Chapter 2. First-Order Equations

Section 2.1. Differential Equations and Solutions

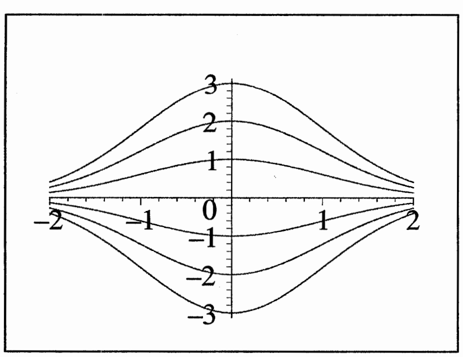
1. $\phi(t, y, y') = t^2 y' + (1+t)y = 0$ must be solved for y'. We get

$$y' = -\frac{(1+t)y}{t^2}.$$

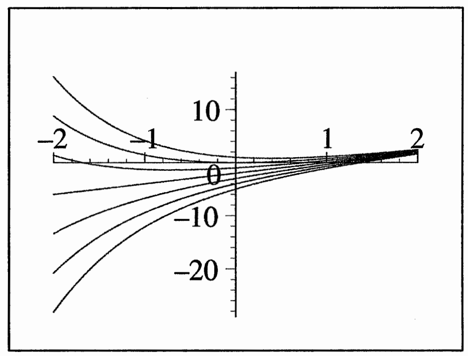
2. $\phi(t, y, y') = ty' - 2y - t^2$ must be solved for y'. We get

$$y' = \frac{2y + t^2}{t}.$$

3. $y'(t) = -Cte^{-(1/2)t^2}$ and $-ty(t) = -tCe^{-(1/2)t^2}$, so $y' = -ty$.

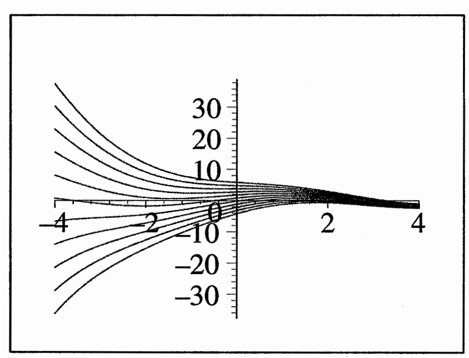


4. $y'(t) + y(t) = (2 - Ce^{-t}) + (2t - 2 + Ce^{-t}) = 2t$



5. If $y(t) = (4/5)\cos t + (8/5)\sin t + Ce^{-(1/2)t}$, then

$$\begin{aligned} &y(t)' + (1/2)y(t) \\ &\quad = [-(4/5)\sin t + (8/5)\cos t - (C/2)e^{-(1/2)t}] \\ &\qquad + (1/2)[(4/5)\cos t + (8/5)\sin t + Ce^{-(1/2)t}] \\ &\quad = 2\cos t. \end{aligned}$$



6. If $y(t) = 4/(1 + Ce^{-4t})$, then

$$y' = \frac{16C\,e^{(-4t)}}{(1 + C\,e^{(-4t)})^2}$$
$$y(4 - y) = \frac{4}{1 + Ce^{-4t}} \times \left[4 - \frac{4}{1 + Ce^{-4t}}\right]$$
$$= \frac{16(1 + Ce^{-4t}) - 16}{(1 + C\,e^{(-4t)})^2}$$
$$= \frac{16C\,e^{(-4t)}}{(1 + C\,e^{(-4t)})^2}$$

7. For $y(t) = 0$, $y'(t) = 0$ and $y(t)(4 - y(t)) = 0(4 - 0) = 0$.

8. (a) If $t^2 + y^2 = C^2$, then

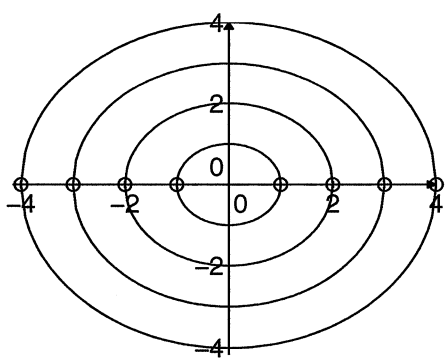
$$\frac{d}{dt}(t^2 + y^2) = \frac{d}{dt}c^2$$
$$2t + 2yy' = 0$$
$$t + yy' = 0$$

(b) If $y(t) = \pm\sqrt{C^2 - t^2}$, then $y' = \mp t/\sqrt{C^2 - t^2}$, and

$$t + yy' = t + [\pm\sqrt{C^2 - t^2}][\mp t/\sqrt{C^2 - t^2}]$$
$$= t - t$$
$$= 0.$$

(c) For $\sqrt{C^2 - t^2}$ to be defined we must have $t^2 \leq C^2$. In order that $y'(t) = \mp t/\sqrt{C^2 - t^2}$, we must restrict the domain further. Hence the interval of existence is $-C < t < C$.

(d)



9. (a) If $t^2 - 4y^2 = c^2$, then

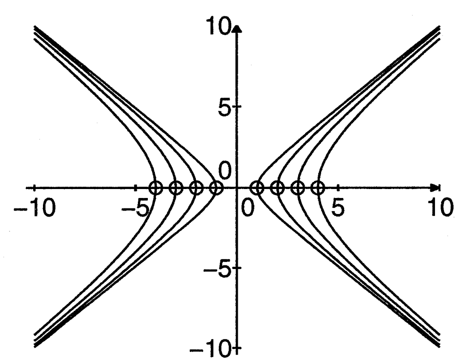
$$\frac{d}{dt}(t^2 - 4y^2) = \frac{d}{dt}t^2$$
$$2t - 8yy' = 0$$
$$t - 4yy' = 0.$$

(b) If $y(t) = \pm\sqrt{t^2 - C^2}/2$, then $y' = \pm t/(2\sqrt{t^2 - C^2})$, and

$$t - 4yy'$$
$$= t - 4[\pm\sqrt{t^2 - C^2}/2][\pm t/(2\sqrt{t^2 - C^2})]$$
$$= t - t$$
$$= 0.$$

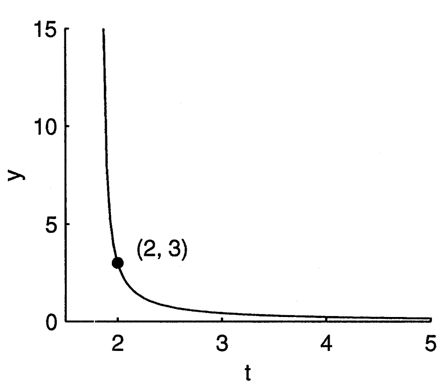
(c) The interval of existence is either $-\infty < t < -C$ or $C < t < \infty$.

(d)

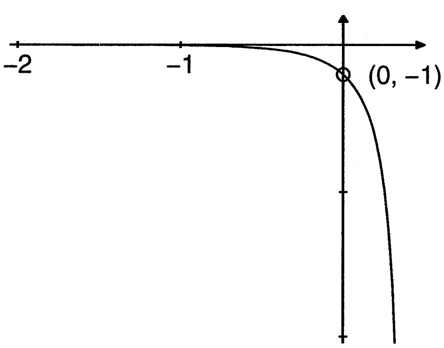


10. If $y(t) = 3/(6t - 11)$, then $y' = -3 \cdot 6/(6t - 11)^2 = -18/(6t - 11)^2$. On the other hand, $-2y^2 = -2[3/(6t - 11)]^2 = -18/(6t - 11)^2$, so we have a solution to the differential equation. Since $y(2) = -3/(12 - 11) = -3$, we have a solution to the initial value problem. The interval of existence is the interval containing 2 where $6t - 11 \neq 0$. This is the

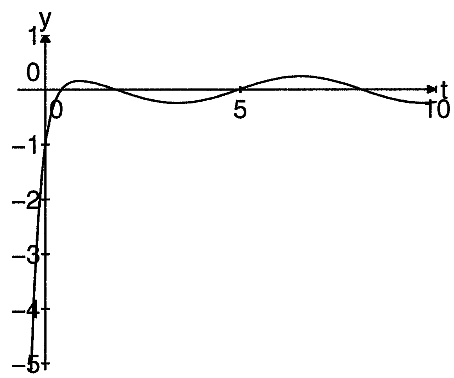
interval $(11/6, \infty)$.



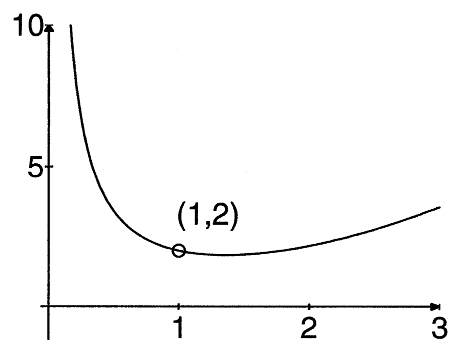
11. See Exercise 6. The interval of existence is $(-\infty, \ln(5)/4)$.



12. $y(t) = (4/17)\cos t + (1/17)\sin t - (21/17)e^{-4t}$ on the interval $(-\infty, \infty)$.

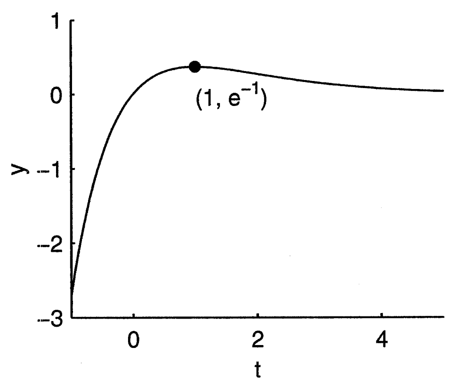


13. $y(t) = \dfrac{1}{3}t^2 + \dfrac{5}{3t}$. The interval of existence is $(0, \infty)$.

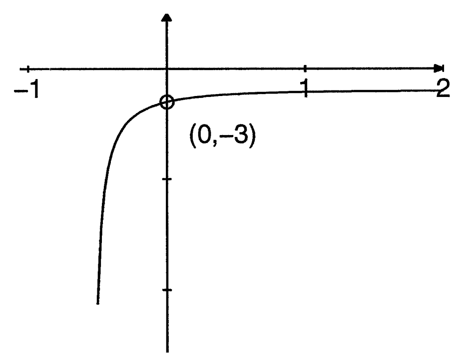


14. We need $e^{-1} = y(1) = e^{-1}(1 + C/(1)) = (1 + C)e^{-1}$. Hence $C = 0$, and our solution is $y(t) = te^{-t}$. This function is defined and differentiable on the whole real line. Hence the interval of existence

is the whole real line.



15. $y(t) = 2/(-1+e^{-2t/3})$. The interval of existence is $(-\ln(3)/2, \infty)$.



16. The initial value problem is

$$y' = \sqrt{y}, \quad y(0) = 1.$$

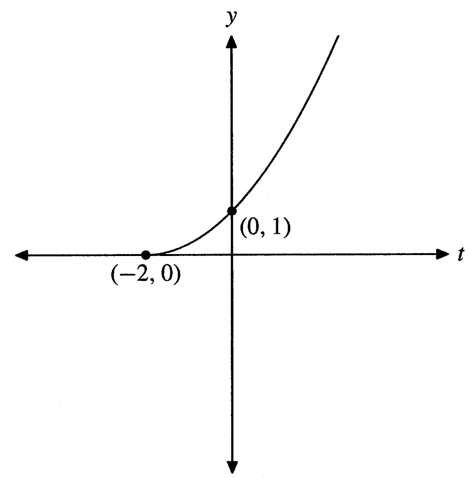
The first solution proposed by Maple, $y(t) = (1/4)(t+2)^2$, satisfies the initial condition, $y(0) = (1/4)(0+2)^2 = 1$. Next,

$$y'(t) = \frac{1}{2}(t+2),$$

and

$$\sqrt{y(t)} = \sqrt{\frac{1}{4}(t+2)^2} = \frac{1}{2}|t+2|.$$

But this equals $(1/2)(t+2)$ only if $t \geq -2$, as shown in the following figure.



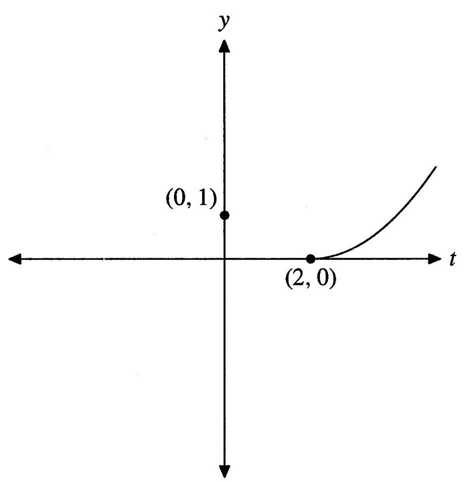
The second solution proposed by Maple, $y(t) = (1/4)(t-2)^2$, satisfies the initial condition, as $y(0) = (1/4)(0-2)^2 = 1$. But

$$y'(t) = \frac{1}{2}(t-2),$$

and

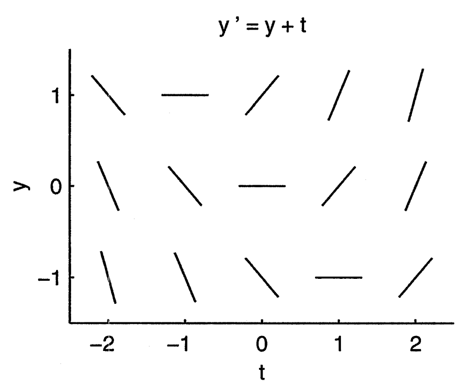
$$\sqrt{y(t)} = \sqrt{\frac{1}{4}(t-2)^2} = \frac{1}{2}|t-2|.$$

But this agrees with $(1/2)(t-2)$ only if $t \geq 2$, as shown in the following figure.



Note that this graph does *not* pass through (0, 1). Hence, $y(t) = (1/4)(t-2)^2$ is *not* a solution of the initial value problem.

17.



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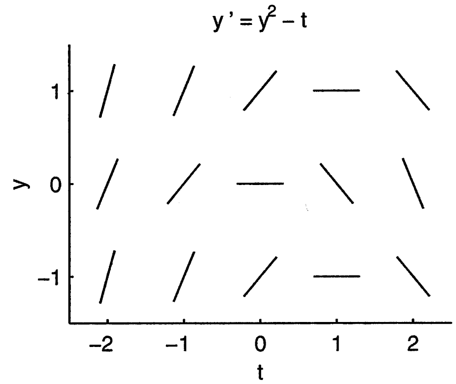

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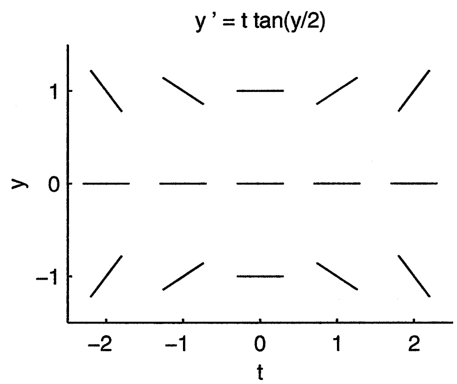

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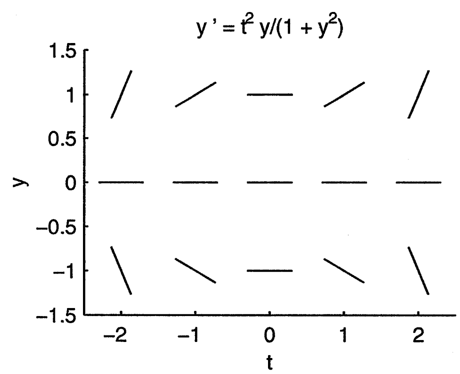

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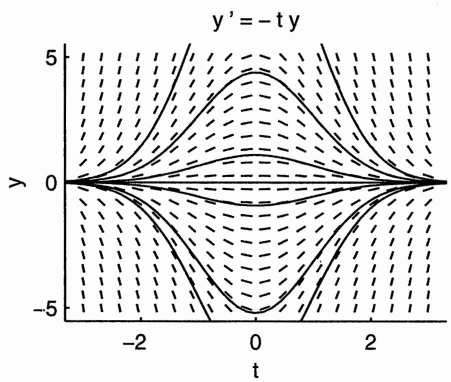

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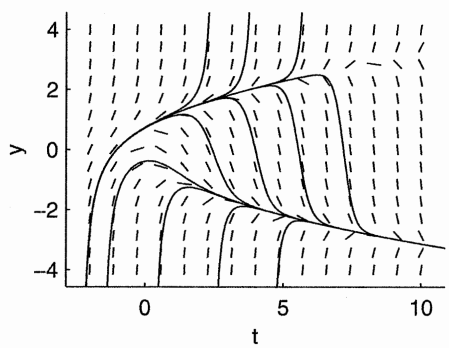

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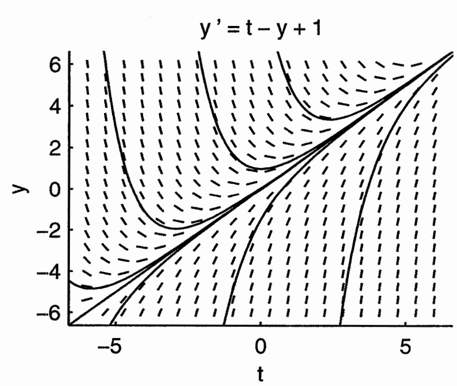
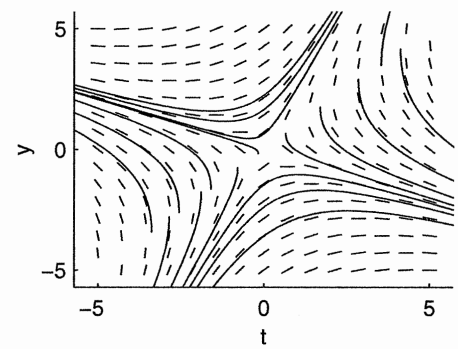


24. Note the difficulty experienced by the solver as it approaches the line $y = t$, where the denominator of the right-hand side of

$$y' = \frac{y+t}{y-t}$$

is undefined.



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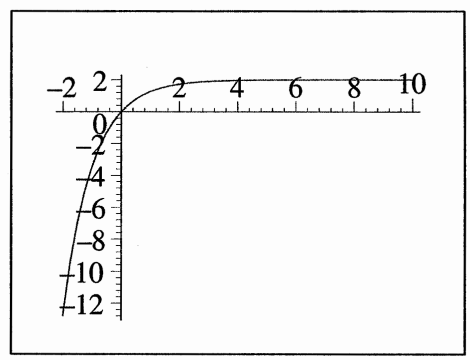

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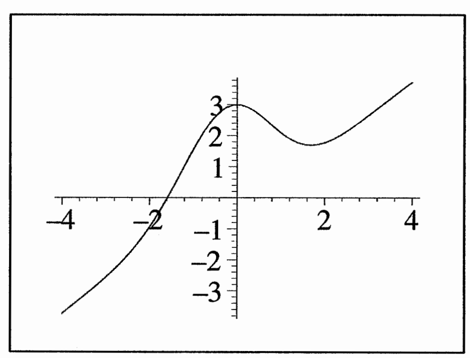

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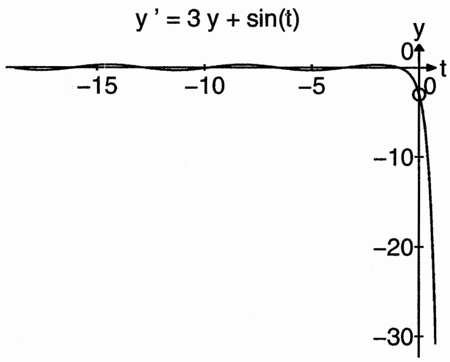

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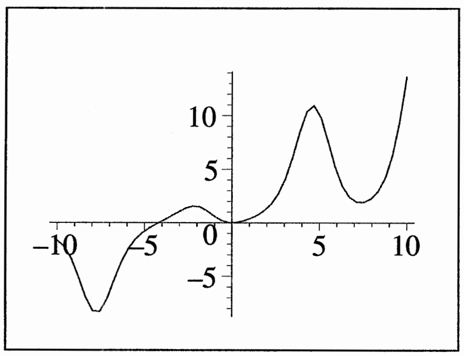

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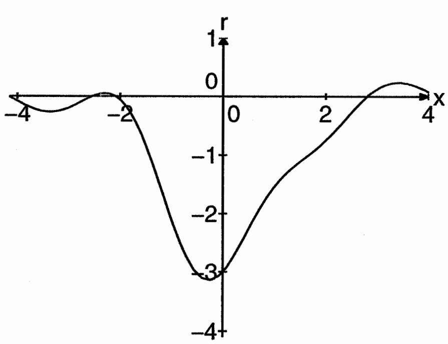

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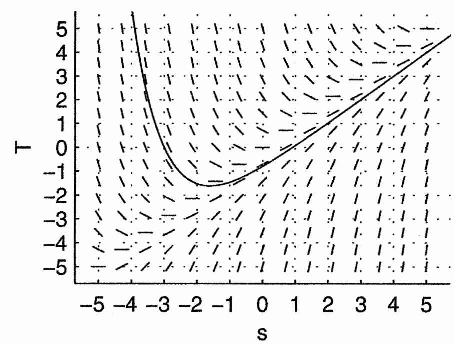

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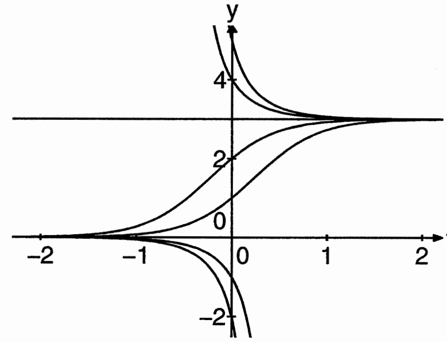

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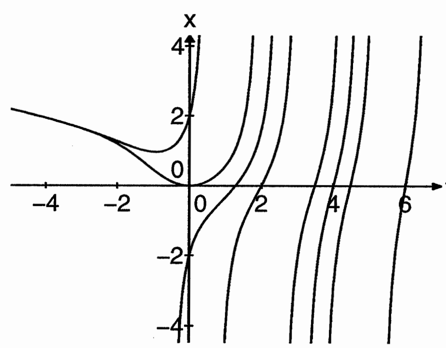

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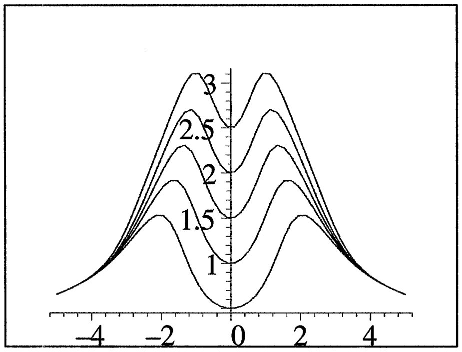

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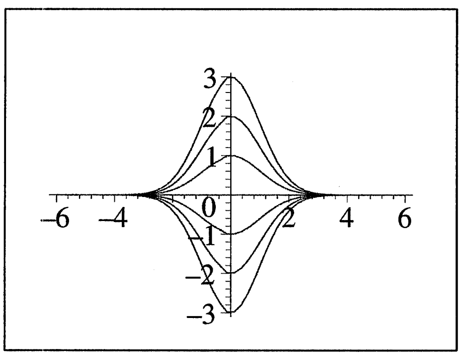
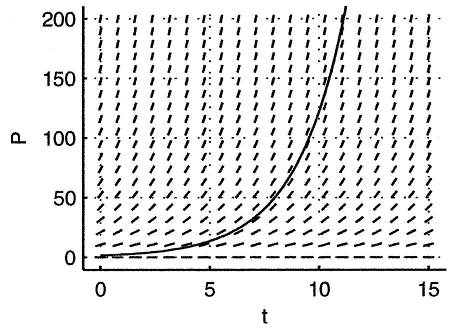


35. We must solve the initial value problem

$$\frac{dP}{dt} = 0.44P, \qquad P(0) = 1.5.$$

Using our numerical solver, we input the equation and initial condition, arriving at the following solution curve.

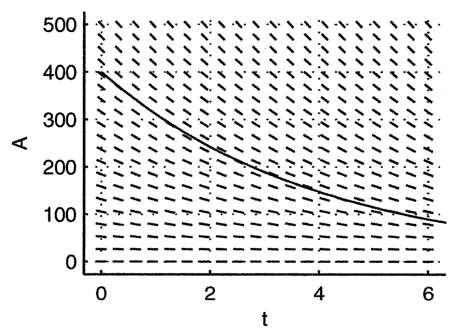


Using the solution curve, we estimate that $P(10) \approx 124$. Thus, there are approximately 124 mg of bacteria present after 10 days.

36. We must solve the initial value problem

$$\frac{dA}{dt} = -0.25A, \qquad A(0) = 400.$$

Using our numerical solver, we input the equation and initial condition, arriving at the following solution curve.

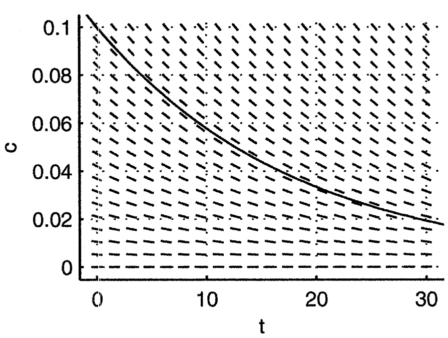


Use the solution curve to estimate $A(4) \approx 150$. Thus, there are approximately 150 mg of material remaining after 4 days.

37. We must solve the initial value problem

$$\frac{dc}{dt} = -0.055c, \qquad c(0) = 0.10.$$

Using our numerical solver, we input the equation and initial condition, arriving at the following solution curve.



Use the solution curve to estimate that it takes a little more than 29 days for the concentration level to dip below 0.02.

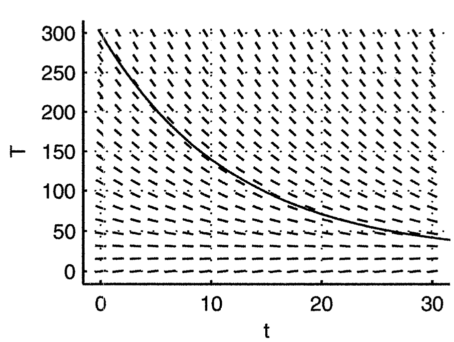
38. The rate at which the rod cools is proportional to the difference between the temperature of the rod and the surrounding air (20° Celsius). Thus,

$$\frac{dT}{dt} = -k(T - 20).$$

With $k = 0.085$ and an initial temperature of 300° Celsius, we must solve the initial value problem

$$\frac{dT}{dt} = -0.085(T - 20), \qquad T(0) = 300,$$

where T is the temperature of the rod at t minutes. Note that since the initial temperature is larger than the surrounding air (20° Celsius), the minus sign insures that the model implies that the rod is cooling. Using our numerical solver, we input the equation and initial condition, arriving at the following solution curve.



Use the solution curve to estimate that it takes a little less than 15 minutes to cool to 100° Celsius.

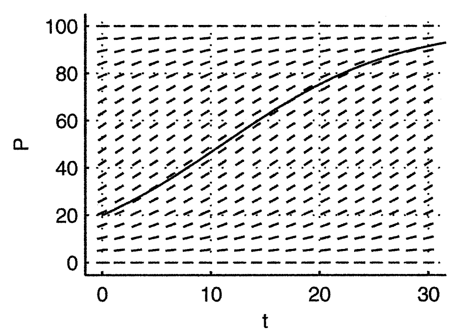
39. The rate at which the population is changing with respect to time is proportional to the product of the population and the number of critters less than the "carrying capacity" (100). Thus,

$$\frac{dP}{dt} = kP(100 - P).$$

With $k = 0.00125$ and an initial population of 20 critters, we must solve the initial value problem

$$\frac{dP}{dt} = 0.00125P(100 - P), \qquad P(0) = 20.$$

Note that the right-hand side of this equation is positive if the number of critters is less than the carrying capacity (100). Thus, we have growth. Using our numerical solver, we input the equation and the initial condition, arriving at the following solution curve.



Use the solution curve to estimate that there are about 91 critters in the environment at the end of 30 days.

Section 2.2. Solutions to Separable Equations

1. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= xy\\ \frac{dy}{y} &= xdx\\ \ln|y| &= \frac{1}{2}x^2 + C\\ |y(x)| &= e^{x^2/2+C}\\ y(x) &= \pm e^C \cdot e^{x^2/2}\\ &= Ae^{x^2/2},\end{aligned}$$

Where the constant $A = \pm e^C$ is arbitrary.

