Chapter 2

First Order Differential Equations

2.1 Separable Equations

1. Rewriting as $ydy = x^4dx$, then integrating both sides, we have $y^2/2 = x^5/5 + c$, or $5y^2 - 2x^5 = c$; $y \neq 0$

2. Rewriting as $ydy = (x^2/(1 + x^3))dx$, then integrating both sides, we obtain that $y^2/2 = \ln|1 + x^3|/3 + c$, or $3y^2 - 2\ln|1 + x^3| = c$; $x \neq -1$, $y \neq 0$.

3. Rewriting as $y^{-3}dy = -\sin x dx$, then integrating both sides, we have $-y^{-2}/2 = \cos x + c$, or $y^{-2} + 2\cos x = c$ if $y \neq 0$. Also, $y = 0$ is a solution.

4. Rewriting as $(7 + 5y)dy = (7x^2 - 1)dx$, then integrating both sides, we obtain $5y^2/2 + 7y - 7x^3/3 + x = c$ as long as $y \neq -7/5$.

5. Rewriting as $\sec^2 y dy = \sin^2 2x dx$, then integrating both sides, we have $\tan y = x/2 - (\sin 4x)/8 + c$, or $8\tan y - 4x + \sin 4x = c$ as long as $\cos y \neq 0$. Also, $y = \pm(2n + 1)\pi/2$ for any integer n are solutions.

6. Rewriting as $(1 - y^2)^{-1/2}dy = dx/x$, then integrating both sides, we have $\arcsin y = \ln|x| + c$. Therefore, $y = \sin(\ln|x| + c)$ as long as $x \neq 0$ and $|y| < 1$. We also notice that $y = \pm 1$ are solutions.

7. Rewriting as $(y/(1 + y^2))dy = xe^{x^2}dx$, then integrating both sides, we obtain $\ln(1 + y^2) = e^{x^2} + c$. Therefore, $y^2 = ce^{e^{x^2}} - 1$.

8. Rewriting as $(y^2 - e^y)dy = (x^2 + e^{-x})dx$, then integrating both sides, we have $y^3/3 - e^y = x^3/3 - e^{-x} + c$, or $y^3 - x^3 - 3(e^y - e^{-x}) = c$ as long as $y^2 - e^y \neq 0$.

9. Rewriting as $(1 + y^2)dy = x^2dx$, then integrating both sides, we have $y + y^3/3 = x^3/3 + c$, or $3y + y^3 - x^3 = c$.

10. Rewriting as $(1 + y^3)dy = \sec^2 x dx$, then integrating both sides, we have $y + y^4/4 = \tan x + c$ as long as $y \neq -1$.

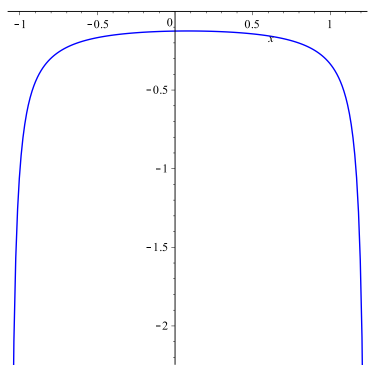
11. Rewriting as $y^{-1/2}dy = 4\sqrt{x}dx$, then integrating both sides, we have $y^{1/2} = 4x^{3/2}/3 + c$, or $y = (4x^{3/2}/3 + c)^2$. Also, $y = 0$ is a solution.

12. Rewriting as $dy/(y - y^2) = xdx$, then integrating both sides, we have $\ln|y| - \ln|1 - y| =$

$x^2/2 + c$, or $y/(1-y) = ce^{x^2/2}$, which gives $y = e^{x^2/2}/(c + e^{x^2/2})$. Also, $y = 0$ and $y = 1$ are solutions.

13.(a) Rewriting as $y^{-2}dy = (1 - 12x)dx$, then integrating both sides, we have $-y^{-1} = x - 6x^2 + c$. The initial condition $y(0) = -1/8$ implies $c = 8$. Therefore, $y = 1/(6x^2 - x - 8)$.

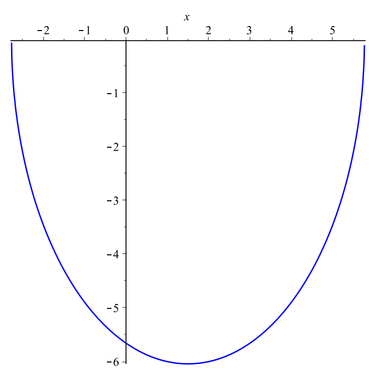
(b)



(c) $(1 - \sqrt{193})/12 < x < (1 + \sqrt{193})/12$

14.(a) Rewriting as $ydy = (3-2x)dx$, then integrating both sides, we have $y^2/2 = 3x - x^2 + c$. The initial condition $y(1) = -6$ implies $c = 16$. Therefore, $y = -\sqrt{-2x^2 + 6x + 32}$.

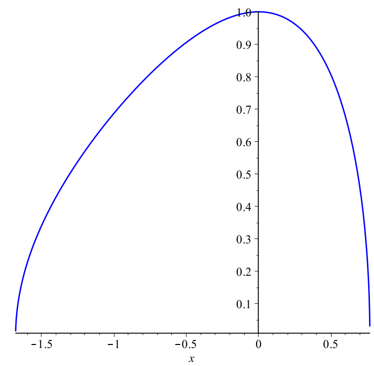
(b)



(c) $(3 - \sqrt{73})/2 < x < (3 + \sqrt{73})/2$

15.(a) Rewriting as $xe^x dx = -ydy$, then integrating both sides, we have $xe^x - e^x = -y^2/2 + c$. The initial condition $y(0) = 1$ implies $c = -1/2$. Therefore, $y = \sqrt{2(1-x)e^x - 1}$.

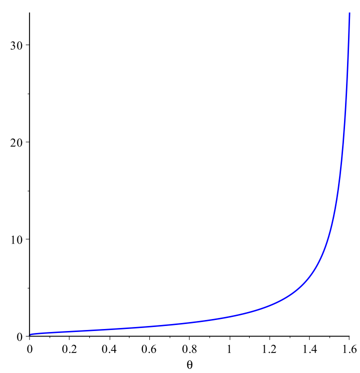
(b)



(c) $-1.68 < x < 0.77$, approximately

16.(a) Rewriting as $r^{-2}dr = \theta^{-1}d\theta$, then integrating both sides, we have $-r^{-1} = \ln|\theta| + c$. The initial condition $r(1) = 2$ implies $c = -1/2$. Therefore, $r = 2/(1 - 2\ln|\theta|)$.

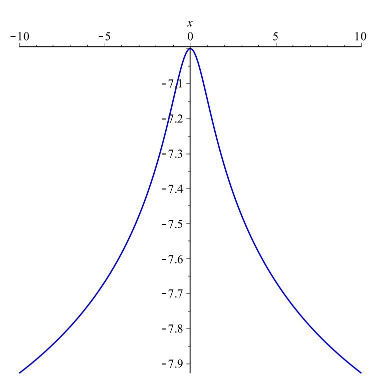
(b)



(c) $0 < \theta < \sqrt{e}$

17.(a) Rewriting as $ydy = 3x/(1+x^2)dx$, then integrating both sides, we have $y^2/2 = 3\ln(1+x^2)/2 + c$. The initial condition $y(0) = -7$ implies $c = 49/2$. Therefore, $y = -\sqrt{3\ln(1+x^2)+49}$.

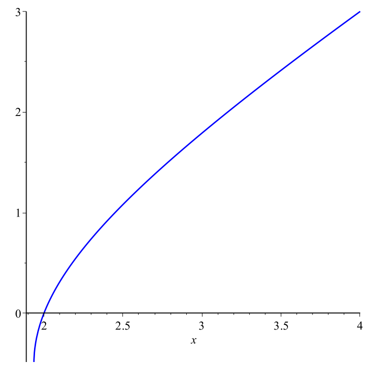
(b)



(c) $-\infty < x < \infty$

18.(a) Rewriting as $(1+2y)dy = 2x dx$, then integrating both sides, we have $y + y^2 = x^2 + c$. The initial condition $y(2) = 0$ implies $c = -4$. Therefore, $y^2 + y = x^2 - 4$. Completing the square, we have $(y+1/2)^2 = x^2 - 15/4$, and, therefore, $y = -1/2 + \sqrt{x^2 - 15/4}$.

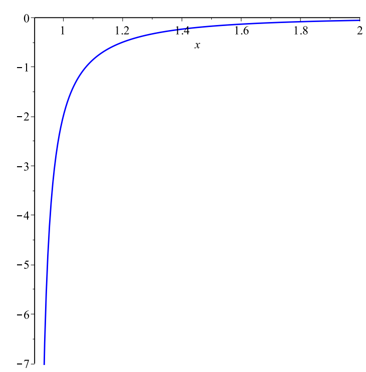
(b)



(c) $\sqrt{15}/2 < x < \infty$

19.(a) Rewriting as $y^{-2}dy = (2x + 4x^3)dx$, then integrating both sides, we have $-y^{-1} = x^2 + x^4 + c$. The initial condition $y(1) = -2$ implies $c = -3/2$. Therefore, $y = 2/(3 - 2x^4 - 2x^2)$.

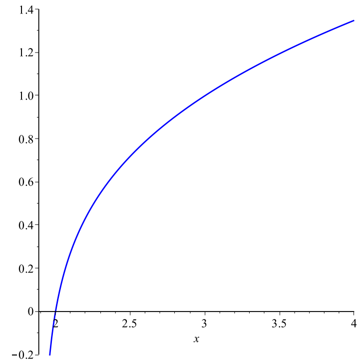
(b)



(c) $\sqrt{(-1+\sqrt{7})/2} < x < \infty$

20.(a) Rewriting as $e^{3y}dy = x^2 dx$, then integrating both sides, we have $e^{3y}/3 = x^3/3 + c$. The initial condition $y(2) = 0$ implies $c = -7/3$. Therefore, $e^{3y} = x^3 - 7$, and $y = \ln(x^3 - 7)/3$.

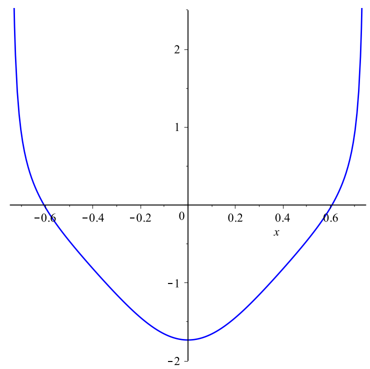
(b)



(c) $\sqrt[3]{7} < x < \infty$

21.(a) Rewriting as $dy/(1+y^2) = \tan 2x dx$, then integrating both sides, we have $\arctan y = -\ln(\cos 2x)/2 + c$. The initial condition $y(0) = -\sqrt{3}$ implies $c = -\pi/3$. Therefore, $y = -\tan(\ln(\cos 2x)/2 + \pi/3)$.

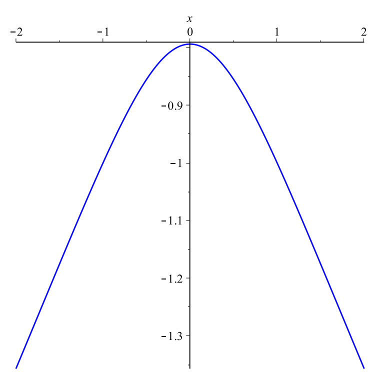
(b)



(c) $-\pi/4 < x < \pi/4$

22.(a) Rewriting as $6y^5 dy = x(x^2+1)dx$, then integrating both sides, we obtain that $y^6 = (x^2+1)^2/4 + c$. The initial condition $y(0) = -1/\sqrt[3]{2}$ implies $c = 0$. Therefore, $y = -\sqrt[3]{(x^2+1)/2}$.

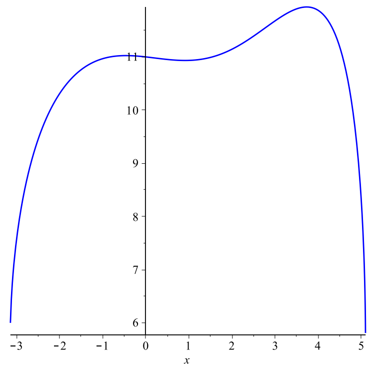
(b)



(c) $-\infty < x < \infty$

23.(a) Rewriting as $(2y-11)dy = (3x^2-e^x)dx$, then integrating both sides, we have $y^2-11y = x^3 - e^x + c$. The initial condition $y(0) = 11$ implies $c = 1$. Completing the square, we have $(y - 11/2)^2 = x^3 - e^x + 125/4$. Therefore, $y = 11/2 + \sqrt{x^3 - e^x + 125/4}$.

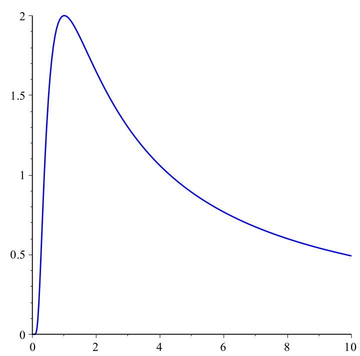
(b)



(c) $-3.14 < x < 5.10$, approximately

24.(a) Rewriting as $dy/y = (1/x^2 - 1/x)dx$, then integrating both sides, we have $\ln|y| = -1/x - \ln|x| + c$. The initial condition $y(1) = 2$ implies $c = 1 + \ln 2$. Therefore, $y = 2e^{1-1/x}/x$.

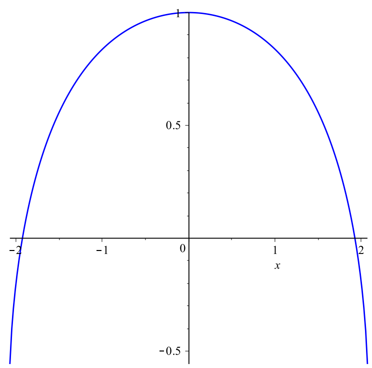
(b)



(c) $0 < x < \infty$

25.(a) Rewriting as $(3+4y)dy = (e^{-x} - e^x)dx$, then integrating both sides, we have $3y+2y^2 = -(e^x + e^{-x}) + c$. The initial condition $y(0) = 1$ implies $c = 7$. Completing the square, we have $(y + 3/4)^2 = -(e^x + e^{-x})/2 + 65/16$. Therefore, $y = -3/4 + (1/4)\sqrt{65 - 8e^x - 8e^{-x}}$.

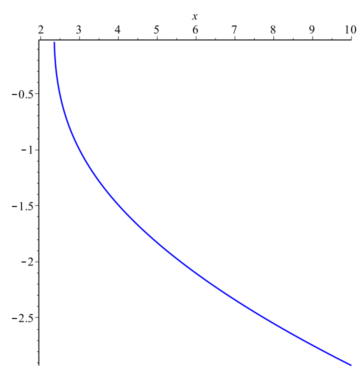
(b)



(c) $-\ln 8 < x < \ln 8$

26.(a) Rewriting as $2ydy = xdx/\sqrt{x^2-4}$, then integrating both sides, we have $y^2 = \sqrt{x^2-4} + c$. The initial condition $y(3) = -1$ implies $c = 1 - \sqrt{5}$. Therefore, $y = -\sqrt{\sqrt{x^2-4} + 1 - \sqrt{5}}$.

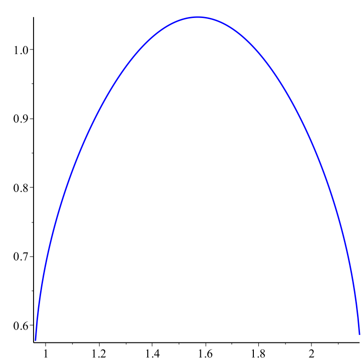
(b)



(c) $2 < x < \infty$

27.(a) Rewriting as $\cos 3ydy = -\sin 2xdx$, then integrating both sides, we have $(\sin 3y)/3 = (\cos 2x)/2 + c$. The initial condition $y(\pi/2) = \pi/3$ implies $c = 1/2$. Thus we obtain that $y = (\pi - \arcsin(3\cos^2 x))/3$.

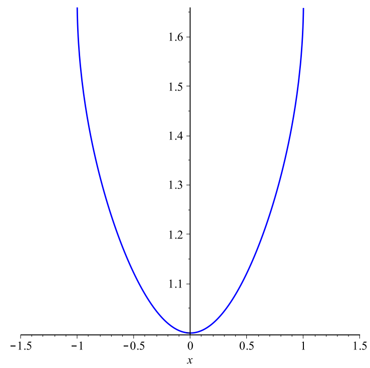
(b)



(c) $\pi/2 - 0.62 < x < \pi/2 + 0.62$, approximately

28.(a) Rewriting as $y^2dy = \arcsin x dx/\sqrt{1-x^2}$, then integrating both sides, we have $y^3/3 = (\arcsin x)^2/2 + c$. The initial condition $y(0) = 1$ implies $c = 1/3$. Thus we obtain that $y = \sqrt[3]{3(\arcsin x)^2/2 + 1}$.

(b)



(c) $-\pi/2 < x < \pi/2$

29. Rewriting the equation as $(12y^2 - 12y)dy = (1 + 3x^2)dx$ and integrating both sides, we have $4y^3 - 6y^2 = x + x^3 + c$. The initial condition $y(0) = 2$ implies $c = 8$. Therefore, $4y^3 - 6y^2 - x - x^3 - 8 = 0$. When $12y^2 - 12y = 0$, the integral curve will have a vertical tangent. This happens when $y = 0$ or $y = 1$. From our solution, we see that $y = 1$ implies $x = -2$; this is the first y value we reach on our solution, therefore, the solution is defined for $-2 < x < \infty$.

30. Rewriting the equation as $(2y^2 - 6)dy = 2x^2dx$ and integrating both sides, we have $2y^3/3 - 6y = 2x^3/3 + c$. The initial condition $y(1) = 0$ implies $c = -2/3$. Therefore, $y^3 - 9y - x^3 = -1$. When $2y^2 - 6 = 0$, the integral curve will have a vertical tangent. This happens when $y = \pm\sqrt{3}$. At these values for y, we have $x = \sqrt[3]{1 \pm 6\sqrt{3}}$. Therefore, the solution is defined on this interval; approximately $-2.11 < x < 2.25$.

31. Rewriting the equation as $y^{-2}dy = (2 + x)dx$ and integrating both sides, we have $-y^{-1} = 2x + x^2/2 + c$. The initial condition $y(0) = 1$ implies $c = -1$. Therefore, $y = -1/(x^2/2 + 2x - 1)$. To find where the function attains it minimum value, we look where $y' = 0$. We see that $y' = 0$ implies $y = 0$ or $x = -2$. But, as seen by the solution formula, y is never zero. Further, it can be verified that $y''(-2) > 0$, and, therefore, the function attains a minimum at $x = -2$.

32. Rewriting the equation as $(3 + 2y)dy = (6 - e^x)dx$ and integrating both sides, we have $3y+y^2 = 6x-e^x+c$. By the initial condition $y(0) = 0$, we have $c = 1$. Completing the square, it follows that $y = -3/2 + \sqrt{6x - e^x + 13/4}$. The solution is defined if $6x - e^x + 13/4 \geq 0$, that is, $-0.43 \leq x \leq 3.08$ (approximately). In that interval, $y' = 0$ for $x = \ln 6$. It can be verified that $y''(\ln 6) < 0$, and, therefore, the function attains its maximum value at $x = \ln 6$.

33. Rewriting the equation as $(10 + 2y)dy = 2\cos 2x dx$ and integrating both sides, we have $10y + y^2 = \sin 2x + c$. By the initial condition $y(0) = -1$, we have $c = -9$. Completing the square, it follows that $y = -5 + \sqrt{\sin 2x + 16}$. To find where the solution attains its

maximum value, we need to check where $y' = 0$. We see that $y' = 0$ when $2\cos 2x = 0$. This occurs when $2x = \pi/2 + 2k\pi$, or $x = \pi/4 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

34. Rewriting this equation as $(1 + y^2)^{-1}dy = 2(1 + x)dx$ and integrating both sides, we have $\arctan y = 2x + x^2 + c$. The initial condition implies $c = 0$. Therefore, the solution is $y = \tan(x^2 + 2x)$. The solution is defined as long as $-\pi/2 < 2x + x^2 < \pi/2$. We note that $2x + x^2 \geq -1$. Further, $2x + x^2 = \pi/2$ for $x \approx -2.6$ and 0.6. Therefore, the solution is valid in the interval $-2.6 < x < 0.6$. We see that $y' = 0$ when $x = -1$. Furthermore, it can be verified that $y''(x) > 0$ for all x in the interval of definition. Therefore, y attains a global minimum at $x = -1$.

35.(a) First, we rewrite the equation as $dy/(y(4-y)) = tdt/3$. Then, using partial fractions, after integration we obtain

$$\left|\frac{y}{y-4}\right| = Ce^{2t^2/3}.$$

From the equation, we see that $y_0 = 0$ implies that $C = 0$, so $y(t) = 0$ for all t. Otherwise, $y(t) > 0$ for all t or $y(t) < 0$ for all t. Therefore, if $y_0 > 0$ and $|y/(y-4)| = Ce^{2t^2/3} \to \infty$, we must have $y \to 4$. On the other hand, if $y_0 < 0$, then $y \to -\infty$ as $t \to \infty$. (In particular, $y \to -\infty$ in finite time.)

(b) For $y_0 = 0.5$, we want to find the time T when the solution first reaches the value 3.98. Using the fact that $|y/(y-4)| = Ce^{2t^2/3}$ combined with the initial condition, we have $C = 1/7$. From this equation, we now need to find T such that $|3.98/.02| = e^{2T^2/3}/7$. Solving this equation, we obtain $T \approx 3.29527$.

36.(a) Rewriting the equation as $y^{-1}(4 - y)^{-1}dy = t(1 + t)^{-1}dt$ and integrating both sides, we have $\ln|y| - \ln|y - 4| = 4t - 4\ln|1 + t| + c$. Therefore, $|y/(y - 4)| = Ce^{4t}/(1 + t)^4 \to \infty$ as $t \to \infty$ which implies $y \to 4$.

(b) The initial condition $y(0) = 2$ implies $C = 1$. Therefore, $y/(y-4) = -e^{4t}/(1+t)^4$. Now we need to find T such that $3.99/-0.01 = -e^{4T}/(1+T)^4$. Solving this equation, we obtain $T \approx 2.84367$.

(c) Using our results from part (b), we note that $y/(y-4) = y_0/(y_0-4)e^{4t}/(1+t)^4$. We want to find the range of initial values y_0 such that $3.99 < y < 4.01$ at time $t = 2$. Substituting $t = 2$ into the equation above, we have $y_0/(y_0-4) = (3/e^2)^4 y(2)/(y(2)-4)$. Since the function $y/(y-4)$ is monotone, we need only find the values y_0 satisfying $y_0/(y_0-4) = -399(3/e^2)^4$ and $y_0/(y_0-4) = 401(3/e^2)^4$. The solutions are $y_0 \approx 3.6622$ and $y_0 \approx 4.4042$. Therefore, we need $3.6622 < y_0 < 4.4042$.

