CHAPTER 2 Linearity and Nonlinearity

2.1 Linear Equations: The Nature of Their Solutions

[redacted]

■ Classification

1. First-order, nonlinear

2. First-order, linear, nonhomogeneous, variable coefficients

3. Second-order, linear, homogeneous, variable coefficients

4. Second-order, linear, nonhomogeneous, variable coefficients

5. Third-order, linear, homogeneous, constant coefficients

6. Third-order, linear, nonhomogeneous, constant coefficients

7. Second-order, linear, nonhomogeneous, variable coefficients

8. Second-order, nonlinear

9. Second-order, linear, homogeneous, variable coefficients

10. Second-order, nonlinear

■ Linear Operation Notation

11. Using the common differential operator notation that $D(y) = \dfrac{dy}{dt}$, we have the following:

(a) $y'' + ty' - 3y = 0$ can be written as $L(y) = 0$ for $L = D^2 + tD - 3$.

(b) $y' + y^2 = 0$ is not a linear DE.

(c) $y' + \sin y = 1$ is not a linear DE.

(d) $y' + t^2 y = 0$ can be written as $L(y) = 0$ for $L = D + t^2$.

(e) $y' + (\sin t)y = 1$ can be written as $L(y) = 1$ for $L = D + \sin t$.

(f) $y'' - 3y' + y = \sin t$ can be written as $L(y) = \sin t$ for $L = D^2 - 3D + 1$.

■ Linear and Nonlinear Operations

12. $L(y) = y' + 2y$

Suppose $y_1,\, y_2$ and y are functions of t and c is any constant. Then

$$\begin{aligned} L(y_1 + y_2) &= (y_1 + y_2)' + 2(y_1 + y_2) \\ &= (y_1' + 2y_1) + (y_2' + 2y_2) \\ &= L(y_1 + y_2) \end{aligned}$$

$$\begin{aligned} L(cy) &= (cy)' + 2(cy) = cy' + 2cy \\ &= c(y' + 2y) = cL(y) \end{aligned}$$

Hence, L is a linear operator.

13. $L(y) = y' + y^2$

To show that $L(y) = y' + y^2$ is not linear we can pick a likely function of t and show that it does not satisfy one of the properties of linearity, equations (2) or (3). Consider the function $y = t$ and the constant $c = 5$:

$$\begin{aligned} L(5t) &= (5t)' + (5t)^2 = 5 + 25t^2 \\ &\neq 5\left((t)' + t^2\right) = 5 + 5t^2 = 5L(t) \end{aligned}$$

Hence, L is not a linear operator.

14. $L(y) = y' + 2ty$

Suppose $y_1,\, y_2$ and y are functions and c is any constant.

$$\begin{aligned} L(y_1 + y_2) &= (y_1 + y_2)' + 2t(y_1 + y_2) \\ &= (y_1' + 2ty_1) + (y_2' + 2ty_2) \\ &= L(y_1) + L(y_2) \end{aligned}$$

$$\begin{aligned} L(cy) &= (cy)' + 2t(cy) = c(y' + 2ty) \\ &= cL(y) \end{aligned}$$

Hence, L is a linear operator. This problem illustrates the fact that the coefficients of a DE can be functions of t and the operator will still be linear.

15. $L(y) = y' - e^t y$

Suppose $y_1,\ y_2$ and y are functions of t and c is any constant.

$$\begin{aligned} L(y_1 + y_2) &= (y_1 + y_2)' - e^t (y_1 + y_2) \\ &= \left(y_1' - e^t y_1\right) + \left(y_2' - e^t y_2\right) \\ &= L(y_1) + L(y_2) \\ L(cy) &= (cy)' - e^t (cy) = c\left(y' - e^t y\right) \\ &= cL(y) \end{aligned}$$

Hence, L is a linear operator. This problem illustrates the fact that a linear operator need not have coefficients that are linear functions of t.

16. $L(y) = y'' + (\sin t) y$

$$\begin{aligned} L(y_1 + y_2) &= (y_1 + y_2)'' + (\sin t)(y_1 + y_2) \\ &= \left\{y_1'' + (\sin t) y_1\right\} + \left\{y_2'' + (\sin t) y_2\right\} \\ &= L(y_1) + L(y_2) \\ L(cy) &= (cy)'' + (\sin t)(cy) \\ &= c\left\{y'' + (\sin t) y\right\} \\ &= cL(y) \end{aligned}$$

Hence, L is a linear operator. This problem illustrates the fact that a linear operator need not have coefficients that are linear functions of t.

17. $L(y) = y'' + \left(1 - y^2\right) y' + y$

$$\begin{aligned} L(cy) &= (cy)'' + \left\{1 - (cy)^2\right\} y' + (cy) \\ &\neq c\left\{y'' + \left(1 - y^2\right) y' + y\right\} \\ &= cL(y) \end{aligned}$$

Hence, $L(y)$ is not a linear operator.

■ Pop Quiz

18. $y' + 2y = 1 \Rightarrow y(t) = ce^{-2t} + \dfrac{1}{2}$

19. $y' + y = 2 \Rightarrow y(t) = ce^{-t} + 2$

20. $y' - 0.08y = 100 \Rightarrow y(t) = ce^{0.08t} - 1250$

21. $y' - 3y = 5 \Rightarrow y(t) = ce^{3t} - \dfrac{5}{3}$

22. $y' + 5y = 1$, $y(1) = 0 \Rightarrow y(t) = ce^{-5t} + \frac{1}{5}$, $y(1) = 0 \Rightarrow c = -\frac{1}{5}e^{5}$. Hence, $y(t) = \frac{1}{5}\left(1 - e^{-5(t-1)}\right)$.

23. $y' + 2y = 4$, $y(0) = 1 \Rightarrow y(t) = ce^{-2t} + 2$, $y(0) = 1 \Rightarrow c = -1$. Hence, $y(t) = 2 - e^{-2t}$.

■ **Superposition Principle**

24. If y_1 and y_2 are solutions of $y' + p(t)y = 0$, then

$$y_1' + p(t)y_1 = 0$$
$$y_2' + p(t)y_2 = 0.$$

Adding these equations gives

$$y_1' + y_2' + p(t)y_1 + p(t)y_2 = 0$$

or

$$(y_1 + y_2)' + p(t)(y_1 + y_2) = 0,$$

which shows that $y_1 + y_2$ is also a solution of the given equation.

If y_1 is a solution, we have

$$y_1' + p(t)y_1 = 0$$

and multiplying by c we get

$$c\left(y_1' + p(t)y_1\right) = 0$$
$$cy_1' + cp(t)y_1 = 0$$
$$(cy_1)' + p(t)(cy_1) = 0,$$

which shows that cy_1 is also a solution of the equation.

■ **Second-Order Superposition Principle**

25. If y_1 and y_2 are solutions of

$$y'' + p(t)y' + q(t)y = 0,$$

we have

$$y_1'' + p(t)y_1' + q(t)y_1 = 0$$
$$y_2'' + p(t)y_2' + q(t)y_2 = 0.$$

Multiplying these equations by c_1 and c_2 respectively, then adding and using properties of the derivative, we arrive at

$$(c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) = 0,$$

which shows that $c_1y_1 + c_2y_2$ is also a solution.

■ Verifying Superposition

26. $y'' - 9y = 0;$ $y_1 = e^{3t} \Rightarrow y_1' = 3e^{3t} \Rightarrow y_1'' = 9e^{3t}$, so that $y_1'' - 9y_1 = 9e^{3t} - 9e^{3t} = 0$.

$y_2 = e^{-3t} \Rightarrow y_2' = -3e^{3t} \Rightarrow y_2'' = 9e^{3t}$, so that $y_2'' - 9y_2 = 9e^{-3t} - 9e^{-3t} = 0$.

Let $y_3 = c_1y_1 + c_2y_2 = c_1e^{3t} + c_2e^{-3t}$, then $y_3' = c_1 3e^{3t} + c_2(-3)e^{-3t}$

$$y_3'' = c_1 9e^{3t} + c_2 9e^{-3t}$$

Thus, $y_3'' - 9y_3 = (c_1 9e^{3t} + c_2 9e^{-3t}) - 9(c_1e^{3t} + c_2e^{-3t}) = 0$

27. $y'' + 4y = 0;$

For $y_1 = \sin 2t$, $y_1{}' = 2\cos 2t \Rightarrow y_1'' = -4\sin 2t$, so that $y_1'' + 4y_1 = -4\sin 2t + 4\sin 2t = 0$.

For $y_2 = \cos 2t$, $y_2{}' = -2\sin 2t \Rightarrow y_2'' = -4\cos 2t$, so that $y_2'' + 4y_2 = -4\cos 2t + 4\cos 2t = 0$.

Let $y_3 = c_1 \sin 2t + c_2 \cos 2t$, then $y_3' = c_1 2\cos 2t + c_2(-2\sin 2t)$

$$y_3'' = c_1(-4\sin 2t) + c_2(-4\cos 2t)$$

Thus, $y_3'' + 4y_3 = c_1(-4\sin 2t) + c_2(-4\cos 2t) + 4(c_1 \sin 2t + c_2 \cos 2t) = 0$.

28. $2y'' + y' - y = 0;$

For $y_1 = e^{t/2}$, $y_1' = \frac{1}{2}e^{t/2}$, $y_1'' = \frac{1}{4}e^{t/2}$.

Substituting: $2\left(\frac{1}{4}e^{t/2}\right) + \frac{1}{2}e^{t/2} - e^{t/2} = 0$.

For $y_2 = e^{-t}$, $y_2' = -e^{-t}$, $y_2'' = e^{-t}$.

Substituting: $2\left(e^{-t}\right) + \left(-e^{-t}\right) - e^{-t} = 0.$

For c_1 and c_2, let $y = c_1e^{t/2} + c_2e^{-t}$.

Substituting: $2\left(c_1 \frac{1}{4}e^{t/2} + c_2e^{-t}\right) + \left(c_1 \frac{1}{2}e^{t/2} - c_2e^{-t}\right) - \left(c_1e^{t/2} + c_2e^{-t}\right) = 0.$

29. $y'' - 5y' + 6y = 0$

For $y_1 = e^{2t}$, $y_1' = 2e^{2t}$, $y_1'' = 4e^{2t}$.

Substituting: $4e^{2t} - 5(2e^{2t}) + 6(e^{2t}) = 0$.

For $y_2 = e^{3t}$, $y_2' = 3e^{3t}$, $y_2'' = 9e^{3t}$.

Substituting: $9e^{3t} - 5(3e^{3t}) + 6e^{3t} = 0$.

For $y = c_1e^{2t} + c_2e^{3t}$,

$$y'' - 5y' + 6y = c_1 4e^{2t} + c_2 9e^{3t} - 5\left(c_1 2e^{2t} + c_2 3e^{3t}\right) + 6\left(c_1 e^{2t} + c_2 e^{3t}\right) = 0.$$

30. $y'' - y' - 6y = 0$

For $y_1 = e^{3t}$, $y_1' = 3e^{3t}$, $y_1'' = 9e^{3t}$

Substituting: $(9e^{3t}) - (3e^{3t)}) - 6(e^{3t}) = 0$

For $y_2 = e^{-2t}$, $y_2' = -2e^{-2t}$, $y_2'' = 4e^{-2t}$

Substituting: $(4e^{-2t}) - (-2e^{-2t}) - 6e^{-2t} = 0$

For $y = c_1e^{3t} + c_2e^{-3t}$

$$y'' - y' - 6y = \left(c_1 9e^{3t} + c_2 4e^{-2t}\right) - \left(c_1 3e^{3t} - 2e^{-2t}\right) - 6\left(c_1 e^{3t} + c_2 e^{-3t}\right) = 0.$$

31. $y'' - 9y = 0$

For $y_1 = \cosh 3t$, $y_1' = 3\sinh 3t$ $y_1'' = 9\cosh 3t$.

Substituting: $(9\cosh 3t) - 9(\cosh 3t) = 0$.

For $y_2 = \sinh 3t$, $y_2' = 3\cosh 3t$, $y_2'' = 9\sinh 3t$.

Substituting: $(9\sinh 3t) - 9(\sinh 3t) = 0$.

For $y = c_1 \cosh 3t + c_2 \sinh 3t$,

$$y'' - 9y = \left(c_1 9\cosh 3t + c_2 9\sinh 3t\right) - 9\left(c_1 \cosh 3t + c_2 \sinh 3t\right) = 0.$$

■ Different Results?

32. The solutions of Problem 31, cosh $3t = \dfrac{3^{3t} + e^{-3t}}{2}$ and sinh $3t = \dfrac{e^{3t} - e^{-3t}}{2}$,

are linear combinations of the solutions of Problem 26 and vice-versa, i.e.,

e^{3t} = cosh $3t$ + sinh $3t$ and e^{-3t} = cosh $3t$ – sinh $3t$.

■ Many from One

33. Because $y(t) = t^2$ is a solution of a linear homogeneous equation, we know by equation (3) that ct^2 is also a solution for any real number c.

■ Guessing Solutions

We can often find a particular solution of a nonhomogeneous DE by inspection (guessing). For the first-order equations given for Problems 27–31 the general solutions come in two parts: solutions to the associated homogeneous equation (which could be found by separation of variables) plus a particular solution of the nonhomogeneous equation. For second-order linear equations as Problems 32–35 we can also sometimes find solutions by inspection.

34. $y' + y = e^t \Rightarrow y(t) = ce^{-t} + \frac{1}{2}e^t$

35. $y' + y = e^{-t} \Rightarrow y(t) = ce^{-t} + te^{-t}$

36. $y' - y = e^t \Rightarrow y(t) = ce^t + te^t$

37. $y' - ty = 0 \Rightarrow y(t) = ce^{t^2/2}$

38. $y' + \dfrac{y}{t} = t^3 \Rightarrow y(t) = \dfrac{c}{t} + \dfrac{t^4}{5}$

39. $y'' - a^2 y = 0 \Rightarrow y(t) = c_1 e^{at} + c_2 e^{-at}$. An alternative form is $y(t) = c_1 \sinh(at) + c_2 \cosh(at)$.

40. $y'' + a^2 y = 0 \Rightarrow y(t) = c_1 \sin at + c_2 \cos at$

41. $y'' + y' = 0 \Rightarrow y(t) = c_1 + c_2 e^{-t}$

42. $y'' - y' = 0 \Rightarrow y(t) = c_1 + c_2 e^t$

■ Nonhomogeneous Principle

In these problems, the verification of y_p is a straightforward substitution. To find the rest of the solution we simply add to y_p all the homogeneous solutions y_h, which we find by inspection or separation of variables.

43. $y' - y = 3e^t$ has general solution $y(t) = y_h + y_p = ce^t + 3te^t$.

44. $y' + 2y = 10 \sin t$ has general solution $y(t) = y_h + y_p = ce^{-2t} + 4 \sin t - 2 \cos t$.

45. $y' - \dfrac{2}{t} y = y^2$ has general solution $y(t) = y_h + y_p = ct^2 + t^3$.

46. $y' + \frac{1}{t+1}y = 2$ has general solution $y(t) = y_h + y_p = \frac{c}{t+1} + \frac{t^2 + 2t}{t+1}$.

■ Third-Order Examples

47. **(a)** For $y_1 = e^t$, we substitute into

$y''' - y'' - y' + y = 0$ to obtain $e^t - e^t - e^t + e^t = 0$.

For $y_2 = te^t$, we obtain $y_2' = te^t + e^t$, $y_2'' = (te^t + e^t) + e^t$, $y_2''' = (te^t + e^t) + 2e^t$,

and substitute to verify

$$(te^t + 3e^t) - (te^t + 2e^t) - (te^t + e^t) + te^t = 0.$$

For $y_3 = e^{-t}$, we obtain $y_3' = -e^{-t}$, $y_3'' = e^{-t}$, $y_3''' = -e^{-t}$,

and substitute to verify

$$(-e^{-t}) - (e^{-t}) - (-e^{-t}) + e^{-t} = 0.$$

(b) $y_h = c_1e^t + c_2te^t + c_3e^{-t}$

(c) Given $y_p = 2t + 1 + e^{2t}$:

$$y_p' = 2 + 2e^{2t}$$

$$y_p'' = 4e^{2t}$$

$$y_p''' = 8e^{2t}$$

To verify:

$$\begin{aligned} y_p''' - y_p'' - y_p' + y_p &= 8e^{2t} - 4e^{2t} - (2 + 2e^{2t}) + (2t + 1 + e^{2t}) \\ &= 2t - 1 + 3e^{2t} \end{aligned}$$

(d) $y(t) = y_h + y_p = c_1e^t + c_2te^t + c_3e^{-t} + 2t + 1 + e^{2t}$

(e) $y' = c_1e^t + c_2(te^t + e^t) - c_3e^{-t} + 2 + 2e^{2t}$

$y'' = c_1e^t + c_2(te^t + 2e^t) + c_3e^t + 4e^{2t}$

$y(0) = 1 = c_1 + c_3 + 2 \Rightarrow c_1 + c_3 = -1$ Equation (1)

$y'(0) = 2 = c_1 + c_2 - c_3 + 4 \Rightarrow c_1 + c_2 - c_3 = -2$ Equation (2)

$y''(0) = 3 = c_1 + 2c_2 + c_3 + 4 \Rightarrow c_1 + 2c_2 + c_3 = -1$ Equation (3)

Add Equation (2) to (1) and (3)

$$\left.\begin{aligned} 2c_1 + c_2 &= -3 \\ 2c_1 + 3c_2 &= -3 \end{aligned}\right\} \Rightarrow c_2 = 0, \qquad c_1 = -\frac{3}{2}, \qquad c_3 = \frac{1}{2}.$$

Thus, $y = -\frac{3}{2}e^t + \frac{1}{2}e^{-t} + 2t + 1 + e^{2t}$.

48. $y''' + y'' - y' - y = 4\sin t + 3$

(a) For $y_1 = e^t$, we obtain by substitution

$$y''' + y'' - y' - y = e^t + e^t - e^t - e^t = 0.$$

For $y = e^{-t}$, we obtain by substitution

$$y''' + y'' - y' - y = -e^{-t} + (e^{-t}) - (-e^{-t}) - e^{-t} = 0.$$

For $y = te^{-t}$ we obtain by substitution

$$y''' + y'' - y' - y = (-te^{-t} + 3e^{-t}) + (te^{-t} - 2e^{-t}) - (-te^{-t} + e^{-t}) - te^{-t} = 0.$$

(b) $y_h = c_1e^t + c_2e^{-t} + c_3te^{-t}$

(c) Given $y_p = \cos t - \sin t - 3$:

$$y_p' = -\sin t - \cos t$$

$$y_p'' = -\cos t + \sin t$$

$$y_p''' = \sin t + \cos t$$

To verify:

$$\begin{aligned} y_p''' + y_p'' - y_p' - y_p &= (\sin t + \cos t) + (-\cos t + \sin t) - (-\sin t - \cos t) - (\cos t - \sin t - 3) \\ &= 4\sin t + 3 \end{aligned}$$

(d) $y(t) = y_h + y_p = c_1e^t + c_2e^{-t} + c_3te^{-t} + \cos t - \sin t - 3$

(e) $y' = c_1e^t - c_2e^{-t} + c_3(-te^{-t} + e^{-t}) - \sin t - \cos t$

$y'' = c_1e^t + c_2e^{-t} + c_3(te^{-t} - 2e^{-t}) - \cos t + \sin t$

$y(0) = 1 = c_1 + c_2 + 1 - 3 \Rightarrow c_1 + c_2 = 3$ Equation (1)

$y'(0) = 2 = c_1 + c_2 - c_3 - 1 \Rightarrow c_1 - c_2 + c_3 = 3$ Equation (2)

$y''(0) = 3 = c_1 + c_2 - 2c_3 - 1 \Rightarrow c_1 + c_2 - 2c_3 = 4$ Equation (3)

Add Equation (2) to (1) and (3)

$$\left.\begin{array}{l} 2c_1 + c_3 = 6 \\ 2c_1 - c_3 = 7 \end{array}\right\} \Rightarrow c_1 = \frac{13}{4},\ c_2 = -\frac{1}{4},\ \ c_3 = -\frac{1}{2}.$$

$$y(t) = \frac{13}{4}e^t - \frac{1}{4}e^{-t} - \frac{1}{2}te^{-t} + \cos t - \sin t - 3.$$

■ **Suggested Journal Entry**

49. Student Project

2.2 Solving the First-Order Linear Differential Equation

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■ General Solutions

The solutions for Problems 1–15 can be found using either the Euler-Lagrange method or the integrating factor method. For problems where we find a particular solution by inspection (Problems 2, 6, 7) we use the Euler-Lagrange method. For the other problems we find it more convenient to use the integrating factor method, which gives both the homogeneous solutions and a particular solution in one swoop. You can use the Euler-Lagrange method to get the same results.

1. $y' + 2y = 0$

By inspection we have $y(t) = ce^{-2t}$.

2. $y' + 2y = 3e^t$

We find the homogeneous solution by inspection as $y_h = ce^{-2t}$. A particular solution on the nonhomogeneous equation can also be found by inspection, and we see $y_p = e^t$. Hence the general solution is $y(t) = ce^{-2t} + e^t$.

3. $y' - y = 3e^t$

We multiply each side of the equation by the integrating factor

$$\mu(t) = e^{\int p(t)dt} = e^{\int(-1)dt} = e^{-t}$$

giving

$$e^{-t}(y' - y) = 3, \text{ or simply } \frac{d}{dt}\left(ye^{-t}\right) = 3.$$

Integrating, we find $ye^{-t} = 3t + c$, or $y(t) = ce^t + 3te^t$.

4. $y' + y = \sin t$

We multiply each side of the equation by the integrating factor $\mu(t) = e^t$, giving

$$e^t(y' + y) = e^t \sin t, \text{ or, } \frac{d}{dt}\left(ye^t\right) = e^t \sin t.$$

Integrating by parts, we get $ye^t = \frac{1}{2}e^t(\sin t - \cos t) + c$.

Solving for y, we find $y(t) = ce^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t$.

5. $y' + y = \dfrac{1}{1+e^t}$

We multiply each side of the equation by the integrating factor $\mu(t) = e^t$, giving

$$e^t\left(y' + y\right) = \frac{e^t}{1+e^t}, \text{ or, } \frac{d}{dt}\left(ye^t\right) = \frac{e^t}{1+e^t}.$$

Integrating, we get $ye^t = \ln\left(1+e^t\right) + c$.

Hence, $y(t) = ce^{-t} + e^{-t}\ln\left(1+e^t\right)$.

6. $y' + 2ty = t$

In this problem we see that $y_p(t) = \frac{1}{2}$ is a solution of the nonhomogeneous equation (there are other single solutions, but this is the easiest to find). Hence, to find the general solution we solve the corresponding homogeneous equation, $y' + 2ty = 0$,

by separation of variables, getting

$$\frac{dy}{y} = -2tdt,$$

which has the general solution $y = ce^{-t^2}$, where c is any constant.

Adding the solutions of the homogeneous equation to the particular solution $y_p = \frac{1}{2}$ we get the general solution of the nonhomogeneous equation:

$$y(t) = ce^{-t^2} + \frac{1}{2}.$$

7. $y' + 3t^2y = t^2$

In this problem we see that $y_p(t) = \frac{1}{3}$ is a solution of the nonhomogeneous equation (there are other single solutions, but this is the easiest to find). Hence, to find the general solution, we solve the corresponding homogeneous equation, $y' + 3t^2y = 0$, by separation of variables, getting

$$\frac{dy}{y} = -3t^2dt,$$

which has the general solution $y(t) = ce^{-t^3}$, where c is any constant. Adding the solutions of the homogeneous equation to the particular solution $y_p = \frac{1}{3}$, we get the general solution of the nonhomogeneous equation

$$y(t) = ce^{-t^3} + \frac{1}{3}.$$

8. $y' + \frac{1}{t}y = \frac{1}{t^2}$, $(t \neq 0)$

We multiply each side of the equation by the integrating factor $\mu(t) = e^{\int dt/t} = e^{\ln t} = t$,

giving

$$t\left(y' + \frac{1}{t}y\right) = \frac{1}{t}, \text{ or, } \frac{d}{dt}(ty) = \frac{1}{t}.$$

Integrating, we find $ty = \ln t + c$.

Solving for *y*, we get $y(t) = \frac{c}{t} + \left(\frac{1}{t}\right)\ln t$.

9. $ty' + y = 2t$

We rewrite the equation as $y' + \frac{1}{t}y = 2$, and multiply each side of the equation by the integrating factor $\mu(t) = e^{\int dt/t} = e^{\ln t} = t$,

giving

$$t\left(y' + \frac{1}{t}y\right) = 2t, \text{ or, } \frac{d}{dt}(ty) = 2t.$$

Integrating, we find $ty = t^2 + c$.

Solving for *y*, we get $y(t) = \frac{c}{t} + t$.

10. $(\cos t)y' + \sin ty = 1$

We rewrite the equation as $y' + (\tan t)y = \sec t$,

and multiply each side of the equation by the integrating factor

$$\mu(t) = e^{\int \tan t dt} = e^{-\ln(\cos t)} = e^{\ln(\cos t)^{-1}} = \sec t,$$

giving

$$\sec t\left(y' + (\tan t)y\right) = \sec^2 t, \text{ or, } \frac{d}{dt}\left((\sec t)y\right) = \sec^2 t.$$

Integrating, we find $(\sec t)y = \tan t + c$. Solving for *y*, we get $y(t) = c\cos t + \sin t$.

11. $y' - \frac{2}{t}y = t^2 \cos t$, $(t \neq 0)$

We multiply each side of the equation by the integrating factor

$$\mu(t) = e^{-\int (2/t)dt} = e^{-2\ln t} = e^{\ln t^{-2}} = t^{-2},$$

giving

$$t^{-2}\left(y' - \frac{2}{t}y\right) = \cos t, \quad \text{or,} \quad \frac{d}{dt}\left(t^{-2}y\right) = \cos t.$$

Integrating, we find $t^{-2}y = \sin t + c$. Solving for *y*, we get $y(t) = ct^2 + t^2 \sin t$.

12. $y' + \frac{3}{t}y = \frac{\sin t}{t^3}$, $(t \neq 0)$

We multiply each side of the equation by the integrating factor

$$\mu(t) = e^{\int (3/t)dt} = e^{3\ln t} = e^{\ln\left(t^3\right)} = t^3,$$

giving

$$t^3\left(y' + \frac{3}{t}y\right) = \sin t, \quad \text{or,} \quad \frac{d}{dt}\left(t^3 y\right) = \sin t.$$

Integrating, we find $t^3 y = -\cos t + c$.

Solving for *y*, we get $y(t) = \frac{c}{t^3} - \frac{1}{t^3}\cos t$.

13. $\left(1 + e^t\right)y' + e^t y = 0$

We rewrite the equation as $y' + \left(\frac{e^t}{1 + e^t}\right)y = 0$,

and then multiply each side of the equation by the integrating factor

$$\mu(t) = e^{\int e^t / \left(1 + e^t\right)dt} = e^{\ln\left(1 + e^t\right)} = 1 + e^t,$$

giving

$$\frac{d}{dt}\left(\left(1 + e^t\right)y\right) = 0.$$

Integrating, we find $\left(1 + e^t\right)y = c$. Solving for *y*, we have $y(t) = \frac{c}{1 + e^t}$.

14. $\left(t^2+9\right)y'+ty=0$

We rewrite the equation as $y'+\left(\dfrac{t}{t^2+9}\right)y=0$,

and then multiply each side of the equation by the integrating factor

$$\mu(t)=e^{\int t/\left(t^2+9\right)dt}=e^{(1/2)\ln\left(t^2+9\right)}=\sqrt{t^2+9},$$

giving $\dfrac{d}{dt}\left(\sqrt{t^2+9}\,y\right)=0$.

Integrating, we find $\sqrt{t^2+9}\,y=c$.

Solving for *y*, we find $y(t)=\dfrac{c}{\sqrt{t^2+9}}$.

15. $y'+\left(\dfrac{2t+1}{t}\right)y=2t$, $(t\neq 0)$

We multiply each side of the equation by the integrating factor

$$\mu(t)=\int\frac{2t+1}{t}dt=\int 2+\frac{1}{t}dt=te^{2t},$$

giving

$$\frac{d}{dt}\left(te^{2t}y\right)=2t^2e^{2t}.$$

Integrating, we find $te^{2t}y=t^2e^{2t}-te^{2t}+\dfrac{1}{2}e^{2t}+c$.

Solving for *y*, we have $y(t)=c\left(\dfrac{e^{-2t}}{t}\right)+\dfrac{1}{2t}+t-1$.

■ Initial-Value Problems

16. $y'-y=1$, $y(0)=1$

By inspection, the homogeneous solutions are $y_h=ce^t$. A particular solution of the nonhomogeneous can also be found by inspection to be $y_p=-1$. Hence, the general solution is

$$y=y_h+y_p=ce^t-1.$$

Substituting $y(0)=1$ gives $c-1=1$ or $c=2$. Hence, the solution of the IVP is

$$y(t)=2e^t-1.$$

17. $y' + 2ty = t^3, \; y(1) = 1$

We can solve the differential equation using either the Euler-Lagrange method or the integrating factor method to get

$$y(t) = \frac{1}{2}t^2 - \frac{1}{2} + ce^{-t^2}.$$

Substituting $y(1) = 1$ we find $ce^{-1} = 1$ or $c = e$. Hence, the solution of the IVP is

$$y(t) = \frac{1}{2}t^2 - \frac{1}{2} + e^{1-t^2}.$$

18. $y' - \left(\frac{3}{t}\right)y = t^3, \; y(1) = 4$

We find the integrating factor to be

$$\mu(t) = e^{-\int(3/t)dt} = e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3}.$$

Multiplying the DE by this, we get

$$\frac{d}{dt}\left(t^{-3}y\right) = 1.$$

Hence, $t^{-3}y = t + c$, or, $y(t) = ct^3 + t^4$.

Substituting $y(1) = 4$ gives $c + 1 = 4$ or $c = 3$. Hence, the solution of the IVP is

$$y(t) = 3t^3 + t^4.$$

19. $y' + 2ty = t, \; y(0) = 1$

We solved this differential equation in Problem 6 using the integrating factor method and found

$$y(t) = ce^{-t^2} + \frac{1}{2}.$$

Substituting $y(0) = 1$ gives $c + \frac{1}{2} = 1$ or $c = \frac{1}{2}$. Hence, the solution of the IVP is

$$y(t) = \frac{1}{2}e^{-t^2} + \frac{1}{2}.$$

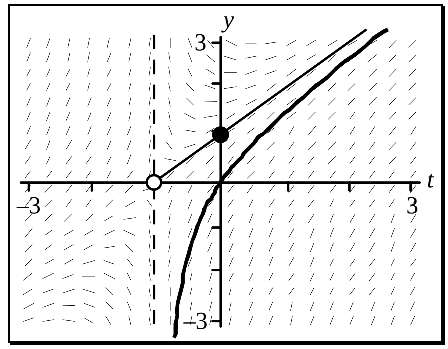
20. $\left(1 + e^t\right)y' + e^t y = 0, \; y(0) = 1$

We solved this DE in Problem 13 and found

$$y(t) = \frac{c}{1 + e^t}.$$

Substituting $y(0) = 1$ gives $\frac{c}{2} = 1$ or $c = 2$. Hence, the solution of the IVP is $y(t) = \frac{2}{1 + e^t}$.