2. Separate the variables and integrate.

$$\begin{aligned}x\frac{dy}{dx} &= 2y\\ \frac{1}{y}dy &= \frac{2}{x}dx\\ \ln|y| &= 2\ln|x| + C\\ |y| &= e^{\ln x^2+C}\\ y(x) &= \pm e^C x^2\end{aligned}$$

Letting $D = \pm e^C$, $y(x) = Dx^2$.

3. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= e^{x-y}\\ e^y dy &= e^x dx\\ e^y &= e^x + C\\ y(x) &= \ln(e^x + C)\end{aligned}$$

4. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= (1+y^2)e^x\\ \frac{1}{1+y^2}dy &= e^x dx\\ \tan^{-1} y &= e^x + C\\ y(x) &= \tan(e^x + C)\end{aligned}$$

5. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= y(x+1)\\ \frac{1}{y}dy &= (x+1)\,dx\\ \ln|y| &= \frac{1}{2}x^2 + x + C\\ |y| &= e^{(1/2)x^2+x+C}\\ y(x) &= \pm e^C e^{(1/2)x^2+x}\end{aligned}$$

Letting $D = \pm e^C$, $y = De^{(1/2)x^2+x}$.

6. Separate the variables and integrate. *Note: Factor the right-hand side.*

$$\begin{aligned}\frac{dy}{dx} &= (e^x+1)(y-2)\\ \frac{1}{y-2}dy &= (e^x+1)\,dx\\ \ln|y-2| &= e^x + x + C\\ |y-2| &= e^{e^x+x+C}\\ y-2 &= \pm e^C e^{e^x+x}\end{aligned}$$

Letting $D = \pm e^C$, $y(x) = De^{e^x+x} + 2$.

7. Separate the variables and integrate,

$$\begin{aligned}\frac{dy}{dx} &= \frac{x}{y+2},\\ (y+2)\,dy &= x\,dx,\\ \frac{1}{2}y^2 + 2y &= \frac{1}{2}x^2 + C,\\ y^2 + 4y - (x^2+D) &= 0,\end{aligned}$$

With D replacing $2C$ in the last step. We can use the quadratic formula to solve for y.

$$\begin{aligned}y(x) &= \frac{-4 \pm \sqrt{16 + 4(x^2+D)}}{2}\\ y(x) &= -2 \pm \sqrt{x^2 + (D+4)}\end{aligned}$$

If we replace $D+4$ with another arbitrary constant E, then $y(x) = -2 \pm \sqrt{x^2+E}$.

8. Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= y\left(\frac{x}{x-1}\right)\\ \frac{1}{y}\,dy &= \left(1+\frac{1}{x-1}\right)dx\\ \ln|y| &= x+\ln|x-1|+C\\ |y| &= e^{x+\ln|x-1|+C}\\ y(x) &= \pm e^{C}e^{x}e^{\ln|x-1|}\end{aligned}$$

Letting $D = \pm e^C$, $y(x) = De^x|x-1|$. It is important to note that this solution is not differentiable at $x = 1$ and further information (perhaps in the form of an initial condition) is needed to remove the absolute value and determine the interval of existence.

9. First a little algebra.

$$\begin{aligned}x^2y' &= y\ln y - y'\\ (x^2+1)y' &= y\ln y\end{aligned}$$

Separate the variables and integrate.

$$\begin{aligned}\frac{1}{y\ln y}\,dy &= \frac{1}{x^2+1}\,dx\\ \frac{1}{u}\,du &= \frac{1}{x^2+1}\,dx,\end{aligned}$$

where $u = \ln y$ and $du = \frac{1}{y}\,dy$. Hence, $\ln|u| = \tan^{-1}x + C$. Solve for u:

$$\begin{aligned}|u| &= e^{\tan^{-1}x+C}\\ u &= \pm e^{C}e^{\tan^{-1}x}\end{aligned}$$

Let $D = \pm e^C$, replace u with $\ln y$, and solve for y.

$$\begin{aligned}\ln y &= De^{\tan^{-1}x}\\ y(x) &= e^{De^{\tan^{-1}x}}\end{aligned}$$

10.

$$\begin{aligned}x\frac{dy}{dx} &= y(1+2x^2)\\ \frac{dy}{y} &= \frac{1+2x^2}{x}\,dx = \left[\frac{1}{x}+2x\right]dx\\ \ln|y| &= \ln|x| + x^2 + C\\ |y(x)| &= e^{\ln|x|+x^2+C} = e^{C}|x|e^{x^2}\\ y(x) &= Axe^{x^2}\end{aligned}$$

11.

$$\begin{aligned}(y^3-2)\frac{dy}{dx} &= x\\ (y^3-2)dy &= x\,dx\\ \frac{y^4}{4} - 2y &= \frac{x^2}{2} + C.\end{aligned}$$

The solution is given implicitly by the equation $y^4 - 8y - 2x^2 = A$, where we have set $A = 4C$.

12.

$$\begin{aligned}\frac{dy}{dx} &= \frac{2x(y+1)}{x^2-1}\\ \frac{dy}{y+1} &= \frac{2x\,dx}{x^2-1}\\ \ln|y+1| &= \ln|x^2-1| + C\\ |y+1| &= e^{\ln|x^2-1|+C} = e^{C}\,|x^2-1|\\ y(x) &= A(x^2-1)-1.\end{aligned}$$

13.

$$\begin{aligned}\frac{dy}{dx} &= \frac{y}{x}\\ \frac{dy}{y} &= \frac{dx}{x}\\ \lambda|y| &= \ln|x| + C\\ |y(x)| &= e^{\ln|x|+C} = e^{C}\,|x|\\ y(x) &= Ax.\end{aligned}$$

The initial condition $y(1) = -2$ gives $A = -2$. The solution is $y(x) = -2x$. The solution is defined for all x, but the differential equation is not defined at $x = 0$ so the interval of existence is $(0, \infty)$.

14.

$$\begin{aligned}\frac{dy}{dt} &= -\frac{2t(1+y^2)}{y}\\ \frac{y\,dy}{1+y^2} &= -2t\,dt\\ \frac{1}{2}\ln(1+y^2) &= -t^2 + C\\ 1+y^2 &= e^{-2t^2+C} = e^{C}e^{-2t^2}\\ 1+y^2 &= Ae^{-2t^2}\end{aligned}$$

With $y(0) = 1$, $1+1^2 = Ae^{-2(0)^2}$ and $A = 2$. Thus,

$$\begin{aligned} 1 + y^2 &= 2e^{-2t^2} \\ y &= \pm\sqrt{2e^{-2t^2} - 1}. \end{aligned}$$

We must choose the branch that contains the initial condition $y(0) = 1$. Thus, $y = \sqrt{2e^{-2t^2} - 1}$. This solution is defined, provided that

$$\begin{aligned} 2e^{-2t^2} - 1 &> 0 \\ e^{-2t^2} &> \frac{1}{2} \\ -2t^2 &> \ln\frac{1}{2} \\ t^2 &< -2\ln\frac{1}{2} \\ t^2 &< \ln 4 \\ |t| &< \sqrt{\ln 4}. \end{aligned}$$

Thus, the interval of existence is $(-\sqrt{\ln 4}, \sqrt{\ln 4})$.

15.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\sin x}{y} \\ y\,dy &= \sin x\,dx \\ \frac{1}{2}y^2 &= -\cos x + C_1 \\ y^2 &= -2\cos x + C \quad (C = 2C_1) \\ y(x) &= \pm\sqrt{C - 2\cos x} \end{aligned}$$

Using the initial condition we notice that we need the plus sign, and $1 = y(\pi/2) = \sqrt{C}$. Thus $C = 1$ and the solution is

$$y(x) = \sqrt{1 - 2\cos x}.$$

The interval of existence will be the interval containing $\pi/2$ where $2\cos x < 1$. This is $\pi/3 < x < 5\pi/3$.

16.

$$\begin{aligned} \frac{dy}{dx} &= e^{x+y} \\ e^{-y}\,dy &= e^x\,dx \\ -e^{-y} &= e^x + C \\ e^{-y} &= -e^x - C \\ -y &= \ln(-e^x - C) \\ y &= -\ln(-e^x - C) \end{aligned}$$

With $y(0) = 1$,

$$\begin{aligned} 1 &= -\ln(-e^0 - C) \\ -1 - C &= e^{-1} \\ C &= -1 - e^{-1}. \end{aligned}$$

Thus,

$$y = -\ln(-e^x + e^{-1} + 1).$$

This solution is defined provided that

$$\begin{aligned} -e^x + e^{-1} + 1 &> 0 \\ e^x &< e^{-1} + 1 \\ x &< \ln(e^{-1} + 1). \end{aligned}$$

Thus, the interval of existence is $(-\infty, \ln(e^{-1}+1))$.

17.

$$\begin{aligned} \frac{dy}{dt} &= 1 + y^2 \\ \frac{dy}{1+y^2} &= dt \\ \tan^{-1}(y) &= t + C \\ y(t) &= \tan(t + C) \end{aligned}$$

For the initial condition we have $1 = y(0) = \tan C$, so $C = \pi/4$ and the solution is $y(t) = \tan(t + \pi/4)$. Since the tangent is continuous on the interval $(-\pi/2, \pi/2)$, the solution $y(t) = \tan(t + \pi/4)$ is continuous on the interval $(-3\pi/4, \pi/4)$.

18.

$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{1+2y} \\ (1 + 2y)\,dy &= x\,dx \\ y + y^2 &= x^2/2 + C \end{aligned}$$

This last equation can be written as $y^2+y-(x^2/2+C)=0$. We solve for y using the quadratic formula

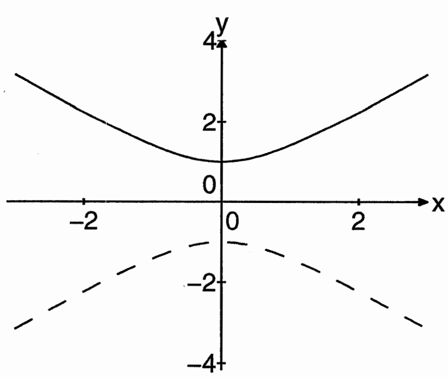
$$\begin{aligned} y(x) &= \left[-1 \pm \sqrt{1+4(x^2/2+C_1)}\right]/2 \\ &= \left[-1 \pm \sqrt{2x^2+C}\right]/2 \quad (C = 1+4C_1) \end{aligned}$$

For the initial condition $y(-1)=0$ we need to take the plus sign in order to counter the -1. Then the initial condition becomes $0=[-1+\sqrt{2+C}]/2$, which means that $C=-1$. Thus the solution is

$$y(x) = \frac{-1+\sqrt{2x^2-1}}{2}.$$

For the interval of existence we need the interval containing -1 where $2x^2-1>0$. This is $-\infty < x < -1/\sqrt{2}$.

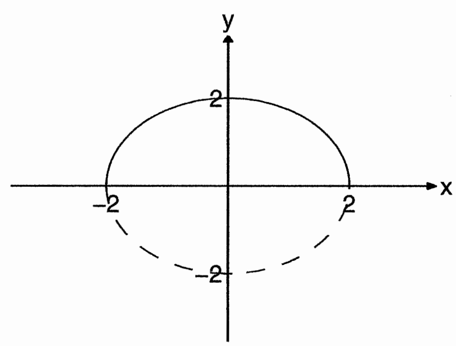
19. With $y(0)=1$, we get the solution $y(x)=\sqrt{1+x^2}$, with interval of existence $(-\infty,\infty)$. This solution is plotted with the solid curve in the following figure. With $y(0)=-1$, we get the solution $y(x)=-\sqrt{1+x^2}$, with interval of existence $(-\infty,\infty)$. This solution is plotted with the dashed curve in the following figure.



20.

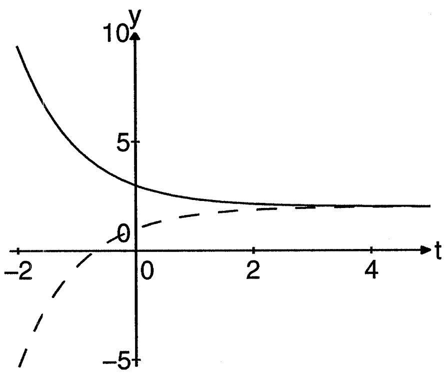
$$\begin{aligned} \frac{dy}{dx} &= -\frac{x}{y} \\ y\,dy &= -x\,dx \\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + C \\ y^2 &= 2C - x^2 \\ y &= \pm\sqrt{2C-x^2} \end{aligned}$$

With $y(0)=2$, we choose the positive branch and $2=\sqrt{2C}$ leads to $C=2$ and the solution $y=\sqrt{4-x^2}$ with interval of existence $(-2,2)$. This solution is plotted as a solid curve in the following figure. With $y(0)=-2$, we choose the negative branch and $-2=-\sqrt{2C}$ leads to $C=2$ and the solution $y-\sqrt{4-x^2}$ with interval of existence $(-2,2)$. This solution is shown as a dashed curve in the figure.



21. With $y(0)=3$ the solution is $y(t)=2+e^{-t}$, on $(-\infty,\infty)$. This solution is plotted is the solid curve in the next figure. With $y(0)=1$ the solution is $y(t)=2-e^{-t}$, on $(-\infty,\infty)$. This solution is plot-

ted is the dashed curve in the next figure.



22.

$$\frac{dy}{dx} = \frac{y^2+1}{y}$$

$$\frac{y\,dy}{y^2+1} = dx$$

$$\frac{1}{2}\ln(y^2+1) = x + C$$

$$y^2 + 1 = e^{2x+2C} = Ae^{2x} \quad (A = e^{2C})$$

$$y^2 = Ae^{2x} - 1$$

$$y(x) = \pm\sqrt{Ae^{2x} - 1}$$

This is the general solution. The initial condition $y(1) = 2$ gives

$$2 = +\sqrt{Ae^2 - 1}$$

$$4 = Ae^2 - 1$$

$$A = 5e^{-2}$$

The particular solution is

$$y(x) = \sqrt{5e^{2x-2} - 1}.$$

The interval of existence requires that

$$5e^{2x-2} - 1 > 0$$

$$2x - 2 > \ln(1/5)$$

$$x > 1 - \ln(5)/2 \approx 0.1953.$$

Thus the interval of existence is $1 - \ln(5)/2 < x < \infty$.

23. We have $N(t) = N_0 e^{-\lambda t}$, and

$$\begin{aligned} N(t + T_{1/2}) &= N_0 e^{-\lambda(t+T_{1/2})} \\ &= N_0 e^{-\lambda t} \cdot e^{-\lambda T_{1/2}} \\ &= N(t) \cdot e^{-\lambda T_{1/2}} \\ &= N(t) \cdot \frac{1}{2} \end{aligned}$$

if $e^{-\lambda T_{1/2}} = 1/2$, or $T_{1/2} = \ln 2/\lambda$.

24. (a) $\lambda = \ln 2/T_{1/2} = 1.5507 \times 10^{-8}$.

(b) We have $N_0 = 1000$ and $N(t) = 100$. Hence $100 = 1000 \cdot e^{-\lambda t}$, or $t = \ln 10/\lambda = 1.4849 \times 10^8$ years.

25. We have $80 = N(4) = 100e^{-4\lambda}$. Hence $\lambda = \ln(100/80)/4 = 0.0558$. Then $T_{1/2} = \ln 2/\lambda = 12.4251$ hours.

26. Using $T_{1/2} = 6$ hours, we have $\lambda = \ln 2/T_{1/2} = 0.1155$. Then $N(9) = 10e^{-9\lambda} = 3.5355$kg.

27. Using $T_{1/2} = 8.04$ days, we have $\lambda = \ln 2/T_{1/2} = 0.0862$. Then $N(20) = 500e^{-20\lambda} = 89.1537$mg.

28. The decay constants are related to the half-lives by $\lambda_{210} = \ln(2)/2.42 = 0.2864$ and $\lambda_{211} = \ln(2)/15 = 0.0462$. The amount of ^{210}Rn is given by $x(t) = x_0 e^{-\lambda_{210} t}$ and of ^{211}Rn by $y(t) = y_0 e^{-\lambda_{211} t}$. The initial condition is that $y(0)/x(0) = y_0/x_0 = 0.2/0.8 = 1/4$, so $4x_0 = 4y_0$. We are looking for a time t when $0.8/0.2 = 4 = y(t)/x(t) = e^{t(\lambda_{210}-\lambda_{211})}/4$. Thus we need $e^{t(\lambda_{210}-\lambda_{211})} = 16$. From this we find that $t = 11.5$ hours.

29. (a) If $N = N_0 e^{-\lambda t}$, then substituting $T_\lambda = 1/\lambda$,

$$\begin{aligned} N &= N_0 e^{-\lambda T_\lambda} \\ &= N_0 e^{-\lambda(1/\lambda)} \\ &= N_0 e^{-1}. \end{aligned}$$

Therefore, after a period of one time constant $T_\lambda = 1/\lambda$, the material remaining is $N_0 e^{-1}$. Thus, the amount of radioactive substance has decreased to e^{-1} of its original value N_0.

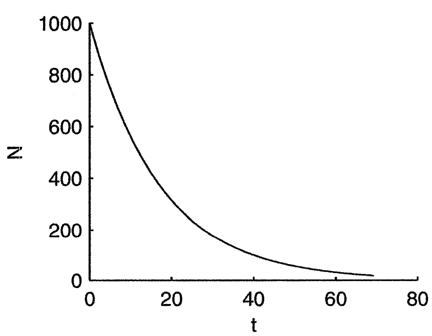
(b) If the half-life is 12 hours, then

$$\begin{aligned}\frac{1}{2}N_0 &= N_0 e^{-\lambda(12)}\\ e^{-12\lambda} &= \frac{1}{2}\\ -12\lambda &= \ln\frac{1}{2}\\ \lambda &= \frac{\ln\frac{1}{2}}{-12}.\end{aligned}$$

Hence, the time constant is

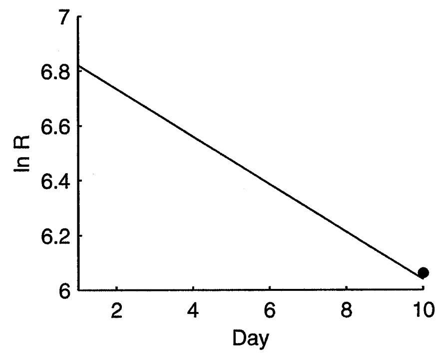
$$T_\lambda = \frac{1}{\lambda} - \frac{-12}{\ln\frac{1}{2}} \approx 17.3\,\text{hr}.$$

(c) If 1000 mg of the substance is present initially, then the amount of substance remaining as a function of time is given by $N = 1000e^{(t\ln(1/2))/12}$. The graph over four time periods ($[0, 4T_\lambda]$) follows.



30. The data is plotted on the following figure. The line is drawn using the slope found by linear regression. It has slope $-\lambda = -0.0869$. Hence the half-life is

$$T_{1/2} = \ln 2/\lambda = 7.927\text{days}.$$



31. The half-life is related to the *decay constant* by

$$T_{1/2} = \frac{\ln 2}{\lambda_{226}}.$$

The *decay rate* is related to the number of atoms present by

$$R = \lambda_{226} N.$$

Substituting,

$$T_{1/2} = \frac{N\ln 2}{R}.$$

Calculate the number of atoms present in the 1g sample.

$$\begin{aligned}N &= 1\text{g} \times \frac{1\text{ mol}}{226\text{ g}} \times \frac{6.02\times 10^{23}\text{ atoms}}{\text{mole}}\\ &= 266\times 10^{21}\text{ atoms}.\end{aligned}$$

Now,

$$T_{1/2} = \frac{(2.66\times 10^{21}\text{ atoms})(\ln 2)}{3.7\times 10^{10}\text{ atom/s}} = 4.99\times 10^{10}\text{s}.$$

In years, $T_{1/2} \approx 1582$ yr. The dedicated reader might check this result in the CRC Table.

32. (a) Because half of the existing ^{14}C decays every 5730 years, there will come a time when physical instruments can no longer measure the remaining ^{14}C. After about 10 half-lives (57300

years), the amount of original material remaining is

$$N_0\left(\frac{1}{2}\right)^{10} \approx 0.00097N_0,$$

a very small amount.

(b) The decay constant is calculated with

$$\lambda = \frac{\ln 2}{T_{1/2}} = \frac{\ln 2}{5568} \approx 0.0001245.$$

We can now write

$$N = N_0 e^{-0.0001245t}.$$

The ratio remaining is 0.617 of the current ratio, so

$$\begin{aligned} 0.617N_0 &= N_0 e^{-0.0001245t} \\ e^{-0.0001245t} &= 0.617 \\ -0.0001245t &= \ln 0.617 \\ t &= \frac{\ln 0.617}{-0.0001245} \end{aligned}$$

Thus, the charcoal is approximately 3879 years old.

33. Let $t = 0$ correspond to midnight. Thus, $T(0) = 31°C$. Because the temperature of the surrounding medium is $A = 21°C$, we can use $T = A + (T_0 - A)e^{-kt}$ and write

$$T = 21 + (31 - 21)e^{-kt} = 21 + 10e^{-kt}.$$

At $t = 1$, $T = 29°C$, which can be used to calculate k.

$$\begin{aligned} 29 &= 21 + 10e^{-k(1)} \\ k &= -\ln(0.8) \\ k &\approx 0.2231 \end{aligned}$$

Thus, $T = 21 + 10e^{-0.2231t}$. To find the time of death, enter "normal" body temperature, $T = 37°C$ and solve for t.

$$\begin{aligned} 37 &= 21 + 10e^{-0.2231t} \\ t &= \frac{\ln 1.6}{-0.2231} \\ t &\approx -2.1 \text{ hrs} \end{aligned}$$

Thus, the murder occurred at approximately 9:54 PM.

34. Let $y(t)$ be the temperature of the beer at time t minutes after being placed into the room. From Newton's law of cooling, we obtain

$$y'(t) = k(70 - y(t)) \quad y(0) = 40$$

Note k is positive since $70 > y(t)$ and $y'(t) > 0$ (the beer is warming up). This equation separates as

$$\frac{dy}{70 - y} = k\,dt$$

which has solution $y = 70 - Ce^{-kt}$. From the initial condition, $y(0) = 40$, $C = 30$. Using $y(10) = 48$, we obtain $48 = 70 - 30e^{-10k}$ or $k = (-1/10)\ln(11/15)$ or $k = .0310$. When $t = 25$, we obtain $y(25) = 70 - 30e^{-.598} \approx 56.18°$.

35. The same differential equation and solution hold as in the previous problem:

$$y(t) = 70 - Ce^{-kt}$$

We let $t = 0$ correspond to when the beer was discovered, so $y(0) = 50$. This means $C = 20$. We also have $y(10) = 60$ or

$$60 = 70 - 20e^{-10k}$$

Therefore, $k = (-1/10)\ln(1/2) \approx .0693$. We want to find the time T when $y(T) = 40$, which gives the equation

$$70 - 20e^{-kT} = 40$$

Since we know k, we can solve this equation for T to obtain

$$T = (-1/k)\ln(3/2) \approx -5.85$$

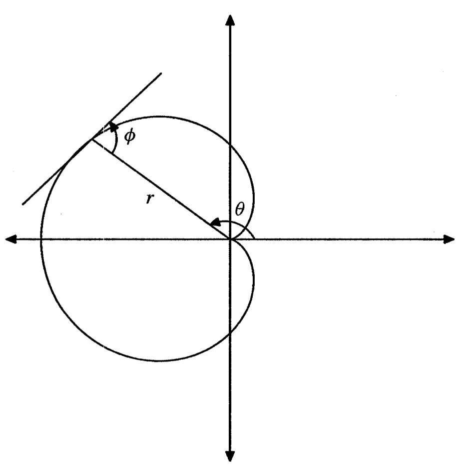
or about 5.85 minutes before the beer was discovered on the counter.

36. $x' = [at+by+c]' = a+by' = a+bf(at+by+c) = a+f(x)$. For the equation $y' = (y+t)^2$ we use $x = t+y$. Then $x' = 1+y' = 1+(y+t)^2 = 1+x^2$. Solving this separable equation in the usual way we get the general solution $x(t) = \tan(t+C)$. In terms of the unknown y, we get $y(t) = x(t) - t = \tan(t+C) - t$.

37. The tangent line at the point (x, y) is $\hat{y} - y = y'(x)(\hat{x} - x)$ (the variables for the tangent line have the hats). The $\hat{x}$ intercept is $\hat{x}_{\text{int}} = -y/y' + x$. Since (x, y) bisects the tangent line, we have $\hat{x}_{\text{int}} = 2x$. Therefore
$$2x = \frac{-y}{y'} + x$$
or
$$\frac{y'}{y} = \frac{-1}{x}$$
This separable differential equation is easily solved to obtain $y(x) = C/x$, where C is an arbitrary constant.

38. With the notation as in the previous problem, the equation of the normal line is
$$\hat{y} - y = \frac{-1}{y'(x)}(\hat{x} - x)$$
The $\hat{x}$-intercept is found to be $\hat{x}_{\text{int}} = yy' + x$. Since $\hat{x}_{\text{int}} - x$ is given to be ± 1, we obtain
$$yy' = \pm 2$$
with solution $y^2 = \pm 4x + C$, where C is an arbitrary constant.

39. Let ϕ be the angle from the radius to the tangent.



From geometry, $\tan\phi = r\,d\theta/dr$. Since $\theta = 2\phi$, we obtain
$$\frac{dr}{d\theta} = r\cot\phi = r\cot(\theta/2)$$
which can be separated as $dr/r = \cot(\theta/2)d\theta$. This can be solved for r as $r(\theta) = C\sin^2(\theta/2)$, where C is a constant.

40. The area under the curve $y = y(t)$ from 0 to x is
$$\int_0^x y(t)\,dt$$
which by assumption, equals $(1/4)xy(x)$ (one-fourth the area of the rectangle). Therefore
$$\int_0^x y(t)\,dt = (1/4)xy(x).$$
Differentiating this equation with respect to x and using the Fundamental Theorem of Calculus for the left side gives
$$y(x) = (1/4)\left(y(x) + xy'(x)\right).$$
This equation separates as
$$\frac{y'}{y} = \frac{3}{x}$$
which has the solution $y(x) = Cx^3$.

41. Center the football at the origin with equation
$$z^2 + x^2 + \frac{y^2}{4} = 4.$$
The top half of the football is the graph of the function
$$z = \sqrt{4 - x^2 - y^2/4}.$$
The (x, y)-components of the path of a rain drop form a curve in the (x, y)- plane which must always point in the direction of the gradient of the function z (the path of steepest descent). The gradient of z is given by
$$\nabla z = \frac{-xi - (y/4)j}{\sqrt{4 - x^2 - y^2/4}}$$
where i and j are the unit vectors parallel to the x and y-axis. Since the path traced by the drop, $y = y(x)$, must point in the direction of ∇z, we must have
$$\frac{dy}{dx} = \text{slope of the gradient} = \frac{z_y}{z_x} = \frac{y}{4x}.$$

This differential equation can be separated and solved as $x = Cy^4$. C can be solved from the initial position of the drop (x_0, y_0) to be $C = x_0/y_0^4$. The final answer is given by inserting $x = Cy^4$ into the expression for z:

$$(x, y, z) = (Cy^4, y, \sqrt{4 - C^2y^8 - y^2/4}),$$

(here, y is the independent variable).

42. Let $y(t)$ be the level of the water and let $V(t)$ be the volume of the water in the bowl at time t. On the one hand, we have dV/dt = cross sectional area of the water level $\times dy/dt$. The cross sectional area of the water level is πx^2, where x is the radius. Using the equation of the side of the bowl ($y = x^2$), we obtain

$$\frac{dV}{dt} = \pi y(t)\frac{dy}{dt}.$$

On the other hand, dV/dt is equal to $\pi a^2 v$ where v is the speed of the water exiting the bowl. From the hint, $v = \sqrt{2gy(t)}$. Thus we obtain the following differential equation:

$$\pi y(t)\frac{dy}{dt} = \frac{dV}{dt} = -\pi a^2\sqrt{2gy(t)}.$$

This equation can be separated and solved for y as

$$y(t) = \left(C - \frac{3}{2}a^2\sqrt{2g}t\right)^{2/3}$$

Since $y(0) = 1$, we obtain $C = 1$. Setting $y(t) = 0$, we obtain

$$t = \frac{2}{3a^2\sqrt{2g}}$$

in units of seconds (here $g = 32$).

