CHAPTER 2
Limits and Their Properties

C H A P T E R 2
Limits and Their Properties

Section 2.1 A Preview of Calculus

1. Precalculus: $(20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

2. Calculus required: Velocity is not constant.

Distance $\approx (20 \text{ ft/sec})(15 \text{ sec}) = 300 \text{ ft}$

3. Calculus required: Slope of the tangent line at $x = 2$ is the rate of change, and equals about 0.16.

4. Precalculus: rate of change $=$ slope $= 0.08$

5. (a) Precalculus: Area $= \frac{1}{2}bh = \frac{1}{2}(5)(4) = 10$ sq. units

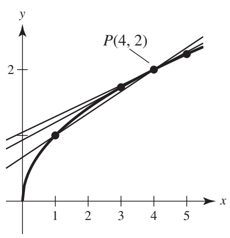
(b) Calculus required: Area $= bh$

$\approx 2(2.5)$

$= 5$ sq. units

6. $f(x) = \sqrt{x}$

(a)



(b) $$\text{slope} = m = \frac{\sqrt{x} - 2}{x - 4} = \frac{\sqrt{x} - 2}{(\sqrt{x} + 2)(\sqrt{x} - 2)} = \frac{1}{\sqrt{x} + 2},\ x \neq 4$$

$$x = 1: m = \frac{1}{\sqrt{1} + 2} = \frac{1}{3}$$

$$x = 3: m = \frac{1}{\sqrt{3} + 2} \approx 0.2679$$

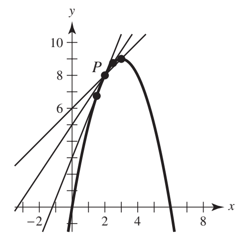
$$x = 5: m = \frac{1}{\sqrt{5} + 2} \approx 0.2361$$

(c) At $P(4, 2)$ the slope is $\frac{1}{\sqrt{4} + 2} = \frac{1}{4} = 0.25$.

You can improve your approximation of the slope at $x = 4$ by considering x-values very close to 4.

7. $f(x) = 6x - x^2$

(a)



(b) $$\text{slope} = m = \frac{(6x - x^2) - 8}{x - 2} = \frac{(x - 2)(4 - x)}{x - 2} = (4 - x),\ x \neq 2$$

For $x = 3, m = 4 - 3 = 1$

For $x = 2.5, m = 4 - 2.5 = 1.5 = \frac{3}{2}$

For $x = 1.5, m = 4 - 1.5 = 2.5 = \frac{5}{2}$

(c) At $P(2, 8)$, the slope is 2. You can improve your approximation by considering values of x close to 2.

8. Answers will vary. *Sample answer*:

The instantaneous rate of change of an automobile's position is the velocity of the automobile, and can be determined by the speedometer.

9. (a) Area $\approx 5 + \frac{5}{2} + \frac{5}{3} + \frac{5}{4} \approx 10.417$

Area $\approx \frac{1}{2}\left(5 + \frac{5}{1.5} + \frac{5}{2} + \frac{5}{2.5} + \frac{5}{3} + \frac{5}{3.5} + \frac{5}{4} + \frac{5}{4.5}\right) \approx 9.145$

(b) You could improve the approximation by using more rectangles.

10. (a) $D_1 = \sqrt{(5-1)^2 + (1-5)^2} = \sqrt{16+16} \approx 5.66$

(b) $D_2 = \sqrt{1 + \left(\frac{5}{2}\right)^2} + \sqrt{1 + \left(\frac{5}{2} - \frac{5}{3}\right)^2} + \sqrt{1 + \left(\frac{5}{3} - \frac{5}{4}\right)^2} + \sqrt{1 + \left(\frac{5}{4} - 1\right)^2}$

$\approx 2.693 + 1.302 + 1.083 + 1.031 \approx 6.11$

(c) Increase the number of line segments.

Section 2.2 Finding Limits Graphically and Numerically

1.

x	3.9	3.99	3.999	4.001	4.01	4.1
$f(x)$	0.2041	0.2004	0.2000	0.2000	0.1996	0.1961

$\lim_{x\to 4} \frac{x-4}{x^2 - 3x - 4} \approx 0.2000 \qquad \left(\text{Actual limit is } \frac{1}{5}.\right)$

2.

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881

$\lim_{x\to 0} \frac{\sqrt{x+1} - 1}{x} \approx 0.5000 \qquad \left(\text{Actual limit is } \frac{1}{2}.\right)$

3.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.9983	0.99998	1.0000	1.0000	0.99998	0.9983

$\lim_{x\to 0} \frac{\sin x}{x} \approx 1.0000 \qquad$ (Actual limit is 1.) (Make sure you use radian mode.)

4.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.0500	0.0050	0.0005	–0.0005	–0.0050	–0.0500

$\lim_{x\to 0} \frac{\cos x - 1}{x} \approx 0.0000 \qquad$ (Actual limit is 0.) (Make sure you use radian mode.)

5.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.9516	0.9950	0.9995	1.0005	1.0050	1.0517

$\lim_{x\to 0} \frac{e^x - 1}{x} \approx 1.0000 \qquad$ (Actual limit is 1.)

6.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	1.0536	1.0050	1.0005	0.9995	0.9950	0.9531

$\lim_{x\to 0} \frac{\ln(x+1)}{x} \approx 1.0000 \qquad$ (Actual limit is 1.)

7.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

$\lim_{x\to 1} \frac{x-2}{x^2+x-6} \approx 0.2500 \qquad \left(\text{Actual limit is } \frac{1}{4}.\right)$

8.

x	–4.1	–4.01	–4.001	–4	–3.999	–3.99	–3.9
$f(x)$	1.1111	1.0101	1.0010	?	0.9990	0.9901	0.9091

$\lim_{x\to -4} \frac{x+4}{x^2+9x+20} \approx 1.0000 \qquad (\text{Actual limit is } 1.)$

9.

x	0.9	0.99	0.999	1.001	1.01	1.1
$f(x)$	0.7340	0.6733	0.6673	0.6660	0.6600	0.6015

$\lim_{x\to 1} \frac{x^4-1}{x^6-1} \approx 0.6666 \qquad \left(\text{Actual limit is } \frac{2}{3}.\right)$

10.

x	–3.1	–3.01	–3.001	–3	–2.999	–2.99	–2.9
$f(x)$	27.91	27.0901	27.0090	?	26.9910	26.9101	26.11

$\lim_{x\to -3} \frac{x^3+27}{x+3} \approx 27.0000 \qquad (\text{Actual limit is } 27.)$

11.

x	–6.1	–6.01	–6.001	–6	–5.999	–5.99	–5.9
$f(x)$	–0.1248	–0.1250	–0.1250	?	–0.1250	–0.1250	–0.1252

$\lim_{x\to -6} \frac{\sqrt{10-x}-4}{x+6} \approx -0.1250 \qquad \left(\text{Actual limit is } -\frac{1}{8}.\right)$

12.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	0.1149	0.115	0.1111	?	0.1111	0.1107	0.1075

$\lim_{x\to 2} \frac{x/(x+1)-2/3}{x-2} \approx 0.1111 \qquad \left(\text{Actual limit is } \frac{1}{9}.\right)$

13.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	1.9867	1.9999	2.0000	2.0000	1.9999	1.9867

$\lim_{x\to 0} \frac{\sin 2x}{x} \approx 2.0000 \qquad (\text{Actual limit is } 2.)$ (Make sure you use radian mode.)

14.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.4950	0.5000	0.5000	0.5000	0.5000	0.4950

$\lim_{x\to 0} \frac{\tan x}{\tan 2x} \approx 0.5000 \qquad \left(\text{Actual limit is } \frac{1}{2}.\right)$

15.

x	1.9	1.99	1.999	2.001	2.01	2.1
$f(x)$	0.5129	0.5013	0.5001	0.4999	0.4988	0.4879

$\lim_{x \to 2} \frac{\ln x - \ln 2}{x - 2} \approx 0.5000$ $\left(\text{Actual limit is } \frac{1}{2}.\right)$

16.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	3.99982	4	4	0	0	0.00018

$\lim_{x \to 0} \frac{4}{1 + e^{1/x}}$ does not exist.

17. $\lim_{x \to 3} (4 - x) = 1$

18. $\lim_{x \to 0} \sec x = 1$

19. $\lim_{x \to 2} f(x) = \lim_{x \to 2} (4 - x) = 2$

20. $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x^2 + 3) = 4$

21. $\lim_{x \to 2} \frac{|x - 2|}{x - 2}$ does not exist.

For values of x to the left of 2, $\frac{|x - 2|}{(x - 2)} = -1$, whereas for values of x to the right of 2, $\frac{|x - 2|}{(x - 2)} = 1$.

22. $\lim_{x \to 0} \frac{4}{2 + e^{1/x}}$ does not exist. The function approaches 2 from the left side of 0 by it approaches 0 from the left side of 0.

23. $\lim_{x \to 0} \cos(1/x)$ does not exist because the function oscillates between –1 and 1 as x approaches 0.

24. $\lim_{x \to \pi/2} \tan x$ does not exist because the function increases without bound as x approaches $\frac{\pi}{2}$ from the left and decreases without bound as x approaches $\frac{\pi}{2}$ from the right.

25. (a) $f(1)$ exists. The black dot at (1, 2) indicates that $f(1) = 2$.

(b) $\lim_{x \to 1} f(x)$ does not exist. As x approaches 1 from the left, $f(x)$ approaches 3.5, whereas as x approaches 1 from the right, $f(x)$ approaches 1.

(c) $f(4)$ does not exist. The hollow circle at $(4, 2)$ indicates that f is not defined at 4.

(d) $\lim_{x \to 4} f(x)$ exists. As x approaches 4, $f(x)$ approaches 2: $\lim_{x \to 4} f(x) = 2$.

26. (a) $f(-2)$ does not exist. The vertical dotted line indicates that f is not defined at –2.

(b) $\lim_{x \to -2} f(x)$ does not exist. As x approaches –2, the values of $f(x)$ do not approach a specific number.

(c) $f(0)$ exists. The black dot at $(0, 4)$ indicates that $f(0) = 4$.

(d) $\lim_{x \to 0} f(x)$ does not exist. As x approaches 0 from the left, $f(x)$ approaches $\frac{1}{2}$, whereas as x approaches 0 from the right, $f(x)$ approaches 4.

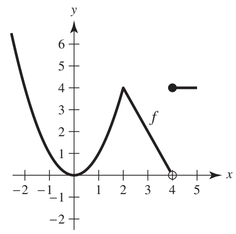
(e) $f(2)$ does not exist. The hollow circle at $\left(2, \frac{1}{2}\right)$ indicates that $f(2)$ is not defined.

(f) $\lim_{x \to 2} f(x)$ exists. As x approaches 2, $f(x)$ approaches $\frac{1}{2}$: $\lim_{x \to 2} f(x) = \frac{1}{2}$.

(g) $f(4)$ exists. The black dot at $(4, 2)$ indicates that $f(4) = 2$.

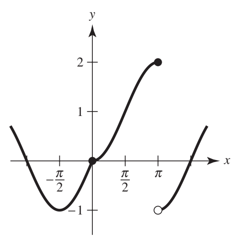
(h) $\lim_{x \to 4} f(x)$ does not exist. As x approaches 4, the values of $f(x)$ do not approach a specific number.

27.



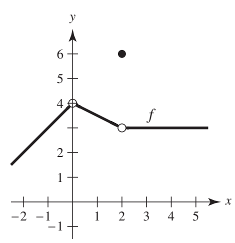
$\lim\limits_{x \to c} f(x)$ exists for all values of $c \neq 4$.

28.

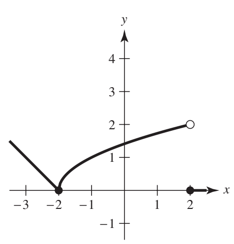


$\lim\limits_{x \to c} f(x)$ exists for all values of $c \neq \pi$.

29. One possible answer is



30. One possible answer is



31. You need $\left|f(x) - 3\right| = \left|(x + 1) - 3\right| = \left|x - 2\right| < 0.4$.

So, take $\delta = 0.4$. If $0 < \left|x - 2\right| < 0.4$, then

$\left|x - 2\right| = \left|(x + 1) - 3\right| = \left|f(x) - 3\right| < 0.4$, as desired.

32. You need $\left|f(x) - 1\right| = \left|\dfrac{1}{x - 1} - 1\right| = \left|\dfrac{2 - x}{x - 1}\right| < 0.01$.

Let $\delta = \dfrac{1}{101}$. If $0 < \left|x - 2\right| < \dfrac{1}{101}$, then

$$-\frac{1}{101} < x - 2 < \frac{1}{101} \Rightarrow 1 - \frac{1}{101} < x - 1 < 1 + \frac{1}{101}$$
$$\Rightarrow \frac{100}{101} < x - 1 < \frac{102}{101}$$
$$\Rightarrow \left|x - 1\right| > \frac{100}{101}$$

and you have

$$\left|f(x) - 1\right| = \left|\frac{1}{x - 1} - 1\right| = \left|\frac{2 - x}{x - 1}\right| < \frac{1/101}{100/101} = \frac{1}{100} = 0.01.$$

33. You need to find δ such that $0 < \left|x - 1\right| < \delta$ implies $\left|f(x) - 1\right| = \left|\dfrac{1}{x} - 1\right| < 0.1$. That is,

$$-0.1 < \frac{1}{x} - 1 < 0.1$$
$$1 - 0.1 < \frac{1}{x} < 1 + 0.1$$
$$\frac{9}{10} < \frac{1}{x} < \frac{11}{10}$$
$$\frac{10}{9} > x > \frac{10}{11}$$
$$\frac{10}{9} - 1 > x - 1 > \frac{10}{11} - 1$$
$$\frac{1}{9} > x - 1 > -\frac{1}{11}.$$

So take $\delta = \dfrac{1}{11}$. Then $0 < \left|x - 1\right| < \delta$ implies

$$-\frac{1}{11} < x - 1 < \frac{1}{11}$$
$$-\frac{1}{11} < x - 1 < \frac{1}{9}.$$

Using the first series of equivalent inequalities, you obtain

$$\left|f(x) - 1\right| = \left|\frac{1}{x} - 1\right| < 0.1.$$

34. You need to find δ such that $0 < |x - 2| < \delta$ implies $|f(x) - 3| = |x^2 - 1 - 3| = |x^2 - 4| < 0.2$. That is,

$$\begin{aligned} -0.2 < x^2 - 4 &< 0.2 \\ 4 - 0.2 < x^2 &< 4 + 0.2 \\ 3.8 < x^2 &< 4.2 \\ \sqrt{3.8} < x &< \sqrt{4.2} \\ \sqrt{3.8} - 2 < x - 2 &< \sqrt{4.2} - 2 \end{aligned}$$

So take $\delta = \sqrt{4.2} - 2 \approx 0.0494$.

Then $0 < |x - 2| < \delta$ implies

$$\begin{aligned} -\left(\sqrt{4.2} - 2\right) < x - 2 &< \sqrt{4.2} - 2 \\ \sqrt{3.8} - 2 < x - 2 &< \sqrt{4.2} - 2. \end{aligned}$$

Using the first series of equivalent inequalities, you obtain

$$|f(x) - 3| = |x^2 - 4| < 0.2.$$

35. $\lim_{x\to 2}(3x + 2) = 3(2) + 2 = 8 = L$

$$\begin{aligned} |(3x + 2) - 8| &< 0.01 \\ |3x - 6| &< 0.01 \\ 3|x - 2| &< 0.01 \\ 0 < |x - 2| &< \tfrac{0.01}{3} \approx 0.0033 = \delta \end{aligned}$$

So, if $0 < |x - 2| < \delta = \frac{0.01}{3}$, you have

$$\begin{aligned} 3|x - 2| &< 0.01 \\ |3x - 6| &< 0.01 \\ |(3x + 2) - 8| &< 0.01 \\ |f(x) - L| &< 0.01. \end{aligned}$$

36. $\lim_{x\to 6}\left(6 - \frac{x}{3}\right) = 6 - \frac{6}{3} = 4 = L$

$$\begin{aligned} \left|\left(6 - \frac{x}{3}\right) - 4\right| &< 0.01 \\ \left|2 - \frac{x}{3}\right| &< 0.01 \\ \left|-\tfrac{1}{3}(x - 6)\right| &< 0.01 \\ |x - 6| &< 0.03 \\ 0 < |x - 6| &< 0.03 = \delta \end{aligned}$$

So, if $0 < |x - 6| < \delta = 0.03$, you have

$$\begin{aligned} \left|-\tfrac{1}{3}(x - 6)\right| &< 0.01 \\ \left|2 - \frac{x}{3}\right| &< 0.01 \\ \left|\left(6 - \frac{x}{3}\right) - 4\right| &< 0.01 \\ |f(x) - L| &< 0.01. \end{aligned}$$

37. $\lim_{x\to 2}(x^2 - 3) = 2^2 - 3 = 1 = L$

$$\begin{aligned} |(x^2 - 3) - 1| &< 0.01 \\ |x^2 - 4| &< 0.01 \\ |(x + 2)(x - 2)| &< 0.01 \\ |x + 2||x - 2| &< 0.01 \\ |x - 2| &< \frac{0.01}{|x + 2|} \end{aligned}$$

If you assume $1 < x < 3$, then $\delta \approx 0.01/5 = 0.002$.

So, if $0 < |x - 2| < \delta \approx 0.002$, you have

$$|x - 2| < 0.002 = \frac{1}{5}(0.01) < \frac{1}{|x + 2|}(0.01)$$

$$\begin{aligned} |x + 2||x - 2| &< 0.01 \\ |x^2 - 4| &< 0.01 \\ |(x^2 - 3) - 1| &< 0.01 \\ |f(x) - L| &< 0.01. \end{aligned}$$

38. $\lim_{x\to 4}(x^2 + 6) = 4^2 + 6 = 22 = L$

$$\begin{aligned} |(x^2 + 6) - 22| &< 0.01 \\ |x^2 - 16| &< 0.01 \\ |(x + 4)(x - 4)| &< 0.01 \\ |x - 4| &< \frac{0.01}{|x + 4|} \end{aligned}$$

If you assume $3 < x < 5$, then $\delta = \frac{0.01}{9} \approx 0.00111$.

So, if $0 < |x - 4| < \delta \approx \frac{0.01}{9}$, you have

$$|x - 4| < \frac{0.01}{9} < \frac{0.01}{|x + 4|}$$

$$\begin{aligned} |(x + 4)(x - 4)| &< 0.01 \\ |x^2 - 16| &< 0.01 \\ |(x^2 + 6) - 22| &< 0.01 \\ |f(x) - L| &< 0.01. \end{aligned}$$

39. $\lim_{x\to 4}(x + 2) = 4 + 2 = 6$

Given $\varepsilon > 0$:

$$\left|(x + 2) - 6\right| < \varepsilon$$
$$\left|x - 4\right| < \varepsilon = \delta$$

So, let $\delta = \varepsilon$. So, if $0 < \left|x - 4\right| < \delta = \varepsilon$, you have

$$\left|x - 4\right| < \varepsilon$$
$$\left|(x + 2) - 6\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

40. $\lim_{x\to -2}(4x + 5) = 4(-2) + 5 = -3$

Given $\varepsilon > 0$:

$$\left|(4x + 5) - (-3)\right| < \varepsilon$$
$$\left|4x + 8\right| < \varepsilon$$
$$4\left|x + 2\right| < \varepsilon$$
$$\left|x + 2\right| < \frac{\varepsilon}{4} = \delta$$

So, let $\delta = \frac{\varepsilon}{4}$.

So, if $0 < \left|x + 2\right| < \delta = \frac{\varepsilon}{4}$, you have

$$\left|x + 2\right| < \frac{\varepsilon}{4}$$
$$\left|4x + 8\right| < \varepsilon$$
$$\left|(4x + 5) - (-3)\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

41. $\lim_{x\to -4}\left(\frac{1}{2}x - 1\right) = \frac{1}{2}(-4) - 1 = -3$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{1}{2}x - 1\right) - (-3)\right| < \varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\tfrac{1}{2}\left|x - (-4)\right| < \varepsilon$$
$$\left|x - (-4)\right| < 2\varepsilon$$

So, let $\delta = 2\varepsilon$.

So, if $0 < \left|x - (-4)\right| < \delta = 2\varepsilon$, you have

$$\left|x - (-4)\right| < 2\varepsilon$$
$$\left|\tfrac{1}{2}x + 2\right| < \varepsilon$$
$$\left|\left(\tfrac{1}{2}x - 1\right) + 3\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

42. $\lim_{x\to 3}\left(\frac{3}{4}x + 1\right) = \frac{3}{4}(3) + 1 = \frac{13}{4}$

Given $\varepsilon > 0$:

$$\left|\left(\tfrac{3}{4}x + 1\right) - \tfrac{13}{4}\right| < \varepsilon$$
$$\left|\tfrac{3}{4}x - \tfrac{9}{4}\right| < \varepsilon$$
$$\tfrac{3}{4}\left|x - 3\right| < \varepsilon$$
$$\left|x - 3\right| < \tfrac{4}{3}\varepsilon$$

So, let $\delta = \frac{4}{3}\varepsilon$.

So, if $0 < \left|x - 3\right| < \delta = \frac{4}{3}\varepsilon$, you have

$$\left|x - 3\right| < \tfrac{4}{3}\varepsilon$$
$$\tfrac{3}{4}\left|x - 3\right| < \varepsilon$$
$$\left|\tfrac{3}{4}x - \tfrac{9}{4}\right| < \varepsilon$$
$$\left|\left(\tfrac{3}{4}x + 1\right) - \tfrac{13}{4}\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

43. $\lim_{x\to 6} 3 = 3$

Given $\varepsilon > 0$:

$$\left|3 - 3\right| < \varepsilon$$
$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$\left|3 - 3\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

44. $\lim_{x\to 2}(-1) = -1$

Given $\varepsilon > 0$: $\left|-1 - (-1)\right| < \varepsilon$

$$0 < \varepsilon$$

So, any $\delta > 0$ will work.

So, for any $\delta > 0$, you have

$$\left|(-1) - (-1)\right| < \varepsilon$$
$$\left|f(x) - L\right| < \varepsilon.$$

45. $\lim_{x\to 0} \sqrt[3]{x} = 0$

Given $\varepsilon > 0$: $\left|\sqrt[3]{x} - 0\right| < \varepsilon$

$$\left|\sqrt[3]{x}\right| < \varepsilon$$

$$|x| < \varepsilon^3 = \delta$$

So, let $\delta = \varepsilon^3$.

So, for $0|x - 0|\delta = \varepsilon^3$, you have

$$|x| < \varepsilon^3$$

$$\left|\sqrt[3]{x}\right| < \varepsilon$$

$$\left|\sqrt[3]{x} - 0\right| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

46. $\lim_{x\to 4} \sqrt{x} = \sqrt{4} = 2$

Given $\varepsilon > 0$: $\left|\sqrt{x} - 2\right| < \varepsilon$

$$\left|\sqrt{x} - 2\right|\left|\sqrt{x} + 2\right| < \varepsilon\left|\sqrt{x} + 2\right|$$

$$|x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$

Assuming $1 < x < 9$, you can choose $\delta = 3\varepsilon$. Then,

$$0 < |x - 4| < \delta = 3\varepsilon \Rightarrow |x - 4| < \varepsilon\left|\sqrt{x} + 2\right|$$

$$\Rightarrow \left|\sqrt{x} - 2\right| < \varepsilon.$$

47. $\lim_{x\to -5}|x - 5| = |(-5) - 5| = |-10| = 10$

Given $\varepsilon > 0$: $\big||x - 5| - 10\big| < \varepsilon$

$$|-(x - 5) - 10| < \varepsilon \qquad (x - 5 < 0)$$

$$|-x - 5| < \varepsilon$$

$$|x - (-5)| < \varepsilon$$

So, let $\delta = \varepsilon$.