■ Synthesizing Facts



21. (a) $y(t)=\dfrac{t^2+2t}{t+1},\ (t>-1)$

(b) $y(t)=t+1,\ (t>-1)$

(c) The algebraic solution given in Example 1 for $k=1$ is

$$y(t)=\frac{t^2+2t+1}{t+1}=\frac{(t+1)^2}{t+1}.$$

Hence, when $t\neq -1$ we have $y=t+1$.

(d) The solution passing through the origin $(0,\ 0)$ asymptotically approaches the line $y=t+1$ as $t\to\infty$, which is the solution passing through $y(0)=1$. The entire line $y=t+1$ is *not* a solution of the DE, as the slope is not defined when $t=-1$. The segment of the line $y=t+1$ for $t>-1$ is the solution passing through $y(0)=1$.

On the other hand, if the initial condition were $y(-5)=-4$, then the solution would be the segment of the line $y=t+1$ for t less than –1. Notice in the direction field the slope element is not defined at $(-1,\ 0)$.

■ Using Integrating Factors

In each of the following equations, we first write in the form $y'+p(t)y=f(t)$ and then identify $p(t)$.

22. $y'+2y=0$

Here $p(t)=2$, therefore the integrating factor is $\mu(t)=e^{\int p(t)dt}=e^{\int 2dt}=e^{2t}$.

Multiplying each side of the equation $y'+2y=0$ by e^{2t} yields

$$\frac{d}{dt}\left(ye^{2t}\right)=0.$$

Integrating gives $ye^{2t}=c$. Solving for y gives $y(t)=ce^{-2t}$.

23. $y'+2y=3e^t$

Here $p(t)=2$, therefore the integrating factor is $\mu(t)=e^{\int p(t)dt}=e^{\int 2dt}=e^{2t}$.

Multiplying each side of the equation $y'+2y=3e^t$ by e^{2t} yields

$$\frac{d}{dt}\left(ye^{2t}\right)=3e^{3t}.$$

Integrating gives $ye^{2t}=e^{3t}+c$. Solving for y gives $y(t)=ce^{-2t}+e^t$.

24. $y' - y = e^{3t}$

Here $p(t) = -1$, therefore the integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{-\int dt} = e^{-t}$.

Multiplying each side of the equation $y' - y = e^{3t}$ by e^{-t} yields

$$\frac{d}{dt}\left(ye^{-t}\right) = e^{2t}.$$

Integrating gives $ye^{-t} = \frac{1}{2}e^{2t} + c$. Solving for *y* gives $y(t) = ce^{t} + \frac{1}{2}e^{3t}$.

25. $y' + y = \sin t$

Here $p(t) = 1$ therefore the integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int dt} = e^{t}$.

Multiplying each side of the equation $y' + y = \sin t$ by e^{t} gives

$$\frac{d}{dt}\left(ye^{t}\right) = e^{t}\sin t.$$

Integrating gives $ye^{t} = \frac{1}{2}e^{t}\left(\sin t - \cos t\right) + c$. Solving for *y* gives $y(t) = \frac{1}{2}\left(\sin t - \cos t\right) + ce^{-t}$.

26. $y' + y = \dfrac{1}{1+e^{t}}$

Here $p(t) = 1$ therefore the integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int dt} = e^{t}$.

Multiplying each side of the equation $y' + y = \dfrac{1}{1+e^{t}}$ by e^{t} yields

$$\frac{d}{dt}\left(ye^{t}\right) = \frac{e^{t}}{1+e^{t}}.$$

Integrating gives $ye^{t} = \ln\left(1+e^{t}\right) + c$. Solving for *y* gives $y(t) = e^{-t}\ln\left(1+e^{t}\right) + ce^{-t}$.

27. $y' + 2ty = t$

Here $p(t) = 2t$, therefore the integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int 2t dt} = e^{t^2}$.

Multiplying each side of the equation $y' + 2ty = t$ by e^{t^2} yields

$$\frac{d}{dt}\left(ye^{t^2}\right) = te^{t^2}.$$

Integrating gives $ye^{t^2} = \frac{1}{2}e^{t^2} + c$. Solving for *y* gives $y(t) = ce^{-t^2} + \frac{1}{2}$.

28. $y' + 3t^2 y = t^2$

Here $p(t) = 3t^2$, therefore the integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int 3t^2 dt} = e^{t^3}$.

Multiplying each side of the equation $y' + 3t^2 y = t^2$ by e^{t^3} yields

$$\frac{d}{dt}\left(ye^{t^3}\right) = t^2 e^{t^3}.$$

Integrating gives $ye^{t^3} = \frac{1}{3}e^{t^3} + c$. Solving for *y* gives $y(t) = ce^{-t^3} + \frac{1}{3}$.

29. $y' + \frac{1}{t}y = \frac{1}{t^2}$

Here $p(t) = \frac{1}{t}$, therefore the integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int (1/t)dt} = e^{\ln t} = t$.

Multiplying each side of the equation $y' + \frac{1}{t}y = \frac{1}{t^2}$ by *t* yields

$$\frac{d}{dt}(ty) = \frac{1}{t}.$$

Integrating gives $ty = \ln t + c$. Solving for *y* gives $y(t) = c\frac{1}{t} + \frac{1}{t}\ln t$.

30. $ty' + y = 2t$

Here $p(t) = \frac{1}{t}$, therefore the integrating factor is $\mu(t) = e^{\int p(t)dt} = e^{\int (1/t)dt} = e^{\ln t} = t$.

Multiplying each side of the equation $y' + \frac{y}{t} = 2$ by *t* yields

$$\frac{d}{dt}(ty) = 2t.$$

Integrating gives $ty = t^2 + c$. Solving for *y* gives $y(t) = c\frac{1}{t} + t$.

■ **Switch for Linearity** $\frac{dy}{dt} = \frac{1}{t+y}$, $y(-1) = 0$

31. Flipping both sides of the equation yields the equivalent linear form $\frac{dt}{dy} = t + y$, or $\frac{dt}{dy} - t = y$.

Solving this equation we get $t(y) = ce^{y} - y - 1$.

Using the condition $y(-1) = 0$, we find $-1 = ce^{0} - 1$, and so $c = 0$. Thus, we have $t = -y - 1$ and solving for *y* gives

$$y(t) = -t - 1.$$

■ **The Tough Made Easy** $\frac{dy}{dt} = \frac{y^2}{e^y - 2ty}$

32. We flip both sides of the equation, getting

$$\frac{dt}{dy} = \frac{e^y - 2ty}{y^2}, \quad \text{or,} \quad \frac{dt}{dy} + \frac{2}{y}t = \frac{e^y}{y^2}.$$

We solve this linear DE for $t(y)$ getting $t(y) = \frac{e^y + c}{y^2}$.

■ **A Useful Transformation**

33. (a) Letting $z = \ln y$, we have $y = e^z$ and $\frac{dy}{dt} = e^z \frac{dz}{dt}$.

Now the equation $\frac{dy}{dt} + ay = by \ln y$ can be rewritten as $e^z \frac{dz}{dt} + ae^z = bze^z$.

Dividing by e^z gives the simple linear equation $\frac{dz}{dt} - bz = -a$.

Solving yields $z = ce^{bt} + \frac{a}{b}$ and using $z = \ln y$, the solution becomes $y(t) = e^{(a/b)+ce^{bt}}$.

(b) If $a = b = 1$, we have $y(t) = e^{\left(1+ce^t\right)}$.

Note that when $c = 0$ we have the constant solution $y = e$.

■ **Bernoulli Equation** $y' + p(t)y = q(t)y^\alpha$, $\alpha \neq 0$, $\alpha \neq 1$

34. (a) We divide by y^α to obtain $y^{-\alpha}y' + p(t)y^{1-\alpha} = q(t)$.

Let $v = y^{1-\alpha}$ so that $v' = (1-\alpha)y^{-a}y'$ and $\frac{v'}{1-\alpha} = y^{-a}y'$.

Substituting into the first equation for $y^{1-\alpha}$ and $y^{-\alpha}y'$, we have

$$\frac{v'}{1-\alpha} + p(t)v = q(t)\text{, a linear DE in v,}$$

which we can now rewrite into standard form as

$$v' + (1-\alpha)p(t)v = (1-\alpha)q(t).$$

(b) $\alpha = 3$, $p(t) = -1$, and $q(t) = 1$; hence $\frac{dv}{dt} + 2v = -2$, which has the general solution

$$v(t) = -1 + ce^{-2t}.$$

Because $v = \frac{1}{y^2}$, this yields $y(t) = \left(-1 + ce^{-2t}\right)^{-1/2}$.

Note, too, that $y = 0$ satisfies the given equation.

(c) When $\alpha = 0$ the Bernoulli equation is

$$\frac{dy}{dt} + p(t)y = q(t),$$

which is the general first-order linear equation we solved by the integrating factor method and the Euler-Lagrange method.

When $\alpha = 1$ the Bernoulli equation is

$$\frac{dy}{dt} + p(t)y = q(t)y, \text{ or, } \frac{dy}{dt} + \left(p(t) - q(t)\right)y = 0,$$

which can be solved by separation of variables.

■ **Bernoulli Practice**

35. $y' + ty = ty^3$ or $y^{-3}y' + ty^{-2} = t$

Let $v = y^{-2}$, so $\frac{dv}{dt} = -2y^{-3}\frac{dy}{dt}$.

Substituting in the DE gives $-\frac{1}{2}\frac{dv}{dt} + tv = t,$ so that $\frac{dv}{dt} - 2tv = -2t,$ which is linear in v, with integrating factor $\mu = e^{\int -2t\,dt} = e^{-t^2}$.

Thus, $e^{-t^2}\frac{dv}{dt} - 2te^{-t^2}v = -2te^{-t^2}$, and $e^{-t^2}v = \int -2te^{-t^2}dt = e^{-t^2} + c,$

so $v = 1 + ce^{t^2}$.

Substituting back for v gives

$$y^2 = \frac{1}{1 + ce^{t^2}}, \quad \text{hence} \quad y(t) = \pm\sqrt{\frac{1}{1 + ce^{t^2}}}.$$

36. $y' - y = e^t y^2$, so that $y^{-2}y' - y^{-1} = e^t$

Let $v = y^{-1}$, so $\dfrac{dv}{dt} = -y^{-2}\dfrac{dy}{dt}$.

Substituting in the DE gives $\dfrac{dv}{dt} + v = -e^t$, which is linear in v with integrating factor $\mu = e^{\int dt} = e^t$.

Thus $e^t \dfrac{dv}{dt} + e^t v = -e^{2t}$, and $e^t v = -\int e^{2t}dt = -\dfrac{e^{2t}}{2} + c$, so $v = -\dfrac{e^t}{2} + ce^{-t}$.

Substituting back for v gives $y^{-1} = -\dfrac{e^t}{2} + c_1 e^{-t}$, or $y(t) = \dfrac{2}{-e^t + c_1 e^{-t}}$.

37. $t^2 y' - 2ty = 3y^4$ or $y^{-4}t^2 y' - 2ty^{-3} = 3$, $t \neq 0$

Let $v = y^{-3}$, so $\dfrac{dv}{dt} = -3y^{-4}\dfrac{dy}{dt}$.

Substituting in the DE gives

$$-\frac{1}{3}t^2\frac{dv}{dt} - 2tv = 3\text{, or, }\frac{dv}{dt} + \frac{6}{t} = -\frac{9}{t^2},$$

which is linear in v, with integrating factor $\mu = e^{\int \frac{6}{t}dt} = e^{6\ln t} = t^6$.

Thus $t^6 \dfrac{dv}{dt} + 6t^5 v = -9t^4$, and $t^6 v = -\int 9t^4 dt = -\dfrac{9}{5}t^5 + c$,

so $v = -\dfrac{9}{5t} + ct^{-5}$. Substituting back for v gives $y^{-3} = -\dfrac{9}{5t} + \dfrac{c}{t^6}$.

Hence $y^3 = \dfrac{1}{-\dfrac{9}{5t} + \dfrac{c}{t^6}} = \dfrac{5t^6}{-9t^5 + 5c}$, so $y(t) = t^2 \sqrt[3]{\dfrac{5}{c_1 - 9t^5}}$.

38. $(1-t^2)y' - ty - ty^2 = 0$ (Assume $|t| < 1$)

$y^{-2}(1-t^2)y' - ty^{-1} = t$

Let $v = y^{-1}$, so $\dfrac{dv}{dt} = -y^{-2}\dfrac{dy}{dt}$.

Substituting in the DE gives

$$-(1-t^2)\frac{dv}{dt} - tv = t,\text{ so that }\frac{dv}{dt} + \frac{t}{1-t^2}v = \frac{-t}{1-t^2},$$

which is linear in v, with integrating factor $\mu = e^{\int \frac{t}{1-t^2}dt} = e^{-\frac{1}{2}\ln(1-t^2)} = (1-t^2)^{-1/2}$.

Thus, $(1-t^2)^{-1/2}\dfrac{dv}{dt}+t(1-t^2)^{-3/2}v=t(1-t^2)^{-3/2}$, and

$$(1-t^2)^{-1/2}v=\int\frac{-t\,dt}{(1-t^2)^{3/2}}=-\frac{1}{2}\int\frac{dw}{w^{3/2}}$$

$$=-\frac{1}{2}\frac{w^{-1/2}}{\left(-\frac{1}{2}\right)}+c$$

$$=w^{-1/2}+c=(1-t^2)^{-1/2}+c$$

(Substitute $w=1-t^2$
$dw=-2tdt$
$-\frac{1}{2}dw=tdt$)

Hence $v = 1 + c(1-t^2)^{1/2}$ and substituting back for v gives $y(t)=\dfrac{1}{1+c(1-t)^{1/2}}$.

39. $y'+\dfrac{y}{t}=\dfrac{y^{-2}}{t}$ $\quad y(1) = 2$

$y^2y'+\dfrac{y^3}{t}=\dfrac{1}{t}$

Let $v = y^3$, so $\dfrac{dv}{dt}=3y^2\dfrac{dy}{dt}$.

Substituting in the DE gives $\dfrac{1}{3}\dfrac{dv}{dt}+\dfrac{1}{t}v=\dfrac{1}{t}$, or $\dfrac{dv}{dt}+\dfrac{3}{t}v=\dfrac{3}{t}$,

which is linear in v, with integrating factors $\mu=e^{\int 3/t\,dt}=e^{3\ln t}=t^3$.

Thus, $t^3\dfrac{dv}{dt}+3t^2v=3t^2$, and $t^3v=\int 3t^2dt=t^3+c$, so $v=1+ct^{-3}$.

Substituting back for v gives $y^3=1+ct^{-3}$ or $y(t)=\sqrt[3]{1+ct^{-3}}$.

For the IVP we substitute the initial condition $y(1) = 2$, which gives $2^3 = 1 + c$, so $c = 7$.

Thus, $y^3 = 1 + 7t^{-3}$ and $y(t)=\sqrt[3]{1+7t^{-3}}$.

40. $3y^2y'-2y^3-t-1=0$

Let $v = y^3$, so $\dfrac{dv}{dt}=3y^2\dfrac{dy}{dt}$, and $\dfrac{dv}{dt}-2v=t+1$,

which is linear in v with integrating factor $\mu=e^{-\int 2dt}=e^{-2t}$.

Thus, $e^{-2t}\dfrac{dv}{dt}-2e^{-2t}v=(t+1)e^{-2t}$, and

$$e^{-2t}v=\int(t+1)e^{-2t}dt$$
$$=-(t+1)\frac{e^{-2t}}{2}-\frac{e^{-2t}}{4}+c. \qquad \text{(Integration by parts)}$$

Hence $v=-\dfrac{t+1}{2}-\dfrac{1}{4}+ce^{2t}=-\dfrac{t}{2}-\dfrac{3}{4}+ce^{2t}$.

Substituting back for v gives $y^3=-\dfrac{t}{2}-\dfrac{3}{4}+ce^{2t}$.

For the IVP, substituting the initial condition $y(0)=2$ gives $8=-\dfrac{3}{4}+c$, $c=\dfrac{35}{4}$.

Hence, $y^3=-\dfrac{t}{2}-\dfrac{3}{4}+\dfrac{35}{4}e^{2t}$, and $y(t)=\sqrt[3]{\dfrac{-t}{2}-\dfrac{3}{4}+\dfrac{35}{4}e^{2t}}$.

■ **Ricatti Equation** $y'=p(t)+q(t)y+r(t)y^2$

41. (a) Suppose y_1 satisfies the DE so that

$$\frac{dy_1}{dt}=p(t)+q(t)y_1+r(t)y_1^2.$$

If we define a new variable $y=y_1+\dfrac{1}{v}$, then $\dfrac{dy}{dt}=\dfrac{dy_1}{dt}-\dfrac{1}{v^2}\left(\dfrac{dv}{dt}\right)$.

Substituting for $\dfrac{dy_1}{dt}$ yields $\dfrac{dy}{dt}=p(t)+q(t)y_1+r(t)y_1^2-\dfrac{1}{v^2}\left(\dfrac{dv}{dt}\right)$.

Now, if we require, as suggested, that v satisfies the linear equation

$$\frac{dv}{dt}=-\big(q(t)+2r(t)y_1\big)v-r(t),$$

then substituting in the previous equation gives

$$\frac{dy}{dt}=p(t)+q(t)y_1+r(t)y_1^2+\frac{q(t)}{v}+\frac{2r(t)y_1}{v}+\frac{r(t)}{v^2},$$

which simplifies to

$$\frac{dy}{dt}=p(t)+q(t)\left(y_1+\frac{1}{v}\right)+r(t)\left(y_1^2+2\left(\frac{y_1}{v}\right)+\frac{1}{v^2}\right)=p(t)+q(t)y+r(t)y^2.$$

Hence, $y=y_1+\dfrac{1}{v}$ satisfies the Ricatti equation as well, as long as v satisfies its given equation.

(b) $y' = -1 + 2y - y^2$

Let $y_1 = 1$ so $y_1' = 0$, and substitution in the DE gives

$$0 = -1 + 2y_1 - y_1^2 = -1 + 2 - 1 = 0.$$

Hence, y_1 satisfies the given equation. To find v and then y, note that

$$p(t) = -1,\ q(t) = 2,\ r(t) = -1.$$

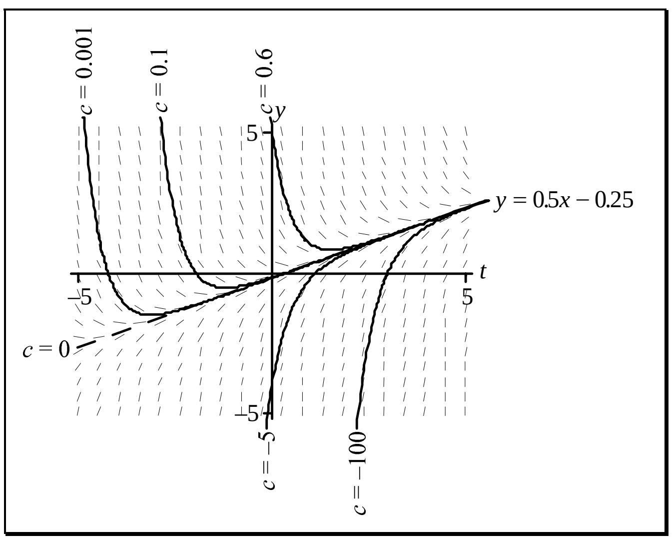
Now find v from the assumed requirement that

$$\frac{dv}{dt} = -\left(2 + 2(-1)(1)\right)v - (-1),$$

which reduces to $\frac{dv}{dt} = 1$. This gives $v(t) = t + c$, hence $y(t) = y_1 + \frac{1}{v} = 1 + \frac{1}{t+c}$.

■ **Computer Visuals**

42. (a) $y' + 2y = t$



(b) $y_h(t) = ce^{-2t}$, $y_p = \frac{1}{2}t - \frac{1}{4}$

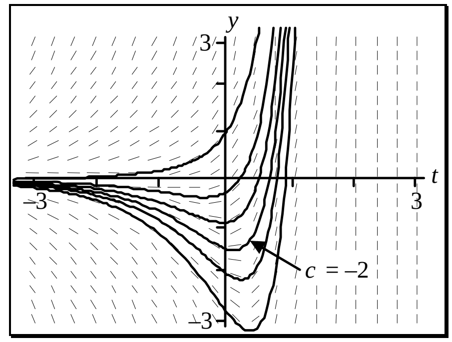
The general solution is

$$y(t) = y_h + y_p = ce^{-2t} + \frac{1}{2}t - \frac{1}{4}.$$

The curves in the figure in part (a) are labeled for different values of c.

(c) The homogeneous solution y_h is transient because $y_h \to 0$ as $t \to \infty$. However, although all solutions are attracted to y_p, we would *not* call y_p a steady-state solution because it is neither constant nor periodic; $y_p \to \infty$ as $t \to \infty$.

43. (a) $y' - y = e^{3t}$



(b) $y_h(t) = ce^t$, $y_p = \frac{1}{2}e^{3t}$.

The general solution is

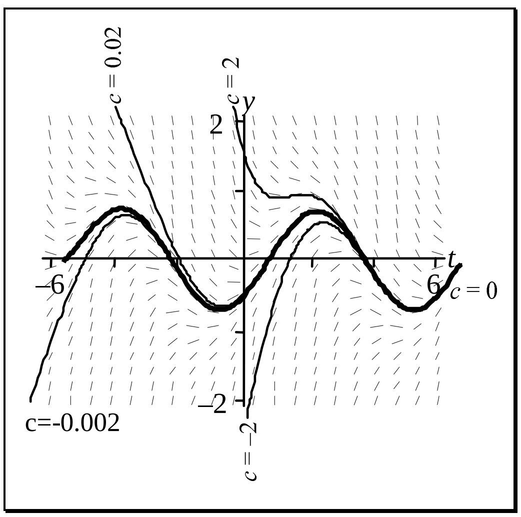
$$y(t) = y_h + y_p = \underbrace{ce^t}_{y_h} + \underbrace{\frac{1}{2}e^{3t}}_{y_p}.$$

(c) There is no steady-state solution because all solutions (including both y_h and y_p) go to ∞ as $t \to \infty$. The c values are approximate:

$$\{0.5, -0.8, -1.5, -2, -2.5, -3.1\}$$

as counted from the top-most curve to the bottom-most one.

44. (a) $y' + y = \sin t$



(b) $y_h(t) = ce^{-t}$, $y_p = \frac{1}{2}\sin t - \frac{1}{2}\cos t$.

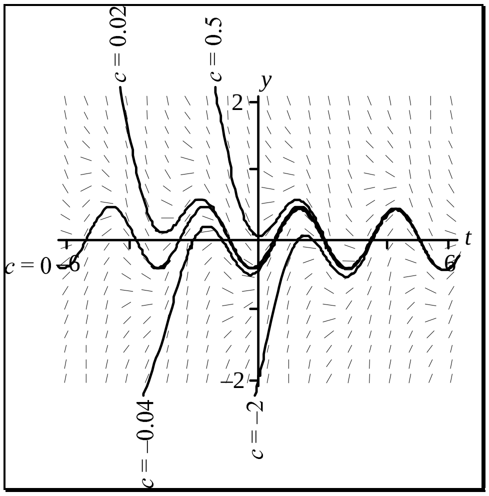
The general solution is $y(t) = y_h + y_p = ce^{-t} + \frac{1}{2}\sin t - \frac{1}{2}\cos t$.

The curves in the figure in part (a) are labeled for different values of c.

(c) The sinusoidal steady-state solution $y_p = \frac{1}{2}\sin t - \frac{1}{2}\cos t$

occurs when $c = 0$. Note that the other solutions approach this solution as $t \to \infty$.

45. (a) $y' + y = \sin 2t$



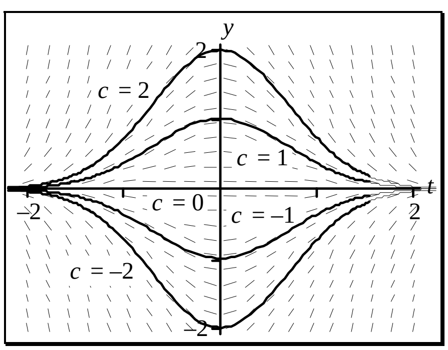
(b) $y_h(t) = ce^{-t}$, $y_p = \frac{1}{5}(\sin 2t - 2\cos 2t)$.

The general solution is $y(t) = \underbrace{ce^{-t}}_{y_h} + \underbrace{\frac{\sin 2t - 2\cos 2t}{5}}_{y_p}$.

(c) The steady-state solution is y_p, which attracts all other solutions.

The transient solution is y_h.

46. (a) $y' + 2ty = 0$

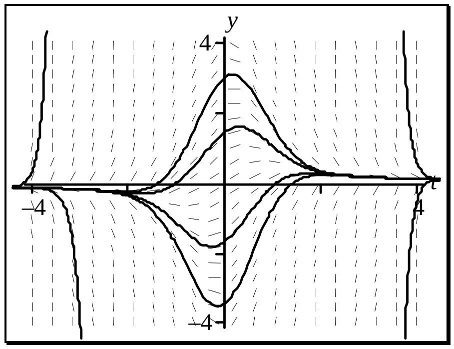


(b) This equation is homogeneous. The general solution is

$$y_h(t) = ce^{-t^2}.$$

(c) The equation has steady-state solution $y = 0$. All solutions tend towards zero as $t \to \infty$.

47. (a) $y' + 2ty = 1$



The approximate c values corresponding to the curves in the center counted from top to bottom, are

$$\{1; -1; 2; -2\}$$

and approximately 50,000 (left curve) and –50,000 (right curve) for the side curves.

(b) $y_h(t) = ce^{-t^2}$, $y_p = e^{-t^2}\int e^{t^2}\,dt$.

The general solution is $y(t) = \underbrace{ce^{-t^2}}_{y_h} + \underbrace{e^{-t^2}\int e^{t^2}\,dt}_{y_p}$.

(c) The steady-state solution is $y(t) = 0$, which is *not* equal to y_p. Both y_h and y_p are transient, but as $t \to \infty$, all solutions approach 0.

■ **Computer Numerics**

48. $y' + 2y = t$, $y(0) = 1$

(a) Using step size $h = 0.1$ and $h = 0.01$ and Euler's method, we compute the following values. In the latter case we print only selected values.

Euler's Method

t	$y(h=0.1)$	$y(h=0.01)$	T	$y(h=0.1)$	$y(h=0.01)$
0	1	1.0000	0.6	0.3777	0.4219
0.1	0.8	0.8213	0.7	0.3621	0.4039
0.2	0.65	0.6845	0.8	0.3597	0.3983
0.3	0.540	0.5819	0.9	0.3678	0.4029
0.4	0.4620	0.5071	1	0.3842	0.4158
0.5	0.4096	0.4552			

By Runge-Kutta (RK4) we obtain $y(1) \approx 0.4192$ for step size $h = 0.1$.

(b) From Problem 42, we found the general solution of DE to be $y(t) = ce^{-2t} + \frac{1}{2}t - \frac{1}{4}$.

Using IC $y(0) = 1$ yields $c = \frac{5}{4}$. The solution of the IVP is $y(t) = \frac{5}{4}e^{-2t} + \frac{1}{2}t - \frac{1}{4}$,

and to 4 places, we have

$$y(1) = \frac{5}{4}e^{-2} + \frac{1}{2} - \frac{1}{4} \approx 0.4192 .$$

(c) The error for $y(1)$ using step size $h = 0.1$ in Euler's approximation is

$$\text{ERROR} = 0.4192 - 0.3842 = 0.035$$

Using step size $h = 0.01$, Euler's method gives

$$\text{ERROR} = 0.4192 - 0.4158 = 0.0034 ,$$

which is much smaller. For step size $h = 0.1$, Runge-Kutta gives $y(1) = 0.4158$ and *zero* error to four decimal places.

(d) The accuracy of Euler's method can be greatly improved by using a smaller step size. The Runge-Kutta method is more accurate for a given step size in most cases.

49. Sample analysis: $y' - y = e^{3t}$, $y(0) = 1$, $y(1)$.

Exact solution is $y = 0.5e^{t} + 0.5e^{3t}$, so $y(1) = 11.4019090461656$ to thirteen decimal places.

(a) $y(1) \approx 9.5944$ by Euler's method for step size $h = 0.1$,

$y(1) \approx 11.401909375$ by Runge-Kutta for step size 0.1 (correct to six decimal places).

(b) From Problem 24, we found the general solution of the DE to be $y = ce^{t} + \frac{1}{2}e^{3t}$.

(c) The accuracy of Euler's method can be greatly improved by using a smaller step size; but it still is not correct to even one decimal place for step size 0.01.

$$y(1) \approx 11.20206 \text{ for step size } h = 0.01$$

(d) MORAL: Euler's method converges ever so slowly to the exact answer—clearly a far smaller step would be necessary to approach the accuracy of the Runge-Kutta method.

50. $y' + 2ty = 1$, $y(0) = 1$

(a) Using step size $h = 0.1$ and $h = 0.01$ and Euler's method, we compute the following values.

Euler's Method

t	$y(h=0.1)$	$y(h=0.01)$	T	$y(h=0.1)$	$y(h=0.01)$
0	1	1.0000	0.6	1.2308	1.1780
0.1	1.1	1.0905	0.7	1.1831	1.1288
0.2	1.178	1.1578	0.8	1.1175	1.0648
0.3	1.2309	1.1999	0.9	1.0387	0.9905
0.4	1.2570	1.2165	1	**0.9517**	**0.9102**
0.5	1.2564	1.2084			

$y(1) \approx 0.905958$ by Runge-Kutta method using step size $h = 0.1$ (correct to six decimal places).

(b) From Problem 47, we found the general solution of DE to be $y(t) = ce^{-t^2} + e^{-t^2}\int e^{t^2}\,dt$. Using IC $y(0) = 1$, yields $c = 1$. The solution of the IVP is

$$y(t) = e^{-t^2}\left(1 + \int_0^t e^{u^2}\,du\right)$$

and so to 10 places, we have $y(1) = 0.9059589485$

(c) The error for $y(1)$ using step size $h = 0.1$ in Euler's approximation is

$$\text{ERROR} = 0.9517 - 0.9059 = 0.0458.$$

Using step size $h = 0.01$, Euler's method gives

$$\text{ERROR} = 0.9102 - 0.9060 = 0.0043,$$

which is much smaller. Using step size $h = 0.1$ in Runge-Kutta method gives ERROR less then 0.000001.

(d) The accuracy of Euler's method can be greatly improved by using a smaller step size, but the Runge-Kutta method has much better performance because of higher degree of accuracy.

■ Direction Field Detective

51. (a) (A) is linear homogeneous, (B) is linear nonhomogeneous, (C) is nonlinear.

(b) If y_1 and y_2 are solutions of a linear homogeneous equation,

$$y' + p(t)y = 0,$$

then $y_1' + p(t)y_1 = 0$, and $y_2' + p(t)y_2 = 0$. We can add these equations, to get

$$\left(y_1' + p(t)y_1\right) + \left(y_2' + p(t)y_2\right) = 0.$$

Because this equation can be written in the equivalent form

$$(y_1 + y_2)' + p(t)(y_1 + y_2) = 0,$$

then $y_1 + y_2$ is also a solution of the given equation.