37. We can write the equation as

$$\left(\frac{cy + d}{ay + b}\right) dy = dx,$$

which gives

$$\left(\frac{cy}{ay + b} + \frac{d}{ay + b}\right) dy = dx.$$

Now we want to rewrite these so in the first component we can simplify by $ay+b$:

$$\frac{cy}{ay+b} = \frac{\frac{1}{a}cay}{ay+b} = \frac{\frac{1}{a}(cay+bc) - bc/a}{ay+b} = \frac{1}{a}c - \frac{\frac{bc}{a}}{ay+b},$$

so we obtain

$$\left(\frac{c}{a} - \frac{bc}{a^2y+ab} + \frac{d}{ay+b}\right) dy = dx.$$

Then integrating both sides, we have

$$\frac{c}{a}y - \frac{bc}{a^2}\ln|a^2y+ab| + \frac{d}{a}\ln|ay+b| = x + C.$$

Simplifying, we have

$$\frac{c}{a}y - \frac{bc}{a^2}\ln|a| - \frac{bc}{a^2}\ln|ay+b| + \frac{d}{a}\ln|ay+b| = x + C,$$

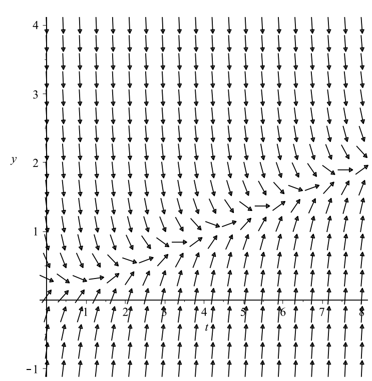
which implies that

$$\frac{c}{a}y + \left(\frac{ad-bc}{a^2}\right)\ln|ay+b| = x + C.$$

Note, in this calculation, since $\frac{bc}{a^2}\ln|a|$ is just a constant, we included it with the arbitrary constant C. This solution will exist as long as $a \neq 0$ and $ay+b \neq 0$.

2.2 Linear Equations: Method of Integrating Factors

1.(a)



(b) All solutions seem to converge to an increasing function as $t \to \infty$.

(c) The integrating factor is $\mu(t) = e^{4t}$. Then

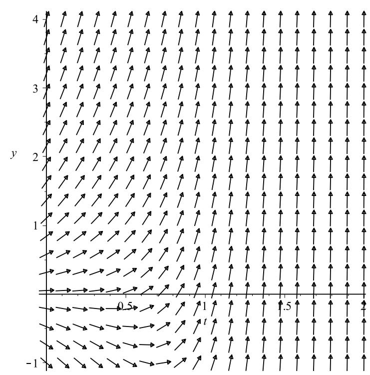
$$e^{4t}y' + 4e^{4t}y = e^{4t}(t + e^{-2t})$$

implies that

$$(e^{4t}y)' = te^{4t} + e^{2t},$$

thus
$$e^{4t}y = \int (te^{4t} + e^{2t})\, dt = \frac{1}{4}te^{4t} - \frac{1}{16}e^{4t} + \frac{1}{2}e^{2t} + c,$$
and then
$$y = ce^{-4t} + \frac{1}{2}e^{-2t} + \frac{t}{4} - \frac{1}{16}.$$
We conclude that y is asymptotic to the linear function $g(t) = t/4 - 1/16$ as $t \to \infty$.

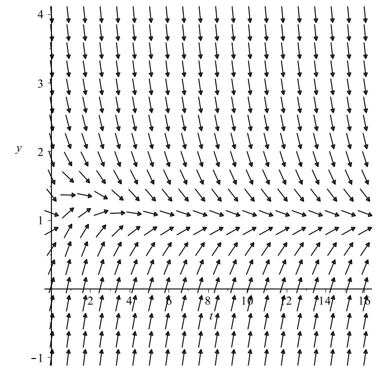
2.(a)



(b) All slopes eventually become positive, so all solutions will eventually increase without bound.

(c) The integrating factor is $\mu(t) = e^{-2t}$. Then
$$e^{-2t}y' - 2e^{-2t}y = e^{-2t}(t^2e^{2t})$$
implies
$$(e^{-2t}y)' = t^2,$$
thus
$$e^{-2t}y = \int t^2\, dt = \frac{t^3}{3} + c,$$
and then
$$y = \frac{t^3}{3}e^{2t} + ce^{2t}.$$
We conclude that y increases exponentially as $t \to \infty$.

3.(a)



(b) All solutions appear to converge to the function $g(t) = 1$.

(c) The integrating factor is $\mu(t) = e^t$. Therefore, $e^t y' + e^t y = t + e^t$, thus $(e^t y)' = t + e^t$, so

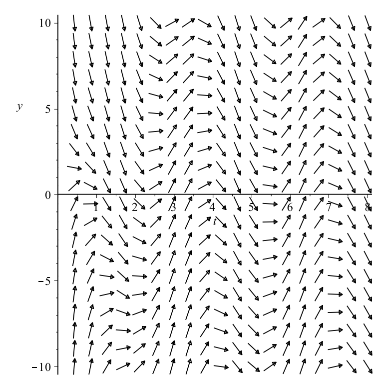
$$e^t y = \int (t + e^t)\, dt = \frac{t^2}{2} + e^t + c,$$

and then

$$y = \frac{t^2}{2} e^{-t} + 1 + ce^{-t}.$$

Therefore, we conclude that $y \to 1$ as $t \to \infty$.

4.(a)



(b) The solutions eventually become oscillatory.

(c) The integrating factor is $\mu(t) = t$. Therefore, $ty' + y = 5t \cos 2t$ implies $(ty)' = 5t \cos 2t$, thus

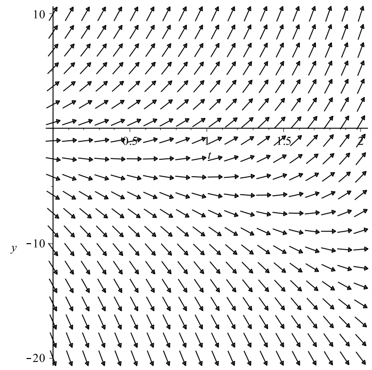
$$ty = \int 5t \cos 2t\, dt = \frac{5}{4} \cos 2t + \frac{5}{2} t \sin 2t + c,$$

and then

$$y = \frac{5 \cos 2t}{4t} + \frac{5 \sin 2t}{2} + \frac{c}{t}.$$

We conclude that y is asymptotic to $g(t) = (5 \sin 2t)/2$ as $t \to \infty$.

5.(a)



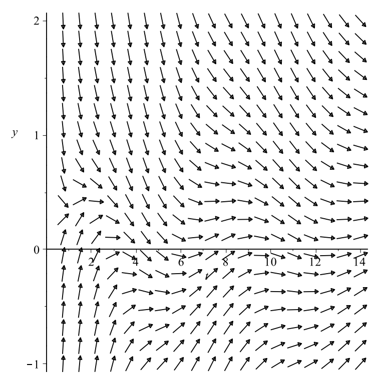
(b) Some of the solutions increase without bound, some decrease without bound.

(c) The integrating factor is $\mu(t) = e^{-2t}$. Therefore, $e^{-2t}y' - 2e^{-2t}y = 3e^{-t}$, which implies $(e^{-2t}y)' = 3e^{-t}$, thus

$$e^{-2t}y = \int 3e^{-t}\,dt = -3e^{-t} + c,$$

and then $y = -3e^{t} + ce^{2t}$. We conclude that y increases or decreases exponentially as $t \to \infty$.

6.(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $g(t) = 0$.

(c) The integrating factor is $\mu(t) = t^2$. Therefore, $t^2y' + 2ty = t\sin t$, thus $(t^2y)' = t\sin t$, so

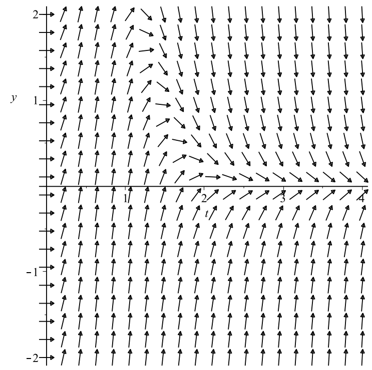
$$t^2y = \int t\sin t\,dt = \sin t - t\cos t + c,$$

and then

$$y = \frac{\sin t - t\cos t + c}{t^2}.$$

We conclude that $y \to 0$ as $t \to \infty$.

7.(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $g(t) = 0$.

(c) The integrating factor is $\mu(t) = e^{t^2}$. Therefore,

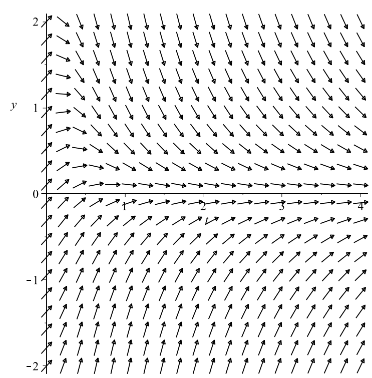
$$(e^{t^2} y)' = e^{t^2} y' + 2tye^{t^2} = 16t,$$

thus

$$e^{t^2} y = \int 16t \, dt = 8t^2 + c,$$

and then $y(t) = 8t^2 e^{-t^2} + ce^{-t^2}$. We conclude that $y \to 0$ as $t \to \infty$.

8.(a)



(b) For $t > 0$, all solutions seem to eventually converge to the function $g(t) = 0$.

(c) The integrating factor is $\mu(t) = (1 + t^2)^2$. Then

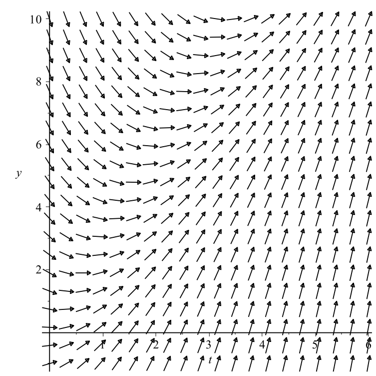
$$(1 + t^2)^2 y' + 4t(1 + t^2)y = \frac{1}{1 + t^2},$$

so

$$((1 + t^2)^2 y) = \int \frac{1}{1 + t^2} \, dt,$$

and then $y = (\arctan t + c)/(1 + t^2)^2$. We conclude that $y \to 0$ as $t \to \infty$.

9.(a)



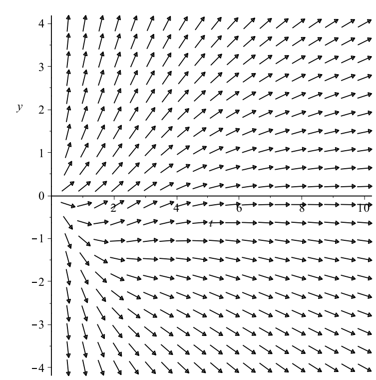
(b) All solutions increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/2}$. Therefore, $2e^{t/2}y' + e^{t/2}y = 3te^{t/2}$, thus

$$2e^{t/2}y = \int 3te^{t/2}\, dt = 6te^{t/2} - 12e^{t/2} + c,$$

and then $y = 3t - 6 + ce^{-t/2}$. We conclude that y is asymptotic to $g(t) = 3t - 6$ as $t \to \infty$.

10.(a)



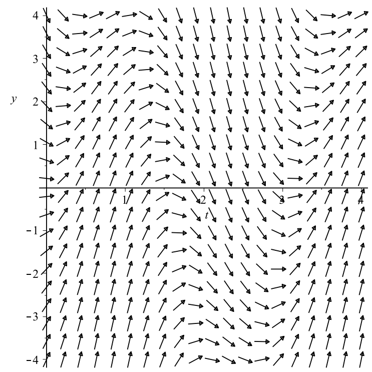
(b) For $y > 0$, the slopes are all positive, and, therefore, the corresponding solutions increase without bound. For $y < 0$ almost all solutions have negative slope and therefore decrease without bound.

(c) By dividing the equation by t, we see that the integrating factor is $\mu(t) = 1/t$. Therefore, $y'/t - y/t^2 = t^2e^{-t}$, thus $(y/t)' = t^2e^{-t}$, so

$$\frac{y}{t} = \int t^2e^{-t}\, dt = -t^2e^{-t} - 2te^{-t} - 2e^{-t} + c,$$

and then $y = -t^3e^{-t} - 2t^2e^{-t} - 2e^{-t} + ct$. We conclude that $y \to \infty$ if $c > 0$, $y \to -\infty$ if $c < 0$ and $y \to 0$ if $c = 0$.

11.(a)



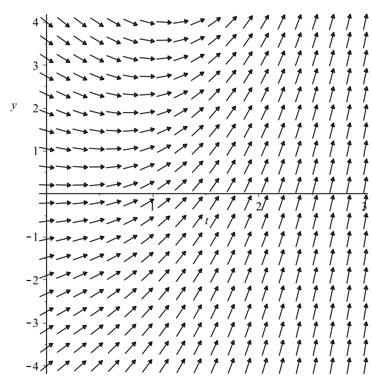
(b) All solutions appear to converge to an oscillatory function.

(c) The integrating factor is $\mu(t) = e^t$. Therefore, $e^t y' + e^t y = 5e^t \sin 2t$, thus $(e^t y)' = 5e^t \sin 2t$, which gives

$$e^t y = \int 5e^t \sin 2t \, dt = -2e^t \cos 2t + e^t \sin 2t + c,$$

and then $y = -2\cos 2t + \sin 2t + ce^{-t}$. We conclude that y is asymptotic to $g(t) = \sin 2t - 2\cos 2t$ as $t \to \infty$.

12.(a)



(b) All solutions increase without bound.

(c) The integrating factor is $\mu(t) = e^{t/2}$. Therefore, $2e^{t/2}y' + e^{t/2}y = 3t^2 e^{t/2}$, thus $(2e^{t/2}y)' = 3t^2 e^{t/2}$, so

$$2e^{t/2}y = \int 3t^2 e^{t/2} \, dt = 6t^2 e^{t/2} - 24te^{t/2} + 48e^{t/2} + c,$$

and then $y = 3t^2 - 12t + 24 + ce^{-t/2}$. We conclude that y is asymptotic to $g(t) = 3t^2 - 12t + 24$ as $t \to \infty$.

13. The integrating factor is $\mu(t) = e^{-t}$. Therefore, $(e^{-t}y)' = 2te^t$, thus

$$y = e^t \int 2te^t \, dt = 2te^{2t} - 2e^{2t} + ce^t.$$

The initial condition $y(0) = 1$ implies $-2 + c = 1$. Therefore, $c = 3$ and $y = 3e^t + 2(t-1)e^{2t}$.

14. The integrating factor is $\mu(t) = e^{2t}$. Therefore, $(e^{2t}y)' = t$, thus

$$y = e^{-2t}\int t\,dt = \frac{t^2}{2}e^{-2t} + ce^{-2t}.$$

The initial condition $y(1) = 0$ implies $e^{-2}/2 + ce^{-2} = 0$. Therefore, $c = -1/2$, and $y = (t^2-1)e^{-2t}/2$.

15. Dividing the equation by t, we see that the integrating factor is $\mu(t) = t^4$. Therefore, $(t^4y)' = t^5 - t^4 + t^3$, thus

$$y = t^{-4}\int (t^5 - t^4 + t^3)\,dt = \frac{t^2}{6} - \frac{t}{5} + \frac{1}{4} + \frac{c}{t^4}.$$

The initial condition $y(1) = 1/4$ implies $c = 1/30$, and $y = (10t^6 - 12t^5 + 15t^4 + 2)/60t^4$.

16. The integrating factor is $\mu(t) = t^2$. Therefore, $(t^2y)' = \cos t$, thus

$$y = t^{-2}\int \cos t\,dt = t^{-2}(\sin t + c).$$

The initial condition $y(\pi) = 0$ implies $c = 0$ and $y = (\sin t)/t^2$.

17. The integrating factor is $\mu(t) = e^{-2t}$. Therefore, $(e^{-2t}y)' = 1$, thus

$$y = e^{2t}\int 1\,dt = e^{2t}(t + c).$$

The initial condition $y(0) = 2$ implies $c = 2$ and $y = (t+2)e^{2t}$.

18. After dividing by t, we see that the integrating factor is $\mu(t) = t^2$. Therefore, $(t^2y)' = t\sin t$, thus

$$y = t^{-2}\int t\sin t\,dt = t^{-2}(\sin t - t\cos t + c).$$

The initial condition $y(\pi/2) = 3$ implies $c = 3(\pi^2/4) - 1$ and $y = t^{-2}(3(\pi^2/4) - 1 - t\cos t + \sin t)$.

19. After dividing by t^3, we see that the integrating factor is $\mu(t) = t^4$. Therefore, $(t^4y)' = te^{-t}$, thus

$$y = t^{-4}\int te^{-t}\,dt = t^{-4}(-te^{-t} - e^{-t} + c).$$

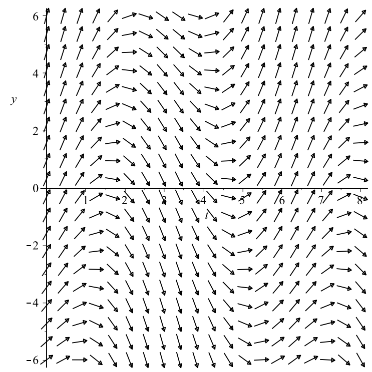
The initial condition $y(-1) = 0$ implies $c = 0$ and $y = -(1+t)e^{-t}/t^4$.

20. After dividing by t, we see that the integrating factor is $\mu(t) = te^t$. Therefore, $(te^ty)' = te^t$, thus

$$y = t^{-1}e^{-t}\int te^t\,dt = t^{-1}e^{-t}(te^t - e^t + c) = t^{-1}(t - 1 + ce^{-t}).$$

The initial condition $y(\ln 2) = 1$ implies $c = 2$ and $y = (t - 1 + 2e^{-t})/t$.

21.(a)

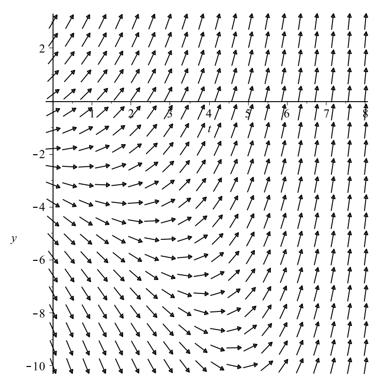


The solutions appear to diverge from an oscillatory solution. It appears that $a_0 \approx -1$. For $a > -1$, the solutions increase without bound. For $a < -1$, the solutions decrease without bound.

(b) The integrating factor is $\mu(t) = e^{-t/3}$. From this, we get the equation $y'e^{-t/3} - ye^{-t/3}/3 = (ye^{-t/3})' = 3e^{-t/3}\cos t$. After integration, $y(t) = (27\sin t - 9\cos t)/10 + ce^{t/3}$, where (using the initial condition) $c = a + 9/10$. The solution will be sinusoidal as long as $c = 0$. Therefore, $a_0 = -9/10$.

(c) y oscillates for $a = a_0$.

22.(a)

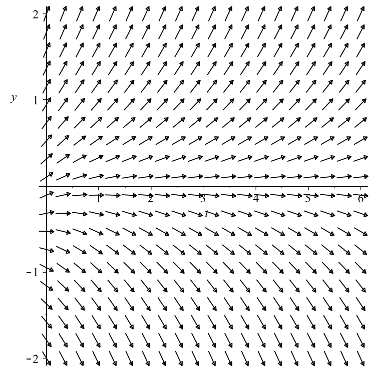


All solutions eventually increase or decrease without bound. The value a_0 appears to be approximately $a_0 = -3$.

(b) The integrating factor is $\mu(t) = e^{-t/2}$. From this, we get the equation $y'e^{-t/2} - ye^{-t/2}/2 = (ye^{-t/2})' = e^{-t/6}/2$. After integration, the general solution is $y(t) = -3e^{t/3} + ce^{t/2}$. The initial condition $y(0) = a$ implies $y = -3e^{t/3} + (a+3)e^{t/2}$. The solution will behave like $(a+3)e^{t/2}$. Therefore, $a_0 = -3$.

(c) $y \to -\infty$ for $a = a_0$.

23.(a)

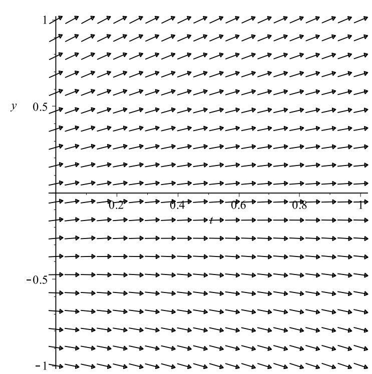


Solutions eventually increase or decrease without bound, depending on the initial value a_0. It appears that $a_0 \approx -1/8$.

(b) Dividing the equation by 3, we see that the integrating factor is $\mu(t) = e^{-2t/3}$. From this, we get the equation $y'e^{-2t/3} - 2ye^{-2t/3}/3 = (ye^{-2t/3})' = 2e^{-\pi t/2-2t/3}/3$. After integration, the general solution is $y(t) = e^{2t/3}(-(2/3)e^{-\pi t/2-2t/3}(1/(\pi/2+2/3)) + c)$. Using the initial condition, we get $y = ((2+a(3\pi+4))e^{2t/3} - 2e^{-\pi t/2})/(3\pi+4)$. The solution will eventually behave like $(2+a(3\pi+4))e^{2t/3}/(3\pi+4)$. Therefore, $a_0 = -2/(3\pi+4)$.

(c) $y \to 0$ for $a = a_0$.

24.(a)

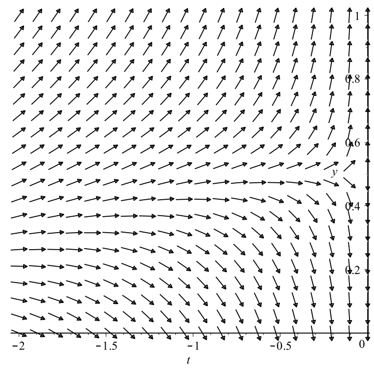


It appears that $a_0 \approx .4$. As $t \to 0$, solutions increase without bound if $y > a_0$ and decrease without bound if $y < a_0$.

(b) The integrating factor is $\mu(t) = te^t$. After multiplication by μ, we obtain the equation $te^ty' + (t+1)e^ty = (te^ty)' = 2t$, so after integration, we get that the general solution is $y = te^{-t} + ce^{-t}/t$. The initial condition $y(1) = a$ implies $y = te^{-t} + (ea-1)e^{-t}/t$. As $t \to 0$, the solution will behave like $(ea-1)e^{-t}/t$. From this, we see that $a_0 = 1/e$.

(c) $y \to 0$ as $t \to 0$ for $a = a_0$.

25.(a)

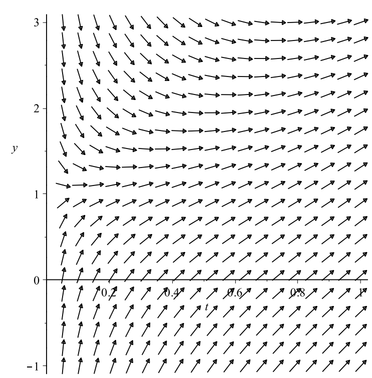


It appears that $a_0 \approx .4$. That is, as $t \to 0$, for $y(-\pi/2) > a_0$, solutions will increase without bound, while solutions will decrease without bound for $y(-\pi/2) < a_0$.

(b) After dividing by t, we see that the integrating factor is $\mu(t) = t^2$. After multiplication by μ, we obtain the equation $t^2y' + 2ty = (t^2y)' = \sin t$, so after integration, we get that the general solution is $y = -\cos t/t^2 + c/t^2$. Using the initial condition, we get the solution $y = -\cos t/t^2 + \pi^2 a/4t^2$. Since $\lim_{t\to 0} \cos t = 1$, solutions will increase without bound if $a > 4/\pi^2$ and decrease without bound if $a < 4/\pi^2$. Therefore, $a_0 = 4/\pi^2$.

(c) For $a_0 = 4/\pi^2$, $y = (1 - \cos t)/t^2 \to 1/2$ as $t \to 0$.

26.(a)



It appears that $a_0 \approx 2$. For $y(1) > a_0$, the solution will increase without bound as $t \to 0$, while the solution will decrease without bound if $y(1) < a_0$.

(b) After dividing by $\sin t$, we see that the integrating factor is $\mu(t) = \sin t$. The equation becomes $(\sin t)y' + (\cos t)y = (y \sin t)' = e^t$, and then after integration, we see that the solution is given by $y = (e^t + c)/\sin t$. Applying our initial condition, we see that our solution is $y = (e^t - e + a \sin 1)/\sin t$. The solution will increase if $1 - e + a \sin 1 > 0$ and decrease if $1 - e + a \sin 1 < 0$. Therefore, we conclude that $a_0 = (e - 1)/\sin 1$.