So for $|x - (-5)| < \delta = \varepsilon$, you have

$$|-(x + 5)| < \varepsilon$$

$$|-(x - 5) - 10| < \varepsilon$$

$$\big||x - 5| - 10\big| < \varepsilon \qquad (\text{because } x - 5 < 0)$$

$$|f(x) - L| < \varepsilon.$$

48. $\lim_{x\to 3}|x - 3| = |3 - 3| = 0$

Given $\varepsilon > 0$: $\big||x - 3| - 0\big| < \varepsilon$

$$|x - 3| < \varepsilon$$

So, let $\delta = \varepsilon$.

So, for $0 < |x - 3| < \delta = \varepsilon$, you have

$$|x - 3| < \varepsilon$$

$$\big||x - 3| - 0\big| < \varepsilon$$

$$|f(x) - L| < \varepsilon.$$

49. $\lim_{x\to 1}\left(x^2 + 1\right) = 1^2 + 1 = 2$

Given $\varepsilon > 0$: $\left|\left(x^2 + 1\right) - 2\right| < \varepsilon$

$$\left|x^2 - 1\right| < \varepsilon$$

$$|(x + 1)(x - 1)| < \varepsilon$$

$$|x - 1| < \frac{\varepsilon}{|x + 1|}$$

If you assume $0 < x < 2$, then $\delta = \varepsilon/3$.

So for $0 < |x - 1| < \delta = \dfrac{\varepsilon}{3}$, you have

$$|x - 1| < \frac{1}{3}\varepsilon < \frac{1}{|x + 1|}\varepsilon$$

$$\left|x^2 - 1\right| < \varepsilon$$

$$\left|\left(x^2 + 1\right) - 2\right| < \varepsilon$$

$$|f(x) - 2| < \varepsilon.$$

50. $\lim_{x\to -4}\left(x^2 + 4x\right) = (-4)^2 + 4(-4) = 0$

Given $\varepsilon > 0$: $\left|\left(x^2 + 4x\right) - 0\right| < \varepsilon$

$$|x(x + 4)| < \varepsilon$$

$$|x + 4| < \frac{\varepsilon}{|x|}$$

If you assume $-5 < x < -3$, then $\delta = \dfrac{\varepsilon}{5}$.

So for $0 < |x - (-4)| < \delta = \dfrac{\varepsilon}{5}$, you have

$$|x + 4| < \frac{\varepsilon}{5} < \frac{1}{|x|}\varepsilon$$

$$|x(x + 4)| < \varepsilon$$

$$\left|\left(x^2 + 4x\right) - 0\right| < \varepsilon$$

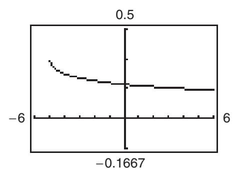
$$|f(x) - L| < \varepsilon.$$

51. $\lim_{x\to\pi} f(x) = \lim_{x\to\pi} 4 = 4$

52. $\lim_{x\to\pi} f(x) = \lim_{x\to\pi} x = \pi$

53. $f(x) = \dfrac{\sqrt{x+5} - 3}{x - 4}$

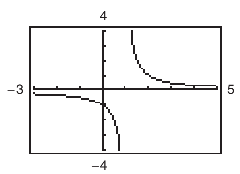
$\lim_{x\to 4} f(x) = \dfrac{1}{6}$



The domain is $[-5, 4) \cup (4, \infty)$. The graphing utility does not show the hole at $\left(4, \dfrac{1}{6}\right)$.

54. $f(x) = \dfrac{x - 3}{x^2 - 4x + 3}$

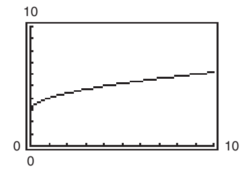
$\lim_{x\to 3} f(x) = \dfrac{1}{2}$



The domain is all $x \neq 1, 3$. The graphing utility does not show the hole at $\left(3, \dfrac{1}{2}\right)$.

55. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}$

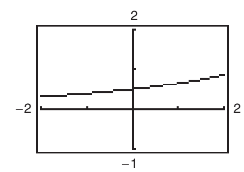
$\lim_{x\to 9} f(x) = 6$



The domain is all $x \geq 0$ except $x = 9$. The graphing utility does not show the hole at $(9, 6)$.

56. $f(x) = \dfrac{e^{x/2} - 1}{x}$

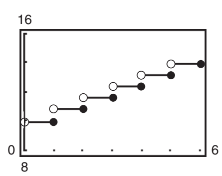
$\lim_{x\to 0} f(x) = \dfrac{1}{2}$



The domain is all $x \neq 0$. The graphing utility does not show the hole at $\left(0, \dfrac{1}{2}\right)$.

57. $C(t) = 9.99 - 0.79[\![-(t - 1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	11.57	12.36	12.36	12.36	12.36	12.36	12.36

$\lim_{t\to 3.5} C(t) = 12.36$

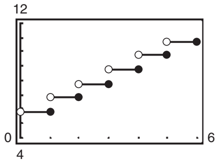
(c)

t	2	2.5	2.9	3	3.1	3.5	4
C	10.78	11.57	11.57	11.57	12.36	12.36	12.36

The $\lim_{t\to 3} C(t)$ does not exist because the values of C approach different values as t approaches 3 from both sides.

58. $C(t) = 5.79 - 0.99[\![-(t-1)]\!]$

(a)



(b)

t	3	3.3	3.4	3.5	3.6	3.7	4
C	7.77	8.76	8.76	8.76	8.76	8.76	8.76

$\lim_{t\to 3.5} C(t) = 8.76$

(c)

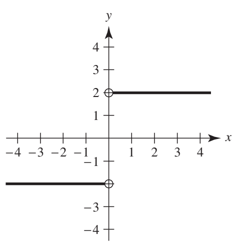
t	2	2.5	2.9	3	3.1	3.5	4
C	6.78	7.77	7.77	7.77	8.76	8.76	8.76

The limit $\lim_{t\to 3} C(t)$ does not exist because the values of C approach different values as t approaches 3 from both sides.

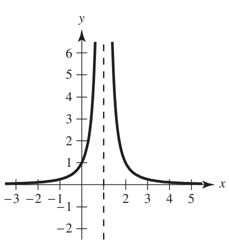
59. $\lim_{x\to 8} f(x) = 25$ means that the values of f approach 25 as x gets closer and closer to 8.

60. In the definition of $\lim_{x\to c} f(x)$, f must be defined on both sides of c, but does not have to be defined at c itself. The value of f at c has no bearing on the limit as x approaches c.

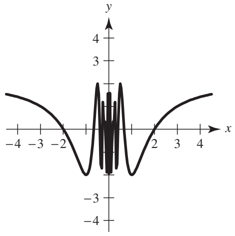
61. (i) The values of f approach different numbers as x approaches c from different sides of c:



(ii) The values of f increase without bound as x approaches c:



(iii) The values of f oscillate between two fixed numbers as x approaches c:



62. (a) No. The fact that $f(2) = 4$ has no bearing on the existence of the limit of $f(x)$ as x approaches 2.

(b) No. The fact that $\lim_{x\to 2} f(x) = 4$ has no bearing on the value of f at 2.

63. (a) $C = 2\pi r$

$$r = \frac{C}{2\pi} = \frac{6}{2\pi} = \frac{3}{\pi} \approx 0.9549 \text{ cm}$$

(b) When $C = 5.5$: $r = \dfrac{5.5}{2\pi} \approx 0.87535$ cm

When $C = 6.5$: $r = \dfrac{6.5}{2\pi} \approx 1.03451$ cm

So $0.87535 < r < 1.03451$.

(c) $\lim_{x\to 3/\pi} (2\pi r) = 6$; $\varepsilon = 0.5$; $\delta \approx 0.0796$

64. $V = \dfrac{4}{3}\pi r^3$, $V = 2.48$

(a) $2.48 = \dfrac{4}{3}\pi r^3$

$$r^3 = \frac{1.86}{\pi}$$

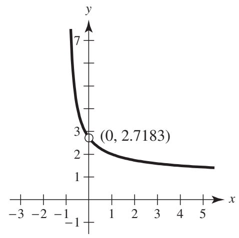
$$r \approx 0.8397 \text{ in.}$$

(b)
$$2.45 \le V \le 2.51$$
$$2.45 \le \frac{4}{3}\pi r^3 \le 2.51$$
$$0.5849 \le r^3 \le 0.5992$$
$$0.8363 \le r \le 0.8431$$

(c) For $\varepsilon = 2.51 - 2.48 = 0.03$, $\delta \approx 0.003$

65. $f(x) = (1+x)^{1/x}$

$$\lim_{x\to 0} (1+x)^{1/x} = e \approx 2.71828$$



x	$f(x)$
–0.1	2.867972
–0.01	2.731999
–0.001	2.719642
–0.0001	2.718418
–0.00001	2.718295
–0.000001	2.718283

x	$f(x)$
0.1	2.593742
0.01	2.704814
0.001	2.716942
0.0001	2.718146
0.00001	2.718268
0.000001	2.718280

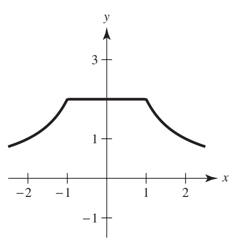
66. $f(x) = \dfrac{|x+1| - |x-1|}{x}$

x	–1	–0.5	–0.1	0	0.1	0.5	1.0
$f(x)$	2	2	2	Undef.	2	2	2

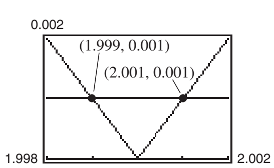
$\lim\limits_{x \to 0} f(x) = 2$

Note that for

$-1 < x < 1, x \neq 0, f(x) = \dfrac{(x+1) + (x-1)}{x} = 2.$



67.



Using the zoom and trace feature, $\delta = 0.001$. So $(2 - \delta, 2 + \delta) = (1.999, 2.001)$.

Note: $\dfrac{x^2 - 4}{x - 2} = x + 2$ for $x \neq 2$.

68. (a) $\lim\limits_{x \to c} f(x)$ exists for all $c \neq -3$.

(b) $\lim\limits_{x \to c} f(x)$ exists for all $c \neq -2, 0$.

69. False. The existence or nonexistence of $f(x)$ at $x = c$ has no bearing on the existence of the limit of $f(x)$ as $x \to c$.

70. True

71. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$f(2) = 0$

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2}(x - 4) = 2 \neq 0$

72. False. Let

$$f(x) = \begin{cases} x - 4, & x \neq 2 \\ 0, & x = 2 \end{cases}$$

$\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2}(x - 4) = 2$ and $f(2) = 0 \neq 2$

73. $f(x) = \sqrt{x}$

$\lim\limits_{x \to 0.25} \sqrt{x} = 0.5$ is true.

As x approaches $0.25 = \frac{1}{4}$ from either side, $f(x) = \sqrt{x}$ approaches $\frac{1}{2} = 0.5$.

74. $f(x) = \sqrt{x}$

$\lim\limits_{x \to 0} \sqrt{x} = 0$ is false.

$f(x) = \sqrt{x}$ is not defined on an open interval containing 0 because the domain of f is $x \geq 0$.

75. Using a graphing utility, you see that

$\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$

$\lim\limits_{x \to 0} \dfrac{\sin 2x}{x} = 2$, etc.

So, $\lim\limits_{x \to 0} \dfrac{\sin nx}{x} = n$.

76. Using a graphing utility, you see that

$\lim\limits_{x \to 0} \dfrac{\tan x}{x} = 1$

$\lim\limits_{x \to 0} \dfrac{\tan 2x}{x} = 2$, etc.

So, $\lim\limits_{x \to 0} \dfrac{\tan(nx)}{x} = n$.

77. If $\lim_{x \to c} f(x) = L_1$ and $\lim_{x \to c} f(x) = L_2$, then for every $\varepsilon > 0$, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that $|x - c| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$ and $|x - c| < \delta_2 \Rightarrow |f(x) - L_2| < \varepsilon$. Let δ equal the smaller of δ_1 and δ_2. Then for $|x - c| < \delta$, you have $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \le |L_1 - f(x)| + |f(x) - L_2| < \varepsilon + \varepsilon$. Therefore, $|L_1 - L_2| < 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $L_1 = L_2$.

78. $f(x) = mx + b, m \ne 0$. Let $\varepsilon > 0$ be given. Take

$$\delta = \frac{\varepsilon}{|m|}.$$

If $0 < |x - c| < \delta = \dfrac{\varepsilon}{|m|}$, then

$$|m||x - c| < \varepsilon$$
$$|mx - mc| < \varepsilon$$
$$|(mx + b) - (mc + b)| < \varepsilon$$

which shows that $\lim_{x \to c} (mx + b) = mc + b$.

79. $\lim_{x \to c} [f(x) - L] = 0$ means that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if

$0 < |x - c| < \delta$,

then

$|(f(x) - L) - 0| < \varepsilon$.

This means the same as $|f(x) - L| < \varepsilon$ when

$0 < |x - c| < \delta$.

So, $\lim_{x \to c} f(x) = L$.

80. (a)
$$\begin{aligned}(3x + 1)(3x - 1)x^2 + 0.01 &= (9x^2 - 1)x^2 + \frac{1}{100}\\ &= 9x^4 - x^2 + \frac{1}{100}\\ &= \frac{1}{100}(10x^2 - 1)(90x^2 - 1)\end{aligned}$$

So, $(3x + 1)(3x - 1)x^2 + 0.01 > 0$ if

$10x^2 - 1 < 0$ and $90x^2 - 1 < 0$.

Let $(a, b) = \left(-\dfrac{1}{\sqrt{90}}, \dfrac{1}{\sqrt{90}}\right)$.

For all $x \ne 0$ in (a, b), the graph is positive. You can verify this with a graphing utility.

(b) You are given $\lim_{x \to c} g(x) = L > 0$. Let $\varepsilon = \frac{1}{2}L$. There exists $\delta > 0$ such that $0 < |x - c| < \delta$ implies that $|g(x) - L| < \varepsilon = \dfrac{L}{2}$. That is,

$$-\frac{L}{2} < g(x) - L < \frac{L}{2}$$
$$\frac{L}{2} < g(x) < \frac{3L}{2}$$

For x in the interval $(c - \delta, c + \delta)$, $x \ne c$, you have $g(x) > \dfrac{L}{2} > 0$, as desired.

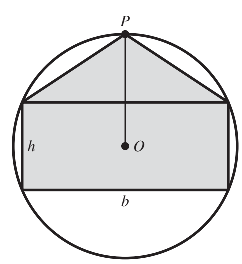
81. The radius OP has a length equal to the altitude z of the triangle plus $\dfrac{h}{2}$. So, $z = 1 - \dfrac{h}{2}$.

Area triangle $= \dfrac{1}{2}b\left(1 - \dfrac{h}{2}\right)$

Area rectangle $= bh$

Because these are equal, $\dfrac{1}{2}b\left(1 - \dfrac{h}{2}\right) = bh$

$$1 - \frac{h}{2} = 2h$$
$$\frac{5}{2}h = 1$$
$$h = \frac{2}{5}.$$



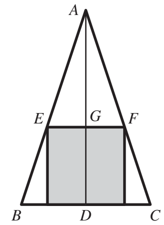
82. Consider a cross section of the cone, where EF is a diagonal of the inscribed cube. $AD = 3$, $BC = 2$.

Let x be the length of a side of the cube. Then $EF = x\sqrt{2}$.

By similar triangles,

$$\frac{EF}{BC} = \frac{AG}{AD}$$

$$\frac{x\sqrt{2}}{2} = \frac{3 - x}{3}$$



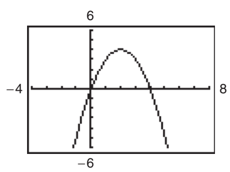
Solving for x, $3\sqrt{2}x = 6 - 2x$

$$\left(3\sqrt{2} + 2\right)x = 6$$

$$x = \frac{6}{3\sqrt{2} + 2} = \frac{9\sqrt{2} - 6}{7} \approx 0.96.$$

Section 2.3 Evaluating Limits Analytically

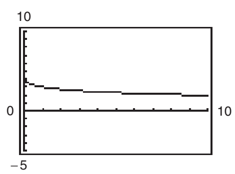
1.



(a) $\lim_{x\to 4} h(x) = 0$

(b) $\lim_{x\to -1} h(x) = -5$

2.

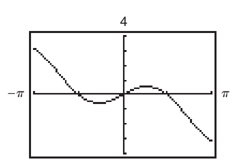


$$g(x) = \frac{12\left(\sqrt{x} - 3\right)}{x - 9}$$

(a) $\lim_{x\to 4} g(x) = 2.4$

(b) $\lim_{x\to 0} g(x) = 4$

3.



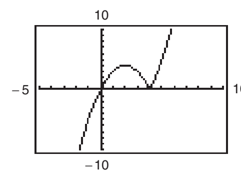
$f(x) = x\cos x$

(a) $\lim_{x\to 0} f(x) = 0$

(b) $\lim_{x\to \pi/3} f(x) \approx 0.524$

$\left(= \dfrac{\pi}{6}\right)$

4.



$f(t) = t\left|t - 4\right|$

(a) $\lim_{t\to 4} f(t) = 0$

(b) $\lim_{t\to -1} f(t) = -5$

5. $\lim_{x\to 2} x^3 = 2^3 = 8$

6. $\lim_{x\to -3} x^4 = (-3)^4 = 81$

7. $\lim_{x\to 0} (2x - 1) = 2(0) - 1 = -1$

8. $\lim_{x\to -4} (2x + 3) = 2(-4) + 3 = -8 + 3 = -5$

9. $\lim_{x\to -3} \left(x^2 + 3x\right) = (-3)^2 + 3(-3) = 9 - 9 = 0$

10. $\lim_{x\to 2} \left(-x^3 + 1\right) = (-2)^3 + 1 = -8 + 1 = -7$

11. $\lim_{x\to -3} \left(2x^2 + 4x + 1\right) = 2(-3)^2 + 4(-3) + 1$

$= 18 - 12 + 1 = 7$

12. $\lim_{x\to 1} \left(2x^3 - 6x + 5\right) = 2(1)^3 - 6(1) + 5$

$= 2 - 6 + 5 = 1$

13. $\lim_{x\to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

14. $\lim_{x\to 2} \sqrt[3]{12x + 3} = \sqrt[3]{12(2) + 3}$

$= \sqrt[3]{24 + 3} = \sqrt[3]{27} = 3$

15. $\lim_{x\to -4}(x+3)^2 = (-4+3)^2 = 1$

16. $\lim_{x\to 0}(3x-2)^4 = (3(0)-2)^4 = (-2)^4 = 16$

17. $\lim_{x\to 2}\frac{1}{x} = \frac{1}{2}$

18. $\lim_{x\to -5}\frac{5}{x+3} = \frac{5}{-5+3} = -\frac{5}{2}$

19. $\lim_{x\to 1}\frac{x}{x^2+4} = \frac{1}{1^2+4} = \frac{1}{5}$

20. $\lim_{x\to 1}\frac{3x+5}{x+1} = \frac{3(1)+5}{1+1} = \frac{3+5}{2} = \frac{8}{2} = 4$

21. $\lim_{x\to 7}\frac{3x}{\sqrt{x+2}} = \frac{3(7)}{\sqrt{7+2}} = \frac{21}{3} = 7$

22. $\lim_{x\to 3}\frac{\sqrt{x+6}}{x+2} = \frac{\sqrt{3+6}}{3+2} = \frac{\sqrt{9}}{5} = \frac{3}{5}$

23. $\lim_{x\to \pi/2}\sin x = \sin\frac{\pi}{2} = 1$

24. $\lim_{x\to \pi}\tan x = \tan\pi = 0$

25. $\lim_{x\to 1}\cos\frac{\pi x}{3} = \cos\frac{\pi}{3} = \frac{1}{2}$

26. $\lim_{x\to 2}\sin\frac{\pi x}{2} = \sin\frac{\pi(2)}{2} = 0$

27. $\lim_{x\to 0}\sec 2x = \sec 0 = 1$

28. $\lim_{x\to \pi}\cos 3x = \cos 3\pi = -1$

29. $\lim_{x\to 5\pi/6}\sin x = \sin\frac{5\pi}{6} = \frac{1}{2}$

30. $\lim_{x\to 5\pi/3}\cos x = \cos\frac{5\pi}{3} = \frac{1}{2}$

31. $\lim_{x\to 3}\tan\left(\frac{\pi x}{4}\right) = \tan\frac{3\pi}{4} = -1$

32. $\lim_{x\to 7}\sec\left(\frac{\pi x}{6}\right) = \sec\frac{7\pi}{6} = \frac{-2\sqrt{3}}{3}$

33. $\lim_{x\to 0}e^x\cos 2x = e^0\cos 0 = 1$

34. $\lim_{x\to 0}e^{-x}\sin\pi x = e^0\sin 0 = 0$

35. $\lim_{x\to 1}\left(\ln 3x + e^x\right) = \ln 3 + e$

36. $\lim_{x\to 1}\ln\left(\frac{x}{e^x}\right) = \ln\left(\frac{1}{e}\right) = \ln e^{-1} = -1$

37. (a) $\lim_{x\to 1}f(x) = 5 - 1 = 4$

(b) $\lim_{x\to 4}g(x) = 4^3 = 64$

(c) $\lim_{x\to 1}g(f(x)) = g(f(1)) = g(4) = 64$

38. (a) $\lim_{x\to -3}f(x) = (-3) + 7 = 4$

(b) $\lim_{x\to 4}g(x) = 4^2 = 16$

(c) $\lim_{x\to -3}g(f(x)) = g(4) = 16$

39. (a) $\lim_{x\to 1}f(x) = 4 - 1 = 3$

(b) $\lim_{x\to 3}g(x) = \sqrt{3+1} = 2$

(c) $\lim_{x\to 1}g(f(x)) = g(3) = 2$

40. (a) $\lim_{x\to 4}f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x\to 21}g(x) = \sqrt[3]{21+6} = 3$

(c) $\lim_{x\to 4}g(f(x)) = g(21) = 3$

41. (a) $\lim_{x\to c}[5g(x)] = 5\lim_{x\to c}g(x) = 5(2) = 10$

(b) $\lim_{x\to c}[f(x)+g(x)] = \lim_{x\to c}f(x) + \lim_{x\to c}g(x) = 3 + 2 = 5$

(c) $\lim_{x\to c}[f(x)g(x)] = \left[\lim_{x\to c}f(x)\right]\left[\lim_{x\to c}g(x)\right] = (3)(2) = 6$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)} = \frac{3}{2}$

42. (a) $\lim_{x\to c}\left[4f(x)\right] = 4\lim_{x\to c} f(x) = 4(2) = 8$

(b) $\lim_{x\to c}\left[f(x) + g(x)\right] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = 2 + \frac{3}{4} = \frac{11}{4}$

(c) $\lim_{x\to c}\left[f(x)g(x)\right] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = 2\left(\frac{3}{4}\right) = \frac{3}{2}$

(d) $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \frac{2}{(3/4)} = \frac{8}{3}$

43. (a) $\lim_{x\to c}\left[f(x)\right]^3 = \left[\lim_{x\to c} f(x)\right]^3 = (4)^3 = 64$

(b) $\lim_{x\to c}\sqrt{f(x)} = \sqrt{\lim_{x\to c} f(x)} = \sqrt{4} = 2$

(c) $\lim_{x\to c}\left[3f(x)\right] = 3\lim_{x\to c} f(x) = 3(4) = 12$

(d) $\lim_{x\to c}\left[f(x)\right]^{3/2} = \left[\lim_{x\to c} f(x)\right]^{3/2} = (4)^{3/2} = 8$

44. (a) $\lim_{x\to c}\sqrt[3]{f(x)} = \sqrt[3]{\lim_{x\to c} f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x\to c}\frac{f(x)}{18} = \frac{\lim_{x\to c} f(x)}{\lim_{x\to c} 18} = \frac{27}{18} = \frac{3}{2}$

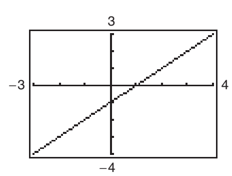
(c) $\lim_{x\to c}\left[f(x)\right]^2 = \left[\lim_{x\to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x\to c}\left[f(x)\right]^{2/3} = \left[\lim_{x\to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

45. $f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1}$ and

$g(x) = x - 1$ agree except at $x = -1$.