(c) The sum of any two solutions follows the direction field only in (A). For the linear homogeneous equation (A) you plot any two solutions y_1 and y_2 by simply following curves in the direction field, and then add these curves, you will see that the sum $y_1 + y_2$ also follows the direction field.

However, in equation (B) you can observe a straight line solution, which is

$$y_1 = \frac{1}{2}t - \frac{1}{4}.$$

If you add this to itself you get $y_1 + y_1 = 2y_1 = t - \frac{1}{2}$, which clearly does not follow the direction field and hence is not a solution. In equation (C) $y_1 = 1$ is a solution but if you add it to itself you can see from the direction field that $y_1 + y_2 = 2$ is not a solution.

■ Recognizing Linear Homogeneous DEs from Direction Fields

52. For (A) and (D): The direction fields appear to represent linear homogeneous DEs because the sum of any two solutions is a solution and a constant times a solution is also a solution. (Just follow the direction elements.)

For (B) , (C) , and (E): These direction fields cannot represent linear homogeneous DEs because the zero function is a solution of linear homogeneous equations, and these direction fields do not indicate that the zero function is a solution. (B) seems to represent a nonlinear DE with more than one equilibrium, while (C) and (E) represent linear but nonhomogeneous DEs.

Note: It may be helpful to look at textbook Figures 2.1.1 and 2.2.2.

■ Suggested Journal Entry

53. Student Project

2.3 Growth and Decay Phenomena

[redacted]

■ Half-Life

1. (a) The half-life t_h is the time required for the solution to reach $\frac{1}{2}y_0$. Hence, $y_0 e^{kt} = \frac{1}{2}y_0$. Solving for t_h, yields $kt_h = -\ln 2$, or

$$t_h = -\frac{1}{k}\ln 2 .$$

(b) The solution to $y' = ky$ is $y(t) = y_0 e^{kt}$ so at time $t = t_1$, we have

$$y(t_1) = y_0 e^{kt_1} = B .$$

Then at $t = t_1 + t_h$ we have

$$y(t_1 + t_h) = y_0 e^{k(t_1 + t_h)} = y_0 e^{kt_1} e^{kt_h} = y_0 e^{kt_1} e^{-k(\ln 2)/k} = Be^{-\ln 2} = Be^{\ln(1/2)} = \frac{1}{2}B .$$

■ Doubling Time

2. For doubling time t_d, we solve $y_0 e^{kt_d} = 2y_0$, which yields

$$t_d = \frac{1}{k}\ln 2 .$$

■ Interpretation of $\frac{1}{k}$

3. If we examine the value of the decay curve

$$y(t) = y_0 e^{kt}$$

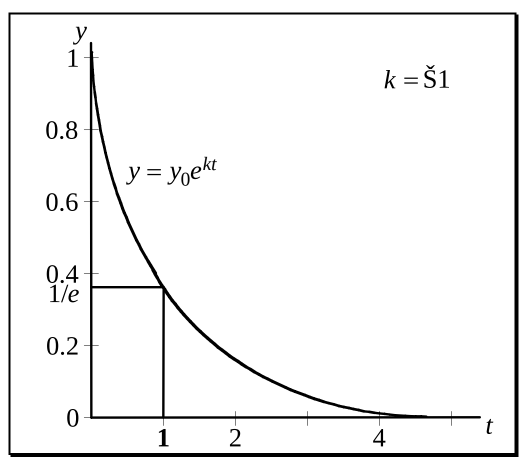
we find

$$y\left(\left|\frac{1}{k}\right|\right) = y_0 e^{k(-1/k)} = y_0 e^{-1} = y_0 (0.3678794\ldots)$$

$$\approx \frac{y_0}{3}.$$

Hence,

$$\left|\frac{1}{k}\right|$$



$y_0 e^{-t}$ falls from y_0 to roughly $\frac{y_0}{3}$ when $t = -\frac{1}{k}$

is a crude approximation of the third-life of a decay curve. In other words, if a substance decays and has a decay constant $k = -0.02$, and time is measured in years, then the third-life of the substance is roughly $\frac{1}{0.02} = 50$ years. That is, every 50 years the substance decays by $\frac{2}{3}$. Note that the curve in the figure falls to $\frac{1}{3}$ of its value in approximately $t = 1$ unit of time.

■ Radioactive Decay

4. $\frac{dQ}{dt} = kQ$ has the general solution

$$Q(t) = ce^{kt}.$$

Initial condition $Q(0) = 100$ gives $Q(t) = 100e^{kt}$ where Q is measured in grams. We also have the initial condition $Q(50) = 75$, from which we find

$$k = \frac{1}{50}\ln\frac{3}{4} \approx -0.0058.$$

The solution is

$$Q(t) \approx 100e^{-0.0058t}$$

where t is measured in years. The half-life is

$$t_h = \frac{\ln 2}{0.0058} \approx 120 \text{ years.}$$

■ Determining Decay from Half-Life

5. $\frac{dQ}{dt} = kQ$ has the general solution $Q(t) = ce^{kt}$. With half-life $t_h = 5$ hours, the decay constant has the value $k = -\frac{1}{5}\ln 2 \approx -0.14$. Hence,

$$Q(t) = Q_0 e^{-0.14t}.$$

Calling t_t the time it takes to decay to $\frac{1}{10}$ the original amount, we have

$$\frac{1}{10}Q_0 = Q_0 e^{-0.14t_t},$$

which we can solve for t_t getting

$$t_t = \frac{5\ln 10}{\ln 2} \approx 16.6 \text{ hours.}$$

■ **Thorium-234**

6. (a) The general decay curve is $Q(t) = ce^{kt}$. With the initial condition $Q(0) = 1$, we have $Q(t) = e^{kt}$. We also are given $Q(1) = 0.8$ so $e^{k} = 0.8$, or $k = \ln(0.8) \approx -0.22$. Hence, we have

$$Q(t) = e^{-0.22t}$$

where Q is measured in grams and t is measured in weeks.

(b) $t_h = -\dfrac{\ln 2}{k} = \dfrac{\ln 2}{0.22} \approx 3.1$ weeks

(c) $Q(10) e^{-0.22(10)} \approx 0.107$ grams

■ **Dating Sneferu's Tomb**

7. The half-life for Carbon-14 is $t_h = 5600$ years, so

$$k = -\frac{1}{t_h}\ln 2 = -\frac{\ln 2}{5600} \approx -0.000124 .$$

Let t_c be the time the wood has been aging, and y_0 be the original amount of carbon. Fifty-five percent of the original amount is $0.55y_0$. The length of time the wood has aged satisfies the equation

$$y_0 e^{-0.000124t_e} = 0.55y_0 .$$

Solving for t_c gives

$$t_c = -\frac{5600 \ln 0.55}{\ln 2} \approx 4830 \text{ years.}$$

■ **Newspaper Announcement**

8. For Carbon-14, $k = \dfrac{-\ln 2}{t_h} = \dfrac{-\ln 2}{5600} \approx -0.000124$. If y_0 is the initial amount, then the final amount of Carbon-14 present after 5000 years will be

$$y_0 e^{5000(-0.000124)} = 0.54y_0 .$$

In other words, 54% of the original carbon was still present.

■ **Radium Decay**

9. 6400 years is 4 half-lives, so that $\dfrac{1}{2^4} \approx 6.25\%$ will be present.

■ **General Half-Life Equation**

10. We are given the two equations

$$Q_1 = Q_0 e^{kt_1}$$
$$Q_2 = Q_0 e^{kt_2}.$$

If we divide, we get

$$\frac{Q_1}{Q_2} = e^{k(t_1 - t_2)}$$

or

$$k(t_1 - t_2) = \ln \frac{Q_1}{Q_2}$$

or

$$k = \frac{\ln \frac{Q_1}{Q_2}}{t_1 - t_2}.$$

Substituting in $t_h = -\frac{1}{k}\ln 2$ yields the general half-life of

$$t_h = -\frac{(t_1 - t_2)\ln 2}{\ln \frac{Q_1}{Q_2}} = \frac{(t_2 - t_1)\ln 2}{\ln \frac{Q_1}{Q_2}}.$$

■ **Nuclear Waste**

11. We have

$$k = -\frac{\ln 2}{t_h} = -\frac{\ln 2}{258} \approx -0.00268$$

and solve for *t* in $y_0 e^{-0.00268t} = 0.05 y_0$. Thus

$$t = \frac{258 \ln 20}{\ln 2} \approx 1{,}115 \text{ years.}$$

■ **Bombarding Plutonium**

12. We are given $k = -\frac{\ln 2}{0.15} \approx -4.6209812$. The differential equation for the amount present is

$$\frac{dA}{dt} = kA + 0.00002, \quad A(0) = 0.$$

Solving this initial-value problem we get the particular solution

$$A(t) = ce^{kt} - \frac{0.00002}{k}$$

where $c = \frac{0.00002}{k} \approx -0.000004$. Plugging in these values gives the total amount

$$A(t) \approx 0.000004\left(1 - e^{-4.6t}\right)$$

measured in micrograms.

■ **Blood Alcohol Levels**

13. (a) First, because the initial blood-alcohol level is 0.2%, we have $P(0) = 0.2$. After one hour, the level reduces by 10%, therefore, we have

$$P(1) = 0.9P(0) = 0.9(0.2).$$

From the decay equation we have $P(1) = 0.2e^k$, hence we have the equation

$$0.2e^k = 0.9(0.2)$$

from which we find $k = \ln 0.9 \approx -0.105$. Thus our decay equation is

$$P(t) = 0.2e^{(\ln 0.9)t} \approx 0.2e^{-0.105t}.$$

(b) The person can legally drive as soon as $P(t) < 0.1$. Setting

$$P(t) = 0.2e^{-0.105t} = 0.1$$

and solving for *t*, yields

$$t = -\frac{\ln 2}{-0.105} \approx 6.6 \text{ hours.}$$

■ **Exxon Valdez Problem**

14. The measured blood-alcohol level was 0.06%, which had been dropping at a rate of 0.015 percentage points per hour for nine hours. This being the case, the captain's initial blood-alcohol level was

$$0.06 + 9(0.015) = 0.195\%.$$

The captain definitely could be liable.

■ **Sodium Pentathol Elimination**

15. The half-life is 10 hours. The decay constant is

$$k = -\frac{\ln 2}{10} \approx -0.069.$$

Ed needs

$$(50 \text{ mg/kg})(100 \text{ kg}) = 5000 \text{ mg}$$

of pentathal to be anesthetized. This is the minimal amount that can be presented in his bloodstream after three hours. Hence,

$$A(3) = A_0 e^{-0.069(3)} \approx 0.813 A_0 = 5000 .$$

Solving for A_0 yields $A_0 = 6{,}155.7$ milligrams or an initial dose of 6.16 grams.

■ **Moonlight at High Noon**

16. Let the initial brightness be I_0. At a depth of $d = 25$ feet, we have that 15% of the light is lost, and so we have $I(25) = 0.85 I_0$. Assuming exponential decay, $I(d) = I_0 e^{kd}$, we have the equation

$$I(25) = I_0 e^{25k} = 0.85 I_0$$

from which we can find

$$k = \frac{\ln 0.85}{25} \approx -0.0065 .$$

To find *d*, we use the equation

$$I_0 e^{-0.0065d} = \frac{1}{300{,}000} I_0 ,$$

from which we determine the depth to be

$$d = \frac{1}{0.0065} \ln(300{,}000) \approx 1940 \text{ feet.}$$

■ **Tripling Time**

17. Here $k = \dfrac{\ln 2}{10}$. We can find the tripling time by solving for *t* in the equation

$$y_0 e^{[(\ln 2)/10]t} = 3 y_0 ,$$

giving $\left(\dfrac{\ln 2}{10}\right) t = \ln 3$ or

$$t = \frac{10 \ln 3}{\ln 2} \approx 15.85 \text{ hours.}$$

■ **Extrapolating the Past**

18. If P_0 is the initial number of bacteria present (in millions), then we are given $P_0 e^{6k} = 5$ and $P_0 e^{9k} = 8$. Dividing one equation by the other we obtain $e^{3k} = \dfrac{8}{5}$, from which we find

$$k = \frac{\ln \frac{8}{5}}{3} .$$

Substituting this value into the first equation gives $P_0 e^{2\ln(8/5)} = 5$, in which we can solve for

$$P_0 = 5e^{-2\ln(8/5)} \approx 1.95 \text{ million bacteria.}$$

■ **Unrestricted Yeast Growth**

19. From Problem 2, we are given

$$k = \frac{\ln 2}{1} = \ln 2$$

with the initial population of $P_0 = 5$ million. The population at time t will be $5e^{(\ln 2)t}$ million, so at $t = 4$ hours, the population will be

$$5e^{4\ln 2} = 5 \cdot 16 = 80 \text{ million.}$$

■ **Unrestricted Bacterial Growth**

20. From Problem 2, we are given $k = \frac{\ln 2}{12}$ so the population equation is

$$P(t) = P_0 e^{t(\ln 2)/12}.$$

In order to have five times the starting value, we require $P_0 e^{t(\ln 2)/12} = 5P_0$, from which we can find

$$t = 12\frac{\ln 5}{\ln 2} \approx 27.9 \text{ hours.}$$

■ **Growth of Tuberculosis Bacteria**

21. We are given the initial number of cells present is $P_0 = 100$, and that $P(1) = 150$ (1.5 times larger), then 100 $e^{k(1)} = 150$, which yields $k = \ln\frac{3}{2}$. Therefore, the population $P(t)$ at any time t is

$$P(t) = 100e^{t\ln(3/2)} \approx 100e^{0.405t} \text{ cells.}$$

■ **Cat and Mouse Problem**

22. (a) For the first 10 years, the mouse population simply had exponential growth

$$M(t) = M_0 e^{kt}.$$

Because the mouse population doubled to 50,000 in 10 years, the initial population must have been 25,000, hence $k = \frac{\ln 2}{10}$. For the first 10 years, the mouse population (in thousands) was

$$M(t) = 25e^{t\ln(2)/10}.$$

Over the next 10 years, the differential equation was $\frac{dM}{dt} = kM - 6$, where $M(0) = 50$; t now measures the number of years after the arrival of the cats. Solving this differential equation yields

$$M(t) = ce^{kt} + \frac{6}{k}.$$

Using the initial condition $M(0) = 50$, we find $c = 50 - \frac{6}{k}$. The number of mice (in thousands) t years after the arrival of the cats is

$$M(t) = \left(50 - \frac{6}{k}\right)e^{kt} + \frac{6}{k}$$

where the constant k is given by $k = \frac{\ln 2}{10} \approx 0.069$. We obtain

$$M(t) = -37e^{0.069t} + 87.$$

(b) $M(10) = 87 - 37e^{0.069(10)} \approx 13.2$ thousand mice.

(c) From part (a), we obtain the value of k for the population growth without harvest, i.e., $k = \frac{\ln 2}{10} \approx 0.0693$. We obtain the new rate of change M' of the mouse population:

$$M' = \frac{\ln 2}{10}M - 0.10M \approx -0.0307M$$
$$M = 25000e^{-0.0307t}$$

After 10 years the mouse population will be 18393 (give or take a mouse or two).

■ **Banker's View of *e***

23. The amount of money in a bank account that collects compound interest with continuous compounding is given by

$$A(t) = A_0e^{rt}$$

where A_0 is the initial amount and r is an annual interest rate. If $A_0 = \$1$ is initially deposited, and if the annual interest rate is $r = 0.10$, then after 10 years the account value will be

$$A(10) = \$1 \cdot e^{0.10(10)} \approx \$2.72.$$

■ **Rule of 70**

24. The doubling time is given in Problem 2 by

$$t_d = \frac{\ln 2}{r} \approx \frac{0.70}{r} = \frac{70}{100r}$$

where $100r$ is an annual interest rate (expressed as a percentage). The rule of 70 makes sense.

■ **Power of Continuous Compounding**

25. The future value of the account will be

$$A(t) = A_0 e^{rt},$$

If $A_0 = \$0.50$, $r = 0.06$ and $t = 160$, then the value of the account after 160 years will be

$$A(160) = 0.5e^{(0.06)(160)} \approx \$7,382.39 .$$

■ **Credit Card Debt**

26. If Meena borrows $A_0 = \$5000$ at an annual interest rate of $r = 0.1995$ (i.e., 19.95%), compounded continuously, then the total amount she owes (initial principle plus interest) after t years is

$$A(t) = A_0 e^{rt} = \$5,000 e^{0.1995t} .$$

After $t = 4$ years, she owes

$$A(4) = \$5,000 e^{0.1995(4)} \approx \$11,105.47 .$$

Hence, she pays $\$11,105.47 - \$5000 = \$6,105.74$ interest for borrowing this money.

■ **Compound Interest Thwarts Hollywood Stunt**

27. The growth rate is $A(t) = A_0 e^{rt}$. In this case $A_0 = 3$, $r = 0.08$, and $t = 320$. Thus, the total bottles of whiskey will be

$$A(320) = 3e^{(0.08)(320)} \approx 393,600,000,000 .$$

That's 393.6 *billion* bottles of whiskey!

■ **It Ain't Like It Use to Be**

28. The growth rate is $A(t) = A_0 e^{rt}$, where $A_0 = 1$, $t = 50$, and $A(50) = 18$ (using thousands of dollars). Hence, we have $18 = e^{50r}$, from which we can find $r = \frac{\ln 18}{50} \approx 0.0578$, or 5.78%.

■ How to Become a Millionaire

29. (a) From Equation (11) we see that the equation for the amount of money is

$$A(t) = A_0 e^{rt} + \frac{d}{r}\left(e^{rt} - 1\right).$$

In this case, $A_0 = 0$, $r = 0.08$, and $d = \$1,000$. The solution becomes

$$A(t) = \frac{1000}{0.08}\left(e^{0.08t} - 1\right).$$

(b) $A(40) = \$1,000,000 = \frac{d}{0.08}\left(e^{0.08(40)} - 1\right)$. Solving for *d*, we get that the required annual deposit $d = \$3,399.55$.

(c) $A(40) = \$1,000,000 = \frac{2500}{r}\left(e^{40r} - 1\right)$. To solve this equation for r we require a computer. Using Maple, we find the interest rates $r = 0.090374$ $(\approx 9.04\%)$. You can confirm this result using direct substitution.

■ Living Off Your Money

30. $A(t) = A_0 e^{rt} - \frac{d}{r}\left(e^{rt} - 1\right)$. Setting $A(t) = 0$ and solving for *t* gives

$$t = \frac{1}{r}\ln\left(\frac{d}{d - rA_0}\right)$$

Notice that when $d = rA_0$ this equation is undefined, as we have division by 0; if $d < rA_0$, this equation is undefined because we have a negative logarithm. For the physical translation of these facts, you must return to the equation for $A(t)$. If $d = rA_0$, you are only withdrawing the interest, and the amount of money in the bank remains constant. If $d < rA_0$, then you aren't even withdrawing the interest, and the amount in the bank increases and $A(t)$ never equals zero.

■ How Sweet It Is

31. From equation (11), we have

$$A(t) = \$1,000,000e^{0.08t} - \frac{100,000}{0.08}\left(e^{0.08t} - 1\right).$$

Setting $A(t) = 0$, and solving for *t*, we have

$$t = \frac{\ln 5}{0.08} \approx 20.1 \text{ years},$$

the time that the money will last.

■ **The Real Value of the Lottery**

32. Following the hint, we let

$$A' = 0.10A - 50,000 .$$

Solving this equation with initial condition $A(0) = A_0$ yields

$$A(t) = (A_0 - 500,000)e^{0.10t} + 500,000 .$$

Setting $A(20) = 0$ and solving for A_0 we get

$$A_0 = \frac{500,000(e^2 - 1)}{e^2} \approx \$432,332 .$$

■ **Continuous Compounding**

33. (a) After one year compounded continuously the value of the account will be

$$S(1) = S_0 e^r .$$

With $r = 0.08$ (8%) interest rate, we have the value

$$S(1) = S_0 e^{0.08} \approx \$1.083287 S_0 .$$

This is equivalent to a single annual compounding at a rate $r_{eff} = 8.329$ %.

(b) If we set the annual yield from a single compounding with interest r_{eff} , $S_0(1 + r_{\text{eff}})$ equal to the annual yield from continuous compounding with interest *r*, $S_0 e^r$, we have

$$S_0(1 + r_{\text{eff}}) = S_0 e^r .$$

Solving for r_{eff} yields $r_{\text{eff}} = e^r - 1$.

(c) $r_{\text{daily}} = \left(1 + \frac{0.08}{365}\right)^{365} - 1 = 0.0832775$ (i.e., 8.328%) effective annual interest rate, which is extremely close to that achieved by continuous compounding as shown in part (a).

■ **Good Test Equation for Computer or Calculator**

34. Student Project.

■ **Your Financial Future**

35. We can write the savings equation (10) as

$$A' = 0.08A + 5000 , \quad A(0) = 0 .$$

The exact solution by (11) is

$$A = \frac{5000}{0.08}(e^{0.08t} - 1) .$$

We list the amounts, rounded to the nearest dollar, for each of the first 20 years.

Year	Amount	Year	Amount
1	5,205	11	88,181
2	10,844	12	100,731
3	16,953	13	114,326
4	23,570	14	129,053
5	30,739	15	145,007
6	38,505	16	162,290
7	46,917	17	181,012
8	56,030	18	201,293
9	65,902	19	223,264
10	76,596	20	247,065

After 20 years at 8%, contributions deposited have totalled $20 \times \$5,000 = \$100,000$ while $147,064 has accumulated in interest, for a total account value of $247,064.

Experiment will show that the interest is the more important parameter over 20 years. This answer can be seen in the solution of the annuity equation

$$A = \frac{5000}{0.08}\left(e^{0.08t} - 1\right).$$

The interest rate occurs in the exponent and the annual deposit simply occurs as a multiplier.

■ Mortgaging a House

36. (a) Since the bank earns 1% monthly interest on the outstanding principle of the loan, and Kelly's group make monthly payments of $2500 to the bank, the amount of money $A(t)$ still owed the bank at time t, where t is measured in months starting from when the loan was made, is given by the savings equation (10) with $a = -2500$.

Thus, we have

$$\frac{dA}{dt} = 0.01A - 2500, \quad A(0) = \$200,000.$$

(b) The solution of the savings equation in (a) was seen (11) to be

$$\begin{aligned} A(t) &= A(0)e^{rt} + \frac{a}{r}\left(e^{rt} - 1\right) \\ &= 200{,}000e^{0.01t} - \frac{2500}{0.01}\left(e^{0.01t} - 1\right) \\ &= -50{,}000e^{0.01t} + \$250{,}000. \end{aligned}$$

(c) To find the length of time for the loan to be paid off, we set $A(t) = 0$, and solve for t. Doing this, we have

$$-50{,}000e^{0.01t} = -\$250{,}000.$$

or

$$0.01t = \ln 5 \text{ or } t = 100 \ln 5 \approx 100(1.609) \approx 161 \text{ months (13 years and 5 months).}$$

■ **Suggested Journal Entry**

37. Student Project

2.4 Linear Models: Mixing and Cooling [redacted]

■ Mixing Details

1. Separating variables, we find

$$\frac{dx}{x}=\frac{2}{t-100}dt$$

from which we get

$$\ln|x|=2\ln|t-100|+c\,.$$

We can solve for $x(t)$ using properties of the logarithm, getting

$$|x|=e^c e^{2\ln|t-100|}=Ce^{\ln(t-100)^2}=C(t-100)^2$$

where $C=e^c>0$ is an arbitrary positive constant. Hence, the final solution is

$$x(t)=\pm C(t-100)^2=c_1(t-100)^2$$

where c_1 is an arbitrary constant.

■ English Brine

2. (a) Salt inflow is

$$(2\text{ lbs/gal})(3\text{ gal/min})=6\text{ lbs/min}\,.$$

Salt outflow is

$$\left(\frac{\text{Q}}{300}\text{ lbs/gal}\right)(3\text{ gal/min})=\frac{\text{Q}}{100}\text{ lbs/min}\,.$$

The differential equation for $Q(t)$, the amount of salt in the tank, is

$$\frac{dQ}{dt}=6-0.01Q\,.$$

Solving this equation with initial condition $Q(0)=50$ yields

$$Q(t)=600-550e^{-0.01t}\,.$$

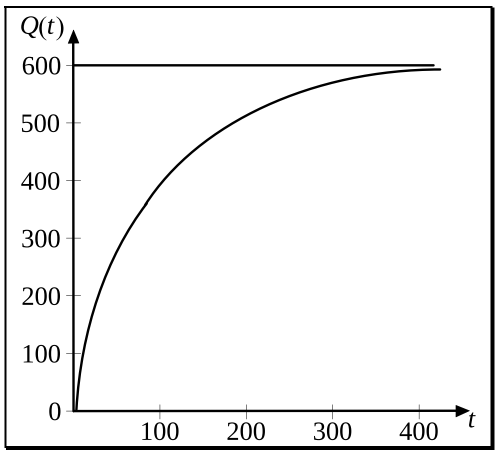
(b) The concentration $conc(t)$ of salt is simply the amount $Q(t)$ divided by the volume (which is constant at 300). Hence the concentration at time t is given by the expression

$$conc(t)=\frac{Q(t)}{300}=2-\frac{11}{6}e^{-0.01t}\,.$$

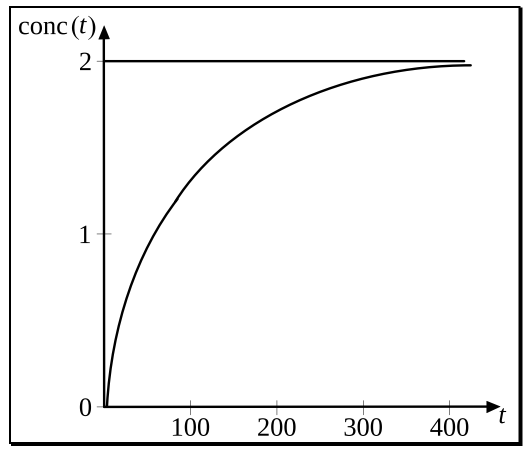
(c) As $t \to \infty$, $e^{-0.01t} \to 0$. Hence $Q(t) \to 600$ lbs of salt in the tank.

(d) Either take the limiting amount and divide by 300, or take the limit as $t \to \infty$ of $conc(t)$. The answer is 2 lbs/gal in either case.

(e) Note that the graphs of $Q(t)$ and of $conc(t)$ differ only in the scales on the vertical axis, because the volume is constant.



Number of lbs of salt in the tank



Concentration of salt in the tank

■ Metric Brine

3. (a) The salt inflow is

$$(0.1 \text{ kg/liter})(4 \text{ liters/min}) = 0.4 \text{ kg/min}.$$

The outflow is $\frac{4}{100}Q \text{ kg/min}$. Thus, the differential equation for the amount of salt is

$$\frac{dQ}{dt} = 0.4 - 0.04Q.$$

Solving this equation with the given initial condition $Q(0) = 50$ gives

$$Q(t) = 10 + 40e^{-0.04t}.$$

(b) The concentration $conc(t)$ of salt is simply the amount $Q(t)$ divided by the volume (which is constant at 100). Hence the concentration at time *t* is given by

$$conc(t) = \frac{Q(t)}{100} = 0.1 - 0.4e^{-0.04t}.$$

(c) As $t \to \infty$, $e^{-0.04t} \to 0$. Hence $Q(t) \to 10 \text{ kg}$ of salt in the tank.

(d) Either take the limiting amount and divide by 100 or take the limit as $t \to \infty$ of $conc(t)$. The answer is 0.1 kg/liter in either case.

■ Salty Goal

4. The salt inflow is given by

$$(2\ \text{lb/gal})(3\ \text{gal/min}) = 6\ \text{lbs/min}\,.$$

The outflow is $\frac{3}{20}Q$. Thus,

$$\frac{dQ}{dt} = 6 - \frac{3}{20}Q\,.$$

Solving this equation with the given initial condition $Q(0) = 5$ yields the amount

$$Q(t) = 40 - 35e^{-3t/20}\,.$$

To determine how long this process should continue in order to raise the amount of salt in the tank to 25 lbs, we set $Q(t) = 25$ and solve for t to get

$$t = \frac{20}{3}\ln\frac{7}{3} \approx 5.6\ \text{minutes.}$$

■ Mysterious Brine

5. Input in lbs/min is $2x$ (where x is the unknown concentration of the brine). Output is

$$\frac{2}{100}Q\ \text{lbs/min}\,.$$

The differential equation is given by

$$\frac{dQ}{dt} = 2x - 0.01Q\,,$$

which has the general solution

$$Q(t) = 200x + ce^{-0.01t}\,.$$

Because the tank had no salt initially, $Q(0) = 0$, which yields $c = -200x$. Hence, the amount of salt in the tank at time t is

$$Q(t) = 200x\left(1 - e^{-0.01t}\right).$$

We are given that

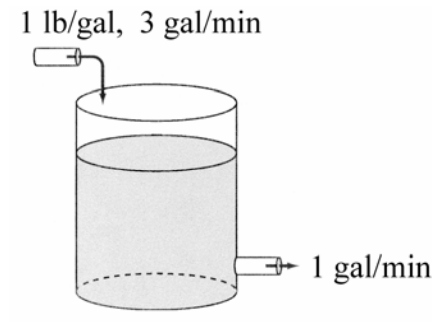
$$Q(120) = (1.4)(200) = 280\,,$$

which we solve for x, to get $x \approx 2.0\ \text{lb/gal}$.

■ **Salty Overflow**

6. Let x = amount of salt in tank at time t.

We have $\dfrac{dx}{dt}=\dfrac{1\text{ lb}}{\text{gal}}\cdot\dfrac{3\text{ gal}}{\text{min}}-\dfrac{(x\text{ lb})\cdot 1\text{ gal/min}}{(300+(3-1)t)\text{gal}}$, with initial volume = 300 gal, capacity = 600 gal.



IVP: $\dfrac{dx}{dt}=3-\dfrac{x}{300+2t}$, $x(0) = 0$

The DE is linear,

$$\frac{dx}{dt}+\frac{x}{300+2t}=3,$$

with integrating factor

$$\mu=e^{\int\frac{1}{300+2t}dt}=e^{\frac{1}{2}\ln(300+2t)} = (300+2t)^{1/2}$$

Thus,

$$(300+2t)^{1/2}\frac{dx}{dt}+\frac{x}{(300+2t)^{1/2}}=3(300+2t)^{1/2}$$

$$(300+2t)^{1/2}x=\int 3(300+2t)^{1/2}\,dt$$

$$=\left(\frac{3}{2}\right)\frac{(300+2t)^{3/2}}{3/2}+c,$$

so

$$x(t)=(300+2t)+c(300+2t)^{-1/2}$$

The initial condition $x(0) = 0$ implies $0 = 300 + \dfrac{c}{\sqrt{300}}$, so $c = -3000\sqrt{3}$.

The solution to the IVP is

$$x(t) = 300 + 2t - 3000\sqrt{3}(300+2t)^{-1/2}$$

The tank will be full when $300 + 2t = 600$, so $t = 150$ min.