(c) If $a_0 = (e - 1) \sin 1$, then $y = (e^t - 1)/\sin t$. As $t \to 0$, $y \to 1$.

27. The integrating factor is $\mu(t) = e^{t/2}$. Therefore, the general solution is $y(t) = (4 \cos t + 8 \sin t)/5 + ce^{-t/2}$. Using our initial condition, we have $y(t) = (4 \cos t + 8 \sin t - 9e^{t/2})/5$.

Differentiating, we obtain

$$y' = (-4\sin t + 8\cos t + 4.5e^{-t/2})/5$$
$$y'' = (-4\cos t - 8\sin t - 2.25e^{t/2})/5.$$

Setting $y' = 0$, the first solution is $t_1 \approx 1.3643$, which gives the location of the first stationary point. Since $y''(t_1) < 0$, the first stationary point is a local maximum. The coordinates of the point are approximately $(1.3643, 0.8201)$.

28. The integrating factor is $\mu(t) = e^{4t/3}$. The general solution of the differential equation is $y(t) = (57 - 12t)/64 + ce^{-4t/3}$. Using the initial condition, we have $y(t) = (57 - 12t)/64 + e^{-4t/3}(y_0 - 57/64)$. This function is asymptotic to the linear function $g(t) = (57 - 12t)/64$ as $t \to \infty$. We will get a maximum value for this function when $y' = 0$, if $y'' < 0$ there. Let us identify the critical points first: $y'(t) = -3/16 + 19e^{-4t/3}/16 - 4y_0e^{-4t/3}y_0/3$; thus setting $y'(t) = 0$, the only solution is $t_1 = \frac{3}{4}\ln((57 - 64y_0)/9)$. Substituting into the solution, the respective value at this critical point is $y(t_1) = \frac{3}{4} - \frac{9}{64}\ln((57 - 64y_0)/9)$. Setting this result equal to zero, we obtain the required initial value $y_0 = (57 - 9e^{16/3})/64 = -28.237$. We can check that the second derivative is indeed negative at this point, thus $y(t)$ has a maximum there and it does not cross the t-axis.

29.(a) The integrating factor is $\mu(t) = e^{t/4}$. The general solution is $y(t) = 12 + (8\cos 2t + 64\sin 2t)/65 + ce^{-t/4}$. Applying the initial condition $y(0) = 0$, we arrive at the specific solution $y(t) = 12 + (8\cos 2t + 64\sin 2t - 788e^{-t/4})/65$. As $t \to \infty$, the solution oscillates about the line $y = 12$.

(b) To find the value of t for which the solution first intersects the line $y = 12$, we need to solve the equation $8\cos 2t + 64\sin 2t - 788e^{-t/4} = 0$. The value of t is approximately 10.0658.

30. The integrating factor is $\mu(t) = e^{-t}$. The general solution is $y(t) = -1 - \frac{3}{2}\cos t - \frac{3}{2}\sin t + ce^t$. In order for the solution to remain finite as $t \to \infty$, we need $c = 0$. Therefore, y_0 must satisfy $y_0 = -1 - 3/2 = -5/2$.

31. The integrating factor is $\mu(t) = e^{-3t/2}$ and the general solution of the equation is $y(t) = -2t - 4/3 - 4e^t + ce^{3t/2}$. The initial condition implies $y(t) = -2t - 4/3 - 4e^t + (y_0 + 16/3)e^{3t/2}$. The solution will behave like $(y_0 + 16/3)e^{3t/2}$ (for $y_0 \neq -16/3$). For $y_0 > -16/3$, the solutions will increase without bound, while for $y_0 < -16/3$, the solutions will decrease without bound. If $y_0 = -16/3$, the solution will decrease without bound as the solution will be $-2t - 4/3 - 4e^t$.

32. By equation (42), we see that the general solution is given by

$$y = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + ce^{-t^2/4}.$$

Applying L'Hôpital's rule,

$$\lim_{t\to\infty}\frac{\int_0^t e^{s^2/4}\,ds}{e^{t^2/4}} = \lim_{t\to\infty}\frac{e^{t^2/4}}{(t/2)e^{t^2/4}} = 0.$$

Therefore, $y \to 0$ as $t \to \infty$.

33. The integrating factor is $\mu(t) = e^{at}$. First consider the case $a \neq \lambda$. Multiplying the equation by e^{at}, we have $(e^{at}y)' = be^{(a-\lambda)t}$, which implies

$$y = e^{-at}\int be^{(a-\lambda)t} = e^{-at}\left(\frac{b}{a-\lambda}e^{(a-\lambda)t} + c\right) = \frac{b}{a-\lambda}e^{-\lambda t} + ce^{-at}.$$

Since a, λ are assumed to be positive, we see that $y \to 0$ as $t \to \infty$. Now if $a = \lambda$ above, then we have $(e^{at}y)' = b$, which implies $y = e^{-at}(bt + c)$ and similarly $y \to 0$ as $t \to \infty$.

34. We notice that $y(t) = ce^{-t} + 3$ approaches 3 as $t \to \infty$. We just need to find a first order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 3$. Then $f' + f = 0$ and $g' + g = 3$. Therefore, $y(t) = ce^{-t} + 3$ satisfies the equation $y' + y = 3$. That is, the equation $y' + y = 3$ has the desired properties.

35. We notice that $y(t) = ce^{-t} + 4 - t$ approaches $4 - t$ as $t \to \infty$. We just need to find a first order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 4 - t$. Then $f' + f = 0$ and $g' + g = -1 + 4 - t = 3 - t$. Therefore, $y(t) = ce^{-t} + 4 - t$ satisfies the equation $y' + y = 3 - t$. That is, the equation $y' + y = 3 - t$ has the desired properties.

36. We notice that $y(t) = ce^{-t} + 2t - 5$ approaches $2t - 5$ as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 2t - 5$. Then $f' + f = 0$ and $g' + g = 2 + 2t - 5 = 2t - 3$. Therefore, $y(t) = ce^{-t} + 2t - 5$ satisfies the equation $y' + y = 2t - 3$. That is, the equation $y' + y = 2t - 3$ has the desired properties.

37. We notice that $y(t) = ce^{-t} + 2 - t^2$ approaches $2 - t^2$ as $t \to \infty$. We just need to find a first-order linear differential equation having that solution. We notice that if $y(t) = f + g$, then $y' + y = f' + f + g' + g$. Here, let $f = ce^{-t}$ and $g(t) = 2 - t^2$. Then $f' + f = 0$ and $g' + g = -2t + 2 - t^2 = 2 - 2t - t^2$. Therefore, $y(t) = ce^{-t} + 2 - t^2$ satisfies the equation $y' + y = 2 - 2t - t^2$. That is, the equation $y' + y = 2 - 2t - t^2$ has the desired properties.

38. Multiplying the equation by $e^{a(t-t_0)}$, we have $e^{a(t-t_0)}y + ae^{a(t-t_0)}y = e^{a(t-t_0)}g(t)$, so $(e^{a(t-t_0)}y)' = e^{a(t-t_0)}g(t)$ and then

$$y(t) = \int_{t_0}^{t} e^{-a(t-s)}g(s)\,ds + e^{-a(t-t_0)}y_0.$$

Assuming $g(t) \to g_0$ as $t \to \infty$, and using L'Hôpital's rule,

$$\lim_{t\to\infty}\int_{t_0}^{t} e^{-a(t-s)}g(s)\,ds = \lim_{t\to\infty}\frac{\int_{t_0}^{t} e^{as}g(s)\,ds}{e^{at}} = \lim_{t\to\infty}\frac{e^{at}g(t)}{ae^{at}} = \frac{g_0}{a}.$$

For an example, let $g(t) = 2 + e^{-t}$. Assume $a \neq 1$. Let us look for a solution of the form $y = ce^{-at} + Ae^{-t} + B$. Substituting a function of this form into the differential equation leads to the equation $(-A + aA)e^{-t} + aB = 2 + e^{-t}$, thus $-A + aA = 1$ and $aB = 2$. Therefore, $A = 1/(a-1)$, $B = 2/a$ and $y = ce^{-at} + e^{-t}/(a-1) + 2/a$. The initial condition $y(0) = y_0$ implies $y(t) = (y_0 - 1/(a-1) - 2/a)e^{-at} + e^{-t}/(a-1) + 2/a \to 2/a$ as $t \to \infty$.

39.(a) The integrating factor is $e^{\int p(t)\,dt}$. Multiplying by the integrating factor, we have

$$e^{\int p(t)\,dt}y' + e^{\int p(t)\,dt}p(t)y = 0.$$

Therefore,

$$\left(e^{\int p(t)\,dt}y\right)' = 0,$$

which implies

$$y(t) = Ae^{-\int p(t)\,dt}$$

is the general solution.

(b) Let $y = A(t)e^{-\int p(t)\,dt}$. Then in order for y to satisfy the desired equation, we need

$$A'(t)e^{-\int p(t)\,dt} - A(t)p(t)e^{-\int p(t)\,dt} + A(t)p(t)e^{-\int p(t)\,dt} = g(t).$$

That is, we need

$$A'(t) = g(t)e^{\int p(t)\,dt}.$$

(c) From equation (iv), we see that

$$A(t) = \int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C.$$

Therefore,

$$y(t) = e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right).$$

40. Here, $p(t) = -6$ and $g(t) = t^6e^{6t}$. The general solution is given by

$$\begin{aligned} y(t) &= e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right) = e^{\int 6\,dt}\left(\int_0^t \tau^6e^{6\tau}e^{\int -6\,d\tau}\,d\tau + C\right) \\ &= e^{6t}\left(\int_0^t \tau^6\,d\tau + C\right) = e^{6t}\left(\frac{t^7}{7} + C\right). \end{aligned}$$

41. Here, $p(t) = 1/t$ and $g(t) = 3\cos 2t$. The general solution is given by

$$\begin{aligned} y(t) &= e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right) = e^{-\int \frac{1}{t}\,dt}\left(\int_0^t 3\cos 2\tau\, e^{\int \frac{1}{\tau}\,d\tau}\,d\tau + C\right) \\ &= \frac{1}{t}\left(\int_0^t 3\tau\cos 2\tau\,d\tau + C\right) = \frac{1}{t}\left(\frac{3}{4}\cos 2t + \frac{3}{2}t\sin 2t + C\right). \end{aligned}$$

42. Here, $p(t) = 2/t$ and $g(t) = \sin t/t$. The general solution is given by

$$\begin{aligned} y(t) &= e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right) = e^{-\int \frac{2}{t}\,dt}\left(\int_0^t \frac{\sin\tau}{\tau}e^{\int \frac{2}{\tau}\,d\tau}\,d\tau + C\right) \\ &= \frac{1}{t^2}\left(\int_0^t \frac{\sin\tau}{\tau}\tau^2\,d\tau + C\right) = \frac{1}{t^2}\left(\int_0^t \tau\sin\tau\,d\tau + C\right) = \frac{1}{t^2}\left(\sin t - t\cos t + C\right). \end{aligned}$$

43. Here, $p(t) = 1/2$ and $g(t) = 3t^2/2$. The general solution is given by

$$\begin{aligned} y(t) &= e^{-\int p(t)\,dt}\left(\int_0^t g(\tau)e^{\int p(\tau)\,d\tau}\,d\tau + C\right) = e^{-\int \frac{1}{2}\,dt}\left(\int_0^t \frac{3\tau^2}{2}e^{\int \frac{1}{2}\,d\tau}\,d\tau + C\right) \\ &= e^{-t/2}\left(\int_0^t \frac{3\tau^2}{2}e^{\tau/2}\,d\tau + C\right) = e^{-t/2}\left(3t^2e^{t/2} - 12te^{t/2} + 24e^{t/2} + C\right) \\ &= 3t^2 - 12t + 24 + Ce^{-t/2}. \end{aligned}$$

2.3 Modeling with First Order Equations

1. Let $Q(t)$ be the quantity of dye in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in} \; - \; \text{rate out}.$$

Here, fresh water is flowing in. Therefore, no dye is coming in. The dye is flowing out at the rate of $(Q/150)$ grams/liters $\cdot$ 3 liters/minute $= (Q/50)$ grams/minute. Therefore,

$$\frac{dQ}{dt} = -\frac{Q}{50}.$$

The solution of this equation is $Q(t) = Ce^{-t/50}$. Since $Q(0) = 450$ grams, $C = 450$. We need to find the time T when the amount of dye present is 2% of what it is initially. That is, we need to find the time T when $Q(T) = 9$ grams. Solving the equation $9 = 450e^{-T/50}$, we conclude that $T = 50\ln(50) \approx 195.6$ minutes.

2. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in} \; - \; \text{rate out}.$$

Here, water containing γ grams/liter of salt is flowing in at a rate of 4 liters/minute. The salt is flowing out at the rate of $(Q/200)$ grams/liter $\cdot$ 4 liters/minute $= (Q/50)$ grams/minute. Therefore,

$$\frac{dQ}{dt} = 4\gamma - \frac{Q}{50}.$$

The solution of this equation is $Q(t) = 200\gamma + Ce^{-t/50}$. Since $Q(0) = 0$ grams, $C = -200\gamma$. Therefore, $Q(t) = 200\gamma(1 - e^{-t/50})$. As $t \to \infty$, $Q(t) \to 200\gamma$.

3. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in} \; - \; \text{rate out}.$$

Here, water containing 1/4 lb/gallon of salt is flowing in at a rate of 4 gallons/minute. The salt is flowing out at the rate of $(Q/160)$ lb/gallon $\cdot$ 4 gallons/minute $= (Q/40)$ lb/minute. Therefore,

$$\frac{dQ}{dt} = 1 - \frac{Q}{40}.$$

The solution of this equation is $Q(t) = 40 + Ce^{-t/40}$. Since $Q(0) = 0$ grams, $C = -40$. Therefore, $Q(t) = 40(1 - e^{-t/40})$ for $0 \leq t \leq 8$ minutes. After 8 minutes, the amount of salt in the tank is $Q(8) = 40(1 - e^{-1/5}) \approx 7.25$ lbs. Starting at that time (and resetting the time variable), the new equation for dQ/dt is given by

$$\frac{dQ}{dt} = -\frac{3Q}{80},$$

since fresh water is being added. The solution of this equation is $Q(t) = Ce^{-3t/80}$. Since we are now starting with 7.25 lbs of salt, $Q(0) = 7.25 = C$. Therefore, $Q(t) = 7.25e^{-3t/80}$. After 8 minutes, $Q(8) = 7.25e^{-3/10} \approx 5.37$ lbs.

4. Let $Q(t)$ be the quantity of salt in the tank. We know that

$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out.}$$

Here, water containing 1 lb/gallon of salt is flowing in at a rate of 3 gallons/minute. The salt is flowing out at the rate of $(Q/(200+t))$ lb/gallon $\cdot$ 2 gallons/minute $= 2Q/(200+t)$ lb/minute. Therefore,

$$\frac{dQ}{dt} = 3 - \frac{2Q}{200+t}.$$

This is a linear equation with integrating factor $\mu(t) = (200+t)^2$. The solution of this equation is $Q(t) = 200 + t + C(200+t)^{-2}$. Since $Q(0) = 100$ lbs, $C = -4{,}000{,}000$. Therefore, $Q(t) = 200 + t - (100(200)^2/(200+t)^2)$. Since the tank has a net gain of 1 gallon of water every minute, the tank will reach its capacity after 300 minutes. When $t = 300$, we see that $Q(300) = 484$ lbs. Therefore, the concentration of salt when it is on the point of overflowing is 121/125 lbs/gallon. The concentration of salt is given by $Q(t)/(200+t)$ (since t gallons of water are added every t minutes). Using the equation for Q above, we see that if the tank had infinite capacity, the concentration would approach 1 lb/gal as $t \to \infty$.

5.(a) Let $Q(t)$ be the quantity of salt in the tank. We know that

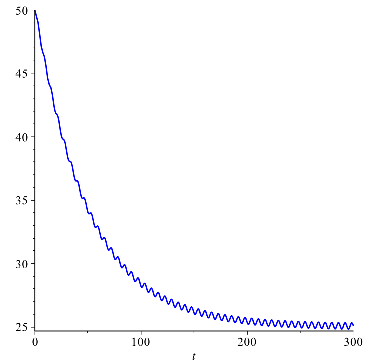
$$\frac{dQ}{dt} = \text{rate in } - \text{ rate out.}$$

Here, water containing $\frac{1}{4}\left(1 + \frac{1}{2}\sin t\right)$ oz/gallon of salt is flowing in at a rate of 2 gal/minute. The salt is flowing out at the rate of $(Q/100)$ oz/gallon$\cdot$2 gallons/minute $= (Q/50)$ oz/minute. Therefore,

$$\frac{dQ}{dt} = \frac{1}{2} + \frac{1}{4}\sin t - \frac{Q}{50}.$$

This is a linear equation with integrating factor $\mu(t) = e^{t/50}$. The solution of this equation is $Q(t) = (12.5\sin t - 625\cos t + 63150e^{-t/50})/2501 + c$. The initial condition, $Q(0) = 50$ oz implies $C = 25$. Therefore, $Q(t) = 25 + (12.5\sin t - 625\cos t + 63150e^{-t/50})/2501$ oz.

(b)



(c) The amount of salt approaches a steady state, which is an oscillation of amplitude $25\sqrt{2501}/5002 \approx 0.24995$ about a level of 25 oz.

6.(a) Using the Principle of Conservation of Energy, we know that the kinetic energy of a particle after it has fallen from a height h is equal to its potential energy at a height t. Therefore, $mv^2/2 = mgh$. Solving this equation for v, we have $v = \sqrt{2gh}$.

(b) The volumetric outflow rate is (outflow cross-sectional area) $\times$ (outflow velocity): $\alpha a\sqrt{2gh}$. The volume of water in the tank at any instant is:

$$V(h) = \int_0^h A(u)\, du$$

where $A(u)$ is the cross-sectional area of the tank at height u. By the chain rule,

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} = A(h)\frac{dh}{dt}.$$

Therefore,

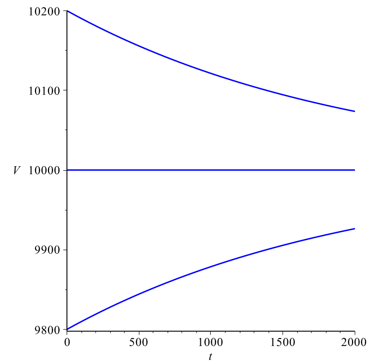
$$\frac{dV}{dt} = A(h)\frac{dh}{dt} = -\alpha a\sqrt{2gh}.$$

(c) The cross-sectional area of the cylinder is $A(h) = \pi(1)^2 = \pi$. The outflow cross-sectional area is $a = \pi(.1)^2 = 0.01\pi$. From part (a), we take $\alpha = 0.6$ for water. Then by part (b), we have

$$\pi\frac{dh}{dt} = -0.006\pi\sqrt{2gh}.$$

This is a separable equation with solution $h(t) = 0.000018gt^2 - 0.006\sqrt{2gh(0)}t + h(0)$. Setting $h(0) = 3$ and $g = 9.8$, we have $h(t) = 0.0001764t^2 - 0.046t + 3$. Then $h(t) = 0$ implies $t \approx 130.4$ seconds.

7.(a) The equation describing the water volume is given by $V' = G - 0.0005V$. Thus the equilibrium volume is $V_e = 2000G$. The figure shows some possible sketches for $V(t)$ when $G = 5$.



(b) The differential equation $V' = G - 0.0005V$ is linear with integrating factor $\mu = e^{t/2000}$. The general solution we obtain is $V(t) = 2000G + ce^{-t/2000}$. If $V(0) = 1.01V_e = 2020G$, then $c = 20G$, and the solution is $V = 2000G + 20Ge^{-t/2000}$.

(c) From part (a), $12000 = V_e = 2000G$, thus $G = 6$ gallons per day.

8.(a) The differential equation describing the rate of change of cholesterol is $c' = r(c_n - c) + k$, where c_n is the body's natural cholesterol level. Thus $c' = -rc + rc_n + k$; this linear equation can be solved by using the integrating factor $\mu = e^{rt}$. We obtain that $c(t) = k/r + c_n + de^{-rt}$; also, $c(0) = k/r + c_n + d$, thus the integration constant is $d = c(0) - k/r - c_n$. The solution is $c(t) = c_n + k/r + (c(0) - c_n - k/r)e^{-rt}$. If $c(0) = 150$, $r = 0.10$, and $c_n = 100$, we obtain that $c(t) = 100 + 10k + (50 - 10k)e^{-t/10}$. Then $c(10) = 100 + 10k + (50 - 10k)e^{-1}$.

(b) The limit of $c(t)$ as $t \to \infty$ is $c_n + k/r = 100 + 25/0.1 = 350$.

(c) We need that $c_n + k/r = 180$, thus $k = 80r = 8$.

9.(a) The differential equation for the amount of poison in the keg is given by $Q' = 5 \cdot 0.5 - 0.5 \cdot Q/500 = 5/2 - Q/1000$. Then using the initial condition $Q(0) = 0$ and the integrating factor $\mu = e^{t/1000}$ we obtain $Q(t) = 2500 - 2500e^{-t/1000}$.

(b) To reach the concentration 0.005 g/L, the amount $Q(T) = 2500(1 - e^{T/1000}) = 2.5$ g. Thus $T = 1000 \ln(1000/999) \approx 1$ minute.

(c) The estimate is 1 minute, because to pour in 2.5 grams of poison without removing the mixture, we have to pour in a half liter of the liquid containing the poison. This takes 1 minute.

10.(a) The equation for S is

$$\frac{dS}{dt} = rS$$

with an initial condition $S(0) = S_0$. The solution of the equation is $S(t) = S_0e^{rt}$. We want to find the time T such that $S(T) = 2S_0$. Our equation becomes $2S_0 = S_0e^{rT}$. Dividing by S_0 and applying the logarithmic function to our equation, we have $rT = \ln(2)$. That is, $T = \ln(2)/r$.

(b) If $r = .08$, then $T = \ln(2)/.08 \approx 8.66$ years.

(c) By part (a), we also know that $r = \ln(2)/T$ where T is the doubling time. If we want the investment to double in $T = 8$ years, then we need $r = \ln(2)/8 \approx 8.66\%$.

(d) For part (b), we get $72/8 = 9$ years. For part (c), we get $72/8 = 9\%$. $\ln(2) \approx 0.693$, or 69.3 for the percentage calculation. A possible reason for choosing 72 is that it has several divisors.

11.(a) The equation for S is given by

$$\frac{dS}{dt} = rS + k.$$

This is a linear equation with solution $S(t) = \frac{k}{r}(e^{rt} - 1)$.

(b) Using the function in part (a), we need to find k so that $S(42) = 1,000,000$ assuming $r = 0.055$. That is, we need to solve

$$1,000,000 = \frac{k}{0.055}(e^{0.055(42)} - 1).$$

The solution of this equation is $k \approx \$6061$.

(c) Now we assume that $k = 4000$ and want to find r. Our equation becomes

$$1,000,000 = \frac{4000}{r}(e^{42r} - 1).$$

The solution of this equation is approximately 6.92%.

12.(a) Let $S(t)$ be the balance due on the loan at time t. To determine the maximum amount the buyer can afford to borrow, we will assume that the buyer will pay \$800 per month. Then

$$\frac{dS}{dt} = 0.09S - 12(800).$$

The solution is given by equation (18), $S(t) = S_0 e^{0.09t} - 106,667(e^{0.09t} - 1)$. If the term of the mortgage is 20 years, then $S(20) = 0$. Therefore, $0 = S_0 e^{0.09(20)} - 106,667(e^{0.09(20)} - 1)$ which implies $S_0 = \$89,034.79$.