$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = \lim_{x\to -1}(x - 1) = -1 - 1 = -2$

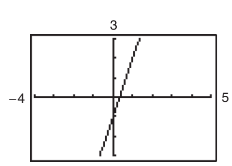


46. $f(x) = \frac{3x^2 + 5x - 2}{x + 2} = \frac{(x + 2)(3x - 1)}{x + 2}$ and

$g(x) = 3x - 1$ agree except at $x = -2$.

$\lim_{x\to -2} f(x) = \lim_{x\to -2} g(x) = \lim_{x\to -2}(3x - 1)$

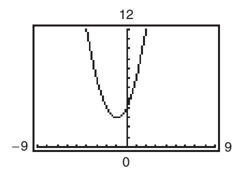
$= 3(-2) - 1 = -7$



47. $f(x) = \frac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except at $x = 2$.

$\lim_{x\to 2} f(x) = \lim_{x\to 2} g(x) = \lim_{x\to 2}\left(x^2 + 2x + 4\right)$

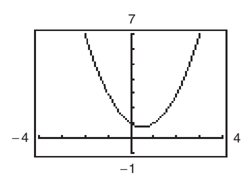
$= 2^2 + 2(2) + 4 = 12$



48. $f(x) = \frac{x^3 + 1}{x + 1}$ and $g(x) = x^2 - x + 1$ agree except at $x = -1$.

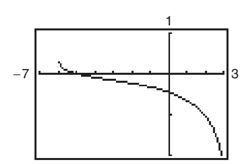
$\lim_{x\to -1} f(x) = \lim_{x\to -1} g(x) = \lim_{x\to -1}\left(x^2 - x + 1\right)$

$= (-1)^2 - (-1) + 1 = 3$



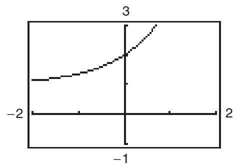
49. $f(x) = \frac{(x + 4)\ln(x + 6)}{x^2 - 16}$ and $g(x) = \frac{\ln(x + 6)}{x - 4}$ agree except at $x = -4$.

$\lim_{x\to -4} f(x) = \lim_{x\to -4} g(x) = \frac{\ln 2}{-8} \approx -0.0866$



50. $f(x) = \dfrac{e^{2x} - 1}{e^x - 1}$ and $g(x) = e^x + 1$ agree except at $x = 0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = e^0 + 1 = 2$$



51. $\displaystyle\lim_{x\to 0}\frac{x}{x^2 - x} = \lim_{x\to 0}\frac{x}{x(x-1)} = \lim_{x\to 0}\frac{1}{x-1} = \frac{1}{0-1} = -1$

52. $\displaystyle\lim_{x\to 0}\frac{2x}{x^2 + 4x} = \lim_{x\to 0}\frac{2x}{x(x+4)} = \lim_{x\to 0}\frac{2}{x+4}$

$$= \frac{2}{0+4} = \frac{2}{4} = \frac{1}{2}$$

53. $\displaystyle\lim_{x\to 4}\frac{x-4}{x^2 - 16} = \lim_{x\to 4}\frac{x-4}{(x+4)(x-4)}$

$$= \lim_{x\to 4}\frac{1}{x+4} = \frac{1}{4+4} = \frac{1}{8}$$

54. $\displaystyle\lim_{x\to 5}\frac{5-x}{x^2 - 25} = \lim_{x\to 5}\frac{-(x-5)}{(x-5)(x+5)}$

$$= \lim_{x\to 5}\frac{-1}{x+5} = \frac{-1}{5+5} = -\frac{1}{10}$$

55. $\displaystyle\lim_{x\to -3}\frac{x^2 + x - 6}{x^2 - 9} = \lim_{x\to -3}\frac{(x+3)(x-2)}{(x+3)(x-3)}$

$$= \lim_{x\to -3}\frac{x-2}{x-3} = \frac{-3-2}{-3-3} = \frac{-5}{-6} = \frac{5}{6}$$

56. $\displaystyle\lim_{x\to 2}\frac{x^2 + 2x - 8}{x^2 - x - 2} = \lim_{x\to 2}\frac{(x-2)(x+4)}{(x-2)(x+1)}$

$$= \lim_{x\to 2}\frac{x+4}{x+1} = \frac{2+4}{2+1} = \frac{6}{3} = 2$$

57. $\displaystyle\lim_{x\to 4}\frac{\sqrt{x+5} - 3}{x-4} = \lim_{x\to 4}\frac{\sqrt{x+5} - 3}{x-4}\cdot\frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3}$

$$= \lim_{x\to 4}\frac{(x+5) - 9}{(x-4)\left(\sqrt{x+5} + 3\right)} = \lim_{x\to 4}\frac{1}{\sqrt{x+5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

58. $\displaystyle\lim_{x\to 3}\frac{\sqrt{x+1} - 2}{x-3} = \lim_{x\to 3}\frac{\sqrt{x+1} - 2}{x-3}\cdot\frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} = \lim_{x\to 3}\frac{x-3}{(x-3)\left[\sqrt{x+1} + 2\right]}$

$$= \lim_{x\to 3}\frac{1}{\sqrt{x+1} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

59. $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+5} - \sqrt{5}}{x} = \lim_{x\to 0}\frac{\sqrt{x+5} - \sqrt{5}}{x}\cdot\frac{\sqrt{x+5} + \sqrt{5}}{\sqrt{x+5} + \sqrt{5}}$

$$= \lim_{x\to 0}\frac{(x+5) - 5}{x\left(\sqrt{x+5} + \sqrt{5}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+5} + \sqrt{5}} = \frac{1}{\sqrt{5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$$

60. $\displaystyle\lim_{x\to 0}\frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{2+x} - \sqrt{2}}{x}\cdot\frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}}$

$$= \lim_{x\to 0}\frac{2 + x - 2}{\left(\sqrt{2+x} + \sqrt{2}\right)x} = \lim_{x\to 0}\frac{1}{\sqrt{2+x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

61. $\displaystyle\lim_{x\to 0}\frac{\frac{1}{3+x} - \frac{1}{3}}{x} = \lim_{x\to 0}\frac{3 - (3+x)}{(3+x)3(x)} = \lim_{x\to 0}\frac{-x}{(3+x)(3)(x)} = \lim_{x\to 0}\frac{-1}{(3+x)3} = \frac{-1}{(3)3} = -\frac{1}{9}$

62. $\displaystyle\lim_{x\to 0}\frac{\frac{1}{x+4} - \frac{1}{4}}{x} = \lim_{x\to 0}\frac{\frac{4 - (x+4)}{4(x+4)}}{x}$

$$= \lim_{x\to 0}\frac{-1}{4(x+4)} = \frac{-1}{4(4)} = -\frac{1}{16}$$

63. $\lim_{\Delta x \to 0} \frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2$

64. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0}(2x + \Delta x) = 2x$

65.
$$\begin{aligned}\lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x} \\ &= \lim_{\Delta x \to 0}(2x + \Delta x - 2) = 2x - 2\end{aligned}$$

66.
$$\begin{aligned}\lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} &= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x\left(3x^2 + 3x\Delta x + (\Delta x)^2\right)}{\Delta x} = \lim_{\Delta x \to 0}\left(3x^2 + 3x\Delta x + (\Delta x)^2\right) = 3x^2\end{aligned}$$

67. $\lim_{x \to 0} \frac{\sin x}{5x} = \lim_{x \to 0}\left[\left(\frac{\sin x}{x}\right)\left(\frac{1}{5}\right)\right] = (1)\left(\frac{1}{5}\right) = \frac{1}{5}$

68. $\lim_{x \to 0} \frac{3(1 - \cos x)}{x} = \lim_{x \to 0}\left[3\left(\frac{(1 - \cos x)}{x}\right)\right] = (3)(0) = 0$

69.
$$\begin{aligned}\lim_{x \to 0} \frac{\sin x(1 - \cos x)}{x^2} &= \lim_{x \to 0}\left[\frac{\sin x}{x} \cdot \frac{1 - \cos x}{x}\right] \\ &= (1)(0) = 0\end{aligned}$$

70. $\lim_{\theta \to 0} \frac{\cos \theta \tan \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$

71. $\lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0}\left[\frac{\sin x}{x} \sin x\right] = (1) \sin 0 = 0$

72.
$$\begin{aligned}\lim_{x \to 0} \frac{\tan^2 x}{x} = \lim_{x \to 0} \frac{\sin^2 x}{x \cos^2 x} &= \lim_{x \to 0}\left[\frac{\sin x}{x} \cdot \frac{\sin x}{\cos^2 x}\right] \\ &= (1)(0) = 0\end{aligned}$$

73.
$$\begin{aligned}\lim_{h \to 0} \frac{(1 - \cos h)^2}{h} &= \lim_{h \to 0}\left[\frac{1 - \cos h}{h}(1 - \cos h)\right] \\ &= (0)(0) = 0\end{aligned}$$

74. $\lim_{\phi \to \pi} \phi \sec \phi = \pi(-1) = -\pi$

75. $\lim_{x \to \pi/2} \frac{\cos x}{\cot x} = \lim_{x \to \pi/2} \sin x = 1$

76.
$$\begin{aligned}\lim_{x \to \pi/4} \frac{1 - \tan x}{\sin x - \cos x} &= \lim_{x \to \pi/4} \frac{\cos x - \sin x}{\sin x \cos x - \cos^2 x} \\ &= \lim_{x \to \pi/4} \frac{-(\sin x - \cos x)}{\cos x(\sin x - \cos x)} \\ &= \lim_{x \to \pi/4} \frac{-1}{\cos x} \\ &= \lim_{x \to \pi/4}(-\sec x) \\ &= -\sqrt{2}\end{aligned}$$

77.
$$\begin{aligned}\lim_{x \to 0} \frac{1 - e^{-x}}{e^x - 1} &= \lim_{x \to 0} \frac{1 - e^{-x}}{e^x - 1} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x \to 0} \frac{(1 - e^{-x})e^{-x}}{1 - e^{-x}} \\ &= \lim_{x \to 0} e^{-x} = 1\end{aligned}$$

78.
$$\begin{aligned}\lim_{x \to 0} \frac{4(e^{2x} - 1)}{e^x - 1} &= \lim_{x \to 0} \frac{4(e^x - 1)(e^x + 1)}{e^x - 1} \\ &= \lim_{x \to 0} 4(e^x + 1) = 4(2) = 8\end{aligned}$$

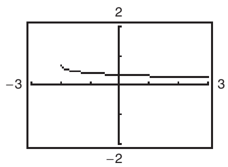
79. $\lim_{t \to 0} \frac{\sin 3t}{2t} = \lim_{t \to 0}\left(\frac{\sin 3t}{3t}\right)\left(\frac{3}{2}\right) = (1)\left(\frac{3}{2}\right) = \frac{3}{2}$

80.
$$\begin{aligned}\lim_{x \to 0} \frac{\sin 2x}{\sin 3x} &= \lim_{x \to 0}\left[2\left(\frac{\sin 2x}{2x}\right)\left(\frac{1}{3}\right)\left(\frac{3x}{\sin 3x}\right)\right] \\ &= 2(1)\left(\frac{1}{3}\right)(1) = \frac{2}{3}\end{aligned}$$

81. $f(x) = \dfrac{\sqrt{x+2} - \sqrt{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	?	0.354	0.353	0.349

It appears that the limit is 0.354.



The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{x+2}-\sqrt{2}}{x}\cdot\frac{\sqrt{x+2}+\sqrt{2}}{\sqrt{x+2}+\sqrt{2}}$

$$= \lim_{x\to 0}\frac{x+2-2}{x\left(\sqrt{x+2}+\sqrt{2}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+2}+\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

82. $f(x) = \dfrac{4-\sqrt{x}}{x-16}$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	–0.1252	–0.125	–0.125	?	–0.125	–0.125	–0.1248

It appears that the limit is –0.125.

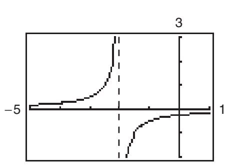
The graph has a hole at $x = 16$.

Analytically, $\displaystyle\lim_{x\to 16}\frac{4-\sqrt{x}}{x-16} = \lim_{x\to 16}\frac{\left(4-\sqrt{x}\right)}{\left(\sqrt{x}+4\right)\left(\sqrt{x}-4\right)} = \lim_{x\to 16}\frac{-1}{\sqrt{x}+4} = -\frac{1}{8}.$

83. $f(x) = \dfrac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.263	–0.251	–0.250	?	–0.250	–0.249	–0.238

It appears that the limit is –0.250.



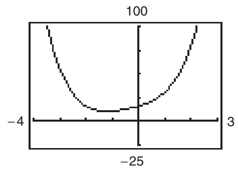
The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x\to 0}\frac{\dfrac{1}{2+x}-\dfrac{1}{2}}{x} = \lim_{x\to 0}\frac{2-(2+x)}{2(2+x)}\cdot\frac{1}{x} = \lim_{x\to 0}\frac{-x}{2(2+x)}\cdot\frac{1}{x} = \lim_{x\to 0}\frac{-1}{2(2+x)} = -\frac{1}{4}.$

84. $f(x) = \dfrac{x^5 - 32}{x - 2}$

x	1.9	1.99	1.999	1.9999	2.0	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41

It appears that the limit is 80.



The graph has a hole at $x = 2$.

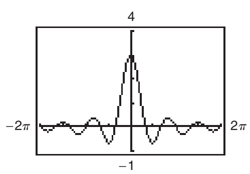
Analytically, $\lim\limits_{x \to 2} \dfrac{x^5 - 32}{x - 2} = \lim\limits_{x \to 2} \dfrac{(x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} = \lim\limits_{x \to 2}(x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$

(*Hint:* Use long division to factor $x^5 - 32$.)

85. $f(t) = \dfrac{\sin 3t}{t}$

t	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

It appears that the limit is 3.



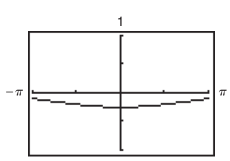
The graph has a hole at $t = 0$.

Analytically, $\lim\limits_{t \to 0} \dfrac{\sin 3t}{t} = \lim\limits_{t \to 0} 3\left(\dfrac{\sin 3t}{3t}\right) = 3(1) = 3.$

86. $f(x) = \dfrac{\cos x - 1}{2x^2}$

x	–1	–0.1	–0.01	0.01	0.1	1
$f(x)$	–0.2298	–0.2498	–0.25	–0.25	–0.2498	–0.2298

It appears that the limit is –0.25.



The graph has a hole at $x = 0$.

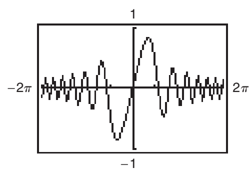
Analytically, $\dfrac{\cos x - 1}{2x^2} \cdot \dfrac{\cos x + 1}{\cos x + 1} = \dfrac{\cos^2 x - 1}{2x^2(\cos x + 1)} = \dfrac{-\sin^2 x}{2x^2(\cos x + 1)} = \dfrac{\sin^2 x}{x^2} \cdot \dfrac{-1}{2(\cos x + 1)}$

$$\lim_{x \to 0}\left[\frac{\sin^2 x}{x^2} \cdot \frac{-1}{2(\cos x + 1)}\right] = 1\left(\frac{-1}{4}\right) = -\frac{1}{4} = -0.25$$

87. $f(x) = \dfrac{\sin x^2}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.099998	–0.01	–0.001	?	0.001	0.01	0.099998

It appears that the limit is 0.



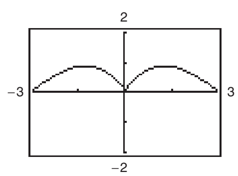
The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0}\dfrac{\sin x^2}{x} = \lim\limits_{x\to 0} x\left(\dfrac{\sin x^2}{x}\right) = 0(1) = 0.$

88. $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

It appears that the limit is 0.

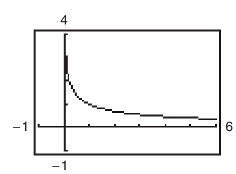


The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt[3]{x}} = \lim\limits_{x\to 0}\sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$

89. $f(x) = \dfrac{\ln x}{x - 1}$

x	0.5	0.9	0.99	1.01	1.1	1.5
$f(x)$	1.3863	1.0536	1.0050	0.9950	0.9531	0.8109

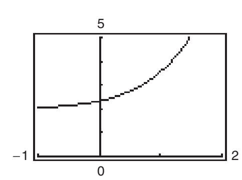


It appears that the limit is 1.

Analytically, $\lim\limits_{x\to 1}\dfrac{\ln x}{x - 1} = 1.$

90. $f(x) = \dfrac{e^{3x} - 8}{e^{2x} - 4}$

x	0.5	0.6	0.69	0.70	0.8	0.9
$f(x)$	2.7450	2.8687	2.9953	3.0103	3.1722	3.3565



It appears that the limit is 3.

Analytically, $\lim\limits_{x\to \ln 2}\dfrac{e^{3x} - 8}{e^{2x} - 4} = \lim\limits_{x\to \ln 2}\dfrac{(e^x - 2)(e^{2x} + 2e^x + 4)}{(e^x - 2)(e^x + 2)} = \lim\limits_{x\to \ln 2}\dfrac{e^{2x} + 2e^x + 4}{e^x + 2} = \dfrac{4 + 4 + 4}{2 + 2} = 3.$

91. $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{3(x+\Delta x)-2-(3x-2)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{3x+3\Delta x-2-3x+2}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{3\Delta x}{\Delta x} = 3$

92. $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{(x+\Delta x)^2-4(x+\Delta x)-(x^2-4x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{x^2+2x\Delta x+\Delta x^2-4x-4\Delta x-x^2+4x}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{\Delta x(2x+\Delta x-4)}{\Delta x} = \lim\limits_{\Delta x\to 0}(2x+\Delta x-4) = 2x-4$

93. $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\dfrac{1}{x+\Delta x+3}-\dfrac{1}{x+3}}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{x+3-(x+\Delta x+3)}{(x+\Delta x+3)(x+3)}\cdot\dfrac{1}{\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{-\Delta x}{(x+\Delta x+3)(x+3)\Delta x}$

$= \lim\limits_{\Delta x\to 0}\dfrac{-1}{(x+\Delta x+3)(x+3)} = \dfrac{-1}{(x+3)^2}$

94. $\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} = \lim\limits_{\Delta x\to 0}\dfrac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\cdot\dfrac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$

$= \lim\limits_{\Delta x\to 0}\dfrac{x+\Delta x-x}{\Delta x\left(\sqrt{x+\Delta x}+\sqrt{x}\right)} = \lim\limits_{\Delta x\to 0}\dfrac{1}{\sqrt{x+\Delta x}+\sqrt{x}} = \dfrac{1}{2\sqrt{x}}$

95. $\lim\limits_{x\to 0}(4-x^2) \le \lim\limits_{x\to 0}f(x) \le \lim\limits_{x\to 0}(4+x^2)$

$4 \le \lim\limits_{x\to 0}f(x) \le 4$

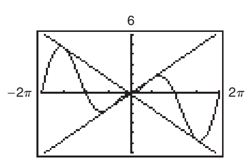
Therefore, $\lim\limits_{x\to 0}f(x) = 4$.

96. $\lim\limits_{x\to a}\left[b-|x-a|\right] \le \lim\limits_{x\to a}f(x) \le \lim\limits_{x\to a}\left[b+|x-a|\right]$

$b \le \lim\limits_{x\to a}f(x) \le b$

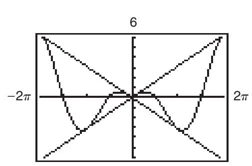
Therefore, $\lim\limits_{x\to a}f(x) = b$.

97. $f(x) = |x|\sin x$



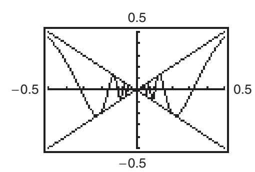
$\lim\limits_{x\to 0}|x|\sin x = 0$

98. $f(x) = |x|\cos x$



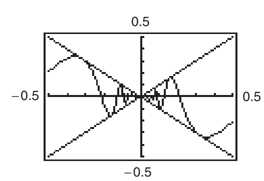
$\lim\limits_{x\to 0}|x|\cos x = 0$

99. $f(x) = x\sin\dfrac{1}{x}$



$\lim\limits_{x\to 0}\left(x\sin\dfrac{1}{x}\right) = 0$

100. $h(x) = x\cos\dfrac{1}{x}$



$\lim\limits_{x\to 0}\left(x\cos\dfrac{1}{x}\right) = 0$

101. (a) Two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

(b) $f(x) = \dfrac{x^2-1}{x-1} = \dfrac{(x+1)(x-1)}{x-1}$ and $g(x) = x+1$ agree at all points except $x = 1$.

(Other answers possible.)

102. An indeterminant form is obtained when evaluating a limit using direct substitution produces a meaningless fractional expression such as $0/0$. That is,

$$\lim_{x\to c}\frac{f(x)}{g(x)}$$

for which $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$

103. If a function f is squeezed between two functions h and g, $h(x) \le f(x) \le g(x)$, and h and g have the same limit L as $x \to c$, then $\lim_{x\to c} f(x)$ exists and equals L

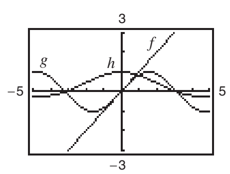
104. (a) Use the dividing out technique because the numerator and denominator have a common factor.

$$\lim_{x\to -2}\frac{x^2+x-2}{x+2} = \lim_{x\to -2}\frac{(x+2)(x-1)}{x+2}$$
$$= \lim_{x\to -2}(x-1) = -2-1 = -3$$

(b) Use the rationalizing technique because the numerator involves a radical expression.

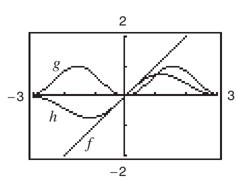
$$\lim_{x\to 0}\frac{\sqrt{x+4}-2}{x} = \lim_{x\to 0}\frac{\sqrt{x+4}-2}{x}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$
$$= \lim_{x\to 0}\frac{(x+4)-4}{x\left(\sqrt{x+4}+2\right)}$$
$$= \lim_{x\to 0}\frac{1}{\sqrt{x+4}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

105. $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$



When the x-values are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. So, $|g|/|f| \approx 1$ when x is "close to" 0.

