Section 3.1

3.1.1: Given $f(x) = 4x - 5$, we have $a = 0$, $b = 4$, and $c = -5$, so $f'(x) = 2ax + b = 4$.

3.1.2: Given $g(t) = -16t^2 + 100$, we have $a = -16$, $b = 0$, and $c = 100$, so $g'(t) = 2at + b = -32t$.

3.1.3: If $h(z) = z(25 - z) = -z^2 + 25z$, then $a = -1$, $b = 25$, and $c = 0$, so $h'(z) = 2az + b = -2z + 25$.

3.1.4: If $f(x) = -49x + 16$, then $a = 0$, $b = -49$, and $c = 16$, so $f'(x) = -49$.

3.1.5: If $y = 2x^2 + 3x - 17$, then $a = 2$, $b = 3$, and $c = -17$, so $\dfrac{dy}{dx} = 2ax + b = 4x + 3$.

3.1.6: If $x = -100t^2 + 16t$, then $a = -100$, $b = 16$, and $c = 0$, so $\dfrac{dx}{dt} = 2at + b = -200t + 16$.

3.1.7: If $z = 5u^2 - 3u$, then $a = 5$, $b = -3$, and $c = 0$, so $\dfrac{dz}{du} = 2au + b = 10u - 3$.

3.1.8: If $v = -5y^2 + 500y$, then $a = -5$, $b = 500$, and $c = 0$, so $\dfrac{dv}{dy} = 2ay + b = -10y + 500$.

3.1.9: If $x = -5y^2 + 17y + 300$, then $a = -5$, $b = 17$, and $c = 300$, so $\dfrac{dx}{dy} = 2ay + b = -10y + 17$.

3.1.10: If $u = 7t^2 + 13t$, then $a = 7$, $b = 13$, and $c = 0$, so $\dfrac{du}{dt} = 2at + b = 14t + 13$.

3.1.11: $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2(x+h) - 1 - (2x-1)}{h} = \lim_{h\to 0} \frac{2x + 2h - 1 - 2x + 1}{h}$$
$$= \lim_{h\to 0} \frac{2h}{h} = \lim_{h\to 0} 2 = 2.$$

3.1.12: $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{2 - 3(x+h) - (2 - 3x)}{h} = \lim_{h\to 0} \frac{2 - 3x - 3h - 2 + 3x}{h}$$
$$= \lim_{h\to 0} \frac{-3h}{h} = \lim_{h\to 0} (-3) = -3.$$

3.1.13: $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 5 - (x^2+5)}{h}$$
$$= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 5 - x^2 - 5}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h} = \lim_{h\to 0} (2x + h) = 2x.$$

3.1.14: $$f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3 - 2(x+h)^2 - (3 - 2x^2)}{h}$$
$$= \lim_{h\to 0} \frac{3 - 2x^2 - 4xh - 2h^2 - 3 + 2x^2}{h} = \lim_{h\to 0} \frac{-4xh - 2h^2}{h} = \lim_{h\to 0} (-4x - 2h) = -4x.$$

3.1.15: $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{2(x+h)+1} - \dfrac{1}{2x+1}}{h}$$

$$= \lim_{h\to 0} \frac{2x+1-(2x+2h+1)}{h(2x+2h+1)(2x+1)} = \lim_{h\to 0} \frac{2x+1-2x-2h-1}{h(2x+2h+1)(2x+1)} = \lim_{h\to 0} \frac{-2h}{h(2x+2h+1)(2x+1)}$$

$$= \lim_{h\to 0} \frac{-2}{(2x+2h+1)(2x+1)} = \frac{-2}{(2x+1)^2}.$$

3.1.16: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\dfrac{1}{3-(x+h)} - \dfrac{1}{3-x}}{h}$

$$= \lim_{h\to 0} \frac{(3-x)-(3-x-h)}{h(3-x-h)(3-x)} = \lim_{h\to 0} \frac{3-x-3+x+h}{h(3-x-h)(3-x)} = \lim_{h\to 0} \frac{h}{h(3-x-h)(3-x)}$$

$$= \lim_{h\to 0} \frac{1}{(3-x-h)(3-x)} = \frac{1}{(3-x)^2}.$$

3.1.17: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h}$

$$= \lim_{h\to 0} \frac{\left(\sqrt{2x+2h+1} - \sqrt{2x+1}\right)\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{(2h+2h+1)-(2x+1)}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)}$$

$$= \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2x+2h+1} + \sqrt{2x+1}\right)} = \lim_{h\to 0} \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}}$$

$$= \frac{2}{2\sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}.$$

3.1.18: $f'(x) = \lim_{h\to 0} \dfrac{1}{h}\left(f(x+h)-f(x)\right) = \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{1}{\sqrt{x+h+1}} - \dfrac{1}{\sqrt{x+1}}\right)$

$$= \lim_{h\to 0} \frac{\sqrt{x+1} - \sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} = \lim_{h\to 0} \frac{\left(\sqrt{x+1} - \sqrt{x+h+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}{h\left(\sqrt{x+h+1}\sqrt{x+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}$$

$$= \lim_{h\to 0} \frac{(x+1)-(x+h+1)}{h\left(\sqrt{x+h+1}\sqrt{x+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}$$

$$= \lim_{h\to 0} \frac{-h}{h\left(\sqrt{x+h+1}\sqrt{x+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}$$

$$= \lim_{h\to 0} \frac{-1}{\left(\sqrt{x+h+1}\sqrt{x+1}\right)\left(\sqrt{x+1} + \sqrt{x+h+1}\right)}$$

$$= \frac{-1}{\left(\sqrt{x+1}\right)^2\left(2\sqrt{x+1}\right)} = -\frac{1}{2(x+1)^{3/2}}.$$