43. Let the unknown curve forming the outside of the bowl be given by $y = y(x)$ (the bowl is then formed by revolving this curve around the y-axis). We can also write this equation as $x = x(y)$ (reversing the roles of the independent and dependent variables). As in the analysis of the last problem, the rate of change of volume, dV/dt is the cross sectional area multiplied by the rate of change in height, dy/dt. The cross sectional area is $\pi x^2 = \pi x(y)^2$. Thus

$$\frac{dV}{dt} = \pi x^2\frac{dy}{dt}$$

From Torricelli's law:

$$\frac{dV}{dt} = -\pi a^2 v = -\pi a^2\sqrt{2gy}$$

Since $dy/dt = C$ (a negative constant), we obtain

$$C\pi x^2 = -\pi a^2\sqrt{2gy}$$

Solving for y, we obtain $y = Kx^4$ where K is a constant.

44. Following the hint, let θ be the polar angle and locate the destroyer at 4 miles along the positive x-axis. The destroyer wants to follow a path so that its arc length is always three times that of the sub. To account for the possibility that the sub heads straight along the positive x-axis, the destroyer should first head from $x = 4$ to $x = 1$ (the sub would move from $x = 0$ to $x = 1$ in this same time frame under this scenario). Now the destroyer must circle around the sub along a polar coordinate path $r = r(\theta)$. We have $r(0) = 1$. If the destroyer intersects the sub at $(\theta, r(\theta))$, then the sub will have traveled $r(\theta)$ and the destroyer would have traveled $\int_0^\theta \sqrt{r'(t)^2 + r(t)^2}\, d\theta$ (arc length along the curve $r = r(\theta)$). Since the speed of the destroyer is three times that of the sub, we obtain

$$3(r(\theta) - 1) = \int_0^\theta \sqrt{r'(t)^2 + r(t)^2}\, d\theta.$$

Differentiating this equation gives

$$3r' = \sqrt{(r')^2 + r^2} \quad \text{or} \quad \frac{dr}{d\theta} = \frac{r(\theta)}{\sqrt{8}} \quad r(0) = 1$$

with solution $r(\theta) = e^{\theta/\sqrt{3}}$.

Section 2.3. Models of Motion

1. We need $gt = c/5$. or $t = c/5g = 612{,}240$ seconds. The distance traveled will be $s = gt^2/2 = 1.84 \times 10^{14}$ meters.

2. We need $0 = -9.8t^2/2 + 15t + 100$. The answer is 6.3 seconds.

3. The depth of the well satisfies $d = 4.9t^2$, where t is the amount of time it takes the stone to hit the water. It also satisfies $d = 340s$, where $s = 8 - t$ is the amount of time it takes for the noise of the splash to reach the ear. Thus we must solve the quadratic equation $4.9t^2 = 340(8 - t)$. The solution is $t = 7.2438$sec. The depth is $d = 340(8 - t) = 257.1$m.

4. In the first 60s the rocket rises to an elevation of $(100 - 9.8)t^2/2 = 162,360$m and achieves a velocity of $v(60) = (100 - 9.8) * 60 = 5412$m/s. After that the velocity is $5412 - 9.8t$. This is zero at the highest point, reached when $t_1 = 552.2$s. The altitude at that point is $162,360 + 5412\,t_1 - 9.8\,t_1^2/2 = 1.657 \times 10^6$m. From there to the ground it takes t_2s, where $4.9t_2^2 = 1.657 \times 10^6$, or $t_2 = 581.5$s. The total trip takes $60 + 552.2 + 581.5 = 1193.7$s.

5. The distance dropped in time t is $4.9t^2$. If T is the time taken for the first half of the trip, then $4.9(T+1)^2 = 2 \times 4.9T^2$, or $4.9(T^2 - 2T - 1) = 0$. Solving we find that $T = 1 + \sqrt{2} = 2.4142$s. So the body fell $2 \times 4.9T^2 = 57.12$m, and it took $T + 1 = 3.4142$s.

6. (a) $v_0^2/2g$
   (b) Both times are equal to v_0/g.
   (c) v_0.

7. The velocities must be changed to ft/s, so $v_0 = 60\,\text{mi/h} = 60 \times 5280/3600 = 88$ft/s, and $v = 30\,\text{mi/h} = 44$ft/s. Then $a = (v^2 - v_0^2)/2(x - x_0) = -5.8\text{ft/s}^2$.

8. We have $v(t) = Ce^{-rt/m} - mg/r$. If $v(0) = 0$ then $C = mg/r$, and $v(t) = mg(e^{-rt/m} - 1)/r$. This is equal to $-mg/2r$ when $e^{-rt/m} = 1/2$. Thus the time required is $t = m\ln(2)/r$. The distance traveled is

$$\begin{aligned} x &= \int_0^t v(s)\,ds \\ &= \frac{mg}{r}\int_0^t (e^{-rs/m} - 1)\,ds \\ &= \frac{mg}{r}\left[\frac{m}{r}(1 - e^{-rt/m}) - t\right] \\ &= \frac{mg}{r}\left[\frac{m}{2r} - \frac{m\ln 2}{r}\right] \\ &= \frac{m^2 g}{r^2}\left[\frac{1}{2} - \ln 2\right] \end{aligned}$$

9. The resistance force has the form $R = -rv$. When $v = 0.2$, $R = -1$ so $r = 5$. The terminal velocity is $v_{\text{term}} = -mg/r = -0.196$m/s.

10. (a) First, the terminal velocity gives us $20 = mg/r$, or $r = mg/20 = 70 \times 9.8/20 = 34.3$. Next, we have $v(t) = Ce^{-rt/m} - mg/r$. Since $v(0) = 0$, $C = mg/r$, and $v(t) = mg(e^{-rt/m} - 1)/r$. Integrating and setting $x(0) = 0$, we get

$$\begin{aligned} x &= \int_0^t v(s)\,ds \\ &= \frac{mg}{r}\int_0^t (e^{-rs/m} - 1)\,ds \\ &= \frac{mg}{r}\left[\frac{m}{r}(1 - e^{-rt/m}) - t\right] \end{aligned}$$

   Hence $v(2) = -12.4938$ and $x(2) = -14.5025$.

   (b) The velocity is 80% of its terminal velocity when $1 - e^{-rt/m} = 0.8$. For the values of $m = 70$ and $r = 34.3$ this becomes $t = 3.2846$s.

11. Without air resistance, $v_0 = \sqrt{2 \times 13.5g} = 16.2665$m/s. With air resistance, v_0 is defined by

$$\int_{v_0}^0 \frac{v\,dv}{v + mg/r} = -\frac{r}{m}\int_{1.5}^{15} dy.$$

Hence,

$$\begin{aligned} &-v_0 + (mg/r)\ln(v_0 + mg/r) - (mg/r)\ln(mg/r) \\ &\quad = -13.5\frac{r}{m} \quad \text{or} \\ &-v_0 + 49\ln(v_0 + 49) - 49\ln(49) = -2.7 \end{aligned}$$

This is an implicit equation for v_0. Solving on a calculator or a computer yields $v_0 = 18.1142$m/s.

12. The impact velocity v_i is defined by

$$\int_0^{v_i} \frac{v\,dv}{v + mg/r} = -\frac{r}{m}\int_{50}^{0} dy$$

From which we get

$$v_i - (mg/r)\ln(v_i + mg/r) + (mg/r)\ln(mg/r) = -50\frac{r}{m}, \quad \text{or}$$

$$v_i - 19.6\ln(v_i + 19.6) + 19.6\ln(19.6) = -25$$

This is an implicit equation for v_i. Solving on a calculator or a computer yields $v_i = -17.3401$m/s.

13. Following the lead of Exercise 11, we find that

$$v\,dv = (-g + R(v)/m)\,dy = \left(-9.8 - 0.5v^2\right)dy$$

Hence if y_1 is the maximum height we have

$$\int_{230}^{0} \frac{v\,dv}{v^2 + 19.6} = -0.5\int_0^{y_1} dy.$$

Hence

$$y_1 = \log(v^2 + 19.6)\Big|_0^{230} = 7.9010.$$

14. (a) Follows from $a = \dfrac{dv}{dt} = \dfrac{dv}{dy}\dfrac{dy}{dt} = v\dfrac{dv}{dy}$.

(b)

$$v\,dv = -\frac{GM}{(R+y)^2}dy$$

$$\int_{v_0}^{v} v\,dv = -\int_0^y \frac{GM}{(R+s)^2}ds$$

$$\frac{1}{2}(v^2 - v_0^2) = -GM\left(\frac{1}{R} - \frac{1}{R+y}\right)$$

$$v^2 = v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right)$$

(c) If y is the maximum height, the corresponding velocity is $v = 0$, so from (3.16)

$$0 = v_0^2 - 2GM\left(\frac{1}{R} - \frac{1}{R+y}\right).$$

Solving for y we get the result.

(d) If $v_0 < \sqrt{2GM/R}$, then $v_0^2 < 2GM/R$, and $2GM/R - v_0^2 > 0$. Hence by (c) the object has a finite maximum height and does not escape. However, when $v_0 = \sqrt{2GM/R}$, $2GM/R - v_0^2 = 0$, and there is no maximum height.

15. Let $x(t)$ be the distance from the mass to the center of the Earth. The force of gravity is kx (proportional to the distance from the center of the Earth). Since the force of gravity at the surface (when $x = R$) is $-mg$, we must have $k = -mg/R$. Newton's law becomes

$$m\frac{d^2x}{dt^2} = \frac{-mgx}{R}$$

Using the reduction of order technique as given in the hint, we obtain

$$v\frac{dv}{dx} = \frac{-gx}{R}$$

which can be separated with a solution given by $v = \sqrt{C - gx^2/R}$. The constant C can be evaluated from the initial condition, $v(x = R) = 0$, to be $C = gR$. When $x = 0$ (the center of the Earth), we obtain $v = \sqrt{C} = \sqrt{gR}$ or approximately 4.93 miles per second.

16. We will use $GM = gR^2$. Once more we use

$$v\,dv = -\frac{GM}{(R+y)^2}dy$$

$$\int_0^v s\,ds = -GM\int_a^0 \frac{dy}{(R+y)^2}$$

$$v^2 = 2GM\left[\frac{1}{R} - \frac{1}{R+a}\right] = \frac{2agR}{a+R}$$

17. The force acting on the chain is the force of gravity applied to the piece of the chain that hangs off

the table. This force is $mgx(t)$ where m is the mass density of the chain. Newton's law gives

$$mx''(t) = mgx(t)$$

Using the hint, $x''(t) = dv/dt = v(dv/dx)$ and so this equation becomes

$$v\frac{dv}{dx} = gx$$

Separating this equation and integrating gives $v^2 = gx^2 + C$. Since $v = 0$ when $x = 2$ (initial velocity is zero), we obtain $C = -4g$. Therefore

$$v = \sqrt{g(x^2 - 4)}.$$

Since $dx/dt = v = \sqrt{g(x^2 - 4)}$, we can separate this equation and integrate to obtain

$$\ln(x + \sqrt{x^2 - 4}) = \sqrt{g}t + K$$

where K is a constant. From the initial condition that $x = 2$ when $t = 0$, we obtain $K = \ln 2$. Inserting $x = 10$ and solving for t, we obtain

$$t = \frac{1}{\sqrt{g}} \ln\left(\frac{10 + \sqrt{96}}{2}\right) \approx .405 \text{ seconds}$$

18. $v = -g - (k/m)v$, v = velocity. Velocity on impact is 58.86 meters/per second downward and 90.9 seconds until he hits the ground.

19. Let x be the height of the parachuter and let v be his velocity. The resistance force is proportional to v and to e^{-ax}. Hence it is given by $R(x, v) = -ke^{ax}v$, where k is a positive constant. Newton's second law gives us $mx'' = -mg - ke^{-ax}x'$, or $mx'' + ke^{ax}x' + mg = 0$.

Section 2.4. Linear Equations

1. Compare $y' = -y + 2$ with $y' = a(t)y + f(t)$ and note that $a(t) = -1$. Consequently, an integrating factor is found with

$$u(t) = e^{\int -a(t)\,dt} = e^{\int 1\,dt} = e^t.$$

Multiply both sides of our equation by this integrating factor and check that the left-hand side of the resulting equation is the derivative of a product.

$$e^t(y' + y) = 2e^t$$
$$(e^t y)' = 2e^t$$

Integrate and solve for y.

$$e^t y = 2e^t + C$$
$$y(t) = 2 + Ce^{-t}$$

2. We have $a(t) = 3$, so $u(t) = e^{-3t}$. Multiplying we see that the equation becomes

$$e^{-3t}y' - 3e^{-3t}y = 5e^{-3t}.$$

We verify that the left-hand side is the derivative of $e^{-3t}y$, so when we integrate we get

$$e^{-3t}y(t) = -\frac{5}{3}e^{-3t} + C.$$

Solving for y, we get

$$y(t) = -\frac{5}{3} + Ce^{3t}.$$

3. Compare $y' + (2/x)y = (\cos x)/x^2$ with $y' = a(x)x + f(x)$ and note that $a(x) = -2/x$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln|x|} = |x|^2 = x^2.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} x^2\left(y' + \frac{2}{x}y\right) &= \cos x \\ x^2y' + 2xy &= \cos x \\ (x^2y)' &= \cos x \end{aligned}$$

Integrate and solve for x.

$$\begin{aligned} x^2y &= \sin x + C \\ y(x) &= \frac{\sin x + C}{x^2} \end{aligned}$$

4. We have $a(t) = -2t$, so $u(t) = e^{t^2}$. Multiplying by u, the equation becomes

$$e^{t^2}y' + 2te^{t^2}y = 5te^{t^2}.$$

We verify that the left-hand side is the derivative of $e^{t^2}y$, so when we integrate we get

$$e^{t^2}y(t) = \frac{5}{2}e^{t^2} + C.$$

Solving for y we get the general solution

$$y(t) = \frac{5}{2} + Ce^{-t^2}.$$

5. Compare $x' - 2x/(t+1) = (t+1)^2$ with $x' = a(t)x + f(t)$ and note that $a(t) = -2/(t+1)$. Consequently, an integrating factor is found with

$$\begin{aligned} u(t) &= e^{\int -a(t)\,dt} = e^{\int -2/(t+1)\,dt} \\ &= e^{-2\ln|t+1|} = |t+1|^{-2} = (t+1)^{-2}. \end{aligned}$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} (t+1)^{-2}\left(x' - \frac{2}{t+1}x\right) &= 1 \\ \left((t+1)^{-2}x\right)' &= 1 \end{aligned}$$

Integrate and solve for x.

$$\begin{aligned} (t+1)^{-2}x &= t + C \\ x(t) &= t(t+1)^2 + C(t+1)^2 \end{aligned}$$

6. If we write the equations as $x' = (4/t)x + t^3$, we see that $a(t) = 4/t$. Thus the integrating factor is

$$u(t) = e^{-\int (4/t)\,dt} = e^{-4\ln t} = t^{-4}.$$

Multiplying by u, the equation becomes

$$t^{-4}x' - 4t^{-5}x = t^{-1}.$$

After verifying that the left-hand side is the derivative of $t^{-4}x$, we can integrate and get

$$t^{-4}x(t) = \ln t + C.$$

Hence the general solution is

$$x(t) = t^4\ln t + Ct^4.$$

7. Divide both sides by $1 + x$ and solve for y'.

$$y' = -\frac{1}{1+x}y + \frac{\cos x}{1+x}$$

Compare this result with $y' = a(x)y + f(x)$ and note that $a(x) = -1/(1+x)$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int 1/(1+x)\,dx} = e^{\ln|1+x|} = |1+x|.$$

If $1 + x > 0$, then $|1+x| = 1 + x$. If $1 + x < 0$, then $|1+x| = -(1+x)$. In either case, if we multiply both sides of our equation by either integrating factor, we arrive at

$$(1+x)y' + y = \cos x.$$

Check that the left-hand side of this result is the derivative of a product, integrate, and solve for y.

$$\begin{aligned} ((1+x)y)' &= \cos x \\ (1+x)y &= \sin x + C \\ y(x) &= \frac{\sin x + C}{1+x} \end{aligned}$$

8. Divide by $1+x^3$ to put the equation into normal form

$$y' = \frac{3x^2}{1+x^3}y + x^2.$$

We see that $a(x) = 3x^2/(1+x^3)$. Hence the integrating factor is

$$u(x) = e^{-\int 3x^2/(1+x^3)\,dx} = e^{-\ln(1+x^3)} = \frac{1}{1+x^3}.$$

Multiplying by this we get

$$\frac{1}{1+x^3}y' - \frac{3x^2}{(1+x^3)^2}y = \frac{x^2}{1+x^3}.$$

We first verify that the left-hand side is the derivative of $(1+x^3)^{-1}y$. Then we integrate, getting

$$\frac{1}{1+x^3}y(x) = \frac{1}{3}\ln(1+x^3) + C.$$

Solving for y, we get

$$y(x) = \frac{1}{3(1+x^3)\ln(1+x^3)} + C(1+x^3).$$

9. Divide both side of this equation by L and solve for di/dt.

$$\frac{di}{dt} = -\frac{R}{L}i + \frac{E}{L}$$

Compare this with $i' = a(t)i + f(t)$ and note that $a(t) = -R/L$. Consequently, an integrating factor is found with

$$u(t) = e^{\int -a(t)\,dx} = e^{\int R/L\,dt} = e^{Rt/L}$$

Multiply both sides of our equation by this integrating factor and note that the resulting left-hand side is the derivative of a product.

$$e^{Rt/L}\left(\frac{di}{dt} + \frac{R}{L}i\right) = \frac{E}{L}e^{Rt/L}$$
$$\left(e^{Rt/L}i\right)' = \frac{E}{L}e^{Rt/L}$$

Integrate and solve for i.

$$e^{Rt/L}i = \frac{E}{R}e^{Rt/L} + C$$
$$i(t) = \frac{E}{R} + Ce^{-Rt/L}$$

10. Compare $y' = my + c_1e^{mx}$ with $y' = a(x)y + f(x)$ and note that $a(x) = m$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int -m\,dx} = e^{-mx}.$$

Multiply both sides of the differential equation by the integrating factor and check that the resulting left-hand side is the derivative of a product.

$$y' - my = c_1e^{mx}$$
$$e^{-mx}(y' - me^{-mx}) = c_1$$
$$\left(e^{-mx}y\right)' = c_1$$

Integrate and solve for y.

$$e^{-mx}y = c_1x + c_2$$
$$y = (c_1x + c_2)e^{mx}$$

11. Compare $y' = \cos x - y\sec x$ with $y' = a(x)y + f(x)$ and note that $a(x) = -\sec x$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int \sec x\,dx}$$
$$= e^{\ln|\sec x + \tan x|} = |\sec x + \tan x|.$$

If $\sec x + \tan x > 0$, then $|\sec x + \tan x| = \sec x + \tan x$. If $\sec x + \tan x < 0$, then $|\sec x + \tan x| = -(\sec x + \tan x)$. In either case, when we multiply both sides of the differential equation by this integrating factor, we arrive at

$$(\sec x + \tan x)(y' + y\sec x) = \cos x(\sec x + \tan x),$$

or

$$(\sec x + \tan x)y' + (\sec^2 x + \sec x\tan x)y = 1 + \sin x$$

Again, check that the left-hand side of this equation is the derivative of a product, then integrate and solve for y.

$$((\sec x + \tan x)y)' = 1 + \sin x$$
$$(\sec x + \tan x)y = x - \cos x + C$$
$$y = \frac{x - \cos x + C}{\sec x + \tan x}$$

12. Compare $x' - (n/t)x = e^t t^n$ with $x' = a(t)x + f(t)$ and note that $a(t) = n/t$. Consequently, an integrating factor is found with

$$u(t) = e^{\int -a(t)\,dt} = e^{\int -n/t\,dt} = e^{-n\ln|t|} = |t|^{-n}.$$

Depending on the sign of t and whether n is even or odd, $|t|^{-n}$ either equals t^{-n} or $-t^{-n}$. In either case, when we multiply our equation by either of these integrating factors, we arrive at

$$t^{-n}x' - nt^{-n-1}x = e^t.$$

Note that the left-hand side of this result is the derivative of a product, integrate, and solve for x.

$$\begin{aligned}\left(t^{-n}x\right)' &= e^t\\ t^{-n}x &= e^t + C\\ x &= t^n e^t + Ct^n\end{aligned}$$

13. (a) Compare $y' + y\cos x = \cos x$ with $y' = a(x)y + f(x)$ and note that $a(x) = -\cos x$. Consequently, an integrating factor is found with

$$u(x) = e^{-\int a(x)\,dx} = e^{\int \cos x\,dx} = e^{\sin x}.$$

Multiply both sides of the differential equation by the integrating factor and check that the resulting left-hand side is the derivative of a product.

$$\begin{aligned}e^{\sin x}(y' + y\cos x) &= e^{\sin x}\cos x\\ \left(e^{\sin x}y\right)' &= e^{\sin x}\cos x\end{aligned}$$

Integrate and solve for y.

$$\begin{aligned}e^{\sin x}y &= e^{\sin x} + C\\ y(x) &= 1 + Ce^{-\sin x}\end{aligned}$$

(b) Separate the variables and integrate.

$$\begin{aligned}\frac{dy}{dx} &= \cos x(1-y)\\ \frac{dy}{1-y} &= \cos x\,dx\\ -\ln|1-y| &= \sin x + C.\end{aligned}$$

Take the exponential of each side.

$$\begin{aligned}|1-y| &= e^{-\sin x - C}\\ 1-y &= \pm e^{-C}e^{-\sin x}\end{aligned}$$

If we let $A = \pm e^{-C}$, then

$$y(x) = 1 - Ae^{-\sin x},$$

where A is any real number, except zero. However, when we separated the variables above by dividing by $y-1$, this was a valid operation only if $y \neq 1$. This hints at another solution. Note that $y = 1$ easily checks in the original equation. Consequently,

$$y(x) = 1 - Ae^{-\sin x},$$

where A is any real number. Note that this will produce the same solutions a $y = 1 + Ce^{-\sin x}$, C any real number, the solution found in part (a).

14. Compare $y' = y + 2xe^{2x}$ with $y' = a(x)y + f(x)$ and note that $a(x) = 1$. Consequently, an integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int -1\,dx} = e^{-x}.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned}e^{-x}y' - e^{-x}y &= 2xe^x\\ \left(e^{-x}y\right)' &= 2xe^x\end{aligned}$$

Integration by parts yields

$$\int 2xe^x\,dx = 2xe^x - \int 2e^x = 2xe^x - 2e^x + C.$$

Consequently,

$$\begin{aligned}e^{-x}y &= 2xe^x - 2e^x + C\\ y(x) &= 2xe^{2x} - 2e^{2x} + ce^x\end{aligned}$$

The initial condition provides

$$3 = y(0) = 2(0)e^{2(0)} - 2e^{2(0)} + Ce^0 = -2 + C.$$

Consequently, $C = 5$ and $y(x) = 2xe^{2x} - 2e^{2x} + 5e^x$.

15. Solve for y'.

$$y' = -\frac{3x}{x^2+1}y + \frac{6x}{x^2+1}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = -3x/(x^2+1)$. Consequently, an integrating factor is found with

$$\begin{aligned} u(x) &= e^{\int -a(x)\,dx} = e^{\int 3x/(x^2+1)\,dx} \\ &= e^{(3/2)\ln(x^2+1)} = (x^2+1)^{3/2}. \end{aligned}$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} (x^2+1)^{3/2}\,y' + 3x(x^2+1)^{1/2}\,y &= 6x(x^2+1)^{1/2} \\ \left((x^2+1)^{3/2}\,y\right)' &= 6x(x^2+1)^{1/2} \end{aligned}$$

Integrate and solve for y.

$$\begin{aligned} (x^2+1)^{3/2}\,y &= 2(x^2+1)^{3/2} + C \\ y &= 2 + C(x^2+1)^{-3/2} \end{aligned}$$

The initial condition gives

$$-1 = y(0) = 2 + C(0^2+1)^{-3/2} = 2 + C.$$

Therefore, $C = -3$ and $y(x) = 2 - 3(x^2+1)^{-3/2}$.

16. Solve for y'.

$$y' = -\frac{4t}{1+t^2}\,y + \frac{1}{1+t^2}$$

Compare this with $y' = a(t)y + f(t)$ and note that $a(t) = -4t/(1+t^2)$. Consequently, an integrating factor is found with

$$\begin{aligned} u(t) &= e^{\int -a(t)\,dx} = e^{\int 4t/(1+t^2)\,dt} \\ &= e^{2\ln|1+t^2|} = (1+t^2)^2. \end{aligned}$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} (1+t^2)^2 y' + 4t(1+t^2) &= \frac{1}{1+t^2} \\ \left((1+t^2)^2 y\right)' &= \frac{1}{1+t^2} \\ (1+t^2)^2 y &= \tan^{-1} t + C \end{aligned}$$

The initial condition $y(1) = 0$ gives

$$(1+1^2)^2(0) = \tan^{-1} 1 + C.$$

Consequently, $C = -\pi/4$ and

$$y(t) = \frac{\tan^{-1} t - \frac{\pi}{4}}{(1+t^2)^2}.$$

17. Compare $x' + x\cos t = (1/2)\sin 2t$ with $x' = a(t)x + f(t)$ and note that $a(t) = -\cos t$. Consequently, an integrating factor is found with

$$u(t) = e^{\int -a(t)\,dt} = e^{\int \cos t\,dt} = e^{\sin t}.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$\begin{aligned} e^{\sin t}x' + e^{\sin t}(\cos t)x &= \frac{1}{2}e^{\sin t}\sin 2t \\ \left(e^{\sin t}x\right)' &= \frac{1}{2}e^{\sin t}\sin 2t \end{aligned}$$

Use $\sin 2t = 2\sin t\cos t$.

$$\left(e^{\sin t}x\right)' = e^{\sin t}\sin t\cos t$$

Let $u = \sin t$ and $dv = e^{\sin t}\cos t\,dt$. Then,

$$\begin{aligned} \int e^{\sin t}\cos t\sin t\,dt &= \int u\,dv \\ &= uv - \int v\,du \\ &= (\sin t)e^{\sin t} - \int e^{\sin t}\cos t\,dt \\ &= (\sin t)e^{\sin t} - e^{\sin t} + C \end{aligned}$$

Therefore,

$$\begin{aligned} e^{\sin t}x &= e^{\sin t}\sin t - e^{\sin t} + C \\ x(t) &= \sin t - 1 + Ce^{-\sin t} \end{aligned}$$

The initial condition gives

$$1 = x(0) = \sin(0) - 1 + Ce^{-\sin(0)} = -1 + C.$$

Consequently, $C = 2$ and $x(t) = \sin t - 1 + 2e^{-\sin t}$.

18. Solve $xy' + 2y = \sin x$ for y'.

$$y' = -\frac{2}{x}y + \frac{\sin x}{x}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = -2/x$ and $f(x) = (\sin x)/x$. It is important to note that neither a nor f is continuous at $x = 0$, a fact that will heavily influence our interval of existence.

An integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln|x|} = |x|^2 = x^2.$$

Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$x^2y' + 2xy = x\sin x$$
$$\left(x^2y\right)' = x\sin x$$

Integration by parts yields

$$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx$$
$$= -x\cos x + \sin x + C.$$

Consequently,

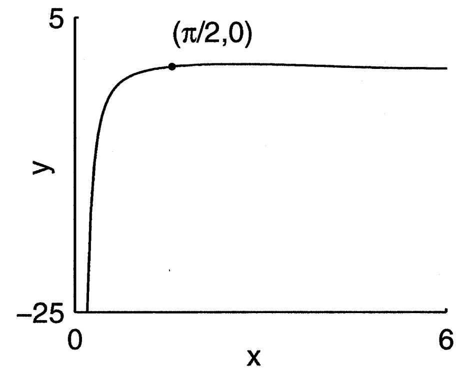
$$x^2y = -x\cos x + \sin x + C,$$
$$y = -\frac{1}{x}\cos x + \frac{1}{x^2}\sin x + \frac{C}{x^2}.$$

The initial condition provides

$$0 = y(\pi/2) = \frac{4}{\pi^2} + \frac{4C}{\pi^2}.$$

Consequently, $C = -1$ and $y = -(1/x)\cos x + (1/x^2)\sin x - 1/x^2$.

We cannot extend any interval to include $x = 0$, as our solution is undefined there. The initial condition $y(\pi/2) = 0$ forces the solution through a point with $x = \pi/2$, a fact which causes us to select $(0, +\infty)$ as the interval of existence. The solution curve is shown in the following figure. Note how it drops to negative infinity as x approaches zero from the right.