(b) Since the homeowner pays \$800 per month for 20 years, he ends up paying a total of \$192,000 for the house. Since the house loan was \$89,034.79, the rest of the amount was interest payments. Therefore, the amount of interest was approximately \$102,965.21.

13.(a) Let $S(t)$ be the balance due on the loan at time t. Taking into account that t is measured in years, we rewrite the monthly payment as $800(1+t/10)$ where t is now in years. The equation for S is given by

$$\frac{dS}{dt} = 0.09S - 12(800)(1 + t/10).$$

This is a linear equation. Its solution is $S(t) = 225185.23 + 10666.67t + ce^{0.09t}$. The initial condition $S(0) = 100,000$ implies $c = -125185.23$. Therefore, the particular solution is $S(t) = 225185.23 + 10666.67t - 125185.23e^{0.09t}$. To find when the loan will be paid, we just need to solve $S(t) = 0$. Solving this equation, we conclude that the loan will be paid off in 11.28 years (135.36 months).

(b) From part (a), we know the general solution is given by $S(t) = 225185.23 + 10666.67t + ce^{0.09t}$. Now we want to find c such that $S(20) = 0$. The solution of this equation is $c = -72486.67$. Therefore, the solution of the equation will be $S(t) = 225185.23 + 10666.67 - 72846.67e^{0.09t}$. Therefore, $S(0) = 225185.23 - 72486.67 = 152,698.56$.

14.(a) If S_0 is the initial balance, then the balance after one month is

$$S_1 = \text{ initial balance + interest - monthly payment } = S_0 + rS_0 - k = (1+r)S_0 - k.$$

Similarly,

$$S_2 = S_1 + rS_1 - k = (1+r)S_1 - k.$$

In general,

$$S_n = (1+r)S_{n-1} - k.$$

(b) $R = 1 + r$ gives $S_n = RS_{n-1} - k$. Therefore,

$$\begin{aligned} S_1 &= RS_0 - k \\ S_2 &= RS_1 - k = R(RS_0 - k) - k = R^2 S_0 - (R+1)k \\ S_3 &= RS_2 - k = R(R^2 S_0 - (R+1)k) - k = R^3 S_0 - (R^2 + R + 1)k. \end{aligned}$$

(c) First we check the base case, $n = 1$. We see that

$$S_1 = RS_0 - k = RS_0 - \left(\frac{R-1}{R-1}\right)k,$$

which implies that that the condition is satisfied for $n = 1$. Then we assume that

$$S_n = R^n S_0 - \frac{R^n - 1}{R-1}k$$

to show that

$$S_{n+1} = R^{n+1} S_0 - \frac{R^{n+1} - 1}{R-1}k.$$

We see that

$$\begin{aligned} S_{n+1} &= RS_n - k \\ &= R\left[R^n S_0 - \frac{R^n - 1}{R-1}k\right] - k \\ &= R^{n+1} S_0 - \left(\frac{R^{n+1} - R}{R-1}\right)k - k \\ &= R^{n+1} S_0 - \left(\frac{R^{n+1} - R}{R-1}\right)k - \left(\frac{R-1}{R-1}\right)k \\ &= R^{n+1} S_0 - \left(\frac{R^{n+1} - R + R - 1}{R-1}\right)k \\ &= R^{n+1} S_0 - \left(\frac{R^{n+1} - 1}{R-1}\right)k. \end{aligned}$$

(d) We are assuming that $S_0 = 20,000$ and $r = 0.08/12$. We need to find k such that $S_{48} = 0$. Our equation becomes

$$S_{48} = R^{48}S_0 - \left(\frac{R^{48}-1}{R-1}\right)k = 0.$$

Therefore,

$$\left(\frac{(1+0.08/12)^{48}-1}{0.08/12}\right)k = \left(1+\frac{0.08}{12}\right)^{48} \cdot 20,000,$$

which implies $k \approx 488.26$, which is very close to the result in Example 2.

15.(a) The general solution is $Q(t) = Q_0e^{-rt}$. If the half-life is 5730, then $Q_0/2 = Q_0e^{-5730r}$ implies $-5730r = \ln(1/2)$. Therefore, $r = 1.2097 \times 10^{-4}$ per year.

(b) Therefore, $Q(t) = Q_0e^{-1.2097\times10^{-4}t}$.

(c) Given that $Q(T) = Q_0/2$, we have the equation $1/2 = e^{-1.2097\times10^{-4}T}$. Solving for T, we have $T = 5,729.91$ years.

16. Let $P(t)$ be the population of mosquitoes at any time t, measured in days. Then

$$\frac{dP}{dt} = rP - 30,000.$$

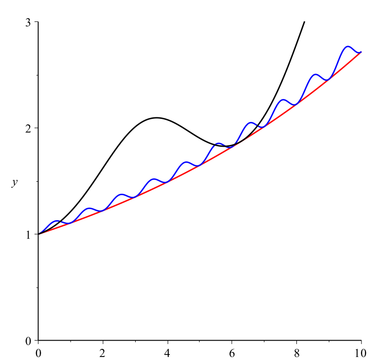
The solution of this linear equation is $P(t) = P_0e^{rt} - \frac{30,000}{r}(e^{rt} - 1)$. In the absence of predators, the equation is $dP_1/dt = rP_1$. The solution of this equation is $P_1(t) = P_0e^{rt}$. Since the population doubles after 7 days, we see that $2P_0 = P_0e^{7r}$. Therefore, $r = \ln(2)/7 = 0.099$ per day. Therefore, the population of mosquitoes at any time t is given by $P(t) = 800,000e^{0.099t} - 303,030(e^{0.099t} - 1)$.

17.(a) The solution of this separable equation is given by $y(t) = \exp(2/10+t/10-2\cos t/10)$. The doubling-time is found by solving the equation $2 = \exp(2/10+t/10-2\cos t/10)$. The solution of this equation is given by $\tau \approx 2.9632$.

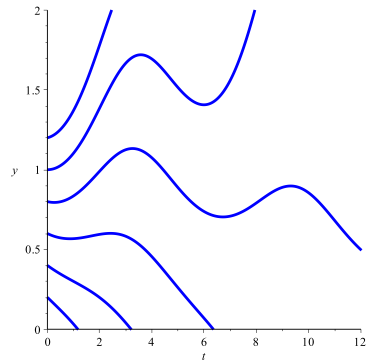
(b) The differential equation will be $dy/dt = y/10$ with solution $y(t) = y(0)e^{t/10}$. The doubling time is found by setting $y(t) = 2y(0)$. In this case, the doubling time is $\tau \approx 6.9315$.

(c) Consider the differential equation $dy/dt = (0.5+\sin(2\pi t))y/5$. This equation is separable with solution $y(t) = \exp((1+\pi t - \cos 2\pi t)/10\pi)$. The doubling time is found by setting $y(t) = 2$. The solution is given by $\tau \approx 6.3804$.

(d)

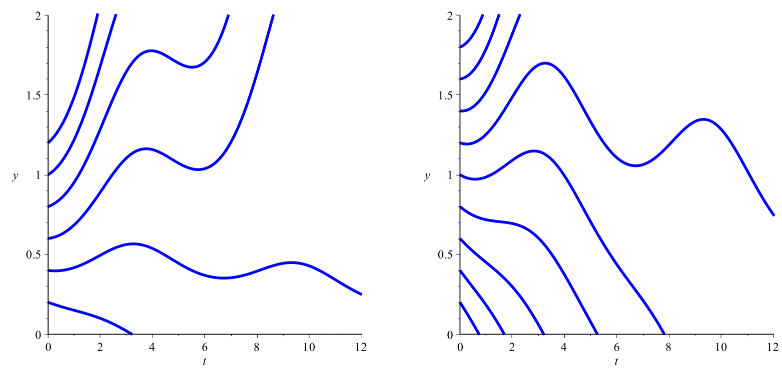


18.(a)



(b) Based on the graph, we estimate that $y_c \approx 0.83$.

(c) We sketch the graphs below for $k = 1/10$ and $k = 3/10$, respectively. Based on these graphs, we estimate that $y_c(1/10) \approx 0.41$ and $y_c(3/10) \approx 1.24$.



(d) From our results from above, we conclude that y_c is a linear function of k.

19. Let $T(t)$ be the temperature of the coffee at time t. The governing equation is given by

$$\frac{dT}{dt} = -k(T - 70).$$

This is a linear equation with solution $T(t) = 70 + ce^{-kt}$. The initial condition $T(0) = 200$ implies $c = 130$. Therefore, $T(t) = 70 + 130e^{-kt}$. Using the fact that $T(1) = 190$, we see that $190 = 70 + 130e^{-k}$ which implies $k = -\ln(12/13) \approx 0.08$ per minute. To find when the temperature reaches 150 degrees, we just need to solve $T(t) = 70 + 130e^{\ln(12/13)t} = 150$. The solution of this equation is $t = \ln(13/8)/\ln(13/12) \approx 6.07$ minutes.

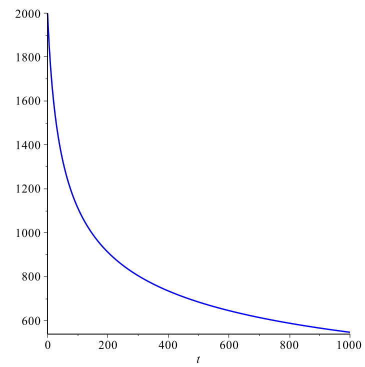
20.(a) The solution of this separable equation is given by

$$u^3 = \frac{u_0^3}{3\alpha u_0^3 t + 1}.$$

Since $u_0 = 2000$, the specific solution is

$$u(t) = \frac{2000}{(6t/125 + 1)^{1/3}}.$$

(b)



(c) We look for τ so that $u(\tau) = 600$. The solution of this equation is $t \approx 750.77$ seconds.

21.(a) The differential equation for Q is

$$\frac{dQ}{dt} = kr + P - \frac{Q(t)}{V}r.$$

Therefore,

$$V\frac{dc}{dt} = kr + P - c(t)r.$$

The solution of this equation is $c(t) = k + P/r + (c_0 - k - P/r)e^{-rt/V}$. Therefore $\lim_{t\to\infty} c(t) = k + P/r$.

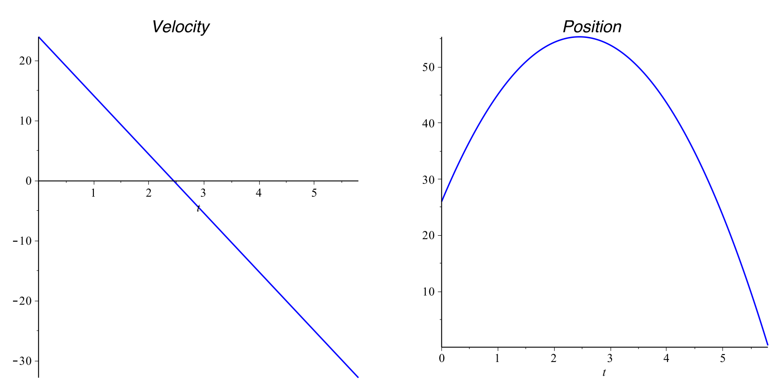
(b) In this case, we will have $c(t) = c_0 e^{-rt/V}$. The reduction times are $T_{50} = \ln(2)V/r$ and $T_{10} = \ln(10)V/r$.

(c) Using the results from part (b), we have: Superior, $T = 430.85$ years; Michigan, $T = 71.4$ years; Erie, $T = 6.05$ years; Ontario, $T = 17.6$ years.

22.(a) Assuming no air resistance, we have $dv/dt = -9.8$. Therefore, $v(t) = -9.8t + v_0 = -9.8t + 24$ and its position at time t is given by $s(t) = -4.9t^2 + 24t + 26$. When the ball reaches its max height, the velocity will be zero. We see that $v(t) = 0$ implies $t = 24/9.8 \approx 2.45$ seconds. When $t = 2.45$, we see that $s(2.45) \approx 55.4$ meters.

(b) Solving $s(t) = -4.9t^2 + 24t + 26 = 0$, we see that $t = 5.81$ seconds.

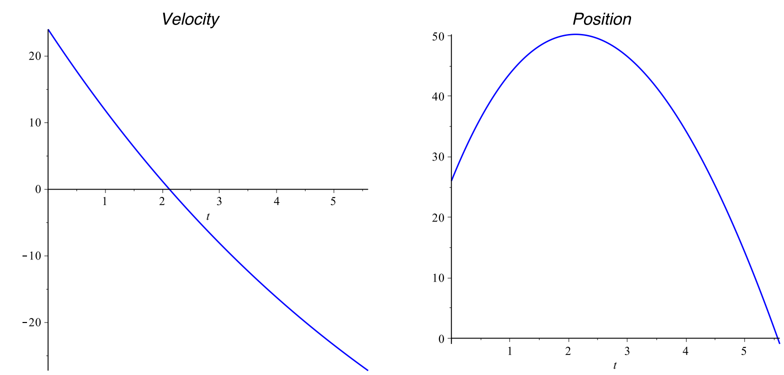
(c)



23.(a) We have $mdv/dt = -v/30 - mg$. Given the conditions from problem 22, we see that the solution is given by $v(t) = -73.5 + 97.5e^{-t/7.5}$. The ball will reach its maximum height when $v(t) = 0$. This occurs at $t = 2.12$ seconds. The height of the ball is given by $s(t) = 757.25 - 73.5t - 731.25e^{-t/7.5}$. When $t = 2.12$ seconds, we have $s(2.12) = 50.24$ meters, the maximum height.

(b) The ball will hit the ground when $s(t) = 0$. This occurs when $t = 5.57$ seconds.

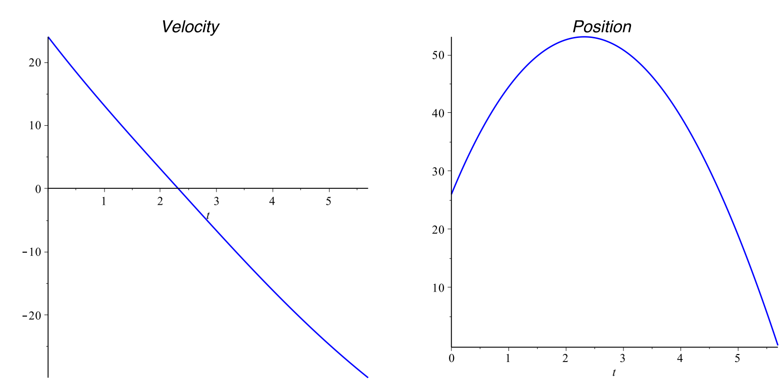
(c)



24.(a) The equation for the upward motion is $mdv/dt = -\mu v^2 - mg$ where $\mu = 1/1325$. Using the data from exercise 22, and the fact that this equation is separable, we see its solution is given by $v(t) = 56.976\tan(0.399 - 0.172t)$. Setting $v(t) = 0$, we see the ball will reach its maximum height at $t = 2.32$ seconds. Integrating $v(t)$, we see the position at time t is given by $s(t) = 331.256\ln(\cos(0.399 - 0.172t)) + 53.1$. Therefore, the maximum height is given by $s(2.32) = 53.1$ meters.

(b) The differential equation for the downward motion is $mdv/dt = \mu v^2 - mg$. The solution of this equation is given by $v(t) = 56.98(1 - e^{0.344t})/(1 + e^{0.344t})$. Integrating $v(t)$, we see that the position is given by $s(t) = 56.98t - 331.279\ln(1 + e^{0.344t}) + 282.725$. Setting $s(t) = 0$, we see that the ball will spend $t = 3.38$ seconds going downward before hitting the ground. Combining this time with the amount of time the ball spends going upward, 2.32 seconds, we conclude that the ball will hit the ground 5.7 seconds after being thrown upward.

(c)



25.(a) Measure the positive direction of motion downward. Then the equation of motion is given by

$$m\frac{dv}{dt} = \begin{cases} -0.75v + mg & 0 < t < 10 \\ -12v + mg & t > 10. \end{cases}$$

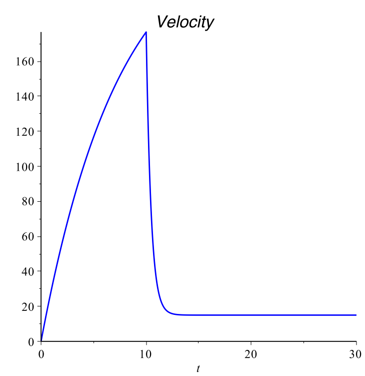
For the first 10 seconds, the equation becomes $dv/dt = -v/7.5 + 32$ which has solution $v(t) = 240(1 - e^{-t/7.5})$. Therefore, $v(10) = 176.7$ ft/s.

(b) Integrating the velocity function from part (a), we see that the height of the skydiver at time t $(0 < t < 10)$ is given by $s(t) = 240t + 1800e^{-t/7.5} - 1800$. Therefore, $s(10) = 1074.5$ feet.

(c) After the parachute opens, the equation for v is given by $dv/dt = -32v/15 + 32$ (as discussed in part (a)). We will reset t to zero. The solution of this differential equation is given by $v(t) = 15 + 161.7e^{-32t/15}$. As $t \to \infty$, $v(t) \to 15$. Therefore, the limiting velocity is $v_l = 15$ feet/second.

(d) Integrating the velocity function from part (c), we see that the height of the sky diver after falling t seconds with his parachute open is given by $s(t) = 15t - 75.8e^{-32t/15} + 1150.3$. To find how long the skydiver is in the air after the parachute opens, we find T such that $s(T) = 0$. Solving this equation, we have $T = 256.6$ seconds.

(e)



26.(a) The equation of motion is given by $dv/dx = -\mu v$.

(b) The speed of the sled satisfies $\ln(v/v_0) = -\mu x$. Therefore, μ must satisfy $\ln(16/160) = -2200\mu$. Therefore, $\mu = \ln(10)/2200$ ft$^{-1} \approx 5.5262$ mi^{-1}.

(c) The solution of $dv/dt = -\mu v^2$ can be expressed as $1/v - 1/v_0 = \mu t$. Using the fact that 1 mi/hour ≈ 1.467 feet/second, the elapsed time is $t \approx 36.64$ seconds.

27.(a) Measure the positive direction of motion upward. The equation of motion is given by $mdv/dt = -kv - mg$. The solution of this equation is given by $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$. Solving $v(t) = 0$, we see that the mass will reach its maximum height $t_m = (m/k)\ln[(mg+kv_0)/mg]$ seconds after being projected upward. Integrating the velocity equation, we see that the position of the mass at this time will be given by the position equation

$$s(t) = -mgt/k + \left[\left(\frac{m}{k}\right)^2 g + \frac{mv_0}{k}\right](1 - e^{-kt/m}).$$

Therefore, the maximum height reached is

$$x_m = s(t_m) = \frac{mv_0}{k} - g\left(\frac{m}{k}\right)^2 \ln\left[\frac{mg+kv_0}{mg}\right].$$

(b) These formulas for t_m and x_m come from the fact that for $\delta << 1$, $\ln(1+\delta) = \delta - \frac{1}{2}\delta^2 + \frac{1}{3}\delta^3 - \frac{1}{4}\delta^4 + \ldots$, which is just Taylor's formula.

(c) Consider the result for t_m in part (b). Multiplying the equation by $\frac{g}{v_0}$, we have

$$\frac{t_m g}{v_0} = \left[1 - \frac{1}{2}\frac{kv_0}{mg} + \frac{1}{3}\left(\frac{kv_0}{mg}\right)^2 - \ldots\right].$$

The units on the left must match the units on the right. Since the units for $t_m g/v_0 = (\mathrm{s}\cdot\mathrm{m}/\mathrm{s}^2)/(\mathrm{m}/\mathrm{s})$, the units cancel. As a result, we can conclude that kv_0/mg is dimensionless.

28.(a) The equation of motion is given by $m\,dv/dt = -kv - mg$. The solution of this equation is given by $v(t) = -mg/k + (v_0 + mg/k)e^{-kt/m}$.

(b) Applying L'Hôpital's rule, as $k \to 0$, we have

$$\lim_{k\to 0} -mg/k + (v_0 + mg/k)e^{-kt/m} = v_0 - gt.$$

(c)

$$\lim_{m\to 0} -mg/k + (v_0 + mg/k)e^{-kt/m} = 0.$$

29.(a) The equation of motion is given by

$$m\frac{dv}{dt} = -6\pi\mu a v + \rho'\frac{4}{3}\pi a^3 g - \rho\frac{4}{3}\pi a^3 g.$$

We can rewrite this equation as

$$v' + \frac{6\pi\mu a}{m}v = \frac{4}{3}\frac{\pi a^3 g}{m}(\rho' - \rho).$$

Multiplying by the integrating factor $e^{6\pi\mu a t/m}$, we have

$$(e^{6\pi\mu a t/m}v)' = \frac{4}{3}\frac{\pi a^3 g}{m}(\rho' - \rho)e^{6\pi\mu a t/m}.$$

Integrating this equation, we have

$$v = e^{-6\pi\mu a t/m}\left[\frac{2a^2 g(\rho' - \rho)}{9\mu}e^{6\pi\mu a t/m} + C\right] = \frac{2a^2 g(\rho' - \rho)}{9\mu} + Ce^{-6\pi\mu a t/m}.$$

Therefore, we conclude that the limiting velocity is $v_L = (2a^2 g(\rho' - \rho))/9\mu$.

(b) By the equation above, we see that the force exerted on the droplet of oil is given by

$$Ee = -6\pi\mu a v + \rho'\frac{4}{3}\pi a^3 g - \rho\frac{4}{3}\pi a^3 g.$$

If $v = 0$, then solving the above equation for e, we have

$$e = \frac{4\pi a^3 g(\rho' - \rho)}{3E}.$$

30.(a) The equation is given by $mdv/dt = -kv - mg$. The solution of this equation is $v(t) = -(mg/k)(1 - e^{-kt/m})$. Integrating, we see that the position function is given by $x(t) = -(mg/k)t + (m/k)^2 g(1 - e^{-kt/m}) + 25$. First, by setting $x(t) = 0$, we see that the ball will hit the ground $t = 2.78$ seconds after it is dropped. Then $v(2.78) = 14.72$ m/second will be the speed when the mass hits the ground.

(b) In terms of displacement, we have $mvdv/dx = -kv + mg$. This equation comes from applying the chain rule: $dv/dt = dv/dx \cdot dx/dt = vdv/dx$. The solution of this differential equation is given by

$$x(v) = -\frac{mv}{k} - \frac{m^2 g}{k^2} \ln\left|\frac{mg - kv}{mg}\right|.$$

Plugging in the given values for k, m, g, we have $x(v) = -2v - 39.2 \ln|0.051v - 1|$. If $v = 8$, then $x(8) = 4.55$ meters.

(c) Using the equation for $x(v)$ above, we set $x(v) = 25$, $v = 8$, $m = 0.4$, $g = 9.8$. Then solving for k, we have $k = 0.49$.

31.(a) The equation of motion is given by $mdv/dt = -GMm/(R + x)^2$. By the chain rule,

$$m\frac{dv}{dx} \cdot \frac{dx}{dt} = -G\frac{Mm}{(R + x)^2}.$$

Therefore,

$$mv\frac{dv}{dx} = -G\frac{Mm}{(R + x)^2}.$$

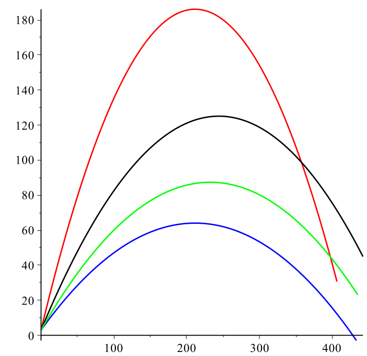
This equation is separable with solution $v^2 = 2GM(R + x)^{-1} + 2gR - 2GM/R$. Here we have used the initial condition $v_0 = \sqrt{2gR}$. From physics, we know that $g = GM/R^2$. Using this substitution, we conclude that $v(x) = \sqrt{2g}\,R/\sqrt{R + x}$.