3.1.19: $f'(x) = \lim_{h\to 0} \dfrac{1}{h}\left(f(x+h)-f(x)\right) = \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{x+h}{1-2(x+h)} - \dfrac{x}{1-2x}\right)$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(1-2x)-(1-2x-2h)(x)}{(1-2x-2h)(1-2x)} = \lim_{h\to 0} \frac{(x-2x^2+h-2xh)-(x-2x^2-2xh)}{h(1-2x-2h)(1-2x)}$$

$$= \lim_{h\to 0} \frac{x-2x^2+h-2xh-x+2x^2+2xh}{h(1-2x-2h)(1-2x)} = \lim_{h\to 0} \frac{h}{h(1-2x-2h)(1-2x)}$$

$$= \lim_{h\to 0} \frac{1}{(1-2x-2h)(1-2x)} = \frac{1}{(1-2x)^2}.$$

3.1.20: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{x+h+1}{x+h-1} - \dfrac{x+1}{x-1}\right)$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h+1)(x-1)-(x+h-1)(x+1)}{(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{(x^2-x+hx-h+x-1)-(x^2+x+hx+h-x-1)}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{x^2-x+hx-h+x-1-x^2-x-hx-h+x+1}{h(x+h-1)(x-1)} = \lim_{h\to 0} \frac{-2h}{h(x+h-1)(x-1)}$$

$$= \lim_{h\to 0} \frac{-2}{(x+h-1)(x-1)} = -\frac{2}{(x-1)^2}.$$

3.1.21: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = -32t$, so $v(t) = 0$ when $t = 0$. The position of the particle then is $x(0) = 100$.

3.1.22: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = -32t + 160$, so $v(t) = 0$ when $t = 5$. The position of the particle then is $x(5) = 425$.

3.1.23: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = -32t + 80$, so $v(t) = 0$ when $t = 2.5$. The position of the particle then is $x(2.5) = 99$.

3.1.24: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = 200t$, so $v(t) = 0$ when $t = 0$. The position of the particle then is $x(0) = 50$.

3.1.25: The velocity of the particle at time t is $\dfrac{dx}{dt} = v(t) = -20 - 10t$, so $v(t) = 0$ when $t = -2$. The position of the particle then is $x(-2) = 120$.

3.1.26: The ball reaches its maximum height when its velocity $v(t) = \dfrac{dy}{dt} = -32t + 160$ is zero, and $v(t) = 0$ when $t = 5$. The height of the ball then is $y(5) = 400$ (ft).

3.1.27: The ball reaches its maximum height when its velocity $v(t) = \dfrac{dy}{dt} = -32t + 64$ is zero, and $v(t) = 0$ when $t = 2$. The height of the ball then is $y(2) = 64$ (ft).

3.1.28: The ball reaches its maximum height when its velocity $v(t) = \dfrac{dy}{dt} = -32t + 128$ is zero, and $v(t) = 0$ when $t = 4$. The height of the ball then is $y(4) = 281$ (ft).

3.1.29: The ball reaches its maximum height when its velocity $v(t) = \dfrac{dy}{dt} = -32t + 96$ is zero, and $v(t) = 0$ when $t = 3$. The height of the ball then is $y(3) = 194$ (ft).

3.1.30: Figure 3.1.22 shows a graph first increasing, then with a horizontal tangent at $x = 0$, then decreasing. Hence its derivative must be first positive, then zero when $x = 0$, then negative. This matches Fig. 3.1.28(c).

3.1.31: Figure 3.1.23 shows a graph first decreasing, then with a horizontal tangent where $x = 1$, then increasing thereafter. So its derivative must be negative for $x < 1$, zero when $x = 1$, and positive for $x > 1$. This matches Fig. 3.1.28(e).

3.1.32: Figure 3.1.24 shows a graph increasing for $x < -1.5$, decreasing for $-1.5 < x < 1.5$, and increasing for $1.5 < x$. So its derivative must be positive for $x < -1.5$, negative for $-1.5 < x < 1.5$, and positive for $1.5 < x$. This matches Fig. 3.1.28(b).

3.1.33: Figure 3.1.25 shows a graph decreasing for $x < -1.5$, increasing for $-1.5 < x < 0$, decreasing for $0 < 1 < 1.5$, and increasing for $1.5 < x$. Hence its derivative is negative for $x < -1.5$, positive for $-1.5 < x < 0$, negative for $0 < x < 1.5$, and positive for $1.5 < x$. Only the graph in Fig. 3.1.28(f) shows these characteristics.

3.1.34: Figure 3.1.26 shows a graph with horizontal tangents near where $x = -3$, $x = 0$, and $x = 3$. So the graph of the derivative must be zero near these three points, and this behavior is matched by Fig. 3.1.28(a).

3.1.35: Figure 3.1.27 shows a graph that increases, first slowly, then rapidly. So its derivative must exhibit the same behavior, and thus its graph is the one shown in Fig. 3.1.28(d).

