we can solve $Q\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ and then solve $R\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, then $QR\mathbf{x} = \mathbf{b}$ as desired. Multiplying $Q\mathbf{y} = \mathbf{b}$ by Q^T and using the fact that $Q^TQ = I_n$, we obtain $\mathbf{y} = Q^T\mathbf{b}$. Therefore, $R\mathbf{x} = \mathbf{y}$ can be replaced by $R\mathbf{x} = Q^T\mathbf{b}$. Therefore, to solve $A\mathbf{x} = \mathbf{b}$, we first determine $\mathbf{y} = Q^T\mathbf{b}$ and then solve the upper triangular system $R\mathbf{x} = Q^T\mathbf{b}$ by back-substitution.

Solutions to Section 2.8

True-False Review:

(a): FALSE. According to the given information, part (c) of the Invertible Matrix Theorem fails, while part (e) holds. This is impossible.

(b): TRUE. This holds by the equivalence of parts (d) and (f) of the Invertible Matrix Theorem.

(c): FALSE. Part (d) of the Invertible Matrix Theorem fails according to the given information, and therefore part (b) also fails. Hence, the equation $A\mathbf{x} = \mathbf{b}$ does not have a unique solution. But it is not valid to conclude that the equation has infinitely many solutions; it could have no solutions. For instance, if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, there are no solutions to $A\mathbf{x} = \mathbf{b}$, although $\text{rank}(A) = 2$.

(d): FALSE. An easy counterexample is the matrix 0_n, which fails to be invertible even though it is upper triangular. Since it fails to be invertible, it cannot e row-equivalent to I_n, by the Invertible Matrix Theorem.

Problems:

1. Since A is an invertible matrix, the only solution to $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$. However, if we assume that $AB = AC$, then $A(B - C) = 0$. If $\mathbf{x}_i$ denotes the ith column of $B - C$, then $\mathbf{x}_i = \mathbf{0}$ for each i. That is, $B - C = 0$, or $B = C$, as required.

2. If $\text{rank}(A) = n$, then the augmented matrix $A^{\#}$ for the system $A\mathbf{x} = \mathbf{0}$ can be reduced to REF such that each column contains a pivot except for the right-most column of all-zeros. Solving the system by back-substitution, we find that $\mathbf{x} = \mathbf{0}$, as claimed.

3. Since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, REF(A) contains a pivot in every column. Therefore, the linear system $A\mathbf{x} = \mathbf{b}$ can be solved by back-substitution for every $\mathbf{b}$ in $\mathbb{R}^n$. Therefore, $A\mathbf{x} = \mathbf{b}$ does have a solution.

Now suppose there are two solutions $\mathbf{y}$ and $\mathbf{z}$ to the system $A\mathbf{x} = \mathbf{b}$. That is, $A\mathbf{y} = \mathbf{b}$ and $A\mathbf{z} = \mathbf{b}$. Subtracting, we find

$$A(\mathbf{y} - \mathbf{z}) = \mathbf{0},$$

and so by assumption, $\mathbf{y} - \mathbf{z} = \mathbf{0}$. That is, $\mathbf{y} = \mathbf{z}$. Therefore, there is only one solution to the linear system $A\mathbf{x} = \mathbf{b}$.

4. If A and B are each invertible matrices, then A and B can each be expressed as a product of elementary matrices, say

$$A = E_1E_2 \dots E_k \qquad \text{and} \qquad B = E_1'E_2' \dots E_l'.$$

Then

$$AB = E_1E_2 \dots E_kE_1'E_2' \dots E_l',$$

so AB can be expressed as a product of elementary matrices. Thus, by the equivalence of (a) and (e) in the Invertible Matrix Theorem, AB is invertible.

5. We are assuming that the equations $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ each have only the trivial solution $\mathbf{x} = \mathbf{0}$. Now consider the linear system

$$(AB)\mathbf{x} = \mathbf{0}.$$

Viewing this equation as

$$A(B\mathbf{x}) = \mathbf{0},$$

we conclude that $B\mathbf{x} = \mathbf{0}$. Thus, $\mathbf{x} = \mathbf{0}$. Hence, the linear equation $(AB)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solutions to Section 2.9

Problems:

1. $A^T - 5B = \begin{bmatrix} -2 & -1 \\ 4 & -1 \\ 2 & 5 \\ 6 & 0 \end{bmatrix} - \begin{bmatrix} -15 & 0 \\ 10 & 10 \\ 5 & -15 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 13 & -1 \\ -6 & -11 \\ -3 & 20 \\ 6 & -5 \end{bmatrix}.$

2. $C^TB = \begin{bmatrix} -5 & -6 & 3 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 2 & 2 \\ 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -20 \end{bmatrix}.$

3. Since A is not a square matrix, it is not possible to compute A^2.

4. $-4A - B^T = \begin{bmatrix} 8 & -16 & -8 & -24 \\ 4 & 4 & -20 & 0 \end{bmatrix} - \begin{bmatrix} -3 & 2 & 1 & 0 \\ 0 & 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & -18 & -9 & -24 \\ 4 & 2 & -17 & -1 \end{bmatrix}.$

5. We have

$$AB = \begin{bmatrix} -2 & 4 & 2 & 6 \\ -1 & -1 & 5 & 0 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 2 & 2 \\ 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 16 & 8 \\ 6 & -17 \end{bmatrix}.$$

Moreover,

$$\mathrm{tr}(AB) = -1.$$

6. We have

$$(AC)(AC)^T = \begin{bmatrix} -2 \\ 26 \end{bmatrix} \begin{bmatrix} -2 & 26 \end{bmatrix} = \begin{bmatrix} 4 & -52 \\ -52 & 676 \end{bmatrix}.$$