(b) By part (a), we know that $dx/dt = v(x) = \sqrt{2g}\,R/\sqrt{R + x}$. We want to solve this differential equation with the initial condition $x(0) = 0$. This equation is separable with solution $x(t) = [\frac{3}{2}(\sqrt{2g}Rt + \frac{2}{3}R^{3/2})]^{2/3} - R$. We want to find the time T such that $x(T) = 240,000$. Solving this equation, we conclude that $T \approx 50.6$ hours.

32.(a) $dv/dt = 0$ implies v is constant, and so using the initial condition we see that $v = u \cos A$. $dw/dt = -g$ implies $w = -gt + c$, but also by the initial condition $w = -gt + u \sin A$.

(b) The equation $dx/dt = v = u \cos A$ along with the initial condition implies $x(t) = (u \cos A)t$. The equation $dy/dt = w = -gt + u \sin A$ along with the initial condition implies $y(t) = -gt^2/2 + (u \sin A)t + h$.

(c) Below we have plotted the trajectory of the ball in the cases $\pi/6$, $\pi/5$, $\pi/4$, and $\pi/3$, respectively.



(d) First, let T be the time it takes for the ball to travel L feet horizontally. Using the equation for x, we know that $x(T) = (u\cos A)T = L$ implies $T = L/u\cos A$. Then, when the ball reaches this wall, we need the height of the ball to be at least H feet. That is, we need $y(T) \geq H$. Now $y(t) = -16t^2 + (u\sin A)t + 3$ implies we need

$$y(T) = -16\frac{L^2}{u^2\cos^2 A} + L\tan A + 3 \geq H.$$

(e) If $L = 350$ and $H = 10$, then our inequality becomes

$$-\frac{1,960,000}{u^2\cos^2 A} + 350\tan A + 3 \geq 10.$$

Now if $u = 110$, then our inequality turns into

$$-\frac{162}{\cos^2 A} + 350\tan A \geq 7.$$

Solving this inequality, we conclude that $0.63 \text{ rad} \leq A \leq 0.96 \text{ rad}$.

(f) We rewrite the inequality in part (e) as

$$\cos^2 A(350\tan A - 7) \geq \frac{1,960,000}{u^2}.$$

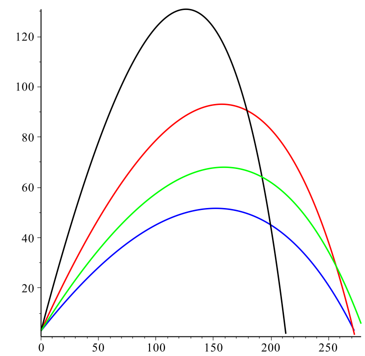
In order to determine the minimum value necessary, we will maximize the function on the left side. Letting $f(A) = \cos^2 A(350\tan A - 7)$, we see that $f'(A) = 350\cos 2A + 7\sin 2A$. Therefore, $f'(A) = 0$ implies $\tan 2A = -50$. For $0 < A < \pi/2$, we see that this occurs at $A = 0.7954$ radians. Substituting this value for A into the inequality above, we conclude that $u^2 \geq 11426.24$. Therefore, the minimum velocity necessary is 106.89 ft/s and the optimal angle necessary is 0.7954 radians.

33.(a) The initial conditions are $v(0) = u\cos A$ and $w(0) = u\sin A$. Therefore, the solutions of the two equations are $v(t) = (u\cos A)e^{-rt}$ and $w(t) = -g/r + (u\sin A + g/r)e^{-rt}$.

(b) Now $x(t) = \int v(t)\,dt = \frac{u}{r}(\cos A)(1 - e^{-rt})$, and

$$y(t) = \int w(t)\,dt = -\frac{gt}{r} + \left(\frac{u}{r}\sin A + \frac{g}{r^2}\right)(1 - e^{-rt}) + h.$$

(c) Below we have plotted the trajectory of the ball in the cases $\pi/6$, $\pi/5$, $\pi/4$, and $\pi/3$, respectively.



(d) Let T be the time it takes the ball to go 350 feet horizontally. Then from above, we see that $e^{-T/5} = 1 - 70/u\cos A$. At the same time, the height of the ball is given by $y(T) = -160T + (800 + 5u\sin A)70/u\cos A + 3$. Therefore, u and A must satisfy the inequality

$$800\ln\left(1 - \frac{70}{u\cos A}\right) + 350\tan A + \frac{56000}{u\cos A} + 3 \geq 10.$$

Using graphical techniques, we identify the minimum velocity necessary is 145.3 ft/s and the optimal angle necessary is 0.644 radians.

34.(a) Solving equation (i), we have $y'(x) = [(k^2 - y)/y]^{1/2}$. The positive answer is chosen since y is an increasing function of x.

(b) $y = k^2\sin^2 t$, thus $dy/dt = 2k^2\sin t\cos t$. Substituting this into the equation in part (a), we have

$$2k^2\sin t\cos t\frac{dt}{dx} = \frac{\cos t}{\sin t}.$$

Therefore, $2k^2\sin^2 t dt = dx$.

(c) Letting $\theta = 2t$, we have $k^2\sin^2(\theta/2)d\theta = dx$. Integrating both sides, we have $x(\theta) = k^2(\theta - \sin\theta)/2$. Further, using the fact that $y = k^2\sin^2 t$, we conclude that $y(\theta) = k^2\sin^2(\theta/2) = k^2(1 - \cos(\theta))/2$.

(d) From part (c), we see that $y/x = (1 - \cos\theta)/(\theta - \sin\theta)$. If $x = 1$ and $y = 2$, the solution of the equation is $\theta \approx 1.401$. Substituting that value of θ into either of the equations in part (c), we conclude that $k \approx 2.193$.

2.4 Differences between Linear and Nonlinear Equations

1. Rewriting the equation as

$$y' + \frac{\ln t}{t-3}y = \frac{2t}{t-3}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $0 < t < 3$.

2. Rewriting the equation as

$$y' + \frac{1}{t(t-4)}y = 0$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $0 < t < 4$.

3. By Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $\pi/2 < t < 3\pi/2$.

4. Rewriting the equation as

$$y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $-\infty < t < -2$.

5. Rewriting the equation as

$$y' + \frac{2t}{4-t^2}y = \frac{3t^2}{4-t^2}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $-2 < t < 2$.

6. Rewriting the equation as

$$y' + \frac{1}{\ln t}y = \frac{\cot t}{\ln t}$$

and using Theorem 2.4.1, we conclude that a solution is guaranteed to exist in the interval $1 < t < \pi$.

7. Using the fact that

$$f = \frac{t-y}{2t+5y} \quad \text{and} \quad f_y = -\frac{7t}{(2t+5y)^2},$$

we see that the hypotheses of Theorem 2.4.2 are satisfied as long as $2t + 5y \neq 0$.

8. Using the fact that

$$f = (1-t^2-y^2)^{1/2} \quad \text{and} \quad f_y = -\frac{y}{(1-t^2-y^2)^{1/2}},$$

we see that the hypotheses of Theorem 2.4.2 are satisfied as long as $t^2 + y^2 < 1$.

9. Using the fact that

$$f = \frac{\ln|ty|}{1-t^2+y^2} \quad \text{and} \quad f_y = \frac{1-t^2+y^2-2y^2\ln|ty|}{y(1-t^2+y^2)^2},$$

we see that the hypotheses of Theorem 2.4.2 are satisfied as long as $y, t \neq 0$ and $1-t^2+y^2 \neq 0$.

10. Using the fact that

$$f = (t^2+y^2)^{3/2} \quad \text{and} \quad f_y = 3y(t^2+y^2)^{1/2},$$

we see that the hypotheses of Theorem 2.4.2 are satisfied for all t and y values.

11. Using the fact that

$$f = \frac{1+t^2}{3y-y^2} \quad \text{and} \quad f_y = -\frac{(1+t^2)(3-2y)}{(3y-y^2)^2},$$

we see that the hypotheses of Theorem 2.4.2 are satisfied as long as $y \neq 0, 3$.

12. Using the fact that

$$f = \frac{(\cot t)y}{1+y} \quad \text{and} \quad f_y = \frac{1}{(1+y)^2},$$

we see that the hypotheses of Theorem 2.4.2 are satisfied as long as $y \neq -1, t \neq n\pi$ for $n = 0, 1, 2 \ldots$.

13.(a) We know that the family of solutions given by equation (19) are solutions of this initial-value problem. We want to determine if one of these passes through the point $(1, 1)$. That is, we want to find $t_0 > 0$ such that if $y = [\frac{2}{3}(t - t_0)]^{3/2}$, then $(t, y) = (1, 1)$. That is, we need to find $t_0 > 0$ such that $1 = \frac{2}{3}(1 - t_0)$. But, the solution of this equation is $t_0 = -1/2$. Therefore the solution does not pass through $(1, 1)$.

(b) From the analysis in part (a), we find a solution passing through $(2, 1)$ by solving $1 = \frac{2}{3}(2 - t_0)$. We obtain $t_0 = 1/2$, and the solution is $y = [\frac{2}{3}(t - 1/2)]^{3/2}$.

(c) Since we need $y_0 = \pm[\frac{2}{3}(2 - t_0)]^{3/2}$, we must have $|y_0| \leq (\frac{4}{3})^{3/2}$.

14.(a) First, it is clear that $y_1(2) = -1 = y_2(2)$. Further,

$$y_1' = -1 = \frac{-t + (t^2 + 4(1-t))^{1/2}}{2} = \frac{-t + [(t-2)^2]^{1/2}}{2}$$

and

$$y_2' = \frac{-t + (t^2 - t^2)^{1/2}}{2}.$$

The function y_1 is a solution for $t \geq 2$. The function y_2 is a solution for all t.

(b) Theorem 2.4.2 requires that f and $\partial f/\partial y$ be continuous in a rectangle about the point $(t_0, y_0) = (2, -1)$. Since f_y is not continuous if $t < 2$ and $y < -1$, the hypotheses of Theorem 2.4.2 are not satisfied.

(c) If $y = ct + c^2$, then

$$y' = c = \frac{-t + [(t+2c)^2]^{1/2}}{2} = \frac{-t + (t^2 + 4ct + 4c^2)^{1/2}}{2}.$$

Therefore, y satisfies the equation for $t \geq -2c$.

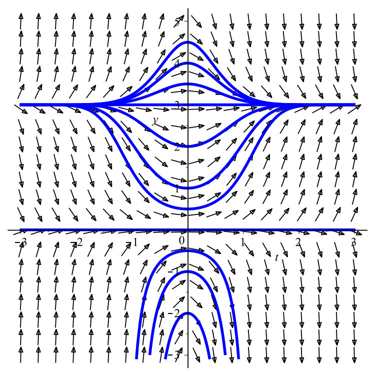
15. The equation is separable, $ydy = -4tdt$. Integrating both sides, we conclude that $y^2/2 = -2t^2 + y_0^2/2$ for $y_0 \neq 0$. The solution is defined for $y_0^2 - 4t^2 \geq 0$.

16. The equation is separable and can be written as $dy/y^2 = 2tdt$. Integrating both sides, we arrive at the solution $y = y_0/(1 - y_0 t^2)$. For $y_0 > 0$, solutions exist as long as $t^2 < 1/y_0$. For $y_0 \leq 0$, solutions exist for all t.

17. The equation is separable and can be written as $dy/y^3 = -dt$. Integrating both sides, we arrive at the solution $y = y_0/(\sqrt{2ty_0^2+1})$. Solutions exist as long as $2y_0t + 1 > 0$. If $y_0 \neq 0$, the solution exists for $t > -\frac{1}{2y_0^2}$ and if $y_0 = 0$, $y(t) = 0$ for all t.

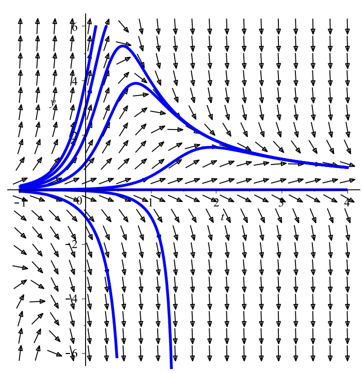
18. The equation is separable and can be written as $ydy = t^2dt/(1+t^3)$. Integrating both sides, we arrive at the solution $y = \pm(\frac{2}{3}\ln|1+t^3| + y_0^2)^{1/2}$. The sign of the solution depends on the sign of the initial data y_0. Solutions exist as long as $\frac{2}{3}\ln|1+t^3| + y_0^2 \geq 0$; that is, as long as $y_0^2 \geq -\frac{2}{3}\ln|1+t^3|$. We can rewrite this inequality as $|1+t^3| \geq e^{-3y_0^2/2}$. In order for the solution to exist, we need $t > -1$ (since the term $t^2/(1+t^3)$ has a singularity at $t = -1$). Therefore, we can conclude that our solution will exist for $[e^{-3y_0^2/2} - 1]^{1/3} < t < \infty$.

19.



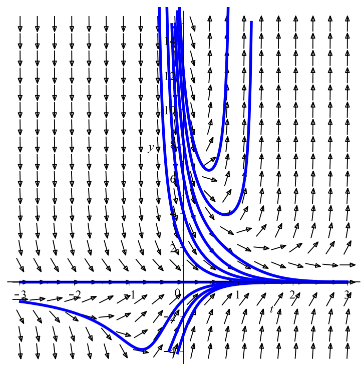
If $y_0 > 0$, then $y \to 3$. If $y_0 = 0$, then $y = 0$. If $y_0 < 0$, then $y \to -\infty$.

20.



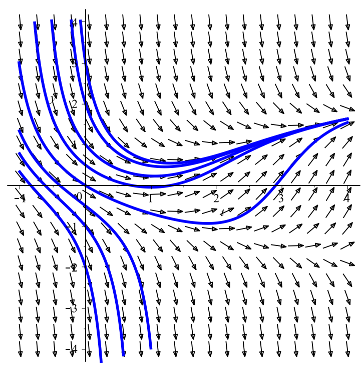
If $y_0 \geq 0$, then $y \to 0$. If $y_0 < 0$, then $y \to -\infty$.

21.



If $y_0 > 9$, then $y \to \infty$. If $y_0 \le 9$, then $y \to 0$.

22.



If $y_0 < y_c \approx -0.019$, then $y \to -\infty$. Otherwise, y is asymptotic to $\sqrt{t-1}$.

23.(a) $\phi(t) = e^{2t}$, thus $\phi' = 2e^{2t}$. Therefore, $\phi' - 2\phi = 0$. Since $(c\phi)' = c\phi'$, we see that $(c\phi)' - 2c\phi = 0$. Therefore, $c\phi$ is also a solution.

(b) $\phi(t) = 1/t$, thus $\phi' = -1/t^2$. Therefore, $\phi' + \phi^2 = 0$. If $y = c/t$, then $y' = -c/t^2$. Therefore, $y' + y^2 = -c/t^2 + c^2/t^2 = 0$ if and only if $c^2 - c = 0$; that is, if $c = 0$ or $c = 1$.

24. If $y = \phi$ satisfies $\phi' + p(t)\phi = 0$, then $y = c\phi$ satisfies $y' + p(t)y = c\phi' + cp(t)\phi = c(\phi' + p(t)\phi) = 0$.

25. Let $y = y_1 + y_2$, then $y' + p(t)y = y_1' + y_2' + p(t)(y_1 + y_2) = y_1' + p(t)y_1 + y_2' + p(t)y_2 = 0$.

26.(a)

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\,ds + c\right] = \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)g(s)\,ds + \frac{c}{\mu(t)}.$$

Therefore, $y_1 = 1/\mu(t)$ and $y_2 = \frac{1}{\mu(t)}\int_{t_0}^{t} \mu(s)g(s)\,ds$.

(b) For $y_1 = 1/\mu(t) = e^{-\int p(t)\,dt}$, we have

$$y_1' + p(t)y_1 = -p(t)e^{-\int p(t)\,dt} + p(t)e^{-\int p(t)\,dt} = 0.$$

(c) For

$$y_2 = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu(s) g(s)\, ds = e^{-\int p(t)\, dt} \int_{t_0}^{t} e^{\int p(s)\, ds} g(s)\, ds,$$

we have

$$\begin{aligned} y_2' + p(t) y_2 &= -p(t) e^{-\int p(t)\, dt} \int_{t_0}^{t} e^{\int p(s)\, ds} g(s)\, ds + e^{-\int p(t)\, dt} e^{\int p(t)\, dt} g(t) \\ &+ p(t) e^{-\int p(t)\, dt} \int_{t_0}^{t} e^{\int p(s)\, ds} g(s)\, ds = g(t). \end{aligned}$$

27. The solution of the initial value problem $y' + 2y = 1$ is $y = 1/2 + ce^{-2t}$. For $y(0) = 0$, we see that $c = -1/2$. Therefore, $y(t) = \frac{1}{2}(1 - e^{-2t})$ for $0 \le t \le 1$. Then $y(1) = \frac{1}{2}(1 - e^{-2})$. Next, the solution of $y' + 2y = 0$ is given by $y = ce^{-2t}$. The initial condition $y(1) = \frac{1}{2}(1 - e^{-2})$ implies $ce^{-2} = \frac{1}{2}(1 - e^{-2})$. Therefore, $c = \frac{1}{2}(e^2 - 1)$ and we conclude that $y(t) = \frac{1}{2}(e^2 - 1)e^{-2t}$ for $t > 1$.

28. The solution of $y' + 2y = 0$ with $y(0) = 1$ is given by $y(t) = e^{-2t}$ for $0 \le t \le 1$. Then $y(1) = e^{-2}$. Then, for $t > 1$, the solution of the equation $y' + y = 0$ is $y = ce^{-t}$. Since we want $y(1) = e^{-2}$, we need $ce^{-1} = e^{-2}$. Therefore, $c = e^{-1}$. Therefore, $y(t) = e^{-1}e^{-t} = e^{-1-t}$ for $t > 1$.

29.(a) Multiplying the equation by $e^{\int_{t_0}^{t} p(s)\, ds}$, we have

$$\left(e^{\int_{t_0}^{t} p(s)\, ds} y\right)' = e^{\int_{t_0}^{t} p(s)\, ds} g(t).$$

Integrating this we obtain

$$e^{\int_{t_0}^{t} p(s)\, ds} y(t) = y_0 + \int_{t_0}^{t} e^{\int_{t_0}^{s} p(r)\, dr} g(s)\, ds,$$

which implies

$$y(t) = y_0 e^{-\int_{t_0}^{t} p(s)\, ds} + \int_{t_0}^{t} e^{-\int_{s}^{t} p(r)\, dr} g(s)\, ds.$$

(b) Assume $p(t) \ge p_0 > 0$ for all $t \ge t_0$ and $|g(t)| \le M$ for all $t \ge t_0$. Therefore,

$$\int_{t_0}^{t} p(s)\, ds \ge \int_{t_0}^{t} p_0\, ds = p_0(t - t_0),$$

which implies

$$e^{-\int_{t_0}^{t} p(s)\, ds} \le e^{-\int_{t_0}^{t} p_0\, ds} = e^{-p_0(t-t_0)} \le 1 \text{ for } t \ge t_0.$$

Also,

$$\begin{aligned} \int_{t_0}^{t} e^{-\int_{s}^{t} p(r)\, dr} g(s)\, ds &\le \int_{t_0}^{t} e^{-\int_{s}^{t} p(r)\, dr} |g(s)|\, ds \le \int_{t_0}^{t} e^{-p_0(t-s)} M\, ds \\ &\le M \left. \frac{e^{-p_0(t-s)}}{p_0} \right|_{t_0}^{t} = M \left[\frac{1}{p_0} - \frac{e^{-p_0(t-t_0)}}{p_0} \right] \le \frac{M}{p_0}. \end{aligned}$$

(c) Let $p(t) = 2t + 1 \geq 1$ for all $t \geq 0$ and let $g(t) = e^{-t^2}$. Therefore, $|g(t)| \leq 1$ for all $t \geq 0$. By the answer to part (a),

$$y(t) = e^{-\int_0^t (2s+1)\,ds} + \int_0^t e^{-\int_s^t (2r+1)\,dr} e^{-s^2}\,ds = e^{-(t^2+t)} + e^{-t^2-t} \int_0^t e^s\,ds = e^{-t^2}.$$

We see that y satisfies the property that y is bounded for all time $t \geq 0$.

2.5 Autonomous Equations and Population Dynamics

1.(a) The equation is separable. Using partial fractions, it can be written as

$$\left(\frac{1}{y} + \frac{1/K}{1 - y/K}\right) dy = r dt.$$

Integrating both sides and using the initial condition $y_0 = K/3$, we know the solution y satisfies

$$\ln\left|\frac{y}{1 - y/K}\right| = rt + \ln\left|\frac{K}{2}\right|.$$

To find the time τ such that $y = 2y_0 = 2K/3$, we substitute $y = 2K/3$ and $t = \tau$ into the equation above. Using the properties of logarithmic functions, we conclude that $\tau = (\ln 4)/r$. If $r = 0.025$, then $\tau \approx 55.452$ years.

(b) Using the analysis from part (a), we know the general solution satisfies

$$\ln\left|\frac{y}{1 - y/K}\right| = rt + c.$$

The initial condition $y_0 = \alpha K$ implies $c = \ln|\alpha K/(1 - \alpha)|$. Therefore,

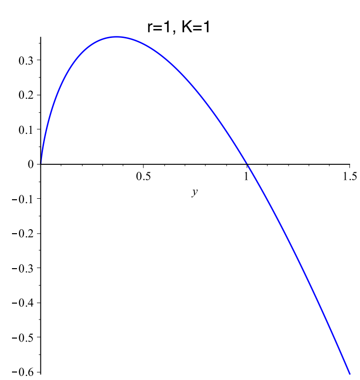
$$\ln\left|\frac{y}{1 - y/K}\right| = rt + \ln\left|\frac{\alpha K}{1 - \alpha}\right|.$$

In order to find the time T at which $y(T) = \beta K$, we use the equation above. We conclude that

$$T = (1/r)\ln|\beta(1 - \alpha)/\alpha(1 - \beta)|.$$

When $\alpha = 0.1$, $\beta = 0.9$, $r = 0.025$, $\tau \approx 175.78$ years.

2.(a) Below we sketch the graph of f for $r = 1 = K$.



The critical points occur at $y^* = 0, K$. Since $f'(0) > 0$, $y^* = 0$ is unstable. Since $f'(K) < 0$, $y^* = K$ is asymptotically stable.

(b) We calculate y''. Using the chain rule, we see that

$$y'' = ry'\left[\ln\left(\frac{K}{y}\right) - 1\right].$$

We see that $y'' = 0$ when $y' = 0$ (meaning $y = 0, K$) or when $\ln(K/y) - 1 = 0$, meaning $y = K/e$. Looking at the sign of y'' in the intervals $0 < y < K/e$ and $K/e < y < K$, we conclude that y is concave up in the interval $0 < y < K/e$ and concave down in the interval $K/e < y < K$.

3.(a) Using the substitution $u = \ln(y/K)$ and differentiating both sides with respect to t, we conclude that $u' = y'/y$. Substitution into the Gompertz equation yields $u' = -ru$. The solution of this equation is $u = u_0 e^{-rt}$. Therefore,

$$\frac{y}{K} = \exp[\ln(y_0/K)e^{-rt}].$$

(b) For $K = 80.5 \times 10^6$, $y_0/K = 0.25$ and $r = 0.71$, we conclude that $y(2) \approx 57.58 \times 10^6$.

(c) Solving the equation in part (a) for t, we see that

$$t = -\frac{1}{r}\ln\left[\frac{\ln(y/K)}{\ln(y_0/K)}\right].$$

Plugging in the given values, we conclude that $\tau \approx 2.21$ years.