At that time, $x(150) = 300 + 2(150) - 3000\sqrt{3}(300+2(150))^{-1/2} \approx 388$ lbs

■ **Cleaning Up Lake Erie**

7. (a) The inflow of pollutant is

$$\left(40\ \text{mi}^3/\text{yr}\right)\left(0.01\%\right)=0.004\ \text{mi}^3/\text{yr},$$

and the outflow is

$$\left(40\,\text{mi}^3/\text{yr}\right)\frac{V(t)\,\text{mi}^3}{100\,\text{mi}^3}=0.4V(t)\,\text{mi}^3/\text{yr}.$$

Thus, the DE is

$$\frac{dV}{dt} = 0.004 - 0.4V$$

with the initial condition

$$V(0) = (0.05\%)\left(100 \text{ mi}^3\right) = 0.05 \text{ mi}^3 .$$

(b) Solving the IVP in part (a) we get the expression

$$V(t) = 0.01 + 0.04e^{-0.4t}$$

where V is the volume of the pollutant in cubic miles.

■ **Correcting a Goof**

8. Input in lbs/min is 0 (she's not adding any salt). Output is $0.03Q$ lbs/min . The differential equation is

$$\frac{dQ}{dt} = -0.03Q ,$$

which has the general solution

$$Q(t) = ce^{-0.03t} .$$

Using the initial condition $Q(0) = 20$, we get the particular solution

$$Q(t) = 20e^{-0.03t} .$$

Because she wants to reduce the amount of salt in the tank to 10 lbs, we set

$$Q(t) = 10 = 20e^{-0.03t} .$$

Solving for t, we get

$$t = \frac{100}{3}\ln 2 \approx 23 \text{ minutes.}$$

(c) A pollutant concentration of 0.02% corresponds to

$$0.02\%\left(100 \text{ mi}^3\right) = 0.02 \text{ mi}^3$$

of pollutant. Finally, setting $V(t) = 0.02$ gives the equation

$$0.02 = 0.01 + 0.04e^{-0.4t} ,$$

which yields

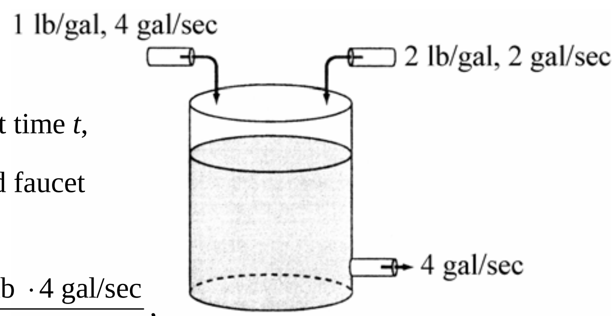
$$t = 2.5\ln(4) \approx 3.5 \text{ years.}$$

■ Changing Midstream

9. Let x = amount of salt in tank at time t.

(a) IVP: $\dfrac{dx}{dt} = \dfrac{1\text{ lb}}{\text{gal}} \cdot \dfrac{4\text{ gal}}{\text{sec}} - \left(\dfrac{x\text{ lb}}{200\text{ gal}}\right)\dfrac{4\text{ gal}}{\text{sec}}$ $\quad x(0) = 0$

(b) $x_{eq} = \dfrac{1\text{ lb}}{\text{gal}} \cdot 200\text{ gal} = 200\text{ lb}$



(c) Now let x = amount of salt in tank at time t, but reset $t = 0$ to be when the second faucet is turned on. This setup gives

$$\frac{dx}{dt} = \frac{4\text{ lb}}{\text{sec}} + \frac{2\text{lb}}{\text{gal}} \cdot \frac{2\text{ gal}}{\text{sec}} - \frac{x\text{ lb} \cdot 4\text{ gal/sec}}{(200 + 2t)\text{gal}},$$

which gives a new IVP:

$$\frac{dx}{dt} = 8 - \frac{4x}{200 + 2t} \qquad x(0) = x_{eq} = 200$$

(d) To find t_f: $200 + 2t_f = 1000 \qquad t_f = 400$ sec

(e) The DE in the new IVP is

$\dfrac{dx}{dt} + \dfrac{2x}{100 + t} = 8$, which is linear with integrating factor

$$\mu = e^{\int \frac{2}{100+t}dt} = e^{\ln(100+t)^2} = (100 + t)^2.$$

Thus, $(100+t)^2 \dfrac{dx}{dt} + 2(100+t)x = 8(100+t)^2$, and

$$(100+t)^2 x = \int 8(100+t)^2\,dt = \frac{8}{3}(100+t)^3 + c,$$

so

$$x(t) = \frac{8}{3}(100+t) + c(100+t)^{-2}.$$

The initial condition $x(0) = 200$ implies $200 = \dfrac{8}{3}(100) + \dfrac{c}{(100)^2}$ or $c = -\dfrac{2}{3}\times 10^6$.

Thus the solution to the new IVP is

$$x = \frac{8}{3}(100+t) - \frac{1}{3}(2\times 10^6)(100+t)^{-2}.$$

When $t_f = 400$, $x(400) = \dfrac{8}{3}(500) - \dfrac{1}{3}\dfrac{(2\times 10^6)}{(500)^2} \approx 1330.7$ lb.

(f) After tank starts to overflow,

Inflow: $\underbrace{\frac{1\text{ lb}}{\text{gal}}\cdot\frac{4\text{ gal}}{\text{sec}}}_{1^{\text{st}}\text{ faucet}} + \underbrace{\frac{2\text{ lb}}{\text{gal}}\cdot\frac{2\text{ gal}}{\text{sec}}}_{2^{\text{nd}}\text{ faucet}} = \frac{8\text{ lbs}}{\text{sec}}$

Outflow: $\Bigg(\underbrace{\frac{4\text{ gal}}{\text{sec}}}_{\text{drain}} + \underbrace{\frac{2\text{ gal}}{\text{sec}}\Bigg)\cdot\frac{x\text{ lb}}{1000\text{ gal}}}_{\text{overflow}} = \frac{6x}{1000}\frac{\text{lbs}}{\text{sec}}$

Hence for $t > 400$ sec, the IVP now becomes

$\frac{dx}{dt} = 8 - \frac{6x}{1000}$, $x(400) = 1330.7$ lb.

■ **Cascading Tanks**

10. (a) The inflow of salt into tank A is zero because fresh water is added. The outflow of salt is

$$\left(\frac{Q_A}{100}\ \text{lbs/gal}\right)(2\ \text{gal/min}) = \frac{1}{50}Q_A\ \text{lb/min}.$$

Tank A initially has a total of $0.5(100) = 50$ pounds of salt, so the initial-value problem is

$$\frac{dQ_A}{dt} = -\frac{Q_A}{50},\ Q_A(0) = 50\ \text{lbs.}$$

(b) Solving for Q_A gives

$$Q_A(t) = ce^{-t/50}$$

and with the initial condition $Q_A(0) = 50$ gives

$$Q_A(t) = 50e^{-t/50}.$$

(c) The input to the second tank is

$$\left(\frac{Q_A}{100}\ \text{lb/gal}\right)(2\ \text{gal/min}) = \frac{1}{50}Q_A\ \text{lb/min} = e^{-t/50}\ \text{lb/min}.$$

The output from tank B is

$$\left(\frac{Q_B}{100}\ \text{lb/gal}\right)(2\ \text{gal/min}) = \frac{1}{50}Q_B\ \text{lbs/min}.$$

Thus the differential equation for tank B is

$$\frac{dQ_B}{dt} = e^{-t/50} - \frac{1}{50}Q_B$$

with initial condition $Q_B(0) = 0$.

(d) Solving the initial-value problem in (c), we get

$$Q_B(t) = te^{-t/50} \text{ pounds.}$$

■ **More Cascading Tanks**

11. (a) Input to the first tank is

$$(0 \text{ gal alch/gal})(1 \text{ gal/min}) = 0 \text{ gal alch/min}.$$

Output is

$$\frac{1}{2}x_0 \text{ gal alch/min}.$$

The tank initially contains 1 gallon of alcohol, or $x_0(0) = 1$. Thus, the differential equation is given by

$$\frac{dx_0}{dt} = -\frac{1}{2}x_0.$$

Solving, we get $x_0(t) = ce^{-t/2}$. Substituting $x_0(0)$, we get $c = 1$, so the first tank's alcohol content is

$$x_0(t) = e^{-t/2}.$$

(b) The first step of a proof by induction is to check the initial case. In our case we check $n = 0$. For $n = 0$, $t^0 = 1$, $0! = 1$, $2^0 = 1$, and hence the given equation yields $x_0(t) = e^{-t/2}$. This result was found in part (a). The second part of an induction proof is to assume that the statement holds for case *n*, and then prove the statement holds for case $n+1$. Hence, we assume

$$x_n(t) = \frac{t^n e^{-t/2}}{n!2^n},$$

which means the concentration flowing into the next tank will be $\frac{x_n}{2}$ (because the volume is 2 gallons). The input of the next tank is $x_n\frac{1}{2}$ and the output $\frac{1}{2}x_{n+1}(t)$. The differential equation for the $(n+1)$ tank will be

$$\frac{dx_{n+1}}{dt}+\frac{1}{2}x_{n+1}=\frac{t^n e^{-t/2}}{n!2^{n+1}},\ x_{n+1}(0)=0.$$

Solving this IVP, we find

$$x_{n+1}(t)=\frac{t^{n+1}e^{-t/2}}{(n+1)!2^{n+1}}$$

which is what we needed to verify. The induction step is complete.

(c) To find the maximum of $x_n(t)$, we take its derivative, getting

$$x_n'=\frac{t^{n-1}e^{-t/2}}{(n-1)!2^n}-\frac{t^n e^{-t/2}}{n!2^{n+1}}.$$

Setting this value to zero, the equation reduces to $2nt^{n-1}-t^n=0$, and thus has roots $t=0$, $2n$. When $t=0$ the function is a minimum, but when $t=2n$, we apply the first derivative test and conclude it is a local maximum point. Substituting this in $x_n(t)$ yields the maximum value

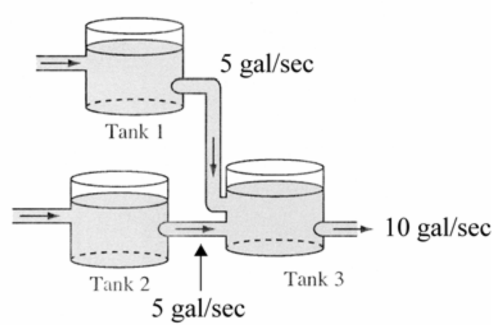
$$x_n(2n)\equiv M_n=\frac{(2n)^n e^{-n}}{n!2^n}=\frac{n^n e^{-n}}{n!}.$$

We can also see that $x_n(t)$ approaches 0 as $t\to\infty$ and so we can be sure this point is a global maximum of $x_n(t)$.

(d) Direct substitution of Stirling's approximation for $n!$ into the formula for M_n in part (c) gives $M_n\approx(2\pi n)^{-1/2}$.

■ **Three Tank Setup**

12. Let x, y, and z be the amounts of salt in Tanks 1, 2, and 3 respectively.



(a) For Tank 1: $\frac{dx}{dt}=\frac{0\text{ lbs}}{\text{gal}}\cdot\frac{5\text{ gal}}{\text{sec}}-\frac{x\text{ lbs}}{200\text{ gal}}\cdot\frac{5\text{ gal}}{\text{sec}}$,

so the IVP for $x(t)$ is

$$\frac{dx}{dt}=\frac{-5x}{200},\qquad x(0)=20.$$

The IVP for the identical Tank 2 is

$$\frac{dy}{dt}=\frac{-5y}{200},\qquad y(0)=20.$$

(b) For Tank 1, $\frac{dx}{dt} = \frac{-x}{40}$, so $x = 20e^{-t/40}$.

For Tank 2, $\frac{dy}{dt} = \frac{-y}{40}$, so $y = 20e^{-t/40}$.

(c)
$$\frac{dz}{dt} = \frac{x}{40} + \frac{y}{40} - \frac{z \text{ lbs}}{500 \text{ gal}} \cdot \frac{10 \text{ gal}}{\text{sec}}$$
$$= \frac{1}{2}e^{-t/40} + \frac{1}{2}e^{-t/40} - \frac{z}{50}.$$

Again we have a linear equation, $\frac{dz}{dt} + \frac{z}{50} = e^{-t/40}$,

with integrating factor $\mu = e^{\int 1/50 dt} = e^{t/50}$.

Thus

$$e^{t/50}\frac{dz}{dt} + \frac{1}{50}e^{t/50}z = e^{-t/40+t/50} = e^{-t/200},$$
$$e^{5/50}z = \int e^{-t/200}dt = -200e^{-t/200} + c,$$

so $z(t) = -200e^{-t/40} + ce^{-t/50}$.

■ Another Solution Method

13. Separating variables, we get $\frac{dT}{T-M} = -kdt$.

Solving this equation yields

$$\ln|T-M| = -kt + c, \text{ or, } |T-M| = e^c e^{-kt}.$$

Eliminating the absolute value, we can write $T - M = \pm e^c e^{-kt} = Ce^{-kt}$

where C is an arbitrary constant. Hence, we have $T(t) = M + Ce^{-kt}$.

Finally, using the condition $T(0) = T_0$ gives $T(t) = M + (T_0 - M)e^{-kt}$.

■ Still Another Approach

14. If $y(t) = T(t) - M$, then $\frac{dy}{dt} = \frac{dT}{dt}$, and $T(t) = y(t) + M$.

Hence the equation becomes

$$\frac{dy}{dt} = -k(y + M - M), \text{ or, } \frac{dy}{dt} = -ky,$$

a decay equation.

■ Using the Time Constant

15. (a) $T(t) = T_0 e^{-kt} + M\left(1 - e^{-kt}\right)$, from Equation (8). In this case, $M = 95$, $T_0 = 75$, and $k = \frac{1}{4}$, yielding the expression

$$T(t) = 75e^{-t/4} + 95\left(1 - e^{-t/4}\right)$$

where t is time measured in hours. Substituting $t = 2$ in this case (2 hours after noon), yields $T(2) \approx 82.9$ °F.

(b) Setting $T(t) = 80$ and simplifying for $T(t)$ yields

$$t = -4\ln\frac{3}{4} \approx 1.15 \text{ hours,}$$

which translates to 1:09 P.M.

■ A Chilling Thought

16. (a) $T(t) = T_0 e^{-kt} + M\left(1 - e^{-kt}\right)$, from Equation (8). In this problem, $T_0 = 75$, $M = 10$, and $T\left(\frac{1}{2}\right) = 50$ (taking time to be in hours). Thus, we have the equation $50 = 10 + 60e^{-k/2}$, from which we can find the rate constant

$$k = -2\ln\frac{2}{3} \approx 0.81.$$

After one hour, the temperature will have fallen to

$$T(1) \approx 10 + 60e^{2\ln(2/3)} = 10 + 60\left(\frac{4}{9}\right) \approx 36.7^\circ \text{ F.}$$

(b) Setting $T(t) = 15$ gives the equation

$$15 = 10 + 60e^{2t\ln(2/3)}.$$

Solving for t gives

$$t = -\frac{\ln 12}{2\ln\left(\frac{2}{3}\right)} \approx 3.06 \text{ hours (3 hrs, 3.6 min).}$$

■ **Drug Metabolism**

17. The drug concentration $C(t)$ satisfies

$$\frac{dC}{dt} = a - bC$$

where *a* and *b* are constants, with $C(0) = 0$. Solving this IVP gives

$$C(t) = \frac{a}{b}\left(1 - e^{-bt}\right).$$

As $t \to \infty$, we have $e^{-bt} \to 0$ (as long as *b* is positive), so the limiting concentration of $C(t)$ is $\frac{a}{b}$. Notice that *b* must be positive or for large *t* we would have $C(t) < 0$, which makes no sense, because $C(t)$ is the amount in the body. To reach one-half of the limiting amount of $\frac{a}{b}$ we set

$$\frac{a}{2b} = \frac{a}{b}\left(1 - e^{-bt}\right)$$

and solve for *t*, getting $t = \frac{\ln 2}{b}$.

■ **Warm or Cold Beer?**

18. Again, we use

$$T(t) = M + \left(T_0 - M\right)e^{-kt}.$$

In this case, $M = 70$, $T_0 = 35$. If we measure *t* in minutes, we have $T(10) = 40$, giving

$$40 = 70 - 35e^{-10k}.$$

Solving for the decay constant *k*, we find

$$k = -\frac{\ln\left(\frac{6}{7}\right)}{10} \approx 0.0154.$$

Thus, the equation for the temperature after *t* minutes is

$$T(t) \approx 70 - 35e^{-0.0154t}.$$

Substituting $t = 20$ gives $T(20) \approx 44.3°$ F.

■ The Coffee and Cream Problem

19. The basic law of heat transfer states that if two substances at different temperatures are mixed together, then the heat (calories) lost by the hotter substance is equal to the heat gained by the cooler substance. The equation expressing this law is

$$M_1 S_1 \Delta t_1 = M_2 S_2 \Delta t_2$$

where M_1 and M_2 are the masses of the substances, S_1 and S_2 are the specific heats, and Δt_1 and Δt_2 are the changes in temperatures of the two substances, respectively.

In this problem we assume the specific heat of coffee (the ability of the substance to hold heat) is the same as the specific heat of cream. Defining

$$\begin{aligned} C(0) &= \text{initial temperature of the coffee} \\ R &= \text{room temperature } (\text{temp of the cream}) \\ T &= \text{temperature of the coffee after the cream is added} \end{aligned}$$

we have

$$M_1\big(C(0) - T\big) = M_2 (T - R).$$

If we assume the mass M_2 of the cream is $\frac{1}{10}$ the mass Mg of the coffee (the exact fraction does not affect the answer), we have

$$10\big(C(0) - T\big) = T - R.$$

The temperature of the coffee after John initially adds the cream is

$$T = \frac{10C(0) + R}{11}.$$

After that John and Maria's coffee cools according to the basic law of cooling, or

$$\text{John: } \left(\frac{10C(0) + R}{11}\right)e^{kt}, \text{ Maria: } C(0)e^{kt}$$

where we measure *t* in minutes. At time $t = 10$ the two coffees will have temperature

$$\text{John: } \left(\frac{10C(0) + R}{11}\right)e^{10k}, \text{ Maria: } C(0)e^{10k}.$$

Maria then adds the same amount of cream to her coffee, which means John and Maria's coffees now have temperature

$$\text{John: } \left(\frac{10C(0)+R}{11}\right)e^{10k}, \text{ Maria: } \left(\frac{10C(0)e^{10k}+R}{11}\right).$$

Multiplying each of these temperatures by 11, subtracting $10C(0)e^{10k}$ and using the fact that $\text{Re}^{10k} > R$, we conclude that John drinks the hotter coffee.

■ **Professor Farlow's Coffee**

20. $T(t) = T_0 e^{-kt} + M\left(1-e^{-kt}\right)$. For this problem, $M = 70$ and $T_0 = 200$ °F. The equation for the coffee temperature is

$$T(t) = 70 + 130e^{-kt}.$$

Measuring *t* in hours, we are given

$$T\left(\frac{1}{4}\right) = 120 = 70 + 130e^{-k/4};$$

so the rate constant is

$$k = -4\ln\frac{5}{13} \approx 3.8.$$

Hence

$$T(t) = 70 + 130e^{-3.8t}.$$

Finally, setting $T(t) = 90$ yields

$$90 = 70 + 130e^{-3.8t},$$

from which we find $t \approx 0.49$ hours, or 29 minutes and 24 seconds.

■ **Case of the Cooling Corpse**

21. (a) $T(t) = T_0 e^{-kt} + M\left(1-e^{-kt}\right)$. We know that $M = 50$ and $T_0 = 98.6$ °F. The first measurement takes place at unknown time t_1 so

$$T(t_1) = 70 = 50 + 48.6e^{-kt_1}$$

or $48.6e^{-kt_1} = 20$. The second measurement is taken two hours later at $t_1 + 2$, yielding

$$60 = 50 + 48.6e^{-k(t_1+2)}$$

or $48.6e^{-k(t_1+2)}=10$. Dividing the second equation by the first equation gives the relationship $e^{-2k}=\frac{1}{2}$ from which $k=\frac{\ln 2}{2}$. Using this value for k the equation for $T(t_1)$ gives

$$70=50+48.6e^{-t_1 \ln 2/2}$$

from which we find $t_1 \approx 2.6$ hours. Thus, the person was killed approximately 2 hours and 36 minutes before 8 P.M., or at 5:24 P.M.

(b) Following exactly the same steps as in part (a) but with T_0 = 98.2° F, the sequence of equations is

$$T(t_1) = 70 = 50 + 48.2e^{-k(t_1)} \Rightarrow 48.2e^{-kt_1} = 20.$$

$$T(t_1 + 2) = 60 = 50 + 48.2e^{-k(t_1+2)} \Rightarrow 48.2e^{-k(t_1+2)} = 10.$$

Dividing the second equation by the first still gives the relationship $e^{-2k} = \frac{1}{2}$,

so $k = \frac{\ln 2}{2}$.

Now we have

$$T(t_1) = 70 = 50 + 48.2e^{-t_1 \ln 2/2}$$

which gives $t_1 \approx 2.54$ hours, or 2 hours and 32 minutes. This estimates the time of the murder at 5.28 PM, only 4 minutes earlier than calculated in part (a).

■ **A Real Mystery**

22. $T(t)=T_0e^{-kt}+M\left(1-e^{-kt}\right)$

While the can is in the refrigerator $T_0=70$ and $M=40$, yielding the equation

$$T(t)=40+30e^{-kt}.$$

Measuring time in minutes, we have

$$T(15)=40+30e^{-15k}=60,$$

which gives $k=-\left(\frac{1}{15}\right)\ln\left(\frac{2}{3}\right)\approx 0.027$. Letting t_1 denote the time the can was removed from the refrigerator, we know that the temperature at that time is given by

$$T(t_1)=40+30e^{-kt_1},$$

which would be the W_0 for the warming equation $W(t)$, the temperature *after* the can is removed from the refrigerator

$$W(t) = 70 + (W_0 - 70)e^{-kt}$$

(the k of the can doesn't change). Substituting W_0 where $t = t_1$ and simplifying, we have

$$W(t) = 70 + 30\left(e^{-kt_1} - 1\right)e^{-kt}.$$

The initial time for this equation is t_1 (the time the can was taken out of the refrigerator), so the time at 2 P.M. will be $60 - t_1$ minutes yielding the equation in t_1:

$$W(60 - t_1) = 60 = 70 + 30\left(e^{-kt_1} - 1\right)e^{-k(60-t_1)}.$$

This simplifies to

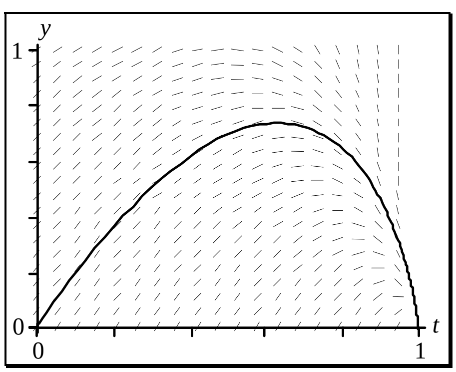
$$\frac{1}{3} = e^{-60k} - e^{-k(60-t_1)},$$

which is relatively easy to solve for t_1 (knowing that $k \approx 0.027$). The solution is $t_1 \approx 37$; hence the can was removed from the refrigerator at 1:37 P.M.

■ Computer Mixing

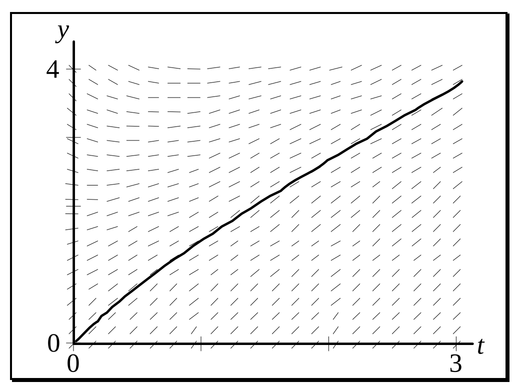
23. $y' + \frac{1}{1-t}y = 2$, $y(0) = 0$

When the inflow is less than the outflow, we note that the amount of salt $y(t)$ in the tank becomes zero when $t = 1$, which is also when the tank is emptied.



24. $y' + \frac{1}{1+t}y = 2$, $y(0) = 0$

When the inflow is greater than the outflow, the amount of dissolved substance keeps growing without end.



■ Suggested Journal Entry

25. Student Project

2.5 Nonlinear Models: Logistic Equation [redacted]

■ Equilibria

Note: Problems 1–6 are all autonomous s equations, so lines of constant slope (isoclines) are horizontal lines.

1. $y' = ay + by^2$, $(a > 0,\ b > 0)$

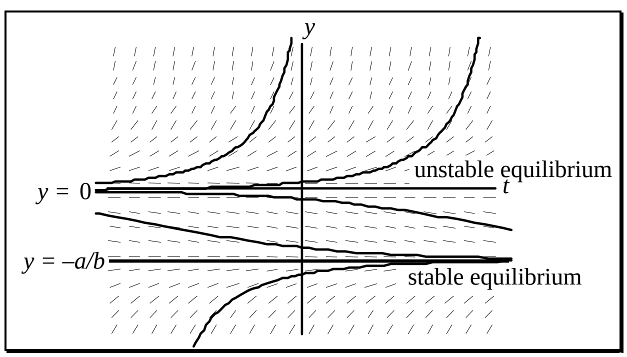
We find the equilibrium points by solving

$$y' = ay + by^2 = 0,$$

getting $y = 0$, $-\frac{a}{b}$. By inspecting

$$y' = y(a + by),$$

we see that solutions have positive slope $(y' > 0)$ when $y > 0$ or $y < -\frac{a}{b}$ and negative slope $(y' < 0)$ for $-\frac{a}{b} < y < 0$. Hence, the equilibrium solution $y(t) \equiv 0$ is unstable, and the equilibrium solution $y(t) \equiv -\frac{a}{b}$ is stable.



2. $y' = ay - by^2$, $(a > 0,\ b > 0)$

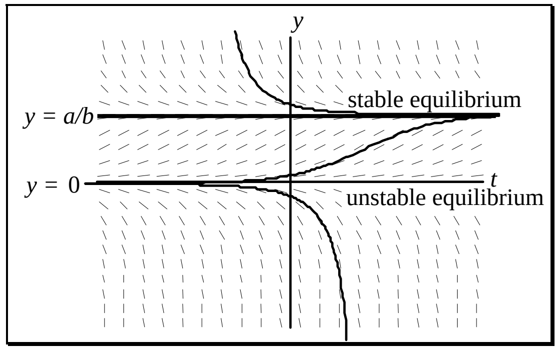
We find the equilibrium points by solving

$$y' = ay - by^2 = 0,$$

getting $y = 0$, $\frac{a}{b}$. By inspecting

$$y' = y(a - by),$$

we see that solutions have negative slope $(y' < 0)$ when $y < 0$ or $y > \frac{a}{b}$ and positive slope $(y' > 0)$ for $0 < y < \frac{a}{b}$. Hence, the equilibrium solution $y(t) \equiv 0$ is unstable, and the equilibrium solution $y(t) \equiv \frac{a}{b}$ is stable.



3. $y' = -ay + by^2$, $(a > 0,\ b > 0)$

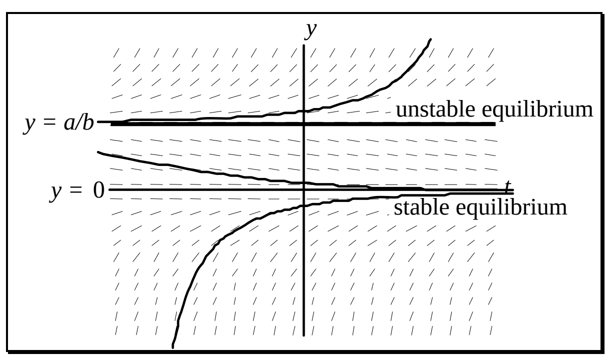
We find the equilibrium points by solving

$$y' = -ay + by^2 = 0,$$

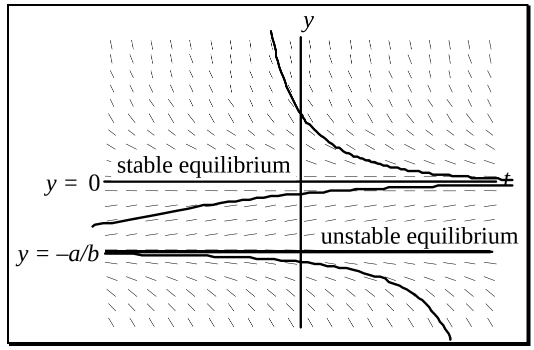
getting $y = 0$, $\frac{a}{b}$. By inspecting

$$y' = y(-a + by),$$

we see that solutions have positive slope when $y < 0$ or $y > \frac{a}{b}$ and negative slope for $0 < y < \frac{a}{b}$. Hence, the equilibrium solution $y(t) \equiv 0$ is stable, and the equilibrium solution $y(t) \equiv \frac{a}{b}$ is unstable.



4. $y' = -ay - by^2$, $(a > 0,\ b > 0)$



We find the equilibrium points by solving

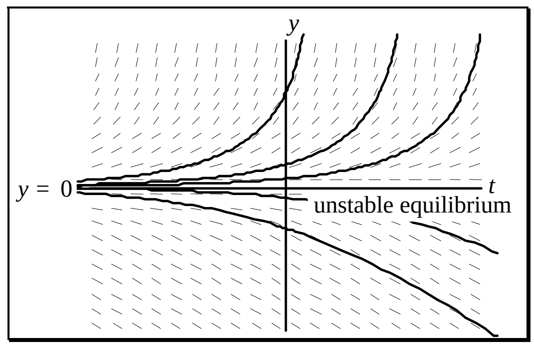
$$y' = -ay - by^2 = 0,$$

getting $y = 0$, $-\frac{a}{b}$. By inspecting

$$y' = -y(a + by),$$

we see that solutions have negative slope when $y > 0$ or $y < -\frac{a}{b}$ and positive slope for $-\frac{a}{b} < y < 0$. Hence, the equilibrium solution $y(t) \equiv 0$ is stable, and the equilibrium solution $y(t) \equiv -\frac{a}{b}$ is unstable.

5. $y' = e^y - 1$

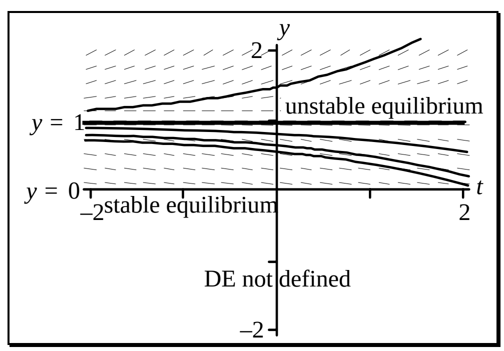


Solving for *y* in the equation

$$y' = e^y - 1 = 0,$$

we get $y = 0$, hence we have one equilibrium (constant) solution $y(t) \equiv 0$. Also $y' > 0$ for *y* positive, and $y' < 0$ for *y* negative. This says that $y(t) \equiv 0$ is an unstable equilibrium point.