7. $(-4B)A = \begin{bmatrix} 12 & 0 \\ -8 & -8 \\ -4 & 12 \\ 0 & -4 \end{bmatrix} \begin{bmatrix} -2 & 4 & 2 & 6 \\ -1 & -1 & 5 & 0 \end{bmatrix} = \begin{bmatrix} -24 & 48 & 24 & 72 \\ 24 & -24 & -56 & -48 \\ -4 & -28 & 52 & -24 \\ 4 & 4 & -20 & 0 \end{bmatrix}.$

8. Using Problem 5, we find that

$$(AB)^{-1} = \begin{bmatrix} 16 & 8 \\ 6 & -17 \end{bmatrix}^{-1} = -\frac{1}{320} \begin{bmatrix} -17 & -8 \\ -6 & 16 \end{bmatrix}.$$

9. We have

$$C^TC = \begin{bmatrix} -5 & -6 & 3 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -6 \\ 3 \\ 1 \end{bmatrix} = [71],$$

and

$$\mathrm{tr}(C^TC) = 71.$$

10.

(a). We have

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix} \begin{bmatrix} 3 & b \\ -4 & a \\ a & b \end{bmatrix} = \begin{bmatrix} 3a-5 & 2a+4b \\ 7a-14 & 5a+9b \end{bmatrix}.$$

In order for this product to equal I_2, we require

$$3a-5=1, \quad 2a+4b=0, \quad 7a-14=0, \quad 5a+9b=1.$$

We quickly solve this for the unique solution: $a = 2$ and $b = -1$.

(b). We have

$$BA = \begin{bmatrix} 3 & -1 \\ -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & -1 & -1 \end{bmatrix}.$$

11. We compute the (i,j)-entry of each side of the equation. We will denote the entries of A^T by a_{ij}^T, which equals a_{ji}. On the left side, note that the (i,j)-entry of $(AB^T)^T$ is the same as the (j,i)-entry of AB^T, and

$$(j,i)\text{-entry of } AB^T = \sum_{k=0}^{n} a_{jk}b_{ki}^T = \sum_{k=0}^{n} a_{jk}b_{ik} = \sum_{k=0}^{n} b_{ik}a_{kj}^T,$$

and the latter expression is the (i, j)-entry of BA^T. Therefore, the (i, j)-entries of $(AB^T)^T$ and BA^T are the same, as required.

12.

(a). The (i, j)-entry of A^2 is

$$\sum_{k=1}^{n} a_{ik}a_{kj}.$$

(b). Assume that A is symmetric. That means that $A^T = A$. We claim that A^2 is symmetric. To see this, note that

$$(A^2)^T = (AA)^T = A^T A^T = AA = A^2.$$

Thus, $(A^2)^T = A^2$, and so A^2 is symmetric.

13. We are assuming that A is skew-symmetric, so $A^T = -A$. To show that $B^T AB$ is skew-symmetric, we observe that

$$(B^T AB)^T = B^T A^T (B^T)^T = B^T A^T B = B^T(-A)B = -(B^T AB),$$

as required.

14. We have

$$A^2 = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

so A is nilpotent.

15. We have

$$A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$A^3 = A^2 A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so A is nilpotent.

16. We have

$$A'(t) = \begin{bmatrix} -3e^{-3t} & -2\sec^2 t \tan t \\ 6t^2 & -\sin t \\ 6/t & -5 \end{bmatrix}.$$

17. We have

$$\int_0^1 B(t)\, dt = \begin{bmatrix} -7t & t^3/3 \\ 6t - t^2/2 & 3t^4/4 + 2t^3 \\ t + t^2/2 & \frac{2}{\pi}\sin(\pi t/2) \\ e^t & t - t^4/4 \end{bmatrix} \Bigg|_0^1 = \begin{bmatrix} -7 & 1/3 \\ 11/2 & 11/4 \\ 3/2 & 2/\pi \\ e-1 & 3/4 \end{bmatrix}.$$

18. Since $A(t)$ is 3×2 and $B(t)$ is 4×2, it is impossible to perform the indicated subtraction.

19. Since $A(t)$ is 3×2 and $B(t)$ is 4×2, it is impossible to perform the indicated subtraction.

20. From the last equation, we see that $x_3 = 0$. Substituting this into the middle equation, we find that $x_2 = 0.5$. Finally, putting the values of x_2 and x_3 into the first equation, we find $x_1 = -6 - 2.5 = -8.5$. Thus, there is a unique solution to the linear system, and the solution set is

$$\{(-8.5, 0.5, 0)\}.$$

21. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{rrr|r} 5 & -1 & 2 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & 11 & 20 & 7 \\ -2 & 6 & 9 & 0 \\ -7 & 5 & -3 & -7 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 11 & 20 & 7 \\ 0 & 28 & 49 & 14 \\ 0 & 82 & 137 & 42 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 11 & 20 & 7 \\ 0 & 1 & 7/4 & 1/2 \\ 0 & 82 & 137 & 42 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{rrr|r} 1 & 11 & 20 & 7 \\ 0 & 1 & 7/4 & 1/2 \\ 0 & 0 & -13/2 & 1 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrr|r} 1 & 11 & 20 & 7 \\ 0 & 1 & 7/4 & 1/2 \\ 0 & 0 & 1 & -2/13 \end{array}\right].$$