3.1.36: Note that

$$C(F) = \frac{5}{9}F - \frac{160}{9} \qquad \text{and so} \qquad F(C) = \frac{9}{5}C + 32.$$

So the rate of change of C with respect to F is

$$C'(F) = \frac{dC}{dF} = \frac{5}{9}$$

and the rate of change of F with respect to C is

$$F'(C) = \frac{dF}{dC} = \frac{9}{5}.$$

3.1.37: Let r note the radius of the circle. Then $A = \pi r^2$ and $C = 2\pi r$. Thus

$$r = \frac{C}{2\pi}, \qquad \text{and so} \qquad A(C) = \frac{1}{4\pi}C^2, \quad C > 0.$$

Therefore the rate of change of A with respect to C is

$$A'(C) = \frac{dA}{dC} = \frac{1}{2\pi}C.$$

3.1.38: Let r denote the radius of the circular ripple in feet at time t (seconds). Then $r = 5t$, and the area within the ripple at time t is $A = \pi r^2 = 25\pi t^2$. The rate at which this area is increasing at time t is $A'(t) = 50\pi t$, so at time $t = 10$ the area is increasing at the rate of $A'(10) = 50\pi \cdot 10 = 500\pi$ (ft^2/s).

3.1.39: The velocity of the car (in feet per second) at time t (seconds) is $v(t) = x'(t) = 100 - 10t$. The car comes to a stop when $v(t) = 0$; that is, when $t = 10$. At that time the car has traveled a distance $x(10) = 500$ (ft). So the car skids for 10 seconds and skids a distance of 500 ft.

3.1.40: Because $V(t) = 10 - \frac{1}{5}t + \frac{1}{1000}t^2$, $V'(t) = -\frac{1}{5} + \frac{1}{500}t$ and so the rate at which the water is leaking out one minute later ($t = 60$) is $V'(60) = -\frac{2}{25}$ (gal/s); that is, -4.8 gal/min. The average rate of change of V from $t = 0$ until $t = 100$ is

$$\frac{V(100) - V(0)}{100 - 0} = \frac{0 - 10}{100} = -\frac{1}{10}.$$

The instantaneous rate of change of V will have this value when $V'(t) = -\frac{1}{10}$, which we easily solve for $t = 50$.

3.1.41: First, $P(t) = 100 + 30t + 4t^2$. The initial population is 100, so doubling occurs when $P(t) = 200$; that is, when $4t^2 + 30t - 100 = 0$. The quadratic formula yields $t = 2.5$ as the only positive solution of this equation, so the population will take two and one-half months to double. Because $P'(t) = 30 + 8t$, the rate of growth of the population when $P = 200$ will be $P'(2.5) = 50$ (chipmunks per month).

3.1.42: In our construction, the tangent line at 1989 passes through the points $(1984, 259)$ and $(1994, 423)$, and so has slope 16.4; this yields a rate of growth of approximately 16.4 thousand per year in 1989. Alternatively, using the *Mathematica* function `Fit` to fit the given data to a sixth-degree polynomial, we find that P (in thousands) is given in terms of t (as a four-digit year) by

$$\begin{aligned} P(t) \approx\,& (1.0453588 \times 10^{-14})x^6 - (4.057899 \times 10^{-11})x^5 + (3.9377735 \times 10^{-8})x^4 \\ &+ (5.93932556 \times 10^{-11})x^3 + (5.972176 \times 10^{-14})x^2 + (5.939427 \times 10^{-17})x + (3.734218 \times 10^{-12}), \end{aligned}$$

and that $P'(1989) \approx 16.4214$. Of course neither method is exact.

3.1.43: On our graph, the tangent line at the point $(20, 810)$ has slope $m_1 \approx 0.6$ and the tangent line at $(40, 2686)$ has slope $m_2 \approx 0.9$ (roughly). A line of slope 1 on our graph corresponds to a velocity of 125 ft/s (because the line through $(0, 0)$ and $(10, 1250)$ has slope 1), and thus we estimate the velocity of the car at time $t = 20$ to be about $(0.6)(125) = 75$ ft/s, and at time $t = 40$ it is traveling at about $(0.9)(125) = 112.5$ ft/s. The method is crude; another person with his own graph measured slightly smaller values of the two slopes and obtained the two velocities 68 ft/s and 102 ft/s shown in the back of the book. When we used the *Mathematica* function `Fit` to fit the data to a sixth-degree polynomial, we obtained

$$\begin{aligned} x(t) \approx\,& 0.0000175721 + (6.500002)x + (1.112083)x^2 + (0.074188)x^3 \\ &- (0.00309375)x^4 + (0.0000481250)x^5 - (0.000000270834)x^6, \end{aligned}$$

which yields $x'(20) \approx 74.3083$ and $x'(40) \approx 109.167$. Of course neither method is highly accurate.

3.1.44: With volume V and edge x, the volume of the cube is given by $V(x) = x^3$. Now $\dfrac{dV}{dx} = 3x^2$, which is indeed half the total surface area $6x^2$ of the cube.

3.1.45: With volume V and radius r, the volume of the sphere is $V(r) = \frac{4}{3}\pi r^3$. Then $\dfrac{dV}{dr} = 4\pi r^2$, and this is indeed the surface area of the sphere.

3.1.46: A right circular cylinder of radius r and height h has volume $V = \pi r^2 h$ and total surface area S obtained by adding the areas of its top, bottom, and curved side: $S = 2\pi r^2 + 2\pi rh$. We are given $h = 2r$, so $V(r) = 2\pi r^3$ and $S(r) = 6\pi r^2$. Also $dV/dr = 6\pi r^2 = S(r)$, so the rate of change of volume with respect to radius is indeed equal to total surface area.