106. $f(x) = x,\ g(x) = \sin^2 x,\ h(x) = \dfrac{\sin^2 x}{x}$



When the x-values are "close to" 0 the magnitude of g is "smaller" than the magnitude of f and the magnitude of g is approaching zero "faster" than the magnitude of f. So, $|g|/|f| \approx 0$ when x is "close to" 0.

107. $s(t) = -16t^2 + 500$

$$\lim_{t\to 2}\frac{s(2)-s(t)}{2-t} = \lim_{t\to 2}\frac{-16(2)^2+500-\left(-16t^2+500\right)}{2-t}$$
$$= \lim_{t\to 2}\frac{436+16t^2-500}{2-t}$$
$$= \lim_{t\to 2}\frac{16\left(t^2-4\right)}{2-t}$$
$$= \lim_{t\to 2}\frac{16(t-2)(t+2)}{2-t}$$
$$= \lim_{t\to 2} -16(t+2) = -64 \text{ ft/sec}$$

The paint can is falling at about 64 feet/second.

108. $s(t) = -16t^2 + 500 = 0$ when $t = \sqrt{\dfrac{500}{16}} = \dfrac{5\sqrt{5}}{2}$ sec. The velocity at time $a = \dfrac{5\sqrt{5}}{2}$ is

$$\begin{aligned}
\lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{s\left(\frac{5\sqrt{5}}{2}\right) - s(t)}{\frac{5\sqrt{5}}{2} - t} &= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{0 - \left(-16t^2 + 500\right)}{\frac{5\sqrt{5}}{2} - t} \\
&= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t^2 - \frac{125}{4}\right)}{\frac{5\sqrt{5}}{2} - t} \\
&= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t + \frac{5\sqrt{5}}{2}\right)\left(t - \frac{5\sqrt{5}}{2}\right)}{\frac{5\sqrt{5}}{2} - t} \\
&= \lim_{t\to\frac{5\sqrt{5}}{2}} \left[-16\left(t + \frac{5\sqrt{5}}{2}\right)\right] = -80\sqrt{5} \text{ ft/sec} \\
&\approx -178.9 \text{ ft/sec.}
\end{aligned}$$

The velocity of the paint can when it hits the ground is about 178.9 ft/sec.

109. $s(t) = -4.9t^2 + 200$

$$\begin{aligned}
\lim_{t\to 3} \frac{s(3) - s(t)}{3 - t} &= \lim_{t\to 3} \frac{-4.9(3)^2 + 200 - \left(-4.9t^2 + 200\right)}{3 - t} \\
&= \lim_{t\to 3} \frac{4.9\left(t^2 - 9\right)}{3 - t} \\
&= \lim_{t\to 3} \frac{4.9(t - 3)(t + 3)}{3 - t} \\
&= \lim_{t\to 3}\left[-4.9(t + 3)\right] \\
&= -29.4 \text{ m/sec}
\end{aligned}$$

The object is falling about 29.4 m/sec.

110. $-4.9t^2 + 200 = 0$ when $t = \sqrt{\dfrac{200}{4.9}} = \dfrac{20\sqrt{5}}{7}$ sec. The velocity at time $a = \dfrac{20\sqrt{5}}{7}$ is

$$\begin{aligned}
\lim_{t\to a} \frac{s(a) - s(t)}{a - t} &= \lim_{t\to a} \frac{0 - \left[-4.9t^2 + 200\right]}{a - t} \\
&= \lim_{t\to a} \frac{4.9(t + a)(t - a)}{a - t} \\
&= \lim_{t\to\frac{20\sqrt{5}}{7}} \left[-4.9\left(t + \frac{20\sqrt{5}}{7}\right)\right] = -28\sqrt{5} \text{ m/sec} \\
&\approx -62.6 \text{ m/sec.}
\end{aligned}$$

The velocity of the object when it hits the ground is about 62.6 m/sec.

111. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to 0} g(x)$ do not exist. However,

$$\lim_{x\to 0}\left[f(x) + g(x)\right] = \lim_{x\to 0}\left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x\to 0}[0] = 0$$

and therefore does not exist.

112. Suppose, on the contrary, that $\lim_{x \to c} g(x)$ exists. Then, because $\lim_{x \to c} f(x)$ exists, so would $\lim_{x \to c}[f(x) + g(x)]$, which is a contradiction. So, $\lim_{x \to c} g(x)$ does not exist.

113. Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - b| < \varepsilon$ whenever $|x - c| < \delta$. Because $|f(x) - b| = |b - b| = 0 < \varepsilon$ for every $\varepsilon > 0$, any value of $\delta > 0$ will work.

114. Given $f(x) = x^n$, n is a positive integer, then

$$\begin{aligned}\lim_{x \to c} x^n &= \lim_{x \to c}\left(xx^{n-1}\right)\\ &= \left[\lim_{x \to c} x\right]\left[\lim_{x \to c} x^{n-1}\right] = c\left[\lim_{x \to c}\left(xx^{n-2}\right)\right]\\ &= c\left[\lim_{x \to c} x\right]\left[\lim_{x \to c} x^{n-2}\right] = c(c)\lim_{x \to c}\left(xx^{n-3}\right)\\ &= \cdots = c^n.\end{aligned}$$

115. If $b = 0$, the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Because $\lim_{x \to c} f(x) = L$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon/|b|$ whenever $0 < |x - c| < \delta$. So, whenever $0 < |x - c| < \delta$, we have

$$|b||f(x) - L| < \varepsilon \quad \text{or} \quad |bf(x) - bL| < \varepsilon$$

which implies that $\lim_{x \to c}[bf(x)] = bL$.

116. Given $\lim_{x \to c} f(x) = 0$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - 0| < \varepsilon$ whenever $0 < |x - c| < \delta$.

Now $|f(x) - 0| = |f(x)| = \big||f(x)| - 0\big| < \varepsilon$ for $|x - c| < \delta$. Therefore, $\lim_{x \to c}|f(x)| = 0$.

117.
$$\begin{aligned}-M|f(x)| &\le f(x)g(x) \le M|f(x)|\\ \lim_{x \to c}\left(-M|f(x)|\right) &\le \lim_{x \to c} f(x)g(x) \le \lim_{x \to c}\left(M|f(x)|\right)\\ -M(0) &\le \lim_{x \to c} f(x)g(x) \le M(0)\\ 0 &\le \lim_{x \to c} f(x)g(x) \le 0\end{aligned}$$

Therefore, $\lim_{x \to c} f(x)g(x) = 0$.

118. (a) If $\lim_{x \to c}|f(x)| = 0$, then $\lim_{x \to c}\left[-|f(x)|\right] = 0$.

$$\begin{aligned}-|f(x)| &\le f(x) \le |f(x)|\\ \lim_{x \to c}\left[-|f(x)|\right] &\le \lim_{x \to c} f(x) \le \lim_{x \to c}|f(x)|\\ 0 &\le \lim_{x \to c} f(x) \le 0\end{aligned}$$

Therefore, $\lim_{x \to c} f(x) = 0$.

(b) Given $\lim_{x \to c} f(x) = L$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. Since $\big||f(x)| - |L|\big| \le |f(x) - L| < \varepsilon$ for $|x - c| < \delta$, then $\lim_{x \to c}|f(x)| = |L|$.

119. Let

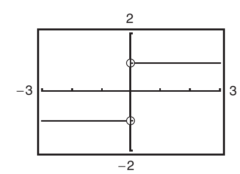
$$f(x) = \begin{cases} 4, & \text{if } x \ge 0\\ -4, & \text{if } x < 0\end{cases}$$

$\lim_{x \to 0}|f(x)| = \lim_{x \to 0} 4 = 4$.

$\lim_{x \to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \ge 0$, $f(x) = 4$.

120. The graphing utility was set in degree mode, instead of *radian* mode.

121. The limit does not exist because the function approaches 1 from the right side of 0 and approaches -1 from the left side of 0.



122. False. $\lim_{x \to \pi} \dfrac{\sin x}{x} = \dfrac{0}{\pi} = 0$

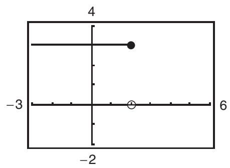
123. True.

124. False. Let

$$f(x) = \begin{cases} x & x \neq 1\\ 3 & x = 1\end{cases}, \quad c = 1.$$

Then $\lim_{x \to 1} f(x) = 1$ but $f(1) \neq 1$.

125. False. The limit does not exist because $f(x)$ approaches 3 from the left side of 2 and approaches 0 from the right side of 2.



126. False. Let $f(x) = \frac{1}{2}x^2$ and $g(x) = x^2$.

Then $f(x) < g(x)$ for all $x \neq 0$. But

$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0.$

127.
$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\\
&= \lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\
&= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\\
&= \left[\lim_{x\to 0}\frac{\sin x}{x}\right]\left[\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right]\\
&= (1)(0) = 0
\end{aligned}$$

128. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

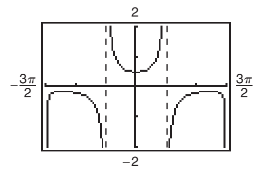
$\lim_{x\to 0} g(x) = 0$

when x is "close to" 0, both parts of the function are "close to" 0.

129. $f(x) = \dfrac{\sec x - 1}{x^2}$

(a) The domain of f is all $x \neq 0, \pi/2 + n\pi$.

(b)



The domain is not obvious. The hole at $x = 0$ is not apparent.

(c) $\lim_{x\to 0} f(x) = \dfrac{1}{2}$

(d)
$$\begin{aligned}
\frac{\sec x - 1}{x^2} &= \frac{\sec x - 1}{x^2}\cdot\frac{\sec x + 1}{\sec x + 1} = \frac{\sec^2 x - 1}{x^2(\sec x + 1)}\\
&= \frac{\tan^2 x}{x^2(\sec x + 1)} = \frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}
\end{aligned}$$

$$\begin{aligned}
\text{So, } \lim_{x\to 0}\frac{\sec x - 1}{x^2} &= \lim_{x\to 0}\frac{1}{\cos^2 x}\left(\frac{\sin^2 x}{x^2}\right)\frac{1}{\sec x + 1}\\
&= 1(1)\left(\frac{1}{2}\right) = \frac{1}{2}.
\end{aligned}$$

130. (a)
$$\begin{aligned}
\lim_{x\to 0}\frac{1-\cos x}{x^2} &= \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\frac{1+\cos x}{1+\cos x}\\
&= \lim_{x\to 0}\frac{1-\cos^2 x}{x^2(1+\cos x)}\\
&= \lim_{x\to 0}\frac{\sin^2 x}{x^2}\cdot\frac{1}{1+\cos x}\\
&= (1)\left(\frac{1}{2}\right) = \frac{1}{2}
\end{aligned}$$

(b) From part (a),

$$\begin{aligned}
\frac{1-\cos x}{x^2} \approx \frac{1}{2} &\Rightarrow 1 - \cos x\\
&\approx \frac{1}{2}x^2 \Rightarrow \cos x\\
&\approx 1 - \frac{1}{2}x^2 \text{ for } x\\
&\approx 0.
\end{aligned}$$

(c) $\cos(0.1) \approx 1 - \frac{1}{2}(0.1)^2 = 0.995$

(d) $\cos(0.1) \approx 0.9950$, which agrees with part (c).

Section 2.4 Continuity and One-Sided Limits

1. (a) $\lim_{x\to 4^+} f(x) = 3$

(b) $\lim_{x\to 4^-} f(x) = 3$

(c) $\lim_{x\to 4} f(x) = 3$

The function is continuous at $x = 4$ and is continuous on $(-\infty, \infty)$.

2. (a) $\lim_{x\to -2^+} f(x) = -2$

(b) $\lim_{x\to -2^-} f(x) = -2$

(c) $\lim_{x\to -2} f(x) = -2$

The function is continuous at $x = -2$.

3. (a) $\lim\limits_{x\to 3^+} f(x) = 0$

(b) $\lim\limits_{x\to 3^-} f(x) = 0$

(c) $\lim\limits_{x\to 3} f(x) = 0$

The function is NOT continuous at $x = 3$.

4. (a) $\lim\limits_{x\to -3^+} f(x) = 3$

(b) $\lim\limits_{x\to -3^-} f(x) = 3$

(c) $\lim\limits_{x\to -3} f(x) = 3$

The function is NOT continuous at $x = -3$ because $f(-3) = 4 \neq \lim\limits_{x\to -3} f(x)$.

5. (a) $\lim\limits_{x\to 2^+} f(x) = -3$

(b) $\lim\limits_{x\to 2^-} f(x) = 3$

(c) $\lim\limits_{x\to 2} f(x)$ does not exist

The function is NOT continuous at $x = 2$.

6. (a) $\lim\limits_{x\to -1^+} f(x) = 0$

(b) $\lim\limits_{x\to -1^-} f(x) = 2$

(c) $\lim\limits_{x\to -1} f(x)$ does not exist.

The function is NOT continuous at $x = -1$.

7. $\lim\limits_{x\to 8^+} \dfrac{1}{x+8} = \dfrac{1}{8+8} = \dfrac{1}{16}$

8. $\lim\limits_{x\to 2^-} \dfrac{2}{x+2} = \dfrac{2}{2+2} = \dfrac{1}{2}$

9.
$$\lim_{x\to 5^+} \frac{x-5}{x^2-25} = \lim_{x\to 5^+} \frac{x-5}{(x+5)(x-5)} = \lim_{x\to 5^+} \frac{1}{x+5} = \frac{1}{10}$$

10.
$$\lim_{x\to 4^+} \frac{4-x}{x^2-16} = \lim_{x\to 4^+} \frac{-(x-4)}{(x+4)(x-4)} = \lim_{x\to 4^+} \frac{-1}{x+4} = \frac{-1}{4+4} = -\frac{1}{8}$$

11. $\lim\limits_{x\to -3^-} \dfrac{x}{\sqrt{x^2-9}}$ does not exist because $\dfrac{x}{\sqrt{x^2-9}}$ decreases without bound as $x \to -3^-$.

12.
$$\lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4^-} \frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

13. $\lim\limits_{x\to 0^-} \dfrac{|x|}{x} = \lim\limits_{x\to 0^-} \dfrac{-x}{x} = -1$

14. $\lim\limits_{x\to 10^+} \dfrac{|x-10|}{x-10} = \lim\limits_{x\to 10^+} \dfrac{x-10}{x-10} = 1$

15.
$$\lim_{\Delta x\to 0^-} \frac{\dfrac{1}{x+\Delta x} - \dfrac{1}{x}}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{x-(x+\Delta x)}{x(x+\Delta x)}\cdot\frac{1}{\Delta x} = \lim_{\Delta x\to 0^-} \frac{-\Delta x}{x(x+\Delta x)}\cdot\frac{1}{\Delta x}$$
$$= \lim_{\Delta x\to 0^-} \frac{-1}{x(x+\Delta x)} = \frac{-1}{x(x+0)} = -\frac{1}{x^2}$$

16.
$$\lim_{\Delta x\to 0^+} \frac{(x+\Delta x)^2 + (x+\Delta x) - (x^2+x)}{\Delta x} = \lim_{\Delta x\to 0^+} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - x^2 - x}{\Delta x}$$
$$= \lim_{\Delta x\to 0^+} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x} = \lim_{\Delta x\to 0^+} (2x + \Delta x + 1) = 2x + 0 + 1 = 2x + 1$$

17. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \frac{x+2}{2} = \frac{5}{2}$

18. $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \left(x^2 - 4x + 6\right) = 9 - 12 + 6 = 3$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \left(-x^2 + 4x - 2\right) = -9 + 12 - 2 = 1$

Since these one-sided limits disagree, $\lim_{x\to 3} f(x)$ does not exist.

19. $\lim_{x\to\pi} \cot x$ does not exist because $\lim_{x\to\pi^+} \cot x$ and $\lim_{x\to\pi^-} \cot x$ do not exist.

20. $\lim_{x\to\pi/2} \sec x$ does not exist because $\lim_{x\to(\pi/2)^+} \sec x$ and $\lim_{x\to(\pi/2)^-} \sec x$ do not exist.

21. $\lim_{x\to 4^-} \left(5[\![x]\!] - 7\right) = 5(3) - 7 = 8$

$\left([\![x]\!] = 3 \text{ for } 3 \le x < 4\right)$

22. $\lim_{x\to 2^+} \left(2x - [\![x]\!]\right) = 2(2) - 2 = 2$

23. $\lim_{x\to 3}\left(2 - [\![-x]\!]\right)$ does not exist because

$\lim_{x\to 3^-} \left(2 - [\![-x]\!]\right) = 2 - (-3) = 5$

and

$\lim_{x\to 3^+} \left(2 - [\![-x]\!]\right) = 2 - (-4) = 6.$

24. $\lim_{x\to 1}\left(1 - \left[\!\left[-\frac{x}{2}\right]\!\right]\right) = 1 - (-1) = 2$

25. $\lim_{x\to 3^+} \ln(x - 3) = \ln 0$

does not exist.

26. $\lim_{x\to 6^-} \ln(6 - x) = \ln 0$

does not exist.

27. $\lim_{x\to 2^-} \ln\left[x^2(3 - x)\right] = \ln\left[4(1)\right] = \ln 4$

28. $\lim_{x\to 5^+} \ln\frac{x}{\sqrt{x - 4}} = \ln\frac{5}{1} = \ln 5$

29. $f(x) = \frac{1}{x^2 - 4}$

has discontinuities at $x = -2$ and $x = 2$ because $f(-2)$ and $f(2)$ are not defined.

30. $f(x) = \frac{x^2 - 1}{x + 1}$

has a discontinuity at $x = -1$ because $f(-1)$ is not defined.

31. $f(x) = \frac{[\![x]\!]}{2} + x$

has discontinuities at each integer k because $\lim_{x\to k^-} f(x) \ne \lim_{x\to k^+} f(x)$.

32. $f(x) = \begin{cases} x, & x < 1 \\ 2, & x = 1 \\ 2x - 1, & x > 1 \end{cases}$ has a discontinuity at $x = 1$ because $f(1) = 2 \ne \lim_{x\to 1} f(x) = 1$.

33. $g(x) = \sqrt{49 - x^2}$ is continuous on $[-7, 7]$.

34. $f(t) = 3 - \sqrt{9 - t^2}$ is continuous on $[-3, 3]$.

35. $\lim_{x\to 0^-} f(x) = 3 = \lim_{x\to 0^+} f(x)$. f is continuous on $[-1, 4]$.

36. $g(2)$ is not defined. g is continuous on $[-1, 2)$.

37. $f(x) = \frac{6}{x}$ has a nonremovable discontinuity at $x = 0$ because $\lim_{x\to 0} f(x)$ does not exist.

38. $f(x) = \frac{4}{x - 6}$ has a nonremovable discontinuity at $x = 6$ because $\lim_{x\to 6} f(x)$ does not exist.

39. $f(x) = 3x - \cos x$ is continuous for all real x.

40. $f(x) = x^2 - 4x + 4$ is continuous for all real x.

41. $f(x) = \frac{1}{4 - x^2} = \frac{1}{(2 - x)(2 + x)}$ has nonremovable discontinuities at $x = \pm 2$ because $\lim_{x\to 2} f(x)$ and $\lim_{x\to -2} f(x)$ do not exist.

42. $f(x) = \cos\frac{\pi x}{2}$ is continuous for all real x.

43. $f(x) = \frac{x}{x^2 - x}$ is not continuous at $x = 0, 1$.

Because $\frac{x}{x^2 - x} = \frac{1}{x - 1}$ for $x \ne 0$, $x = 0$ is a removable discontinuity, whereas $x = 1$ is a nonremovable discontinuity.

44. $f(x) = \dfrac{x}{x^2 - 4}$ has nonremovable discontinuities at $x = 2$ and $x = -2$ because $\lim_{x \to 2} f(x)$ and $\lim_{x \to -2} f(x)$ do not exist.

45. $f(x) = \dfrac{x}{x^2 + 1}$ is continuous for all real x.

46. $f(x) = \dfrac{x - 5}{x^2 - 25} = \dfrac{x - 5}{(x + 5)(x - 5)}$

has a nonremovable discontinuity at $x = -5$ because $\lim_{x \to -5} f(x)$ does not exist, and has a removable discontinuity at $x = 5$ because

$\lim_{x \to 5} f(x) = \lim_{x \to 5} \dfrac{1}{x + 5} = \dfrac{1}{10}.$

47. $f(x) = \dfrac{x + 2}{x^2 - 3x - 10} = \dfrac{x + 2}{(x + 2)(x - 5)}$

has a nonremovable discontinuity at $x = 5$ because $\lim_{x \to 5} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$\lim_{x \to -2} f(x) = \lim_{x \to -2} \dfrac{1}{x - 5} = -\dfrac{1}{7}.$

48. $f(x) = \dfrac{x + 2}{x^2 - x - 6} = \dfrac{x + 2}{(x - 3)(x + 2)}$

has a nonremovable discontinuity at $x = 3$ because $\lim_{x \to 3} f(x)$ does not exist, and has a removable discontinuity at $x = -2$ because

$\lim_{x \to -2} f(x) = \lim_{x \to -2} \dfrac{1}{x - 3} = -\dfrac{1}{5}.$

49. $f(x) = \dfrac{|x + 7|}{x + 7}$

has a nonremovable discontinuity at $x = -7$ because $\lim_{x \to -7} f(x)$ does not exist.

50. $f(x) = \dfrac{|x - 5|}{x - 5}$

has a nonremovable discontinuity at $x = 5$ because $\lim_{x \to 5} f(x)$ does not exist.

51. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1$

2. $\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} x = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x^2 = 1 \end{aligned}\right\} \lim_{x \to 1} f(x) = 1$

3. $f(-1) = \lim_{x \to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

52. $f(x) = \begin{cases} -2x + 3, & x < 1 \\ x^2, & x \ge 1 \end{cases}$

has a **possible** discontinuity at $x = 1$.

1. $f(1) = 1^2 = 1$

2. $\left.\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} (-2x + 3) = 1 \\ \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^+} x^2 = 1 \end{aligned}\right\} \lim_{x \to 1} f(x) = 1$

3. $f(1) = \lim_{x \to 1} f(x)$

f is continuous at $x = 1$, therefore, f is continuous for all real x.

53. $f(x) = \begin{cases} \dfrac{x}{2} + 1, & x \le 2 \\ 3 - x, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = \dfrac{2}{2} + 1 = 2$

2. $\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} \left(\frac{x}{2} + 1\right) = 2 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (3 - x) = 1 \end{aligned}\right\} \lim_{x \to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

54. $f(x) = \begin{cases} -2x, & x \le 2 \\ x^2 - 4x + 1, & x > 2 \end{cases}$

has a **possible** discontinuity at $x = 2$.

1. $f(2) = -2(2) = -4$

2. $\left.\begin{array}{l} \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (-2x) = -4 \\ \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x^2 - 4x + 1) = -3 \end{array}\right\} \lim_{x\to 2} f(x)$ does not exist.

Therefore, f has a nonremovable discontinuity at $x = 2$.

55. $f(x) = \begin{cases} \tan\dfrac{\pi x}{4}, & |x| < 1 \\ x, & |x| \ge 1 \end{cases}$

$= \begin{cases} \tan\dfrac{\pi x}{4}, & -1 < x < 1 \\ x, & x \le -1 \text{ or } x \ge 1 \end{cases}$

has **possible** discontinuities at $x = -1, x = 1$.