From the last row, we conclude that $x_3 = -2/13$, and using the middle row, we can solve for x_2: we have $x_2 + \frac{7}{4} \cdot \left(-\frac{2}{13}\right) = \frac{1}{2}$, so $x_2 = \frac{20}{26} = \frac{10}{13}$. Finally, from the first row we can get x_1: we have $x_1 + 11 \cdot \frac{10}{13} + 20 \cdot \left(-\frac{2}{13}\right) = 7$, and so $x_1 = \frac{21}{13}$. So there is a unique solution:

$$\left\{\left(\frac{21}{13}, \frac{10}{13}, -\frac{2}{13}\right)\right\}.$$

1. $A_{21}(2)$ **2.** $A_{12}(2)$, $A_{13}(7)$ **3.** $M_2(1/28)$ **4.** $A_{23}(-82)$ **5.** $M_3(-2/13)$

22. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 1 & 0 & 1 & 5 \\ 4 & 4 & 0 & 12 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & -4 & 4 & 8 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & -4 & 4 & 8 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From this row-echelon form, we see that z is a free variable. Set $z = t$. Then from the middle row of the matrix, $y = t - 2$, and from the top row, $x + 2(t-2) - t = 1$ or $x = -t + 5$. So the solution set is

$$\{(-t+5, t-2, t) : t \in \mathbb{R}\} = \{(5, -2, 0) + t(-1, 1, 1) : t \in \mathbb{R}\}.$$

1. $A_{12}(-1)$, $A_{13}(-4)$ **2.** $M_2(-1/2)$ **3.** $A_{23}(4)$

23. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{rrrr|r} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

The bottom row of this matrix shows that this system has no solutions.

1. $A_{12}(2), A_{13}(-3)$ **2.** $A_{23}(1)$ **3.** $M_2(1/3), M_3(1/3)$

24. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccccc|c} 3 & 0 & -1 & 2 & -1 & 1 \\ 1 & 3 & 1 & -3 & 2 & -1 \\ 4 & -2 & -3 & 6 & -1 & 5 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 3 & 0 & -1 & 2 & -1 & 1 \\ 4 & -2 & -3 & 6 & -1 & 5 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -9 & -4 & 11 & -7 & 4 \\ 0 & -14 & -7 & 18 & -9 & 9 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 28 & 14 & -36 & 18 & -18 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \stackrel{4}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right]$$

$$\stackrel{5}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & -27 & -12 & 33 & -21 & 12 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \stackrel{6}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 42 & -48 & -102 & -150 \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right] \stackrel{7}{\sim} \left[\begin{array}{ccccc|c} 1 & 3 & 1 & -3 & 2 & -1 \\ 0 & 1 & 2 & -3 & -3 & -6 \\ 0 & 0 & 1 & -\frac{8}{7} & -\frac{17}{7} & -\frac{25}{7} \\ 0 & 0 & 0 & 1 & 4 & -2 \end{array}\right].$$

We see that $x_5 = t$ is the only free variable. Back substitution yields the remaining values:

$$x_5 = t, \quad x_4 = -4t - 2, \quad x_3 = -\frac{41}{7} - \frac{15}{7}t, \quad x_2 = -\frac{2}{7} - \frac{33}{7}t, \quad x_1 = -\frac{2}{7} + \frac{16}{7}t.$$

So the solution set is

$$\left\{\left(-\frac{2}{7} + \frac{16}{7}t, -\frac{2}{7} - \frac{33}{7}t, -\frac{41}{7} - \frac{15}{7}t, -4t - 2, t\right) : t \in \mathbb{R}\right\}$$
$$= \left\{t\left(\frac{16}{7}, -\frac{33}{7}, -\frac{15}{7}, -4, 1\right) + \left(-\frac{2}{7}, -\frac{2}{7}, -\frac{41}{7}, -2, 0\right) : t \in \mathbb{R}\right\}.$$

1. P_{12} **2.** $A_{12}(-3), A_{13}(-4)$ **3.** $M_2(3), M_3(-2)$ **4.** $A_{23}(1)$ **5.** P_{23} **6.** $A_{23}(27)$ **7.** $M_3(1/42)$

25. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 1 & 1 & 1 & 2 & -5 & 8 \\ 2 & 3 & 1 & 4 & -9 & 17 \\ 2 & 2 & 2 & 3 & -8 & 14 \end{array}\right] \stackrel{1}{\sim} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & -1 & 2 & -2 \end{array}\right] \stackrel{2}{\sim} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\stackrel{3}{\sim} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & -3 & 6 \\ 0 & 1 & -1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

From this row-echelon form, we see that $x_5 = t$ and $x_3 = s$ are free variables. Furthermore, solving this system by back-substitution, we see that

$$x_5 = t, \quad x_4 = 2t + 2, \quad x_3 = s, \quad x_2 = s - t + 1, \quad x_1 = 2t - 2s + 3.$$

So the solution set is

$$\{(2t-2s+3, s-t+1, s, 2t+2, t) : s,t \in \mathbb{R}\} = \{t(2,-1,0,2,1)+s(-2,1,1,0,0)+(3,1,0,2,0) : s,t \in \mathbb{R}\}.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-1),\ \mathrm{A}_{13}(-2),\ \mathrm{A}_{14}(-2) \quad \textbf{2. } \mathrm{A}_{24}(1) \quad \textbf{3. } \mathrm{P}_{23}}$$

26. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{rrr|r} 1 & -3 & 2i & 1 \\ -2i & 6 & 2 & -2 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 1 & -3 & 2i & 1 \\ 0 & 6-6i & -2 & -2+2i \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & -3 & 2i & 1 \\ 0 & 1 & -\frac{1}{6}(1+i) & -\frac{1}{3} \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(2i) \quad \textbf{2. } \mathrm{M}_2(\tfrac{1}{6-6i})}$$