19. Solve for y'.

$$y' = \frac{1}{2x+3}y + (2x+3)^{-1/2}$$

Compare this with $y' = a(x)y + f(x)$ and note that $a(x) = 1/(2x+3)$ and $f(x) = (2x+3)^{-1/2}$. It is important to note that a is continuous everywhere except $x = -3/2$, but f is continuous only on $(-3/2, +\infty)$, facts that will heavily influence our interval of existence.

An integrating factor is found with

$$u(x) = e^{\int -a(x)\,dx} = e^{\int -1/(2x+3)\,dx}$$
$$= e^{-(1/2)\ln|2x+3|} = |2x+3|^{-1/2}.$$

However, we will assume that $x > -3/2$ (a domain where both a and f are defined), so $u(x) = (2x+3)^{-1/2}$. Multiply both sides of our equation by the integrating factor and note that the left-hand side of the resulting equation is the derivative of a product.

$$(2x+3)^{-1/2}y' - (2x+3)^{-3/2}y = (2x+3)^{-1}$$
$$\left((2x+3)^{-1/2}y\right)' = (2x+3)^{-1}$$

Integrate and solve for y.

$$(2x+3)^{-1/2}y = \frac{1}{2}\ln(2x+3) + C,$$

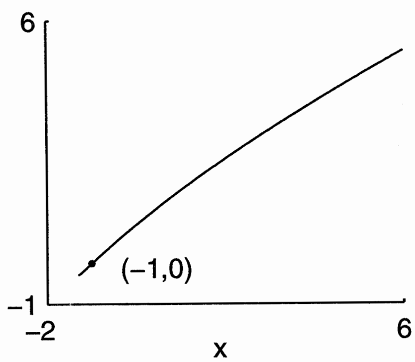
or

$$y = \frac{1}{2}(2x+3)^{1/2}\ln(2x+3) + C(2x+3)^{1/2}$$

The initial condition provides

$$0 = y(-1) = C.$$

Consequently, $y = (1/2)(2x+3)^{1/2}\ln(2x+3)$. The interval of existence is $(-3/2, +\infty)$ and the solution curve is shown in the following figure.



20. Compare $y' = \cos x - y\sec x$ with $y' = a(x)y + f(x)$ and note that $a(x) = -\sec x$ and $f(x) = \cos x$. Although f is continuous everywhere, a has discontinuities at $x = \pi/2 + k\pi$, k an integer.

An integrating factor is found with

$$\begin{aligned} u(x) &= e^{\int -a(x)\,dx} = e^{\int \sec x\,dx} \\ &= e^{\ln|\sec x + \tan x|} = |\sec x + \tan x|. \end{aligned}$$

If $\sec x + \tan x > 0$, the $|\sec x + \tan x| = \sec x + \tan x$. If $\sec x + \tan x < 0$, the $|\sec x + \tan x| = -(\sec x + \tan x)$. Multiplying our equation by either integrating factor produces the same result.

$$\begin{aligned} &(\sec x + \tan x)y' + (\sec x \tan x + \sec^2 x)y \\ &\quad = 1 + \sin x, \end{aligned}$$

From which follows:

$$\begin{aligned} ((\sec x + \tan x)y)' &= 1 + \sin x \\ (\sec x + \tan x)y &= x - \cos x + C \end{aligned}$$

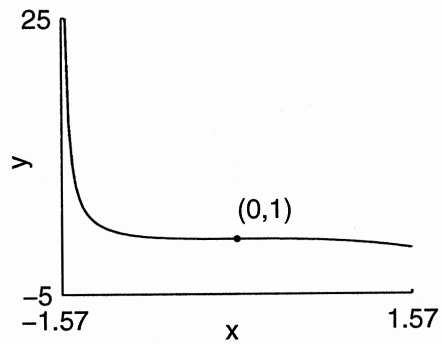
Use the initial condition, $y(0) = 1$.

$$(\sec 0 + \tan 0)(1) = 0 - \cos(0) + C$$

Consequently, $C = 2$ and

$$y = \frac{x - \cos x + 2}{\sec x + \tan x}.$$

The initial condition forces the graph to pass through $(0, 1)$, but $a(x)$ has nearby discontinuities at $x = -\pi/2$ and $x = \pi/2$. Consequently, the interval of existence is maximally extended to $(-\pi/2, \pi/2)$, as shown in the following figure.



21. Solve for x.

$$x' = -\frac{1}{1+t}x + \frac{\cos t}{1+t}$$

Compare this result with $x' = a(t)x + f(t)$ and note that $a(t) = -1/(1+t)$ and $f(t) = \cos t/(1+t)$, neither of which are continuous at $t = -1$. An integrating factor is found with

$$u(t) = e^{\int -a(t)\,dt} = e^{\int 1/(t+1)} = e^{\ln|1+t|} = |1+t|.$$

However, the initial condition dictates that our solution pass through the point $(-\pi/2, 0)$. Because of the discontinuity at $t = -1$, our solution must remain to the left of $t = -1$. Consequently, with $t < -1$, $u(t) = -(1+t)$. However, multiplying our equation by $u(t)$ produces

$$\begin{aligned} (1+t)x' + x &= \cos t, \\ ((1+t)x)' &= \cos t, \\ (1+t)x &= \sin t + C. \end{aligned}$$

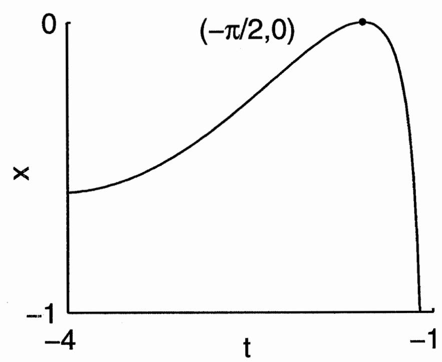
Use the initial condition.

$$\left(1 - \frac{\pi}{2}\right)(0) = \sin\left(-\frac{\pi}{2}\right) + C$$

Consequently, $C = 1$ and

$$x = \frac{1+\sin t}{1+t}.$$

The interval of existence is maximally extended to $(-\infty, -1)$, as shown in the following figure.



22. Let $z = x^{1-n}$. Then

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} = (1-n)x^{-n}\frac{dx}{dt}.$$

This motivates multiplying our equation by $(1 - n)x^{-n}$ to produce

$$(1-n)x^{-n}\frac{dx}{dt} = (1-n)a(t)x^{1-n} + (1-n)f(t).$$

Replacing $(1-n)x^{-n}dx/dt$ with dz/dt and x^{1-n} with z produces the desired result.

$$\frac{dz}{dt} = (1-n)a(t)z + (1-n)f(t)$$

23. In this case $n = 2$, so we set $z = y^{-1}$. Then

$$\begin{aligned}\frac{dz}{dx} &= \frac{dz}{dy}\frac{dy}{dx}\\ &= -y^{-2}\left[xy^2 - \frac{y}{x}\right]\\ &= -x + \frac{1}{xy}\\ &= -x + \frac{z}{x}.\end{aligned}$$

This is a linear equation for z. The integrating factor is $1/x$, so we have

$$\begin{aligned}\frac{1}{x}\left[\frac{dz}{dx} - \frac{z}{x}\right] &= \frac{1}{x}(-x)\\ \left[\frac{z}{x}\right]' &= -1\\ \frac{z}{x} &= -x + C\\ z(x) &= x(C-x)\end{aligned}$$

Since $z = 1/y$, our solution is $y(x) = \dfrac{1}{x(C-x)}$.

24. In this case $n = 2$, so we set $z = y^{-1}$. Then

$$\begin{aligned}\frac{dz}{dx} &= \frac{dz}{dy}\frac{dy}{dx}\\ &= y^{-2}\left[y^2 - y\right]\\ &= -1 + \frac{1}{y}\\ &= -1 + z\end{aligned}$$

This is a linear equation for z. The integrating factor is e^{-x}, so we have

$$\begin{aligned}e^{-x}\left[\frac{dz}{dx} - z\right] &= -e^{-x}\\ \left[e^{-x}z\right]' &= -e^{-x}\\ e^{-x}z &= e^{-x} + C\\ z(x) &= 1 + Ce^{x}.\end{aligned}$$

Since $z = 1/y$ our solution is $y(x) = \dfrac{1}{1+Ce^x}$.

25. Solve for y'.

$$y' = -\frac{1}{x}y + x^3y^3$$

Compare this with $y' = a(x)y + f(x)y^n$ and note that this has the form of Bernoulli's equation with $n = 3$. Let $z = y^{1-3} = y^{-2}$. Then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = -2y^{-3}\frac{dy}{dx}.$$

Multiply the equation by $-2y^{-3}$.

$$-2y^{-3}\frac{dy}{dx} = \frac{2}{x}y^{-2} - 2x^3$$

Replace $-2y^{-3}(dy/dx)$ with dz/dx and y^{-2} with z.

$$\frac{dz}{dx} = \frac{2}{x}z - 2x^3$$

This equation is linear with integrating factor

$$u(x) = e^{\int -a(x)\,dx} = e^{\int -2/x\,dx} = e^{-2\ln|x|} = x^{-2}.$$

Multiply by the integrating factor and integrate.

$$\begin{aligned} x^{-2}z' - 2x^{-3}z &= -2x \\ \left(x^{-2}z\right)' &= -2x \\ x^{-2}z &= -x^2 + C \\ z &= -x^4 + Cx^2 \end{aligned}$$

Replace z with y^{-2} and solve for y.

$$\begin{aligned} y^{-2} &= -x^4 + Cx^2 \\ y &= \pm 1/\sqrt{Cx^2 - x^4} \end{aligned}$$

26. Compare this with $P' = a(t)P + f(t)P^n$ and note that this has the form of Bernoulli's equation with $n = 2$. Let $z = P^{1-2} = P^{-1}$. Then

$$\frac{dz}{dt} = \frac{dz}{dP}\frac{dP}{dt} = -P^{-2}\frac{dP}{dt}.$$

Multiply the equation by $-P^{-2}$.

$$-P^{-2}\frac{dP}{dt} = -aP^{-1} + b$$

Replace $-P^{-2}(dP/dt)$ with dz/dt and P^{-1} with z.

$$\frac{dz}{dt} = -az + b$$

This equation is linear with integrating factor

$$u(t) = e^{\int -a(t)\,dt} = e^{\int a\,dt} = e^{at}.$$

Multiply by the integrating factor and integrate.

$$\begin{aligned} e^{at}\frac{dz}{dt} + ae^{at}z &= be^{at} \\ \frac{d}{dt}\left(e^{at}z\right) &= be^{at} \\ e^{at}z &= \frac{b}{a}e^{at} + C \\ z &= \frac{b}{a} + Ce^{-at} \end{aligned}$$

Replace z with P^{-1} and solve for y.

$$\begin{aligned} P^{-1} &= \frac{b}{a} + Ce^{-at} \\ P &= \frac{1}{b/a + Ce^{-at}} \end{aligned}$$

27. (a) Since $y = y_1 + z$, $y^2 = y_1^2 + 2y_1z + z^2$. Hence

$$\begin{aligned} z' &= y' - y_1' \\ &= -[\psi y^2 + \phi y + \chi] + [\psi y_1^2 + \phi y_1 + \chi] \\ &= \psi[y_1^2 - y^2] + \phi[y_1 - y] \\ &= -\psi[2y_1z + z^2] - \phi z \\ &= -(2y_1\psi + \phi)z - \psi z^2 \end{aligned}$$

(b) Since $y_1 = 1/t$ is a solution, we set $z = y + 1/t$. Then $y = z - 1/t$, and $y^2 = z^2 - 2z/t + 1/t^2$, so

$$\begin{aligned} z' &= y' - \frac{1}{t^2} \\ &= -\frac{1}{t^2} - \frac{y}{t} + y^2 - \frac{1}{t^2} \\ &= -\frac{3z}{t} + z^2. \end{aligned}$$

This is a Bernoulli equation with $n = 2$. Thus we set $w = 1/z$. Differentiating, we get

$$\begin{aligned} w' &= -\frac{1}{t^2}z' \\ &= -\frac{1}{t^2}\left[-\frac{3z}{t} + z^2\right] \\ &= \frac{3}{tz} - 1 \\ &= \frac{3w}{t} - 1. \end{aligned}$$

This is a linear equation, and t^{-3} is an integrating factor.

$$\begin{aligned} t^{-3}\left[w' - \frac{3w}{t}\right] &= -t^{-3} \\ \left[t^{-3}w\right] &= -t^{-3} \\ t^{-3}w(t) &= \frac{1}{2}t^{-2} + C \\ w(t) &= \frac{1}{2}t + Ct^3 \end{aligned}$$

Now it is a matter of unravelling the changes of variable. First

$$z(t) = \frac{1}{w(t)} = \frac{2}{t + 2Ct^3} = \frac{2}{t + Bt^3},$$

Where we have set $B = 2C$. Then

$$y(t) = z(t) - \frac{1}{t} = \frac{2}{t + Bt^3} - \frac{1}{t}.$$

This looks a little better if we use partial fractions to write

$$\begin{aligned} y(t) &= \frac{2}{t + Bt^3} - \frac{1}{t} \\ &= \frac{2}{t} - \frac{2Bt}{1 + Bt^2} - \frac{1}{t} \\ &= \frac{1}{t} - \frac{2Bt}{1 + Bt^2}. \end{aligned}$$

28. The model is $N' = kN(1000 - N)$, where the proportionality constant k is yet to be determined. Since we know that when $N = 100$, the rate of infection is 90/day, we have $k \cdot 100 \cdot 900 = 90$, we find that $k = 1 \times 10^{-3}$. Hence the model equation is $N' = N - N^2/1000$. This is a Bernoulli equation, with n=2. Accordingly we set $x = 1/N$. Then

$$\begin{aligned} x' &= -N'/N^2 \\ &= -1/N + 10^{-3} \\ &= -x + 10^{-3}. \end{aligned}$$

Solving this linear equation, we get $x(t) = 10^{-3}[1 + Ce^{-t}]$. Hence $N(t) = 1/x(t) = 1000/[1 + Ce^{-t}]$. At $t = 0$, $N = 20 = 1/[1 + C]$. Hence $C = 49$, and the solution is $N(t) = 1000/[1 + 49e^{-t}]$.

We have $N(t) = 0.9 \times 1000 = 900$ when $t = 6.089$ days.

29. Newton's law of cooling says the rate of change of temperature is equal to k times the difference between the current temperature and the ambient temperature. In this case the ambient temperature is decreasing from 0°C, and at a constant rate of 1°C per hour. Hence the model equation is $T' = -k(T + t)$, where we are taking $t = 0$ to be midnight. This is a linear equation. The solution is $T(t) = -t + 1/k + Ce^{-kt}$. Since $T(0) = 31$, the constant evaluates to $C = 31 - 1/k$. The solution is $T(t) = -t + 1/k + (31 - 1/k)e^{-kt}$.

We need to compute the time t_0 when $T(t_0) = 37$, using $k \approx 0.2231$ from Exercise 35 of Section 2. This is a nonlinear equation, but using a calculator or a computer we can find that $t_0 \approx -0.8022$. Since $t = 0$ corresponds to midnight, this means that the time of death is approximately 11:12 PM.

30. The homogeneous equation, $y' = -3y$ has solution $y_h(t) = e^{-3t}$. We look for a particular solution in the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - 3vy_h \\ &= v'y_h - 3y_p, \end{aligned}$$

and $y_p' = -3y_p + 4$, we have $v' = 4/y_h = 4e^{3t}$. Integrating we see that $v(t) = 4e^{3t}/3$, and

$$y_p(t) = v(t)y_h(t) = \frac{4}{3}e^{3t} \cdot e^{-3t} = \frac{4}{3}.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = \frac{4}{3} + Ce^{-3t}.$$

31. The homogeneous equation, $y' = -2y$ has solution $y_h(t) = e^{-2t}$. We look for a particular solution in the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - 2vy_h \\ &= v'y_h - 2y_p, \end{aligned}$$

and $y_p' = -2y_p + 5$, we have $v' = 5/y_h = 5e^{2t}$. Integrating we see that $v(t) = 5e^{2t}/2$, and

$$y_p(t) = v(t)y_h(t) = \frac{5}{2}e^{2t} \cdot e^{-2t} = \frac{5}{2}.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = \frac{5}{2} + Ce^{-2t}.$$

32. The homogeneous equation, $y' = -(2/x)y$ has solution $y_h(x) = x^{-2}$. We look for a particular solution in the form $y_p(x) = v(x)y_h(x)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - 2y_p/x, \end{aligned}$$

and $y_p' = -(2/x)y_p + 8x$, we have $v' = 8x/y_h = 8x^3$. Hence $v(x) = 2x^4$, and

$$y_p(x) = v(x)y_h(x) = 2x^2.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = 2x^2 + Cx^{-2}.$$

33. The homogeneous equation, $y' = -y/t$, has solution $y_h(t) = 1/t$. We look for a particular solution in the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - 2vy_h \\ &= v'y_h - y_p/t, \end{aligned}$$

and $y_p' = -y_p/t + 4t$, we have $v' = 4t/y_h = 4t^2$. Integrating we get $v(t) = 4t^3/3$, and

$$y_p(t) = v(t)y_h(t) = \frac{4}{3}t^2.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = \frac{4}{3}t^2 + C/t.$$

34. The homogeneous equation, $x' = -2x$ has solution $x_h(t) = e^{-2t}$. We look for a particular solution in the form $x_p(t) = v(t)x_h(t)$, where v is an unknown function. Since

$$\begin{aligned} x_p' &= v'x_h + vx_h' \\ &= v'x_h - 2vx_h \\ &= v'x_h - 2x_p, \end{aligned}$$

and $x_p' = -2x_p + t$, we have $v' = t/x_h = te^{2t}$. Integrating, we get $v(t) = (t/2 - 1/4)e^{2t}$, and

$$x_p(t) = v(t)x_h(t) = \frac{1}{4}[2t - 1].$$

$$x(t) = x_p(t) + Cx_h(t) = \frac{1}{4}[2t - 1] + Ce^{-2t}.$$

35. The homogeneous equation, $y' = -2xy$ has solution $y_h(x) = e^{-x^2}$. We look for a particular solution in the form $y_p(x) = v(x)y_h(x)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - 2xy_p, \end{aligned}$$

and $y_p' = -2xy_p + 4x$, we have $v' = 4x/y_h = 4xe^{x^2}$. Hence $v(x) = 2e^{x^2}$, and

$$y_p(x) = v(x)y_h(x) = 2.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = 2 + Ce^{x^2}.$$

36. The homogeneous equation, $y' = 3y$ has solution $y_h(t) = e^{3t}$. We look for a particular solution in the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h + 3vy_h \\ &= v'y_h + 3y_p, \end{aligned}$$

and $y_p' = 3y_p + 4$, we have $v' = 4/y_h = 4e^{-3t}$. Integrating we see that $v(t) = -4e^{3t}/3$, and

$$y_p(t) = v(t)y_h(t) = -\frac{4}{3}e^{-3t} \cdot e^{3t} = -\frac{4}{3}.$$

The general solution is

$$y(t) = y_p(t) + Cy_h(t) = -\frac{4}{3} + Ce^{3t}.$$

Since $y(0) = 2$, we must have $2 = -4/3 + C$, or $C = 10/3$. Thus the solution is

$$y(t) = \left(-4 + 10e^{3t}\right)/3.$$

37. The homogeneous equation $y' = -y/2$ has solution $y_h(t) = e^{-t/2}$. We look for a particular solution of the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - vy_h/2 \\ &= v'y_h - y_p/2, \end{aligned}$$

and $y_p' = -y_p/2 + t$, we have $v' = t/y_h(t) = te^{t/2}$. Integrating we find that $v(t) = (2t-4)e^{t/2}$, and $y_p(t) = v(t)y_h(t) = (2t-4)$. The general solution is $y(t) = y_p(t) + Cy_h(t) = (2t-4) + Ce^{-t/2}$. From $y(0) = 1$ we compute that $C = 5$, so the solution is

$$y(t) = (2t-4) + 5e^{-t/2}.$$

38. The homogeneous equation $y' = -y$ has solution $y_h(t) = e^{-t}$. We look for a particular solution of the form $y_p(t) = v(t)y_h(t)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - vy_h \\ &= v'y_h - y_p, \end{aligned}$$

and $y_p' = -y_p + e^t$, we see that $v' = e^t/y_h = e^{2t}$. Integrating we get $v(t) = e^{2t}/2$, and $y_p(t) = e^t/2$. The general solution is $y(t) = y_p(t) + Cy_h(t) = e^t/2 + Ce^{-t}$. From $y(0) = 1$, we compute that $C = 1/2$, so the solution is

$$y(t) = \frac{1}{2}\left(e^t + e^{-t}\right).$$

39. The homogeneous equation $y' = -2xy$ has solution $y_h(x) = e^{-x^2}$. We look for a particular solution of the form $y_p(x) = v(x)y_h(x)$, where v is an unknown function. Since

$$\begin{aligned} y_p' &= v'y_h + vy_h' \\ &= v'y_h - 2xvy_h \\ &= v'y_h - 2xy_p, \end{aligned}$$

and $y_p' = -2xy_p + 2x^3$, we see that $v' = 2x^3e^{x^2}$. Integrating we get $v(x) = (x^2-1)e^{x^2}$, and $y_p(x) = x^2 - 1$. The general solution is $y(x) = y_p(x) + Cy_h(x) = x^2 - 1 + Ce^{-x^2}$. Since $y(0) = -1$, we have $C = 0$, and the solution is

$$y(x) = x^2 - 1.$$

40. The homogeneous equation $x' = (2/t^2)x$ has solution $x_h(t) = e^{-2/t}$. We look for a particular solution of the form $x_p(t) = v(t)x_h(t)$, where v is an unknown function. Since

$$\begin{aligned} x_p' &= v'x_h + vx_h' \\ &= v'x_h + 2vx_h/t^2 \\ &= v'x_h + 2x_p/t^2, \end{aligned}$$

and $x_p' = 2x_p/t^2 + 1/t^2$, we have $v' = 1/(t^2x_h) = e^{2/t}/t^2$. Integrating we find that $v(t) = -e^{2/t}/2$, and $x_p(t) = -1/2$. The general solution is $x(t) = x_p(t) + Cx_h(t) = -1/2 + Ce^{-2/t}$. Since $x(1) = 0$, we find that $C = e^2/2$, and the solution is

$$x(t) = \frac{1}{2}\left(-1 + e^{2(1-1/t)}\right).$$

41. The homogeneous equation $x' = -4tx/(1+t^2)$ has solution $x_h(t) = (1+t^2)^{-2}$. We look for a particular solution of the form $x_p(t) = v(t)x_h(t)$, where v is an unknown function. Since

$$\begin{aligned} x_p' &= v'x_h + vx_h' \\ &= v'x_h - 4tx_p/(1+t^2), \end{aligned}$$

and $x_p' = -4tx_p/(1+t^2) + t/(1+t^2)$, so

$$v' = \frac{t}{1+t^2} \cdot \frac{1}{x_h(t)} = t(1+t^2).$$

Integrating, we get $v(t) = t^2/2 + t^4/4$. Thus

$$x_p(t) = v(t)x_h(t) = \frac{2t^2 + t^4}{4(1+t^2)^2}.$$

The general solution is

$$x(t) = x_p(t) + Cx_h(t) = \frac{4C + 2t^2 + t^4}{4(1+t^2)^2}.$$

The initial condition $x(0) = 1$ implies that $C = 1$, so the solution is

$$y(t) = \frac{4 + 2t^2 + t^4}{4(1+t^2)^2}.$$

42. (a) The equation $T + kT = 0$ is separable, with solution $T_h = Ce^{-kt}$, C an arbitrary constant.

(b) The equation $T' = -k(T - A)$ is autonomous. We seek a constant solution (see Section 2.9) that makes the right side equal to zero. Hence, $T_p = A$ is a particular solution of the inhomogeneous equation.

(c) The general solution is $T = T_h + T_p = Ce^{-kt} + A$, with C an arbitrary constant.

(d) Again, the solution of the homogeneous equation $T' + kT = 0$ is $T_h = Ce^{-kt}$, with C an arbitrary constant. The inhomogeneous equation $T' = -k(T - A) + H$ is also autonomous (the right side is independent of t). We seek a constant solution by setting the right side equal to zero.

$$\begin{aligned} -k(T_p - A) &= H \\ T_p - A &= \frac{H}{K} \\ T_p &= A + \frac{H}{K} \end{aligned}$$

Hence, the general solution is given by the equation

$$T = T_h + T_p = Ce^{-kt} + \left(A + \frac{H}{K}\right).$$

43. (a) The solution of the homogeneous equation $T' + kT = 0$ is $T_h = Fe^{-kt}$, with F an arbitrary constant.

(b) We guess that $T_p = C\cos\omega t + D\sin\omega t$ is a particular solution. Substituting T_p and $T_p' = -C\omega\sin\omega t + D\omega\cos\omega t$ in the left side of $T' + kT = kA\sin\omega t$, then gathering coefficients of $\cos\omega t$ and $\sin\omega t$, we obtain

$$T_p' + kT_p = (-\omega C + kD)\sin\omega t + (kC + \omega D)\cos\omega t. \quad (4.1)$$

Comparing this with the right side of $T_p' + kT_p = kA\sin\omega t$, we see that

$$-\omega C + kD = kA \quad \text{and} \quad kC + \omega D = 0.$$

(c) Solving these equations simultaneously (for example, multiply the first equation by k, the second by ω, then add the equations to eliminate C) provides

$$C = -\frac{\omega k A}{k^2 + \omega^2} \quad \text{and} \quad D = \frac{k^2 A}{k^2 + \omega^2}.$$

Substituting these results in $T_p = C\cos\omega t + D\sin\omega t$ provides the particular solution

$$T_p = -\frac{\omega k A}{k^2 + \omega^2}\cos\omega t + \frac{k^2 A}{k^2 + \omega^2}\sin\omega t.$$

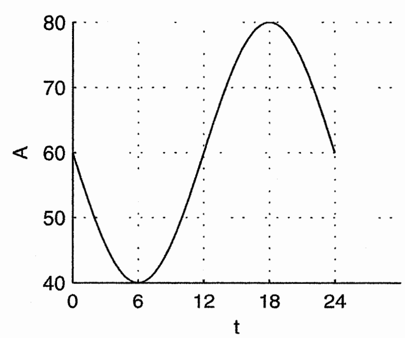
Hence, the general solution is

$$\begin{aligned} T &= T_h + T_p \\ &= Fe^{-kt} + \frac{kA}{k^2 + \omega^2}\left[k\sin\omega t - \omega\cos\omega t\right]. \end{aligned}$$

44. (a) If the period of the ambient temperature is 24 hours, then the computation

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{24} = \frac{\pi}{12}$$

gives the angular frequency. Because the sinusoid has a maximum of 80° F and a minimum of 40° F, the amplitude will be half of the difference, or 20. A sketch of the ambient temperature follows.



Note the minimum at 6 am, then the maximum at 6 pm. What we have is an upside-down sine, with angular frequency $\pi/12$, that is shifted upward 60° F. Thus, the equation for the ambient temperature must be

$$A = 60 - 20\sin\frac{\pi t}{12}.$$

Therefore, the model, adjusted for this ambient temperature, becomes

$$\frac{dT}{dt} = -\frac{1}{2}\left(T - 60 + 20\sin\frac{\pi t}{12}\right). \quad (4.2)$$

(b) The homogeneous equation $T' + (1/2)T = 0$ has solution $T_h = Ce^{-t/2}$, where C is an arbitrary constant. Now, consider the inhomogeneous equation

$$T' + \frac{1}{2}T = 30 - 10\sin\frac{\pi t}{12}. \quad (4.3)$$

Note that the right hand side consists of a constant and a sinusoid. Let's try a particular solution having the form

$$T_p = D + E\cos\frac{\pi t}{12} + F\sin\frac{\pi t}{12}. \quad (4.4)$$

Substitute this guess and its derivative into the left hand side of equation (4.4) and collect coefficients to get

$$T' + \frac{1}{2}T = \frac{1}{2}D + \left(-\frac{\pi}{12}E + \frac{1}{2}F\right)\sin\frac{\pi t}{12} + \left(\frac{1}{2}E + \frac{\pi}{12}F\right)\cos\frac{\pi t}{12} \quad (4.5)$$

Comparing this with the right-hand side of equation (4.3), we see that

$$\frac{1}{2}D = 30,$$
$$-\frac{\pi}{12}E + \frac{1}{2}F = -10, \quad \text{and}$$
$$\frac{1}{2}E + \frac{\pi}{12}F = 0.$$

Clearly, $D = 60$, and solving the remaining two equations simultaneously, we obtain

$$E = \frac{120\pi}{36+\pi^2} \quad \text{and} \quad F = \frac{-720}{36+\pi^2}.$$

These values of D, E, and F, when inserted into equation (4.4), provide the particular solution

$$T_p = 60 + \frac{120\pi}{36+\pi^2}\cos\frac{\pi t}{12} - \frac{720}{36+\pi^2}\sin\frac{\pi t}{12}.$$

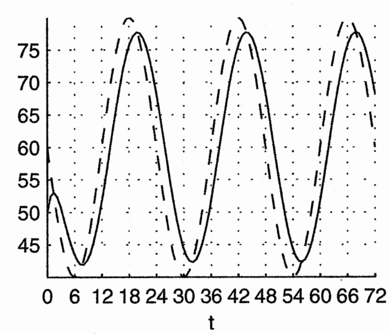
Thus, the general solution is $T = T_h + T_p$, or

$$T_p = Ce^{-t/2} + 60 + \frac{120}{36+\pi^2}\left[\pi\cos\frac{\pi t}{12} - 6\sin\frac{\pi t}{12}\right]. \quad (4.6)$$

Now, when the initial condition $T(0) = 50$ is substituted into equation (4.6), we obtain $C = -10 - 120\pi/(36+\pi^2)$. Thus, the general solution becomes

$$T = -\left(10 + \frac{120\pi}{36+\pi^2}\right)e^{-t/2} + 60 + \frac{120}{36+\pi^2}\left[\pi\cos\frac{\pi t}{12} - 6\sin\frac{\pi t}{12}\right]. \quad (4.7)$$

(c) The plot of the ambient temperature is shown as a dashed curve in the following figure. The temperature T inside the cabin is shown as a solid curve.