3.1.47: We must compute dV/dt when $t = 30$; $V(r) = \frac{4}{3}\pi r^3$ is the volume of the balloon when its radius is r. We are given $r = \dfrac{60-t}{12}$, and thus

$$V(t) = \frac{4}{3}\pi\left(\frac{60-t}{12}\right)^3 = \frac{\pi}{1296}(216000 - 10800t + 180t^2 - t^3).$$

Therefore

$$\frac{dV}{dt} = \frac{\pi}{1296}(-10800 + 360t - 3t^2),$$

and so $V'(t) = -\dfrac{25\pi}{12}$ in.^{3}/s; that is, air is leaking out at approximately 6.545 in.^{3}/s.

3.1.48: From $V(p) = \dfrac{1.68}{p}$ we derive $V'(p) = -\dfrac{1.68}{p^2}$. The rate of change of V with respect to p when $p = 2$ (atm) is then $V'(2) = -0.42$ (liters/atm).

3.1.49: Let $V(t)$ denote the volume (in cm^3) of the snowball at time t (in hours) and let $r(t)$ denote its radius then. From the data given in the problem, $r = 12 - t$. The volume of the snowball is

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(12-t)^3 = \frac{4}{3}\pi\left(1728 - 432t + 36t^2 - t^3\right),$$

so its instantaneous rate of change is

$$V'(t) = \frac{4}{3}\pi\left(-432 + 72t - 3t^2\right).$$

Hence its rate of change of volume when $t = 6$ is $V'(6) = -144\pi$ cm^3/h. Its average rate of change of volume from $t = 3$ to $t = 9$ in cm^3/h is

$$\frac{V(9) - V(3)}{9 - 3} = \frac{36\pi - 972\pi}{6} = -156\pi \ (\text{cm}^3/\text{h}).$$

3.1.50: The velocity of the ball at time t is $\dfrac{dy}{dt} = -32t + 96$, which is zero when $t = 3$. So the maximum height of the ball is $y(3) = 256$ (ft). It hits the ground when $y(t) = 0$; that is, when $-16t^2 + 96t + 112 = 0$. The only positive solution of this equation is $t = 7$, so the impact speed of the ball is $|y'(7)| = 128$ (ft/s).

3.1.51: The spaceship hits the ground when $25t^2 - 100t + 100 = 0$, which has solution $t = 2$. The velocity of the spaceship at time t is $y'(t) = 50t - 100$, so the speed of the spaceship at impact is (fortunately) zero.

3.1.52: Because $P(t) = 100 + 4t + \frac{3}{10}t^2$, we have $P'(t) = 4 + \frac{3}{5}t$. The year 1986 corresponds to $t = 6$, so the rate of change of P then was $P'(6) = 7.6$ (thousands per year). The average rate of change of P from 1983 ($t = 3$) to 1988 ($t = 8$) was

$$\frac{P(8) - P(3)}{8 - 3} = \frac{151.2 - 114.7}{5} = 7.3 \text{ (thousands per year).}$$

3.1.53: The average rate of change of the population from January 1, 1990 to January 1, 2000 was

$$\frac{P(10) - P(0)}{10 - 0} = \frac{6}{10} = 0.6 \quad \text{(thousands per year).}$$

The instantaneous rate of change of the population (in thousands per year, again) at time t was

$$P'(t) \;=\; 1 - (0.2)t + (0.018)t^2.$$

Using the quadratic formula to solve the equation $P'(t) = 0.6$, we find two solutions:

$$t = \frac{50 - 10\sqrt{7}}{9} \approx 2.6158318766 \quad \text{and} \quad t = \frac{50 + 10\sqrt{7}}{9} \approx 8.4952792345.$$

These values of t correspond to August 12, 1992 and June 30, 1998, respectively.

3.1.54: (a) If $f(x) = |x|$, then

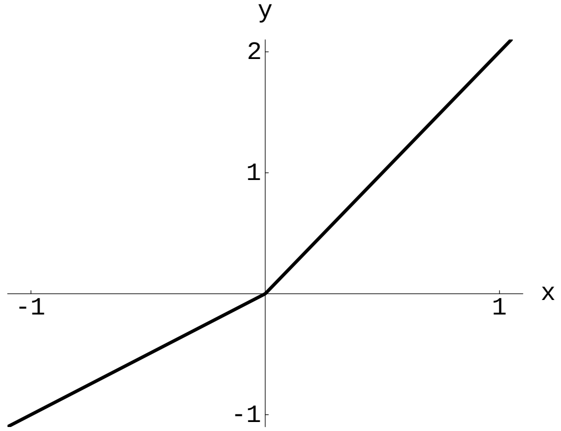
$$f'_-(0) = \lim_{h \to 0^-} \frac{|h| - 0}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1;$$

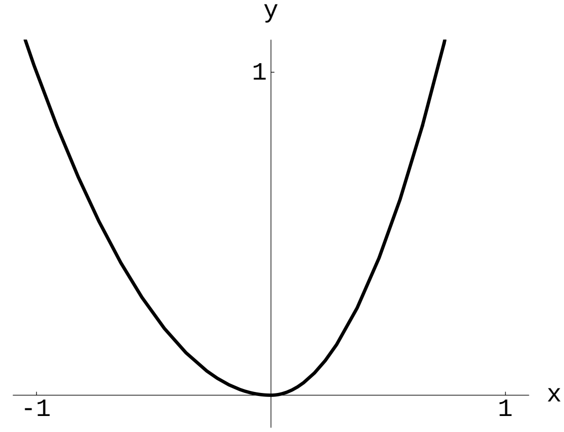
similarly, $f'_+(0) = 1$. (b) The function $f(x) = |2x - 10|$ is not differentiable at $x = 5$. Its right-hand derivative there is

$$f'_+(5) = \lim_{h \to 0^+} \frac{|2 \cdot (5 + h) - 10| - 0}{h} = \lim_{h \to 0^+} \frac{10 + 2h - 10}{h} = 2.$$

Similarly, $f'_-(5) = -2$.