From the last augmented matrix above, we see that x_3 is a free variable. Let us set $x_3 = t$, where t is a complex number. Then we can solve for x_2 using the equation corresponding to the second row of the row-echelon form: $x_2 = -\frac{1}{3}+\frac{1}{6}(1+i)t$. Finally, using the first row of the row-echelon form, we can determine that $x_1 = \frac{1}{2}t(1-3i)$. Therefore, the solution set for this linear system of equations is

$$\{(\frac{1}{2}t(1-3i), -\frac{1}{3}+\frac{1}{6}(1+i)t, t) : t \in \mathbb{C}\}.$$

27. We reduce the corresponding linear system as follows:

$$\left[\begin{array}{rr|r} 1 & -k & 6 \\ 2 & 3 & k \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rr|r} 1 & -k & 6 \\ 0 & 3+2k & k-12 \end{array}\right].$$

If $k \neq -\frac{3}{2}$, then each column of the row-reduced coefficient matrix will contain a pivot, and hence, the linear system will have a unique solution. If, on the other hand, $k = -\frac{3}{2}$, then the system is inconsistent, because the last row of the row-echelon form will have a pivot in the right-most column. Under no circumstances will the linear system have infinitely many solutions.

28. First observe that if $k = 0$, then the second equation requires that $x_3 = 2$, and then the first equation requires $x_2 = 2$. However, x_1 is a free variable in this case, so there are infinitely many solutions.

Now suppose that $k \neq 0$. Then multiplying each row of the corresponding augmented matrix for the linear system by $1/k$ yields a row-echelon form with pivots in the first two columns only. Therefore, the third variable, x_3, is free in this case. So once again, there are infinitely many solutions to the system.

We conclude that the system has infinitely many solutions for all values of k.

29. Since this linear system is homogeneous, it already has at least one solution: $(0,0,0)$. Therefore, it only remains to determine the values of k for which this will be the only solution. We reduce the corresponding matrix as follows:

$$\left[\begin{array}{rrr|r} 10 & k & -1 & 0 \\ k & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{rrr|r} 10k & k^2 & -k & 0 \\ 10k & 10 & -10 & 0 \\ 1 & 1/2 & -1/2 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{rrr|r} 1 & 1/2 & -1/2 & 0 \\ 10k & 10 & -10 & 0 \\ 10k & k^2 & -k & 0 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{rrr|r} 1 & 1/2 & -1/2 & 0 \\ 0 & 10-5k & 5k-10 & 0 \\ 0 & k^2-5k & 4k & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{rrr|r} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & k^2-5k & 4k & 0 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{rrr|r} 1 & 1/2 & -1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k^2-k & 0 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{M}_1(k),\ \mathrm{M}_2(10),\ \mathrm{M}_3(1/2) \quad \textbf{2. } \mathrm{P}_{13} \quad \textbf{3. } \mathrm{A}_{12}(-10k),\ \mathrm{A}_{13}(-10k) \quad \textbf{4. } \mathrm{M}_2(\tfrac{1}{10-5k}) \quad \textbf{5. } \mathrm{A}_{23}(5k-k^2)}$$

Note that the steps above are not valid if $k = 0$ or $k = 2$ (because Step 1 is not valid with $k = 0$ and Step 4 is not valid if $k = 2$). We will discuss those special cases individually in a moment. However if $k \neq 0, 2$, then the steps are valid, and we see from the last row of the last matrix that if $k = 1$, we have infinitely many solutions. Otherwise, if $k \neq 0, 1, 2$, then the matrix has full rank, and so there is a unique solution to the linear system.

If $k = 2$, then the last two rows of the original matrix are the same, and so the matrix of coefficients of the linear system is not invertible. Therefore, the linear system must have infinitely many solutions.

If $k = 0$, we reduce the original linear system as follows:

$$\left[\begin{array}{ccc|c} 10 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 1 & -1 & 0 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -4/5 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -1/10 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1/5 & 0 \end{array}\right].$$

The last matrix has full rank, so there will be a unique solution in this case.

$$\boxed{\textbf{1. } \mathrm{M}_1(1/10) \quad \textbf{2. } \mathrm{A}_{13}(-2) \quad \textbf{3. } \mathrm{A}_{23}(-1)}$$

To summarize: The linear system has infinitely many solutions if and only if $k = 1$ or $k = 2$. Otherwise, the system has a unique solution.

30. To solve this system, we need to reduce the corresponding augmented matrix for the linear system to row-echelon form. This gives us

$$\left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 1 & 0 & k & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 0 & k & k-k^2 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & k & k-k^2 & 0 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{ccc|c} 1 & -k & k^2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2k-k^2 & -k \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-1) \quad \textbf{2. } \mathrm{P}_{23} \quad \textbf{3. } \mathrm{A}_{23}(-k)}$$

Now provided that $2k - k^2 \neq 0$, the system can be solved without free variables via back-substitution, and therefore, there is a unique solution. Consider now what happens if $2k - k^2 = 0$. Then either $k = 0$ or $k = 2$. If $k = 0$, then only the first two columns of the last augmented matrix above are pivoted, and we have a free variable corresponding to x_3. Therefore, there are infinitely many solutions in this case. On the other hand, if $k = 2$, then the last row of the last matrix above reflects an inconsistency in the linear system, and there are no solutions.

To summarize, the system has no solutions if $k = 2$, a unique solution if $k \neq 0$ and $k \neq 2$, and infinitely many solutions if $k = 0$.