Note that the transient part of the solution dies out quickly. Indeed, because of the factor of $e^{-t/2}$, the time constant (See Section 2.2, Exercise **??**) is $T_c = 2$ hr. Thus, in about four time constants, or 8 hours, this part of the temperature solution is negligible. Finally, note how the temperature in the cabin reacts to and trails the ambient temperature outside, which makes sense.

Section 2.5. Mixing Problems

1. (a) Let $S(t)$ denote the amount of sugar in the tank, measured in pounds. The rate in is 3 gal/min $\times$ 0.2 lb/gal $=$ 0.6 lb/min. The rate out is 3 gal/min $\times$ $S/100$ lb/gal $=$ $3S/100$ lb/min. Hence

$$\begin{aligned}\frac{dS}{dt} &= \text{rate in} - \text{rate out}\\ &= 0.6 - 3S/100\end{aligned}$$

This linear equation can be solved using the integrating factor $u(t) = e^{3t/100}$ to get the general solution $S(t) = 20 + Ce^{-3t/100}$. Since $S(0) = 0$, the constant $C = -20$ and the solution is $S(t) = 20(1 - e^{-3t/100})$.
$S(20) = 10(1 - e^{-0.6}) \approx 9.038$lb.

(b) $S(t) = 15$ when $e^{-3t/100} = 1 - 15/20 = 1/4$. Taking logarithms this translates to $t = (100 \ln 4)/3 \approx 46.2098$m.

(c) As $t \to \infty$ $S(t) \to 20$.

2. (a) Let $x(t)$ represent the number of pounds of sugar in the tank at time t. The rate in is 0, and the rate out is 2 gal/min$\cdot$ $x/50$ lb/gal $= x/25$ lb/min. Hence the model equation is $x' = -x/25$. The general solution is $x(t) = Ae^{-t/25}$. The initial condition implies that $A = x(0) = 50$ gal $\times$ 2 lb/gal $=$ 100 lb. Hence the solution is $x(t) = 100e^{-t/25}$. After 10 minutes we have $x(10) = 67.032$ lb of sugar in the tank.

(b) We have to find t such that $x(t) = 100e^{-t/25} = 20$. This comes to $t = 25 \ln 5 \approx 40.2359$ min.

(c) $x(t) = 100e^{-t/25} \to 0$ as $t \to \infty$.

3. (a) Let $x(t)$ represent the number of pounds of salt in the tank at time t. The rate at which the salt in the tank is changing with respect to time is equal to the rate at which salt enters the tank minus the rate at which salt leaves the tank, i.e.,

$$\frac{dx}{dt} = \text{rate in} - \text{rate out}.$$

In order that the units match in this equation, dx/dt, the rate In, and the rate Out must each be measured in pounds per minute (lb/min).

Solution enters the tank at 5 gal/min, but the concentration of this solution is 1/4 lb/gal. Consequently,

$$\text{rate in} = 5\ \text{gal/min} \times \frac{1}{4}\ \text{lb/gal} = \frac{5}{4}\ \text{lb/min}.$$

Solution leaves the tank at 5 gal/min, but at what concentration? Assuming perfect mixing, the concentration of salt in the solution is found by dividing the amount of salt by the volume of solution, $c(t) = x(t)/100$. Consequently,

$$\text{rate out} = 5\ \text{gal/min} \times \frac{x(t)}{100}\ \text{lb/gal} = \frac{1}{20}x(t)\ \text{lb/min}.$$

As there are 2 lb of salt present in the solution initially, $x(0) = 2$ and

$$\frac{dx}{dt} = \frac{5}{4} - \frac{1}{20}x, \quad x(0) = 2.$$

Multiply by the integrating factor, $e^{(1/20)t}$, and integrate.

$$\begin{aligned}\left(e^{(1/20)t}x\right)' &= \frac{5}{4}e^{(1/20)t}\\ e^{(1/20)t}x &= 25e^{(1/20)t} + C\\ x &= 25 + Ce^{-(1/20)t}\end{aligned}$$

The initial condition $x(0) = 2$ gives $C = -23$ and

$$x(t) = 25 - 23e^{-(1/20)t}.$$

Thus, the concentration at time t is given by

$$c(t) = \frac{x(t)}{100} = \frac{25 - 23e^{-(1/20)t}}{100},$$

and the eventual concentration can be found by taking the limit as $t \to +\infty$.

$$\lim_{t\to+\infty} \frac{25 - 23e^{-(1/20)t}}{100} = \frac{1}{4}\ \text{lb/gal}$$

Note that this answer is quite reasonable as the concentration of solution entering the tank is also $1/4$ lb/gal.

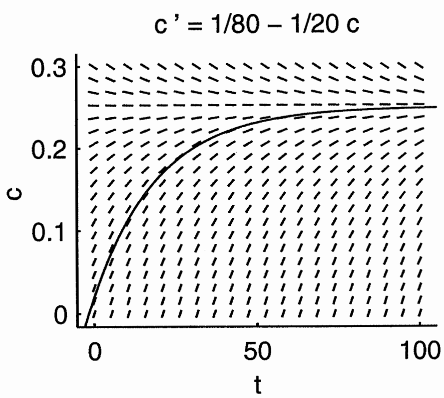
(b) We found it convenient to manipulate our original differential equation before using our solver. The key idea is simple: we want to sketch the concentration $c(t)$, not the salt content $x(t)$. However,

$$c(t) = \frac{x(t)}{100} \quad \text{or} \quad x(t) = 100c(t).$$

Consequently, $x'(t) = 100c'(t)$. Substituting these into our balance equation gives

$$\begin{aligned} x' &= \frac{5}{4} - \frac{1}{20}x, \\ 100c' &= \frac{5}{4} - \frac{1}{20}(100c), \\ c' &= \frac{1}{80} - \frac{1}{20}c, \end{aligned}$$

with $c(0) = x(0)/100 = 2/100 = 0.02$. The numerical solution of this ODE is presented in the following figure. Note how the concentration approaches 0.25 lb/gal.



4. Let $x(t)$ represent the amount of salt in the solution at time t. Let r represent the rate (gal/min) that water enters (and leaves) the tank. Consequently, the rate at which salt enters the tank is 0 gal/min, but the

$$\text{rate out} = r \text{ gal/min} \times \frac{x(t)}{500} \text{ lb/gal} = \frac{r}{500}x(t) \text{ lb/min}.$$

Thus,

$$\begin{aligned} \frac{dx}{dt} &= \text{rate in} - \text{rate out}, \\ \frac{dx}{dt} &= -\frac{r}{500}x. \end{aligned}$$

Let $c(t)$ represent the concentration at time t. Thus, $c(t) = x(t)/500$, or $500c(t) = x(t)$ and $500c'(t) = x'(t)$. Substitute these into the rate equation to produce

$$\begin{aligned} 500c' &= -\frac{r}{500}(500c), \\ c' &= -\frac{r}{500}c. \end{aligned}$$

This equation is separable, with solution $c = Ae^{-(r/500)t}$. Use the initial concentration, $c(0) = .05$ lb/gal, to produce

$$c = 0.05e^{-(r/500)t}.$$

The concentration must reach 1% in one hour (60 min), so $c(60) = 0.01$ and

$$\begin{aligned} 0.01 &= 0.05e^{-(r/500)(60)}, \\ \frac{1}{5} &= e^{-(3/25)r}, \\ r &= \frac{25}{3}\ln 5, \\ r &\approx 13.4 \text{ gal/min}. \end{aligned}$$

5. The volume is increasing at the rate of 2 gal/min, so the volume at time t is $V(t) = 20 + 2t$. The tank is full when $V(t) = 50$, or when $t = 15$ min. If $x(t)$ is the amount of salt in the tank at time t, then the concentration is $x(t)/V(t)$. The rate in is 4 gal/min $\cdot$ 0.5 lb/gal $= 2$ lb/min. The rate out is 2 gal/min $\cdot x/V$ lb/gal. Hence the model equation is

$$x' = 2 - 2x/V = 2 - \frac{x}{10+t}.$$

This linear equation can be solved using the integrating factor $u(t) = 10 + t$, giving the general solution $x(t) = 10 + t + C/(10 + t)$. The initial condition $x(0) = 0$ enables us to compute that $C = -100$, so the solution is $x(t) = 10 + t - 100/(10 + t)$. At $t = 15$, when the tank is full, we have $x(15) = 21$ lb.

6. The volume in the tank is decreasing at 1 gal/min, so the volume is $V(t) = 100 - t$. There is no sugar coming in, and the rate out is 3 gal/min $\times$ $S(t)/V(t)$ lb/gal. Hence the differential equation is

$$\frac{dS}{dt} = \frac{-3S}{100-t}.$$

This equation is linear and homogeneous. It can be solved by separating variables. The general solution is $S(t)+A(100-t)^3$. Since $S(0) = 100\times 0.05 = 5$, we see that $A = 5\times 10^{-6}$, and the solution is $S(t) = 5\times 10^{-6}\times(100-t)^3$.
When $V(t) = 100 - t = 50$ gal,

$$S(t) = 5\times 10^{-6}\times 50^3 = 0.625\,\text{lb}.$$

7. (a) The volume of liquid in the tank is increasing by 2 gal/min. Hence the volume is $V(t) = 100 + 2t$ gal. Let $x(t)$ be the amount of pollutant in the tank, measured in lbs. The rate in during this initial period is 6 gal/min $\cdot$ 0.5 lb/gal = 3 lb/gal. The rate out is 8 gl/min $\cdot x/V = 4x/(50+t)$. Hence the model equation is

$$x' = 3 - 4x/(50+t).$$

This linear equation can be solved using the integrating factor $u(t) = (50+t)^4$. The general solution is $x(t) = 3(50+t)/5 + C(50+t)^{-4}$. The initial condition $x(0) = 0$ allows us to compute the constant to be $C = -1.875\times 10^8$. Hence the solution is

$$x(t) = \frac{3t}{5} + 30 - \frac{1.875\times 10^8}{(50+t)^4}.$$

After 10 minutes the tank contains $x(10) = 21.5324$ lb of salt.

(b) Now the volume is decreasing at the rate of 4 gal/min from the initial volume of 120 gal. Hence if we start with $t = 0$ at the 10 minute mark, the volume is $V(t) = 120-4t$ gal. Now the rate in is 0, and the rate out is 8 gal/min $\cdot x/V = 2x/(30-t)$. Hence the model equation is

$$x' = -\frac{2x}{30-t}.$$

This homogeneous linear equation can solved by separating variables to find the general solution $x(t) = A(30-t)^2$. At $t = 0$ we have $x(0) = 21.5342$, from which we find that $A = 21.5342/900$, and the solution is

$$x(t) = \frac{21.5342}{900}(30-t)^2.$$

We are asked to find when this is one-half of 21.5342. This happens when $(30-t)^2 = 450$ or at $t = 8.7868$ min.

8. Let $x(t)$ represent the amount of drug in the organ at time t. The rate at which the drug enters the organ is

$$\text{rate in} = a\ \text{cm}^3/\text{s}\times\kappa\ \text{g/cm}^3 = a\kappa\ \text{g/s}.$$

The rate at which the drug leaves the organ equals the rate at which fluid leaves the organ, multiplied by the concentration of the drug in the fluid at that time. Hence,

$$\text{rate out} = b\ \text{cm}^3/\text{s}\times\frac{x(t)}{V_0+rt}\ \text{g/cm}^3 = \frac{b}{V_0+rt}x(t)\ \text{g/s}.$$

Consequently,

$$\frac{dx}{dt} = a\kappa - \frac{b}{V_0+rt}x.$$

The integrating factor is

$$u(t) = e^{\int b/(V_0+rt)\,dt} = e^{(b/r)\ln(V_0+rt)} = (V_0+rt)^{b/r}.$$

Multiply by the integrating factor and integrate.

$$\begin{aligned}\left((V_0+rt)^{b/r}x\right)' &= a\kappa(V_0+rt)^{b/r}\\ (V_0+rt)^{b/r}x &= \frac{a\kappa}{r(b/r+1)}(V_0+rt)^{b/r+1} + L\\ x &= \frac{a\kappa}{b+r}(V_0+rt) + L(V_0+rt)^{-b/r}\end{aligned}$$

No drug in the system initially gives $x(0) = 0$ and $L = -a\kappa V_0^{b/r+1}/(b+r)$. Consequently,

$$\begin{aligned}x &= \frac{a\kappa}{b+r}(V_0+rt) - \frac{a\kappa V_0^{b/r+1}}{b+r}(V_0+rt)^{-b/r},\\ x &= \frac{a\kappa}{b+r}(V_0+rt)\left[1 - V_0^{b/r+1}(V_0+rt)^{-b/r-1}\right],\\ x &= \frac{a\kappa}{b+r}(V_0+rt)\left[1-\left(\frac{V_0}{V_0+rt}\right)^{b/r+1}\right].\end{aligned}$$

The concentration is found by dividing $x(t)$ by $V(t) = V_0 + rt$. Consequently,

$$c(t) = \frac{a\kappa}{b+r}\left[1-\left(\frac{V_0}{V_0+rt}\right)^{(b+r)/r}\right].$$

9. (a) The rate at which pollutant enters the lake is

$$\text{rate in} = p\,\text{km}^3/\text{yr}.$$

The rate at which the pollutant leaves the lake is found by multiplying the flow rate by the concentration of pollutant in the lake.

$$\begin{aligned}\text{rate out} &= (r+p)\,\text{km}^3/\text{yr} \times \frac{x(t)}{V}\,\text{km}^3/\text{km}^3\\ &= \frac{r+p}{V}x(t)\,\text{km}^3/\text{yr}\end{aligned}$$

Consequently,

$$\frac{dx}{dt} = p - \frac{r+p}{V}x.$$

But $c(t) = x(t)/V$, so $Vc'(t) = x'(t)$ and

$$\begin{aligned}Vc' &= p - \frac{r+p}{V}(Vc),\\ c' + \frac{r+p}{V}c &= \frac{p}{V}.\end{aligned}$$

(b) With $r = 50$ and $p = 2$, the equation becomes

$$\begin{aligned}c' &= \frac{2}{100} - \frac{50+2}{100}c,\\ c' &= 0.02 - 0.52c.\end{aligned}$$

This is linear and solved in the usual manner.

$$\begin{aligned}\left(e^{0.52t}c\right)' &= 0.02e^{0.052t}\\ e^{0.52t}c &= \frac{0.02}{0.52}e^{0.052t} + K\\ c &= \frac{1}{26} + Ke^{-0.52t}\end{aligned}$$

The initial concentration is zero, so $c(0) = 0$ produces $K = -1/26$ and

$$c = \frac{1}{26} - \frac{1}{26}e^{-0.52t}.$$

The question asks when the concentration reaches 2%, or when $c(t) = 0.02$. Thus,

$$\begin{aligned}0.02 &= \frac{1}{26}\left(1 - e^{-0.52t}\right),\\ e^{-0.52t} &= 0.48,\\ t &= -\frac{\ln 0.48}{0.52},\\ t &\approx 1.41\text{ years.}\end{aligned}$$

10. Because the factory stops putting pollutant in the lake, $p = 0$ and $c' + ((r+p)/V)c = p/V$ becomes

$$c' + \frac{50}{100}c = 0.$$

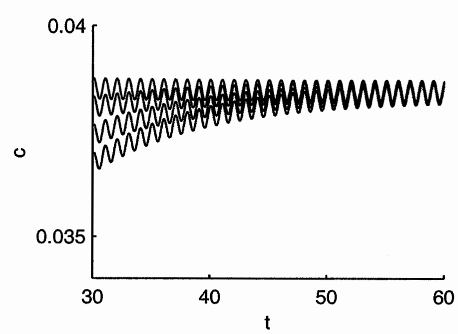
Note that we carried $r = 50$ from Exercise 9. This equation is separable, with general solution $c = Ke^{-(1/2)t}$. The initial concentration is 3.5%, so $c(0) = 0.035$ produces $K = 0.035$ and

$$c(t) = 0.035e^{-(1/2)t}.$$

The question asks for the time required to lower the concentration to 2%. That is, when does $c(t) = 0.02$?

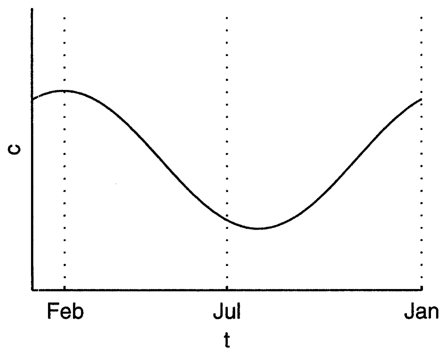
$$\begin{aligned}0.02 &= 0.035e^{-(1/2)t}\\ -\frac{1}{2}t &= \ln\frac{0.02}{0.035}\\ t &= 2\ln\frac{0.035}{0.02}\\ t &\approx 1.1\text{ years}\end{aligned}$$

11. (a) The concentrations are plotted in the following figure. In steady-state the concentration varies periodically.



(b) The following figure shows one year of the oscillation, and indicates that the maximum concentration occurs early in February. This is four months after the time of the minimum flow.

Thus there is a shift of phase between the cause and the effect.



12. For Tank A we have a constant volume of 100 gal. Let $x(t)$ denote the amount of salt in Tank A. The rate into Tank A is 0, and the rate out is 5 gal/s $\times$ $x/100$ lb/gal $= x/20$ lb/s. Hence the model equation is
$$x' = -\frac{x}{20}.$$
The solution with initial value $x(0) = 20$ is $x(t) = 20e^{-t/20}$.

The volume of solution in Tank B is increasing at 2.5 gal/s. Hence the volume at time t is $200 + 2.5t$. Let $y(t)$ denote the amount of salt in Tank B. Then the rate into Tank B is the same as the rate out of Tank A, $x/20$. The rate out of Tank B is 2.5 gal/s $\times$ $y/(200 + 2.5t)$ lb/gal $= y/(80 + t)$ lb/s. Hence the model equation is
$$y' = \frac{x}{20} - \frac{y}{80+t} = e^{-t/20} - \frac{y}{80+t}.$$
This linear equation can be solved using the integrating factor $u(t) = 80 + t$. The general solution is
$$y(t) = \frac{C}{80+t} - 20e^{-t/20} - \frac{400}{80+t}e^{-t/20}.$$
Since $y(0) = 40$, we can compute that $C = 65 \times 80 = 5200$. Hence the solution is
$$y(t) = \frac{5200}{80+t} - 20e^{-t/20} - \frac{400}{80+t}e^{-t/20}.$$
Tank B will contain 250 gal when $t = 20$. At this point we have $y(20) = 43.1709$ lb.

13. (a) Let $x(t)$ be the amount of pollutant (measured in km^3) in Lake Happy Times. The rate in for Lake Happy Times is 2 km^3/yr. The rate out is 52 km^3/yr $\times$ $x/100 = 0.52x$ km^3/yr. Hence the model equation is
$$x' = 2 - 0.52x.$$
This linear equation can be solved using the integrating factor $u(t) = e^{0.52t}$. With the initial condition $x(0) = 0$ we find the solution $x(t) = 2[1 - e^{-0.52t}]/0.52$.

Let $y(t)$ be the amount of pollutant (measured in km^3) in Lake Sad Times. The rate into lake Sad Times is the same as the rate out of Lake Happy Times, or $0.52x$ km^3/yr. The rate out is 52 km^3/yr $\times$ $y/100 = 0.52y$ km^3/yr. Hence the model equation is
$$y' = 0.52(x - y) = 2[1 - e^{-0.52t}] - 0.52y.$$
This linear equation can also be solved using the integrating factor $u(t) = e^{0.52t}$. With the initial condition $y(0) = 0$ we find the solution
$$\begin{aligned} y(t) &= 2[1 - e^{-0.52t}]/0.52 - 2te^{-0.52t} \\ &= x(t) - 2te^{-0.52t}. \end{aligned}$$
After 3 months, when $t = 1/4$, we have $x(1/4) = 0.4689\text{km}^3$ and $y(1/4) = 0.0298\text{km}^3$.

(b) If the factory is shut down, then the flow of pollutant at the rate of 2 km^3/yr is stopped. This means that the flow between the lakes and that out of Lake Sad Times will be reduced to 50 km^3/yr in order to maintain the volumes. We will start time over at this point and we have the initial conditions $x(0) = x_1 = 0.4689\text{km}^3$, and $y(0) = y_1 = 0.0298\text{km}^3$.

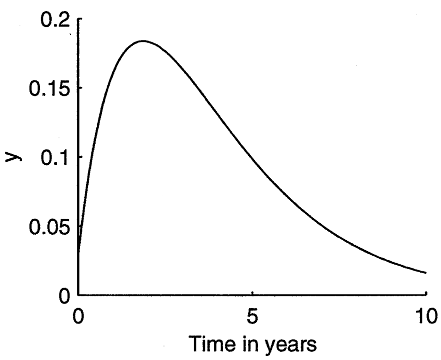
Now there is no flow of pollutant into Lake Happy Times, and the rate out is $x/2$ km^3/yr. Hence the model equation is $x' = -x/2$. The solution is $x(t) = x_1e^{-t/2}$.

The rate into Lake Sad times is $x/2$ km^3/yr, and the rate out is
$y/2$ km^3/yr. The model equation is $y' = (x - y)/2 = x_1e^{-t/2}/2 - y/2$. this time we

use the integrating factor $u(t) = e^{t/2}$ and find the solution

$$y(t) = [x_1 t/2 + y_1]e^{-t/2}.$$

The plot of the solution over 10 years is shown in the following figure. It is perhaps a little surprising to see that the level of pollution in Lake Sad Times continues to rise for some time after the factory is closed.



Using a computer or a calculator, we find that $y(t) = y_1/2$ when $t = 10.18$ yrs.

14. Let $x(t)$ represent the amount of salt in Tank I at time t. The rate at which salt enters Tank I is

$$\text{rate In I} = a \text{ lb/gal} \times b \text{ gal/min} = ab \text{ lb/min}.$$

Salt leaves Tank II at

$$\begin{aligned}\text{rate out I} &= x(t)/V \text{ lb/gal} \times b \text{ gal/min}\\ &= (b/V)x(t) \text{ lb/min}.\end{aligned}$$

Consequently,

$$\frac{dx}{dt} = ab - \frac{b}{V}x.$$

This equation is linear with general solution

$$x = aV + Ke^{-(b/V)t}.$$

Initially, there is no salt in Tank I, so $x(0) = 0$ produces $K = -aV$ and

$$x = aV - aVe^{-(b/V)t}.$$

Let $y(t)$ represent the amount of salt in Tank II at time t. Salt enters Tank II at the same rate as it leaves Tank I. Consequently,

$$\text{rate in II} = (b/V)x(t) \text{ lb/min}.$$

Salt leaves Tank II at

$$\begin{aligned}\text{rate out II} &= y(t)/V \text{ lb/gal} \times b \text{ gal/min}\\ &= (b/V)y(t) \text{ lb/min}.\end{aligned}$$

Consequently,

$$\frac{dy}{dt} = \frac{b}{V}x - \frac{b}{V}y.$$

Substitute the solution found for x.

$$\begin{aligned}\frac{dy}{dt} &= \frac{b}{V}\left(aV - aVe^{-(b/V)t}\right) - \frac{b}{V}y,\\ \frac{dy}{dt} &= -\frac{b}{V}y + \left(ab - abe^{-(b/V)t}\right).\end{aligned}$$

This equation is also linear, with integrating factor $e^{(b/V)t}$, so

$$\begin{aligned}\left(e^{(b/V)t}y\right)' &= ab\left(e^{(b/V)t} - 1\right),\\ e^{(b/V)t}y &= aVe^{(b/V)t} - abt + L,\\ y &= aV - abte^{-(b/V)t} + Le^{-(b/V)t}.\end{aligned}$$

Initially, there is no salt in Tank II, so $y(0) = 0$ produces $L = -aV$ and

$$y = aV - abte^{-(b/V)t} - aVe^{-(b/V)t}.$$

Section 2.6. Exact Differential Equations

1. $dF = 2ydx + (2x + 2y)dy$

2. $dF = (2x - 2y)dx + (-x + 2y)dy$

3. $dF = \dfrac{xdx + ydy}{\sqrt{x^2 + y^2}}$

4. $dF = \dfrac{-xdx - ydy}{(x^2 + y^2)^{3/2}}$

5.
$$dF = \frac{1}{x^2 + y^2}\big(x^2ydx + y^3dx - ydx + x^3dy + xy^2dy + xdy\big)$$

6. $dF = (1/x + 2xy^3)dx + (1/y + 3x^2y^2)dy$

7.
$$dF = \left(\frac{2x}{x^2 + y^2} + 1/y\right)dx + \left(\frac{2y}{x^2 + y^2} - \frac{x}{y^2}\right)dy$$

8. $dF = \dfrac{ydx - xdy + 4x^2y^3dy + 4y^5dy}{x^2 + y^2}$

9. With $P = 2x + y$ and $Q = x - 6y$, we see that
$$\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x}$$
so the equation is exact. We solve by setting
$$F(x, y) = \int P(x, y)\,dx = \int (2x + y)\,dx = x^2 + xy + \phi(y).$$
To find ϕ, we differentiate
$$Q(x, y) = \frac{\partial F}{\partial y} = x + \phi'(y).$$
Hence $\phi' = -6y$, and we can take $\phi(y) = -3y^2$. Hence the solution is $F(x, y) = x^2 + xy - 3y^2 = C$.

10. With $P = 1 - y\sin x$ and $Q = \cos x$, we see that
$$\frac{\partial P}{\partial y} = -\sin x = \frac{\partial Q}{\partial x},$$
so the equation is exact. We solve by setting
$$F(x, y) = \int P(x, y)\,dx = \int (1 - y\sin x)\,dx = x + y\cos x + \phi(y).$$
To find ϕ, we differentiate
$$Q(x, y) = \frac{\partial F}{\partial y} = \cos x + \phi'(y).$$
Thus $\phi' = 0$, so we can take $\phi = 0$. Hence the solution is $F(x, y) = x + y\cos x = C$.

11. With $P = 1 + \dfrac{y}{x}$ and $Q = -\dfrac{1}{x}$, we compute
$$\frac{\partial P}{\partial y} = \frac{1}{x} \neq \frac{\partial Q}{\partial x} = \frac{1}{x^2}.$$
Hence the equation is not exact.

12. With $P = \dfrac{x}{\sqrt{x^2 + y^2}}$ and $Q = \dfrac{y}{\sqrt{x^2 + y^2}}$, we compute
$$\frac{\partial P}{\partial y} = \frac{-2xy}{(x^2 + y^2)^{3/2}} = \frac{\partial Q}{\partial x},$$
so the equation is exact. To find the solution we integrate
$$F(x, y) = \int P(x, y)\,dx = \int \frac{x}{\sqrt{x^2 + y^2}}\,dx = \sqrt{x^2 + y^2} + \phi(y).$$
To find ϕ, we differentiate
$$Q(x, y) = \frac{\partial F}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \phi'(y).$$
Thus $\phi' = 0$, so we can take $\phi = 0$. Hence the solution is $F(x, y) = \sqrt{x^2 + y^2} = C$.