1. $f(-1) = -1$ $\quad$ $f(1) = 1$

2. $\lim_{x\to -1} f(x) = -1$ $\quad$ $\lim_{x\to 1} f(x) = 1$

3. $f(-1) = \lim_{x\to -1} f(x)$ $\quad$ $f(1) = \lim_{x\to 1} f(x)$

f is continuous at $x = \pm 1$, therefore, f is continuous for all real x.

56. $f(x) = \begin{cases} \csc\dfrac{\pi x}{6}, & |x - 3| \le 2 \\ 2, & |x - 3| > 2 \end{cases}$

$= \begin{cases} \csc\dfrac{\pi x}{6}, & 1 \le x \le 5 \\ 2, & x < 1 \text{ or } x > 5 \end{cases}$

has **possible** discontinuities at $x = 1, x = 5$.

1. $f(1) = \csc\dfrac{\pi}{6} = 2$ $\quad$ $f(5) = \csc\dfrac{5\pi}{6} = 2$

2. $\lim_{x\to 1} f(x) = 2$ $\quad$ $\lim_{x\to 5} f(x) = 2$

3. $f(1) = \lim_{x\to 1} f(x)$ $\quad$ $f(5) = \lim_{x\to 5} f(x)$

f is continuous at $x = 1$ and $x = 5$, therefore, f is continuous for all real x.

57. $f(x) = \begin{cases} \ln(x + 1), & x \ge 0 \\ 1 - x^2, & x < 0 \end{cases}$

has a **possible** discontinuity at $x = 0$.

1. $f(0) = \ln(0 + 1) = \ln 1 = 0$

2. $\left.\begin{array}{l} \lim_{x\to 0^-} f(x) = 1 - 0 = 1 \\ \lim_{x\to 0^+} f(x) = 0 \end{array}\right\} \lim_{x\to 0} f(x)$ does not exist.

So, f has a nonremovable discontinuity at $x = 0$.

58. $f(x) = \begin{cases} 10 - 3e^{5-x}, & x > 5 \\ 10 - \frac{3}{5}x, & x \le 5 \end{cases}$

has a **possible** discontinuity at $x = 5$.

1. $f(5) = 7$

2. $\left.\begin{array}{l} \lim_{x\to 5^+} f(x) = 10 - 3e^{5-5} = 7 \\ \lim_{x\to 5^-} f(x) = 10 - \frac{3}{5}(5) = 7 \end{array}\right\} \lim_{x\to 5} f(x) = 7$

3. $f(5) = \lim_{x\to 5} f(x)$

f is continuous at $x = 5$, so, f is continuous for all real x.

59. $f(x) = \csc 2x$ has nonremovable discontinuities at integer multiples of $\pi/2$.

60. $f(x) = \tan\dfrac{\pi x}{2}$ has nonremovable discontinuities at each $2k + 1$, k is an integer.

61. $f(x) = [\![x - 8]\!]$ has nonremovable discontinuities at each integer k.

62. $f(x) = 5 - [\![x]\!]$ has nonremovable discontinuities at each integer k.

63. $f(1) = 3$

Find a so that $\lim_{x\to 1^-}(ax - 4) = 3$

$$a(1) - 4 = 3$$
$$a = 7.$$

64. $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} \dfrac{4\sin x}{x} = 4$

$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+}(a - 2x) = a$

Let $a = 4$.

65. Find a and b such that $\lim_{x\to -1^+}(ax + b) = -a + b = 2$ and $\lim_{x\to 3^-}(ax + b) = 3a + b = -2$.

$$\begin{aligned} a - b &= -2 \\ (+)3a + b &= -2 \\ \hline 4a &= -4 \\ a &= -1 \\ b &= 2 + (-1) = 1 \end{aligned}$$

$$f(x) = \begin{cases} 2, & x \le -1 \\ -x + 1, & -1 < x < 3 \\ -2, & x \ge 3 \end{cases}$$

66. $\lim_{x\to a} g(x) = \lim_{x\to a} \dfrac{x^2 - a^2}{x - a}$

$= \lim_{x\to a}(x + a) = 2a$

Find a such $2a = 8 \Rightarrow a = 4$.

67. $f(1) = \arctan(1 - 1) + 2 = 2$

Find a such that $\lim_{x\to 1^-}\left(ae^{x-1} + 3\right) = 2$

$$ae^{1-1} + 3 = 2$$
$$a + 3 = 2$$
$$a = -1.$$

68. $f(4) = 2e^{4a} - 2$

Find a such that $\lim_{x\to 4^+} \ln(x - 3) + x^2 = 2e^{4a} - 2$

$$\begin{aligned} \ln(4 - 3) + 4^2 &= 2e^{4a} - 2 \\ 16 &= 2e^{4a} - 2 \\ 9 &= e^{4a} \\ \ln 9 &= 4a \\ a &= \frac{\ln 9}{4} = \frac{\ln 3^2}{4} = \frac{\ln 3}{2}. \end{aligned}$$

69. $f(g(x)) = (x - 1)^2$

Continuous for all real x

70. $f(g(x)) = \dfrac{1}{\sqrt{x - 1}}$

Nonremovable discontinuity at $x = 1$; continuous for all $x > 1$

71. $f(g(x)) = \dfrac{1}{(x^2 + 5) - 6} = \dfrac{1}{x^2 - 1}$

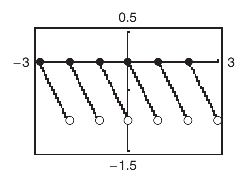
Nonremovable discontinuities at $x = \pm 1$

72. $f(g(x)) = \sin x^2$

Continuous for all real x

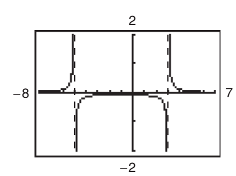
73. $y = [\![x]\!] - x$

Nonremovable discontinuity at each integer



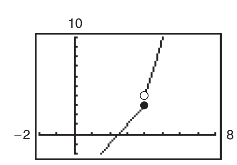
74. $h(x) = \dfrac{1}{x^2 + 2x - 15} = \dfrac{1}{(x + 5)(x - 3)}$

Nonremovable discontinuities at $x = -5$ and $x = 3$



75. $g(x) = \begin{cases} x^2 - 3x, & x > 4 \\ 2x - 5, & x \le 4 \end{cases}$

Nonremovable discontinuity at $x = 4$

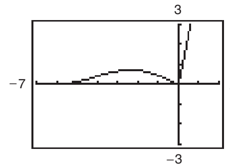


76. $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x < 0 \\ 5x, & x \ge 0 \end{cases}$

$f(0) = 5(0) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \dfrac{(\cos x - 1)}{x} = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (5x) = 0$



Therefore, $\lim_{x\to 0} f(x) = 0 = f(0)$ and f is continuous on the entire real line.

$(x = 0$ was the only possible discontinuity.$)$

77. $f(x) = \dfrac{x}{x^2 + x + 2}$

Continuous on $(-\infty, \infty)$

78. $f(x) = \dfrac{x + 1}{\sqrt{x}}$

Continuous on $(0, \infty)$

79. $f(x) = 3 - \sqrt{x}$

Continuous on $[0, \infty)$

80. $f(x) = x\sqrt{x + 3}$

Continuous on $[-3, \infty)$

81. $f(x) = \sec\dfrac{\pi x}{4}$

Continuous on:

$\ldots, (-6, -2), (-2, 2), (2, 6), (6, 10), \ldots$

82. $f(x) = \cos\dfrac{1}{x}$

Continuous on $(-\infty, 0)$ and $(0, \infty)$

83. $f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \ne 1 \\ 2, & x = 1 \end{cases}$

Since $\lim_{x\to 1} f(x) = \lim_{x\to 1} \dfrac{x^2 - 1}{x - 1} = \lim_{x\to 1} \dfrac{(x - 1)(x + 1)}{x - 1}$

$= \lim_{x\to 1} (x + 1) = 2,$

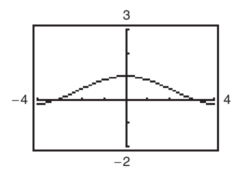
f is continuous on $(-\infty, \infty)$.

84. $f(x) = \begin{cases} 2x - 4, & x \ne 3 \\ 1, & x = 3 \end{cases}$

Since $\lim_{x\to 3} f(x) = \lim_{x\to 3} (2x - 4) = 2 \ne 1$,

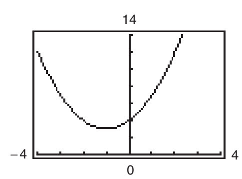
f is continuous on $(-\infty, 3)$ and $(3, \infty)$.

85. $f(x) = \dfrac{\sin x}{x}$



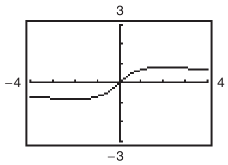
The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(0)$ is not defined, you know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

86. $f(x) = \dfrac{x^3 - 8}{x - 2}$



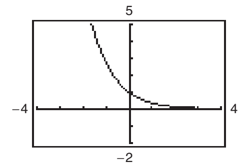
The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(2)$ is not defined, you know that f has a discontinuity at $x = 2$. This discontinuity is removable so it does not show up on the graph.

87. $f(x) = \dfrac{\ln(x^2 + 1)}{x}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(0)$ is not defined, you know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

88. $f(x) = \dfrac{-e^{-x} + 1}{e^x - 1}$



The graph **appears** to be continuous on the interval $[-4, 4]$. Because $f(0)$ is not defined, you know that f has a discontinuity at $x = 0$. This discontinuity is removable so it does not show up on the graph.

89. $f(x) = \frac{1}{12}x^4 - x^3 + 4$ is continuous on the interval $[1, 2]$. $f(1) = \frac{37}{12}$ and $f(2) = -\frac{8}{3}$. By the Intermediate Value Theorem, there exists a number c in $[1, 2]$ such that $f(c) = 0$.

90. $f(x) = -\dfrac{5}{x} + \tan\left(\dfrac{\pi x}{10}\right)$ is continuous on the interval $[1, 4]$.

$f(1) = -5 + \tan\left(\dfrac{\pi}{10}\right) \approx -4.7$ and $f(4) = -\dfrac{5}{4} + \tan\left(\dfrac{2\pi}{5}\right) \approx 1.8$. By the Intermediate Value Theorem, there exists a number c in $[1, 4]$ such that $f(c) = 0$.

91. h is continuous on the interval $\left[0, \dfrac{\pi}{2}\right]$. $h(0) = -2 < 0$ and $h\left(\dfrac{\pi}{2}\right) \approx 0.91 > 0$. By the Intermediate Value Theorem, there exists a number c in $\left[0, \dfrac{\pi}{2}\right]$ such that $h(c) = 0$.

92. g is continuous on the interval $[0, 1]$. $g(0) \approx -2.77 < 0$ and $g(1) \approx 1.61 > 0$. By the Intermediate Value Theorem, there exists a number c in $[0, 1]$ such that $g(c) = 0$.

93. $f(x) = x^3 + x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 1$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.68$. Using the *root* feature, you find that $x \approx 0.6823$.

94. $f(x) = x^4 - x^2 + 3x - 1$

$f(x)$ is continuous on $[0, 1]$.

$f(0) = -1$ and $f(1) = 2$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.37$. Using the *root* feature, you find that $x \approx 0.3733$.

95. $g(t) = 2\cos t - 3t$

g is continuous on $[0, 1]$.

$g(0) = 2 > 0$ and $g(1) \approx -1.9 < 0$.

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(t)$, you find that $t \approx 0.56$. Using the *root* feature, you find that $t \approx 0.5636$.

96. $h(\theta) = \tan\theta + 3\theta - 4$ is continuous on $[0, 1]$.

$h(0) = -4$ and $h(1) = \tan(1) - 1 \approx 0.557$.

By the Intermediate Value Theorem, $h(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $h(\theta)$, you find that $\theta \approx 0.91$. Using the *root* feature, you obtain $\theta \approx 0.9071$.

97. $f(x) = x + e^x - 3$

f is continuous on $[0, 1]$.

$f(0) = e^0 - 3 = -2 < 0$ and

$f(1) = 1 + e - 3 = e - 2 > 0.$

By the Intermediate Value Theorem, $f(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $f(x)$, you find that $x \approx 0.79$. Using the *root* feature, you find that $x \approx 0.7921$.

98. $g(x) = 5\ln(x + 1) - 2$

g is continuous on $[0, 1]$.

$g(0) = 5\ln(0 + 1) - 2 = -2$ and

$g(1) = 5\ln(2) - 2 > 0.$

By the Intermediate Value Theorem, $g(c) = 0$ for at least one value of c between 0 and 1. Using a graphing utility to zoom in on the graph of $g(x)$, you find that $x \approx 0.49$. Using the *root* feature, you find that $x \approx 0.4918$.

99. $f(x) = x^2 + x - 1$

f is continuous on $[0, 5]$.

$f(0) = -1$ and $f(5) = 29$

$-1 < 11 < 29$

The Intermediate Value Theorem applies.

$$\begin{aligned} x^2 + x - 1 &= 11 \\ x^2 + x - 12 &= 0 \\ (x + 4)(x - 3) &= 0 \\ x = -4 \text{ or } x &= 3 \end{aligned}$$

$c = 3$ ($x = -4$ is not in the interval.)

So, $f(3) = 11$.

100. $f(x) = x^2 - 6x + 8$

f is continuous on $[0, 3]$.

$f(0) = 8$ and $f(3) = -1$

$-1 < 0 < 8$

The Intermediate Value Theorem applies.

$$\begin{aligned} x^2 - 6x + 8 &= 0 \\ (x - 2)(x - 4) &= 0 \\ x = 2 \text{ or } x &= 4 \end{aligned}$$

$c = 2$ ($x = 4$ is not in the interval.)

So, $f(2) = 0$.

101. $f(x) = x^3 - x^2 + x - 2$

f is continuous on $[0, 3]$.

$f(0) = -2$ and $f(3) = 19$

$-2 < 4 < 19$

The Intermediate Value Theorem applies.

$$\begin{aligned} x^3 - x^2 + x - 2 &= 4 \\ x^3 - x^2 + x - 6 &= 0 \\ (x - 2)(x^2 + x + 3) &= 0 \\ x &= 2 \end{aligned}$$

($x^2 + x + 3$ has no real solution.)

$$c = 2$$

So, $f(2) = 4$.

102. $f(x) = \dfrac{x^2 + x}{x - 1}$

f is continuous on $\left[\frac{5}{2}, 4\right]$. The nonremovable discontinuity, $x = 1$, lies outside the interval.

$f\left(\frac{5}{2}\right) = \frac{35}{6}$ and $f(4) = \frac{20}{3}$

$\frac{35}{6} < 6 < \frac{20}{3}$

The Intermediate Value Theorem applies.

$$\begin{aligned} \frac{x^2 + x}{x - 1} &= 6 \\ x^2 + x &= 6x - 6 \\ x^2 - 5x + 6 &= 0 \\ (x - 2)(x - 3) &= 0 \\ x = 2 \text{ or } x &= 3 \end{aligned}$$

$c = 3$ ($x = 2$ is not in the interval.)

So, $f(3) = 6$.

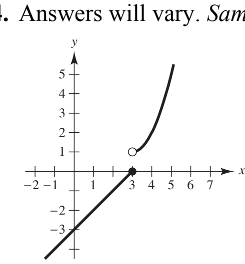
103. (a) The limit does not exist at $x = c$.

(b) The function is not defined at $x = c$.

(c) The limit exists at $x = c$, but it is not equal to the value of the function at $x = c$.

(d) The limit does not exist at $x = c$.

104. Answers will vary. *Sample answer*:



The function is not continuous at $x = 3$ because $\lim_{x \to 3^+} f(x) = 1 \neq 0 = \lim_{x \to 3^-} f(x)$.

105. If f and g are continuous for all real x, then so is $f + g$ (Theorem 2.11, part 2). However, f/g might not be continuous if $g(x) = 0$. For example, let $f(x) = x$ and $g(x) = x^2 - 1$. Then f and g are continuous for all real x, but f/g is not continuous at $x = \pm 1$.

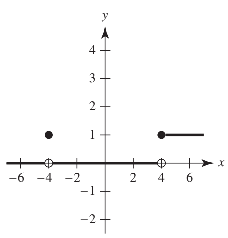
106. A discontinuity at c is removable if the function f can be made continuous at c by appropriately defining (or redefining) $f(c)$. Otherwise, the discontinuity is nonremovable.

(a) $f(x) = \dfrac{|x - 4|}{x - 4}$

(b) $f(x) = \dfrac{\sin(x + 4)}{x + 4}$

(c) $f(x) = \begin{cases} 1, & x \geq 4 \\ 0, & -4 < x < 4 \\ 1, & x = -4 \\ 0, & x < -4 \end{cases}$

$x = 4$ is nonremovable, $x = -4$ is removable



107. True

1. $f(c) = L$ is defined.

2. $\lim_{x \to c} f(x) = L$ exists.

3. $f(c) = \lim_{x \to c} f(x)$

All of the conditions for continuity are met.

108. True. If $f(x) = g(x)$, $x \neq c$, then $\lim_{x \to c} f(x) = \lim_{x \to c} g(x)$ (if they exist) and at least one of these limits then does not equal the corresponding function value at $x = c$.

109. False. A rational function can be written as $P(x)/Q(x)$ where P and Q are polynomials of degree m and n, respectively. It can have, at most, n discontinuities.

110. False. $f(1)$ is not defined and $\lim_{x \to 1} f(x)$ does not exist.

111. The functions agree for integer values of x:

$$\left.\begin{aligned} g(x) &= 3 - [\![-x]\!] = 3 - (-x) = 3 + x \\ f(x) &= 3 + [\![x]\!] = 3 + x \end{aligned}\right\} \text{for } x \text{ an integer}$$

However, for non-integer values of x, the functions differ by 1.

$$f(x) = 3 + [\![x]\!] = g(x) - 1 = 2 - [\![-x]\!].$$

For example,

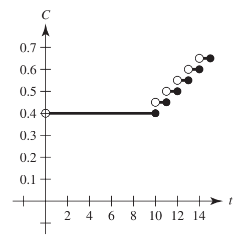
$$f\left(\tfrac{1}{2}\right) = 3 + 0 = 3,\ g\left(\tfrac{1}{2}\right) = 3 - (-1) = 4.$$

112. $\lim_{t \to 4^-} f(t) \approx 28$

$\lim_{t \to 4^+} f(t) \approx 56$

At the end of day 3, the amount of chlorine in the pool has decreased to about 28 oz. At the beginning of day 4, more chlorine was added, and the amount is now about 56 oz.

113. $C(t) = \begin{cases} 0.40, & 0 < t \leq 10 \\ 0.40 + 0.05[\![t - 9]\!], & t > 10, t \text{ not an integer} \\ 0.40 + 0.05(t - 10), & t > 10, t \text{ an integer} \end{cases}$



There is a nonremovable discontinuity at each integer greater than or equal to 10.

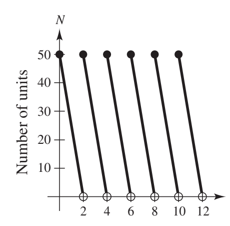
Note: You could also express C as

$$C(t) = \begin{cases} 0.40, & 0 < t \leq 10 \\ 0.40 - 0.05[\![10 - t]\!], & t > 10 \end{cases}$$

114. $N(t) = 25\left(2\left[\!\!\left[\dfrac{t + 2}{2}\right]\!\!\right] - t\right)$

t	0	1	1.8	2	3	3.8
$N(t)$	50	25	5	50	25	5

Discontinuous at every positive even integer. The company replenishes its inventory every two months.



115. Let $s(t)$ be the position function for the run up to the campsite. $s(0) = 0$ ($t = 0$ corresponds to 8:00 A.M., $s(20) = k$ (distance to campsite)). Let $r(t)$ be the position function for the run back down the mountain: $r(0) = k$, $r(10) = 0$. Let $f(t) = s(t) - r(t)$.

When $t = 0$ (8:00 A.M.),
$f(0) = s(0) - r(0) = 0 - k < 0$.

When $t = 10$ (8:00 A.M.), $f(10) = s(10) - r(10) > 0$.

Because $f(0) < 0$ and $f(10) > 0$, then there must be a value t in the interval $[0, 10]$ such that $f(t) = 0$. If $f(t) = 0$, then $s(t) - r(t) = 0$, which gives us $s(t) = r(t)$. Therefore, at some time t, where $0 \le t \le 10$, the position functions for the run up and the run down are equal.

116. Let $V = \frac{4}{3}\pi r^3$ be the volume of a sphere with radius r.

V is continuous on $[5, 8]$. $V(5) = \dfrac{500\pi}{3} \approx 523.6$ and

$V(8) = \dfrac{2048\pi}{3} \approx 2144.7$. Because

$523.6 < 1500 < 2144.7$, the Intermediate Value Theorem guarantees that there is at least one value r between 5 and 8 such that $V(r) = 1500$. (In fact, $r \approx 7.1012$.)

117. Suppose there exists x_1 in $[a, b]$ such that $f(x_1) > 0$ and there exists x_2 in $[a, b]$ such that $f(x_2) < 0$. Then by the Intermediate Value Theorem, $f(x)$ must equal zero for some value of x in $[x_1, x_2]$ (or $[x_2, x_1]$ if $x_2 < x_1$). So, f would have a zero in $[a, b]$, which is a contradiction. Therefore, $f(x) > 0$ for all x in $[a, b]$ or $f(x) < 0$ for all x in $[a, b]$.

118. Let c be any real number. Then $\lim\limits_{x \to c} f(x)$ does not exist because there are both rational and irrational numbers arbitrarily close to c. Therefore, f is not continuous at c.

119. If $x = 0$, then $f(0) = 0$ and $\lim\limits_{x \to 0} f(x) = 0$. So, f is continuous at $x = 0$.

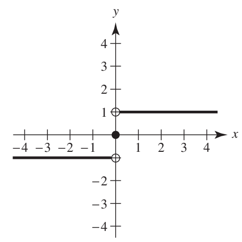
If $x \ne 0$, then $\lim\limits_{t \to x} f(t) = 0$ for x rational, whereas $\lim\limits_{t \to x} f(t) = \lim\limits_{t \to x} kt = kx \ne 0$ for x irrational. So, f is not continuous for all $x \ne 0$.

120. $\operatorname{sgn}(x) = \begin{cases} -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \\ 1, & \text{if } x > 0 \end{cases}$

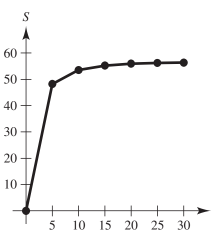
(a) $\lim\limits_{x \to 0^-} \operatorname{sgn}(x) = -1$

(b) $\lim\limits_{x \to 0^+} \operatorname{sgn}(x) = 1$

(c) $\lim\limits_{x \to 0} \operatorname{sgn}(x)$ does not exist.

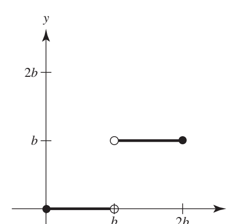


121. (a)



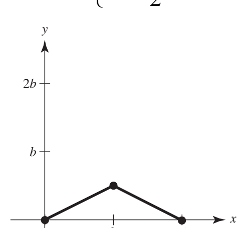
(b) There appears to be a limiting speed and a possible cause is air resistance.

122. (a) $f(x) = \begin{cases} 0, & 0 \le x < b \\ b, & b < x \le 2b \end{cases}$



NOT continuous at $x = b$.

(b) $g(x) = \begin{cases} \dfrac{x}{2}, & 0 \le x \le b \\ b - \dfrac{x}{2}, & b < x \le 2b \end{cases}$



Continuous on $[0, 2b]$.