31. No, there are no common points of intersection. A common point of intersection would be indicated by a solution to the linear system consisting of the equations of the three planes. However, the corresponding augmented matrix can be row-reduced as follows:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 1 & 3 & 0 & 0 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -4 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -5 \end{array}\right].$$

The last row of this matrix shows that the linear system is inconsistent, and so there are no points common to all three planes.

$$\boxed{\textbf{1. } \mathrm{A}_{13}(-1) \quad \textbf{2. } \mathrm{A}_{23}(-1)}$$

32.

(a). We have

$$\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 7/4 \\ -2 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 7/4 \\ 0 & 17/2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 7/4 \\ 0 & 1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{M}_1(1/4) \quad \textbf{2. } \mathrm{A}_{12}(2) \quad \textbf{3. } \mathrm{M}_2(2/17)}$$

(b). We have: $\text{rank}(A) = 2$, since the row-echelon form of A in (a) consists two nonzero rows.

(c). We have

$$\left[\begin{array}{cc|cc} 4 & 7 & 1 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cc|cc} 1 & 7/4 & 1/4 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cc|cc} 1 & 7/4 & 1/4 & 0 \\ 0 & 17/2 & 1/2 & 1 \end{array}\right] \overset{3}{\sim} \left[\begin{array}{cc|cc} 1 & 7/4 & 1/4 & 0 \\ 0 & 1 & 1/17 & 2/17 \end{array}\right]$$

$$\overset{4}{\sim} \left[\begin{array}{cc|cc} 1 & 0 & 5/34 & -7/34 \\ 0 & 1 & 1/17 & 2/17 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{M}_1(1/4) \quad \textbf{2. } \mathrm{A}_{12}(2) \quad \textbf{3. } \mathrm{M}_2(2/17) \quad \textbf{4. } \mathrm{A}_{21}(-7/4)}$$

Thus,

$$A^{-1} = \begin{bmatrix} \frac{5}{34} & -\frac{7}{34} \\ \frac{1}{17} & \frac{2}{17} \end{bmatrix}.$$

33.

(a). We have

$$\begin{bmatrix} 2 & -7 \\ -4 & 14 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & -7 \\ 0 & 0 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -7/2 \\ 0 & 0 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(2) \quad \textbf{2. } \mathrm{M}_1(1/2)}$$

(b). We have: $\text{rank}(A) = 1$, since the row-echelon form of A in (a) has one nonzero row.

(c). Since $\text{rank}(A) < 2$, A is not invertible.

34.

(a). We have

$$\begin{bmatrix} 3 & -1 & 6 \\ 0 & 2 & 3 \\ 3 & -5 & 0 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 2 & 3 \\ 1 & -5/3 & 0 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 2 & 3 \\ 0 & -4/3 & -2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & -1/3 & 2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{M}_1(1/3), \mathrm{M}_3(1/3) \quad \textbf{2. } \mathrm{A}_{13}(-1) \quad \textbf{3. } \mathrm{A}_{23}(2/3) \quad \textbf{4. } \mathrm{M}_2(1/2)}$$

(b). We have: $\text{rank}(A) = 2$, since the row-echelon form of A in (a) consists of two nonzero rows.

(c). Since $\text{rank}(A) < 3$, A is not invertible.

35.

(a). We have

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$\overset{4}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 7 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $\text{A}_{12}(-2)$, $\text{A}_{34}(-1)$ **3.** P_{34} **4.** $\text{M}_2(-1/3)$, $\text{A}_{34}(-3)$ **5.** $\text{M}_4(1/7)$

(b). We have: $\text{rank}(A) = 4$, since the row-echelon form of A in (a) consists of four nonzero rows.

(c). We have

$$\left[\begin{array}{cccc|cccc} 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 3 & 4 & 0 & 0 & 1 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 7 & 0 & 0 & 4 & -3 \end{array}\right]$$

$$\overset{5}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2/3 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 4/7 & -3/7 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 2/3 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -3/7 & 4/7 \\ 0 & 0 & 0 & 1 & 0 & 0 & 4/7 & -3/7 \end{array}\right].$$

1. P_{12} **2.** $\text{A}_{12}(-2)$, $\text{A}_{34}(-1)$ **3.** P_{34} **4.** $\text{A}_{34}(-3)$, $\text{M}_2(-1/3)$ **5.** $\text{M}_4(1/7)$, $\text{A}_{21}(-2)$ **6.** $\text{A}_{43}(1)$

Thus,

$$A^{-1} = \begin{bmatrix} 2/3 & -1/3 & 0 & 0 \\ -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & -3/7 & 4/7 \\ 0 & 0 & 4/7 & -3/7 \end{bmatrix}.$$

36.

(a). We have

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & -1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \overset{5}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{M}_1(1/3) \quad \textbf{2. } \mathrm{A}_{13}(-1) \quad \textbf{3. } \mathrm{P}_{23} \quad \textbf{4. } \mathrm{A}_{23}(2) \quad \textbf{5. } \mathrm{M}_2(-1), \mathrm{M}_3(1/3)}$$

(b). We have: $\mathrm{rank}(A) = 3$, since the row-echelon form of A in (a) has 3 nonzero rows.