3.1.55: The graphs of the function of part (a) is shown next, on the left; the graph of the function of part (b) is on the right.





(a) $f'_-(0) = 1$ while $f'_+(0) = 2$. Hence f is not differentiable at $x = 0$. (b) In contrast,

$$f'_-(0) = \lim_{h \to 0^-} \frac{(0+h)^2 - 2 \cdot 0^2}{h} = 0$$

and

$$f'_+(0) = \lim_{h \to 0^+} \frac{2 \cdot (0+h)^2 - 2 \cdot 0^2}{h} = 0;$$

therefore f is differentiable at $x = 0$ and $f'(0) = 0$.

3.1.56: The function f is clearly differentiable except possibly at $x = 1$. But

$$f'_-(1) = \lim_{h \to 0^-} \frac{2 \cdot (1+h) + 1 - 3}{h} = 2$$

and

$$f'_+(1) = \lim_{h \to 0^+} \frac{4 \cdot (1+h) - (1+h)^2 - 3}{h} = \lim_{h \to 0^+} \frac{4 + 4h - 1 - 2h - h^2 - 3}{h} = \lim_{h \to 0^+} (2-h) = 2.$$

Therefore f is differentiable at $x = 1$ as well.

3.1.57: Clearly f is differentiable except possibly at $x = 3$. Moreover,

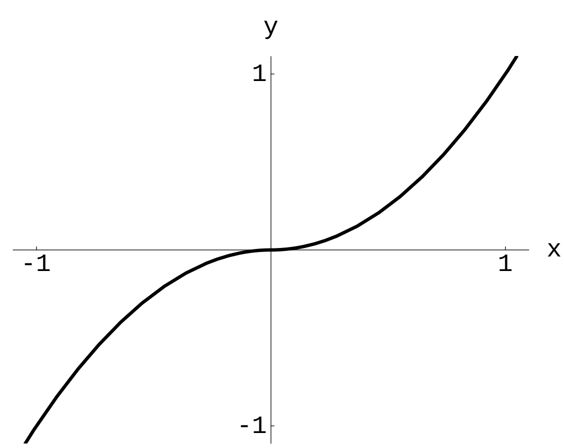
$$\begin{aligned} f'_-(3) &= \lim_{h \to 0^-} \frac{11 + 6 \cdot (3+h) - (3+h)^2 - 20}{h} \\ &= \lim_{h \to 0^-} \frac{11 + 18 + 6h - 9 - 6h - h^2 - 20}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = 0 \end{aligned}$$

and

$$\begin{aligned} f'_+(3) &= \lim_{h \to 0^+} \frac{(3+h)^2 - 6 \cdot (3+h) + 29 - 20}{h} \\ &= \lim_{h \to 0^+} \frac{9 + 6h + h^2 - 18 - 6h + 29 - 20}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = 0. \end{aligned}$$

Therefore the function f is also differentiable at $x = 3$; moreover, $f'(3) = 0$.

3.1.58: The graph of $f(x) = x \cdot |x|$ is shown next.



Because $f(x) = x^2$ if $x > 0$ and $f(x) = -x^2$ if $x < 0$, f is differentiable except possibly at $x = 0$. But

$$f'_-(0) = \lim_{h \to 0^-} \frac{(0+h) \cdot |0+h|}{h} = \lim_{h \to 0^-} \frac{-h^2}{h} = 0$$

and

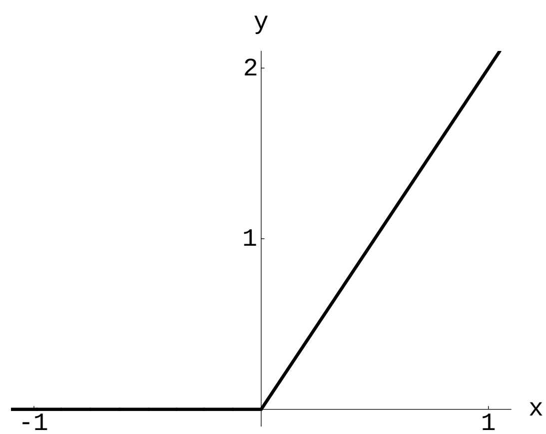
$$f'_+(0) = \lim_{h \to 0^+} \frac{(0+h) \cdot |0+h|}{h} = \lim_{h \to 0^+} \frac{h^2}{h} = 0.$$

Therefore f is differentiable at $x = 0$ and $f'(0) = 0$. Because

$$f'(x) = \begin{cases} 2x & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -2x & \text{if } x < 0, \end{cases}$$

we see that $f'(x) = 2|x|$ for all x.

3.1.59: The graph of $f(x) = x + |x|$ is shown next.