13. Exact $x^3 + xy - y^3 = C$

14. Not exact

15. Exact $u^2/2 + vu - v^2/2 = C$

16. Exact $\ln(u^2 + v^2) = C$

17. Not exact

18. Exact. $F(u, y) = y \ln u - 2u = C$

19. Exact $x \sin 2t - t^2 = C$

20. Exact $x^2y^2 + x^4 = C$

21. Not exact.

22. $-x/y + \ln x = C$

23. $x^2y^2/2 - \ln x + \ln y = C$

24. $\dfrac{-(y+1)^2}{x^3} = C$

25. $x - (1/2)\ln(x^2 + y^2) = C$

26. $\mu(x) = 1/x^2$. $F(x, y) = \dfrac{xy^2/2 - y}{x} = C$

27. $\mu(x) = \dfrac{1}{x}$. $F(x, y) = xy - \ln x - \dfrac{y^2}{2} = C$.

28. $\mu(y) = 1/\sqrt{y}$. $F(x, y) = 2x\sqrt{y} + (2/3)y^{3/2} = C$

29. $\mu(y) = 1/y^2$. $F(x, y) = \dfrac{yx + x^2}{y} = C$

30. $F(x, y) = x^2y^3 = C$

31. $x + y$ and $x - y$ are homogeneous of degree one.

32. $x^2 - xy - y^2$ and $4xy$ are homogeneous of degree two.

33. $x - \sqrt{x^2 + y^2}$ and $-y$ are homogenous of degree one.

34. $\ln x - \ln y$ and 1 are homogeneous of degree zero.

35. $x^2 - Cx = y^2$

36.
$$F(x, y) = -(1/2)\ln\left(\frac{x^2 + y^2}{x^2}\right) + \arctan(y/x) - \ln x = C$$

37. $F(x, y) = xy + (3/2)x^2 = C$.

38. $y^2x^2 - 4\ln y - 2\ln x = C$

39. $y(x) = \dfrac{x + Cx^4}{1 - 2Cx^3}$

40. $y(x) = x\ln(C + 2\ln x)$

41. (a) First,
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v_0 \sin\theta - \omega}{v_0 \cos\theta}.$$
However, $\cos\theta = x/\sqrt{x^2 + y^2}$ and $\sin\theta = y/\sqrt{x^2 + y^2}$, so
$$\frac{dy}{dx} = \frac{\dfrac{v_0 y}{\sqrt{x^2 + y^2}} - \omega}{\dfrac{v_0 x}{\sqrt{x^2 + y^2}}} = \frac{v_0 y - \omega\sqrt{x^2 + y^2}}{v_0 x}.$$
Divide top and bottom by v_0 and replace ω/v_0 with k.
$$\frac{dy}{dx} = \frac{y - \dfrac{\omega}{v_0}\sqrt{x^2 + y^2}}{x} = \frac{y - k\sqrt{x^2 + y^2}}{x}.$$
(b) Write the equation
$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2 + y^2}}{x}$$
in the form
$$(y - k\sqrt{x^2 + y^2})\,dx - x\,dy = 0.$$

Both terms are homogeneous (degree 1), so make substitutions $y = xv$ and $dy = x\,dv + v\,dx$.

$$(xv - k\sqrt{x^2 + x^2v^2})\,dx - x(x\,dv + v\,dx) = 0$$

After cancelling the common factor x and combining terms,

$$k\sqrt{1+v^2}\,dx - x\,dv = 0.$$

Separate variables and integrate.

$$\frac{k\,dx}{x} - \frac{dv}{\sqrt{1+v^2}} = 0$$
$$k\ln x - \ln(\sqrt{1+v^2} + v) = C$$

Note the initial condition $(x, y) = (a, 0)$. Because $y = xv$, v must also equal zero at this point. Thus, $(x, v) = (a, 0)$ and

$$k\ln a - \ln(\sqrt{1 - 0^2} + 0) = C$$
$$C = k\ln a.$$

Therefore,

$$k\ln x - \ln(\sqrt{1+v^2} + v) = k\ln a.$$

Taking the exponential of both sides,

$$e^{\ln x^k - \ln(\sqrt{1+v^2}+v)} = e^{\ln a^k}$$
$$\frac{x^k}{v + \sqrt{1+v^2}} = a^k$$
$$\left(\frac{x}{a}\right)^k = v + \sqrt{1+v^2}.$$

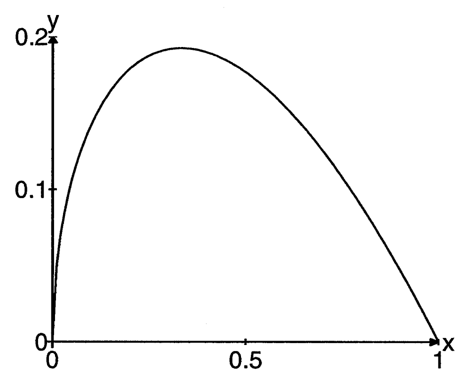
Solve for v.

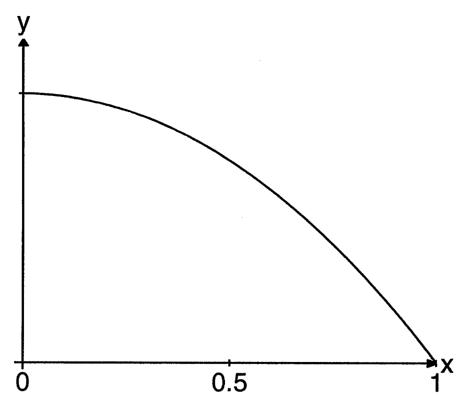
$$\left(\frac{x}{a}\right)^k - v = \sqrt{1+v^2}$$
$$\left(\frac{x}{a}\right)^{2k} - 2v\left(\frac{x}{a}\right)^k + v^2 = 1 + v^2$$
$$\left(\frac{x}{a}\right)^{2k} - 1 = 2v\left(\frac{x}{a}\right)^k$$
$$\frac{1}{2}\left[\left(\frac{x}{a}\right)^k - \left(\frac{x}{a}\right)^{-k}\right] = v$$

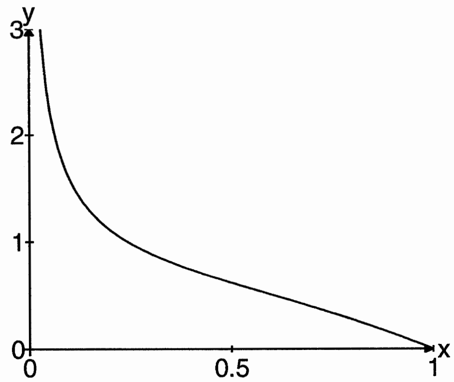
Finally, recall that $y = xv$, so

$$\frac{y}{x} = \frac{1}{2}\left[\left(\frac{x}{a}\right)^k - \left(\frac{x}{a}\right)^{-k}\right]$$
$$y = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1+k} - \left(\frac{x}{a}\right)^{1-k}\right].$$

(c) The following three graphs show the cases where $a = 1$, and $k = 1/2,\ 1,\ 3/2$. When $0 < k < 1$, the wind speed is less than that of the goose and the goose flies home. When $k = 1$ the two speeds are equal, and try as he might, the goose can't get home. Instead he approaches a point due north of the nest. When $k > 1$ the wind speed is greater, so the goose loses ground and keeps getting further from the nest.



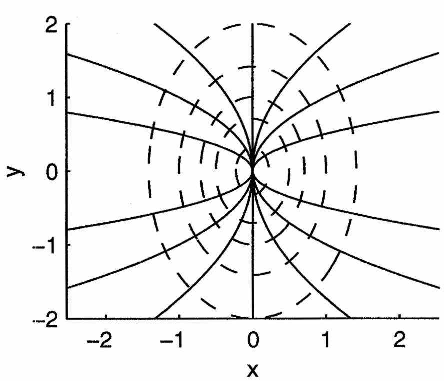




42. The hyperbolas with $F(x, y) = y^2/x = C$ are the solid curves in the following figure. The orthogonal family must satisfy

$$\frac{dy}{dx} = \frac{\partial F}{\partial y} \bigg/ \frac{\partial F}{\partial x} = -\frac{2x}{y}.$$

The solution to this separable equation is found to be given implicitly by $G(x, y) = 2x^2 + y^2 = C$. These curves are the dashed ellipses in the accompanying figure. They do appear to be orthogonal.



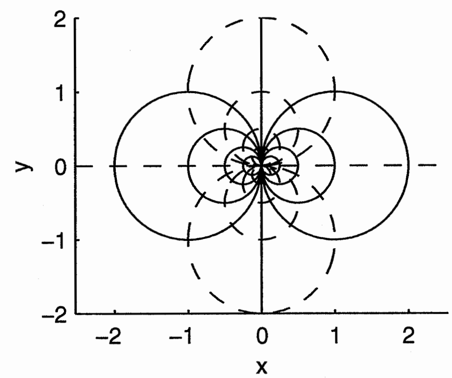
43. (a) The curves are defined by the equation $F(x, y) = x/(x^2 + y^2) = c$. Hence the orthogonal family must satisfy

$$\frac{dy}{dx} = \frac{\partial F}{\partial y} \bigg/ \frac{\partial F}{\partial x} = \frac{2xy}{x^2 - y^2}.$$

(b) The differential equation is homogeneous. Solving in the usual way we find that the orthogonal family is defined implicitly by

$$G(x, y) = \frac{y}{x^2 + y^2} = C.$$

The original curves are the solid curves in the following figure, and the orthogonal family is dashed.



44. $\ln(y^2 + x^2) - (2/3)y^3 = C$

45. $\arctan(y/x) - y^4/4 = C$

46. Assuming that $m \neq n - 1$, divide both sides of

$$x\,dy + y\,dx = x^m y^n\,dx$$

by $x^n y^n$ to obtain

$$\frac{x\,dy + y\,dx}{(xy^n)} = x^{m-n}\,dx$$

$$\frac{d((xy)^{1-n})}{1-n} = x^{m-n}\,dx.$$

Thus, because $m - n + 1 \neq 0$,

$$\frac{(xy)^{1-n}}{1-n} = \frac{x^{m-n+1}}{m-n+1} + C$$

$$(m - n + 1)(xy)^{1-n} - (1 - n)x^{m-n+1} = C.$$

47. $\arctan(y/x) - (1/4)(y^2 + x^2)^2 = C$

48. $x/y - \ln(xy + 1) = C$

49. $Cxy = \dfrac{x+y}{x-y}$

50. (a) An exterior angle of a triangle equals the sum of its two remote interior angles, so $\theta = \phi + \alpha$. We're given that $\alpha = \beta$ and ϕ and β are corresponding angles on the same side of a transversal cutting parallel lines, so $\phi = \beta$. Thus, $\theta = \phi + \alpha = \beta + \beta = 2\beta$ and

$$\tan\theta = \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta}.$$

However, $\tan\theta = y/x$ and $\tan\beta$ equals the slope of the tangent line to $y = y(x)$ at the point (x, y); i.e., $\tan\beta = y'$. Thus,

$$\frac{y}{x} = \frac{2y'}{1-(y')^2}.$$

(b) Use the result from part (a) and cross multiply.

$$y - y(y')^2 = 2xy'$$
$$0 = y(y')^2 + 2xy' - y$$

Use the quadratic formula to solve for y'.

$$y' = \frac{-x \pm \sqrt{x^2+y^2}}{y}$$

Rearranging,

$$\frac{dy}{dx} = \frac{-x \pm \sqrt{x^2+y^2}}{y}$$

becomes

$$\pm\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = dx.$$

The trick now is to recognize that the left-hand side equals $\pm d(\sqrt{x^2+y^2})$. Thus, when we integrate,

$$\pm d(\sqrt{x^2+y^2}) = dx$$
$$\pm\sqrt{x^2+y^2} = x + C.$$

Square, then solve for y^2.

$$x^2 + y^2 = x^2 + 2Cx + C^2$$
$$y^2 = 2Cx + C^2$$

This, as was somewhat expected, is the equation of a parabola.

Section 2.7. Existence and Uniqueness of Solutions

1. The right hand side of the equation is $f(t, y) = 4 + y^2$. f is continuous in the whole plane. Its partial derivative $\partial f/\partial y = 2y$ is also continuous on the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.

2. The right hand side of the equation is $f(t, y) = \sqrt{y}$. f is defined only where $y \geq 0$, and it is continuous there. However, $\partial f/\partial y = 1/(2\sqrt{y})$, which is only continuous for $y > 0$. Our initial condition is at $y_0 = 0$, and $t_0 = 4$. There is no rectangle containing (t_0, y_0) where both f and $\partial f/\partial y$ are defined and continuous. Consequently the hypotheses of the theorem are not satisfied.

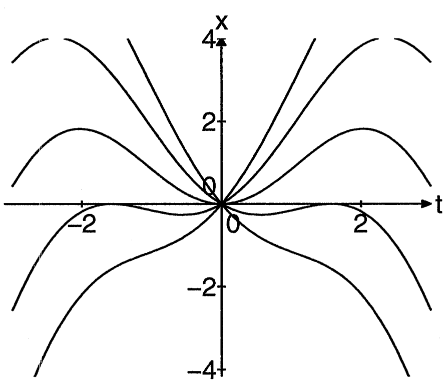
3. The right hand side of the equation is $f(t, y) = t\tan^{-1} y$, which is continuous in the whole plane. $\partial f/\partial y = t/(1 + y^2)$ is also continuous in the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.

4. The right hand side of the equation is $f(s, \omega) = \omega \sin\omega + s$, which is continuous in the whole plane. $\partial f/\partial\omega = \sin\omega + \omega\cos\omega$ is also continuous in the whole plane. Hence the hypotheses are satisfied and the theorem guarantees a unique solution.

5. The right hand side of the equation is $f(t, x) = t/(x + 1)$, which is continuous in the whole plane, except where $x = -1$. $\partial f/\partial x = -t/(x + 1)^2$ is also continuous in the whole plane, except where $x = -1$. Hence the hypotheses are satisfied in a rectangle containing the initial point $(0, 0)$, so the theorem guarantees a unique solution.

6. The right hand side of the equation is $f(x, y) = y/x + 2$, which is continuous in the whole plane, except where $x = 0$. Since the initial point is $(0, 1)$, f is discontinuous there. Consequently there is no rectangle containing this point in which f is continuous. The hypotheses are not satisfied, so the theorem does not guarantee a unique solution.

7. The equation is linear. The general solution is $y(t) = t \sin t + Ct$. Several solutions are plotted in the following figure.

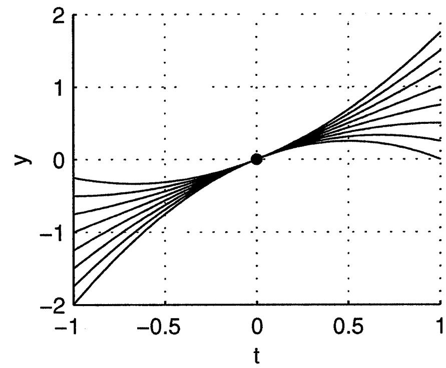


Since every solution satisfies $y(0) = 0$, there is no solution with $y(0) = -3$. If we put the equation into normal form

$$\frac{dy}{dt} = \frac{1}{t}y + t\cos t,$$

we see that the right hand side $f(t, y)$ fails to be continuous at $t = 0$. Consequently the hypotheses of the existence theorem are not satisfied.

8. The equation is linear. The general solution is $y(t) = t + 2Ct^2$. Several solutions are plotted in the following figure.



Since the general solution is $y(t) = t + 2Ct^2$, every solution satisfies $y(0) = 0$. There is no solution with $y(0) = 2$. If we put the equation into normal form

$$\frac{dy}{dt} = \frac{2y - t}{t},$$

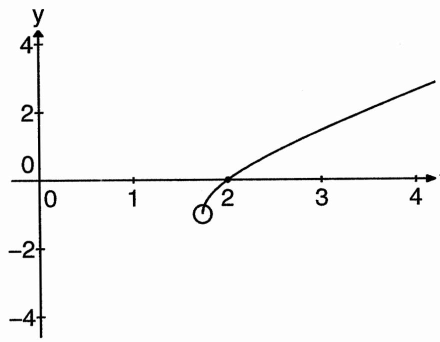
we see that the right hand side $f(t, y) = (2y - t)/t$ fails to be continuous at $t = 0$. Consequently the hypotheses of the existence theorem are not satisfied.

9. The y-derivative of the right hand side $f(t, y) = 3y^{2/3}$ is $2y^{-1/3}$, which is not continuous at $y = 0$. Hence the hypotheses of Theorem 7.16 are not satisfied.

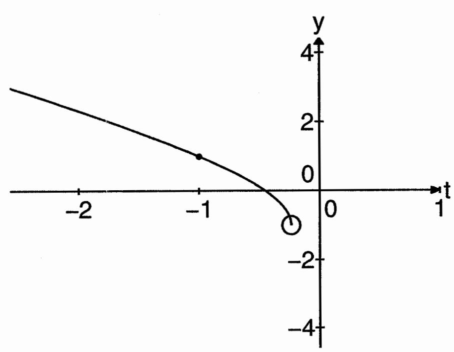
10. The y-derivative of the right hand side $f(t, y) = ty^{1/2}$ is $ty^{-1/2}/2$ which is not continuous at $y = 0$. Hence the hypotheses of Theorem 7.16 are not satisfied.

11. The exact solution is $y(t) = -1 + \sqrt{t^2 - 3}$. The interval of solution is $(\sqrt{3}, \infty)$. The solver has trouble near $\sqrt{3}$. The point where the difficulty arises is

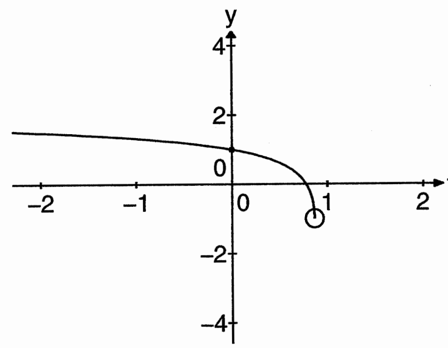
circled in the following figure.



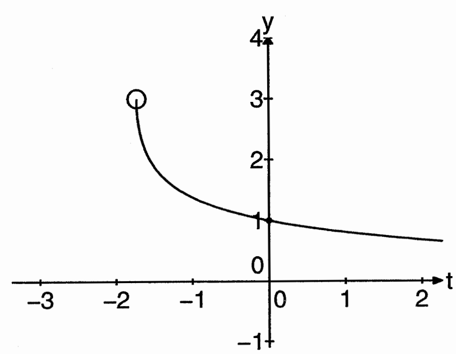
12. The exact solution is $y(t) = -1+\sqrt{t^2-4t-1}$. The interval of existence is $(-\infty, 2-\sqrt{5})$. The solver has trouble near $2-\sqrt{5} \approx -0.2361$. The point where the difficulty arises is circled in the following figure.



13. The exact solution is $y(t) = -1+\sqrt{4+2\ln(1-t)}$. The interval of existence is $(-\infty, 1-e^{-2})$. The solver has trouble near $1-e^{-2} \approx 0.8647$. The point where the difficulty arises is circled in the following figure.

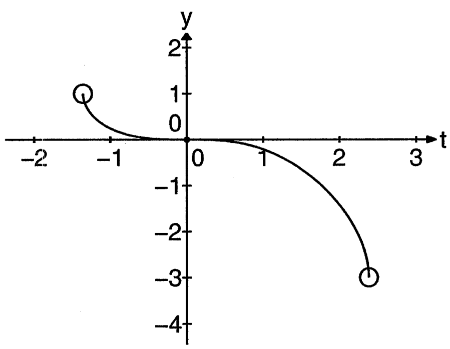


14. The exact solution is $y(t) = 3 - \sqrt{4+2\ln(t+2)-2\ln 2}$. The interval of existence is $(-2+2e^{-2}, \infty)$. The solver has trouble near $-2+2e^{-2} \approx -1.7293$. The point where the difficulty arises is circled in the following figure.

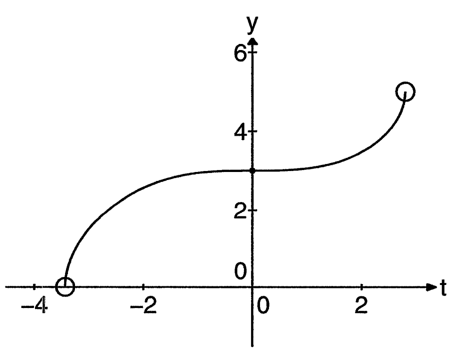


15. The solution is defined implicitly by the equation $y^3/3+y^2-3y = 2t^3/3$. The solver has trouble near $(t_1, 1)$, where $t_1 = -(5/2)^{1/3} \approx -1.3572$, and also near $(t_2, -3)$, where $t_2 = (27/2)^{1/3} \approx 2.3811$. The points where the difficulty arises are circled in the

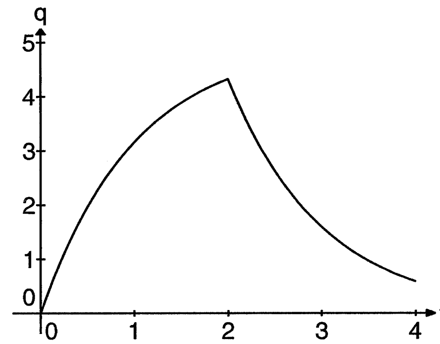
following figure.



16. The solution is defined implicitly by the equation $2y^3 - 15y^2 + 2t^3 = -81$. The solver that trouble near $(t_1, 0)$, where $t_1 = -(81/2)^{1/3} \approx -3.4341$, and also near $(t_2, 5)$, where $t_2 = 22^{1/3} \approx 2.8020$. The points where the difficulty arises are circled in the following figure.



17. The computed solution is shown in the following figure.

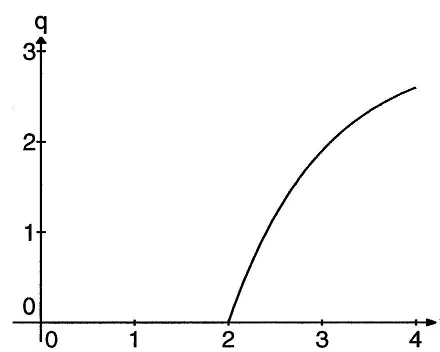


The exact solution is

$$q(t) = \begin{cases} 5 - 5e^{-t}, & \text{if } 0 < t < 2, \\ 5(1 - e^{-2})e^{2-t}, & \text{if } t \ge 2 \end{cases}$$

Hence $q(4) = 5(1 - e^{-2})e^{-2} \approx 0.5851$.

18. The computed solution is shown in the following figure.



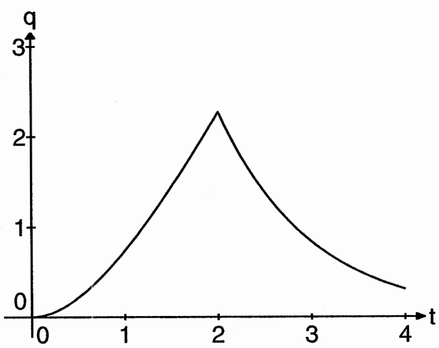
The exact solution is

$$q(t) = \begin{cases} 0, & \text{if } 0 < t < 2, \\ 3(1 - e^{2-t}), & \text{if } t \ge 2 \end{cases}$$

Hence $q(4) = 3(1 - e^{-2}) \approx 2.5940$.

19. The computed solution is shown in the following fig-

ure.

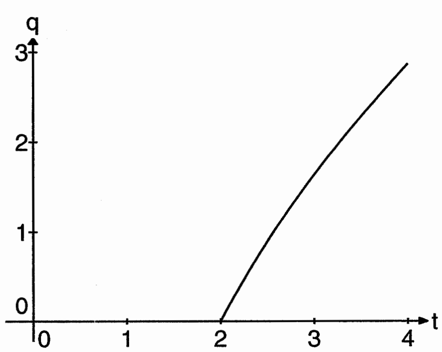


The exact solution is

$$q(t) = \begin{cases} 2(t - 1 + e^{-t}), & \text{if } 0 < t < 2, \\ 2(1 + e^{-2})e^{2-t}, & \text{if } t \geq 2 \end{cases}$$

Hence $q(4) = 2(1 + e^{-2})e^{-2} \approx 0.3073$.

20. The computed solution is shown in the following figure.



The exact solution is

$$q(t) = \begin{cases} 0, & \text{if } 0 < t < 2, \\ t - 1 - e^{2-t}, & \text{if } t \geq 2 \end{cases}$$

Hence $q(4) = 3 - e^{-2} \approx 2.8647$.

21. (a) If

$$y(t) = \begin{cases} 0, & \text{if } t \leq t_0 \\ (t - t_0)^3, & \text{if } t > t_0, \end{cases}$$

then

$$\begin{aligned} y'(t_0^+) &= \lim_{t \searrow t_0} \frac{y(t) - y(t_0)}{t - t_0} \\ &= \lim_{t \to t_0^+} \frac{(t - t_0)^3 - 0}{t - t_0} \\ &= \lim_{t \to t_0^+} (t - t_0)^2 \\ &= 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} y'(t_0^-) &= \lim_{t \nearrow t_0} \frac{y(t) - y(t_0)}{t - t_0} \\ &= \lim_{t \to t_0^-} \frac{0 - 0}{t - t_0} \\ &= 0. \end{aligned}$$

Therefore, $y'(t_0) = 0$, since both the left and right-hand derivatives equal zero.

(b) The right hand side of the equation, $f(t, y) = 3y^{2/3}$, is continuous, but $\partial f/\partial y = 2y^{-1/3}$ is not continuous where $y = 0$. Hence the hypotheses of Theorem 7.16 are not satisfied.

22. (a) If

$$y(t) = \begin{cases} 3e^{-2t}, & \text{if } t < 1 \\ 5/2 + (3 - 5e^2/2)e^{-2t}, & \text{if } t \geq 1 \end{cases}$$

then it is easily seen that

$$y'(t) = \begin{cases} -6e^{-2t}, & t < 1 \\ (-6 + 5e^2)e^{-2t} & t > 1. \end{cases}$$

It remains to find the derivative at $t = 1$. Remember, because of the "cusp" at $t = 1$, we suspect that this derivative will not exist. First, the derivative from the right,

$$\begin{aligned}
y'_+(1) &= \lim_{t\to1^+} \frac{y(t)-y(1)}{t-1} \\
&= \lim_{t\to1^+} \frac{1}{t-1}\left[\left(\frac{5}{2}+\left(3-\frac{5e^2}{2}\right)e^{-2t}\right) - \left(\frac{5}{2}+\left(3-\frac{5e^2}{2}\right)e^{-2}\right)\right] \\
&= \lim_{t\to1^+} \frac{\left(3-\frac{5e^2}{2}\right)e^{-2t} - \left(3-\frac{5e^2}{2}\right)e^{-2}}{t-1}
\end{aligned}$$

Note the indeterminant form 0/0, so l'Hôpital's rule applies.

$$\begin{aligned}
y'_+ &= \lim_{t\to1^+}\left[-2\left(3-\frac{5e^2}{2}\right)e^{-2t}\right] \\
&= 5-6e^{-2}
\end{aligned}$$

However, the derivative from the left,

$$\begin{aligned}
y'_-(1) &= \lim_{t\to1^-} \frac{y(t)-y(1)}{t-1} \\
&= \lim_{t\to1^-} \frac{3e^{-2t} - \left(\frac{5}{2}+\left(3-\frac{5e^2}{2}\right)e^{-2}\right)}{t-1} \\
&= \lim_{t\to1^-} \frac{3e^{-2t}-3e^{-2}}{t-1}.
\end{aligned}$$

Again, an indeterminant form 0/0, so we apply l'Hôpital's rule.

$$\begin{aligned}
y'_-(1) &= \lim_{t\to1^-}(-6e^{-2t}) \\
&= -6e^{-2}.
\end{aligned}$$

(b) The derivative from the left doesn't equal the derivative from the right. The function $y = y(t)$ is not differentiable at $t = 1$ and cannot be a solution of the differential equation on any interval containing $t = 1$.

(c) We have that

$$y'(t) = \begin{cases} -6e^{-2t}, & t < 1 \\ (-6+5e^2)e^{-2t} & t > 1. \end{cases}$$

If

$$f(t) = \begin{cases} 0, & \text{if } t < 1 \\ 5, & \text{if } t > 1, \end{cases}$$

then

$$-2y+f(t) = \begin{cases} -6e^{-2t}, & t < 1 \\ -5+(-6+5e^2)e^{-2t}, & t > 1. \end{cases}$$

Thus, $y = y(t)$ is a solution of $y' = -2y+f(t)$ on any interval not containing $t = 1$.