(c). We have

$$\left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & -1/3 & 0 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & -1 & 2 & -1/3 & 0 & 1 \\ 0 & 2 & -1 & 0 & 1 & 0 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & -1 & 2 & -1/3 & 0 & 1 \\ 0 & 0 & 3 & -2/3 & 1 & 2 \end{array}\right]$$

$$\overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & -2 & 1/3 & 0 & -1 \\ 0 & 0 & 1 & -2/9 & 1/3 & 2/3 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/3 & 0 & 0 \\ 0 & 1 & 0 & -1/9 & 2/3 & 1/3 \\ 0 & 0 & 1 & -2/9 & 1/3 & 2/3 \end{array}\right].$$

$$\boxed{\textbf{1. } \mathrm{M}_1(1/3) \quad \textbf{2. } \mathrm{A}_{13}(-1) \quad \textbf{3. } \mathrm{P}_{23} \quad \textbf{4. } \mathrm{A}_{23}(2) \quad \textbf{5. } \mathrm{M}_2(-1), \mathrm{M}_3(1/3) \quad \textbf{6. } \mathrm{A}_{32}(2)}$$

Hence,

$$A^{-1} = \begin{bmatrix} 1/3 & 0 & 0 \\ -1/9 & 2/3 & 1/3 \\ -2/9 & 1/3 & 2/3 \end{bmatrix}.$$

37.

(a). We have

$$\begin{bmatrix} -2 & -3 & 1 \\ 1 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ -2 & -3 & 1 \\ 0 & 5 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 5 & 3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 0 & -2 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(2) \quad \textbf{3. } \mathrm{A}_{23}(-1) \quad \textbf{4. } \mathrm{M}_2(1/5), \mathrm{M}_3(-1/2)}$$

(b). We have: $\mathrm{rank}(A) = 3$, since the row-echelon form of A in (a) consists of 3 nonzero rows.

(c). We have

$$\left[\begin{array}{ccc|ccc} -2 & -3 & 1 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ -2 & -3 & 1 & 1 & 0 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 5 & 5 & 1 & 2 & 0 \\ 0 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 5 & 5 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 & -2 & 1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1/5 & 2/5 & 0 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array}\right]$$

$$\overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -4/5 & -3/5 & 0 \\ 0 & 1 & 1 & 1/5 & 2/5 & 0 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array}\right] \overset{6}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 7/5 & -1 \\ 0 & 1 & 0 & -3/10 & -3/5 & 1/2 \\ 0 & 0 & 1 & 1/2 & 1 & -1/2 \end{array}\right].$$

1. P_{12} **2.** $A_{12}(2)$ **3.** $A_{23}(-1)$ **4.** $M_2(1/5)$, $M_3(-1/2)$ **5.** $A_{21}(-4)$ **6.** $A_{31}(2)$, $A_{32}(-1)$

Thus,

$$A^{-1} = \begin{bmatrix} 1/5 & 7/5 & -1 \\ -3/10 & -3/5 & 1/2 \\ 1/2 & 1 & -1/2 \end{bmatrix}.$$

38. We use the Gauss-Jordan method to find A^{-1}:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 4 & -3 & 13 & 0 & 1 & 0 \\ 1 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \overset{1}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right] \overset{2}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & -1 & 7 & -2 & 1 \end{array}\right]$$

$$\overset{3}{\sim} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -7 & 2 & -1 \end{array}\right] \overset{4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & -3 & 1 & 0 \\ 0 & 1 & 1 & -4 & 1 & 0 \\ 0 & 0 & 1 & -7 & 2 & -1 \end{array}\right] \overset{5}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 25 & -7 & 4 \\ 0 & 1 & 0 & 3 & -1 & 1 \\ 0 & 0 & 1 & -7 & 2 & -1 \end{array}\right].$$

1. $A_{12}(-4)$, $A_{13}(-1)$ **2.** $A_{23}(-2)$ **3.** $M_3(-1)$ **4.** $A_{21}(1)$ **5.** $A_{31}(-4)$, $A_{32}(-1)$

Thus,

$$A^{-1} = \begin{bmatrix} 25 & -7 & 4 \\ 3 & -1 & 1 \\ -7 & 2 & -1 \end{bmatrix}.$$

Now $\mathbf{x}_i = A^{-1}\mathbf{e}_i$ for each i. So

$$\mathbf{x}_1 = A^{-1}\mathbf{e}_1 = \begin{bmatrix} 25 \\ 3 \\ -7 \end{bmatrix}, \quad \mathbf{x}_2 = A^{-1}\mathbf{e}_2 = \begin{bmatrix} -7 \\ -1 \\ 2 \end{bmatrix}, \quad \mathbf{x}_3 = A^{-1}\mathbf{e}_3 = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}.$$

39. We have $\mathbf{x}_i = A^{-1}\mathbf{b}_i$, where

$$A^{-1} = -\frac{1}{39}\begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}.$$

Therefore,

$$\mathbf{x}_1 = A^{-1}\mathbf{b}_1 = -\frac{1}{39}\begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = -\frac{1}{39}\begin{bmatrix} -12 \\ -3 \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 12 \\ 3 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 4 \\ 1 \end{bmatrix},$$

$$\mathbf{x}_2 = A^{-1}\mathbf{b}_2 = -\frac{1}{39}\begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = -\frac{1}{39}\begin{bmatrix} -23 \\ -22 \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 23 \\ 22 \end{bmatrix},$$

and

$$\mathbf{x}_3 = A^{-1}\mathbf{b}_3 = -\frac{1}{39}\begin{bmatrix} -2 & -5 \\ -7 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 5 \end{bmatrix} = -\frac{1}{39}\begin{bmatrix} -21 \\ 24 \end{bmatrix} = \frac{1}{39}\begin{bmatrix} 21 \\ -24 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 7 \\ -8 \end{bmatrix}.$$

40.