4.(a) The surface area of the cone is given by

$$\begin{aligned} S &= \pi a\sqrt{h^2+a^2} + \pi a^2 = \pi a^2(\sqrt{(h/a)^2+1}+1) = \frac{\pi a^2 h}{3}\cdot\frac{3}{h}\left(\sqrt{(h/a)^2+1}+1\right) \\ &= c\pi\left(\frac{\pi a^2 h}{3}\right)^{2/3}\cdot\left(\frac{3a}{\pi h}\right)^{2/3} = c\pi\left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3}. \end{aligned}$$

Therefore, if the rate of evaporation is proportional to the surface area, then rate out $= \alpha\pi(3a/\pi h)^{2/3}V^{2/3}$. Thus

$$\frac{dV}{dt} = \text{rate in} - \text{rate out} = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3}\left(\frac{\pi}{3}a^2 h\right)^{2/3} = k - \alpha\pi\left(\frac{3a}{\pi h}\right)^{2/3} V^{2/3}.$$

(b) The equilibrium volume can be found by setting $dV/dt = 0$. We see that the equilibrium volume is

$$V = \left(\frac{k}{\alpha\pi}\right)^{3/2}\left(\frac{\pi h}{3a}\right).$$

To find the equilibrium height, we use the fact that the height and radius of the conical pond maintain a constant ratio. Therefore, if h_e, a_e represent the equilibrium values for the

h and a, we must have $h_e/a_e = h/a$. Further, we notice that the equilibrium volume can be written as

$$V = \frac{\pi}{3}\left(\frac{k}{\alpha\pi}\right)\left(\frac{k}{\alpha\pi}\right)^{1/2}\left(\frac{h}{a}\right) = \frac{\pi}{3}a_e^2 h_e^2,$$

where $h_e = (k/\alpha\pi)^{1/2}(h/a)$ and $a_e = (k\alpha\pi)^{1/2}$. $f'(V) = -\frac{2}{3}\alpha\pi(3a/\pi h)^{2/3}V^{-1/3} < 0$, thus the equilibrium is asymptotically stable.

(c) In order to guarantee that the pond does not overflow, we need the rate of water in to be less than or equal to the rate of water out. Therefore, we need $k - \alpha\pi a^2 \leq 0$.

5.(a) The rate of increase of the volume is given by

$$\frac{dV}{dt} = k - \alpha a\sqrt{2gh}.$$

Since the cross-section is constant, $dV/dt = A dh/dt$. Therefore,

$$\frac{dh}{dt} = (k - \alpha a\sqrt{2gh})/A.$$

(b) Setting $dh/dt = 0$, we conclude that the equilibrium height of water is

$$h_e = \frac{1}{2g}\left(\frac{k}{\alpha a}\right)^2.$$

Since $f'(h_e) < 0$, the equilibrium height is stable.

6.(a) The equilibrium points are $y^* = 0, 1$. Since $f'(0) = \alpha > 0$, the equilibrium solution $y^* = 0$ is unstable. Since $f'(1) = -\alpha < 0$, the equilibrium solution $y^* = 1$ is asymptotically stable.

(b) The equation is separable. The solution is given by

$$y(t) = \frac{y_0}{e^{-\alpha t} - y_0 e^{-\alpha t} + y_0} = \frac{y_0}{e^{-\alpha t} + y_0(1 - e^{-\alpha t})}.$$

We see that $\lim_{t\to\infty} y(t) = 1$.

7.(a) The solution of the separable equation is $y(t) = y_0 e^{-\beta t}$.

(b) Using the result from part (a), we see that $dx/dt = -\alpha x y_0 e^{-\beta t}$. This equation is separable with solution $x(t) = x_0 exp[-\alpha y_0(1 - e^{-\beta t})/\beta]$.

(c) As $t \to \infty$, $y \to 0$ and $x \to x_0 \exp(-\alpha y_0/\beta)$.

8.(a) Letting $' = d/dt$, we have

$$z' = \frac{nx' - xn'}{n^2} = \frac{-\beta nx - \mu nx + \nu\beta x^2 + \mu nx}{n^2} = -\beta\frac{x}{n} + \nu\beta\left(\frac{x}{n}\right)^2 = -\beta z + \nu\beta z^2 = -\beta z(1 - \nu z).$$

(b) First, we rewrite the equation as $z' + \beta z = \beta\nu z^2$. This is a Bernoulli equation with $n = 2$. Let $w = z^{1-n} = z^{-1}$. Then, our equation can be written as $w' - \beta w = -\beta\nu$. This is a linear equation with solution $w = \nu + ce^{\beta t}$. Then, using the fact that $z = 1/w$, we see that $z = 1/(\nu + ce^{\beta t})$. Finally, the initial condition $z(0) = 1$ implies $c = 1 - \nu$. Therefore, $z(t) = 1/(\nu + (1-\nu)e^{\beta t})$.

(c) Evaluating $z(20)$ for $\nu = \beta = 1/8$, we conclude that $z(20) = 0.0927$.

9.(a) Since the critical points are $x^* = p, q$, we will look at their stability. Since $f'(x) = -\alpha q - \alpha p + 2\alpha x^2$, we see that $f'(p) = \alpha(2p^2 - q - p)$ and $f'(q) = \alpha(2q^2 - q - p)$. Now if $p > q$, then $-p < -q$, and, therefore, $f'(q) = \alpha(2q^2 - q - p) < \alpha(2q^2 - 2q) = 2\alpha q(q-1) < 0$ since $0 < q < 1$. Therefore, if $p > q$, $f'(q) < 0$, and, therefore, $x^* = q$ is asymptotically stable. Similarly, if $p < q$, then $x^* = p$ is asymptotically stable, and therefore, we can conclude that $x(t) \to \min\{p, q\}$ as $t \to \infty$.
The equation is separable. It can be solved by using partial fractions as follows. We can rewrite the equation as

$$\left(\frac{1/(q-p)}{p-x} + \frac{1/(p-q)}{q-x}\right) dx = \alpha dt,$$

which implies

$$\ln\left|\frac{p-x}{q-x}\right| = (p-q)\alpha t + C.$$

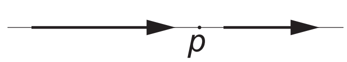
The initial condition $x_0 = 0$ implies $C = \ln|p/q|$, and, therefore,

$$\ln\left|\frac{q(p-x)}{p(q-x)}\right| = (p-q)\alpha t.$$

Applying the exponential function and simplifying, we conclude that

$$x(t) = \frac{pq(e^{(p-q)\alpha t} - 1)}{pe^{(p-q)\alpha t} - q}.$$

(b) In this case, $x^* = p$ is the only critical point. Since $f(x) = \alpha(p-x)^2$ is concave up, we conclude that $x^* = p$ is semistable. Further, if $x_0 = 0$, we can conclude that $x \to p$ as $t \to \infty$. The phase line is shown below.



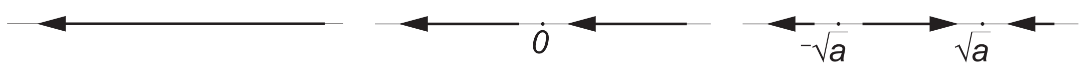
This equation is separable. Its solution is given by

$$x(t) = \frac{p^2 \alpha t}{p\alpha t + 1}.$$

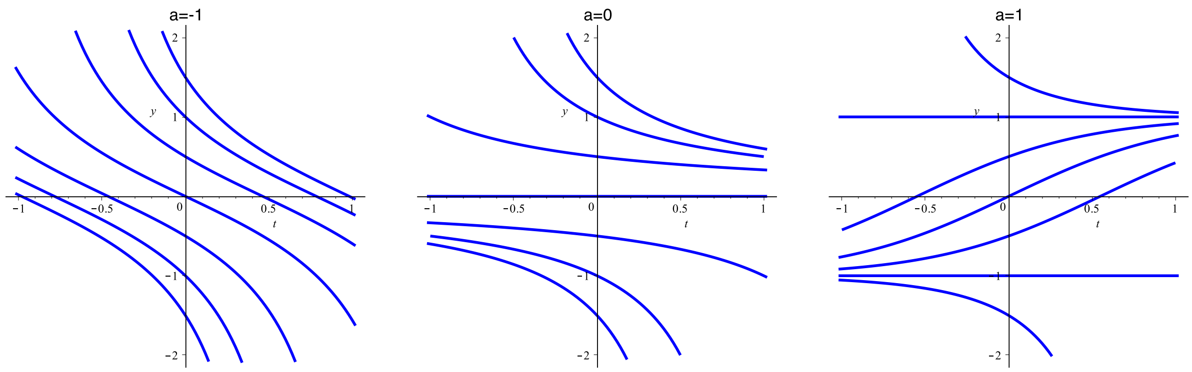
10.(a) The critical points occur when $a - y^2 = 0$. If $a < 0$, there are no critical points. If $a = 0$, then $y^* = 0$ is the only critical point. If $a > 0$, then $y^* = \pm\sqrt{a}$ are the two critical points.

(b) We note that $f'(y) = -2y$. Therefore, $f'(\sqrt{a}) < 0$ which implies that $\sqrt{a}$ is asymptotically stable; $f'(-\sqrt{a}) > 0$ which implies $-\sqrt{a}$ is unstable; the behavior of f' around $y^* = 0$ implies that $y^* = 0$ is semistable. The phase lines are shown below.

$a < 0$ $a = 0$ $a > 0$

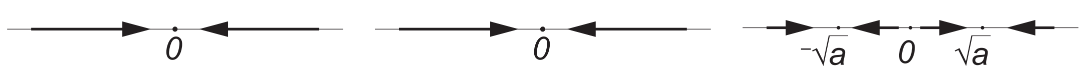


(c) Below, we graph solutions in the case $a = -1$, $a = 0$, and $a = 1$, respectively.

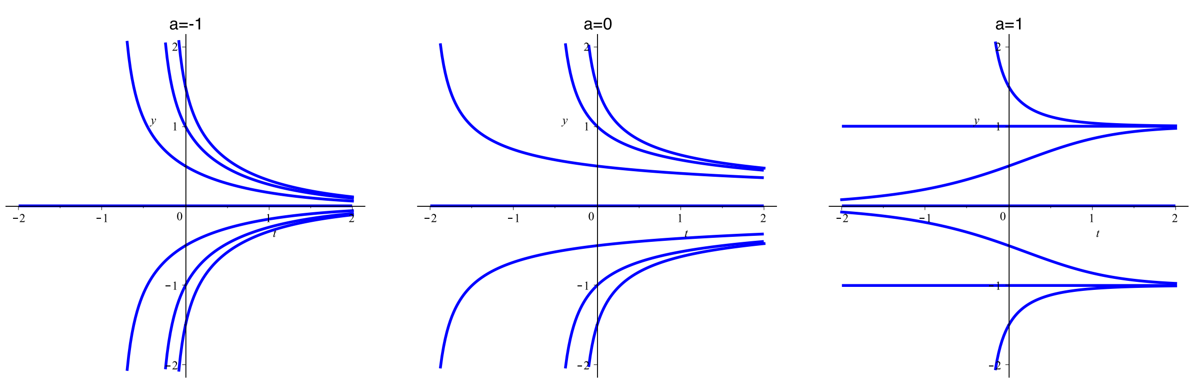


11.(a) First, for $a < 0$, the only critical point is $y^* = 0$. Second, for $a = 0$, the only critical point is $y^* = 0$. Third, for $a > 0$, the critical points are at $y^* = 0, \pm\sqrt{a}$. Here, $f'(y) = a - 3y^2$. If $a < 0$, then $f'(y) < 0$ for all y, and, therefore, $y^* = 0$ will be asymptotically stable. If $a = 0$, then $f'(0) = 0$. From the behavior on either side of $y^* = 0$, we see that $y^* = 0$ will be asymptotically stable. If $a > 0$, then $f'(0) = a > 0$ which implies that $y^* = 0$ is unstable for $a > 0$. Further, $f'(\pm\sqrt{a}) = -2a < 0$. Therefore, $y^* = \pm\sqrt{a}$ are asymptotically stable for $a > 0$. The phase lines are shown below.

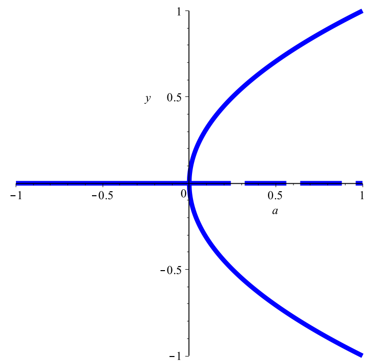
$a < 0$ $a = 0$ $a > 0$



(b) Below, we graph solutions in the case $a = -1$, $a = 0$, and $a = 1$, respectively.

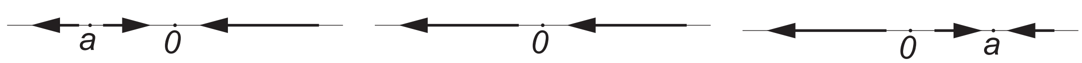


(c)

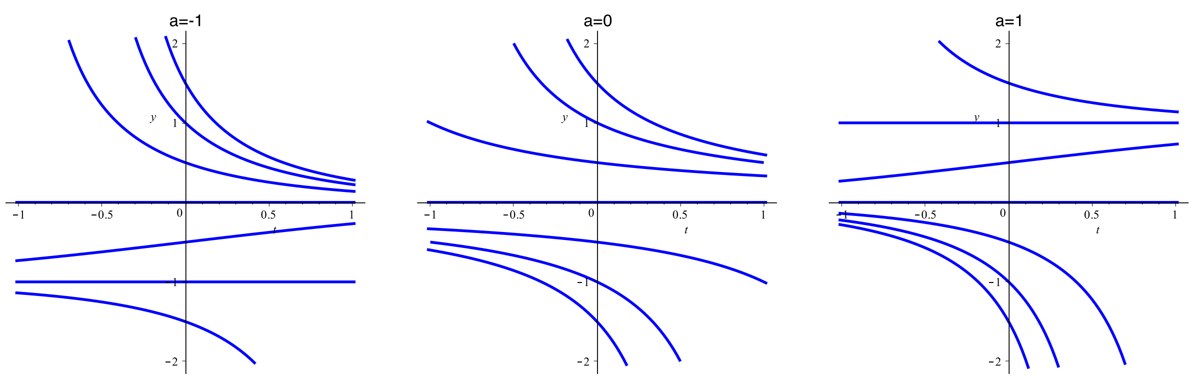


12.(a) For $a < 0$, the critical points are $y^* = 0, a$. Since $f'(y) = a - 2y$, $f'(0) = a < 0$ and $f'(a) = -a > 0$. Therefore, $y^* = 0$ is asymptotically stable and $y^* = a$ is unstable for $a < 0$. For $a = 0$, the only critical point is $y^* = 0$. which is semistable since $f(y) = -y^2$ is concave down. For $a > 0$, the critical points are $y^* = 0, a$. Since $f'(0) = a > 0$ and $f'(a) = -a < 0$, the critical point $y^* = 0$ is unstable while the critical point $y^* = a$ is asymptotically stable for $a > 0$. The phase lines are shown below.

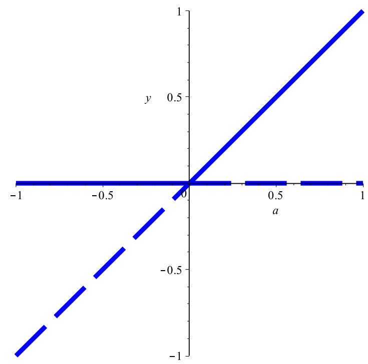
$a < 0$ $\qquad$ $a = 0$ $\qquad$ $a > 0$



(b) Below, we graph solutions in the case $a = -1$, $a = 0$, and $a = 1$, respectively.



(c)

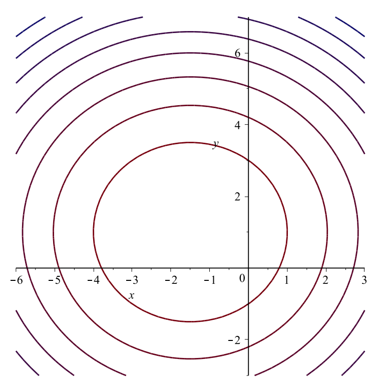


2.6 Exact Equations and Integrating Factors

1.(a) Here $M(x,y) = 2x+3$ and $N(x,y) = 2y-2$. Since $M_y = N_x = 0$, the equation is exact.

(b) Since $\psi_x = M = 2x+3$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = x^2+3x+h(y)$. Then $\psi_y = h'(y) = N = 2y-2$ implies $h(y) = y^2-2y$. Therefore, $\psi(x,y) = x^2+3x+y^2-2y = c$.

(c)

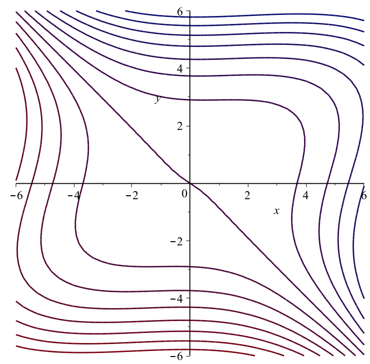


2.(a) Here $M(x,y) = 2x+4y$ and $N(x,y) = 2x-2y$. Since $M_y \neq N_x$, the equation is not exact.

3.(a) Here $M(x,y) = 3x^2-2xy+2$ and $N(x,y) = 6y^2-x^2+3$. Since $M_y = -2x = N_x$, the equation is exact.

(b) Since $\psi_x = M = 3x^2-2xy+2$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = x^3-x^2y+2x+h(y)$. Then $\psi_y = -x^2+h'(y) = N = 6y^2-x^2+3$ implies $h'(y) = 6y^2+3$. Therefore, $h(y) = 2y^3+3y$ and $\psi(x,y) = x^3-x^2y+2x+2y^3+3y = c$.

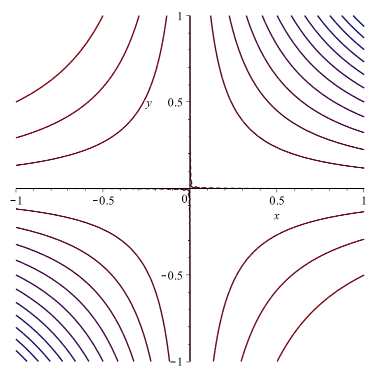
(c)



4.(a) Here $M(x,y) = 2xy^2 + 2y$ and $N(x,y) = 2x^2y + 2x$. Since $M_y = 4xy + 2 = N_x$, the equation is exact.

(b) Since $\psi_x = M = 2xy^2 + 2y$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = x^2y^2 + 2xy + h(y)$. Then $\psi_y = 2x^2y + 2x + h'(y) = N = 2x^2y + 2x$ implies $h'(y) = 0$. Therefore, $h(y) = c$ and $\psi(x,y) = x^2y^2 + 2xy = c$.

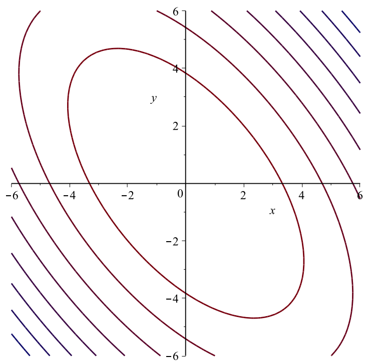
(c)



5.(a) Here $M(x,y) = 4x + 2y$ and $N(x,y) = 2x + 3y$. Since $M_y = 2 = N_x$, the equation is exact.

(b) Since $\psi_x = M = 4x + 2y$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = 2x^2 + 2xy + h(y)$. Then $\psi_y = 2x + h'(y) = N = 2x + 3y$ implies $h'(y) = 3y$. Therefore, $h(y) = 3y^2/2$ and $\psi(x,y) = 2x^2 + 2xy + 3y^2/2 = k$.

(c)

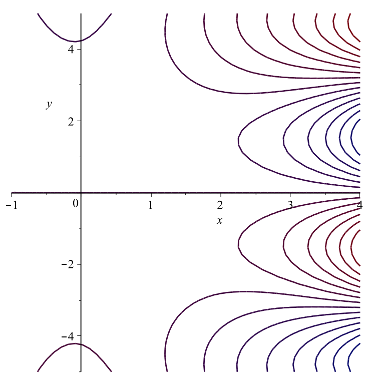


6.(a) Here $M = 4x - 2y$ and $N = 2x - 3y$. Since $M_y = -2$ and $N_x = 2$, the equation is not exact.

7.(a) Here $M(x, y) = e^x \sin y - 2y \sin x$ and $N(x, y) = e^x \cos y + 2\cos x$. Since $M_y = e^x \cos y - \sin x = N_x$, the equation is exact.

(b) Since $\psi_x = M = e^x \sin y - 2y \sin x$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = e^x \sin y + 2y \cos x + h(y)$. Then $\psi_y = e^x \cos y + 2\cos x + h'(y) = N = e^x \cos y + 2\cos x$ implies $h'(y) = 0$. Therefore, $h(y) = c$ and $\psi(x, y) = e^x \sin y + 2y \cos x = c$.

(c)

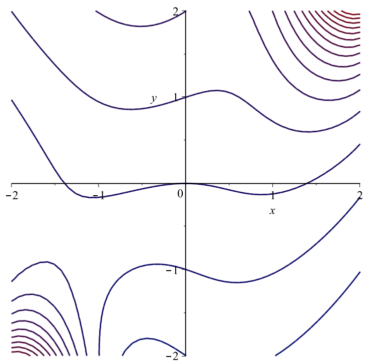


8.(a) Here $M = e^x \sin y + 3y$ and $N = -3x + e^x \sin y$. Therefore, $M_y = e^x \cos y + 3$ and $N_x = -3 + e^x \sin y$. Since $M_y \neq N_x$, therefore, the equation is not exact.

9.(a) Here $M(x, y) = ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x$ and $N(x, y) = xe^{xy} \cos 2x - 3$. Since $M_y = e^{xy} \cos 2x + xye^{xy} \cos 2x - 2xe^{xy} \sin 2x = N_x$, the equation is exact.

(b) Since $\psi_x = M = ye^{xy} \cos 2x - 2e^{xy} \sin 2x + 2x$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = e^{xy} \cos 2x + x^2 + h(y)$. Then $\psi_y = xe^{xy} \cos 2x + h'(y) = N = xe^{xy} \cos 2x - 3$ implies $h'(y) = -3$. Therefore, $h(y) = -3y$ and $\psi(x, y) = e^{xy} \cos 2x + x^2 - 3y = c$.

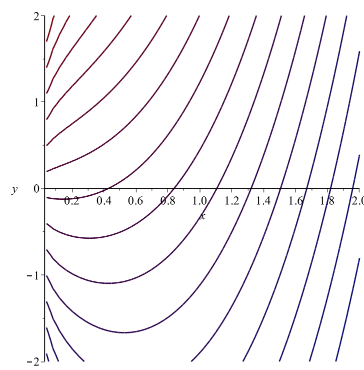
(c)



10.(a) Here $M(x,y) = y/x + 6x$ and $N(x,y) = \ln x - 2$. Since $M_y = 1/x = N_x$, the equation is exact.

(b) Since $\psi_x = M = y/x + 6x$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = y \ln x + 3x^2 + h(y)$. Then $\psi_y = \ln x + h'(y) = N = \ln x - 2$ implies $h'(y) = -2$. Therefore, $h(y) = -2y$ and $\psi(x,y) = y \ln x + 3x^2 - 2y = c$.

(c)

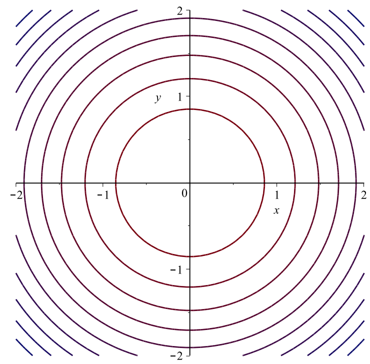


11.(a) Here $M(x,y) = x \ln y + xy$ and $N(x,y) = y \ln x + xy$. Since $M_y = x/y + x$ and $N_x = y/x + y$, we conclude that the equation is not exact.