Because $f(x) = 0$ if $x < 0$ and $f(x) = 2x$ if $x > 0$, clearly f is differentiable except possibly at $x = 0$. Next,

$$f'_-(0) = \lim_{h \to 0^-} \frac{0 + h + |0+h|}{h} = \lim_{h \to 0^-} \frac{h - h}{h} = 0,$$

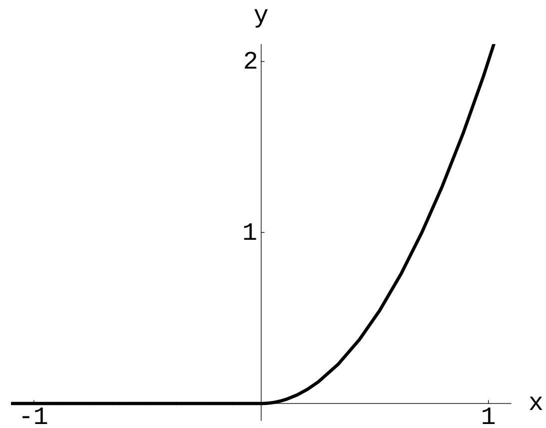
whereas

$$f'_+(0) = \lim_{h \to 0^+} \frac{0 + h + |0+h|}{h} = \lim_{h \to 0^+} \frac{2h}{h} = 2 \neq 0.$$

Therefore f is not differentiable at $x = 0$. In summary, $f'(x) = 2$ if $x > 0$ and $f'(x) = 0$ if $x < 0$. For a "single-formula" version of the derivative, consider

$$f'(x) = 1 + \frac{|x|}{x}.$$

3.1.60: The graph of $f(x) = x \cdot (x + |x|)$, is shown next.



Because $f(x) = 2x^2$ if $x > 0$ and $f(x) = 0$ if $x < 0$, $f'(x)$ exists except possibly at $x = 0$. But

$$f'_-(0) = \lim_{h \to 0^-} \frac{(0+h)\cdot(0+h+|0+h|)}{h} = \lim_{h \to 0^-} \frac{h \cdot (h-h)}{h} = 0$$

and

$$f'_+(0) = \lim_{h \to 0^+} \frac{(0+h)\cdot(0+h+|0+h|)}{h} = \lim_{h \to 0^+} \frac{h \cdot (h+h)}{h} = 0.$$

Therefore f is differentiable at $x = 0$ and $f'(0) = 0$. Finally, $f'(x) = 4x$ if $x > 0$ and $f'(x) = 0$ if $x < 0$. For a "single-formula" version of the derivative, consider $f'(x) = 2 \cdot (x + |x|)$.

Section 3.2

3.2.1: Given: $f(x) = 3x^2 - x + 5$. We apply the rule for differentiating a linear combination and the power rule to obtain

$$f'(x) = 3D_x(x^2) - D_x(x) + D_x(5) = 3 \cdot 2x - 1 + 0 = 6x - 1.$$

3.2.2: Given: $g(t) = 1 - 3t^2 - 2t^4$. We apply the rule for differentiating a linear combination and the power rule to obtain

$$g'(t) = D_t(1) - 3D_t(t^2) - 2D_t(t^4) = 0 - 3 \cdot 2t - 2 \cdot 4t^3 = -6t - 8t^3.$$

3.2.3: Given: $f(x) = (2x+3)(3x-2)$. We apply the product rule to obtain

$$f'(x) = (2x+3)D_x(3x-2) + (3x-2)D_x(2x+3) = (2x+3) \cdot 3 + (3x-2) \cdot 2 = 12x + 5.$$

3.2.4: Given: $g(x) = (2x^2-1)(x^3+2)$. We apply the product rule, the rule for differentiating a linear combination, and the power rule to obtain

$$g'(x) = (2x^2-1)D_x(x^3+2) + (x^3+2)D_x(2x^2-1) = (2x^2-1)(3x^2) + (x^3+2)(4x) = 10x^4 - 3x^2 + 8x.$$

3.2.5: Given: $h(x) = (x+1)^3$. We rewrite $h(x)$ in the form

$$h(x) = (x+1)(x+1)(x+1)$$

and then apply the extended product rule in Eq. (16) to obtain

$$\begin{aligned} h'(x) &= (x+1)(x+1)\,D_x(x+1) + (x+1)(x+1)\,D_x(x+1) + (x+1)(x+1)\,D_x(x+1) \\ &= (x+1)(x+1)(1) + (x+1)(x+1)(1) + (x+1)(x+1)(1) = 3(x+1)^2. \end{aligned}$$

Alternatively, we could rewrite $h(x)$ in the form

$$h(x) = x^3 + 3x^2 + 3x + 1$$

and then apply the power rule and the rule for differentiating a linear combination to obtain

$$h'(x) = D_x(x^3) + 3\,D_x(x^2) + 3\,D_x(x) + D_x(1) = 3x^2 + 6x + 3.$$

The first method gives the answer in a more useful factored form that makes it easier to determine where $h'(x)$ is positive, where negative, and where zero.

3.2.6: Given: $g(t) = (4t-7)^2 = (4t-7)\cdot(4t-7)$. We apply the product rule and the rule for differentiating a linear combination to obtain

$$g'(t) = (4t-7)\,D_t(4t-7) + (4t-7)\,D_t(4t-7) = 4\cdot(4t-7) + 4\cdot(4t-7) = 8\cdot(4t-7) = 32t - 56.$$

3.2.7: Given: $f(y) = y(2y-1)(2y+1)$. We apply the extended product rule in Eq. (16) to obtain

$$\begin{aligned} f'(y) &= (2y-1)(2y+1)\,D_y(y) + y(2y+1)\,D_y(2y-1) + y(2y-1)\,D_y(2y+1) \\ &= (2y-1)(2y+1)\cdot 1 + y(2y+1)\cdot 2 + y(2y-1)\cdot 2 = 4y^2 - 1 + 4y^2 + 2y + 4y^2 - 2y = 12y^2 - 1. \end{aligned}$$

Alternatively, we could first expand: $f(y) = 4y^3 - y$. Then we could apply the rule for differentiating a linear combination and the power rule to obtain $f'(y) = 4\,D_y(y^3) - D_y(y) = 12y^2 - 1$.