(a). We have

$$(A^{-1}B)(B^{-1}A) = A^{-1}(BB^{-1})A = A^{-1}I_nA = A^{-1}A = I_n$$

and

$$(B^{-1}A)(A^{-1}B) = B^{-1}(AA^{-1})B = B^{-1}I_nB = B^{-1}B = I_n.$$

Therefore,

$$(B^{-1}A)^{-1} = A^{-1}B.$$

(b). We have

$$(A^{-1}B)^{-1} = B^{-1}(A^{-1})^{-1} = B^{-1}A,$$

as required.

41(a). We have $B^4 = (S^{-1}AS)(S^{-1}AS)(S^{-1}AS)(S^{-1}AS) = S^{-1}A(SS^{-1})A(SS^{-1})A(SS^{-1})AS = S^{-1}AIAIAIAS = S^{-1}A^4S$, as required.

41(b). We can prove this by induction on k. For $k = 1$, the result is $B = S^{-1}AS$, which was already given. Now assume that $B^k = S^{-1}A^kS$. Then $B^{k+1} = BB^k = S^{-1}AS(S^{-1}A^kS) = S^{-1}A(SS^{-1})A^kS = S^{-1}AIA^kS = S^{-1}A^{k+1}S$, which completes the induction step.

42.

(a). We reduce A to the identity matrix:

$$\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & \frac{7}{4} \\ -2 & 5 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & \frac{17}{2} \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & 1 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

1. $\mathrm{M}_1(\frac{1}{4})$ **2.** $\mathrm{A}_{12}(2)$ **3.** $\mathrm{M}_2(\frac{2}{17})$ **4.** $\mathrm{A}_{21}(-\frac{7}{4})$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{17} \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -\frac{7}{4} \\ 0 & 1 \end{bmatrix}.$$

We have $E_4E_3E_2E_1A = I_2$, so that

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{17}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{7}{4} \\ 0 & 1 \end{bmatrix},$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i.

(b). We can reduce A to upper triangular form by the following elementary row operation:

$$\begin{bmatrix} 4 & 7 \\ -2 & 5 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 4 & 7 \\ 0 & \frac{17}{2} \end{bmatrix}.$$

1. $\mathrm{A}_{12}(\frac{1}{2})$

Therefore we have the multiplier $m_{12} = -\frac{1}{2}$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 4 & 7 \\ 0 & \frac{17}{2} \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be easily verified by direct multiplication.

43.

(a). We reduce A to the identity matrix:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{8}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

1. P_{12} **2.** $A_{12}(-2)$ **3.** $M_2(-\frac{1}{3})$ **4.** $A_{21}(-2)$ **5.** $M_3(\frac{1}{3})$
6. $A_{34}(-4)$ **7.** $M_4(-\frac{3}{7})$ **8.** $A_{43}(-\frac{4}{3})$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{bmatrix}, \quad E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{3}{7} \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have

$$E_8E_7E_6E_5E_4E_3E_2E_1A = I_4$$

so that

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdots \\ &\quad \cdots \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{4}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i.

(b). We can reduce A to upper triangular form by the following elementary row operations:

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 4 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{A}_{12}(-\tfrac{1}{2}) \quad \textbf{2. } \mathrm{A}_{34}(-\tfrac{4}{3})}$$

Therefore, the nonzero multipliers are $m_{12} = \frac{1}{2}$ and $m_{34} = \frac{4}{3}$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & -\frac{7}{3} \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be easily verified by direct multiplication.

44.

(a). We reduce A to the identity matrix:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 3 & 0 & 0 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -1 \\ 0 & 3 & -6 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 3 & -6 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & -\frac{9}{2} \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \overset{8}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boxed{\begin{array}{c} \textbf{1. } \mathrm{P}_{13} \quad \textbf{2. } \mathrm{A}_{13}(-3) \quad \textbf{3. } \mathrm{M}_2(\tfrac{1}{2}) \quad \textbf{4. } \mathrm{A}_{23}(-3) \quad \textbf{5. } \mathrm{M}_3(-\tfrac{2}{9}) \\ \textbf{6. } \mathrm{A}_{21}(1) \quad \textbf{7. } \mathrm{A}_{31}(-\tfrac{3}{2}) \quad \textbf{8. } \mathrm{A}_{32}(\tfrac{1}{2}) \end{array}}$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{2}{9} \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_7 = \begin{bmatrix} 1 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}.$$

We have

$$E_8E_7E_6E_5E_4E_3E_2E_1A = I_3$$

so that

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1} \\ &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \cdots \\ &\qquad \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{9}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i.

(b). We can reduce A to upper triangular form by the following elementary row operations:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix}.$$

$$\boxed{\textbf{1. } \mathrm{A}_{13}(-\tfrac{1}{3}) \quad \textbf{2. } \mathrm{A}_{23}(\tfrac{1}{2})}$$

Therefore, the nonzero multipliers are $m_{13} = \frac{1}{3}$ and $m_{23} = -\frac{1}{2}$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & \frac{3}{2} \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be verified by direct multiplication.

45.