123. $f(x) = \begin{cases} 1 - x^2, & x \le c \\ x, & x > c \end{cases}$

f is continuous for $x < c$ and for $x > c$. At $x = c$, you need $1 - c^2 = c$. Solving $c^2 + c - 1$, you obtain

$$c = \frac{-1 \pm \sqrt{1 + 4}}{2} = \frac{-1 \pm \sqrt{5}}{2}.$$

124. Let y be a real number. If $y = 0$, then $x = 0$. If $y > 0$, then let $0 < x_0 < \pi/2$ such that $M = \tan x_0 > y$ (this is possible since the tangent function increases without bound on $[0, \pi/2)$). By the Intermediate Value Theorem, $f(x) = \tan x$ is continuous on $[0, x_0]$ and $0 < y < M$, which implies that there exists x between 0 and x_0 such that $\tan x = y$. The argument is similar if $y < 0$.

125. $f(x) = \dfrac{\sqrt{x + c^2} - c}{x}, c > 0$

Domain: $x + c^2 \ge 0 \Rightarrow x \ge -c^2$ and $x \ne 0$, $[-c^2, 0) \cup (0, \infty)$

$$\lim_{x\to 0}\frac{\sqrt{x + c^2} - c}{x} = \lim_{x\to 0}\frac{\sqrt{x + c^2} - c}{x}\cdot\frac{\sqrt{x + c^2} + c}{\sqrt{x + c^2} + c} = \lim_{x\to 0}\frac{(x + c^2) - c^2}{x\left[\sqrt{x + c^2} + c\right]} = \lim_{x\to 0}\frac{1}{\sqrt{x + c^2} + c} = \frac{1}{2c}$$

Define $f(0) = 1/(2c)$ to make f continuous at $x = 0$.

126. **1.** $f(c)$ is defined.

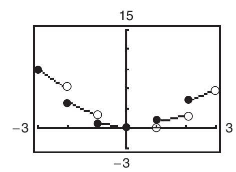
2. $\lim_{x\to c} f(x) = \lim_{\Delta x\to 0} f(c + \Delta x) = f(c)$ exists.

[Let $x = c + \Delta x$. As $x \to c, \Delta x \to 0$]

3. $\lim_{x\to c} f(x) = f(c)$.

Therefore, f is continuous at $x = c$.

127. $h(x) = x[\![x]\!]$



h has nonremovable discontinuities at $x = \pm 1, \pm 2, \pm 3, \ldots$.

128. (a) Define $f(x) = f_2(x) - f_1(x)$. Because f_1 and f_2 are continuous on $[a, b]$, so is f.

$f(a) = f_2(a) - f_1(a) > 0$ and $f(b) = f_2(b) - f_1(b) < 0$

By the Intermediate Value Theorem, there exists c in $[a, b]$ such that $f(c) = 0$.

$f(c) = f_2(c) - f_1(c) = 0 \Rightarrow f_1(c) = f_2(c)$

(b) Let $f_1(x) = x$ and $f_2(x) = \cos x$, continuous on $[0, \pi/2]$, $f_1(0) < f_2(0)$ and $f_1(\pi/2) > f_2(\pi/2)$.

So by part (a), there exists c in $[0, \pi/2]$ such that $c = \cos(c)$.

Using a graphing utility, $c \approx 0.739$.

129. The statement is true.

If $y \ge 0$ and $y \le 1$, then $y(y - 1) \le 0 \le x^2$, as desired. So assume $y > 1$. There are now two cases.

Case 1: If $x \le y - \frac{1}{2}$, then $2x + 1 \le 2y$ and

$$\begin{aligned} y(y - 1) &= y(y + 1) - 2y \\ &\le (x + 1)^2 - 2y \\ &= x^2 + 2x + 1 - 2y \\ &\le x^2 + 2y - 2y \\ &= x^2 \end{aligned}$$

Case 2: If $x \ge y - \frac{1}{2}$

$$\begin{aligned} x^2 &\ge \left(y - \tfrac{1}{2}\right)^2 \\ &= y^2 - y + \tfrac{1}{4} \\ &> y^2 - y \\ &= y(y - 1) \end{aligned}$$

In both cases, $y(y - 1) \le x^2$.

130. $P(1) = P(0^2 + 1) = P(0)^2 + 1 = 1$

$P(2) = P(1^2 + 1) = P(1)^2 + 1 = 2$

$P(5) = P(2^2 + 1) = P(2)^2 + 1 = 5$

Continuing this pattern, you see that $P(x) = x$ for infinitely many values of x.

So, the finite degree polynomial must be constant: $P(x) = x$ for all x.

Section 2.5 Infinite Limits

1. $\lim\limits_{x\to -2^+} 2\left|\dfrac{x}{x^2 - 4}\right| = \infty$

$\lim\limits_{x\to -2^-} 2\left|\dfrac{x}{x^2 - 4}\right| = \infty$

2. $\lim\limits_{x\to -2^+} \dfrac{1}{x + 2} = \infty$

$\lim\limits_{x\to -2^-} \dfrac{1}{x + 2} = -\infty$

3. $\lim\limits_{x\to -2^+} \tan \dfrac{\pi x}{4} = -\infty$

$\lim\limits_{x\to -2^-} \tan \dfrac{\pi x}{4} = \infty$

4. $\lim\limits_{x\to -2^+} \sec \dfrac{\pi x}{4} = \infty$

$\lim\limits_{x\to -2^-} \sec \dfrac{\pi x}{4} = -\infty$

5. $f(x) = \dfrac{1}{x - 4}$

As x approaches 4 from the left, $x - 4$ is a small negative number. So,

$\lim\limits_{x\to 4^-} f(x) = -\infty$

As x approaches 4 from the right, $x - 4$ is a small positive number. So,

$\lim\limits_{x\to 4^+} f(x) = \infty$

6. $f(x) = \dfrac{-1}{x - 4}$

As x approaches 4 from the left, $x - 4$ is a small negative number. So,

$\lim\limits_{x\to 4^-} f(x) = \infty.$

As x approaches 4 from the right, $x - 4$ is a small positive number. So,

$\lim\limits_{x\to 4^+} f(x) = -\infty.$

7. $f(x) = \dfrac{1}{(x - 4)^2}$

As x approaches 4 from the left or right, $(x - 4)^2$ is a small positive number. So,

$\lim\limits_{x\to 4^+} f(x) = \lim\limits_{x\to 4^-} f(x) = \infty.$

8. $f(x) = \dfrac{-1}{(x - 4)^2}$

As x approaches 4 from the left or right, $(x - 4)^2$ is a small positive number. So,

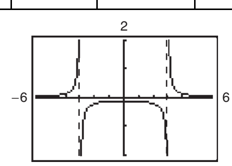
$\lim\limits_{x\to 4^-} f(x) = \lim\limits_{x\to 4^+} f(x) = -\infty.$

9. $f(x) = \dfrac{1}{x^2 - 9}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	0.308	1.639	16.64	166.6	–166.7	–16.69	–1.695	–0.364

$\lim\limits_{x\to -3^-} f(x) = \infty$

$\lim\limits_{x\to -3^+} f(x) = -\infty$

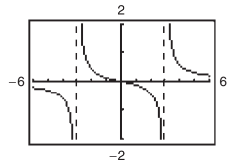


10. $f(x) = \dfrac{x}{x^2 - 9}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	–1.077	–5.082	–50.08	–500.1	499.9	49.92	4.915	0.9091

$\lim\limits_{x\to -3^-} f(x) = -\infty$

$\lim\limits_{x\to -3^+} f(x) = \infty$

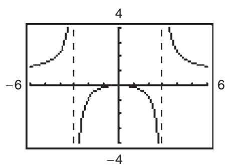


11. $f(x) = \dfrac{x^2}{x^2 - 9}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	3.769	15.75	150.8	1501	–1499	–149.3	–14.25	–2.273

$\lim\limits_{x\to -3^-} f(x) = \infty$

$\lim\limits_{x\to -3^+} f(x) = -\infty$

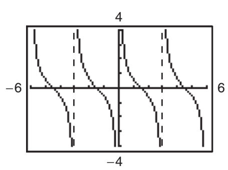


12. $f(x) = \cot\dfrac{\pi x}{3}$

x	–3.5	–3.1	–3.01	–3.001	–2.999	–2.99	–2.9	–2.5
$f(x)$	–1.7321	–9.514	–95.49	–954.9	954.9	95.49	9.514	1.7321

$\lim\limits_{x\to -3^-} f(x) = -\infty$

$\lim\limits_{x\to -3^+} f(x) = \infty$



13. $f(x) = \dfrac{1}{x^2}$

$\lim\limits_{x\to 0^+} \dfrac{1}{x^2} = \infty = \lim\limits_{x\to 0^-} \dfrac{1}{x^2}$

Therefore, $x = 0$ is a vertical asymptote.

14. $f(x) = \dfrac{2}{(x - 3)^3}$

$\lim\limits_{x\to 3^-} \dfrac{2}{(x - 3)^3} = -\infty$

$\lim\limits_{x\to 3^+} \dfrac{2}{(x - 3)^3} = \infty$

Therefore, $x = 3$ is a vertical asymptote.

15. $f(x) = \dfrac{x^2}{x^2 - 4} = \dfrac{x^2}{(x + 2)(x - 2)}$

$\lim\limits_{x\to -2^-} \dfrac{x^2}{x^2 - 4} = \infty$ and $\lim\limits_{x\to -2^+} \dfrac{x^2}{x^2 - 4} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim\limits_{x\to 2^-} \dfrac{x^2}{x^2 - 4} = -\infty$ and $\lim\limits_{x\to 2^+} \dfrac{x^2}{x^2 - 4} = \infty$

Therefore, $x = 2$ is a vertical asymptote.

16. $f(x) = \dfrac{3x}{x^2 + 9}$

No vertical asymptotes because the denominator is never zero.

17. $g(t) = \dfrac{t-1}{t^2+1}$

No vertical asymptotes because the denominator is never zero.

18. $h(s) = \dfrac{3s+4}{s^2-16} = \dfrac{3s+4}{(s-4)(s+4)}$

$\lim\limits_{s\to 4^-} \dfrac{3s+4}{s^2-16} = -\infty$ and $\lim\limits_{s\to 4^+} \dfrac{3s+4}{s^2-16} = \infty$

Therefore, $s = 4$ is a vertical asymptote.

$\lim\limits_{s\to -4^-} \dfrac{3s+4}{s^2-16} = -\infty$ and $\lim\limits_{s\to -4^+} \dfrac{3s+4}{s^2-16} = \infty$

Therefore, $s = -4$ is a vertical asymptote.

19. $f(x) = \dfrac{3}{x^2+x-2} = \dfrac{3}{(x+2)(x-1)}$

$\lim\limits_{x\to -2^-} \dfrac{3}{x^2+x-2} = \infty$ and $\lim\limits_{x\to -2^+} \dfrac{3}{x^2+x-2} = -\infty$

Therefore, $x = -2$ is a vertical asymptote.

$\lim\limits_{x\to 1^-} \dfrac{3}{x^2+x-2} = -\infty$ and $\lim\limits_{x\to 1^+} \dfrac{3}{x^2+x-2} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

20. $g(x) = \dfrac{x^3-8}{x-2} = \dfrac{(x-2)(x^2+2x+4)}{x-2}$

$= x^2 + 2x + 4,\ x \neq 2$

$\lim\limits_{x\to 2} g(x) = 4 + 4 + 4 = 12$

There are no vertical asymptotes. The graph has a hole at $x = 2$.

21. $f(x) = \dfrac{x^2-2x-15}{x^3-5x^2+x-5}$

$= \dfrac{(x-5)(x+3)}{(x-5)(x^2+1)}$

$= \dfrac{x+3}{x^2+1},\ x \neq 5$

$\lim\limits_{x\to 5} f(x) = \dfrac{5+3}{5^2+1} = \dfrac{15}{26}$

There are no vertical asymptotes. The graph has a hole at $x = 5$.

22. $h(x) = \dfrac{x^2-9}{x^3+3x^2-x-3}$

$= \dfrac{(x-3)(x+3)}{(x-1)(x+1)(x+3)}$

$= \dfrac{x-3}{(x+1)(x-1)},\ x \neq -3$

$\lim\limits_{x\to -1^-} h(x) = -\infty$ and $\lim\limits_{x\to -1^+} h(x) = \infty$

Therefore, $x = -1$ is a vertical asymptote.

$\lim\limits_{x\to 1^-} h(x) = \infty$ and $\lim\limits_{x\to 1^+} h(x) = -\infty$

Therefore, $x = 1$ is a vertical asymptote.

$\lim\limits_{x\to -3} h(x) = \dfrac{-3-3}{(-3+1)(-3-1)} = -\dfrac{3}{4}$

Therefore, the graph has a hole at $x = -3$.

23. $f(x) = \dfrac{e^{-2x}}{x-1}$

$\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} = \infty$

Therefore, $x = 1$ is a vertical asymptote.

24. $g(x) = xe^{-2x}$

The function is continuous for all x. Therefore, there are no vertical asymptotes.

25. $h(t) = \dfrac{\ln(t^2+1)}{t+2}$

$\lim\limits_{t\to -2^-} h(t) = -\infty$ and $\lim\limits_{t\to -2^+} = \infty$

Therefore, $t = -2$ is a vertical asymptote.

26. $f(z) = \ln(z^2-4) = \ln[(z+2)(z-2)]$

$= \ln(z+2) + \ln(z-2)$

The function is undefined for $-2 < z < 2$.

Therefore, the graph has holes at $z = \pm 2$.

27. $f(x) = \dfrac{1}{e^x-1}$

$\lim\limits_{x\to 0^-} f(x) = -\infty$ and $\lim\limits_{x\to 0^+} f(x) = \infty$

Therefore, $x = 0$ is a vertical asymptote.

28. $f(x) = \ln(x+3)$

$\lim\limits_{x\to -3} f(x) = -\infty$

Therefore, $x = -3$ is a vertical asymptote.

29. $f(x) = \csc \pi x = \dfrac{1}{\sin \pi x}$

Let n be any integer.

$\lim\limits_{x \to n} f(x) = -\infty$ or ∞

Therefore, the graph has vertical asymptotes at $x = n$.

30. $f(x) = \tan \pi x = \dfrac{\sin \pi x}{\cos \pi x}$

$\cos \pi x = 0$ for $x = \dfrac{2n + 1}{2}$, where n is an integer.

$\lim\limits_{x \to \frac{2n+1}{2}} f(x) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at

$x = \dfrac{2n + 1}{2}$.

31. $s(t) = \dfrac{t}{\sin t}$

$\sin t = 0$ for $t = n\pi$, where n is an integer.

$\lim\limits_{t \to n\pi} s(t) = \infty$ or $-\infty$ (for $n \neq 0$)

Therefore, the graph has vertical asymptotes at $t = n\pi$, for $n \neq 0$.

$\lim\limits_{t \to 0} s(t) = 1$

Therefore, the graph has a hole at $t = 0$.

32. $g(\theta) = \dfrac{\tan \theta}{\theta} = \dfrac{\sin \theta}{\theta \cos \theta}$

$\cos \theta = 0$ for $\theta = \dfrac{\pi}{2} + n\pi$, where n is an integer.

$\lim\limits_{\theta \to \frac{\pi}{2} + n\pi} g(\theta) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at

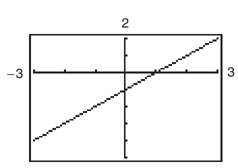
$\theta = \dfrac{\pi}{2} + n\pi$.

$\lim\limits_{\theta \to 0} g(\theta) = 1$

Therefore, the graph has a hole at $\theta = 0$.

33. $\lim\limits_{x \to -1} \dfrac{x^2 - 1}{x + 1} = \lim\limits_{x \to -1} (x - 1) = -2$

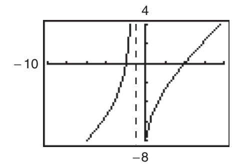
Removable discontinuity at $x = -1$



34. $\lim\limits_{x \to -1^-} \dfrac{x^2 - 2x - 8}{x + 1} = \infty$

$\lim\limits_{x \to -1^+} \dfrac{x^2 - 2x - 8}{x + 1} = -\infty$

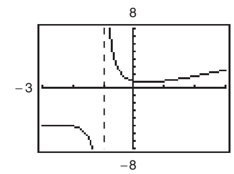
Vertical asymptote at $x = -1$



35. $\lim\limits_{x \to -1^+} \dfrac{x^2 + 1}{x + 1} = \infty$

$\lim\limits_{x \to -1^-} \dfrac{x^2 + 1}{x + 1} = -\infty$

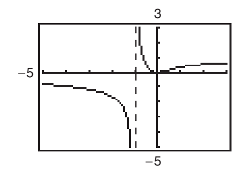
Vertical asymptote at $x = -1$



36. $\lim\limits_{x \to -1^+} \dfrac{\ln(x^2 + 1)}{x + 1} = \infty$

$\lim\limits_{x \to -1^-} \dfrac{\ln(x^2 + 1)}{x + 1} = -\infty$

Vertical asymptote at $x = -1$



37. $\lim\limits_{x \to -1^+} \dfrac{1}{x + 1} = \infty$

38. $\lim\limits_{x \to 1^-} \dfrac{-1}{(x - 1)^2} = -\infty$

39. $\lim\limits_{x \to 2^+} \dfrac{x}{x - 2} = \infty$

40. $\lim\limits_{x \to 2^-} \dfrac{x^2}{x^2 + 4} = \dfrac{4}{4 + 4} = \dfrac{1}{2}$

41. $\lim\limits_{x \to -3^-} \dfrac{x + 3}{(x^2 + x - 6)} = \lim\limits_{x \to -3^-} \dfrac{x + 3}{(x + 3)(x - 2)}$

$= \lim\limits_{x \to -3^-} \dfrac{1}{x - 2} = -\dfrac{1}{5}$

42. $\lim\limits_{x \to -(1/2)^+} \dfrac{6x^2 + x - 1}{4x^2 - 4x - 3} = \lim\limits_{x \to -(1/2)^+} \dfrac{(3x - 1)(2x + 1)}{(2x - 3)(2x + 1)}$

$= \lim\limits_{x \to -(1/2)^+} \dfrac{3x - 1}{2x - 3} = \dfrac{5}{8}$

43. $\lim\limits_{x \to 0^-} \left(1 + \dfrac{1}{x}\right) = -\infty$

44. $\lim\limits_{x \to 0^+} \left(6 - \dfrac{1}{x^3}\right) = -\infty$

45. $\lim\limits_{x \to -4^-} \left(x^2 + \dfrac{2}{x + 4}\right) = -\infty$

46. $\lim\limits_{x \to 3^+} \left(\dfrac{x}{3} + \cot \dfrac{\pi x}{2}\right) = \infty$

47. $\lim_{x \to 0^+} \dfrac{2}{\sin x} = \infty$

48. $\lim_{x \to (\pi/2)^+} \dfrac{-2}{\cos x} = \infty$

49. $\lim_{x \to 8^-} \dfrac{e^x}{(x - 8)^3} = -\infty$

50. $\lim_{x \to 4^+} \ln(x^2 - 16) = -\infty$

51. $\lim_{x \to (\pi/2)^-} \ln|\cos x| = \ln\left|\cos\dfrac{\pi}{2}\right| = \ln 0 = -\infty$

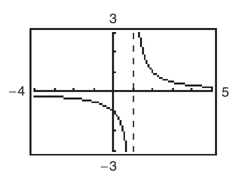
52. $\lim_{x \to 0^+} e^{-0.5x} \sin x = 1(0) = 0$

53. $\lim_{x \to (1/2)^-} x \sec \pi x = \lim_{x \to (1/2)^-} \dfrac{x}{\cos \pi x} = \infty$

54. $\lim_{x \to (1/2)^+} x^2 \tan \pi x = -\infty$

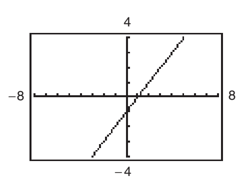
55. $f(x) = \dfrac{x^2 + x + 1}{x^3 - 1} = \dfrac{x^2 + x + 1}{(x - 1)(x^2 + x + 1)}$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \dfrac{1}{x - 1} = \infty$



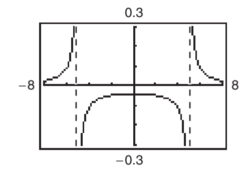
56. $f(x) = \dfrac{x^3 - 1}{x^2 + x + 1} = \dfrac{(x - 1)(x^2 + x + 1)}{x^2 + x + 1}$

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x - 1) = 0$



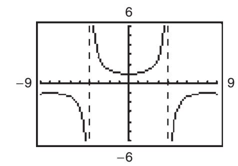
57. $f(x) = \dfrac{1}{x^2 - 25}$

$\lim_{x \to 5^-} f(x) = -\infty$



58. $f(x) = \sec \dfrac{\pi x}{8}$

$\lim_{x \to 4^+} f(x) = -\infty$



59. A limit in which $f(x)$ increases or decreases without bound as x approaches c is called an infinite limit. ∞ is not a number. Rather, the symbol

$\lim_{x \to c} f(x) = \infty$

says how the limit fails to exist.

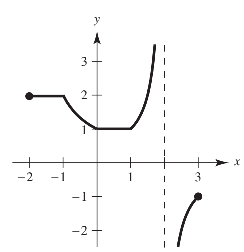
60. The line $x = c$ is a vertical asymptote if the graph of f approaches $\pm\infty$ as x approaches c.

61. One answer is

$f(x) = \dfrac{x - 3}{(x - 6)(x + 2)} = \dfrac{x - 3}{x^2 - 4x - 12}.$

62. No. For example, $f(x) = \dfrac{1}{x^2 + 1}$ has no vertical asymptote.

63.

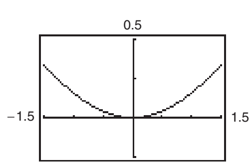


64. $m = \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}}$

$\lim_{v \to c^-} m = \lim_{v \to c^-} \dfrac{m_0}{\sqrt{1 - (v^2/c^2)}} = \infty$

65. (a)

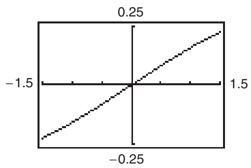
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0411	0.0067	0.0017	≈ 0	≈ 0	≈ 0



$\lim_{x \to 0^+} \dfrac{x - \sin x}{x} = 0$

(b)

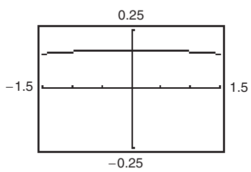
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.0823	0.0333	0.0167	0.0017	≈ 0	≈ 0



$$\lim_{x\to 0^+} \frac{x - \sin x}{x^2} = 0$$

(c)

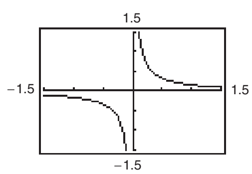
x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.1646	0.1663	0.1666	0.1667	0.1667	0.1667



$$\lim_{x\to 0^+} \frac{x - \sin x}{x^3} = 0.1667\ (1/6)$$

(d)

x	1	0.5	0.2	0.1	0.01	0.001	0.0001
$f(x)$	0.1585	0.3292	0.8317	1.6658	16.67	166.7	1667.0



$$\lim_{x\to 0^+} \frac{x - \sin x}{x^4} = \infty \text{ or } n > 3,\ \lim_{x\to 0^+} \frac{x - \sin x}{x^n} = \infty.$$

66. $\lim_{V\to 0^+} P = \infty$

As the volume of the gas decreases, the pressure increases.