3.2.8: Given: $f(x) = 4x^4 - \dfrac{1}{x^2}$. We apply various rules, including the reciprocal rule, to obtain

$$f'(x) = 4\,D_x(x^4) - \left(-\frac{D_x(x^2)}{(x^2)^2}\right) = 4\cdot 4x^3 + \frac{2x}{x^4} = 16x^3 + \frac{2}{x^3}.$$

Alternatively, we could rewrite: $f(x) = 4x^4 - x^{-2}$. Then we could apply the rule for differentiating a linear combination and the power rule (both for positive and for negative integral exponents) to obtain

$$f'(x) = 4\,D_x(x^4) - D_x(x^{-2}) = 4\cdot 4x^3 - (-2)x^{-3} = 16x^3 + 2x^{-3} = 16x^3 + \frac{2}{x^3}.$$

3.2.9: We apply the rule for differentiating a linear combination and the reciprocal rule (twice) to obtain

$$g'(x) = D_x\left(\frac{1}{x+1}\right) - D_x\left(\frac{1}{x-1}\right)$$
$$= -\frac{D_x(x+1)}{(x+1)^2} + \frac{D_x(x-1)}{(x-1)^2} = -\frac{1}{(x+1)^2} + \frac{1}{(x-1)^2}.$$

Looking ahead to later sections and chapters—in which we will want to find where $g'(x)$ is positive, negative, or zero—it would be good practice to simplify $g'(x)$ to

$$g'(x) = \frac{(x+1)^2 - (x-1)^2}{(x+1)^2(x-1)^2} = \frac{4x}{(x+1)^2(x-1)^2}.$$

3.2.10: We apply the reciprocal rule to $f(t) = \dfrac{1}{4-t^2}$ to obtain

$$f'(t) = -\frac{D_t(4-t^2)}{(4-t^2)^2} = -\frac{-2t}{(4-t^2)^2} = \frac{2t}{(4-t^2)^2}.$$

3.2.11: First write (or think of) $h(x)$ as

$$h(x) = 3 \cdot \frac{1}{x^2+x+1},$$

then apply the rule for differentiating a linear combination and the reciprocal rule to obtain

$$h'(x) = 3 \cdot \left(-\frac{D_x(x^2+x+1)}{(x^2+x+1)^2}\right) = \frac{-3 \cdot (2x+1)}{(x^2+x+1)^2}.$$

Alternatively apply the quotient rule directly to obtain

$$h'(x) = \frac{(x^2+x+1)D_x(3) - 3\,D_x(x^2+x+1)}{(x^2+x+1)^2} = \frac{-3 \cdot (2x+1)}{(x^2+x+1)^2}.$$

3.2.12: Multiply numerator and denominator in $f(x)$ by x to obtain

$$f(x) = \frac{1}{1 - \dfrac{2}{x}} = \frac{x}{x-2}.$$

Then apply the quotient rule to obtain

$$f'(x) = \frac{(x-2)\,D_x(x) - x\,D_x(x-2)}{(x-2)^2} = \frac{(x-2) \cdot 1 - x \cdot 1}{(x-2)^2} = \frac{-2}{(x-2)^2}.$$

3.2.13: Given $g(t) = (t^2+1)(t^3+t^2+1)$, apply the product rule, the rule for differentiating a linear combination, and the power rule to obtain

$$g'(t) = (t^2+1)\,D_t(t^3+t^2+1) + (t^3+t^2+1)\,D_t(t^2+1)$$
$$= (t^2+1)(3t^2+2t+0) + (t^3+t^2+1)(2t+0)$$
$$= (3t^4+2t^3+3t^2+2t) + (2t^4+2t^3+2t) = 5t^4+4t^3+3t^2+4t.$$

Alternatively, first expand: $g(t) = t^5 + t^4 + t^3 + 2t^2 + 1$, then apply the rule for differentiating a linear combination and the power rule.

3.2.14: Given $f(x) = (2x^3 - 3)(17x^4 - 6x + 2)$, apply the product rule, the rule for differentiating a linear combination, and the power rule to obtain

$$\begin{aligned} f'(x) &= (2x^3 - 3)(68x^3 - 6) + (6x^2)(17x^4 - 6x + 2) \\ &= (136x^6 - 216x^3 + 18) + (102x^6 - 36x^3 + 12x^2) = 238x^6 - 252x^3 + 12x^2 + 18. \end{aligned}$$

Alternatively, first expand $f(x)$, then apply the linear combination rule and the power rule.

3.2.15: The easiest way to find $g'(z)$ is first to rewrite $g(z)$:

$$g(z) = \frac{1}{2z} - \frac{1}{3z^2} = \frac{1}{2}z^{-1} - \frac{1}{3}z^{-2}.$$

Then apply the linear combination rule and the power rule (for negative integral exponents) to obtain

$$g'(z) = \frac{1}{2}(-1)z^{-2} - \frac{1}{3}(-2)z^{-3} = -\frac{1}{2z^2} + \frac{2}{3z^3} = \frac{4 - 3z}{6z^3}.$$

The last step is advisable should it be necessary to find where $g'(z)$ is positive, where it is negative, and where it is zero.