(a). We reduce A to the identity matrix:

$$\begin{bmatrix} -2 & -3 & 1 \\ 1 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ -2 & -3 & 1 \\ 0 & 5 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 5 & -3 \end{bmatrix} \overset{3}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 5 & 5 \\ 0 & 1 & -8 \end{bmatrix} \overset{4}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -8 \\ 0 & 5 & 5 \end{bmatrix}$$

$$\overset{5}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -8 \\ 0 & 0 & 45 \end{bmatrix} \overset{6}{\sim} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \overset{7}{\sim} \begin{bmatrix} 1 & 0 & 34 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \overset{8}{\sim} \begin{bmatrix} 1 & 0 & 34 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overset{9}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\boxed{\begin{array}{c} \textbf{1. } \mathrm{P}_{12} \quad \textbf{2. } \mathrm{A}_{12}(2) \quad \textbf{3. } \mathrm{A}_{23}(-1) \quad \textbf{4. } \mathrm{P}_{23} \quad \textbf{5. } \mathrm{A}_{23}(-5) \\ \textbf{6. } \mathrm{M}_3(\tfrac{1}{45}) \quad \textbf{7. } \mathrm{A}_{21}(-4) \quad \textbf{8. } \mathrm{A}_{32}(8) \quad \textbf{9. } \mathrm{A}_{31}(-34) \end{array}}$$

The elementary matrices corresponding to these row operations are

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}, \quad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{45} \end{bmatrix},$$

$$E_7 = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_9 = \begin{bmatrix} 1 & 0 & -34 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We have

$$E_9E_8E_7E_6E_5E_4E_3E_2E_1A = I_3$$

so that

$$\begin{aligned} A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}E_8^{-1}E_9^{-1} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdots \\ &\qquad \cdots \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 45 \end{bmatrix} \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 34 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \end{aligned}$$

which is the desired expression since E_i^{-1} is an elementary matrix for each i.

(b). We can reduce A to upper triangular form by the following elementary row operations:

$$\begin{bmatrix} -2 & -3 & 1 \\ 1 & 4 & 2 \\ 0 & 5 & 3 \end{bmatrix} \overset{1}{\sim} \begin{bmatrix} -2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 5 & 3 \end{bmatrix} \overset{2}{\sim} \begin{bmatrix} -2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -2 \end{bmatrix}.$$

Therefore, the nonzero multipliers are $m_{12} = -\frac{1}{2}$ and $m_{23} = 2$. Hence, setting

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} -2 & -3 & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \\ 0 & 0 & -2 \end{bmatrix},$$

we have the LU factorization $A = LU$, which can be verified by direct multiplication.

46(a). Using the distributive laws of matrix multiplication, first note that

$$(A+2B)^2 = (A+2B)(A+2B) = A(A+2B)+2B(A+2B) = A^2+A(2B)+(2B)A+(2B)^2 = A^2+2AB+2BA+4B^2.$$

Thus, we have

$$\begin{aligned} (A+2B)^3 &= (A+2B)(A+2B)^2 \\ &= A(A+2B)^2 + 2B(A+2B)^2 \\ &= A(A^2+2AB+2BA+4B^2) + 2B(A^2+2AB+2BA+4B^2) \\ &= A^3 + 2A^2B + 2ABA + 4AB^2 + 2BA^2 + 4BAB + 4B^2A + 8B^3, \end{aligned}$$

as needed.

46(b). Each occurrence of B in the answer to part (a) must now be accompanied by a minus sign. Therefore, all terms containing an odd number of Bs will experience a sign change. The answer is

$$(A-2B)^3 = A^3 - 2A^2B - 2ABA - 2BA^2 + 4AB^2 + 4BAB + 4B^2A - 8B^3.$$

47. The answer is 2^k, because each term in the expansion of $(A+B)^k$ consists of a string of k matrices, each of which is either A or B (2 possibilities for each matrix in the string). Multiplying the possibilities for each position in the string of length k, we get 2^k different strings, and hence 2^k different terms in the expansion of $(A+B)^k$. So, for instance, if $k=4$, we expect 16 terms, corresponding to the 16 strings $AAAA$, $AAAB$, $AABA$, $ABAA$, $BAAA$, $AABB$, $ABAB$, $ABBA$, $BAAB$, $BABA$, $BBAA$, $ABBB$, $BABB$, $BBAB$, $BBBA$, and $BBBB$. Indeed, one can verify that the expansion of $(A+B)^4$ is precisely the sum of the 16 terms we just wrote down.

48. We claim that

$$\begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix}.$$

To see this, simply note that

$$\begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} = I_{n+m}$$

and

$$\begin{pmatrix} A^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & I_m \end{pmatrix} = I_{n+m}.$$

49. For a 2×4 matrix, the leading ones can occur in 6 different positions:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a 3×4 matrix, the leading ones can occur in 4 different positions:

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For a 4×6 matrix, the leading ones can occur in 15 different positions:

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For an $m \times n$ matrix with $m \leq n$, the answer is the binomial coefficient

$$C(n,m) = \left(\begin{array}{c} n \\ m \end{array} \right) = \frac{n!}{m!(n-m)!}.$$

This represents n "choose" m, which is the number of ways to choose m columns from the n columns of the matrix in which to put the leading ones. This choice then determines the structure of the matrix.

50. We claim that the inverse of A^{10} is B^5. To prove this, use the fact that $A^2B = I$ to observe that

$$A^{10}B^5 = A^2A^2A^2A^2(A^2B)BBBB = A^2A^2A^2A^2IBBBB = A^2A^2A^2(A^2B)BBB$$

$$= A^2A^2A^2IBBB = A^2A^2(A^2B)BB = A^2A^2IBB = A^2(A^2B)B = A^2IB = A^2B = I,$$

as required.

51. We claim that the inverse of A^9 is B^6. To prove this, use the fact that $A^3B^2 = I$ to observe that

$$A^9B^6 = A^3A^3(A^3B^2)B^2B^2 = A^3A^3IB^2B^2 = A^3(A^3B^2)B^2 = A^3IB^2 = A^3B^2 = I,$$

as required.