67. (a) $r = \dfrac{2(7)}{\sqrt{625 - 49}} = \dfrac{7}{12}$ ft/sec

(b) $r = \dfrac{2(15)}{\sqrt{625 - 225}} = \dfrac{3}{2}$ ft/sec

(c) $\lim_{x\to 25^-} \dfrac{2x}{\sqrt{625 - x^2}} = \infty$

68. (a) Average speed $= \dfrac{\text{Total distance}}{\text{Total time}}$

$$50 = \frac{2d}{(d/x) + (d/y)}$$

$$50 = \frac{2xy}{y + x}$$

$$50y + 50x = 2xy$$

$$50x = 2xy - 50y$$

$$50x = 2y(x - 25)$$

$$\frac{25x}{x - 25} = y$$

Domain: $x > 25$

(b)

x	30	40	50	60
y	150	66.667	50	42.857

(c) $\lim_{x\to 25^+} \dfrac{25x}{\sqrt{x - 25}} = \infty$

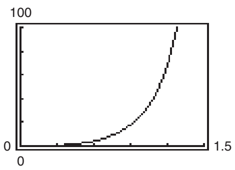
As x gets close to 25 mi/h, y becomes larger and larger.

69. (a) $A = \frac{1}{2}bh - \frac{1}{2}r^2\theta = \frac{1}{2}(10)(10\tan\theta) - \frac{1}{2}(10)^2\theta = 50\tan\theta - 50\theta$

Domain: $\left(0, \frac{\pi}{2}\right)$

(b)

θ	0.3	0.6	0.9	1.2	1.5
$f(\theta)$	0.47	4.21	18.0	68.6	630.1



(c) $\lim_{\theta\to\pi/2^-} A = \infty$

70. (a) Because the circumference of the motor is half that of the saw arbor, the saw makes $1700/2 = 850$ revolutions per minute.

(b) The direction of rotation is reversed.

(c) $2(20\cot\phi) + 2(10\cot\phi)$: straight sections. The angle subtended in each circle is $2\pi - \left(2\left(\frac{\pi}{2} - \phi\right)\right) = \pi + 2\phi$.

So, the length of the belt around the pulleys is $20(\pi + 2\phi) + 10(\pi + 2\phi) = 30(\pi + 2\phi)$.

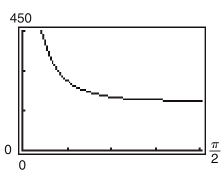
Total length $= 60\cot\phi + 30(\pi + 2\phi)$

Domain: $\left(0, \frac{\pi}{2}\right)$

(d)

ϕ	0.3	0.6	0.9	1.2	1.5
L	306.2	217.9	195.9	189.6	188.5

(e)



(f) $\lim_{\phi\to(\pi/2)^-} L = 60\pi \approx 188.5$

(All the belts are around pulleys.)

(g) $\lim_{\phi\to 0^+} L = \infty$

71. False. For instance, let

$f(x) = \frac{x^2 - 1}{x - 1}$ or

$g(x) = \frac{x}{x^2 + 1}$.

72. True

73. False. The graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$ and $y = \csc x$ have vertical asymptotes.

74. False. Let

$$f(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 3, & x = 0. \end{cases}$$

The graph of f has a vertical asymptote at $x = 0$, but $f(0) = 3$.

75. Let $f(x) = \dfrac{1}{x^2}$ and $g(x) = \dfrac{1}{x^4}$, and $c = 0$.

$\lim\limits_{x\to 0}\dfrac{1}{x^2} = \infty$ and $\lim\limits_{x\to 0}\dfrac{1}{x^4} = \infty$, but $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{x^4}\right) = \lim\limits_{x\to 0}\left(\dfrac{x^2 - 1}{x^4}\right) = -\infty \neq 0$.

76. Given $\lim\limits_{x\to c} f(x) = \infty$ and $\lim\limits_{x\to c} g(x) = L$:

(1) Difference:

Let $h(x) = -g(x)$. Then $\lim\limits_{x\to c} h(x) = -L$, and $\lim\limits_{x\to c}\left[f(x) - g(x)\right] = \lim\limits_{x\to c}\left[f(x) + h(x)\right] = \infty$, by the Sum Property.

(2) Product:

If $L > 0$, then for $\varepsilon = L/2 > 0$ there exists $\delta_1 > 0$ such that $\left|g(x) - L\right| < L/2$ whenever $0 < \left|x - c\right| < \delta_1$. So, $L/2 < g(x) < 3L/2$. Because $\lim\limits_{x\to c} f(x) = \infty$ then for $M > 0$, there exists $\delta_2 > 0$ such that $f(x) > M(2/L)$ whenever $\left|x - c\right| < \delta_2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < \left|x - c\right| < \delta$, you have $f(x)g(x) > M(2/L)(L/2) = M$. Therefore $\lim\limits_{x\to c} f(x)g(x) = \infty$. The proof is similar for $L < 0$.

(3) Quotient: Let $\varepsilon > 0$ be given.

There exists $\delta_1 > 0$ such that $f(x) > 3L/2\varepsilon$ whenever $0 < \left|x - c\right| < \delta_1$ and there exists $\delta_2 > 0$ such that $\left|g(x) - L\right| < L/2$ whenever $0 < \left|x - c\right| < \delta_2$. This inequality gives us $L/2 < g(x) < 3L/2$. Let δ be the smaller of δ_1 and δ_2. Then for $0 < \left|x - c\right| < \delta$, you have

$$\left|\frac{g(x)}{f(x)}\right| < \frac{3L/2}{3L/2\varepsilon} = \varepsilon.$$

Therefore, $\lim\limits_{x\to c}\dfrac{g(x)}{f(x)} = 0$.

77. Given $\lim\limits_{x\to c} f(x) = \infty$, let $g(x) = 1$. Then

$\lim\limits_{x\to c}\dfrac{g(x)}{f(x)} = 0$ by Theorem 1.15.

78. Given $\lim\limits_{x\to c}\dfrac{1}{f(x)} = 0$. Suppose $\lim\limits_{x\to c} f(x)$ exists and equals L.

Then, $\lim\limits_{x\to c}\dfrac{1}{f(x)} = \dfrac{\lim\limits_{x\to c} 1}{\lim\limits_{x\to c} f(x)} = \dfrac{1}{L} = 0$.

This is not possible. So, $\lim\limits_{x\to c} f(x)$ does not exist.

79. $f(x) = \dfrac{1}{x - 3}$ is defined for all $x > 3$.

Let $M > 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{1}{x - 3} > M$ whenever $3 < x < 3 + \delta$.

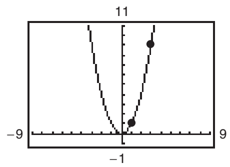
Equivalently, $x - 3 < \dfrac{1}{M}$ whenever $\left|x - 3\right| < \delta, x > 3$.

So take $\delta = \dfrac{1}{M}$. Then for $x > 3$ and $\left|x - 3\right| < \delta, \dfrac{1}{x - 3} > \dfrac{1}{8} = M$ and so $f(x) > M$.

80. $f(x) = \dfrac{1}{x - 5}$ is defined for all $x < 5$. Let $N < 0$ be given. You need $\delta > 0$ such that $f(x) = \dfrac{1}{x - 5} < N$ whenever $5 - \delta < x < 5$. Equivalently, $x - 5 > \dfrac{1}{N}$ whenever $\left|x - 5\right| < \delta, x < 5$. Equivalently, $\dfrac{1}{\left|x - 5\right|} < -\dfrac{1}{N}$ whenever $\left|x - 5\right| < \delta, x < 5$. So take $\delta = -\dfrac{1}{N}$. Note that $\delta > 0$ because $N < 0$. For $\left|x - 5\right| < \delta$ and $x < 5, \dfrac{1}{\left|x - 5\right|} > \dfrac{1}{\delta} = -N$, and $\dfrac{1}{x - 5} = -\dfrac{1}{\left|x - 5\right|} < N$.

Review Exercises for Chapter 2

1. Calculus required. Using a graphing utility, you can estimate the length to be 8.3. Or, the length is slightly longer than the distance between the two points, approximately 8.25.

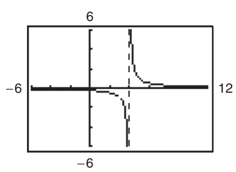


2. Precalculus. $L = \sqrt{(9-1)^2 + (3-1)^2} \approx 8.25$

3. $f(x) = \dfrac{x-3}{x^2 - 7x + 12}$

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$f(x)$	–0.9091	–0.9901	–0.9990	?	–1.0010	–1.0101	–1.1111

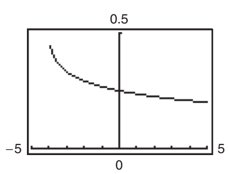
$\lim\limits_{x\to 3} f(x) \approx -1.0000$ (Actual limit is -1.)



4. $f(x) = \dfrac{\sqrt{x+4} - 2}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.2516	0.2502	0.2500	?	0.2500	0.2498	0.2485

$\lim\limits_{x\to 0} f(x) \approx 0.2500$ $\left(\text{Actual limit is } \frac{1}{4}.\right)$



5. $h(x) = \dfrac{4x - x^2}{x} = \dfrac{x(4-x)}{x} = 4 - x,\ x \neq 0$

(a) $\lim\limits_{x\to 0} h(x) = 4 - 0 = 4$

(b) $\lim\limits_{x\to -1} h(x) = 4 - (-1) = 5$

6. $f(t) = \dfrac{\ln(t+2)}{t}$

(a) $\lim\limits_{t\to 0} f(t)$ does not exist because $\lim\limits_{t\to 0^-} f(t) = -\infty$ and $\lim\limits_{t\to 0^+} f(t) = \infty$.

(b) $\lim\limits_{t\to -1} f(t) = \dfrac{\ln 1}{-1} = 0$

7. $\lim_{x\to 1}(x+4) = 1 + 4 = 5$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then for $0 < |x-1| < \delta = \varepsilon$, you have

$$\begin{aligned} |x-1| &< \varepsilon \\ |(x+4)-5| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

8. $\lim_{x\to 9}\sqrt{x} = \sqrt{9} = 3$

Let $\varepsilon > 0$ be given. You need

$\left|\sqrt{x}-3\right| < \varepsilon \Rightarrow \left|\sqrt{x}+3\right|\left|\sqrt{x}-3\right| < \varepsilon\left|\sqrt{x}+3\right| \Rightarrow |x-9| < \varepsilon\left|\sqrt{x}+3\right|$.

Assuming $4 < x < 16$, you can choose $\delta = 5\varepsilon$.

So, for $0 < |x-9| < \delta = 5\varepsilon$, you have

$$\begin{aligned} |x-9| < 5\varepsilon &< \left|\sqrt{x}+3\right|\varepsilon \\ \left|\sqrt{x}-3\right| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

9. $\lim_{x\to 2}(1-x^2) = 1 - 2^2 = -3$

Let $\varepsilon > 0$ be given. You need

$\left|1-x^2-(-3)\right| < \varepsilon \Rightarrow \left|x^2-4\right| = |x-2||x+2| < \varepsilon \Rightarrow |x-2| < \dfrac{1}{|x+2|}\varepsilon$

Assuming $1 < x < 3$, you can choose $\delta = \dfrac{\varepsilon}{5}$.

So, for $0 < |x-2| < \delta = \dfrac{\varepsilon}{5}$, you have

$$\begin{aligned} |x-2| < \frac{\varepsilon}{5} &< \frac{\varepsilon}{|x+2|} \\ |x-2||x+2| &< \varepsilon \\ \left|x^2-4\right| &< \varepsilon \\ \left|4-x^2\right| &< \varepsilon \\ \left|(1-x^2)-(-3)\right| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

10. $\lim_{x\to 5} 9 = 9$. Let $\varepsilon > 0$ be given. δ can be any positive number. So, for $0 < |x-5| < \delta$, you have

$$\begin{aligned} |9-9| &< \varepsilon \\ |f(x)-L| &< \varepsilon. \end{aligned}$$

11. $\lim_{x\to -6} x^2 = (-6)^2 = 36$

12. $\lim_{x\to 0}(5x-3) = 5(0) - 3 = -3$

13. $\lim_{x\to 6}(x-2)^2 = (6-2)^2 = 16$

14. $\lim_{x\to -5}\sqrt[3]{x-3} = \sqrt[3]{(-5)-3} = \sqrt[3]{-8} = -2$

15. $\lim_{x\to 4}\dfrac{4}{x-1} = \dfrac{4}{4-1} = \dfrac{4}{3}$

16. $\lim_{x\to 2}\dfrac{x}{x^2+1} = \dfrac{2}{2^2+1} = \dfrac{2}{4+1} = \dfrac{2}{5}$

17. $\lim_{t\to-2}\dfrac{t+2}{t^2-4} = \lim_{t\to-2}\dfrac{1}{t-2} = -\dfrac{1}{4}$

18. $\lim_{t\to4}\dfrac{t^2-16}{t-4} = \lim_{t\to4}\dfrac{(t-4)(t+4)}{t-4}$

$= \lim_{t\to4}(t+4) = 4+4 = 8$

19. $\lim_{x\to4}\dfrac{\sqrt{x-3}-1}{x-4} = \lim_{x\to4}\dfrac{\sqrt{x-3}-1}{x-4}\cdot\dfrac{\sqrt{x-3}+1}{\sqrt{x-3}+1}$

$= \lim_{x\to4}\dfrac{(x-3)-1}{(x-4)\left(\sqrt{x-3}+1\right)}$

$= \lim_{x\to4}\dfrac{1}{\sqrt{x-3}+1} = \dfrac{1}{2}$

20. $\lim_{x\to0}\dfrac{\sqrt{4+x}-2}{x} = \lim_{x\to0}\dfrac{\sqrt{4+x}-2}{x}\cdot\dfrac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = \lim_{x\to0}\dfrac{1}{\sqrt{4+x}+2} = \dfrac{1}{4}$

21. $\lim_{x\to0}\dfrac{\left[1/(x+1)\right]-1}{x} = \lim_{x\to0}\dfrac{1-(x+1)}{x(x+1)} = \lim_{x\to0}\dfrac{-1}{x+1} = -1$

22. $\lim_{s\to0}\dfrac{\left(1/\sqrt{1+s}\right)-1}{s} = \lim_{s\to0}\left[\dfrac{\left(1/\sqrt{1+s}\right)-1}{s}\cdot\dfrac{\left(1/\sqrt{1+s}\right)+1}{\left(1/\sqrt{1+s}\right)+1}\right]$

$= \lim_{s\to0}\dfrac{\left[1/(1+s)\right]-1}{s\left[\left(1/\sqrt{1+s}\right)+1\right]} = \lim_{s\to0}\dfrac{-1}{(1+s)\left[\left(1/\sqrt{1+s}\right)+1\right]} = -\dfrac{1}{2}$

23. $\lim_{x\to0}\dfrac{1-\cos x}{\sin x} = \lim_{x\to0}\left(\dfrac{x}{\sin x}\right)\left(\dfrac{1-\cos x}{x}\right) = (1)(0) = 0$

24. $\lim_{x\to(\pi/4)}\dfrac{4x}{\tan x} = \dfrac{4(\pi/4)}{1} = \pi$

25. $\lim_{x\to1} e^{x-1}\sin\dfrac{\pi x}{2} = e^0\sin\dfrac{\pi}{2} = 1$

26. $\lim_{x\to2}\dfrac{\ln(x-1)^2}{\ln(x-1)} = \lim_{x\to2}\dfrac{2\ln(x-1)}{\ln(x-1)} = \lim_{x\to2} 2 = 2$

27. $\lim_{\Delta x\to0}\dfrac{\sin\left[(\pi/6)+\Delta x\right]-(1/2)}{\Delta x} = \lim_{\Delta x\to0}\dfrac{\sin(\pi/6)\cos\Delta x+\cos(\pi/6)\sin\Delta x-(1/2)}{\Delta x}$

$= \lim_{\Delta x\to0}\dfrac{1}{2}\cdot\dfrac{(\cos\Delta x-1)}{\Delta x} + \lim_{\Delta x\to0}\dfrac{\sqrt{3}}{2}\cdot\dfrac{\sin\Delta x}{\Delta x} = 0 + \dfrac{\sqrt{3}}{2}(1) = \dfrac{\sqrt{3}}{2}$

28. $\lim_{\Delta x\to0}\dfrac{\cos(\pi+\Delta x)+1}{\Delta x} = \lim_{\Delta x\to0}\dfrac{\cos\pi\cos\Delta x-\sin\pi\sin\Delta x+1}{\Delta x}$

$= \lim_{\Delta x\to0}\left[-\dfrac{(\cos\Delta x-1)}{\Delta x}\right] - \lim_{\Delta x\to0}\left[\sin\pi\dfrac{\sin\Delta x}{\Delta x}\right]$

$= -0-(0)(1) = 0$

29. $\lim_{x\to c}\left[f(x)g(x)\right] = \left[\lim_{x\to c}f(x)\right]\left[\lim_{x\to c}g(x)\right]$

$= (-6)\left(\tfrac{1}{2}\right) = -3$

30. $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)} = \dfrac{-6}{\left(\frac{1}{2}\right)} = -12$

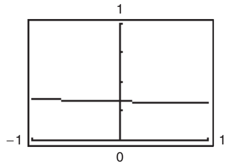
31. $\lim_{x\to c}\left[f(x)+2g(x)\right] = \lim_{x\to c}f(x) + 2\lim_{x\to c}g(x)$

$= -6+2\left(\tfrac{1}{2}\right) = -5$

32. $\lim_{x\to c}\left[f(x)\right]^2 = \left[\lim_{x\to c}f(x)\right]^2$

$= (-6)^2 = 36$

33. $f(x) = \dfrac{\sqrt{2x + 9} - 3}{x}$



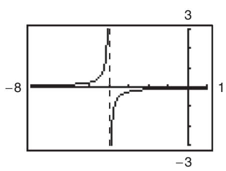
The limit appears to be $\frac{1}{3}$.

x	–0.01	–0.001	0	0.001	0.01
$f(x)$	0.3335	0.3333	?	0.3333	0.331

$\lim_{x\to 0} f(x) \approx 0.3333$

$$\lim_{x\to 0} \frac{\sqrt{2x + 9} - 3}{x} \cdot \frac{\sqrt{2x + 9} + 3}{\sqrt{2x + 9} + 3} = \lim_{x\to 0} \frac{(2x + 9) - 9}{x\left[\sqrt{2x + 9} + 3\right]} = \lim_{x\to 0} \frac{2}{\sqrt{2x + 9} + 3} = \frac{2}{\sqrt{9} + 3} = \frac{1}{3}$$

34. $f(x) = \dfrac{\left[1/(x + 4)\right] - (1/4)}{x}$



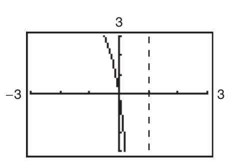
The limit appears to be $-\frac{1}{16}$

x	–0.01	–0.001	0	0.001	0.01
$f(x)$	–0.0627	–0.0625	?	–0.0625	–0.0623

$\lim_{x\to 0} f(x) \approx -0.0625 = -\dfrac{1}{16}$

$$\lim_{x\to 0} \frac{\frac{1}{x + 4} - \frac{1}{4}}{x} = \lim_{x\to 0} \frac{4 - (x + 4)}{(x + 4)4(x)} = \lim_{x\to 0} \frac{-1}{(x + 4)4} = -\frac{1}{16}$$

35. $f(x) = \lim_{x\to 0} \dfrac{20\left(e^{x/2} - 1\right)}{x - 1}$



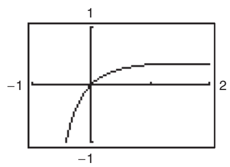
The limit appears to be 0.

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	0.8867	0.0988	0.0100	–0.0100	–0.1013	–1.1394

$\lim_{x\to 0} f(x) \approx 0.0000$

$$\lim_{x\to 0} \frac{20\left(e^{x/2} - 1\right)}{x - 1} = \frac{20\left(e^{0} - 1\right)}{0 - 1} = \frac{0}{-1} = 0$$

36. $f(x) = \dfrac{\ln(x+1)}{x+1}$



The limit appears to be 0.

x	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(x)$	−0.1171	−0.0102	−0.0010	?	0.0010	0.0099	0.0866

$\lim\limits_{x\to 0} f(x) \approx 0.0000$

$$\lim_{x\to 0} \frac{\ln(x+1)}{x+1} = \frac{\ln 1}{1} = \frac{0}{1} = 0$$

37.
$$\begin{aligned} v &= \lim_{t\to 4} \frac{s(4) - s(t)}{4 - t} \\ &= \lim_{t\to 4} \frac{\left[-4.9(16) + 250\right] - \left[-4.9t^2 + 250\right]}{4 - t} \\ &= \lim_{t\to 4} \frac{4.9\left(t^2 - 16\right)}{4 - t} \\ &= \lim_{t\to 4} \frac{4.9(t - 4)(t + 4)}{4 - t} \\ &= \lim_{t\to 4} \left[-4.9(t + 4)\right] = -39.2 \text{ m/sec} \end{aligned}$$

The object is falling at about 39.2 m/sec.

38. $-4.9t^2 + 250 = 0 \Rightarrow t = \dfrac{50}{7}$ sec

When $a = \dfrac{50}{7}$, the velocity is

$$\begin{aligned} \lim_{t\to a} \frac{s(a) - s(t)}{a - t} &= \lim_{t\to a} \frac{\left[-4.9a^2 + 250\right] - \left[-4.9t^2 + 250\right]}{a - t} \\ &= \lim_{t\to a} \frac{4.9\left(t^2 - a^2\right)}{a - t} \\ &= \lim_{t\to a} \frac{4.9(t - a)(t + a)}{a - t} \\ &= \lim_{t\to a} \left[-4.9(t + a)\right] \\ &= -4.9(2a) \qquad \left(a = \frac{50}{7}\right) \\ &= -70 \text{ m/sec.} \end{aligned}$$

The velocity of the object when it hits the ground is about 70 m/sec.

39. $\lim\limits_{x\to 3^+} \dfrac{1}{x+3} = \dfrac{1}{3+3} = \dfrac{1}{6}$

40.
$$\begin{aligned} \lim_{x\to 6^-} \frac{x - 6}{x^2 - 36} &= \lim_{x\to 6^-} \frac{x - 6}{(x - 6)(x + 6)} \\ &= \lim_{x\to 6^-} \frac{1}{x + 6} \\ &= \frac{1}{12} \end{aligned}$$

41. $$\lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4^-} \frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4^-} \frac{x-4}{(x-4)\left(\sqrt{x}+2\right)} = \lim_{x\to 4^-} \frac{1}{\sqrt{x}+2} = \frac{1}{4}$$

42. $\lim_{x\to 3^-} \frac{|x-3|}{x-3} = \lim_{x\to 3^-} \frac{-(x-3)}{x-3} = -1$

43. $\lim_{x\to 2^-} \left(2[\![x]\!]+1\right) = 2(1)+1 = 3$

44. $\lim_{x\to 4} [\![x-1]\!]$ does not exist. There is a break in the graph at $x = 4$.

45. $\lim_{x\to 2} f(x) = 0$

46. $\lim_{x\to 1^+} g(x) = 1+1 = 2$

47. $\lim_{t\to 1} h(t)$ does not exist because $\lim_{t\to 1^-} h(t) = 1+1 = 2$ and $\lim_{t\to 1^+} h(t) = \frac{1}{2}(1+1) = 1$.

48. $\lim_{s\to -2} f(s) = 2$

49. $f(x) = x^2 - 4$ is continuous for all real x.