12.(a) Here $M(x,y) = x/(x^2+y^2)^{3/2}$ and $N(x,y) = y/(x^2+y^2)^{3/2}$. Since $M_y = N_x$, the equation is exact.

(b) Since $\psi_x = M = x/(x^2+y^2)^{3/2}$, to solve for ψ, we integrate M with respect to x. We conclude that $\psi = -1/(x^2+y^2)^{1/2} + h(y)$. Then $\psi_y = y/(x^2+y^2)^{3/2} + h'(y) = N = y/(x^2+y^2)^{3/2}$ implies $h'(y) = 0$. Therefore, $h(y) = 0$ and $\psi(x,y) = -1/(x^2+y^2)^{1/2} = c$ or $x^2 + y^2 = k$.

(c)



13. Here $M(x,y) = 2x - y$ and $N(x,y) = 2y - x$. Therefore, $M_y = N_x = -1$ which implies that the equation is exact. Integrating M with respect to x, we conclude that $\psi = x^2 - xy + h(y)$. Then $\psi_y = -x + h'(y) = N = 2y - x$ implies $h'(y) = 2y$. Therefore, $h(y) = y^2$ and we conclude that $\psi = x^2 - xy + y^2 = c$. The initial condition $y(1) = 3$ implies $c = 7$. Therefore, $x^2 - xy + y^2 = 7$. Solving for y, we conclude that $y = (x + \sqrt{28 - 3x^2})/2$. Therefore, the solution is valid for $3x^2 \leq 28$, i.e. for $-\sqrt{28/3} < x < \sqrt{28/3}$.

14. Here $M(x,y) = 9x^2 + y - 1$ and $N(x,y) = -4y + x$. Therefore, $M_y = N_x = 1$ which implies that the equation is exact. Integrating M with respect to x, we conclude that $\psi = 3x^3 + xy - x + h(y)$. Then $\psi_y = x + h'(y) = N = -4y + x$ implies $h'(y) = -4y$. Therefore, $h(y) = -2y^2$ and we conclude that $\psi = 3x^3 + xy - x - 2y^2 = c$. The initial condition $y(1) = 0$ implies $c = 2$. Therefore, $3x^3 + xy - x - 2y^2 = 2$. Solving for y, we conclude that $y = (x - (24x^3 + x^2 - 8x - 16)^{1/2})/4$. The solution is valid for $x > 0.9846$.

15. Here $M(x,y) = xy^2 + bx^2y$ and $N(x,y) = x^3 + x^2y$. Therefore, $M_y = 2xy + bx^2$ and $N_x = 3x^2 + 2xy$. In order for the equation to be exact, we need $b = 3$. Taking this value for b, we integrating M with respect to x. We conclude that $\psi = x^2y^2/2 + x^3y + h(y)$. Then $\psi_y = x^2y + x^3 + h'(y) = N = x^3 + x^2y$ implies $h'(y) = 0$. Therefore, $h(y) = c$ and $\psi(x,y) = x^2y^2/2 + x^3y = c$. That is, the solution is given implicitly as $x^2y^2 + 2x^3y = k$.

16. Here $M(x,y) = ye^{2xy} + x$ and $N(x,y) = bxe^{2xy}$. Then $M_y = e^{2xy} + 2xye^{2xy}$ and $N_x = be^{2xy} + 2bxye^{2xy}$. The equation will be exact as long as $b = 1$. Integrating M with respect to x, we conclude that $\psi = e^{2xy}/2 + x^2/2 + h(y)$. Then $\psi_y = xe^{2xy} + h'(y) = N = xe^{2xy}$ implies $h'(y) = 0$. Therefore, $h(y) = 0$ and we conclude that the solution is given implicitly by the equation $e^{2xy} + x^2 = c$.

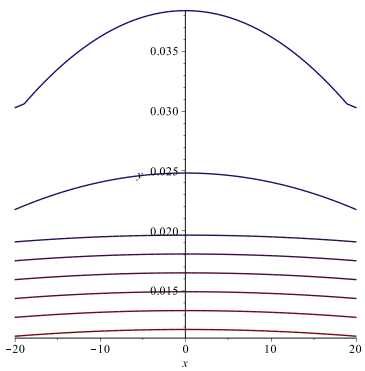
17. We notice that $\psi(x,y) = f(x) + g(y)$. Therefore, $\psi_x = f'(x)$ and $\psi_y = g'(y)$. That is, $\psi_x = M(x, y_0)$, and $\psi_y = N(x_0, y)$. Furthermore, $\psi_{xy} = M_y$ and $\psi_{yx} = N_x$. Based on the hypothesis, $\psi_{xy} = \psi_{yx}$ and $M_y = N_x$.

18. We notice that $(M(x))_y = 0 = (N(y))_x$. Therefore, the equation is exact.

19.(a) Here $M(x,y) = x^2y^3$ and $N(x,y) = x + xy^2$. Therefore, $M_y = 3x^2y^2$ and $N_x = 1 + y^2$. We see that the equation is not exact. Now, multiplying the equation by $\mu(x,y) = 1/xy^3$, the equation becomes $xdx + (1 + y^2)/y^3 dy = 0$. Now we see that for this equation $M = x$ and $N = (1 + y^2)/y^3$. Therefore, $M_y = 0 = N_x$.

(b) Integrating M with respect to x, we see that $\psi = x^2/2 + h(y)$. Further, $\psi_y = h'(y) = N = (1+y^2)/y^3 = 1/y^3 + 1/y$. Therefore, $h(y) = -1/2y^2 + \ln y$ and we conclude that the solution of the equation is given implicitly by $x^2 - 1/y^2 + 2\ln y = c$ and $y = 0$.

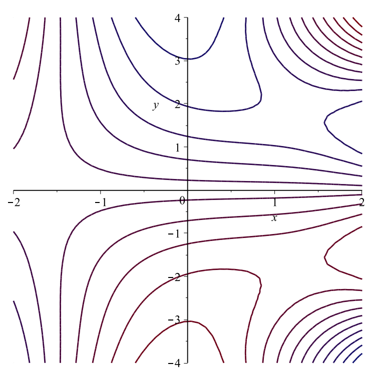
(c)



20.(a) We see that $M_y = (y\cos y - \sin y)/y^2$ while $N_x = 2e^{-x}\sin x - 2e^{-x}\cos x$. Therefore, $M_y \neq N_x$. However, multiplying the equation by $\mu(x,y) = ye^x$, the equation becomes $(e^x\sin y - 2y\sin x)dx + (e^x\cos y + 2\cos x)dy = 0$. Now we see that for this equation $M = e^x\sin y - 2y\sin x$ and $N = e^x\cos y + 2\cos x$. Therefore, $M_y = e^x\cos y - 2\sin x = N_x$.

(b) Integrating M with respect to x, we see that $\psi = e^x\sin y + 2y\cos x + h(y)$. Further, $\psi_y = e^x\cos y + 2\cos x + h'(y) = N = e^x\cos y + 2\cos x$. Therefore, $h(y) = 0$ and we conclude that the solution of the equation is given implicitly by $e^x\sin y + 2y\cos x = c$.

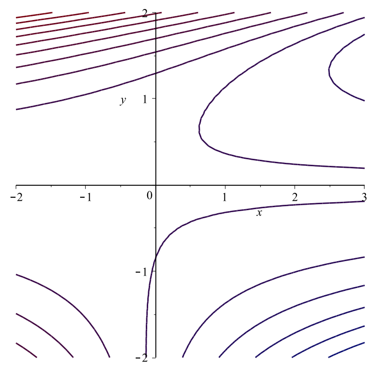
(c)



21.(a) We see that $M_y = 1$ while $N_x = 2$. Therefore, $M_y \neq N_x$. However, multiplying the equation by $\mu(x,y) = y$, the equation becomes $y^2dx + (2xy - y^2e^y)dy = 0$. Now we see that for this equation $M = y^2$ and $N = 2xy - y^2e^y$. Therefore, $M_y = 2y = N_x$.

(b) Integrating M with respect to x, we see that $\psi = xy^2 + h(y)$. Further, $\psi_y = 2xy + h'(y) = N = 2xy - y^2e^y$. Therefore, $h'(y) = -y^2e^y$ which implies that $h(y) = -e^y(y^2 - 2y + 2)$, and we conclude that the solution of the equation is given implicitly by $xy^2 - e^y(y^2 - 2y + 2) = c$.

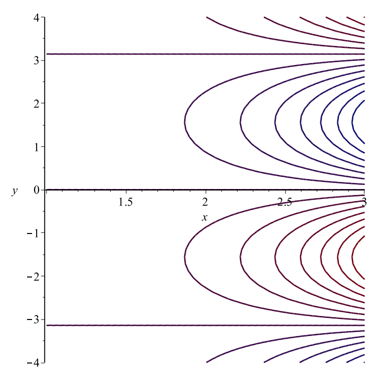
(c)



22.(a) We see that $M_y = (x+2)\cos y$ while $N_x = \cos y$. Therefore, $M_y \neq N_x$. However, multiplying the equation by $\mu(x,y) = xe^x$, the equation becomes $(x^2+2x)e^x \sin y dx + x^2 e^x \cos y dy = 0$. Now we see that for this equation $M_y = (x^2+2x)e^x \cos y = N_x$.

(b) Integrating M with respect to x, we see that $\psi = x^2 e^x \sin y + h(y)$. Further, $\psi_y = x^2 e^x \cos y + h'(y) = N = x^2 e^x \cos y$. Therefore, $h'(y) = 0$ which implies that the solution of the equation is given implicitly by $x^2 e^x \sin y = c$.

(c)



23. Suppose μ is an integrating factor which will make the equation exact. Then multiplying the equation by μ, we have $\mu M dx + \mu N dy = 0$. Then we need $(\mu M)_y = (\mu N)_x$. That is, we need $\mu_y M + \mu M_y = \mu_x N + \mu N_x$. Then we rewrite the equation as $\mu(N_x - M_y) = \mu_y M - \mu_x N$. Suppose μ does not depend on x. Then $\mu_x = 0$. Therefore, $\mu(N_x - M_y) = \mu_y M$. Using the assumption that $(N_x - M_y)/M = Q(y)$, we can find an integrating factor μ by choosing μ which satisfies $\mu_y/\mu = Q$. We conclude that $\mu(y) = \exp \int Q(y)\, dy$ is an integrating factor of the differential equation.

24. Suppose μ is an integrating factor which will make the equation exact. Then multiplying the equation by μ, we have $\mu M dx + \mu N dy = 0$. Then we need $(\mu M)_y = (\mu N)_x$. That is, we need $\mu_y M + \mu M_y = \mu_x N + \mu N_x$. Then we rewrite the equation as $\mu(N_x - M_y) = \mu_y M - \mu_x N$. By the given assumption, we need μ to satisfy $\mu R(xM - yN) = \mu_y M - \mu_x N$. This equation is satisfied if $\mu_y = (\mu x)R$ and $\mu_x = (\mu y)R$. Consider $\mu = \mu(xy)$. Then $\mu_x = \mu' y$ and $\mu_y = \mu' x$

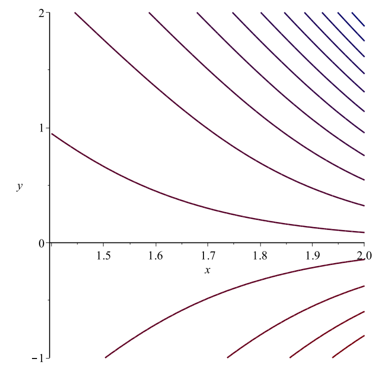
where $' = d/dz$ for $z = xy$. Therefore, we need to choose μ to satisfy $\mu' = \mu R$. This equation is separable with solution $\mu = \exp(\int R(z)\,dz)$.

25.(a) Since $(M_y - N_x)/N = 3$ is a function of x only, we know that $\mu = e^{3x}$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$e^{3x}(3x^2y + 2xy + y^3)dx + e^{3x}(x^2 + y^2)dy = 0.$$

Then $M_y = e^{3x}(3x^2 + 2x + 3y^2) = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = (x^2y + y^3/3)e^{3x} + h(y)$. Then $\psi_y = (x^2 + y^2)e^{3x} + h'(y) = N = e^{3x}(x^2 + y^2)$. Therefore, $h'(y) = 0$ and we conclude that the solution is given implicitly by $(3x^2y + y^3)e^{3x} = c$.

(b)

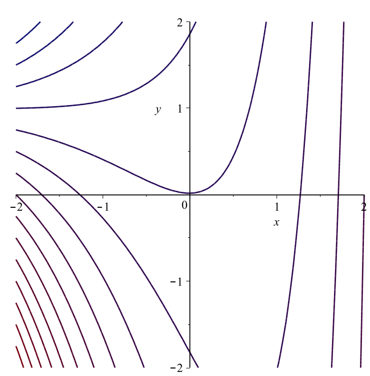


26.(a) Since $(M_y - N_x)/N = -1$ is a function of x only, we know that $\mu = e^{-x}$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$(e^{-x} - e^{x} - ye^{-x})dx + e^{-x}dy = 0.$$

Then $M_y = -e^{-x} = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = -e^{-x} - e^{x} + ye^{-x} + h(y)$. Then $\psi_y = e^{-x} + h'(y) = N = e^{-x}$. Therefore, $h'(y) = 0$ and we conclude that the solution is given implicitly by $-e^{-x} - e^{x} + ye^{-x} = c$.

(b)

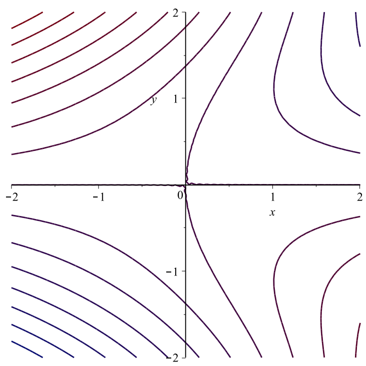


27.(a) Since $(N_x - M_y)/M = 1/y$ is a function of y only, we know that $\mu(y) = e^{\int 1/y\,dy} = y$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$ydx + (x - y\sin y)dy = 0.$$

Then for this equation, $M_y = 1 = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = xy + h(y)$. Then $\psi_y = x + h'(y) = N = x - y\sin y$. Therefore, $h'(y) = -y\sin y$ which implies that $h(y) = -\sin y + y\cos y$, and we conclude that the solution is given implicitly by $xy - \sin y + y\cos y = c$.

(b)

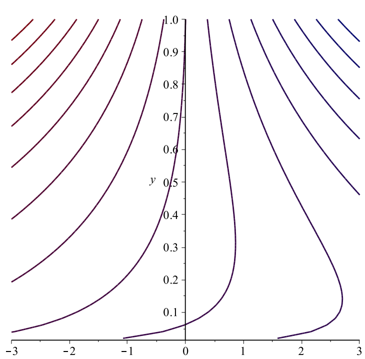


28.(a) Since $(N_x - M_y)/M = (2y-1)/y$ is a function of y only, we know that $\mu(y) = e^{\int 2-1/y\,dy} = e^{2y}/y$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$e^{2y}dx + (2xe^{2y} - 1/y)dy = 0.$$

Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = xe^{2y} + h(y)$. Then $\psi_y = 2xe^{2y} + h'(y) = N = 2xe^{2y} - 1/y$. Therefore, $h'(y) = -1/y$ which implies that $h(y) = -\ln y$, and we conclude that the solution is given implicitly by $xe^{2y} - \ln y = c$ or $y = e^2x + ce^x + 1$.

(b)

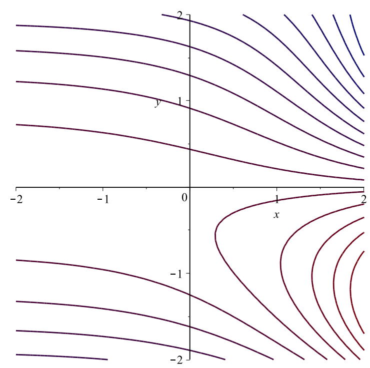


29.(a) Since $(N_x - M_y)/M = \cot y$ is a function of y only, we know that $\mu(y) = e^{\int \cot(y)\,dy} = \sin y$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$e^x \sin y dx + (e^x \cos y + 2y)dy = 0.$$

Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = e^x \sin y + h(y)$. Then $\psi_y = e^x \cos y + h'(y) = N = e^x \cos y + 2y$. Therefore, $h'(y) = 2y$ which implies that $h(y) = y^2$, and we conclude that the solution is given implicitly by $e^x siny + y^2 = c$.

(b)

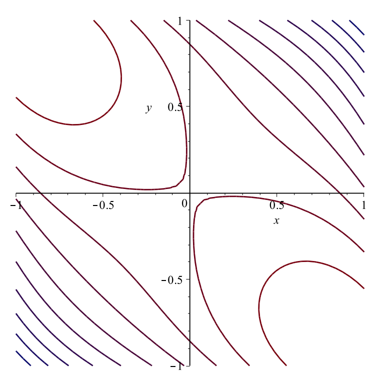


30. Since $(N_x - M_y)/M = 2/y$ is a function of y only, we know that $\mu(y) = e^{\int 2/y\,dy} = y^2$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$(4x^3 + 3y)dx + (3x + 4y^3)dy = 0.$$

Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = x^4 + 3xy + h(y)$. Then $\psi_y = 3x + h'(y) = N = 3x + 4y^3$. Therefore, $h'(y) = 4y^3$ which implies that $h(y) = y^4$, and we conclude that the solution is given implicitly by $x^4 + 3xy + y^4 = c$.

(b)

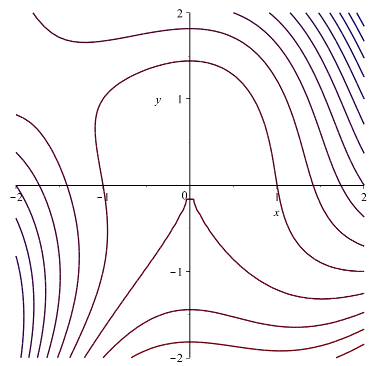


31. Since $(N_x - M_y)/(xM - yN) = 1/xy$ is a function of xy only, we know that $\mu(xy) = e^{\int 1/xy\,dy} = xy$ is an integrating factor for this equation. Multiplying the equation by μ, we have

$$(3x^2y + 6x)dx + (x^3 + 3y^2)dy = 0.$$

Then for this equation, $M_y = N_x$. Therefore, this new equation is exact. Integrating M with respect to x, we conclude that $\psi = x^3y + 3x^2 + h(y)$. Then $\psi_y = x^3 + h'(y) = N = x^3 + 3y^2$. Therefore, $h'(y) = 3y^2$ which implies that $h(y) = y^3$, and we conclude that the solution is given implicitly by $x^3y + 3x^2 + y^3 = c$.

(b)



32. Using the integrating factor $\mu = [xy(2x+y)]^{-1}$, this equation can be rewritten as

$$\left(\frac{2}{x} + \frac{2}{2x+y}\right) dx + \left(\frac{1}{y} + \frac{1}{2x+y}\right) dy = 0.$$

Integrating M with respect to x, we see that $\psi = 2\ln|x| + \ln|2x+y| + h(y)$. Then $\psi_y = (2x+y)^{-1} + h'(y) = N = (2x+y)^{-1} + 1/y$. Therefore, $h'(y) = 1/y$ which implies that $h(y) = \ln|y|$. Therefore, $\psi = 2\ln|x| + \ln|2x+y| + \ln|y| = c$. Applying the exponential function, we conclude that the solution is given implicitly be $2x^3y + x^2y^2 = c$.

2.7 Substitution Methods

1.(a) $f(x,y) = (x+1)/y$, thus $f(\lambda x, \lambda y) = (\lambda x + 1)/\lambda y \neq (x+1)/y$. The equation is not homogeneous.

2.(a) $f(x,y) = (x^4+1)/(y^4+1)$, thus $f(\lambda x, \lambda y) = (\lambda^4 x^4 + 1)/(\lambda^4 y^4 + 1) \neq (x^4+1)/(y^4+1)$. The equation is not homogeneous.

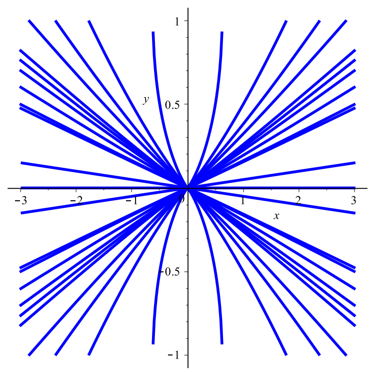
3.(a) $f(x,y) = (3x^2y + y^3)/(3x^3 - xy^2)$ satisfies $f(\lambda x, \lambda y) = f(x,y)$. The equation is homogeneous.

(b) The equation is $y' = (3x^2y + y^3)/(3x^3 - xy^2) = (3(y/x) + (y/x)^3)/(3 - (y/x)^2)$. Let $y = ux$. Then $y' = u'x + u$, thus $u'x = (3u + u^3)/(3 - u^2) - u = 2u^3/(3 - u^2)$. We obtain

$$\int \frac{3-u^2}{2u^3}\, du = -\frac{3}{4}u^{-2} - \frac{1}{2}\ln|u| = \int \frac{1}{x}\, dx = \ln|x| + c.$$

Therefore, the solution is given implicitly by $-(3/4)x^2/y^2 - (1/2)\ln|y/x| = \ln|x| + c$. Also, $u = 0$ solves the equation, thus $y = 0$ is a solution as well.

(c)



4.(a) $f(x,y) = y(y+1)/x(x-1)$, thus $f(\lambda x, \lambda y) = \lambda y(\lambda y+1)/\lambda x(\lambda x+1) \neq y(y+1)/x(x-1)$. The equation is not homogeneous.

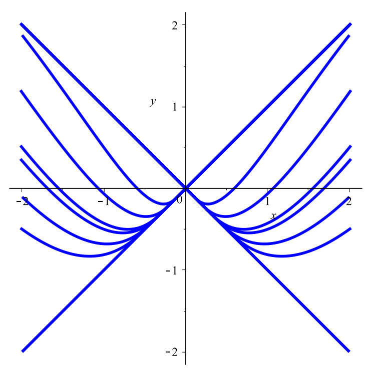
5.(a) $f(x,y) = (\sqrt{x^2-y^2}+y)/x$ satisfies $f(\lambda x, \lambda y) = f(x,y)$. The equation is homogeneous.

(b) The equation is $y' = (\sqrt{x^2-y^2}+y)/x = \sqrt{1-(y/x)^2} + y/x$. Let $y = ux$. Then $y' = u'x + u$, thus $u'x = \sqrt{1-u^2} + u - u = \sqrt{1-u^2}$. We obtain

$$\int \frac{1}{\sqrt{1-u^2}}\, du = \arcsin u = \int \frac{1}{x}\, dx = \ln|x| + c.$$

Therefore, the solution is given implicitly by $\arcsin(y/x) = \ln|x| + c$, thus $y = x\sin(\ln|x| + c)$. Also, $y = x$ and $y = -x$ are solutions.

(c)



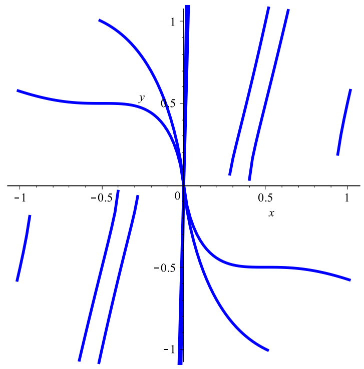
6.(a) $f(x,y) = (x+y)^2/xy$ satisfies $f(\lambda x, \lambda y) = f(x,y)$. The equation is homogeneous.