23. If

$$y(t) = \begin{cases} 0, & t < 0 \\ t^4, & t \geq 0, \end{cases}$$

then it is easily seen that

$$y'(t) = \begin{cases} 0, & t < 0 \\ 4t^3, & t > 0. \end{cases}$$

It remains to check the existence of $y'(0)$. First, the left-derivative.

$$y'_-(0) = \lim_{t\to0^-}\frac{y(t)-y(0)}{t-0} = \lim_{t\to0^-}\frac{0-0}{t} = 0.$$

Secondly,

$$y'_+(0) = \lim_{t\to0^+}\frac{y(t)-y(0)}{t-0} = \lim_{t\to0^+}\frac{t^4}{t} = \lim_{t\to0^+} t^3 = 0.$$

Thus, $y'(0) = 0$ and we can write

$$y'(t) = \begin{cases} 0, & t < 0 \\ 4t^3, & t \geq 0. \end{cases}$$

Now,

$$ty'(t) = \begin{cases} 0, & t < 0 \\ 4t^4, & t \geq 0, \end{cases}$$

and

$$4y(t) = \begin{cases} 0, & t < 0 \\ 4t^4, & t \geq 0, \end{cases}$$

so $y = y(t)$ is a solution of $ty' = 4y$. Finally, $y(0) = 0$. In a similar manner, it is not difficult to show that

$$\omega(t) = \begin{cases} 0, & t < 0 \\ 5t^4, & t \geq 0 \end{cases}$$

is also a solution of the initial value problem $ty' = 4y$, $y(0) = 0$. At first glance, it would appear that we have contradicted uniqueness. However, if $ty' = 4y$ is written in normal form,

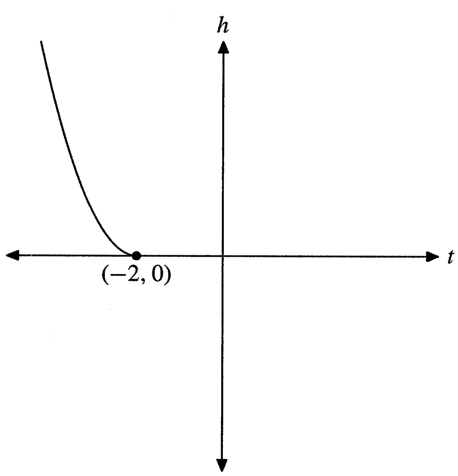
$$y' = \frac{4y}{t},$$

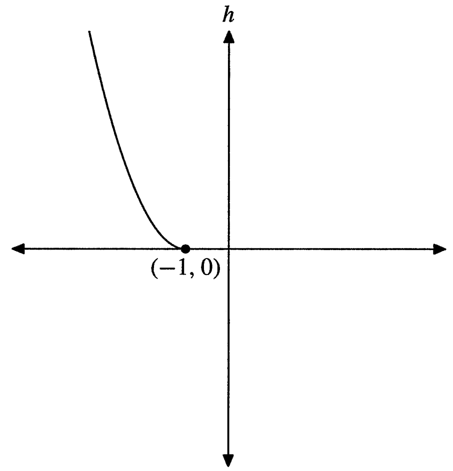
then

$$\frac{\partial}{\partial y}\left(\frac{4y}{t}\right) = \frac{4}{t}$$

is *not* continuous on any rectangular region containing the vertical axis (where $t = 0$), so the hypotheses of the Uniqueness Theorem are not satisfied. There is no contradiction of uniqueness.

24. (a) The point here is the fact that you don't know the moment the water completely drained. Here are two possibilities.





(b) Let A represent the cross area of the drum and h the height of the water in the drum. Then Δh represents the change in height and $A\Delta h$ the volume of water that has left the drum. A particle of water leaving the drain at speed v travels a distance $v\Delta t$ in a time Δt. Because a is the cross section of the drain, the volume of water leaving the drain in time Δt is $av\Delta t$. Because the water leaving the drum in time Δt must exit the drain,

$$A\Delta h = av\Delta t$$
$$A\frac{\Delta h}{\Delta t} = av.$$

Taking the limit as $\Delta t \to 0$,

$$A\frac{dh}{dt} = av.$$

Using $v^2 = 2gh$, $v = \sqrt{2gh}$ and

$$\frac{dh}{dt} = -\frac{a}{A}\sqrt{2gh}.$$

The minus sign is present because the drum is *draining*.

(c) If we let $\omega = \alpha h$ and $s = \beta t$, then by the chain rule

$$\frac{d\omega}{ds} = \frac{d\omega}{dh}\cdot\frac{dh}{dt}\cdot\frac{dt}{ds} = \frac{\alpha}{\beta}\frac{dh}{dt}.$$

Multiply both sides of our equation by α/β,

$$\frac{\alpha}{\beta}\frac{dh}{dt} = -\left(\frac{a}{A}\right)\left(\frac{\alpha}{\beta}\right)\sqrt{2gh}.$$

Replace $(\alpha/\beta)(dh/dt)$ with $d\omega/ds$ and h with ω/α.

$$\frac{d\omega}{ds} = -\left(\frac{a}{A}\right)\left(\frac{\alpha}{\beta}\right)\sqrt{\frac{2g\omega}{\alpha}}$$
$$\frac{d\omega}{ds} = -\frac{1}{\beta}\left(\frac{a}{A}\right)\sqrt{2g\alpha}\sqrt{\omega}.$$

Let h_0 represent the height of a full tank. This motivates the selection of $\alpha = 1/h_0$ and $\omega = h/h_0$, as $\omega = 0$ when the tank is empty and $\omega = 1$ when the tank is full. Thus,

$$\frac{d\omega}{ds} = -\frac{1}{\beta}\left(\frac{a}{A}\right)\sqrt{\frac{2g}{h_0}}\sqrt{\omega},$$

which motivates the selection of

$$\beta = \left(\frac{a}{A}\right)\sqrt{\frac{2g}{h_0}},$$

which upon substitution, gives us

$$\frac{d\omega}{ds} = -\sqrt{\omega}.$$

(d) Separate the variables and integrate.

$$\omega^{-1/2}\,d\omega = -ds$$
$$2\omega^{1/2} = -s + C$$
$$\omega^{1/2} = \frac{1}{2}(C - s)$$
$$\omega = \frac{1}{4}(C - s)^2$$

However, as evidenced in part (a), we only want the left half of this parabola. After the drum empties, it remains empty for all time. Thus, for any $C < s_0$,

$$\omega(s) = \begin{cases} \frac{1}{4}(C - s)^2, & s < C \\ 0, & s \geq C, \end{cases}$$

is a solution of $\omega' = -\sqrt{\omega}$, $\omega(s_0) = 0$. Finally, this emergence of multiple solutions does not contradict uniqueness, because in

$$\frac{\partial}{\partial\omega}(-\sqrt{\omega}) = -\frac{1}{2\sqrt{\omega}}$$

is not continuous on any rectangle containing the horizontal axis (defined by $\omega = 0$).

25. The equation $x' = f(t, x)$ satisfies the hypotheses of the uniqueness theorem. Notice that $x_1(0) = x_2(0) = 0$. If they were both solutions $x' = f(t, x)$ near $t = 0$, then by the uniqueness theorem they would have to be equal everywhere. Since they are not, they cannot both be solutions of the differential equation.

26. The equation $x' = f(t, x)$ satisfies the hypotheses of the uniqueness theorem. Notice that $x_1(\pi/2) = x_2(\pi/2) = 0$. If they were both solutions $x' = f(t, x)$ near $t = \pi/2$, then by the uniqueness theorem they would have to be equal everywhere. Since they are not, they cannot both be solutions of the differential equation.

27. Notice that $x_1(t) = 0$ is a solution to the same differential equation with initial value $x_1(0) = 0 < 1 = x(0)$. The right hand side of the differential equation, $f(t, x) = x\cos^2 t$ and $\partial f/\partial x = \cos^2 t$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for x and x_1 cannot cross. Hence we must have $x(t) > x_1(t) = 0$ for all t.

28. Notice that $y_1(t) = 3$ is a solution to the same differential equation with initial value $y_1(1) = 3 > 1 = y(1)$. The right hand side of the differential equation, $f(t, y) = (y - 3)e^{\cos(ty)}$ and $\partial f/\partial y = e^{\cos(ty)}[1 - t(y - 3)\sin(ty)]$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for y and y_1 cannot cross. Hence we must have $y(t) < y_1(t) = 3$ for all t.

29. Notice that the right hand side of the equation is $f(t, y) = (y^2 - 1)e^{ty}$ and f is continuous on the whole plane. Its partial derivative $\partial f/\partial y = 2ye^{ty} + t(y^2 - 1)e^{ty}$ is also continuous on the whole plane. Thus the hypotheses of the uniqueness theorem are satisfied. By direct substitution we discover

that $y_1(t) = -1$ and $y_2(t) = 1$ are both solutions to the differential equation. If y is a solution and satisfies $y(1) = 0$, then $y_1(1) < y(1) < y_2(1)$. By the uniqueness theorem we must have $y_1(t) < y(t) < y_2(t)$ for all t for which y is defined. Hence $-1 < y(t) < 1$ for all t for which y is defined.

30. Notice that $x_1(t) = 0$ and $x_2(t) = 1$ are solutions to the same differential equation with initial values $x_1(0) = 0 < 1/2 = x(0) < 1 = x_2(0)$. The right hand side of the differential equation, $f(t, x) = (x^3 - x)/(1 + t^2x^2)$, and

$$\frac{\partial f}{\partial x} = \frac{(3x^2 - 1)(1 + t^2x^2) - 2t^2x(x^3 - x)}{(1 + t^2x^2)^2},$$

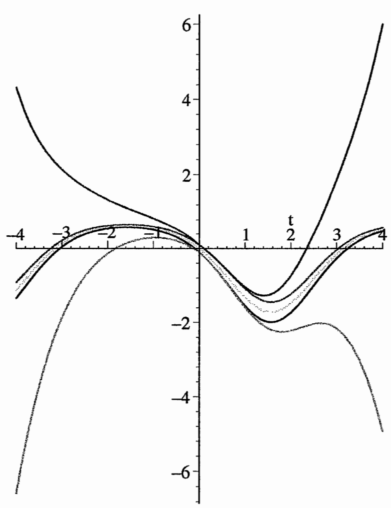
are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for x, x_1, and x_2 cannot cross. Hence we must have $0 = x_1(t) < x(t) < x_2(t) = 1$ for all t.

31. Notice that $x_1(t) = t^2$ is a solution to the same differential equation with initial value $x_1(0) = 0 < 1 = x(0)$. The right hand side of the differential equation, $f(t, x) = x - t^2 + 2t$ and $\partial f/\partial x = 1$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for x and x_1 cannot cross. Hence we must have $t^2 = x_1(t) < x(t)$ for all t.

32. Notice that $y_1(t) = \cos t$ is a solution to the same differential equation with initial value $y_1(0) = 1 < 2 = y(0)$. The right hand side of the differential equation, $f(t, y) = y^2 - \cos^2 t - \sin t$ and $\partial f/\partial y = 2y$ are both continuous on the whole plane. Consequently the uniqueness theorem applies, so the solution curves for y and y_1 cannot cross. Hence we must have $y(t) > y_1(t) = \cos t$ for all t.

Section 2.8. Dependence of Solutions on Initial Conditions

1. $x(0) = 0.8009$
2. $x(0) = .9084$
3. $x(0) = 0.9596$
4. $x(0) = 0.9826$
5. $x(0) = 0.7275$
6. $x(0) = 0.72897$
7. $x(0) = 0.7290106$
8. $x(0) = 0.729011125$
9. $x(0) = -3.2314$
10. $x(0) = -3.23208$
11. $x(0) = -3.2320923$
12. $x(0) = -3.23092999999$
13. Ten! :-)
14. $1 - e^{\sin t} - (1/10)e^{|t|} \le y(t) \le 1 - e^{\sin t} + (1/10)e^{|t|}$

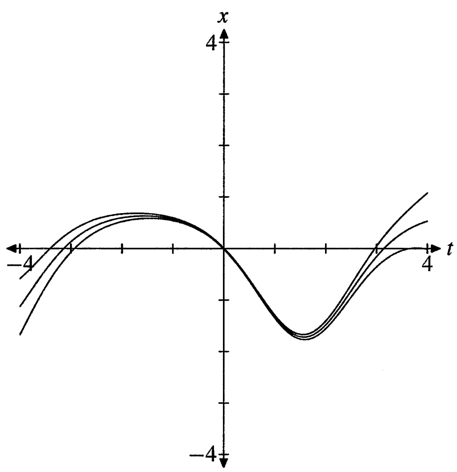


The three middle curves are the solutions to the differential equation corresponding to the initial conditions $x(0) = -.1,\ 0\ .1$, and the outside curves are the graphs of e_L and e_H. Note how the solutions of the differential equation remain inside the graphs of e_L and e_H.

15. The only adjustment from the previous exercise is that we now want $|x_0 - y_0| < 0.01$. This leads to

$$1 - e^{\sin t} - 0.01e^{|t|} \le y(t) \le 1 - e^{\sin t} + 0.01e^{|t|}$$

and this image.



16. (a) The right hand side of the equation is $f(t, x) = (x - 1)\cos t$. Thus $\partial f/\partial x = \cos t$, and $\max|\partial f/\partial x| = \max|\cos t| = 1$. Hence Theorem 7.15 predicts that $|x(t) - y(t)| \le |x(0) - y(0)|e^{|t|}$.

(b) The equation is separable and linear, and the solutions are $x(t) = 1 - e^{\sin t}$ and $y(t) = 1 - 9e^{\sin t}/10$. Hence the separation is $x(t)-y(t) = e^{\sin t}/10$. Since $\sin t \le |t|$, we see that

$$|x(t)-y(t)| = e^{\sin t}/10 \le e^{|t|}/10 = |x(0)-y(0)|e^{|t|}.$$

(c) Since $\sin t < |t|$ except at $t = 0$, we have $|x(t) - y(t)| < e^{|t|}/10$, except at $t = 0$.

17. (a) The right hand side of the equation is $f(t, x) = -2x + \sin t$, and $\partial f/\partial x = -2$. Hence $M = \max(|\partial f/\partial x|) = 2$, and Theorem 7.15 predicts that $|y(t) - x(t)| \le |y(0) - x(0)|e^{M|t|} = |y(0) - x(0)|e^{2|t|}$.

(b) The equation is linear, and we find that $x(t) = [2\sin t - \cos t]/5$, and $y(t) = [2\sin t - \cos t]/5 - e^{-2t}/10$. Hence

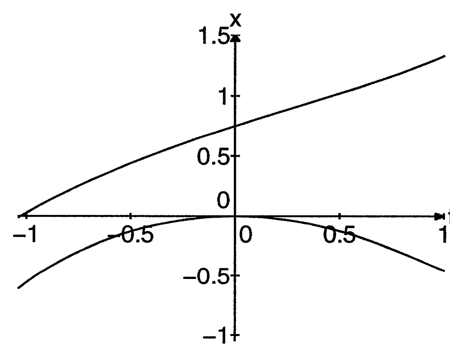
$$x(t) - y(t) = e^{-2t}/10 = |x(0) - y(0)|e^{-2t} \le |y(0) - x(0)|e^{2|t|}.$$

(c) Since $e^{-2t} = e^{2|t|}$ for $t < 0$, we see that the maximum predicted error is achieved for all $t < 0$.

18. The right hand side of the equation is $f(t, x) = x^2 - t$, and $\partial f/\partial x = 2x$. On the rectangle R we have $|x| \le 2$, so $M = \max|\partial f/\partial x| = \max|2x| = 4$. Thus the bound predicted by Theorem 7.15 is

$$|x_1(t) - x_2(t)| \le |x_1(0) - x_2(0)|e^{4|t|} = 3e^{4|t|}/4.$$

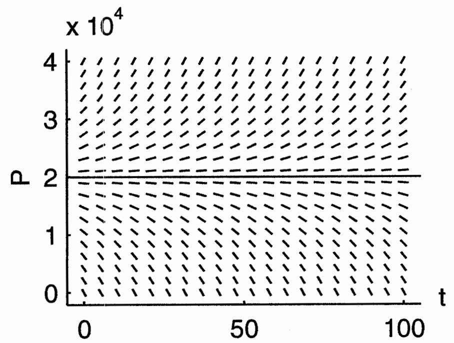
The maximum predicted error is where $|t| = 1$, and it is 40.9486. the two solutions are plotted in the following figure.



The actual bound is about 2, which is much less than 41, the theoretical bound.

Section 2.9. Autonomous Equations and Stability

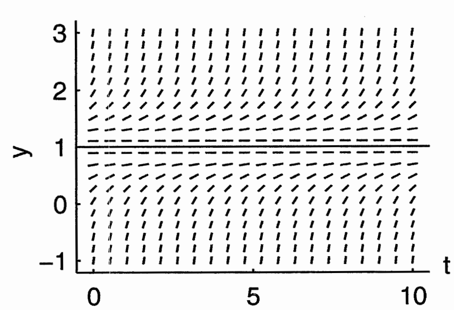
1. Note that $P' = 0.05P - 1000$ is autonomous, having form $P' = f(P)$. Solving the equation $0 = f(P) = 0.05P - 1000$, we find the equilibrium point $P = 20000$. Thus, $P(t) = 20000$ is an unstable equilibrium solution, as shown in the following figure.



2. Note that $y' = 1 - 2y + y^2$ is autonomous, having form $y' = f(y)$. Solve the equation $f(y) = 0$ to find the equilibrium points.

$$1 - 2y + y^2 = 0$$
$$y = 1.$$

Thus, $y(t) = 1$ is an unstable equilibrium solution, as shown in the following figure.

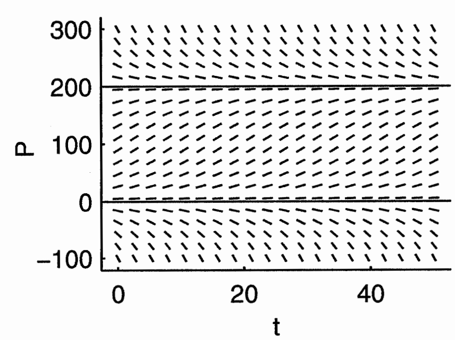


3. Note that $x' = t^2 - x^2$ is not autonomous, having form $x' = f(t, x)$, where $f(t, x) = t^2 - x^2$. The explicit dependence of the right-hand side of this differential equation on the independent variable t causes the equation to be non-autonomous.

4. Note that $P' = 0.13P(1 - P/200)$ is autonomous, having form $P' = f(P)$. Solve the equation $f(P) = 0$ to find the equilibrium points.

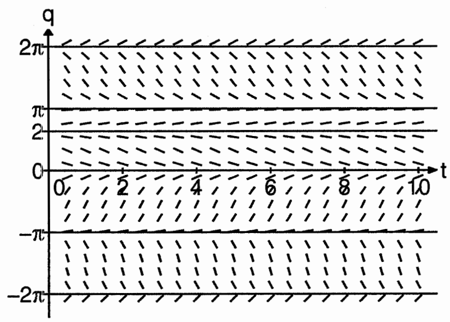
$$0.13P\left(1 - \frac{P}{200}\right) = 0$$
$$P = 0 \quad \text{or} \quad P = 200$$

Thus, $P(t) = 0$ and $P(t) = 200$ are equilibrium solutions, as shown in the following figure. $P = 0$ unstable and $P = 200$ is asymptotically stable.



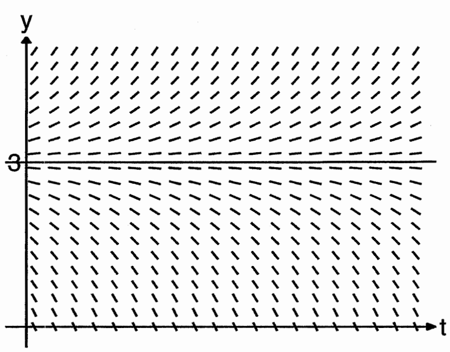
5. The equation is autonomous. The point $q = 2$ is an unstable equilibrium point, as the following figure shows. In addition every solution of $\sin q = 0$ is an equilibrium point. These are the points $k\pi$, where k is any integer, positive of negative. The stability of the equilibrium points alternates between asymptotic

stable and unstable, as is seen in the figure.

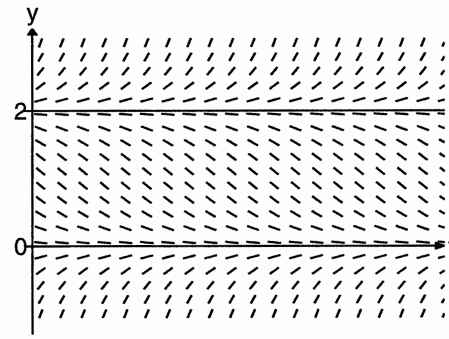


6. The equation is not autonomous because of the $\cos t$ term.

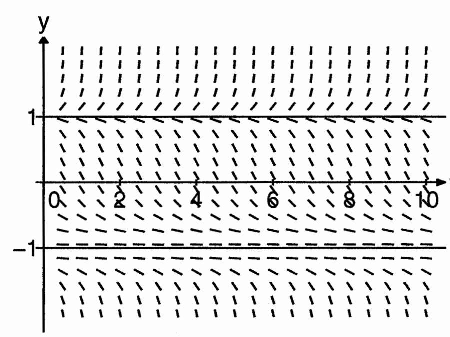
7. Note that the graph of $f(y)$ intercepts the y-axis at $y = 3$. Consequently, $y = 3$ is an equilibrium point ($f(3) = 0$) and $y(t) = 3$ is an equilibrium solution, shown in the following figure. The solution $y = 3$ is unstable.



8. Note that the graph of $f(y)$ intercepts the y-axis at $y = 0$ and $y = 2$. Consequently, $y = 0$ and $y = 2$ are equilibrium points ($f(0) = 0$ and $f(2) = 0$) and $y(t) = 0$ and $y(t) = 2$ are equilibrium solutions, shown in the following figure. The solution $y = 0$ is asymptotically stable and $y = 2$ is unstable.

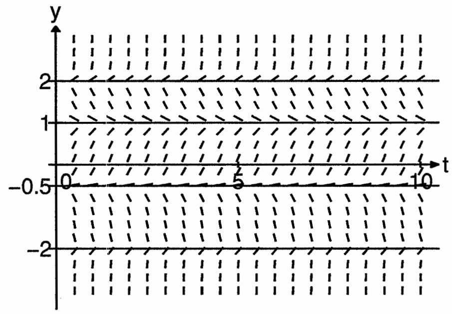


9. Since $f(y)$ has zeros at $y = -1$ and $y = 1$, these are equilibrium points. Correspondingly, $y(t) = -1$ and $y(t) = 1$ are equilibrium solutions, and are plotted in the following figure. Both are unstable.

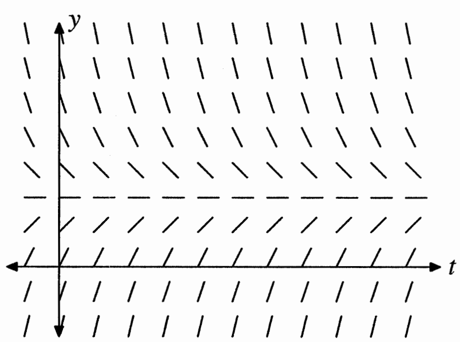


10. Since $f(y)$ has zeros at $y = -2$, $y = -1/2$, $y = 1$, and at $y = 2$, all four are equilibrium points. Correspondingly, $y(t) = -2$, $y(t) = -1/2$, $y(t) = 1$, and $y(t) = 2$ are equilibrium solutions, and are plotted in the following figure. $y = -2$ and $y = 1$ are

asymptotically stable and the other two are unstable.

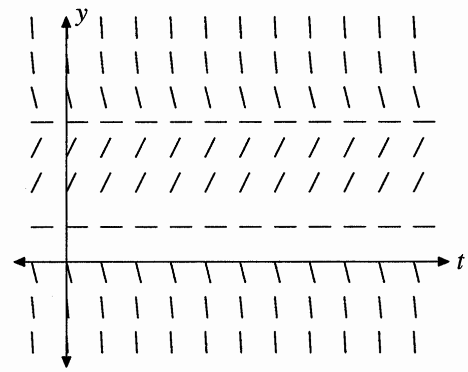


11. Because the differential equation $y' = f(y)$ is autonomous, the slope at any point (t, x) in the direction field does not depend on t, only on y, as shown in the following figure.



The equilibrium point is asymptotically stable.

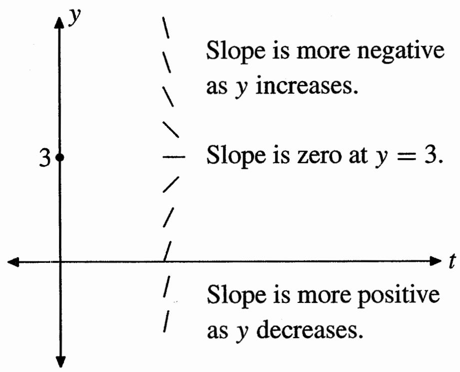
12. Because the differential equation $y' = f(y)$ is autonomous, the slope at any point (t, x) in the direction field does not depend on t, only on y, as shown in the following figure.



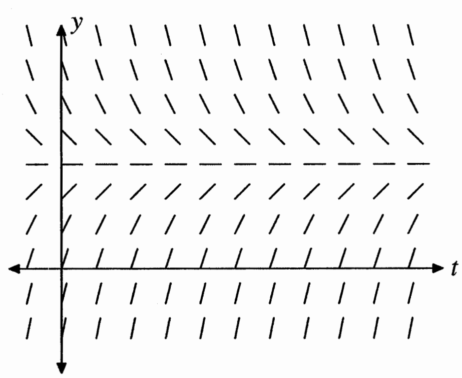
There are two equilibrium points. The smaller of them is unstable and the other is asymptotically stable.

13. The key thing to note is the fact that y' and $f(y)$ are equal. Consequently, the value of $f(y)$ is the slope of the direction line positioned at (t, y).

- At $y = 3$, $f(y) = 0$ and the slope is zero. Thus $y = 3$ is an equilibrium point. This is shown in the following figure.
- To the right of $y = 3$, note that the graph of f dips below the y-axis. Therefore, as y increases beyond 3, the slope becomes increasingly negative. This is also shown in the following figure.
- To the left of $y = 3$, note that the graph of f rise above the y-axis. Therefore, as y decreases below 3, the slope becomes increasingly positive. This is also shown in the following figure. In particular, this means that the equilibrium point $y = 3$ is asymptotically stable.



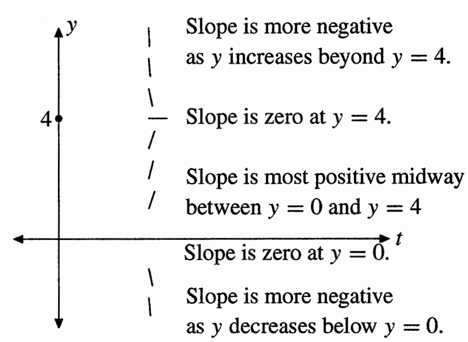
Finally, because the equation $y' = f(y)$ is autonomous, the slope of a direction line positioned at (t, y) depends only on y and not on t. Consequently, the rest of the direction field is easily completed, as shown in the next figure.



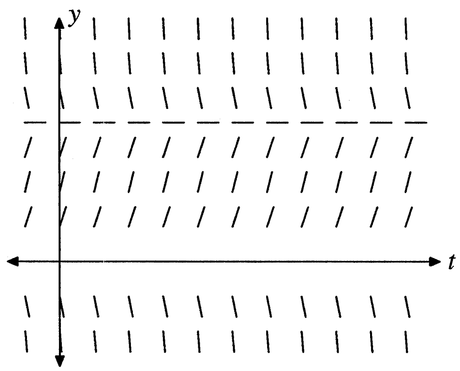
14. The key thing to note is the fact that y' and $f(y)$ are equal. Consequently, the value of $f(y)$ is the slope of the direction line positioned at (t, y).

   - At $y = 0$ and $y = 4$, $f(y) = 0$, so the slope of the direction lines at these y-values is zero. These points are the equilibrium points. This is shown in the following figure.
   - To the right of $y = 4$, note that the graph of f dips below the y-axis. Furthermore, as y increases beyond 4, $f(y)$ (the slope of the direction line at (t, y)) becomes increasingly negative. This is also shown in the following figure.
   - Between $y = 0$ and $y = 4$, the graph of f lies above the y-axis. Consequently, $f(y)$ is positive for $0 < y < 4$. Moreover, the graph of f has a maximum about halfway between $y = 0$ and $y = 4$. Consequently, the slope of the direction field lines will be positive between $y = 0$ and $y = 4$, with a maximum positive slope occurring about halfway between $y = 0$ and $y = 4$. This is shown in the following figure.
   - To the left of $y = 0$, note that the graph of f falls below the y-axis. Furthermore, as y decreases below 0, $f(y)$ (the slope of the direction line at (t, y)) becomes increasingly negative. This is also shown in the next figure.