50. $f(x) = x^2 - x + 20$ is continuous for all real x.

51. $f(x) = \frac{4}{x-5}$ has a nonremovable discontinuity at $x = 5$ because $\lim_{x\to 5} f(x)$ does not exist.

52. $f(x) = \frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)}$

has nonremovable discontinuities at $x = \pm 3$ because $\lim_{x\to 3} f(x)$ and $\lim_{x\to -3} f(x)$ do not exist.

53. $f(x) = \frac{x}{x^3-x} = \frac{x}{x(x^2-1)} = \frac{1}{(x-1)(x+1)},\ x \neq 0$

has nonremovable discontinuities at $x = \pm 1$ because $\lim_{x\to -1} f(x)$ and $\lim_{x\to 1} f(x)$ do not exist, and has a removable discontinuity at $x = 0$ because

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{(x-1)(x+1)} = -1.$$

54. $$f(x) = \frac{x+3}{x^2-3x-18} = \frac{x+3}{(x+3)(x-6)} = \frac{1}{x-6},\ x \neq -3$$

has a nonremovable discontinuity at $x = 6$ because $\lim_{x\to 6} f(x)$ does not exist, and has a removable discontinuity at $x = -3$ because

$$\lim_{x\to -3} f(x) = \lim_{x\to -3} \frac{1}{x-6} = -\frac{1}{9}.$$

55. $f(2) = 5$

Find c so that $\lim_{x\to 2^+} (cx+6) = 5$.

$$c(2)+6 = 5$$
$$2c = -1$$
$$c = -\frac{1}{2}$$

56. $\lim_{x\to 1^+} (x+1) = 2$

$\lim_{x\to 3^-} (x+1) = 4$

Find b and c so that $\lim_{x\to 1^-} \left(x^2+bx+c\right) = 2$ and $\lim_{x\to 3^+} \left(x^2+bx+c\right) = 4$.

Consequently you get $1+b+c = 2$ and $9+3b+c = 4$.

Solving simultaneously, $b = -3$ and $c = 4$.

57. $f(x) = -3x^2 + 7$

Continuous on $(-\infty, \infty)$

58. $f(x) = \frac{4x^2+7x-2}{x+2} = \frac{(4x-1)(x+2)}{x+2}$

Continuous on $(-\infty, -2) \cup (-2, \infty)$. There is a removable discontinuity at $x = -2$.

59. $f(x) = \sqrt{x - 4}$

Continuous on $[4, \infty)$

60. $f(x) = [\![x + 3]\!]$

$\lim\limits_{x \to k^+} [\![x + 3]\!] = k + 3$ where k is an integer.

$\lim\limits_{x \to k^-} [\![x + 3]\!] = k + 2$ where k is an integer.

Nonremovable discontinuity at each integer k

Continuous on $(k, k + 1)$ for all integers k

61. $g(x) = 2e^{[\![x]\!]/4}$ is continuous on all intervals $(n, n + 1)$, where n is an integer. g has nonremovable discontinuities at each n.

62. $h(x) = -2 \ln |5 - x|$

Because $|5 - x| > 0$ except for $x = 5$, h is continuous on $(-\infty, 5) \cup (5, \infty)$.

63. $f(x) = \dfrac{3x^2 - x - 2}{x - 1} = \dfrac{(3x + 2)(x - 1)}{x - 1}$

$\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} (3x + 2) = 5$

Removable discontinuity at $x = 1$

Continuous on $(-\infty, 1) \cup (1, \infty)$

64. $f(x) = \begin{cases} 5 - x, & x \le 2 \\ 2x - 3, & x > 2 \end{cases}$

$\lim\limits_{x \to 2^-} (5 - x) = 3$

$\lim\limits_{x \to 2^+} (2x - 3) = 1$

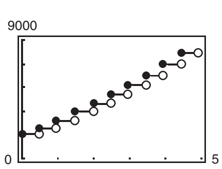
Nonremovable discontinuity at $x = 2$

Continuous on $(-\infty, 2) \cup (2, \infty)$

65. f is continuous on $[1, 2]$. $f(1) = -1 < 0$ and $f(2) = 13 > 0$. Therefore by the Intermediate Value Theorem, there is at least one value c in $(1, 2)$ such that $2c^3 - 3 = 0$.

66. $A = 5000(1.06)^{[\![2t]\!]}$

Nonremovable discontinuity every 6 months



67. $f(x) = \dfrac{x^2 - 4}{|x - 2|} = (x + 2)\left[\dfrac{x - 2}{|x - 2|}\right]$

(a) $\lim\limits_{x \to 2^-} f(x) = -4$

(b) $\lim\limits_{x \to 2^+} f(x) = 4$

(c) $\lim\limits_{x \to 2} f(x)$ does not exist.

68. $f(x) = \sqrt{(x - 1)x}$

(a) Domain: $(-\infty, 0] \cup [1, \infty)$

(b) $\lim\limits_{x \to 0^-} f(x) = 0$

(c) $\lim\limits_{x \to 1^+} f(x) = 0$

69. $f(x) = \dfrac{3}{x}$

$\lim\limits_{x \to 0^-} \dfrac{3}{x} = -\infty$

$\lim\limits_{x \to 0^+} \dfrac{3}{x} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

70. $f(x) = \dfrac{5}{(x - 2)^4}$

$\lim\limits_{x \to 2^-} \dfrac{5}{(x - 2)^4} = \infty = \lim\limits_{x \to 2^+} \dfrac{5}{(x - 2)^4}$

Therefore, $x = 2$ is a vertical asymptote.

71. $f(x) = \dfrac{x^3}{x^2 - 9} = \dfrac{x^3}{(x + 3)(x - 3)}$

$\lim\limits_{x \to -3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim\limits_{x \to -3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = -3$ is a vertical asymptote.

$\lim\limits_{x \to -3^-} \dfrac{x^3}{x^2 - 9} = -\infty$ and $\lim\limits_{x \to 3^+} \dfrac{x^3}{x^2 - 9} = \infty$

Therefore, $x = 3$ is a vertical asymptote.

72. $f(x) = \dfrac{6x}{36 - x^2} = -\dfrac{6x}{(x + 6)(x - 6)}$

$\lim\limits_{x \to -6^-} \dfrac{6x}{36 - x^2} = \infty$ and $\lim\limits_{x \to -6^+} \dfrac{6x}{36 - x^2} = -\infty$

Therefore, $x = -6$ is a vertical asymptote.

$\lim\limits_{x \to 6^-} \dfrac{6x}{36 - x^2} = \infty$ and $\lim\limits_{x \to 6^+} \dfrac{6x}{36 - x^2} = -\infty$

Therefore, $x = 6$ is a vertical asymptote.

73. $g(x) = \dfrac{2x+1}{x^2-64} = \dfrac{2x+1}{(x+8)(x-8)}$

$\lim\limits_{x\to-8^-} \dfrac{2x+1}{x^2-64} = -\infty$ and $\lim\limits_{x\to-8^+} \dfrac{2x+1}{x^2-64} = \infty$

Therefore, $x = -8$ is a vertical asymptote.

$\lim\limits_{x\to8^-} \dfrac{2x+1}{x^2-64} = -\infty$ and $\lim\limits_{x\to8^+} \dfrac{2x+1}{x^2-64} = \infty$

Therefore, $x = 8$ is a vertical asymptote.

74. $f(x) = \csc \pi x = \dfrac{1}{\sin \pi x}$

$\sin \pi x = 0$ for $x = n$, where n is an integer.

$\lim\limits_{x\to n} f(x) = \infty$ or $-\infty$

Therefore, the graph has vertical asymptotes at $x = n$.

75. $g(x) = \ln(25 - x^2) = \ln[(5+x)(5-x)]$

$\lim\limits_{x\to5} \ln(25 - x^2) = 0$

$\lim\limits_{x\to-5} \ln(25 - x^2) = 0$

Therefore, the graph has holes at $x = \pm5$. The graph does not have any vertical asymptotes.

76. $f(x) = 7e^{-3/x}$

$\lim\limits_{x\to0^-} 7e^{-3/x} = \infty$

Therefore, $x = 0$ is a vertical asymptote.

77. $\lim\limits_{x\to1^-} \dfrac{x^2+2x+1}{x-1} = -\infty$

78. $\lim\limits_{x\to(1/2)^+} \dfrac{x}{2x-1} = \infty$

79. $\lim\limits_{x\to-1^+} \dfrac{x+1}{x^3+1} = \lim\limits_{x\to-1^+} \dfrac{1}{x^2-x+1} = \dfrac{1}{3}$

80. $\lim\limits_{x\to-1^-} \dfrac{x+1}{x^4-1} = \lim\limits_{x\to-1^-} \dfrac{1}{(x^2+1)(x-1)} = -\dfrac{1}{4}$

81. $\lim\limits_{x\to0^+} \left(x - \dfrac{1}{x^3}\right) = -\infty$

82. $\lim\limits_{x\to2^-} \dfrac{1}{\sqrt[3]{x^2-4}} = -\infty$

83. $\lim\limits_{x\to0^+} \dfrac{\sin 4x}{5x} = \lim\limits_{x\to0^+} \left[\dfrac{4}{5}\left(\dfrac{\sin 4x}{4x}\right)\right] = \dfrac{4}{5}$

84. $\lim\limits_{x\to0^+} \dfrac{\sec x}{x} = \infty$

85. $\lim\limits_{x\to0^+} \dfrac{\csc 2x}{x} = \lim\limits_{x\to0^+} \dfrac{1}{x \sin 2x} = \infty$

86. $\lim\limits_{x\to0^-} \dfrac{\cos^2 x}{x} = -\infty$

87. $\lim\limits_{x\to0^+} \ln(\sin x) = -\infty$

88. $\lim\limits_{x\to0^-} 12e^{-2/x} = \infty$

89. $C = \dfrac{80{,}000p}{100-p}, 0 \le p < 100$

(a) $C(15) \approx \$14{,}117.65$

(b) $C(50) = \$80.000$

(c) $C(90) = \$720{,}000$

(d) $\lim\limits_{p\to100^-} \dfrac{80{,}000p}{100-p} = \infty$

90. $f(x) = \dfrac{\tan 2x}{x}$

(a)

x	–0.1	–0.01	–0.001	0.001	0.01	0.1
$f(x)$	2.0271	2.0003	2.0000	2.0000	2.0003	2.0271

$\lim\limits_{x\to0} \dfrac{\tan 2x}{x} = 2$

(b) Yes, define $f(x) = \begin{cases} \dfrac{\tan 2x}{x}, & x \ne 0 \\ 2, & x = 0 \end{cases}.$

Now $f(x)$ is continuous at $x = 0$.

Problem Solving for Chapter 2

1. (a) $\text{Perimeter } \Delta PAO = \sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + y^2} + 1$

$$= \sqrt{x^2 + (x^2-1)^2} + \sqrt{x^2 + x^4} + 1$$

$\text{Perimeter } \Delta PBO = \sqrt{(x-1)^2 + y^2} + \sqrt{x^2 + y^2} + 1$

$$= \sqrt{(x-1)^2 + x^4} + \sqrt{x^2 + x^4} + 1$$

(b) $$r(x) = \frac{\sqrt{x^2 + (x^2-1)^2} + \sqrt{x^2 + x^4} + 1}{\sqrt{(x-1)^2 + x^4} + \sqrt{x^2 + x^4} + 1}$$

x	4	2	1	0.1	0.01
Perimeter ΔPAO	33.02	9.08	3.41	2.10	2.01
Perimeter ΔPBO	33.77	9.60	3.41	2.00	2.00
$r(x)$	0.98	0.95	1	1.05	1.005

(c) $$\lim_{x\to 0^+} r(x) = \frac{1+0+1}{1+0+1} = \frac{2}{2} = 1$$

2. (a) $\text{Area } \Delta PAO = \frac{1}{2}bh = \frac{1}{2}(1)(x) = \frac{x}{2}$

$\text{Area } \Delta PBO = \frac{1}{2}bh = \frac{1}{2}(1)(y) = \frac{y}{2} = \frac{x^2}{2}$

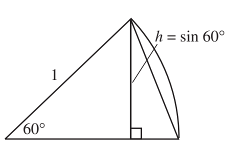
(b) $$a(x) = \frac{\text{Area } \Delta PBO}{\text{Area } \Delta PAO} = \frac{x^2/2}{x/2} = x$$

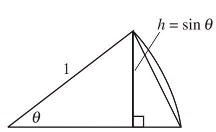
x	4	2	1	0.1	0.01
Area ΔPAO	2	1	1/2	1/20	1/200
Area ΔPBO	8	2	1/2	1/200	1/20,000
$a(x)$	4	2	1	1/10	1/100

(c) $$\lim_{x\to 0^+} a(x) = \lim_{x\to 0^+} x = 0$$

3. (a) There are 6 triangles, each with a central angle of $60° = \pi/3$. So,

$$\text{Area hexagon} = 6\left[\frac{1}{2}bh\right] = 6\left[\frac{1}{2}(1)\sin\frac{\pi}{3}\right] = \frac{3\sqrt{3}}{2} \approx 2.598.$$





$$\text{Error} = \text{Area (Circle)} - \text{Area (Hexagon)} = \pi - \frac{3\sqrt{3}}{2} \approx 0.5435$$

(b) There are n triangles, each with central angle of $\theta = 2\pi/n$. So,

$$A_n = n\left[\frac{1}{2}bh\right] = n\left[\frac{1}{2}(1)\sin\frac{2\pi}{n}\right] = \frac{n\sin(2\pi/n)}{2}.$$

(c)

n	6	12	24	48	96
A_n	2.598	3	3.106	3.133	3.139

(d) As n gets larger and larger, $2\pi/n$ approaches 0. Letting $x = 2\pi/n$, $A_n = \dfrac{\sin(2\pi/n)}{2/n} = \dfrac{\sin(2\pi/n)}{(2\pi/n)}\pi = \dfrac{\sin x}{x}\pi$

which approaches $(1)\pi = \pi$.

4. (a) Slope $= \dfrac{4-0}{3-0} = \dfrac{4}{3}$

(b) Slope $= -\dfrac{3}{4}$ Tangent line: $y - 4 = -\dfrac{3}{4}(x-3)$

$$y = -\frac{3}{4}x + \frac{25}{4}$$

(c) Let $Q = (x, y) = \left(x, \sqrt{25 - x^2}\right)$

$$m_x = \frac{\sqrt{25-x^2} - 4}{x-3}$$

(d)
$$\begin{aligned}\lim_{x\to 3} m_x &= \lim_{x\to 3}\frac{\sqrt{25-x^2}-4}{x-3}\cdot\frac{\sqrt{25-x^2}+4}{\sqrt{25-x^2}+4}\\ &= \lim_{x\to 3}\frac{25-x^2-16}{(x-3)\left(\sqrt{25-x^2}+4\right)}\\ &= \lim_{x\to 3}\frac{(3-x)(3+x)}{(x-3)\left(\sqrt{25-x^2}+4\right)}\\ &= \lim_{x\to 3}\frac{-(3+x)}{\sqrt{25-x^2}+4} = \frac{-6}{4+4} = -\frac{3}{4}\end{aligned}$$

This is the slope of the tangent line at P.

5. (a) Slope $= -\dfrac{12}{5}$

(b) Slope of tangent line is $\dfrac{5}{12}$.

$$y + 12 = \frac{5}{12}(x-5)$$

$$y = \frac{5}{12}x - \frac{169}{12} \text{ Tangent line}$$

(c) $Q = (x, y) = \left(x, -\sqrt{169 - x^2}\right)$

$$m_x = \frac{-\sqrt{169-x^2}+12}{x-5}$$

(d)
$$\begin{aligned}\lim_{x\to 5} m_x &= \lim_{x\to 5}\frac{12-\sqrt{169-x^2}}{x-5}\cdot\frac{12+\sqrt{169-x^2}}{12+\sqrt{169-x^2}}\\ &= \lim_{x\to 5}\frac{144-(169-x^2)}{(x-5)\left(12+\sqrt{169-x^2}\right)}\\ &= \lim_{x\to 5}\frac{x^2-25}{(x-5)\left(12+\sqrt{169-x^2}\right)}\\ &= \lim_{x\to 5}\frac{(x+5)}{12+\sqrt{169-x^2}} = \frac{10}{12+12} = \frac{5}{12}\end{aligned}$$

This is the same slope as part (b).

6. $$\frac{\sqrt{a+bx}-\sqrt{3}}{x} = \frac{\sqrt{a+bx}-\sqrt{3}}{x}\cdot\frac{\sqrt{a+bx}+\sqrt{3}}{\sqrt{a+bx}+\sqrt{3}} = \frac{(a+bx)-3}{x\left(\sqrt{a+bx}+\sqrt{3}\right)}$$

Letting $a = 3$ simplifies the numerator.

So, $\displaystyle\lim_{x\to 0}\frac{\sqrt{3+bx}-\sqrt{3}}{x} = \lim_{x\to 0}\frac{bx}{x\left(\sqrt{3+bx}+\sqrt{3}\right)} = \lim_{x\to 0}\frac{b}{\sqrt{3+bx}+\sqrt{3}}$

Setting $\dfrac{b}{\sqrt{3}+\sqrt{3}} = \sqrt{3}$, you obtain $b = 6$. So, $a = 3$ and $b = 6$.

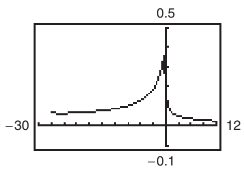
7. (a) $3 + x^{1/3} \geq 0$

$x^{1/3} \geq -3$

$x \geq -27$

Domain: $x \geq -27, x \neq 1$ or $[-27, 1) \cup (1, \infty)$

(b)



(c) $\lim_{x \to -27^+} f(x) = \dfrac{\sqrt{3 + (-27)^{1/3}} - 2}{-27 - 1} = \dfrac{-2}{-28} = \dfrac{1}{14}$

≈ 0.0714

(d)
$$\begin{aligned}
\lim_{x \to 1} f(x) &= \lim_{x \to 1} \frac{\sqrt{3 + x^{1/3}} - 2}{x - 1} \cdot \frac{\sqrt{3 + x^{1/3}} + 2}{\sqrt{3 + x^{1/3}} + 2}\\
&= \lim_{x \to 1} \frac{3 + x^{1/3} - 4}{(x - 1)\left(\sqrt{3 + x^{1/3}} + 2\right)}\\
&= \lim_{x \to 1} \frac{x^{1/3} - 1}{\left(x^{1/3} - 1\right)\left(x^{2/3} + x^{1/3} + 1\right)\left(\sqrt{3 + x^{1/3}} + 2\right)}\\
&= \lim_{x \to 1} \frac{1}{\left(x^{2/3} + x^{1/3} + 1\right)\left(\sqrt{3 + x^{1/3}} + 2\right)}\\
&= \frac{1}{(1 + 1 + 1)(2 + 2)} = \frac{1}{12}
\end{aligned}$$

8. $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \left(a^2 - 2\right) = a^2 - 2$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \dfrac{ax}{\tan x} = a\left(\text{because } \lim_{x \to 0} \dfrac{\tan x}{x} = 1\right)$

Thus, $a^2 - 2 = a$

$a^2 - a - 2 = 0$

$(a - 2)(a + 1) = 0$

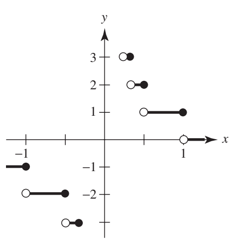
$a = -1, 2$

9. (a) $\lim_{x \to 2} f(x) = 3$: g_1, g_4

(b) f continuous at 2: g_1

(c) $\lim_{x \to 2^-} f(x) = 3$: g_1, g_3, g_4

10.



(a) $f\left(\frac{1}{4}\right) = [\![4]\!] = 4$

$f(3) = \left[\!\left[\frac{1}{3}\right]\!\right] = 0$

$f(1) = [\![1]\!] = 1$

(b) $\lim_{x \to 1^-} f(x) = 1$

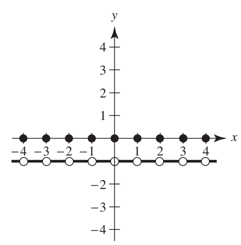
$\lim_{x \to 1^+} f(x) = 0$

$\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0^+} f(x) = \infty$

(c) f is continuous for all real numbers except $x = 0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$

11.



(a) $f(1) = [\![1]\!] + [\![-1]\!] = 1 + (-1) = 0$

$f(0) = 0$

$f\left(\frac{1}{2}\right) = 0 + (-1) = -1$

$f(-2.7) = -3 + 2 = -1$

(b) $\lim_{x \to 1^-} f(x) = -1$

$\lim_{x \to 1^+} f(x) = -1$

$\lim_{x \to 1/2} f(x) = -1$

(c) f is continuous for all real numbers except $x = 0, \pm 1, \pm 2, \pm 3, \ldots$

12. (a)
$$v^2 = \frac{192{,}000}{r} + v_0^2 - 48$$
$$\frac{192{,}000}{r} = v^2 - v_0^2 + 48$$
$$r = \frac{192{,}000}{v^2 - v_0^2 + 48}$$
$$\lim_{v\to 0} r = \frac{192{,}000}{48 - v_0^2}$$

Let $v_0 = \sqrt{48} = 4\sqrt{3}$ mi/sec.

(b)
$$v^2 = \frac{1920}{r} + v_0^2 - 2.17$$
$$\frac{1920}{r} = v^2 - v_0^2 + 2.17$$
$$r = \frac{1920}{v^2 - v_0^2 + 2.17}$$
$$\lim_{v\to 0} r = \frac{1920}{2.17 - v_0^2}$$

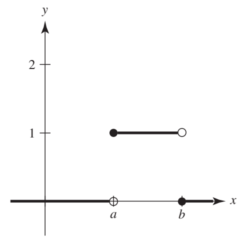
Let $v_0 = \sqrt{2.17}$ mi/sec $\quad (\approx 1.47$ mi/sec$)$.

(c)
$$r = \frac{10{,}600}{v^2 - v_0^2 + 6.99}$$
$$\lim_{v\to 0} r = \frac{10{,}600}{6.99 - v_0^2}$$

Let $v_0 = \sqrt{6.99} \approx 2.64$ mi/sec.

Because this is smaller than the escape velocity for Earth, the mass is less.

13. (a)



(b) (i) $\lim_{x\to a^+} P_{a,b}(x) = 1$

(ii) $\lim_{x\to a^-} P_{a,b}(x) = 0$

(iii) $\lim_{x\to b^+} P_{a,b}(x) = 0$

(iv) $\lim_{x\to b^-} P_{a,b}(x) = 1$

(c) $P_{a,b}$ is continuous for all positive real numbers except $x = a, b$.

(d) The area under the graph of U, and above the x-axis, is 1.

14. Let $a \neq 0$ and let $\varepsilon > 0$ be given. There exists $\delta_1 > 0$ such that if $0 < |x - 0| < \delta_1$ then $|f(x) - L| < \varepsilon$. Let $\delta = \delta_1/|a|$. Then for $0 < |x - 0| < \delta = \delta_1/|a|$, you have
$$|x| < \frac{\delta_1}{|a|}$$
$$|ax| < \delta_1$$
$$|f(ax) - L| < \varepsilon.$$

As a counterexample, let
$$a = 0 \text{ and } f(x) = \begin{cases} 1, & x \neq 0 \\ 2, & x = 0 \end{cases}.$$

Then $\lim_{x\to 0} f(x) = 1 = L$, but
$$\lim_{x\to 0} f(ax) = \lim_{x\to 0} f(0) = \lim_{x\to 0} 2 = 2.$$