(b) The equation is $y' = (x^2+2xy+y^2)/xy = x/y + 2 + y/x$. Let $y = ux$. Then $y' = u'x + u$, thus $u'x = 1/u + 2 + u - u = 1/u + 2 = (1+2u)/u$. We obtain

$$\int \frac{u}{1+2u}\, du = \frac{u}{2} - \frac{1}{4}\ln|1+2u| = \int \frac{1}{x}\, dx = \ln|x| + c.$$

Therefore, the solution is given implicitly by $y/2x - (1/4)\ln|1+2y/x| = \ln|x| + c$. Also, $y = -x/2$ is a solution.

(c)



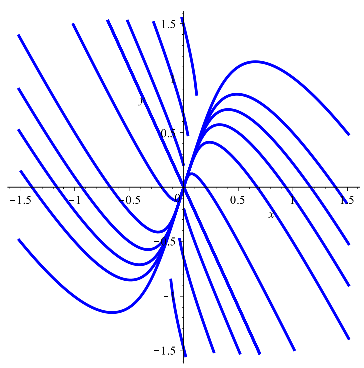
7.(a) $f(x,y) = (4y-7x)/(5x-y)$ satisfies $f(\lambda x, \lambda y) = f(x,y)$. The equation is homogeneous.

(b) The equation is $y' = (4y-7x)/(5x-y) = (4y/x-7)/(5-y/x)$. Let $y = ux$. Then $y' = u'x + u$, thus $u'x = (4u-7)/(5-u) - u = (u^2-u-7)/(5-u)$. We obtain

$$\int \frac{5-u}{u^2-u-7}\,du = \frac{9\sqrt{29}-29}{58}\ln|1+\sqrt{29}-2u| - \frac{29+9\sqrt{29}}{58}\ln|-1+\sqrt{29}+2u| = \ln|x| + c.$$

The solution is given implicitly by substituting back $u = y/x$. Also, $y = x(1\pm\sqrt{29})/2$ are solutions.

(c)



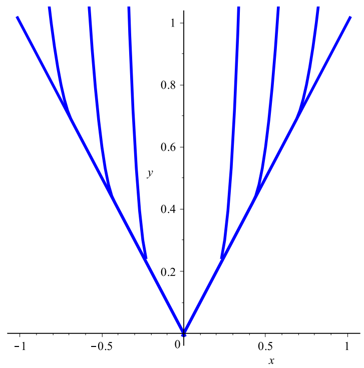
8.(a) $f(x,y) = (4\sqrt{y^2-x^2}+y)/x$ satisfies $f(\lambda x, \lambda y) = f(x,y)$. The equation is homogeneous.

(b) The equation is $y' = (4\sqrt{y^2-x^2}+y)/x = 4\sqrt{(y/x)^2-1}+y/x$. Let $y = ux$. Then $y' = u'x+u$, thus $u'x = 4\sqrt{u^2-1}+u-u = 4\sqrt{u^2-1}$. We obtain

$$\int \frac{1}{4\sqrt{u^2-1}}\,du = \frac{1}{4}\ln|u+\sqrt{u^2-1}| = \int \frac{1}{x}\,dx = \ln|x| + c.$$

Therefore, the solution is given implicitly by $\ln|y/x+\sqrt{(y/x)^2-1}| = \ln x^4 + c$. Also, $y = x$ and $y = -x$ are solutions.

(c)



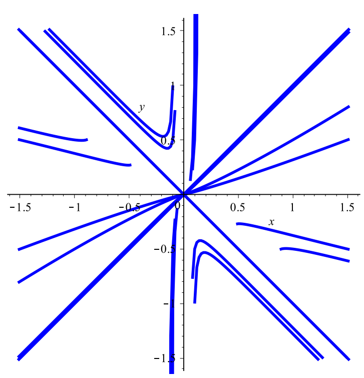
9.(a) $f(x,y) = (y^4+2xy^3-3x^2y^2-2x^3y)/(2x^2y^2-2x^3y-2x^4)$ satisfies $f(\lambda x, \lambda y) = f(x,y)$. The equation is homogeneous.

(b) The equation is $y' = (y^4+2xy^3-3x^2y^2-2x^3y)/(2x^2y^2-2x^3y-2x^4) = ((y/x)^4 + 2(y/x)^3 - 3(y/x)^2 - 2(y/x))/(2(y/x)^2-2(y/x)-2)$. Let $y = ux$. Then $y' = u'x+u$, thus $u'x = (u^4+2u^3-3u^2-2u)/(2u^2-2u-2) - u = (u^4-u^2)/(2u^2-2u-2)$. We obtain

$$\int \frac{2u^2-2u-2}{u^4-u^2}\,du = -\frac{2}{u} + 2\ln|u| - \ln|1-u^2| = \ln|x| + c.$$

The solution is given implicitly by $\ln|1-y^2/x^2| + 2x/y + \ln|x| = c$. Also, $y = x$, $y = -x$ and $y = 0$ are solutions.

(c)



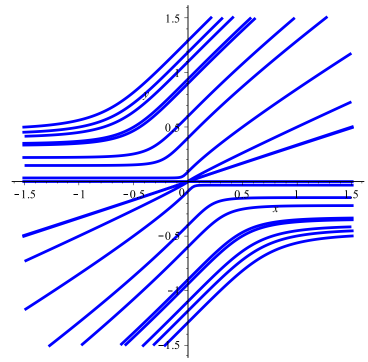
10.(a) $f(x,y) = (y+xe^{x/y})/ye^{x/y}$ satisfies $f(\lambda x, \lambda y) = f(x,y)$. The equation is homogeneous.

(b) The equation is $dx/dy = (y+xe^{x/y})/ye^{x/y} = e^{-x/y}+x/y$. Let $x = uy$. Then $x' = u'y+u$, thus $u'y = e^{-u}+u-u = e^{-u}$. We obtain

$$\int e^u\,du = e^u = \int \frac{1}{y}\,dy = \ln|y| + c.$$

Therefore, the solution is given by $x/y = \ln(\ln|y|+c)$, i.e. $x = y\ln(\ln|y|+c)$. Also, $y = 0$ is a solution.

(c)



11. The equation is homogeneous. Let $y = ux$; we obtain $y' = u'x + u = 1/u + u$, thus $udu = (1/x)dx$ and we obtain $(y/x)^2 = u^2 = 2(\ln x + c)$. The initial condition gives $1/4 = 2(\ln 2 + c)$, thus $c = 1/8 - \ln 2$ and the solution is $y = x\sqrt{2\ln x + 1/4 - \ln 4}$. The solution exists on the interval $(2e^{-1/8}, \infty)$.

12. The equation is homogeneous. Let $y = ux$; we obtain $y' = u'x + u = (1+u)/(1-u)$, i.e. $u'x = (1+u^2)/(1-u)$. Integration gives

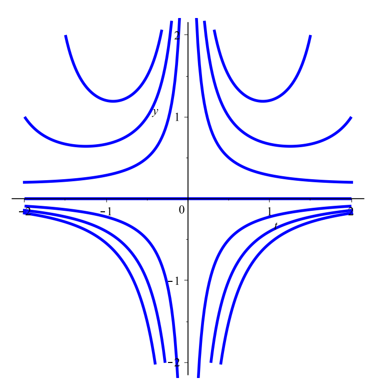
$$\int \frac{1-u}{1+u^2}\,du = \arctan u - \frac{1}{2}\ln(1+u^2) = \int \frac{1}{x}\,dx = \ln|x| + c.$$

The initial condition implies that $\arctan(8/5) - \ln\sqrt{1+64/25} = \ln 5 + c$. The solution is given implicitly by $\arctan(y/x) - \ln\sqrt{1+y^2/x^2} - \ln|x| = c$. The solution exists on the interval $(-128.1, 5.3)$, approximately.

13.(a) $y' + (1/t)y = ty^2$

(b) Here $n = 2$, thus we set $u = y^{-1}$. The equation becomes $u' - (1/t)u = -t$; the integrating factor is $\mu = 1/t$ and we obtain $(u/t)' = -1$. After integration, we get $u/t = -t + c$, thus $u = -t^2 + ct$ and then $y = 1/(ct - t^2)$. Also, $y = 0$ is a solution.

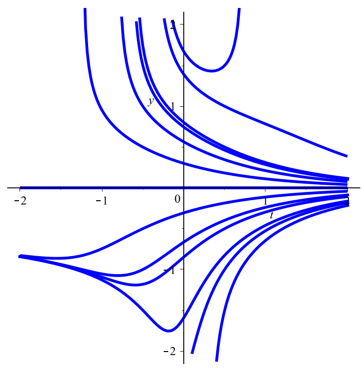
(c)



14.(a) $y' + y = ty^4$

(b) Here $n = 4$, thus we set $u = y^{-3}$. The equation becomes $u' - 3u = -t$; the integrating factor is $\mu = e^{-3t}$ and we obtain $(ue^{-3t})' = -te^{-3t}$. After integration, $ue^{-3t} = (t/3)e^{-3t} + e^{-3t}/9 + c$, thus $u = t/3 + 1/9 + ce^{3t}$ and then $y = (t/3 + 1/9 + ce^{3t})^{-1/3}$. Also, $y = 0$ is a solution.

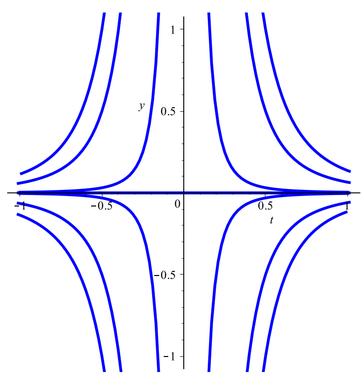
(c)



15.(a) $y' + (3/t)y = t^2y^2$

(b) Here $n = 2$, thus we set $u = y^{-1}$. The equation becomes $u' - (3/t)u = -t^2$; the integrating factor is $\mu = 1/t^3$ and we obtain $(u/t^3)' = -1/t$. After integration, we get $u/t^3 = -\ln t + c$, thus $u = -t^3 \ln t + ct^3$ and then $y = 1/(ct^3 - t^3 \ln t)$. Also, $y = 0$ is a solution.

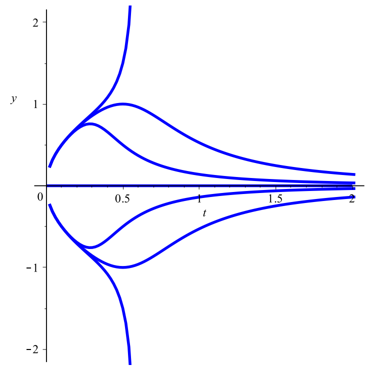
(c)



16.(a) $y' + (2/t)y = (1/t^2)y^3$

(b) Here $n = 3$, thus we set $u = y^{-2}$. The equation becomes $u' - (4/t)u = -2/t^2$; the integrating factor is $\mu = 1/t^4$ and we obtain $(u/t^4)' = -2/t^6$. After integration, $u/t^4 = 2t^{-5}/5 + c$, thus $u = 2t^{-1}/5 + ct^4$ and then $y = (2t^{-1}/5 + ct^4)^{-1/2}$. Also, $y = 0$ is a solution.

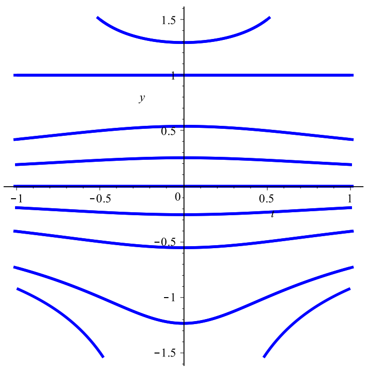
(c)



17.(a) $y' + (4t/5(1+t^2))y = (4t/5(1+t^2))y^4$

(b) Here $n = 4$, thus we set $u = y^{-3}$. The equation becomes $u' - 12t/5(1+t^2)u = -12t/5(1+t^2)$; the integrating factor is $\mu = (1+t^2)^{-6/5}$ and we obtain $(u\mu)' = -12t(1+t^2)^{-11/5}/5$. After integration, $u = 1 + c(1+t^2)^{6/5}$, thus $y = (1 + c(1+t^2)^{6/5})^{-1/3}$. Also, $y = 0$ is a solution.

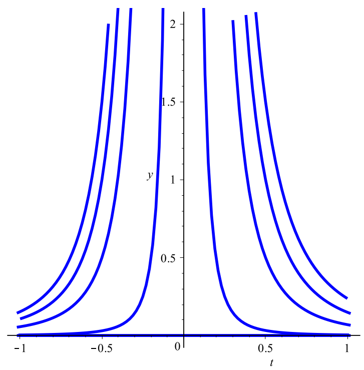
(c)



18.(a) $y' + (3/t)y = (2/3)y^{5/3}$

(b) Here $n = 5/3$, thus we set $u = y^{-2/3}$. The equation becomes $u' - (2/t)u = -4/9$; the integrating factor is $\mu = 1/t^2$ and we obtain $(u/t^2)' = -4/9t^2$. After integration, $u/t^2 = 4/9t + c$, thus $u = 4t/9 + ct^2$ and then $y = (4t/9 + ct^2)^{-3/2}$. Also, $y = 0$ is a solution.

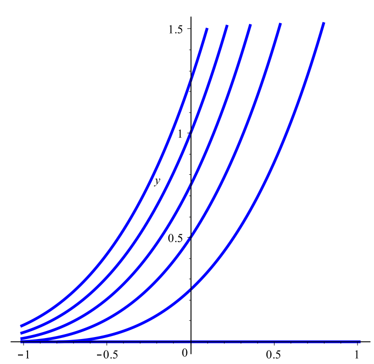
(c)



19.(a) $y' - y = y^{1/2}$

(b) Here $n = 1/2$, thus we set $u = y^{1/2}$. The equation becomes $u' - u/2 = 1/2$; the integrating factor is $\mu = e^{-t/2}$ and we obtain $(ue^{-t/2})' = e^{-t/2}/2$. After integration, $ue^{-t/2} = -e^{-t/2} + c$, thus $u = ce^{t/2} - 1$ and then $y = (ce^{t/2} - 1)^2$. Also, $y = 0$ is a solution.

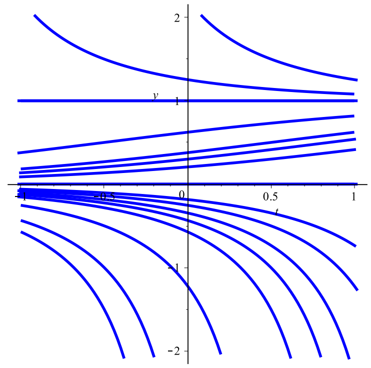
(c)



20.(a) $y' - ry = -ky^2$

(b) Here, $n = 2$. Therefore, let $u = y^{-1}$. Making this substitution, we see that u satisfies the equation $u' + ru = k$. This equation is linear with integrating factor e^{rt}. Therefore, we have $(e^{rt}u)' = ke^{rt}$. The solution of this equation is given by $u = (k + cre^{-rt})/r$. Then, using the fact that $y = 1/u$, we conclude that $y = r/(k + cre^{-rt})$. Also, $y = 0$ is a solution.

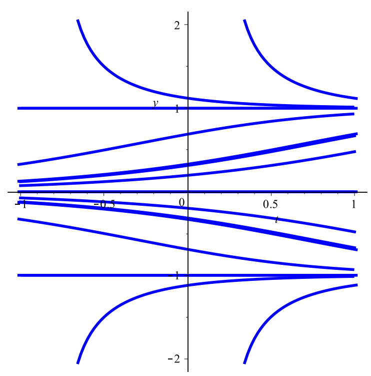
(c) The figure shows the solutions for $r = 1$, $k = 1$.



21.(a) $y' - \epsilon y = -\sigma y^3$

(b) Here $n = 3$. Therefore, $u = y^{-2}$ satisfies $u' + 2\epsilon u = 2\sigma$. This equation is linear with integrating factor $e^{2\epsilon t}$. Its solution is given by $u = (\sigma + c\epsilon e^{-2\epsilon t})/\epsilon$. Then, using the fact that $y^2 = 1/u$, we see that $y = \pm\sqrt{\epsilon}/\sqrt{\sigma + c\epsilon e^{-2\epsilon t}}$.

(c) The figure shows the solutions for $\epsilon = 1$, $\sigma = 1$.



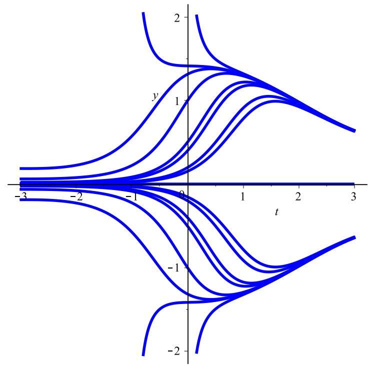
22.(a) $y' - (\Gamma \cos t + T)y = -y^3$

(b) Here $n = 3$. Therefore, $u = y^{-2}$ satisfies $u' + 2(\Gamma \cos t + T)u = 2$. This equation is linear with integrating factor $e^{2(\Gamma \sin t + Tt)}$. Therefore, $\left(e^{2(\Gamma \sin t + Tt)}u\right)' = 2e^{2(\Gamma \sin t + Tt)}$, which implies

$$u = 2e^{-2(\Gamma \sin t + Tt)} \int_{t_0}^{t} \exp(2(\Gamma \sin s + Ts))\, ds + ce^{-2(\Gamma \sin t + Tt)}.$$

Then $u = y^{-2}$ implies $y = \pm\sqrt{1/u}$.

(c) The figure shows the solutions for $\Gamma = 1$, $T = 1$.



23.(a) Assume y_1 solves the equation: $y_1' = A + By_1 + Cy_1^2$. Let $y = y_1 + v$; we obtain $y_1' + v' = y' = A + By + Cy^2 = A + B(y_1+v) + C(y_1+v)^2 = A + By_1 + Bv + Cy_1^2 + 2Cy_1v + Cv^2$. Then $v' = Bv + 2Cy_1v + Cv^2$, i.e. $v' - (B + 2Cy_1)v = Cv^2$ which is a Bernoulli equation with $n = 2$.

(b) If $y_1 = 4t$, then $y_1' = 4$ and $4 + 3t \cdot 4t = 4 - 4t^2 + (4t)^2$. Using the previous idea, let $y = y_1 + v$; we obtain $4 + v' = y_1' + v' = y' = 4 - 4t^2 + y^2 - 3ty = 4 - 4t^2 + (4t + v)^2 - 3t(4t + v)$, i.e. $v' = -4t^2 + 16t^2 + 8tv + v^2 - 12t^2 - 3tv = 5tv + v^2$. Let $u = v^{-1}$, then we obtain $u' + 5tu = -1$. The integrating factor is $\mu = e^{5t^2/2}$, and we obtain $u = -e^{-5t^2/2} \int_0^t e^{5s^2/2}\, ds$. Thus $y = 4t - (e^{-5t^2/2} \int_0^t e^{5s^2/2}\, ds)^{-1}$.

24.(a) Homogeneous.

(b) Setting $y = ux$, we obtain $y' = u'x + u = (u - 3)/(9u - 2)$, i.e. $u'x = 3(-1 + u - 3u^2)/(9u - 2)$. After integration, we obtain the implicit solution $(3/2)\ln(1 - y/x + 3y^2/x^2) - \arctan((-1 + 6y/x)/\sqrt{11})/\sqrt{11} + 3\ln x = c$.

25.(a) Linear.

(b) Consider the equation so that $x = x(y)$. Then $dx/dy = -2x + 3e^y$; the integrating factor is $\mu = e^{2y}$, we obtain $(e^{2y}x)' = 3e^{3y}$. After integration, $e^{2y}x = e^{3y} + c$, thus $x = e^y + ce^{-2y}$.

26.(a) Bernoulli.

(b) Let $u = y^{-1}$. The equation turns into $u' + u = -4e^x$; integrating factor is $\mu = e^x$. We obtain $(ue^x)' = -4e^{2x}$, after integration $ue^x = -2e^{2x} + c$, thus $u = -2e^x + ce^{-x}$ and then $y = 1/(ce^{-x} - 2e^x)$.

27.(a) Linear.

(b) The integrating factor is $\mu = e^{x + \ln x} = xe^x$; the equation turns into $(xe^xy)' = xe^x$, after integration $xe^xy = xe^x - e^x + c$, and then $y = 1 - 1/x + ce^{-x}/x$.

28.(a) Exact.

(b) The equation is $(\frac{1}{2}\sin 2x - xy^2)dx + (1 - x^2)ydy$. We need $\psi(x, y)$ so that $\psi_x = \frac{1}{2}\sin 2x - xy^2$; thus $\psi(x, y) = -\frac{1}{4}\cos 2x - \frac{1}{2}x^2y^2 + h(y)$. Now $\psi_y = -x^2y + h'(y) = -x^2y + y$, thus $h(y) = y^2/2$. We obtain the implicitly defined solution $\cos 2x + 2x^2y^2 - 2y^2 = c$.

29.(a) Separable, linear.

(b) Separation of variables gives $dy/y = dx/\sqrt{x}$; after integration, we get $\ln y = 2\sqrt{x} + c$ and then $y = ce^{2\sqrt{x}}$.

30.(a) Separable, exact.

(b) Write $(5xy^2 + 5y)dx + (5x^2y + 5x)dy = 0$. We need $\psi(x, y)$ so that $\psi_x = 5xy^2 + 5y$; thus $\psi(x, y) = 5x^2y^2/2 + 5xy + h(y)$. Now $\psi_y = 5x^2y + 5x + h'(y) = 5x^2y + 5x$, thus we obtain that the solution is given implicitly as $5x^2y^2 + 10xy = 5xy(xy + 2) = c$. We can see that this is the same as $xy = C$.

31.(a) Exact, Bernoulli.

(b) Write $(y^2 + 1 + \ln x)dx + 2xydy = 0$. We need $\psi(x, y)$ so that $\psi_x = y^2 + 1 + \ln x$; thus $\psi(x, y) = y^2x + x\ln x + h(y)$. Now $\psi_y = 2xy + h'(y) = 2xy$, thus we obtain that the solution is given implicitly as $y^2x + x \ln x = c$.

32.(a) Linear, exact.

(b) Write $(-y - 2(2 - x)^5)dx + (2 - x)dy = 0$. We need $\psi(x, y)$ so that $\psi_x = -y - 2(2 - x)^5$; thus $\psi(x, y) = -yx + (2 - x)^6/3 + h(y)$. Now $\psi_y = -x + h'(y) = 2 - x$, and then $h(y) = 2y$. We obtain that the solution is given implicitly as $-3yx + (2 - x)^6 + 6y = c$.

33.(a) Separable, autonomous (if viewed as dx/dy).

(b) $dy/dx = -x/\ln x$, thus after integration, $y = -\ln \ln x + C$.

34.(a) Homogeneous.

(b) Setting $y = ux$, we obtain $y' = u'x + u = (3u^2 + 2u)/(2u + 1)$. This implies that $u'x = (u + u^2)/(1 + 2u)$. After integration, we obtain that the implicit solution is given by $\ln(y/x) + \ln(1 + y/x) = \ln x + c$, i.e. $y/x^2 + y^2/x^3 = C$.

35.(a) Bernoulli, homogeneous.

(b) Let $u = y^2$. Then $u' = 2yy' = 4x + (5/2x)y^2 = 4x + (5/2x)u$; we get the linear equation $u' - (5/2x)u = 4x$. The integrating factor is $\mu = x^{-5/2}$, and the equation turns into $(ux^{-5/2})' = 4x^{-3/2}$. After integration, we get $u = y^2 = -8x^2 + cx^{5/2}$.

36.(a) Autonomous, separable, Bernoulli.

(b) Let $u = y^{3/4}$. Then $u' = (3/4)y^{-1/4}y' = (3/4)y^{-1/4}(y^{1/4} - y) = 3/4 - 3u/4$. The integrating factor is $\mu = e^{3x/4}$, and we get $(ue^{3x/4})' = 3e^{3x/4}/4$. After integration, $ue^{3x/4} = e^{3x/4} + c$, and then $u = y^{3/4} = 1 + ce^{-3x/4}$. We get $y = (1 + ce^{-3x/4})^{4/3}$.