6. $y' = y - \sqrt{y}$



Setting $y' = 0$ we find equilibrium points at

$$y = 0 \text{ and } 1.$$

The equilibrium at $y = 0$ is stable; that at $y = 1$ is unstable. Note also that the DE is only defined when $y \geq 0$.

■ Nonautonomous Sketching

For nonautonomous equations, the lines of constant slope are not horizontal lines as they were in the autonomous equations in Problems 1–6.

7. $y' = y(y - t)$

In this equation we observe that $y' = 0$ when $y = 0$, and when $y = t$; $y \equiv 0$ is equilibrium, but $y = t$ is just an isocline of horizontal slopes. We can draw these lines in the *ty*-plane with horizontal elements passing through them.

We then observe from the DE that when

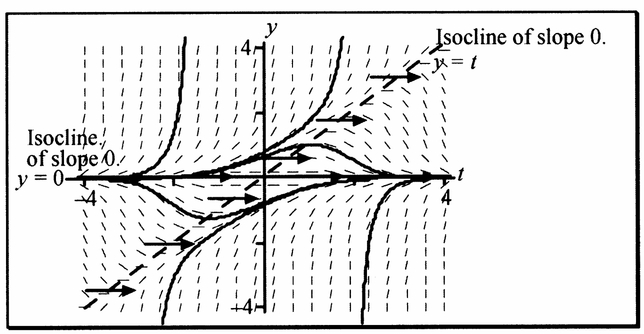
$y > 0$ and $y > t$ the slope is positive

$y > 0$ and $y < t$ the slope is negative

$y < 0$ and $y > t$ the slope is negative

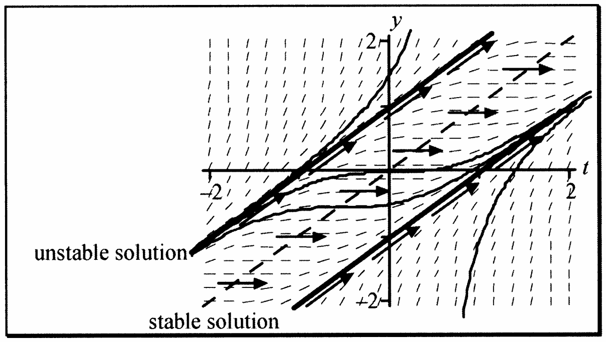
$y < 0$ and $y < t$ the slope is positive.

From the preceding facts, we surmise that the solutions behave according to our simple analysis of the sign y'. As can be seen from this figure, the equilibrium $y \equiv 0$ is stable at $t > 0$ and unstable at $t < 0$.

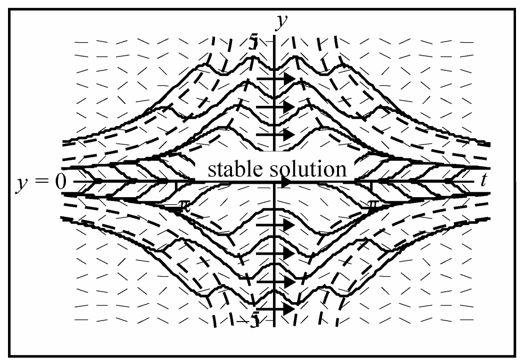


8. $y' = (y-t)^2$

In this equation we observe that $y' = 0$ when $y = t$. We can draw isoclines $y - t = c$ and elements with slope $y' = c^2$ passing through them. Note that the solutions for $c = \pm 1$ are also *solutions* to the DE. Note also that for this DE the slopes are all positive.



9. $y' = \sin(yt)$



Isoclines of horizontal slopes (dashed) are hyperbolas $yt = \pm n\pi$ for $n = 0, 1, 2, \ldots$. On the computer drawn graph you can sketch the hyperbolas for isoclines and verify the alternating occurrence of positive and negative slopes between them as specified by the DE.

Only $y \equiv 0$ is an equilibrium (unstable for $t < 0$, stable for $t > 0$).

■ Inflection Points

10. $y' = r\left(1 - \frac{y}{L}\right)y$

We differentiate with respect to *t* (using the chain rule), and then substitute for $\frac{dy}{dt}$ from the DE. This gives

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = ry\left(-\frac{1}{L}\right)\frac{dy}{dt} + r\left(1 - \frac{y}{L}\right)\frac{dy}{dt} = \left(-\frac{2ry}{L} + r\right)ry\left(1 - \frac{y}{L}\right).$$

Setting $\frac{d^2y}{dt^2} = 0$ and solving for *y* yields $y = 0$, L, $\frac{L}{2}$. Values $y = 0$ and $y = L$ are equilibrium points; $y = \frac{L}{2}$ is an inflection point. See text Figure 2.5.8.

11. $y' = -r\left(1 - \frac{y}{T}\right)y$

We differentiate with respect to t (using the chain rule), and then substitute for $\frac{dy}{dt}$ from the DE. This gives

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = -ry\left(-\frac{1}{T}\right)\frac{dy}{dt} - r\left(1 - \frac{y}{T}\right)\frac{dy}{dt} = -\left(\frac{2ry}{T} - r\right)r\left(1 - \frac{y}{T}\right)y .$$

Setting $\frac{d^2y}{dt^2} = 0$ and solving for y yields $y = 0$, T, $\frac{T}{2}$. Values $y = 0$ and $y = T$ are equilibrium points; $y = \frac{T}{2}$ is an inflection point. See text Figure 2.5.9.

12. $y' = \cos(y - t)$

We differentiate y' with respect to t (using the chain rule), and then substitute for $\frac{dy}{dt}$ from the DE. This gives

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = -\sin(y - t)\frac{dy}{dt} = -\sin(y - t)\cos(y - t) .$$

Setting $\frac{d^2y}{dt^2} = 0$ and solving for y yields $y - t = n\pi$, $y - t = \frac{n\pi}{2} + n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$. Note the inflection points change with t in this nonautonomous case. See text Figure 2.5.3, graph for (2), to see that the inflection points occur only when $y = -1$, so they lie along the lines $y = t + m\pi$ where m is an odd integer.

■ **Logistic Equation** $y' = r\left(1 - \frac{y}{L}\right)y$

13. (a) We rewrite the logistic DE by separation of variables and partial fractions to obtain

$$\left(\frac{1}{y} + \frac{\frac{1}{L}}{1 - \frac{y}{L}}\right)dy = rdt .$$

Integrating gives

$$\ln|y| - \ln\left|1 - \frac{y}{L}\right| = rt + c .$$

If $y_0 > L$, we know by qualitative analysis (see text Figure 2.5.8) that $y > L$ for all future time. Thus $\ln\left|1-\frac{y}{L}\right| = \ln\left(\frac{y}{L}-1\right)$ in this case, and the implicit solution (8) becomes

$$\frac{y}{\frac{y}{L}-1} = Ce^{rt}\text{, with } C = \frac{y_0}{\frac{y_0}{L}-1}.$$

Substitution of this new value of *C* and solving for *y* gives $y_0 > L$ gives

$$y(t) = \frac{L}{1+\left(\frac{L}{y_0}-1\right)e^{-rt}}$$

which turns out (surprise!) to match (10) for $y_0 < L$. You must show the algebraic steps to confirm this fact.

(b) The derivation of formula (10) required $\ln\left|1-\frac{y}{L}\right|$, which is undefined if $y = L$. Thus, although formula (10) happens to evaluate also to $y \equiv L$ if $y \equiv L$, our derivation of the formula is not legal in that case, so it is not legitimate to use (10).

However the original logistic DE $y' = r\left(1-\frac{y}{L}\right)y$ is fine if $y \equiv L$ and reduces in that case to $\frac{dy}{dt} = 0$, so $y = \text{a constant}$ (which must be *L* if $y(0) = L$).

(c) The solution formula (10) states

$$y(t) = \frac{L}{1+\left(\frac{L}{y_0}-1\right)e^{-rt}}.$$

If $0 < y_0 < L$, the denominator is greater than 1 and as *t* increases, $y(t)$ approaches *L* from below.

If $y_0 > L$, the denominator is less than 1 and as *t* increases, $y(t)$ approaches *L* from above.

If $y_0 = L$, $y(t) = L$. These implications of the formula are confirmed by the graph of Figure 2.5.8.

(d) By straightforward though lengthy computations, taking the second derivative of

$$y(t) = \frac{L}{1+\left(\frac{L}{y_0}-1\right)e^{-rt}} \qquad (10)$$

gives

$$y'' = \frac{L\left(\frac{L}{y_0}-1\right)r^2 e^{-rt}\left\{2\left(\frac{L}{y_0}-1\right)e^{-rt}-\left[1+\left(\frac{L}{y_0}-1\right)e^{-rt}\right]\right\}}{\left[1+\left(\frac{L}{y_0}-1\right)e^{-rt}\right]^3}.$$

Setting $y''=0$, we get

$$2\left(\frac{L}{y_0}-1\right)e^{-rt}-\left[1+\left(\frac{L}{y_0}-1\right)e^{-rt}\right]=0$$

or

$$\left(\frac{L}{y_0}-1\right)e^{-rt}=1.$$

Solving for *t*, we get $t^*=\frac{1}{r}\ln\left(\frac{L}{y_0}-1\right)$. Substituting this value into the analytical solution for the logistic equation, we get $y(t^*)=\frac{L}{2}$.

At t^* the rate y' is

$$r\left(1-\frac{\frac{L}{2}}{L}\right)\left(\frac{L}{2}\right)=\frac{r}{2}\left(\frac{L}{2}\right)=\frac{rL}{4}.$$

■ Fitting the Logistic Law

14. The logistic equation is

$$y'=ry\left(1-\frac{y}{L}\right).$$

If initially the population doubles every hour, we have

$$t_d=\frac{\ln 2}{r}=1$$

which gives the growth rate $r=\frac{1}{\ln 2}\approx 1.4$. We are also given $L=5\times 10^9$. The logistic curve after 4 hrs is calculated from the analytic solution formula,

$$y(t)=\frac{L}{1+\left(\frac{L}{y_0}-1\right)e^{-rt}}=\frac{5\times 10^9}{1+\left(\frac{5\times 10^9}{10^9}-1\right)e^{-1.4(4)}}=\frac{5\times 10^9}{1+4e^{-5.6}}\approx 4.9\times 10^9.$$

■ Culture Growth

15. Let y = population at time t, so $y(0) = 1000$ and $L = 100{,}000$.

The DE solution, from equation (10), is

$$y = \frac{100{,}000}{1 + \left(\frac{100{,}000}{1000} - 1\right)e^{-rt}}.$$

To evaluate r, substitute the given fact that when $t = 1$, population has doubled.

$$y(1) = 2(1000) = \frac{100{,}000}{1 + (100 - 1)e^{-r}}$$

$$2(1 + 99e^{-r}) = 100$$

$$198e^{-r} = 98$$

$$e^{-r} = \frac{98}{198}$$

$$-r = \ln\left(\frac{98}{198}\right)$$

$$r = .703$$

Thus $y(t) = \dfrac{100{,}000}{1 + 99e^{-.703t}}$.

(a) After 5 days: $y(5) = \dfrac{100{,}000}{1 + 99e^{-(.703)5}} = 25{,}348$ cells

(b) When $y = 50{,}000$, find t:

$$50{,}000 = \frac{100{,}000}{1 + 99e^{-.703t}}$$

$$1 + 99e^{-.703t} = 2$$

$$t \approx 6.536 \text{ days}$$

■ Logistic Model with Harvesting

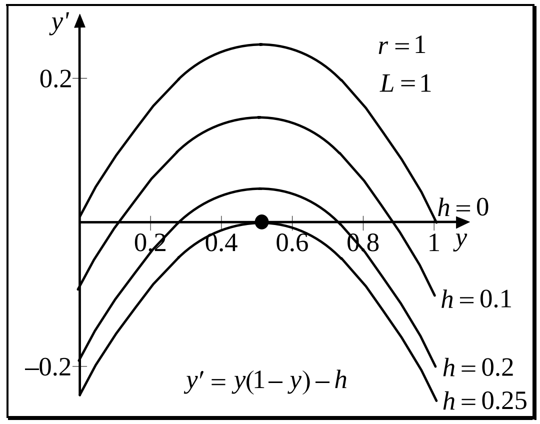
16. $y' = ry\left(1 - \frac{y}{L}\right) - h(t)$

(a) Graphs of y' versus y for different values of harvesting h are shown. Feasible harvests are those values of h that keep the slope y' positive for some

$$0 < y < L .$$

Because the curve y' versus y is always a maximum at

$$y = \frac{L}{2},$$



we find the value of h that gives $y'\left(\frac{L}{2}\right) = 0$; this will be the maximum sustainable harvesting value h_{max}. By setting $y'\left(\frac{L}{2}\right) = 0$, we find $h_{max} = \frac{rL}{4}$.

(b) As a preliminary to graphing, we find the equilibrium solutions under harvesting by setting $y' = 0$ in the equation

$$y' = r\left(1 - \frac{y}{L}\right)y - h$$

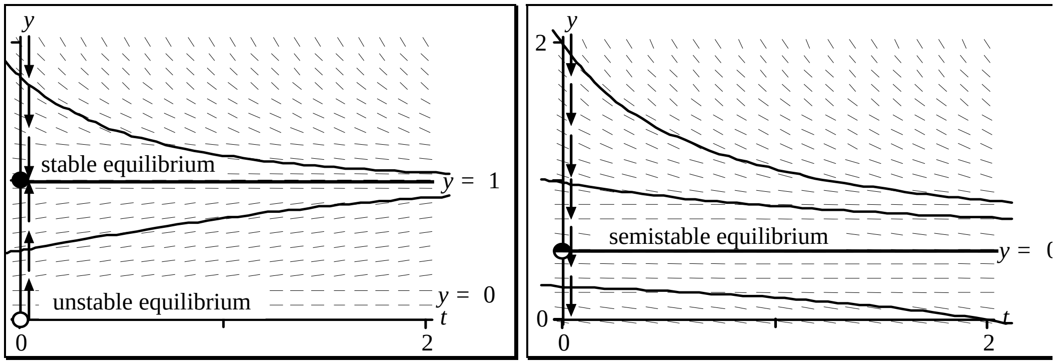
getting

$$y^2 - Ly + \left(\frac{h}{R}\right)L = 0 .$$

Solving for y, we get

$$y_1,\ y_2 = \frac{L \pm \sqrt{L^2 - \left(\frac{4hL}{r}\right)}}{2}$$

where both roots are positive. The smaller root represents the smaller equilibrium (constant) solution, which is unstable, and the larger root is the larger stable equilibrium solution. As we say in part (a), harvesting (measured in fish/day or some similar unit) must satisfy $h < \frac{rL}{4}$.



Straight logistic

$y' = y(1-y)$

Logistic with maximum sustainable harvesting

$y' = y(1-y) - 0.25$

Note that the equilibrium value with harvesting $h = 0.25$ is lower than the equilibrium value without harvesting. Note further that maximum harvesting has changed the phase line and the direction of solutions below equilibrium. The harvesting graph implies that fishing is fine when the population is above equilibrium, but wipes out the population when it is below equilibrium.

■ Campus Rumor

17. Let x = number in thousands of people who have heard the rumor.

$\frac{dx}{dt} = kx(80 - x)$ $\qquad x(0) = 1 \qquad x(1) = 10$

Rearranging the DE to standard logistic form (6) gives $\frac{dx}{dt} = 80k\left(1 - \frac{x}{80}\right)x$.

With $r = 80k$, the solution, by equation (10), is $x(t) = \dfrac{80}{1 + \left(\frac{80}{1} - 1\right)e^{-rt}}$.

To evaluate r, substitute the given fact that when $t = 1$, ten thousand people have heard the rumor.

$$10 = \frac{80}{1 + 79e^{-r}}$$

$$1 + 79e^{-r} = 8 \quad \Rightarrow \quad e^{-r} = \frac{7}{79} \quad \Rightarrow \quad r \approx 2.4235.$$

Thus $x(t) = \dfrac{80}{1 + 79e^{-2.4235t}}$.

■ **Water Rumor**

18. Let N be the number of people who have heard the rumor at time t

(a) $$\frac{dN}{dt} = kN(200,000 - N) = 200,000k\left(1 - \frac{N}{200,000}\right)N$$

(b) Yes, this is a logistic model.

(c) Set $\frac{dN}{dt} = 0.$ Equilibrium solutions: $N = 0$, $N = 200,000$.

(d) Let $r = 200,000k$. Assume $N(0) = 1$.

Then

$$N = \frac{200,000}{1 + \left(\frac{200,000}{1} - 1\right)e^{-rt}}$$

At $t = 1$ week,

$$1000 = \frac{200,000}{1 + 199,999e^{-r}}$$

$$1 + 199,999e^{-r} = 200$$

$$e^{-r} = \frac{199}{199,999} \quad \Rightarrow \quad r = 6.913.$$

Thus

$$N(t) = \frac{200,000}{1 + 199,999e^{-6.913t}}.$$

To find t when $N = 100,000$:

$$100,000 = \frac{200,000}{1 + 199,999e^{-6.913t}}$$

$$\Downarrow$$

$1 + 199,999e^{-6.913t} = 2$, $\quad e^{-6.913t} = \frac{1}{199,999}$, and $t = 1.77$ weeks $= 12.4$ days.

(e) We assume the same population. Let $t_N > 0$ be the time the article is published.

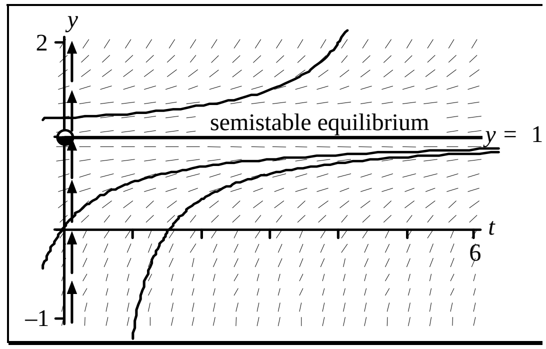
Let P = number of people who are aware of the counterrumor.

Let P_0 be the number of people who became aware of the counterrumor at time t_N.

$\frac{dP}{dt} = aP(200,000 - P)$ $\quad P(t_N) = P_0$, and a is a constant of proportionality.

■ Semistable Equilibrium

19. $y' = (1-y)^2$



We draw upward arrows on the y-axis for

$$y \neq 1$$

to indicate the solution is increasing. When

$$y = 1$$

we have a constant solution.

Because the slope lines have positive slope both above and below the constant solution $y(t) \equiv 1$, we say that the solution $y(t) \equiv 1$, or the point 1, is semistable (stable from below, unstable from above). In other words, if a solution is perturbed from the value of 1 to a value below 1, the solution will move back towards 1, but if the constant solution $y(t) \equiv 1$ is perturbed to a value larger than 1, it will not move back towards 1. Semistable equilibria are customarily noted with half-filled circles.

■ Gompertz Equation $\frac{dy}{dt} = y(1 - b\ln y)$

20. (a) Letting $z = \ln y$ and using the chain rule we get

$$\frac{dz}{dt} = \left(\frac{dz}{dy}\right)\left(\frac{dy}{dt}\right) = \left(\frac{1}{y}\right)\left(\frac{dy}{dt}\right).$$

Hence, the Gompertz equation becomes

$$\frac{dz}{dt} = a - bz .$$

(b) Solving this DE for z we find

$$z(t) = Ce^{-bt} + \frac{a}{b} .$$

Substituting back $z = \ln y$ gives

$$y(t) = e^{a/b} e^{Ce^{-bt}} .$$

Using the initial condition $y(0) = y_0$, we finally get $C = \ln y_0 - \frac{a}{b}$.

(c) From the solution in part (b), $\lim_{t\to\infty} y(t) = e^{a/b}$ when $b > 0$, $y(t) \to \infty$ when $b < 0$.

■ Fitting the Gompertz Law

21. (a) From Problem 20,

$$y(t) = e^{a/b} e^{ce^{-bt}}$$

where $c = \ln y_0 - \frac{a}{b}$. In this case $y(0) = 1$, $y(2) = 2$. We note $y(24) \approx y(28) \approx 10$ means the limiting value $e^{a/b}$ has been reached. Thus

$$e^{a/b} = 10\,,$$

so

$$\frac{a}{b} = \ln 10 \approx 2.3\,.$$

The constant $c = \ln 1 - \frac{a}{b} = 0 - 2.3 = -2.3$. Hence,

$$y(t) = 10e^{-2.3e^{-bt}}$$

and

$$y(2) = 10e^{-2.3e^{-2b}} = 2\,.$$

Solving for *b*:

$$\begin{aligned} -2.3e^{-2b} &= \ln\frac{2}{10} \approx -1.609 \\ e^{-2b} &\approx -\frac{1.609}{2.3} \approx 0.6998 \\ -2b &= \ln(0.6998) \approx -0.357 \\ b &\approx 0.1785 \end{aligned}$$

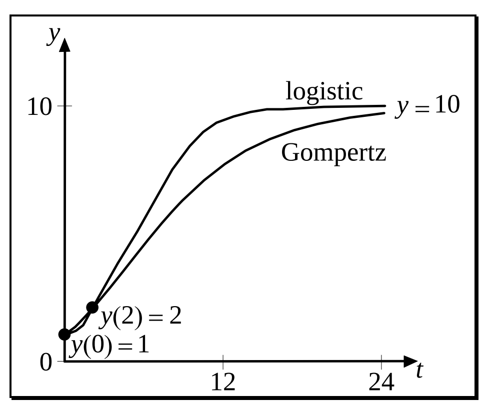
and $a = 2.3b$ gives $a \approx 0.4105$.

(b) The logistic equation

$$y' = ry\left(1 - \frac{y}{L}\right)$$

has solution

$$y(t) = \frac{L}{1 + \left(\frac{L}{y_0} - 1\right)e^{-rt}}\,.$$



We have $L = 10$ and $y_0 = 1$, so

$$y(t) = \frac{10}{1 + 9e^{-rt}}$$

and

$$y(2) = \frac{10}{1 + 9e^{-2r}} = 2\,.$$

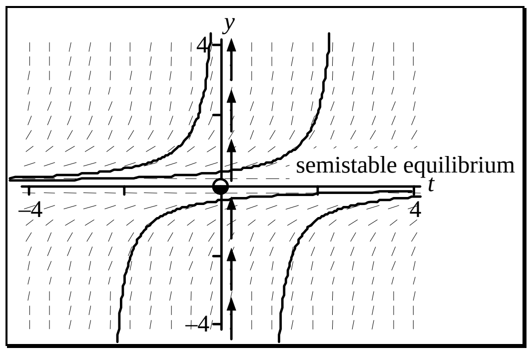
Solving for r

$$\begin{aligned} 9e^{-2r} &= \frac{10}{2} - 1 = 4 \\ -2r &= \ln\frac{4}{9} \approx -0.8109 \\ r &= \frac{-0.8109}{-2} \approx 0.405. \end{aligned}$$

■ Autonomous Analysis

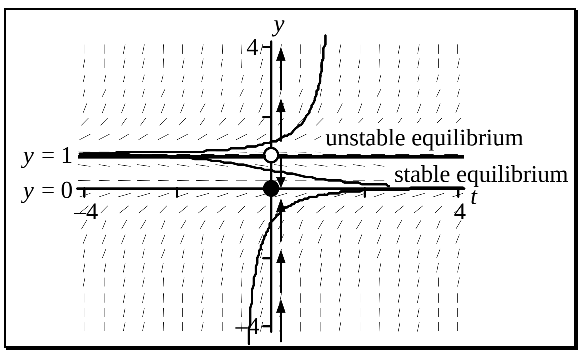
22. (a) $y' = y^2$

(b) One semistable equilibrium at $y(t) \equiv 0$ is stable from below, unstable from above.



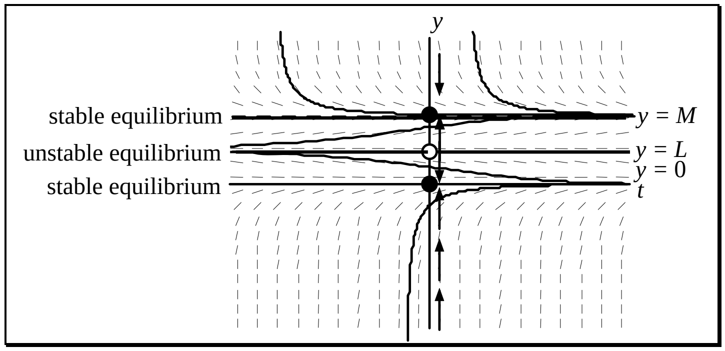
23. (a) $y' = -y(1 - y)$

(b) The equilibrium solutions are $y(t) \equiv 0$, $y(t) \equiv 1$. The solution $y(t) \equiv 0$ is stable. The solution $y(t) \equiv 1$ is unstable.



24. (a) $y' = -y\left(1-\frac{y}{L}\right)\left(1-\frac{y}{M}\right),\ y' = -y(1-y)(1-0.5y)$

(b) The equilibrium points are $y=0$, L, M. $y=0$ is stable. $y=M$ is stable if $M>L$ and unstable if $M<L$. $y=L$ is stable if $M<L$ and unstable if $M>L$.

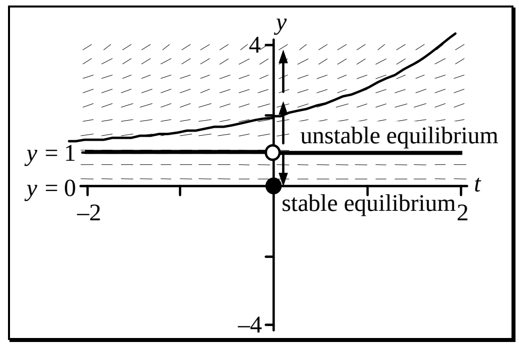


25. (a) $y' = y - \sqrt{y}$

Note that the DE is only defined for

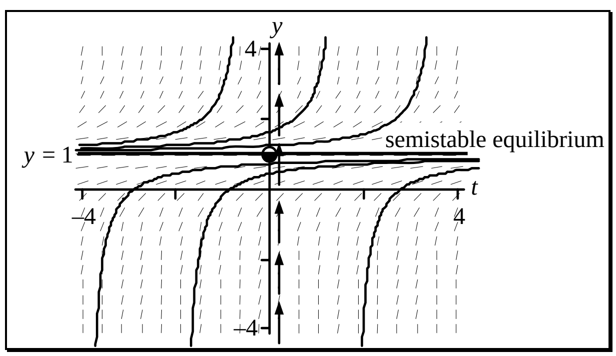
$y \geq 0$.

(b) The constant solution $y(t) \equiv 0$ is stable, the solution $y(t) \equiv 1$ is unstable.



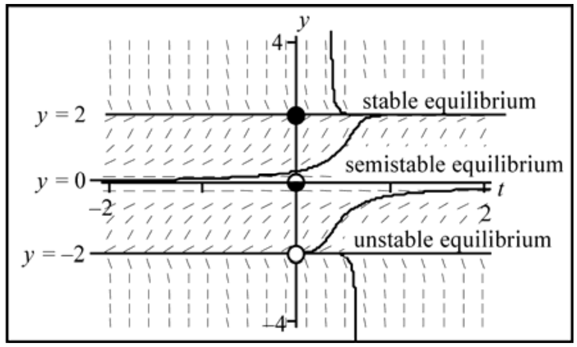
26. (a) $y' = k(1-y)^2,\ k>0$

(b) The constant solution $y(t) \equiv 1$ is semi-stable (unstable above, stable below).



27. (a) $y' = y^2\left(4-y^2\right)$

(b) The equilibrium solution $y(t) \equiv 2$ is stable, the solution $y(t) \equiv -2$ is unstable and the solution $y(t) \equiv 0$ is semistable.



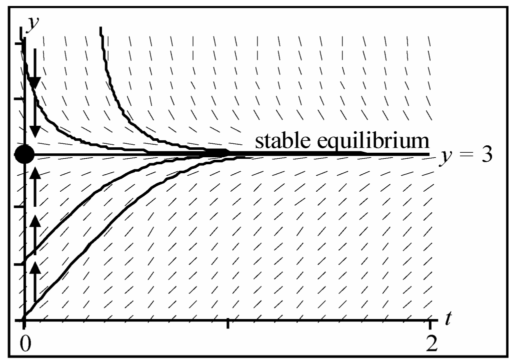
■ **Stefan's Law Again**

28. $T' = k\left(M^4 - T^4\right)$

The equation tells us that when

$$0 < T < M,$$

the solution $T = T(t)$ increases because $T' > 0$, and when $M < T$ the solution decreases because $T' < 0$. Hence, the equilibrium point $T(t) \equiv M$ is stable. We have drawn the directional field of Stefan's equation for $M = 3$, $k = 0.05$.



$$\frac{dT}{dt} = 0.05\left(3^4 - T^4\right)$$

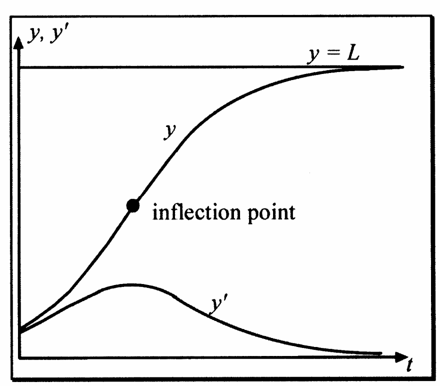
To $> M$ gives solutions falling to M.

To $< M$ gives solutions rising to M.

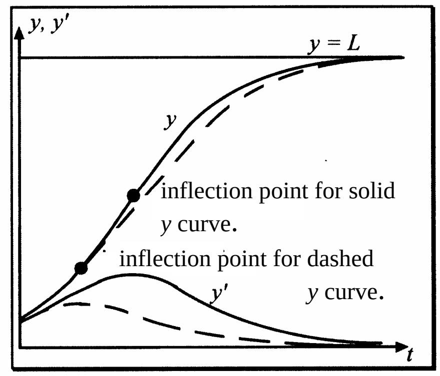
These observations actions match intuition and experiment.

■ **Hubbert Peak**

29. (a) From even a hand-sketched logistic curve you can graph its slope y' and find a roughly bell-shaped curve for $y'(t)$. Depending on the scales used, it may be steeper or flatter than the bell curve shown in Fig. 1.3.5.



(b) For a pure logistic curve, the inflection point always occurs when $y = \frac{L}{2}$. However, if we consider models different from the logistic model that still show similar solutions between 0 and the nonzero equilibrium, it is possible for the inflection point to be closer to 0. When this happens oil recovery reaches the maximum production rate much earlier.



Of course the logistic model is a crude model of oil production. For example it doesn't take into consideration the fact that when oil prices are high, many oil wells are placed into production.

If the inflection point is lower than halfway on an approximately logistic curve, the peak on the y' curve occurs sooner and lower creating an asymmetric curve for y'.

(c) These differences may or may not be significant to people studying oil production; it depends on what they are looking for. The long-term behavior, however, is the same; the peak just occurs sooner. After the peak occurs, if the model holds, it is downhill insofar as oil production is concerned. Typical skew of peak position is presented on the figures above.