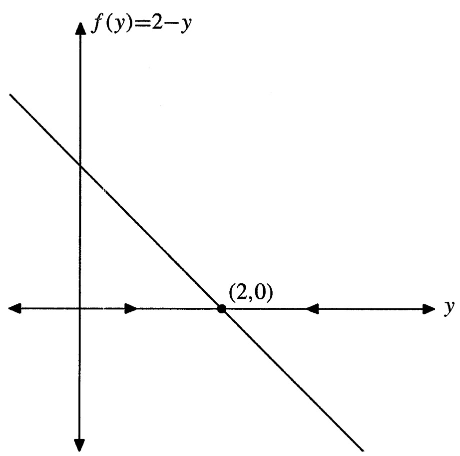
From these considerations we see that the equilibrium point $y = 0$ is unstable, and $y = 4$ is asymptotically stable.



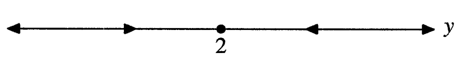
Finally, because the equation $y' = f(y)$ is autonomous, the slope of a direction line positioned at (t, y) depends only on y and not on t. Consequently, the rest of the direction field is easily completed, as shown in the next figure.



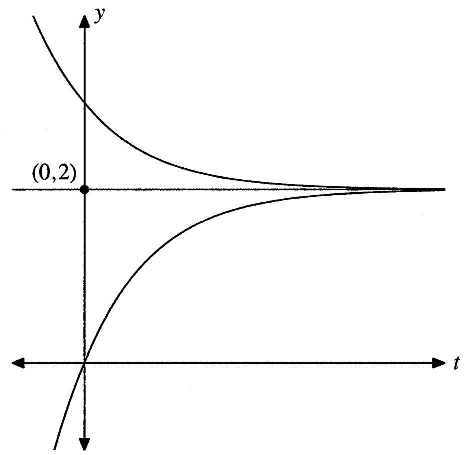
15. (i) In this case, $f(y) = 2 - y$, whose graph is shown in the following figure.



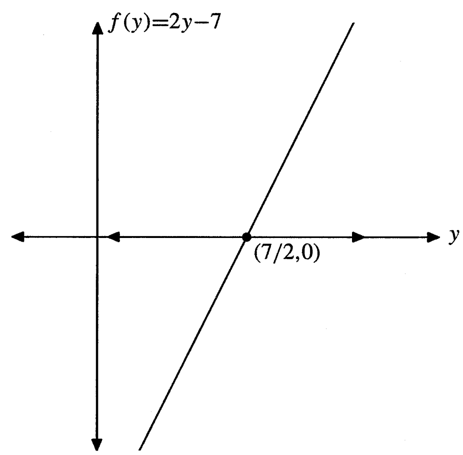
(ii) The phase line is easily captured from the previous figure, and is shown in the following figure.



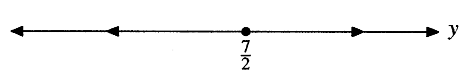
(iii) The phase line in the second figure indicates that solutions increase if $y < 2$ and decrease if $y > 2$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the stable equilibrium solution, $y(t) = 2$.



16. (i) In this case, $f(y) = 2y - 7$, whose graph is shown in the next figure.

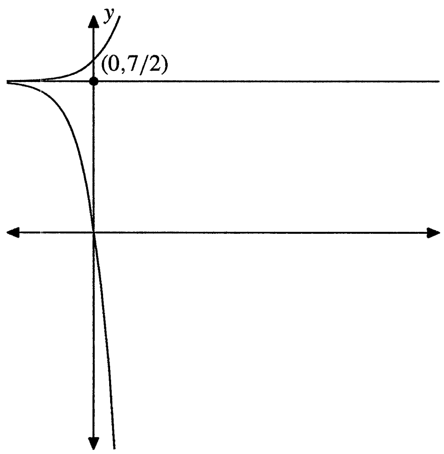


(ii) The phase line is easily captured from this figure, and is shown in next figure.

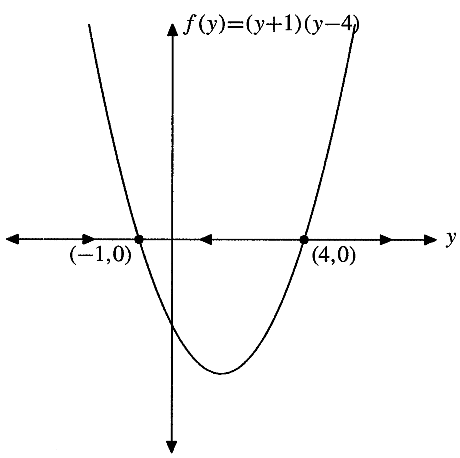


(iii) The phase line in the second figure indicates that solutions decrease if $y < 7/2$ and increase if

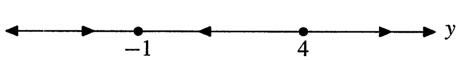
$y > 7/2$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = 7/2$.



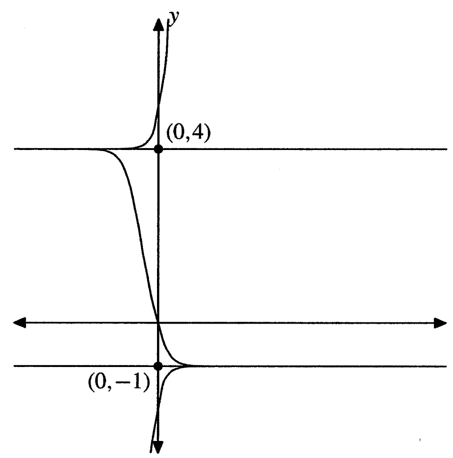
17. (i) In this case, $f(y) = (y+1)(y-4)$, whose graph is shown in the next figure.



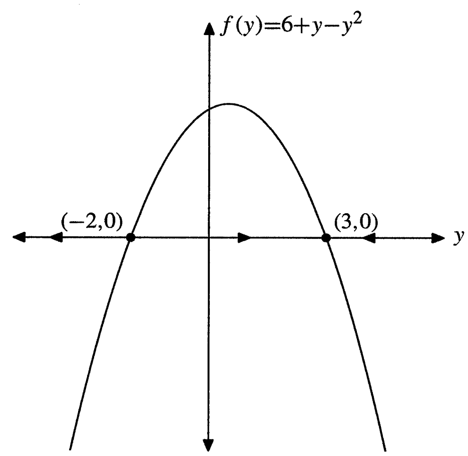
(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.



(iii) The phase line in the second figure indicates that solutions increase if $y < -1$, decrease for $-1 < y < 4$, and increase if $y > 4$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = 4$, and the stable equilibrium solution, $y(t) = -1$.

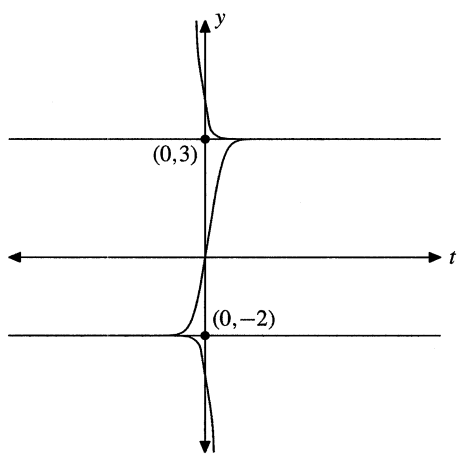


18. (i) In this case, $f(y) = 6 + y - y^2$ factors as $f(y) = (2+y)(3-y)$, whose graph is shown in the next figure.

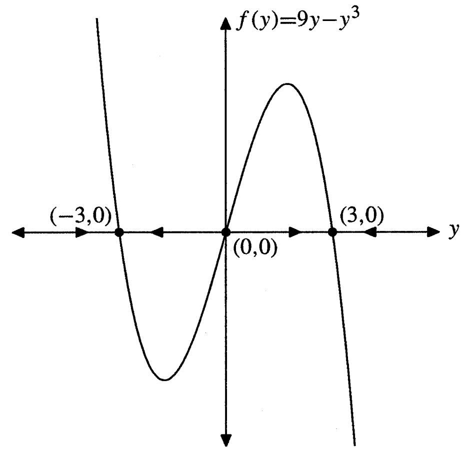


(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.

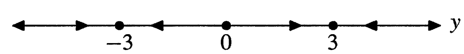
(iii) The phase line in the second figure indicates that solutions decrease if $y < -2$, increase for $-2 < y < 3$, and decrease if $y > 3$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = -2$, and the stable equilibrium solution, $y(t) = 3$.



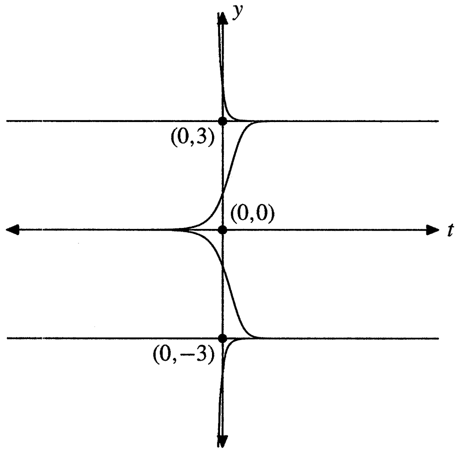
19. (i) In this case, $f(y) = 9y - y^3$ factors as $f(y) = y(y+3)(y-3)$, whose graph is shown in the next figure.



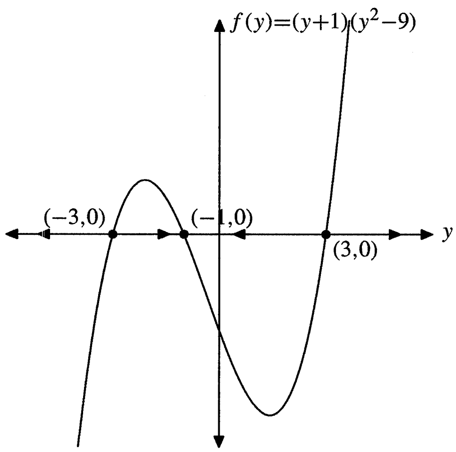
(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.



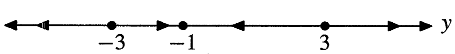
(iii) The phase line in the second figure indicates that solutions increase if $y < -3$, decrease for $-3 < y < 0$, increase if $0 < y < 3$, and decrease for $y > 3$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the stable equilibrium solution, $y(t) = -3$, the unstable equilibrium solution, $y(t) = 0$, and the stable equilibrium solution, $y(t) = 3$.



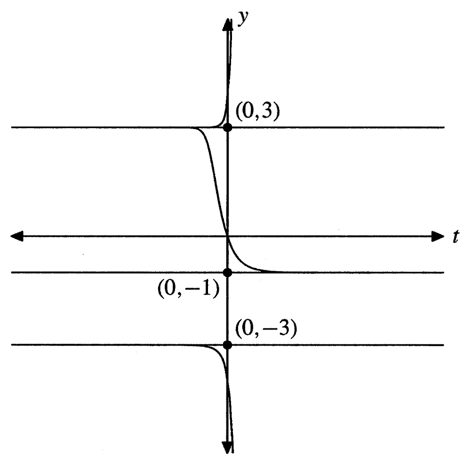
20. (i) In this case, $f(y) = (y + 1)(y^2 - 9)$ factors as $f(y) = (y+1)(y-3)(y+3)$, whose graph is shown in the next figure.



(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.

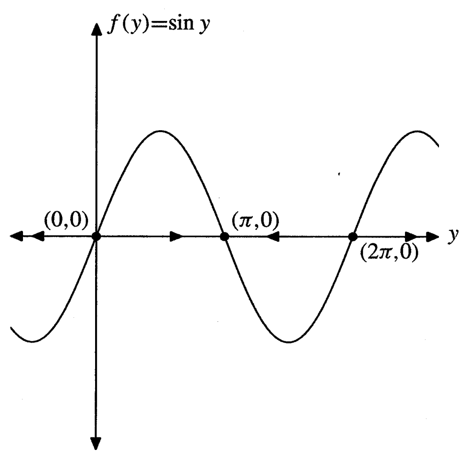


(iii) The phase line in the second figure indicates that solutions decrease if $y < -3$, increase for $-3 < y < -1$, decrease if $-1 < y < 3$, and increase for $y > 3$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = -3$, the stable equilibrium solution, $y(t) = -1$, and the unstable equilibrium solution, $y(t) = 3$.

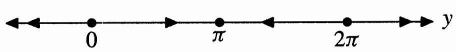


21. Due to the periodic nature of this equation, we sketch only a few regions. You can easily use the periodicity to produce more regions.

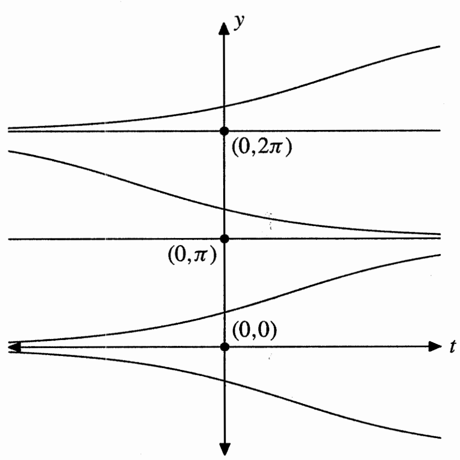
(i) In this case, $f(y) = \sin y$, whose graph is shown in the next figure.



(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.

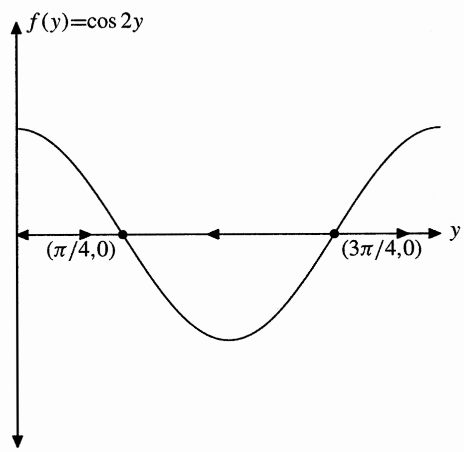


(iii) The phase line in the second figure indicates that solutions decrease if $-\pi < y < 0$, increase for $0 < y < \pi$, decrease if $\pi < y < 2\pi$, and increase for $2\pi < y < 3\pi$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the unstable equilibrium solution, $y(t) = 0$, the stable equilibrium solution, $y(t) = \pi$, and the unstable equilibrium solution, $y(t) = 2\pi$.

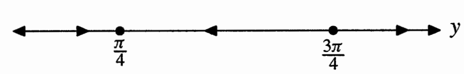


22. Due to the periodic nature of this equation, we sketch only a few regions. You can easily use the periodicity to produce more regions.

(i) In this case, $f(y) = \cos 2y$, whose graph is shown in the next figure.

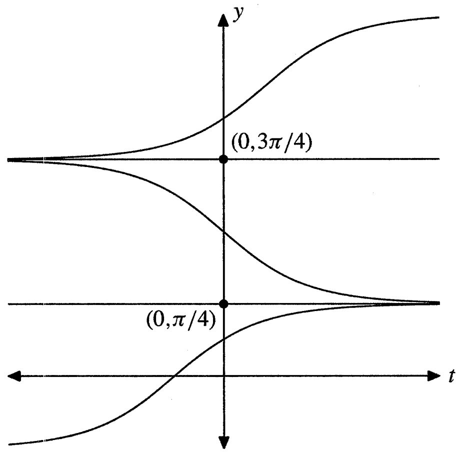


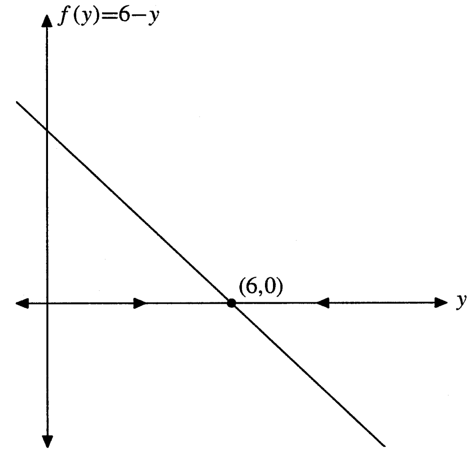
(ii) The phase line is easily captured from the previous figure, and is shown in the next figure.



(iii) The phase line in the second figure indicates that solutions increase if $-\pi/4 < y < \pi/4$, decrease for $\pi/4 < y < 3\pi/4$, and increase if $3\pi/4 < y < 5\pi/4$. This allows us to easily construct the phase portrait shown in the ty plane in the next figure. Note the stable equilibrium solution, $y(t) = \pi/4$, and the unstable equilibrium solution,

$y(t) = 3\pi/4$.





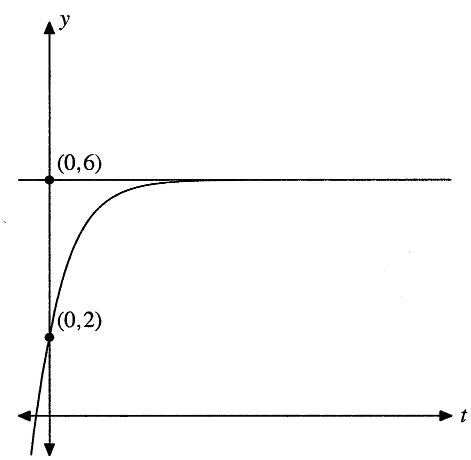
23. The equation is linear, so multiply by the integrating factor and integrate.

$$\begin{aligned}\left(e^t y\right)' &= 6e^t\\ e^t y &= 6e^t + C\\ y(t) &= 6 + Ce^{-t}\end{aligned}$$

The initial condition $y(0) = 2$ produces $C = -4$ and $y(t) = 6 - 4e^{-t}$. Now, e^{-t} approaches zero as $t \to +\infty$, so

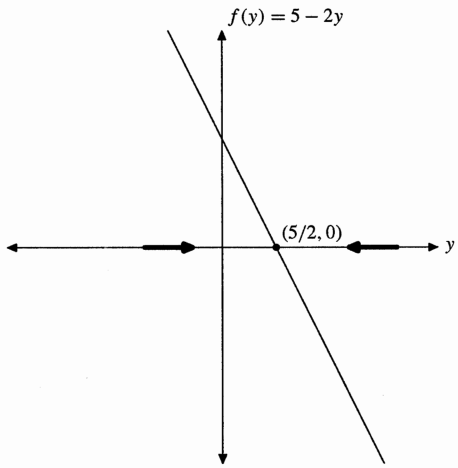
$$\lim_{t\to+\infty} y(t) = \lim_{t\to+\infty} \left(6 - 4e^{-t}\right) = 6.$$

Compare $y' = f(y)$ with $y' = 6 - y$. Then $f(y) = 6 - y$, whose graph is shown in the first figure below. The phase line on the y-axis in this figure shows that $y = 6$ is a stable equilibrium point, so a trajectory with initial condition $y(0) = 2$ should approach the stable equilibrium solution $y(t) = 6$, as shown in the second figure. This agrees nicely with the analytical solution.



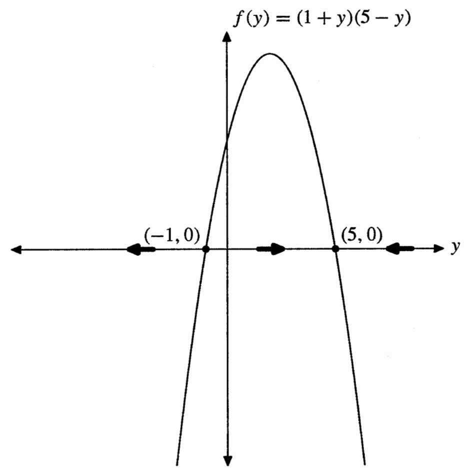
24. Writing the equation as $y' = 5 - 2y$, we see that the right hand side is $f(y) = 5 - 2y$. The graph of f is in the next figure. We have also indicated the direction of the solutions on the y-axis, which shows that $y = 5/2$ is an asymptotically stable equilibrium point. Thus any solution curve will approach

$y = 5/2$ as t increases.



The exact solution can be found since the equations is separable (and linear). With some work we find that it is $y(t) = 5[1 - e^{-2t}]/2$. Clearly the solution has the indicated limiting behavior.

25. The equation has the form $y' = f(y)$, where $f(y) = (1 + y)(5 - y)$. The graph of f is in the next figure. We have also indicated the direction of the solutions on the y-axis. This shows that $y = -1$ is an unstable equilibrium point, and $y = 5$ is an asymptotically stable equilibrium point. Therefore, a solution starting with $y(0) = 2$ will increase and approach $y = 5$ as t increases.



To find the exact solution, we separate variables and use partial fractions to get

$$\frac{1}{6}\left[\frac{1}{1+y} + \frac{1}{5-y}\right] dy = dt$$

Integrate,

$$\begin{aligned}\frac{1}{6}\ln|1+y| - \frac{1}{6}\ln|5-y| &= t + C,\\ \ln|1+y| - \ln|5-y| &= 6t + 6C,\\ \ln\left|\frac{y+1}{y-5}\right| &= 6t + 6C\\ \frac{y+1}{y-5} &= Ae^{6t},\end{aligned}$$

where $A = \pm e^{6C}$. Using the initial condition $y(0) = 2$ we see that $A = -1$, so

$$\frac{y+1}{y-5} = -e^{6t}.$$

Solving for y, we find that

$$y(t) = \frac{5e^{6t} - 1}{1 + e^{6t}} = \frac{5 - e^{-6t}}{1 + e^{-6t}}.$$

From this we see that $y(t) \to 5$ as $t \to \infty$, agreeing with what we discovered earlier.

26. Separating variables,

$$\frac{dy}{dt} = (3+y)(1-y)$$
$$\frac{dy}{(3+y)(1-y)} = dt.$$

A partial fraction decomposition allows us to continue.

$$\frac{1}{4}\left[\frac{1}{3+y} + \frac{1}{1-y}\right] dy = dt$$
$$\ln|3+y| - \ln|1-y| = 4t + C$$
$$\ln\left|\frac{3+y}{1-y}\right| = 4t + C$$
$$\left|\frac{3+y}{1-y}\right| = e^C e^{4t}$$
$$\frac{3+y}{1-y} = Ae^{4t}$$

with $y(0) = 2$,

$$\frac{3+2}{1-2} = Ae^{4(0)} \Rightarrow A = -5$$

and

$$\frac{3+y}{1-y} = -5e^{4t}$$
$$3 + y = -5e^{4t} + 5ye^{4t}$$
$$3 + 5e^{4t} = y(5e^{4t} - 1)$$
$$y = \frac{3+5e^{4t}}{5e^{4t}-1}.$$

Multiply top and bottom by e^{-4t}.

$$y = \frac{3e^{-4t}+5}{5-e^{-4t}}$$

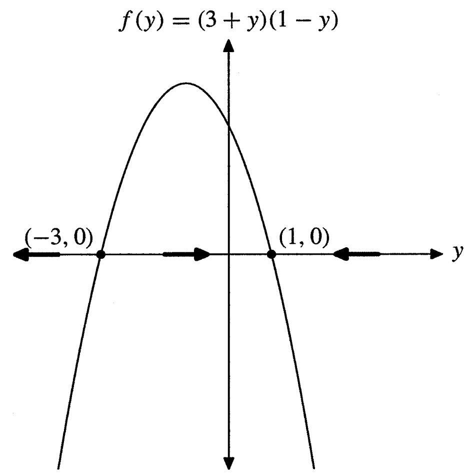
Thus,

$$\lim_{t\to\infty} y = \frac{0+5}{5-0} = 1.$$

Using qualitative analysis, plot the graph of the right-hand side of

$$\frac{dy}{dt} = (3+y)(1-y)$$

versus y.



Note the equilibrium points at $y = -3$ and $y = 1$. Moreover, note that between -3 and 2, solutions increase to the stable point at $y = 1$. Thus,

$$\lim_{t\to\infty} y(t) = 1.$$

27. We have the equation $x' = f(x) = 4 - x^2$. The equilibrium points are at $x = \pm 2$, where $f(x) = 0$. We have $f'(x) = -2x$. Since $f'(-2) = 4 > 0$, $x = -2$ is unstable. Since $f'(2) = -4 < 0$, $x = 2$ is asymptotically stable.

28. We have the equation $x' = f(x) = x(x-1)(x+2)$. The equilibrium points are at $x = 0$, 1, and -2, where $f(x) = 0$. We have $f'(x) = 3x^2 + 2x - 2$. Since $f'(0) = -2 < 0$, $x = 0$ is asymptotically stable. Because $f'(1) = 3 > 0$, $x = 1$ is unstable. Finally, because $f'(-2) = 2 > 0$, $x = -2$ is also unstable.

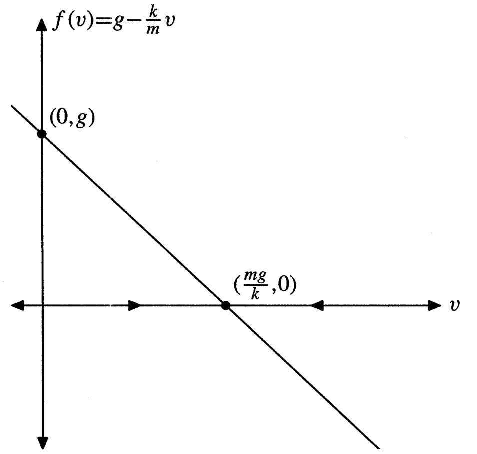
29. (a) $f(x) = x^2$, $f(x) = x^3$, or $f(x) = x^4$.
(b) $f(x) = -x^3$, $f(x) = -x^5$, or $f(x) = -x^7$.

30. Notice that we are measuring the displacement as positive below the plane. First divide through by m to get

$$\frac{dv}{dt} = g - \frac{k}{m}v.$$

Note that this equation is autonomous, having form $v' = f(v)$. The graph of f is a line, with slope

$-k/m$ and intercept g, as shown in the following figure.



The phase line on the v-axis in this figure shows that $v = (mg)/k$ is a stable equilibrium point. Our sky-diver starts from rest, so the solution trajectory with $v(0) = 0$ should approach the stable equilibrium solution, $v(t) = (mg)/k$. Consequently, the terminal velocity is $(mg)/k$.

31. Let $x(t)$ represent the amount of salt in the tank at time t. The rate at which solution enters the tank is given by

$$\text{Rate In} = 2\,\text{gal/min} \times 3\,\text{lb/gal} = 6\,\text{lb/min}.$$

The rate at which solution leaves the tank is

$$\text{Rate Out} = 2\,\text{gal/min} \times \frac{x}{100}\,\text{lb/gal} = \frac{1}{50}x\,\text{lb/min}.$$

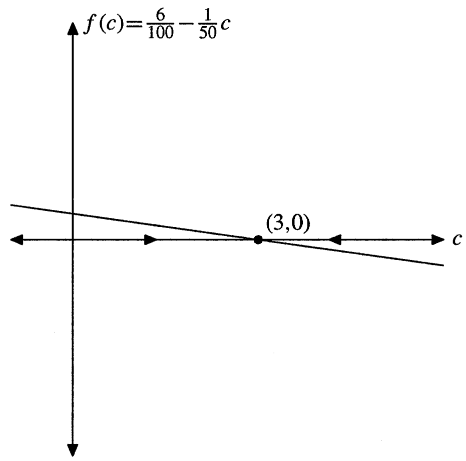
Consequently,

$$\frac{dx}{dt} = 6 - \frac{1}{50}x.$$

Let $c(t)$ represent the concentration of salt in the solution at time t. Thus, $c(t) = x(t)/100$ and $100c' = x'$.

$$100c' = 6 - \frac{1}{50}(100c)$$
$$c' = \frac{6}{100} - \frac{1}{50}c$$

Let $f(c) = 6/100 - (1/50)c$. Setting $f(c) = 0$ produces the equilibrium point $c = 3$, as shown in the following figure.



The phase line on the c-axis in this figure shows that $c = 3$ is a stable equilibrium point so a trajectory with initial condition $c(0) = 0$ (the initial concentration of salt is zero) should approach the stable equilibrium solution $c(t) = 3$.