■ **Useful Transformation**

30. $y' = ky(1-y)$

Letting $z = \frac{y}{1-y}$ yields

$$\frac{dz}{dt} = \left(\frac{dz}{dy}\right)\left(\frac{dy}{dt}\right) = \frac{1}{(1-y)^2}\left(\frac{dy}{dt}\right).$$

Substituting for $\frac{dy}{dt}$ from the original DE yields a new equation

$$(1-y)^2 \frac{dz}{dt} = ky(1-y),$$

which gives the result

$$\frac{dz}{dt} = \frac{ky}{1-y} = kz.$$

Solving this first-order equation for $z = z(t)$, yields $z(t) = ce^{kt}$ and substituting this in the transformation $z = \dfrac{y}{1-y}$, we get $\dfrac{y}{1-y} = ce^{kt}$.

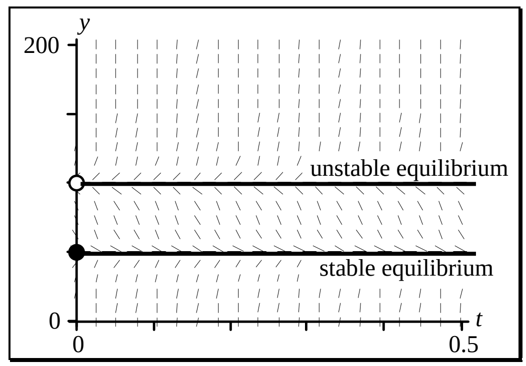
Finally, solving this for y gives $y(t) = \dfrac{1}{1 + \frac{1}{c}e^{-kt}} = \dfrac{1}{1 + c_1 e^{-kt}}$,

where $c_1 = \dfrac{1}{y_0} - 1$.

■ Chemical Reactions

31. $x = k(100 - x)(50 - x)$

The solutions for the given initial conditions are shown on the graph. Note that all behaviors are at equilibrium or flown off scale before $t = 0.1$!



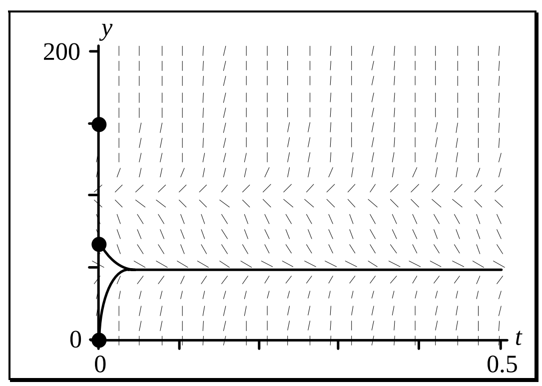
Direction field and equilbrium

The solution curve for $x(0) = 150$ is almost vertical.

(a) A solution starting at $x(0) = 0$ increases and approaches 50.

(b) A solution starting at $x(0) = 75$ decreases and approaches 50.

(c) A solution starting at $x(0) = 150$ increases without bound.



Solutions for three given initial conditions.

Noting the location of equilibrium and the direction field as shown in a second graph leads to the following conclusions: Any $x(0) > 100$ causes $x(t)$ to increase without bound and fly off scale very quickly. On the other hand, for any $x(0) \in (0,100)$ the solution will approach an equilibrium value of 50, which implies the tiniest amount is sufficient to start the reaction.

If you are looking for a different scenario, you might consider some other modeling options that appear in Problem 32.

■ **General Chemical Reaction of Two Substances** $\dfrac{dx}{dt} = k(a-x)^m (b-x)^n, a < b$

32. (a), (b) We consider the four cases when the exponents are *even* positive integers and/or *odd* positive integers. In each case, we analyze the sign of the derivative for different values of *x*. For convenience we pick $a = 1$, $b = 2$, $k = 1$.

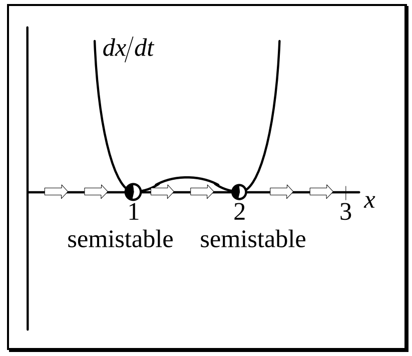
- $$\frac{dx}{dt} = (1-x)^{\text{even}} (2-x)^{\text{even}}.$$

We have drawn a graph of $\dfrac{dx}{dt}$ versus *x*. By drawing arrows to the right when

$\dfrac{dx}{dt}$ is positive

and arrows to the left when

$\dfrac{dx}{dt}$ is negative,

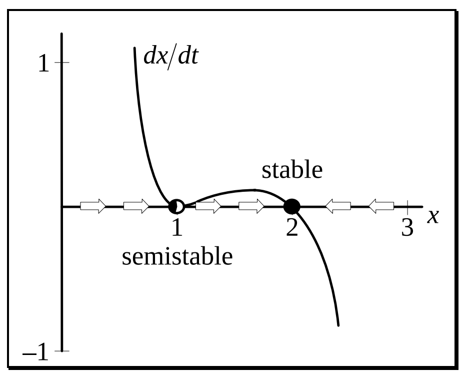


Both even exponents

we have a horizontal phase line for $x(t)$. We also see that both equilibrium solutions $x(t) \equiv 1$, $x(t) \equiv 2$ are unstable; although both are semistable; stable from below and unstable from above.

- $$\frac{dx}{dt} = (1-x)^{\text{even}} (2-x)^{\text{odd}}.$$

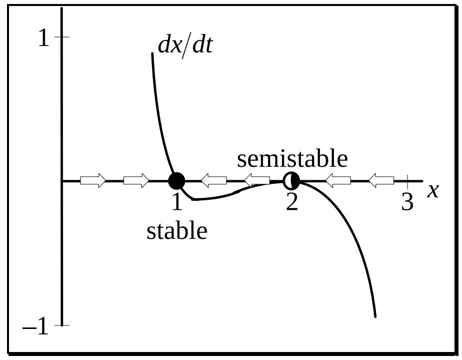
Here $x(t) \equiv 1$ is unstable although it is stable from below. The solution $x(t) \equiv 2$ is stable.



Even and odd exponents

- $$\frac{dx}{dt} = (1-x)^{\text{odd}}(2-x)^{\text{even}}.$$

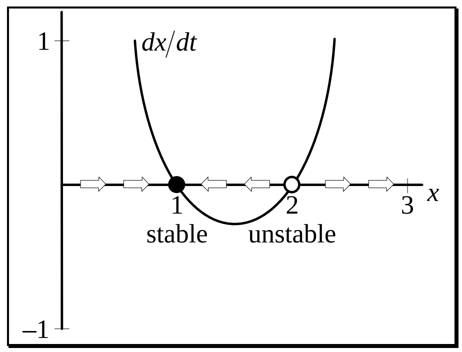
Here $x(t) \equiv 2$ is semistable, stable from above and unstable from below. The solution $x(t) \equiv 1$ is stable.



Odd and even exponents

- $$\frac{dx}{dt} = (1-x)^{\text{odd}}(2-x)^{\text{odd}}.$$

Here the smaller of the two solutions, $x(t) \equiv 1$, is stable; the larger solution, $x(t) \equiv 2$, is unstable.



Both odd exponents

■ Solving the Threshold Equation

33. $y' = -ry\left(1 - \frac{y}{T}\right)$

Introducing backwards time $\tau = -t$, yields

$$\frac{dy}{dt} = \frac{dy}{d\tau}\frac{d\tau}{dt} = -\frac{dy}{d\tau}.$$

Hence, if we run the threshold equation

$$\frac{dy}{dt} = -ry\left(1 - \frac{y}{T}\right)$$

backwards, we get

$$-\frac{dy}{d\tau} = -ry\left(1 - \frac{y}{T}\right).$$

Equivalently it also yields the first-order equation

$$\frac{dy}{d\tau} = ry\left(1 - \frac{y}{T}\right),$$

which is the logistic equation with $L = T$ and $t = \tau$. We know the solution of this logistic equation to be

$$y(\tau) = \frac{T}{1 + \left(\frac{T}{y_0} - 1\right)e^{-r\tau}}.$$

We can now find the solution of the threshold equation by replacing τ by $-t$, yielding

$$y(t) = \frac{T}{1 + \left(\frac{T}{y_0} - 1\right)e^{rt}}.$$

■ Limiting Values for the Threshold Equation

34. $y' = -ry\left(1 - \frac{y}{T}\right)$

(a) For $0 < y_0 < T$ as $t \to \infty$ the denominator of

$$y(t) = \frac{T}{1 + \left(\frac{T}{y_0} - 1\right)e^{rt}}$$

goes to plus infinity and so $y(t)$ goes to zero.

(b) For $y_0 > T$ the denominator of

$$y(t) = \frac{T}{1 + \left(\frac{T}{y_0} - 1\right)e^{rt}}$$

will reach zero (causing the solution to "blow up") when

$$1 + \left(\frac{T}{y_0} - 1\right)e^{rt} = 0.$$

Solving for *t* gives the location of a vertical asymptote on the *ty* graph

$$t^* = \frac{1}{r}\ln\left(\frac{y_0}{y_0 - T}\right).$$

■ Pitchfork Bifurcation

35. $y' = \alpha y - y^3 = y\left(\alpha - y^2\right)$

(a) For $\alpha \le 0$ the only real root of $y\left(\alpha - y^2\right) = 0$ is $y = 0$. Because

$$y' = y\left(\alpha - y^2\right) < 0 \text{ for } y > 0$$

and

$$y' = y\left(\alpha - y^2\right) > 0 \text{ for } y < 0,$$

the equilibrium solution $y = 0$ is stable.

(b) When $\alpha > 0$ the equation

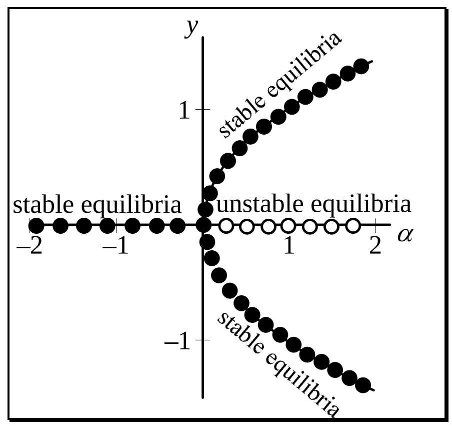
$$y' = y\left(\alpha - y^2\right) = 0$$

has roots

$$y = 0,\ \pm\sqrt{\alpha}\,.$$

The points $y = \pm\sqrt{\alpha}$ are stable, but $y = 0$ is unstable as illustrated by graphing a phase line or direction field of $y' = -y\left(1 - y^2\right)$.

(c)



Pitchfork bifurcation at $(0,\ 0)$

■ Another Bifurcation

36. $y' = y^2 + by + 1$

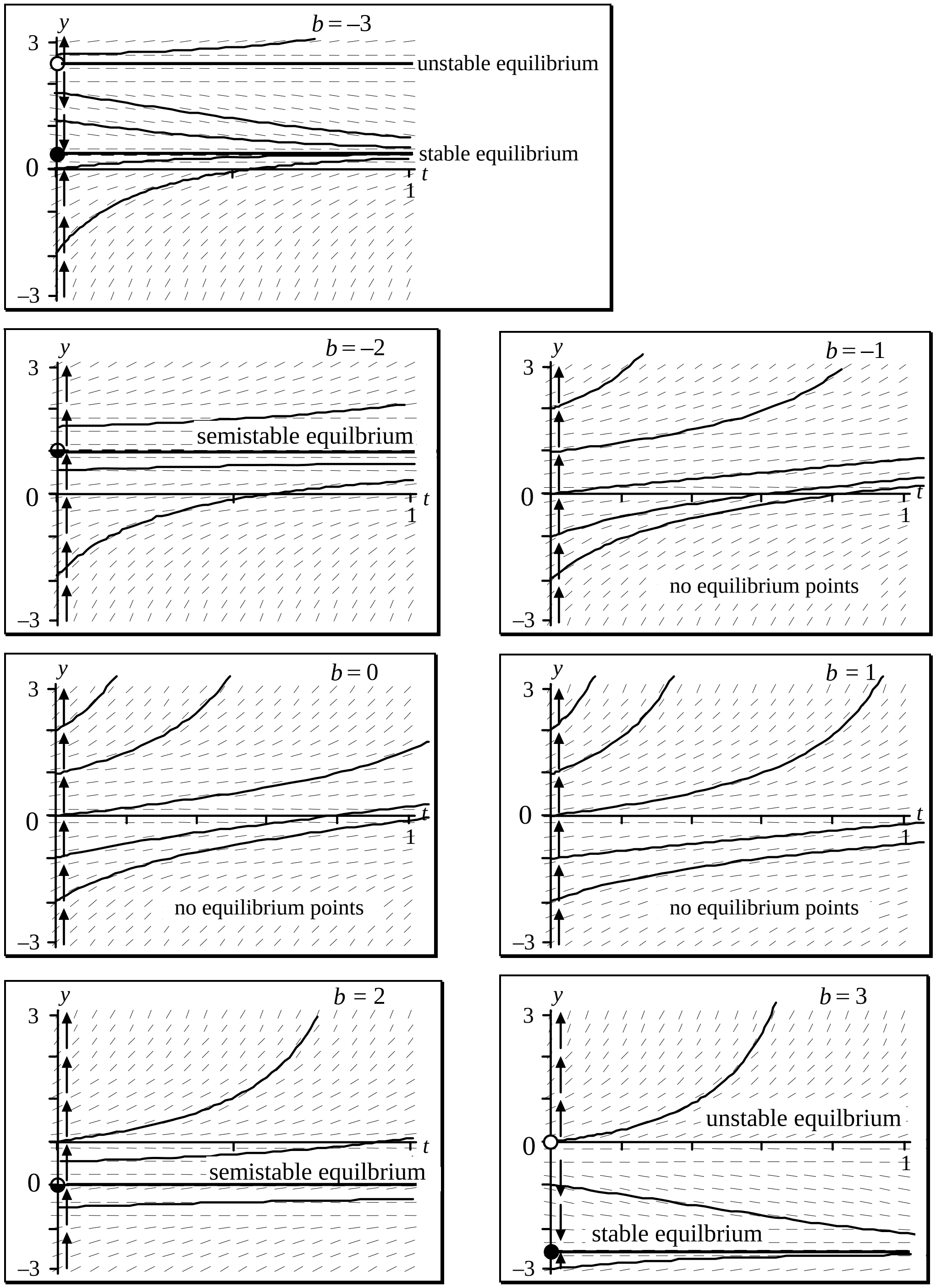
(a) We find the equilibrium points of the equation by setting $y' = 0$ and solving for *y*. Doing this we get

$$y = \frac{-b \pm \sqrt{b^2 - 4}}{2}.$$

We see that for $-2 < b < 2$ there are no (real) solutions, and thus no equilibrium solutions. For $b = -2$ we have the equilibrium solution +1, and for $b = +2$ we have equilibrium solution –1. For each $|b| \ge 2$ we have two equilibrium solutions.

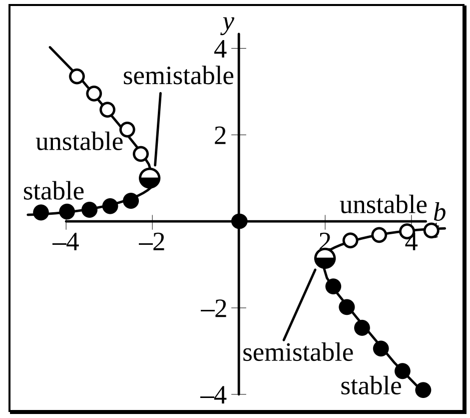
(b) The bifurcation points are at $b = -2$ and $b = +2$. As *b* passes through –2 (increasing), the number of equilibrium solutions changes from 2 to 1 to 0, and when *b* passes through +2, the number of equilibrium solutions changes from 0 to 1 to 2.

(c) We have drawn some solution for each of the values $b = -3$, –2, –1, 0, 1, 2, and 3.



(d) For $b = 2$ and $b = -2$ the single equilibrium is semistable. (Solutions above are repelled; those below are attracted.) For $|b| > 2$ there are two equilibria; the larger one is unstable and the smaller one is stable. For $|b| < 2$ there are *no* equilibria.

(e) The bifurcation diagram shows the location of equilibrium points for y versus the parameter value b. Solid circles represent stable equilibria; open circles represent unstable equilibria.



Equilibria of $y' = y^2 + by + 1$ versus b

■ Computer Lab: Bifurcation

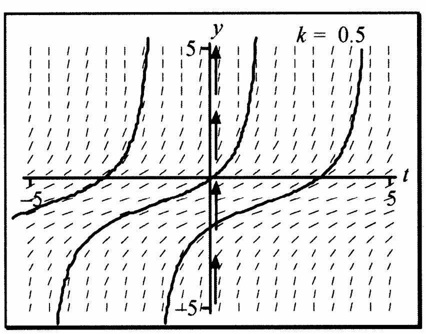
37. $y' = ky^2 + y + 1$

(a) Setting $y' = 0$ yields *two* equilibria,

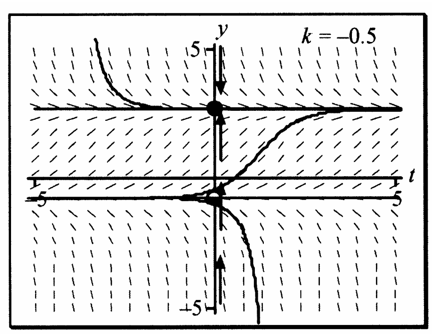
$$y_e = \frac{-1 \pm \sqrt{1-4k}}{2k}$$

for $k < \frac{1}{4}$; *none* for $k > \frac{1}{4}$; *one* for $k = \frac{1}{4}$.

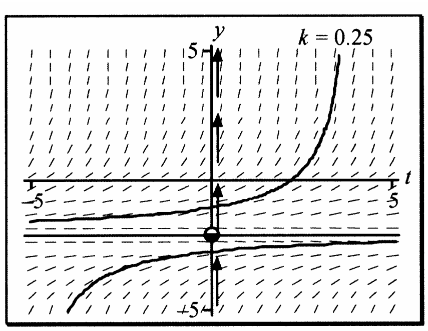
(b) The following phase-plane graphs illustrate the bifurcation.



No equilibrium point



Two equilibrium points



One semistable equilibrium point

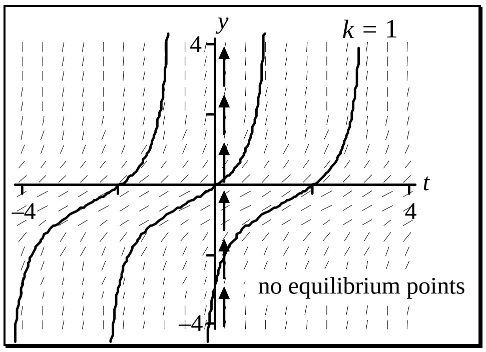
38. $y' = y^2 + y + k$

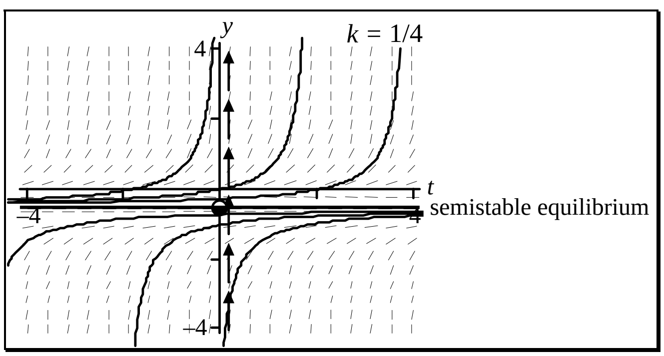
(a) Setting $y' = 0$ yields *two* equilibria,

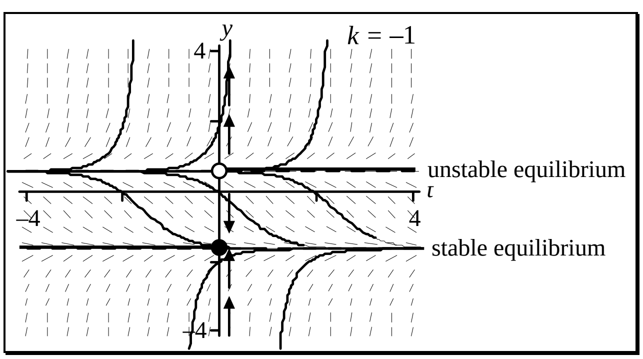
$$y_e = \frac{-1 \pm \sqrt{1-4k}}{2},$$

for $k < \frac{1}{4}$; *none* for $k > \frac{1}{4}$; *one* for $k = \frac{1}{4}$.

(b) The following phase-plane graphs illustrate the bifurcation.



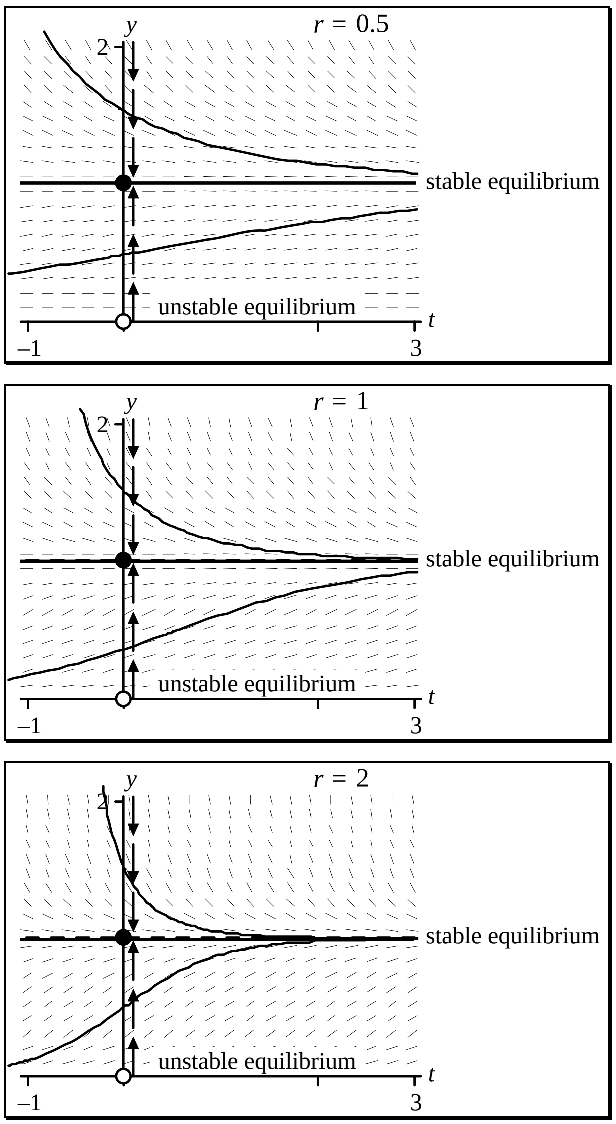




■ Computer Lab: Growth Equations

39. $y' = ry\left(1 - \dfrac{y}{L}\right)$

We graph the direction field of this equation for $L = 1$, $r = 0.5$, 1, 2, and 5. We keep L fixed because all it does is raise or lower the steady-state solution to $y = L$. We see that the larger the parameter r, the faster the solution approaches the steady-state L.



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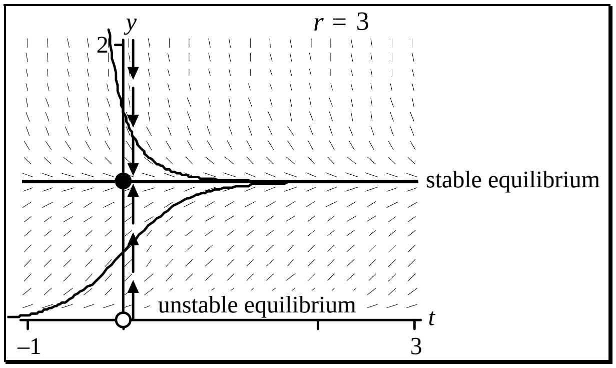


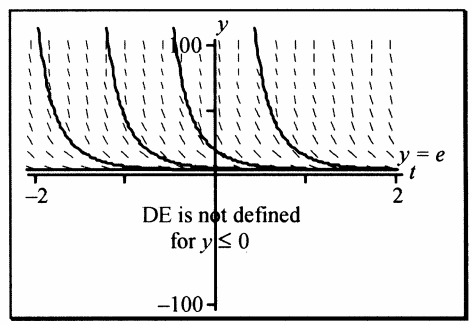
40. $y' = -r\left(1 - \dfrac{y}{T}\right)y$

See text Figure 2.5.9.

The parameter r governs the steepness of the solution curves; the higher r the more steeply y leaves the threshold level T.

41. $y' = r\left(1 - \dfrac{\ln y}{L}\right)y$

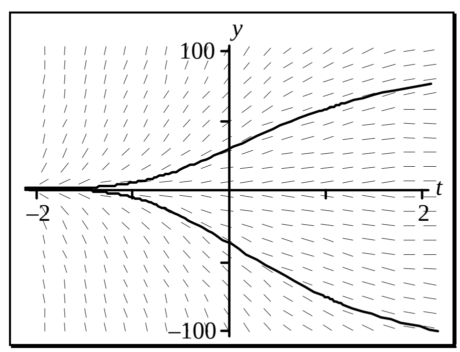
Equilibrium at $y = e^L$; higher r values give steeper slopes.



$\ln y$ requires $y > 0$

42. $y' = re^{-\beta t}y$

For larger β or for larger r, solution curves fall more steeply. Unstable equilibrium $r = 1$, $\beta = 1$



■ **Suggested Journal Entry**

43. Student Project

2.6 Systems of DEs: A First Look [redacted]

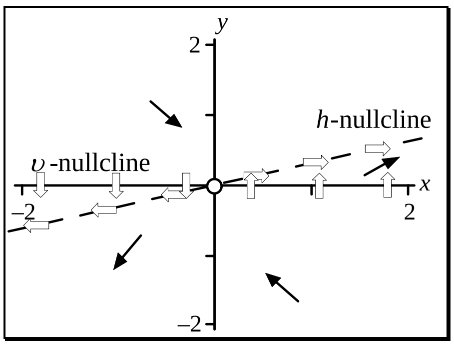
■ Predicting System Behavior

1. (a) $x' = y$

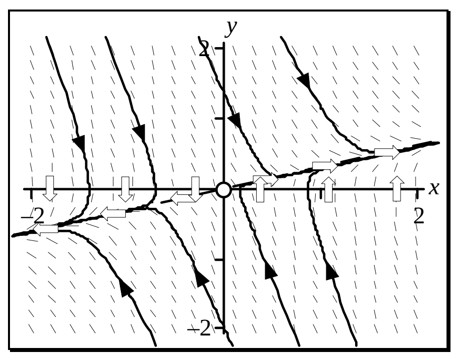
$y' = x - 3y$

This (linear) system has one equilibrium point at the origin, $(x, y) = (0, 0)$, as do all linear systems. The υ- and h-nullclines are respectively, as shown in part (b).

(b)



(c) A few solutions along with the vertical and horizontal nullclines are drawn.



(d) The equilibrium point (0, 0) is unstable.

All solutions tend quickly to $y = \dfrac{x}{3}$ then move gradually towards $+\infty$ or $-\infty$ asymptotically along that line. Whether the motion is left or right depends on the initial conditions.

2. (a) $x' = 1 - x - y$

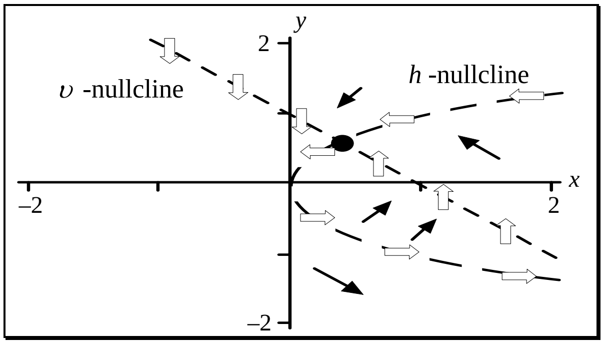
$y' = x - y^2$

Setting $x' = 0$ and $y' = 0$ gives

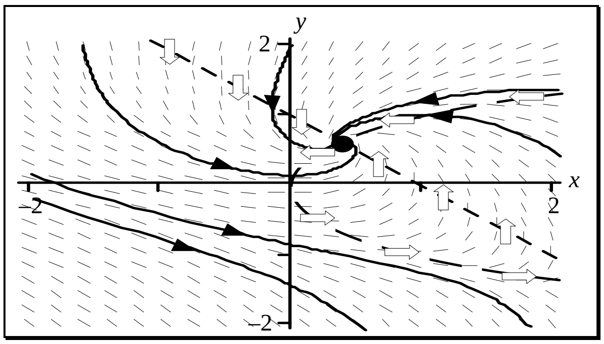
$$\upsilon\text{-nullcline} \quad 1 - x - y = 0$$
$$h\text{-nullcline} \quad x - y^2 = 0.$$

From the intersection of the two nullclines we find two equilibrium points shown in the following figures. We can locate them graphically far more easily than algebraically!

(b)



(c)



(d) The lower equilibrium point at

$$\left[\frac{1}{4}(1+\sqrt{5})^2,\ \frac{1}{2}(-1-\sqrt{5})\right]$$

is unstable and the upper equilibrium at

$$\left[\frac{1}{4}(1-\sqrt{5})^2,\ \frac{1}{2}(\sqrt{5}-1)\right]$$

is stable.

Most trajectories spiral counterclockwise toward the first quadrant equilibrium point. However, if the initial condition is somewhat left or below the 4th quadrant equilibrium, they shoot down towards $-\infty$. We suspect a dividing line between these behaviors, and we will find it in Chapter 6.

3. (a) $x' = 1 - x - y$

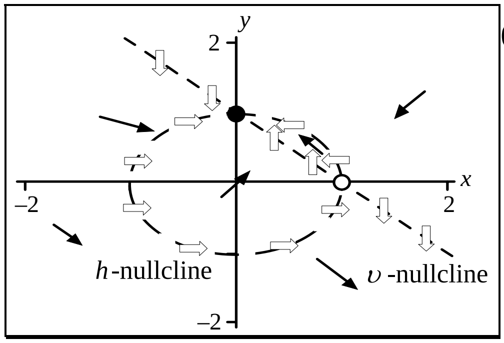
$y' = 1 - x^2 - y^2$

Setting $x' = 0$ and $y' = 0$ gives

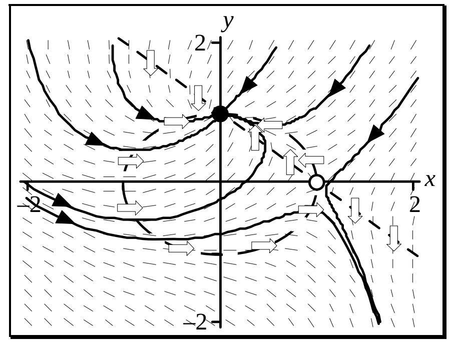
$$h\text{-nullcline} \quad x^2 + y^2 = 1$$
$$\upsilon\text{-nullcline} \quad x + y = 1.$$

From the intersection of the two nullclines we find two equilibrium points (0, 1), (1, 0).

(b)



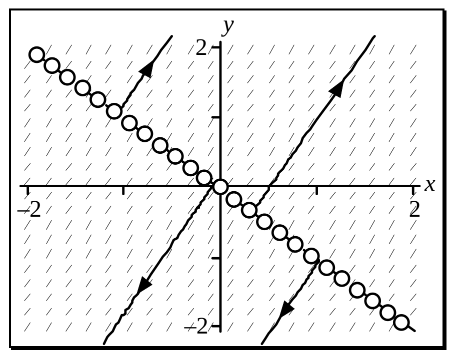
(c)



(d) The equilibrium at (1, 0) is unstable; the equilibrium at (0, 1) is stable. Most trajectories seem to be attracted to the stable equilibrium, but those that approach the lower unstable equilibrium from below or form the right will turn down toward the lower right.

4. $x' = x + y$

$y' = 2x + 2y$



(a) This (singular) system has an entire line of equilibrium points on the line

$$x + y = 0.$$

(b) The direction field and the line of unstable equilibrium points are shown at the right.

(c) We superimpose on the direction field a few solutions.

(d) From part (c) we see the equilibrium points on the line $x + y = 0$ are all unstable.

All nonequilibrium trajectories shoot away from the equilibria along straight lines (of slope 2), towards $+\infty$ if the IC is above the line $x + y = 0$ and toward $-\infty$ if the IC is below $x + y = 0$.

5. (a) $x' = 4 - x - y$

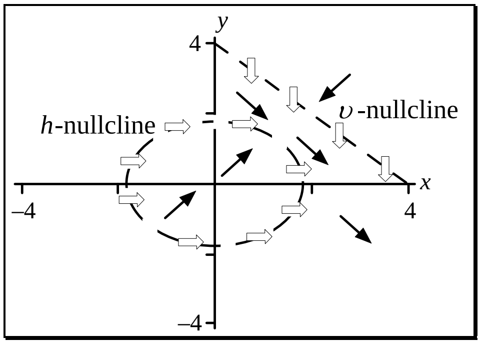
$y' = 3 - x^2 - y^2$

Setting $x' = 0$ and $y' = 0$ gives the intersection of the nullclines:

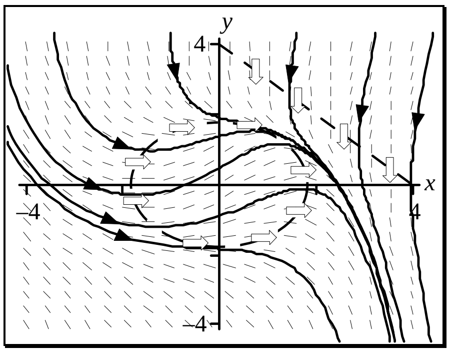
$$\begin{array}{ll} h\text{-nullcline} & x^2 + y^2 = 3 \\ \upsilon\text{-nullcline} & y = 4 - x. \end{array}$$

We find no equilibria because the nullclines do not intersect.

(b)



(c)



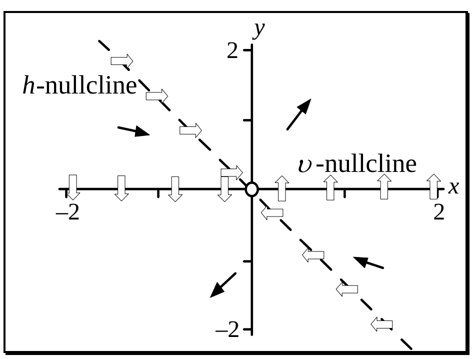
(d) There are no equilibria—all solutions head down to lower right.

6. (a)

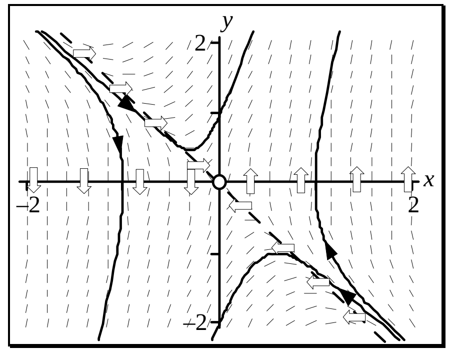
$$x' = y$$
$$y' = 5x + 3y$$

This (linear) system has one equilibrium point at $(x,\ y) = (0,\ 0)$ as do all linear systems. The 64-dollar question is: Is it stable? The υ- and h-nullclines: $y = 0$, $5x + 3y = 0$, are shown following and indicate that the origin (0, 0) is unstable. Hence, points starting near the origin will leave the origin. We will see later other ways for showing that (0, 0) is unstable.

(b)



(c) The direction field and a few solutions are drawn. Note how the solutions cross the vertical and horizontal nullclines



(d) We see from the preceding figure that solutions come from infinity along a line (that is, not a nulllcline), and then if they are not *exactly* on the line head off either upwards and to the left or downwards and go to the left on another line. Whether they go up or down depends on whether they initially start above or below the line. It appears that points that start exactly on the line will go to (0, 0). We will see later in Chapter 6 when we study linear systems using eigenvalues and eigenvectors that the solutions come from infinity on one eigenvector and go to infinity on another eigenvector.

7. (a) $x' = 1 - x - y$

$y' = x - |y|$

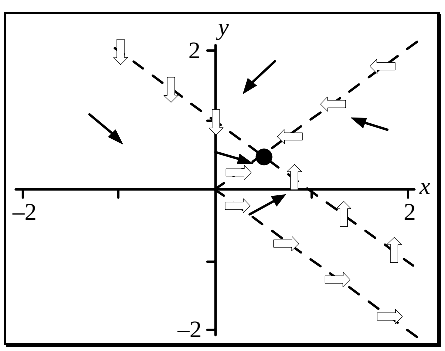
Setting $x' = y' = 0$ and finding the intersection of the nullclines:

h-nullcline $|y| = x$

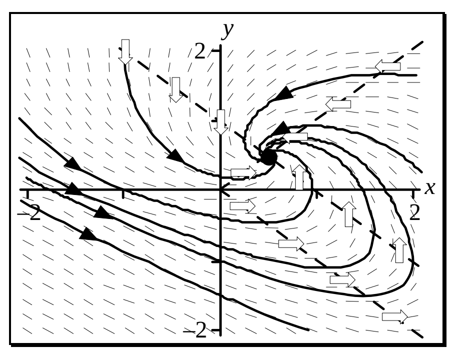
υ-nullcline $y = 1 - x$

we find one equilibrium point $\left(\frac{1}{2}, \frac{1}{2}\right)$. The arrows indicate that it is a stable equilibrium.

(b)



(c)

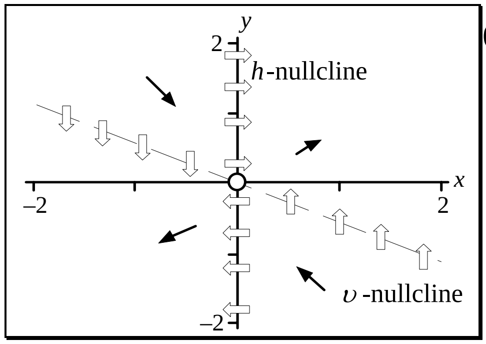


(d) The equilibrium is stable; all other solutions spiral into it.

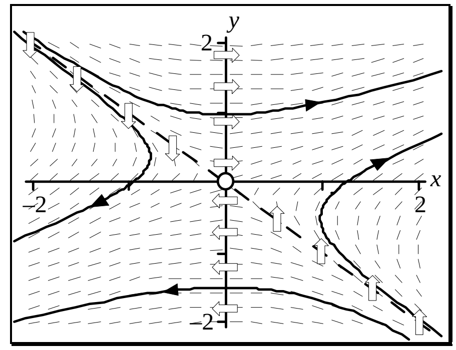
8. (a) $x' = x + 2y$
$y' = x$

This (linear) system has one equilibrium point at the origin (0, 0), as do all linear systems. The υ- and h-nullclines: $x + 2y = 0$, $x = 0$, are shown in part (b) and indicate that the origin (0, 0) is unstable. We will see later other ways to show that the system is unstable.

(b)



(c)



(d) The equilibrium point (0, 0) is unstable. Other solutions come from upper left or the lower right, heading toward origin but veers off towards $\pm\infty$ in the upper right or lower left.

■ **Creating a Predator-Prey Model**

9. (a) $\dfrac{dR}{dt} = 0.15R - 0.00015RF$

$\dfrac{dF}{dt} = -0.25F + 0.00003125RF$

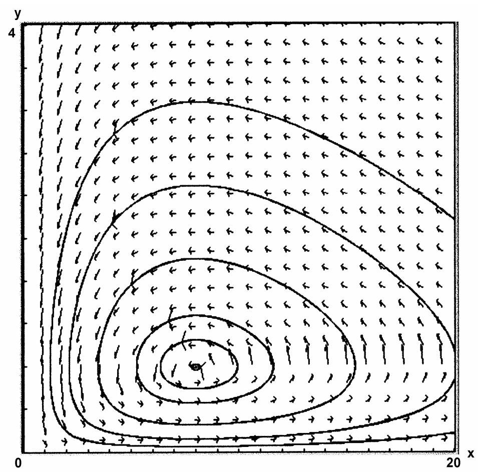
The rabbits reproduce at a natural rate of 15%; their population is diminished by meetings with foxes. The fox population is diminishing at a rate of 25%; this decline is mitigated only slightly by meeting rabbits as prey. Comparing the predator-prey rates in the two populations shows a much larger effect on the rabbit population, which is consistent with the fact that each fox needs several rabbits to survive.

(b) $\dfrac{dR}{dt} = 0.15R - 0.00015RF - 0.1R = 0.05R - 0.00015RF$

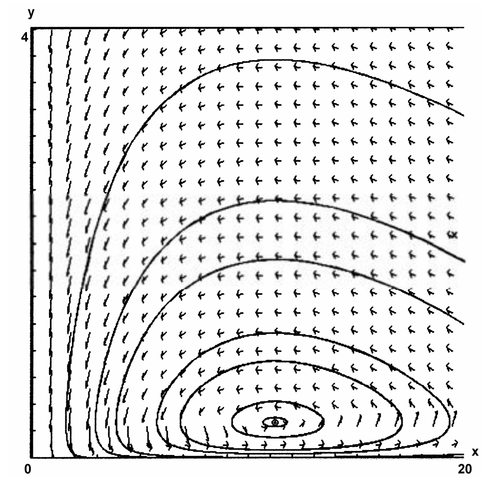
$\dfrac{dF}{dt} = -0.25F + 0.00003125RF - 0.1F = -0.35F + 0.00003125RF$

Both populations are diminished by the harvesting. The equilibrium populations move from (8000, 1000) in Part (a) to (11200, 333) in Part (b), i.e., more rabbits and fewer foxes if both populations are harvested at the same rate. Figures on the next page.

In figures, x and y are measured in thousands. Note that the vertical axes have different scales from the horizontal axes.



9(a) Equilbrium at (8, 1)



9(b) Equilbrium at (11.2, 0.3)

■ Sharks and Sardines with Fishing

10. (a) With fishing the equilibrium point of the system

$$x' = x(a - by - f)$$
$$y' = y(-c + dx - f)$$

is

$$\tilde{x}_e = \frac{c+f}{d} = \frac{c}{d} + \frac{f}{d}$$
$$\tilde{y}_e = \frac{a-f}{b} = \frac{a}{b} - \frac{f}{b}.$$

With fishing we increase the equilibrium of the prey x_e by $\frac{f}{d}$ and decrease the equilibrium of the predator y_e by $\frac{f}{b}$.

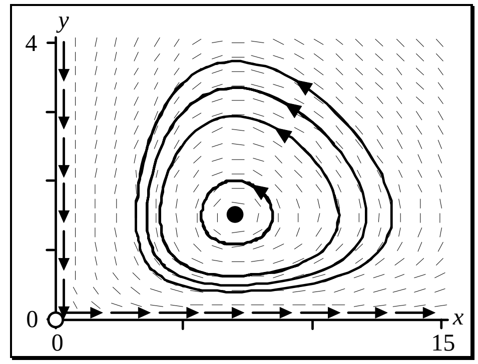
Using the parameters from Example 3 we set

$$a = 2,\ b = 1,\ c = 3,\ d = 0.5;$$

the new equilibrium point of the fished model is

$$\tilde{x}_e = \frac{c+f}{d} = \frac{c}{d} + \frac{f}{d} = 6 + 2f$$

$$\tilde{y}_e = \frac{a-f}{b} = \frac{a}{b} - \frac{f}{b} = 2 - f.$$

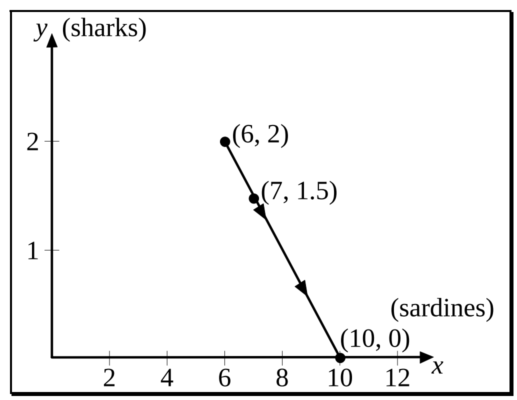


Shark (y) and sardine (x) trajectories

The trajectories are closed curves representing periodic motion of both sharks and sardines. The trajectories look like the trajectories of the unfished case in Example 3 except the equilibrium point has moved to the right (more prey) and down (fewer predators).

(b) With the parameters in part (a) and $f = 0.5$ the equilibrium point is (7, 1.5). This compares with the equilibrium point (6, 2) in the unfished case.

As the fishing rate f increases from 0 to 2, the equilibrium point moves along the line from the unfished equilibrium at (6, 2) to (10, 0). Hence, the fishing of each population at the same rate benefits the sardines (x) and drives the sharks (y) to extinction. This is illustrated in the figure.



(c) You should fish for sardines when the sardine population is increasing and sharks when the shark population is increasing. In both cases, more fishing tends to move the populations closer to equilibrium while maintaining higher populations in the low parts of the cycle.

(d) If we look at the insecticide model and assume both the good guys (predators) and bad guys (prey) are harvested at the same rate, the good guys will also be diminished and the bad guys peak again. As $f \to 1$ (try $f = 0.8$) the predators get decimated first, then the prey can peak again. If you look at part (a), you see that the predator/prey model does not allow either population to go below zero, as the x- and y-axes are solutions and the solutions move along the axes, thus it is impossible for other solutions to cross either of these axes. You might continue this exploration with the IDE tool, Lotka-Volterra with Harvest, as in Problem 24.

■ Analyzing Competition Models

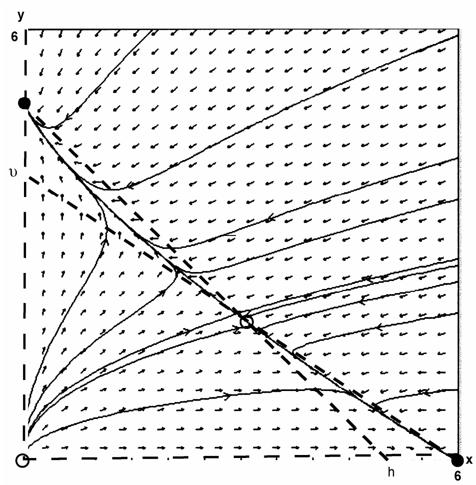
11. $\dfrac{dR}{dt} = R(1200 - 2R - 3S)$, $\dfrac{dS}{dt} = S(500 - R - S)$

Rabbits are reproducing at the astonishing rate of 1200 per rabbit per unit time, in the absence of competition. However, crowding of rabbits decreases the population at a rate double the population. Furthermore, competition by sheep for the same resources diminishes the rabbit population by three times the number of sheep!

Sheep on the other hand reproduce at a far slower (but still astonishing) rate of 500 per sheep per unit time. Competition among themselves and with rabbits diminishes merely one to one with the number of rabbits and sheep.

Equilibria occur at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 500 \end{bmatrix}, \begin{bmatrix} 600 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 300 \\ 200 \end{bmatrix}$.

The equilibria on the axes that are not the origin are the points toward which the populations head. Which species dies out depends on where they start. See Figure, where x and y are measured in hundreds.



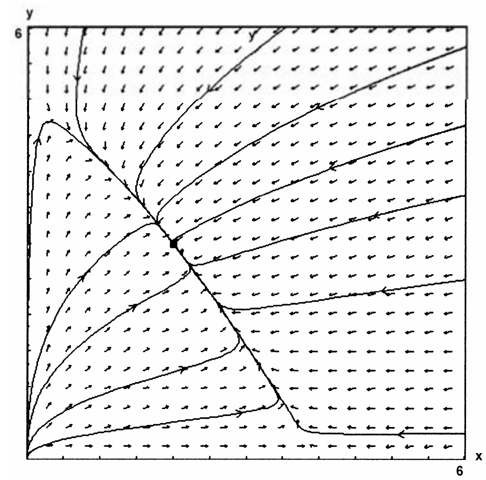
12. $\dfrac{dR}{dt} = R(1200 - 3R - 2S)$

$\dfrac{dS}{dt} = S(500 - R - S)$

The explanations of the equations are the same as those in Problem 11 except that the rabbit population is affected more by the crowding of its own population, less by the number of sheep.

Equilibria occur at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 500 \end{bmatrix}, \begin{bmatrix} 400 \\ 0 \end{bmatrix}$, or $\begin{bmatrix} 200 \\ 300 \end{bmatrix}$.

In this system the equilibria on the axes are all unstable, so the populations always head toward a coexistence equilibrium at $\begin{bmatrix} 200 \\ 300 \end{bmatrix}$. See Figure, where x and y are measured in hundreds.



Finding the Model

Example of appropriate models are as follows, with real positive coefficients.

13. $$\begin{aligned} x' &= ax - bx^2 - dxy - fx \\ y' &= -cy + dxy \end{aligned}$$

14. $$\begin{aligned} x' &= ax + bxy \\ y' &= cy - dxy + eyz \\ z' &= fz - gx^2 - hyz \end{aligned}$$

15. $$\begin{aligned} x' &= ax - bx^2 - cxy - dxz \\ y' &= ey - fy^2 + gxy \\ z' &= -hz + kxz \end{aligned}$$

Host-Parasite Models

16. (a) A suggested model is

$$\begin{aligned} H' &= aH - c\frac{H}{1+P} \\ P' &= -bP + dHP \end{aligned}$$

where *a*, *b*, *c*, and *d* are positive parameters. Here a species of beetle (parasite) depends on a certain species of tree (host) for survival. Note that if the beetle were so effective as to wipe out the entire population of trees, then it would die out itself, which is reflected in our model (note the differential equation in *P*). On the other hand, in the absence of the beetle, the host tree may or may not die out depending on the size of the parameters *a* and *c*. We would probably pick $a > c$, so the host population would increase in the absence of the parasite. Note too that model says that when the parasite (*P*) population gets large, it will not destroy the host entirely, as $\frac{H}{1+P}$ becomes small. The modeler might want to estimate the values of parameters a, b, c, d so the solution fits observed data. The modeler would also like to know the qualitative behavior of $(P,\ H)$ in the PH plane.

Professor Larry Turyn of Wright State University argues for a different model,

$$H' = -\frac{CH}{HP},$$

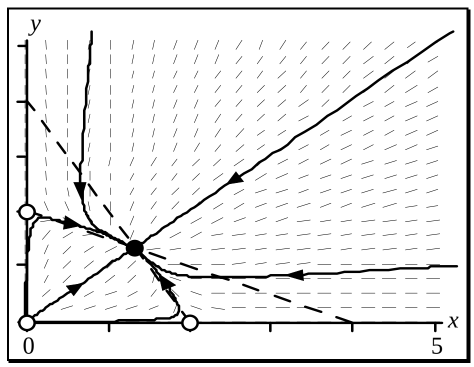
to better account for the case of very small P.

(b) Many bacteria are parasitic on external and internal body surfaces; some invading inner tissue causing diseases such as typhoid fever, tuberculosis, and pneumonia. It is important to construct models of the dynamics of these complex organisms.

■ **Competition**

17. (a)

$$x' = x(4 - 2x - y)$$
$$y' = y(4 - x - 2y)$$



Setting $x' = 0$ and $y' = 0$ we find

υ-nullclines $2x + y = 4,\ x = 0$
h-nullclines $x + 2y = 4,\ y = 0.$

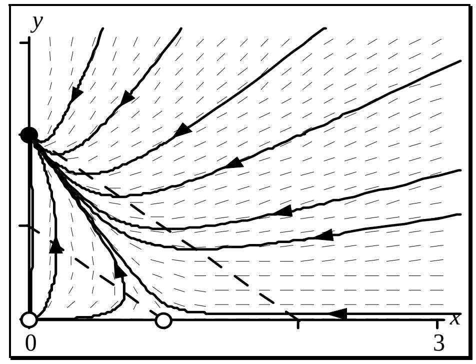
Equilibrium points: (0, 0), (0, 2), (2, 0), $\left(\frac{4}{3},\ \frac{4}{3}\right)$. The directions of the solution curves are shown in the figure.

(b) It can be seen from the figure, that the equilibrium points (0, 0), (0, 2) and (2, 0) are unstable. Only the point $\left(\frac{4}{3},\ \frac{4}{3}\right)$ is stable because all solution curves nearby point toward it.

(c) Some solution curves are shown in the figure.

(d) Because all the solution curves eventually reach the stable equilibrium at $\left(\frac{4}{3},\ \frac{4}{3}\right)$, the two species described by this model can coexist.

18. (a) $x' = x(1 - x - y)$
$y' = y(2 - x - y)$



Setting $x' = 0$ and $y' = 0$ we find

υ-nullclines $x + y = 1,\ x = 0$
h-nullclines $x + y = 2,\ y = 0$.

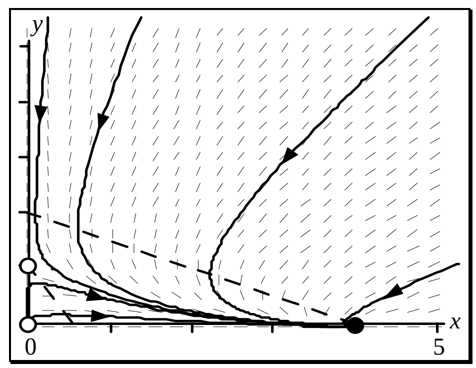
Equilibrium points: (0, 0), (0, 2), (1, 0). The directions of the solution curves are shown in the figure.

(b) It can be seen from the figure, that the equilibrium points (0, 0) and (1, 0) are unstable; the point (0, 2) is stable because all solution curves nearby point toward it.

(c) Some solution curves are shown in the figure.

(d) Because all the solution curves eventually reach the stable equilibrium at (0, 2), the x species always die out and the two species described by this model cannot coexist.

19. (a) $x' = x(4 - x - 2y)$
$y' = y(1 - 2x - y)$



Setting $x' = 0$ and $y' = 0$ we find

υ-nullclines $x + 2y = 4,\ x = 0$
h-nullclines $2x + y = 1,\ y = 0$.

Equilibrium points: (0, 0), (0, 1), (4, 0). The directions of the solution curves are shown in the figure.

(b) It can be seen from the figure, that the equilibrium points (0, 0) and (0, 1) are unstable; the point (4, 0) is stable because all solution curves nearby point toward it.

(c) Some solution curves are shown in the figure.

(d) Because all the solution curves eventually reach the stable equilibrium at (4, 0), the y species always die out and the two species described by this model cannot coexist.

20. (a) $x' = x(2 - x - 2y)$

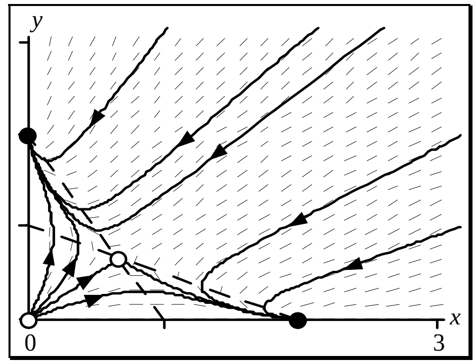
$y' = y(2 - 2x - y)$

Setting $x' = 0$ and $y' = 0$ we find

υ-nullclines $x + 2y = 2,\ x = 0$

h-nullclines $2x + y = 2,\ y = 0.$

Equilibrium points: (0, 0), (0, 2), (2, 0), $\left(\frac{2}{3}, \frac{2}{3}\right)$. The directions of the solution curves are shown in the figure.



(b) It can be seen from the figure, that the equilibrium points (0, 0) and $\left(\frac{2}{3}, \frac{2}{3}\right)$ are unstable; the points (0, 2) and (2, 0) are stable because all nearby arrows point toward them.

(c) Some solution curves are shown in the figure.

(d) Because all the solution curves eventually reach one of the stable equilibria at (0, 2) or (2, 0), the two species described by this model cannot coexist, unless they are exactly at the unstable equilibrium point $\left(\frac{2}{3}, \frac{2}{3}\right)$. Which species dies out is determined by the initial conditions.

■ **Simpler Competition**

21. $x' = x(a - by)$

$y' = y(c - dx)$

Setting $x' = 0$, we find the υ-nullclines are the vertical line $x = 0$ and the horizontal line $y = \frac{a}{b}$. Setting $y' = 0$, we find the h-nullclines are the horizontal $y = 0$ and vertical line $x = \frac{c}{d}$. The equilibrium points are (0, 0) and $\left(\frac{c}{d}, \frac{a}{b}\right)$. By observing the signs of x', y' we find

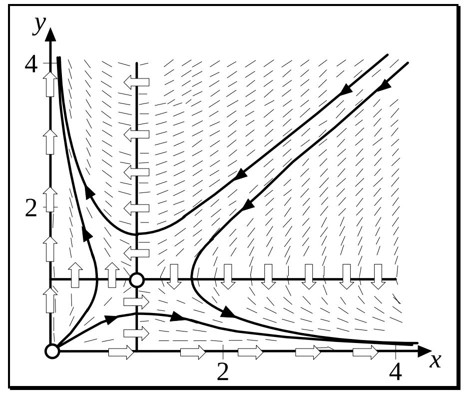
$$x' > 0,\ y' > 0 \text{ when } x < \frac{c}{d},\ y < \frac{a}{b}$$

$$x' < 0,\ y' < 0 \text{ when } x > \frac{c}{d},\ y > \frac{a}{b}$$

$$x' < 0,\ y' > 0 \text{ when } x > \frac{c}{d},\ y < \frac{a}{b}$$

$$x' > 0,\ y' < 0 \text{ when } x < \frac{c}{d},\ y > \frac{a}{b}$$

Hence, both equilibrium points are unstable. We can see from the following direction field (with $a = b = c = d = 1$) that one of two species, depending on the initial conditions, goes to infinity and the other toward extinction.

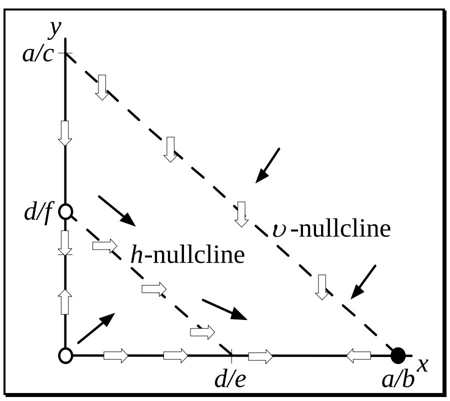


One can get the initial values for these curves directly from the graph.

■ Nullcline Patterns

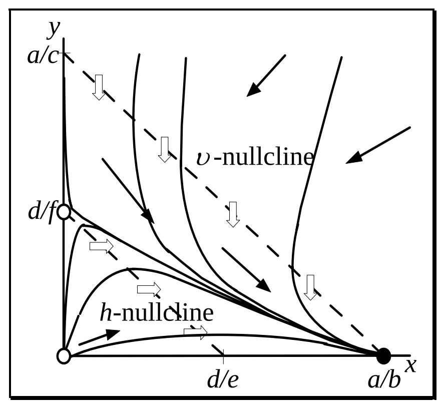
22. (a–e) When the υ-nullcline lies above the *h*-nullcline, there are three equilibrium points in the first quadrant: (0, 0), $\left(0, \dfrac{d}{f}\right)$ and $\left(\dfrac{a}{b}, 0\right)$. The points (0, 0), $\left(0, \dfrac{d}{f}\right)$ are unstable and $\left(\dfrac{a}{b}, 0\right)$ is stable. Hence, only population *x* survives.

(a)–(b)



Nullclines and equilibria

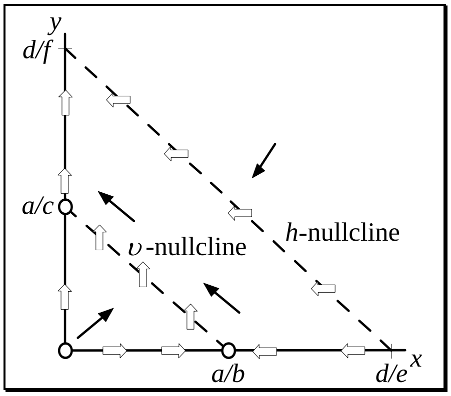
(c)–(d)



Sample trajectories when the υ-nullcline is above the *h*-nullcline

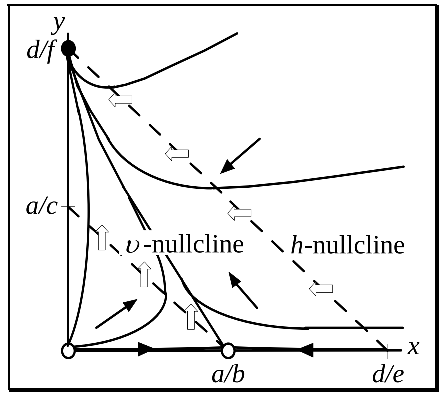
23. (a–e) When the h-nullcline lies above the υ-nullcline, there are three equilibrium points in the first quadrant: (0, 0), $\left(\frac{a}{b},\ 0\right)$ and $\left(0,\ \frac{d}{f}\right)$. The points (0, 0), $\left(\frac{a}{b},\ 0\right)$ are unstable and $\left(0,\ \frac{d}{f}\right)$ stable. Hence, only population y survives.

(a)–(b)



Nullclines

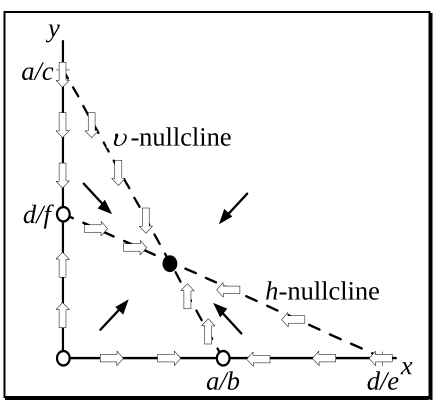
(c)–(d)



Sample trajectories when the h-nullcline is above the υ-nullcline

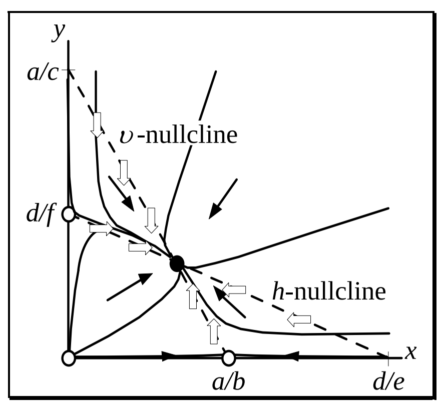
24. (a–e) When the two nullclines intersect as they do in the figure, then there are four equilibrium points in the first quadrant: (0, 0), $\left(\frac{a}{b},\ 0\right)$, $\left(0,\ \frac{d}{f}\right)$, and $\left(x_e,\ y_e\right)$, where $\left(x_e,\ y_e\right)$ is the intersection of the lines $bx + cy = a$, $ex + fy = d$. Analyzing the sign of the derivatives in the four regions of the first quadrant, we find that $\left(x_e,\ y_e\right)$ is stable and the others unstable. Hence, the two populations can coexist.

(a)–(b)



Nullclines and equilibria

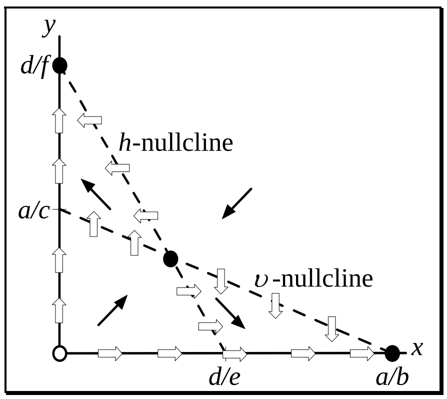
(c)–(d)



Typical trajectories when the nullclines intersect and the slope of the vertical nullcline is more negative.

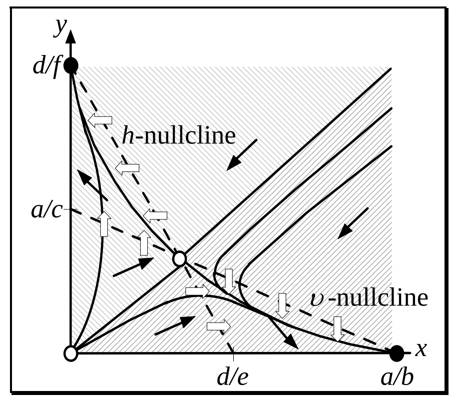
25. (a–e) When the two nullclines intersect as they do in the figure, then there are four equilibrium points in the first quadrant: (0, 0), $\left(\frac{a}{b},\ 0\right)$, $\left(0,\ \frac{d}{f}\right)$, and $\left(x_e,\ y_e\right)$, where $\left(x_e,\ y_e\right)$ is the intersection of the lines $bx+cy=a$, $ex+fy=d$. Analyzing the sign of the derivatives in the four regions of the first quadrant, we find $\left(\frac{a}{b},\ 0\right)$ and $\left(0,\ \frac{d}{f}\right)$ are stable and the other two unstable. Hence, only one of the two populations survives, and *which* survives depends on the initial conditions. See Figures. For initial conditions in the upper region y survives; for initial conditions in the lower region, x survives.

(a)–(b)



Nullclines and equilibria

(c)–(d)



Typical trajectories when the nullclines intersect and the slope of the h-nullcline is more negative.

■ **Unfair Competition**

26. $x' = ax(1-bx) - cxy$
$y' = dy - exy$

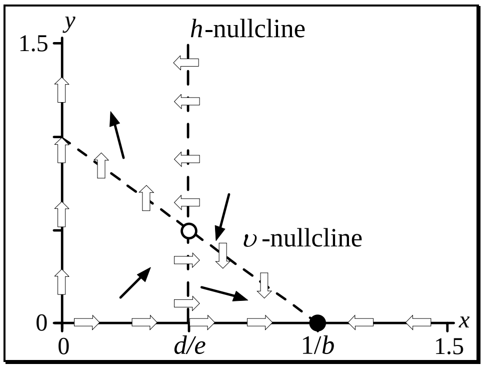
Setting $x' = y' = 0$, we find three equilibrium points:

$$(0, 0),\ \left(\frac{1}{b},\ 0\right),\ \text{and}\ \left(\frac{d}{e},\ a\frac{e-bd}{ce}\right).$$

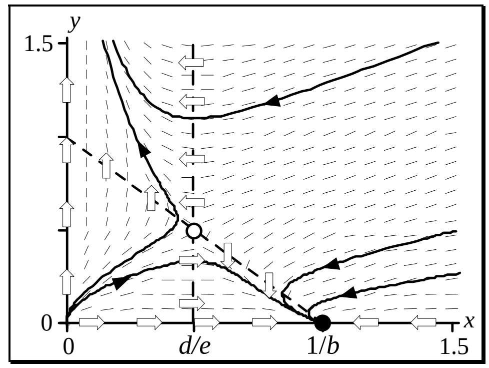
The point (0, 0) corresponds to both populations becoming extinct, the point $\left(\frac{1}{b},\ 0\right)$ corresponds to the second population becoming extinct, and the point $\left(\frac{d}{e},\ a\frac{e-bd}{ce}\right)$ corresponds to either a stable coexistent point or an unstable point. If we take the special case where $\frac{1}{b} > \frac{d}{e}$, e.g.,

$$x' = x(1-x) - xy$$
$$y' = 0.5y - xy$$

where $a = b = c = e = 1$, $d = 0.5$, we have the equilibrium points (0, 0), (1, 0), and (0.5, 0.5). If we draw two nullclines; v-nullcline: $y = 1 - x$, h-nullcline: $x = 0.5$, as shown following we see that the equilibrium point (0.5, 0.5) is unstable. Hence, the two species cannot coexist.



Nullclines and equilibria for $\frac{1}{b} > \frac{d}{e}$



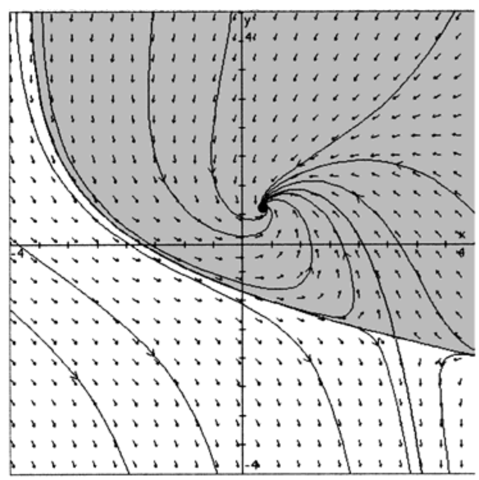
Sample trajectories for $\frac{1}{b} > \frac{d}{e}$

The reader must check separately the cases where $\frac{1}{b} = \frac{d}{e}$ or $\frac{1}{b} < \frac{d}{e}$.

■ Basins of Attraction

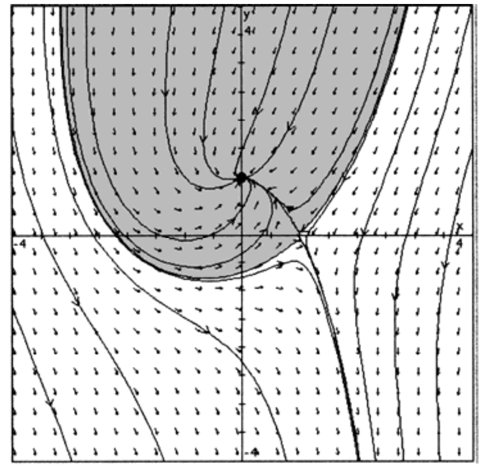
27. Adding shading to the graph obtained in Problem 2 shows the basis of the stable equilibrium at

$$\left(\frac{1}{4}\left(1-\sqrt{5}\right)^2, \frac{1}{2}\left(\sqrt{5}-1\right)\right) \approx (0.38, 0.60).$$

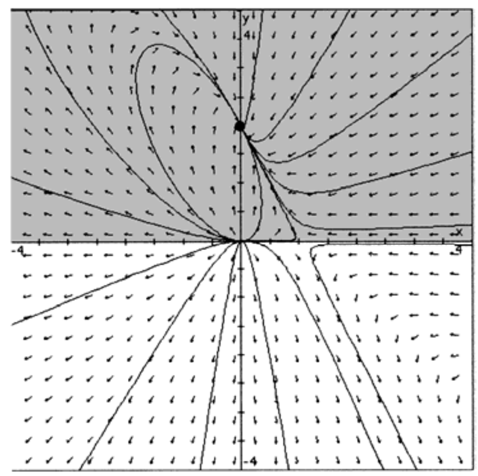


28. Adding shading to the graph obtained in Problem 3 shows the basis of the stable equilibrium at

(0, 1).



29. Adding shading to the graph obtained in Problem 18 shows that the entire first and second quadrants are the basin of attraction for the stable equilbrium at
(0, 2).

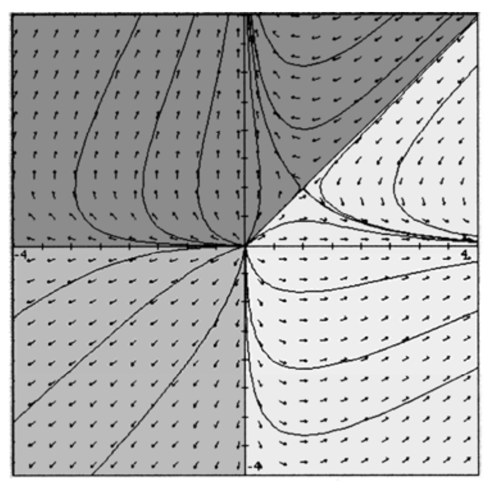


30. The graph obtained in Problem 21 has *no* stable equilibrium, but we say that there are three basins:

For $x > y$ and $y > 0$, trajectories are attracted to $(0, \infty)$.

For $x > 0$ and $x < y$, trajectories are attracted to $(\infty, 0)$.

For $x < 0$ and $y < 0$, trajectories are attracted to $(-\infty, -\infty)$.



■ Computer Lab: Parameter Investigation

31. Hold three of the parameters constant and observe how the fourth parameter affects the behavior of the two species. See if the behavior makes sense in your understanding of the model. Keep in mind that parameter a_R is a measure of how well the prey grows in the absence of the predator (large a_R for rabbits), a_F is a measure of how fast the predator population will decline when the prey is absent (large a_F if the given prey is the only source of food for the predator), c_R is a measure of how fast the prey's population declines per number of prey and predators, and c_F is a measure of how fast the predator's population increases per number of prey and predators. Even if you are not a biology major, you may still ponder the relative sizes of the four parameters in the two predator–prey systems: foxes and rabbits, and ladybugs and aphids. You can use these explanations to reach the same conclusions as in Problem 9.

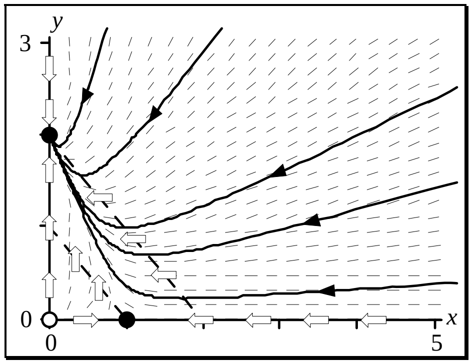
■ Computer Lab: Competition Outcomes

32. (a) Using the IDE software, hold all parameters fixed except one and observe how the last parameters affects the solution. See if the behavior of the two species makes sense in your understanding of the model. Play a mind game and predict if there will be coexistence between the species, whether one becomes extinct, and so on, before you make the parameter change.

Note that in the IDE tool, Competitive Exclusion, there are six parameters; K_1, B_1, r_1, K_2, B_2, and r_2. The parameters in our text called a_R, b_R, c_R, a_F, b_F, and c_F and enter the equations slightly differently. The reason for this discrepancy is due to the way the parameters in the IDE software affect the two isoclines, called the N_1 and N_2 isoclines in the IDE software.

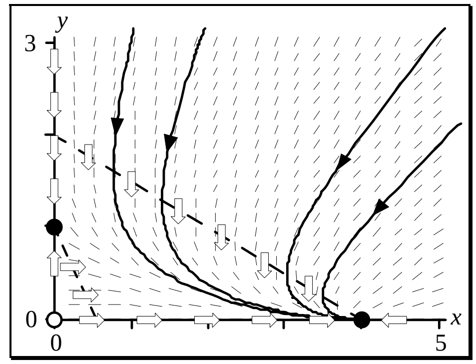
By changing the parameters K_1 and K_2 in the IDE software, you simply move the respective isoclines in a parallel direction. The parameters B_1, B_2 change the slopes of the nullclines. And finally, the parameters r_1, r_2 do not affect the nullclines, but affect the direction field or the transient part of the solution.

Your hand-sketched phase plane for the four cases should qualitatively look like the following four pictures, with the basins of attraction colored for each equilibrium.



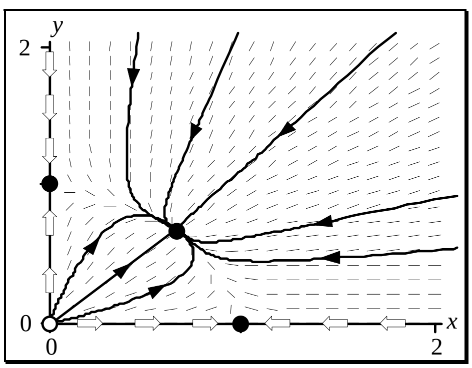
Case 1: Population x dies out

$$x' = x(1 - x - y)$$
$$y' = y(2 - x - y)$$



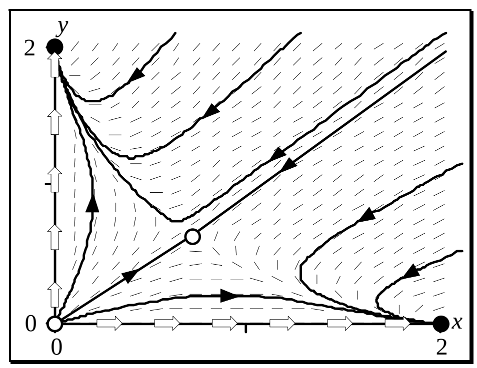
Case 2: Population y dies out

$$x' = x(4 - x - 2y)$$
$$y' = y(1 - 2x - y)$$



Case 3: Populations coexist

$$x' = x(2 - 2x - y)$$
$$y' = y(2 - x - 2y)$$

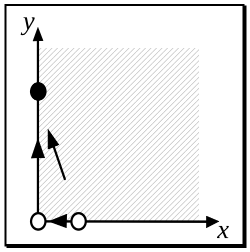


Case 4: One of the populations dies out

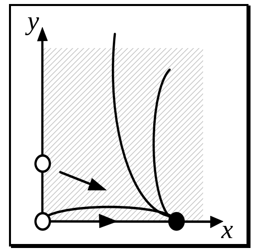
$$x' = x(2 - x - 2y)$$
$$y' = y(2 - 2x - y)$$

Of the four different scenarios to the competitive model, in only one (Case 3) can both species coexist. In the other three cases one of the two dies out. In Case 4 the species that dies out depends on the initial conditions, and in Case 1 and 2 one species will die out regardless of the initial condition. Note too that in all four cases if one population initially starts at zero, it remains at zero.

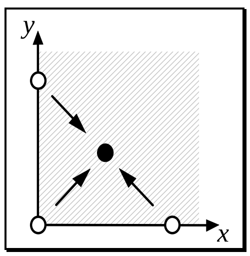
(b) The basins of attraction for each stable equilibrium are shown for each of the four cases. Compare with the figures in part (a).



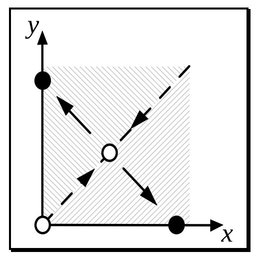
Case 1



Case 2



Case 3



Case 4

■ **Suggested Journal Entry**

33. Student Project

CAS Project 2A Heating and Cooling: Transient and Steady-State Solutions

In this project you will use your CAS to study solution curves when Newton's Law of Cooling is used to model temperature changes inside a house. A useful solution formula is derived and then used to illustrate the notions of transient and steady-state temperature and the time constant associated with the model. Using `Maple` the relevant concepts and commands are function definitions and fine tuning of plots.

> **User defined functions**
> A function f is defined using the arrow construct as in `f := x -> x*sin(x)` which defines $f(x) = x\sin x$. One can also use the **unapply** command to convert, say, an expression named "Harry" into a function of x named "f" as in `f := unapply( Harry, x)` which constructs $f(x) = \text{Harry}$.

Newton's Law of Cooling models temperature change using the differential equation $y'(t) = k[A(t) - y(t)]$. Here $y(t)$ is the temperature inside the house at time t and $A(t)$ is the ambient temperature, i.e. the temperature outside the house. The letter k represents a positive constant. If there is a heat source (or sink) in the house that causes a change in the temperature at the rate $H(t)$ the model is

$$y'(t) = k[A(t) - y(t)] + H(t).$$

The Solution Formula. This is a linear equation:

$$y'(t) + ky(t) = kA(t) + H(t).$$

The integrating factor is $\mu = e^{kt}$ so, after multiplying the model equation by e^{kt}, it simplifies to

$$\frac{d}{dt}\left[e^{kt}y(t)\right] = e^{kt}\left[kA(t) + H(t)\right].$$

Integrate with respect to t to obtain

$$e^{kt}y(t) = \int e^{kt}\left[kA(t) + H(t)\right]dt + C$$

and the general solution is

$$y(t) = Ce^{-kt} + e^{-kt}\int e^{kt}\left[kA(t) + H(t)\right]dt.$$

For our purposes it will be convenient to make the antiderivative explicit and write the general solution in the form $y(t) = Ce^{-kt} + \int_0^t e^{-k(t-\tau)}\left[kA(\tau) + H(\tau)\right]d\tau$. This is especially nice because substitution of $t = 0$ reveals that $C = y(0)$. Denoting $y(0)$ as y_0 we have arrived at a solution formula that is as useful as it is beautiful.

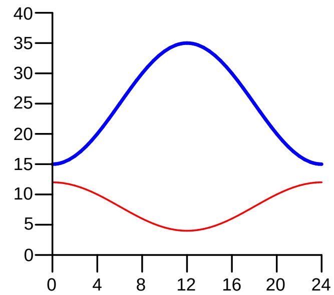
$$y(t) = y_0e^{-kt} + \int_0^t e^{-k(t-\tau)}\left[kA(\tau) + H(\tau)\right]d\tau.$$

The Time Constant. The first term in the solution formula is there to satisfy the initial condition. It is referred to as a *transient term* because every $1/k$ seconds it decreases by 37% ($e^{-1} = 0.368$). In $5 \cdot \frac{1}{k}$ seconds this term will be less than 0.7% of its initial value, eventually disappearing altogether. The number $1/k$ is called the *time constant.* In any plot the transients in the solution will disappear in about $5 \cdot \frac{1}{k}$ seconds.

Example. It's a cold winter evening. At midnight the house temperature is 25° F. Assume that the outside temperature varies daily from a low of 15° at midnight to a high of 35° at 12 noon. The furnace is set to cycle on and off periodically providing enough heat to make the temperature inside the house increase at the rate of 12° an hour at midnight and at the rate of 4° an hour at noon. Use the solution formula derived above to obtain plots of the inside temperature over a 72 hour period beginning at 12 midnight ($t = 0$). Assume that $k = 0.2$.

The first entry defines the functions $A(t)$ and $H(t)$ and plots them over a 24 hour period.

```
> A := t -> 25 - 10*cos(2*Pi*t/24):
  H := t -> 8 + 4*cos(2*Pi*t/24):
  plot( [A(t),H(t)], t=0..24, 0..40, color=[blue,red], thickness=[2,1],
        tickmarks=[[4*n $ n=0..6],[5*n $ n=0..8]] );
```



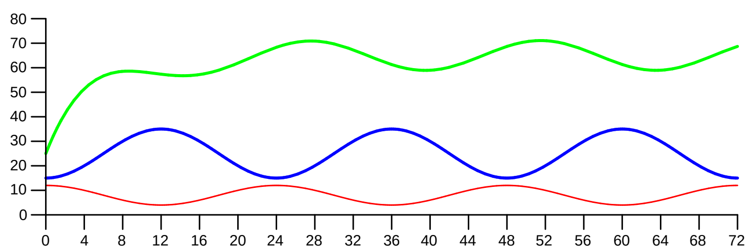
The ambient temperature is the upper (blue) curve. The lower (red) curve is the rate of change of the inside temperature over a 24 hour time period.

Now define y_0 and k and evaluate the solution formula. The **unapply** command makes the solution expression into the function y. The solution formula is then displayed using 3 digit approximations for the constants. The graph of the inside temperature follows. We have also included the graph of the outside temperature and the curve showing the furnace's contribution to rate of change of $y(t)$.

```
> y0 := 25: k := 0.2:
> y0*exp(-k*t) + int(exp(-k*(t-tau))*(k*A(tau) + H(tau)),tau=0..t):
  y := unapply(%,t):                          #The percent sign refers to the last output.
  'y(t)' = evalf[3](y(t));
```

$$y(t) = 25e^{-0.2t} - 68.7e^{-0.200t} + 65.0 + 3.69\cos(0.262t) + 4.84\sin(0.262t)$$

```
> plot( [y(t),A(t),H(t)], t=0..72, 0..80, color=[green,blue,red],
        thickness=[2,2,1], tickmarks=[[4*n $ n=0..18],[10*n $ n=0..8]]);
```



The highest (green) curve is $y(t)$, the temperature inside the house at time t. The middle (blue) and lowest (red) curves are as above.

Observe that

- Integration produced another transient in the solution formula. It is also there to accommodate initial conditions and has the same time constant: $\frac{1}{0.2} = 5$ hours. The transients vanish in about 25 hours.
- The terms that remain after the transients are gone are referred to as the *steady-state* terms in the solution. In this example we see a sinusoid oscillating around 65° with an amplitude of $\sqrt{3.69^2 + 4.84^2} = 6.07$. Thus the inside steady-state temperature varies from a high of 71° at hour 28 (4 AM) to a low of 59° 12 hours later (4 PM).

Things for you to do.

1. Get the code working. The cycling of the furnace was set up to make the house warm quickly.
   a) It is clear from the solution curve that once steady-state is attained, the house will be too warm

at night and too cold during the day. Change the furnace cycle to start at 4° per hour at midnight ($t = 0$) and cycle to 12° per hour at noon. Comment on the effect this has on the house temperature, both transient and steady-state.

b) Experiment with the timing of the furnace cycling (always between 12 and 4 degrees per hour) to attain a steady-state house temperature that is maximum close to 12 noon and minimum close to 12 midnight. Are the maximum and the minimum values for $y(t)$ always the same? Hint. This process will require a time shift in the formula for $H(t)$.

2. Change the initial house temperature to 70°. What effect does this change have on the steady-state temperature? What effect does it have on the time it takes for $y(t)$ to reach steady-state?

3. Return to $y_0 = 25$ and the original $H(t)$. There are furnace cycling parameters (max/min) that yield a constant steady-state house temperature. They can be found experimentally or by making a careful analysis of the integral formula for the solution that is obtained by substituting $H(t) = 8 + b_0 \cos(2\pi t/24)$. Find these parameters and display the solution curve.

4. Try to obtain a furnace schedule that will bring the house to a constant temperature of 68 degrees. Do it by experimentation (CAS), then verify by obtaining the solution formula for this cycling schedule (paper and pencil).

5. Explore solution curves when the ambient temperature has a varying average value. That is, $A(t) = a(t) - 10\cos(2\pi t/24)$ for some function $a(t)$. For example, start by assuming that $a(t)$ is increasing at the rate of 2 degrees per day: $a(t) = 25 + \frac{2}{24}t$. What effect does this have on the behavior of $y(t)$?

CAS Project 2B Growth Equations and Bifurcations

In this project you will use your CAS to study growth equations and the effect of parameters on the solutions of first-order, non-linear equations. See Exercises 35–42 in Section 2.5 of the text. Using `Maple` the relevant concepts and commands are

> **DEplot**
> The procedure that draws direction fields and numerically generated solution curves. See CAS Project 1A.
>
> **for..do loops**
> Simple looping constructs can be made easily. The data that is created in a **for..do loop** can be stored in a table. The syntax is self-explanatory.
>
> **dsolve**
> The command that finds symbolic solutions for a differential equation and numeric solutions for an initial value problem. See CAS Project 1B.

Example 1. Consider the Gompertz growth equation $y' = r\big(1 - \frac{\ln y}{L}\big)y$. Use direction fields and solution curves to determine the significance of the parameters r and L. Obtain a symbolic solution, if possible.

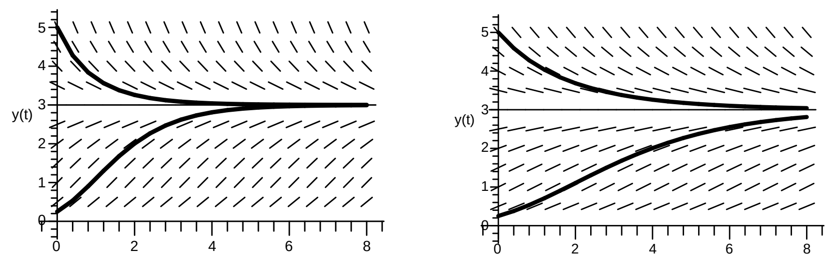
Observe that $y(t) > 0$ for all t. It can be seen by inspection that there is one equilibrium solution, $y(t) \equiv e^L$. Thus the value of the parameter L determines the position of the equilibrium. We investigate the effect of the parameter r by examining the direction fields and solution curves corresponding to $y(0) = 0.25$ and $y(0) = 5$ when $r = 1$ and when $r = 0.5$. In both cases, $L = \ln 3$, so $y(t) \equiv 3$ is the equilibrium solution.

The first entries below load the **DEplot** package, define the DE, then define L and r for the first plot. The second plot was created after changing r to 0.5. The two plots are placed side by side for easy comparison. From the graph we see that the value of r determines the rate at which the solutions approach the equilibrium point.

```
> with(DEtools):
> DE := diff(y(t),t) = r*(1-ln(y(t))/L)*y(t);
```

$$DE := \frac{d}{dt}y(t) = r\left(1 - \frac{\ln(y(t))}{L}\right)y(t)$$

```
> L := ln(3): r := 1:
  DEplot( DE, y(t), t=0..8, y=0..5, arrows=line,
          dirgrid=[18,11], color=black, scaling=constrained,
          [[y(0)=0.25],[y(0)=5]], linecolor=black);
```



Direction field and solution, $r = 1$. Direction field and solution, $r = 0.5$.

Symbolic solutions can be found. `Maple`'s solution for the initial value $y(0) = 0.25$, arbitrary r, and $L = \ln 3$ is displayed below. Check it by substituting into the differential equation.

```
> unassign('r');
  dsolve( {DE,y(0)=0.25} );
```

$$y(t) = 3e^{-e^{-\frac{r\,t}{\ln(3)}+\ln(\ln(12))}}$$

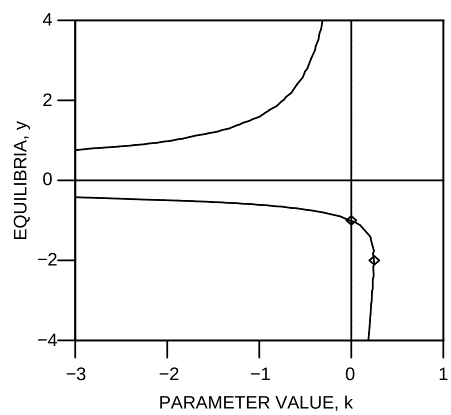
Example 2. Consider the differential equation $y' = ky^2 + y + 1$. Determine the significance of the parameter k on the nature of the equilibrium points. Obtain symbolic solutions, if possible.

For a fixed k solve the equation $ky^2 + y + 1 = 0$ for y to see that if $k \neq 0$, then the equilibrium points are $y = \frac{-1 \pm \sqrt{1-4k}}{2k}$.

- If $k > 1/4$, then there are no equilibrium solutions. All solutions increase without bound as $t \to \infty$.
- If $k = 1/4$, then $y(t) \equiv -2$ is the only equilibrium solution. Moreover, the differential equation is $y' = \frac{1}{4}(y+2)^2$ so all other solutions are increasing and this is a semistable equilibrium point. Solutions approach $y = -2$ from below and flow away from above.
- If $1/4 > k > 0$, then there are two equilibrium points, $y_1 < y_2 < 0$. Since the differential equation has the form $y' = k(y - y_1)(y - y_2)$ with k positive, y_1 is a stable equilibrium and y_2 is unstable.
- If $k = 0$, there is one equilibrium, $y_0 = -1$. Since the differential equation is $y' = y + 1$, this equilibrium is unstable.
- If $0 > k > -\infty$, then there are two equilibrium points once more, $y_1 < 0 < y_2$. The differential equation has the form $y' = k(y - y_1)(y - y_2)$ with k negative, so y_1 is an unstable equilibrium and y_2 is stable.

The plot below, called a *bifurcation diagram*, shows where these points are for each k value. Note that the curve is simply the implicit plot of the equation $ky^2 + y + 1 = 0$ in the (k, y)-plane. To interpret this plot, sketch a vertical line at any k-value. Where that line intersects the curves you will find the y-values that are the equilibria for the DE with that k-value.

```
> with(plots):
> display( implicitplot( k*y^2 + y + 1 = 0, k=-3..1, y=-4..4, color=black),
           plot( [[0,t,t=-4..4],[t,0,t=-4..1],[[0.25,-2]],[[0,-1]]],
                 style=[line,line,point,point], symbolsize=18, color=black),
                 view=[-3..1,-4..4], tickmarks=[4,4], axes=boxed,
                 labels=["PARAMETER VALUE, k","EQUILIBRIA, y"],
                 labeldirections=[horizontal,vertical]);
```



The bifurcation diagram for $y' = ky^2 + y + 1$.

Since the number and the nature of the equilibrium points changes at the values $k = 0$ and $k = 1/4$, these are the bifurcation points. See the corresponding points in the diagram. The following table summarizes this information.

Interval	Number of Equilibria	Nature
$\infty > k > \frac{1}{4}$	none	
$k = \frac{1}{4}$	one	*semistable*
$\frac{1}{4} > k > 0$	two	*stable* $<$ *unstable*
$k = 0$	one	*unstable*
$0 > k > -\infty$	two	*unstable* $<$ *stable*

`Maple`'s **dsolve** procedure comes up with the following symbolic solution to this equation.

```
> dsolve( diff(y(t),t) = k*y(t)^2 + y(t) + 1 );
```

$$y(t) = \frac{1}{2}\frac{-1 + \tan\left(\frac{1}{2}t\sqrt{4k-1} + \frac{1}{2}\_C1\sqrt{4k-1}\right)\sqrt{4k-1}}{k}$$

Things for you to do.

1. Use direction fields and solution curves similar to the ones shown in Example 1 to determine the significance of the parameters in the threshold equation and the equation for decaying exponential rate in Exercises 40 and 42 of Section 2.5. Obtain symbolic solutions if you can.
2. Make a bifurcation analysis for the parameter k in the equation $y' = y^2 + y + k$ (Exercise 38 in Section 2.5). Include the calculation of the equilibrium solutions for each k, a bifurcation diagram, and a bifurcation table similar to the ones shown in Example 2. Obtain a symbolic solution if you can.