3.2.16: The quotient rule yields

$$\begin{aligned} f'(x) &= \frac{x^2\,D_x(2x^3 - 3x^2 + 4x - 5) - (2x^3 - 3x^2 + 4x - 5)\,D_x(x^2)}{(x^2)^2} \\ &= \frac{(x^2)(6x^2 - 6x + 4) - (2x^3 - 3x^2 + 4x - 5)(2x)}{x^4} = \frac{(6x^4 - 6x^3 + 4x^2) - (4x^4 - 6x^3 + 8x^2 - 10x)}{x^4} \\ &= \frac{6x^4 - 6x^3 + 4x^2 - 4x^4 + 6x^3 - 8x^2 + 10x}{x^4} = \frac{2x^4 - 4x^2 + 10x}{x^4} = \frac{2x^3 - 4x + 10}{x^3}. \end{aligned}$$

But if the objective is to obtain the correct answer as quickly as possible, regardless of its appearance, you could proceed as follows (using the linear combination rule and the power rule for negative integral exponents):

$$f(x) = 2x - 3 + 4x^{-1} - 5x^{-2}, \quad \text{so} \quad f'(x) = 2 - 4x^{-2} + 10x^{-3}.$$

3.2.17: Apply the extended product rule in Eq. (16) to obtain

$$\begin{aligned} g'(y) &= (3y^2 - 1)(y^2 + 2y + 3)\,D_y(2y) + (2y)(y^2 + 2y + 3)\,D_y(3y^2 - 1) + (2y)(3y^2 - 1)\,D_y(y^2 + 2y + 3) \\ &= (3y^2 - 1)(y^2 + 2y + 3)(2) + (2y)(y^2 + 2y + 3)(6y) + (2y)(3y^2 - 1)(2y + 2) \\ &= (6y^4 + 12y^3 + 18y^2 - 2y^2 - 4y - 6) + (12y^4 + 24y^3 + 36y^2) + (12y^4 - 4y^2 + 12y^3 - 4y) \\ &= 30y^4 + 48y^3 + 48y^2 - 8y - 6. \end{aligned}$$

Or if you prefer, first expand $g(y)$, then apply the linear combination rule and the power rule to obtain

$$g(y) = (6y^3 - 2y)(y^2 + 2y + 3) = 6y^5 + 12y^4 + 16y^3 - 4y^2 - 6y, \quad \text{so}$$

$$g'(y) = 30y^4 + 48y^3 + 48y^2 - 8y - 6.$$

3.2.18: By the quotient rule,

$$f'(x) = \frac{(x^2+4)\,D_x(x^2-4) - (x^2-4)\,D_x(x^2+4)}{(x^2+4)^2} = \frac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2}.$$

3.2.19: Apply the quotient rule to obtain

$$g'(t) = \frac{(t^2+2t+1)\,D_t(t-1) - (t-1)\,D_t(t^2+2t+1)}{(t^2+2t+1)^2} = \frac{(t^2+2t+1)(1) - (t-1)(2t+2)}{[\,(t+1)^2]^2}$$

$$= \frac{(t^2+2t+1) - (2t^2-2)}{(t+1)^4} = \frac{3+2t-t^2}{(t+1)^4} = -\frac{(t+1)(t-3)}{(t+1)^4} = \frac{3-t}{(t+1)^3}.$$

3.2.20: Apply the reciprocal rule to obtain

$$u'(x) = -\frac{D_x(x^2+4x+4)}{(x+2)^4} = -\frac{2x+4}{(x+2)^4} = -\frac{2}{(x+2)^3}.$$

3.2.21: Apply the reciprocal rule to obtain

$$v'(t) = -\frac{D_t(t^3-3t^2+3t-1)}{(t-1)^6} = -\frac{3t^2-6t+3}{(t-1)^6} = -\frac{3(t-1)^2}{(t-1)^6} = -\frac{3}{(t-1)^4}.$$

3.2.22: The quotient rule yields

$$h(x) = \frac{(2x-5)\,D_x(2x^3+x^2-3x+17) - (2x^3+x^2-3x+17)\,D_x(2x-5)}{(2x-5)^2}$$

$$= \frac{(2x-5)(6x^2+2x-3) - (2x^3+x^2-3x+17)(2)}{(2x-5)^2}$$

$$= \frac{(12x^3-26x^2-16x+15) - (4x^3+2x^2-6x+34)}{(2x-5)^2}$$

$$= \frac{12x^3-26x^2-16x+15-4x^3-2x^2+6x-34}{(2x-5)^2} = \frac{8x^3-28x^2-10x-19}{(2x-5)^2}.$$

3.2.23: The quotient rule yields

$$g'(x) = \frac{(x^3+7x-5)(3) - (3x)(3x^2+7)}{(x^3+7x-5)^2} = \frac{3x^3+21x-15-9x^3-21x}{(x^3+7x-5)^2} = -\frac{6x^3+15}{(x^3+7x-5)^2}.$$

3.2.24: First expand the denominator, then multiply numerator and denominator by t^2, to obtain

$$f(t) = \frac{1}{\left(t + \dfrac{1}{t}\right)^2} = \frac{1}{t^2 + 2 + \dfrac{1}{t^2}} = \frac{t^2}{t^4 + 2t^2 + 1}.$$

Then apply the quotient rule to obtain

$$f'(t) = \frac{(t^4 + 2t^2 + 1)(2t) - (t^2)(4t^3 + 4t)}{\left[\,(t^2 + 1)^2\right]^2} = \frac{2t^5 + 4t^3 + 2t - 4t^5 - 4t^3}{(t^2 + 1)^4} = \frac{2t - 2t^5}{(t^2 + 1)^4}.$$

A modest simplification is possible:

$$f'(t) = -\frac{2t(t^4 - 1)}{(t^2 + 1)^4} = -\frac{2t(t^2 + 1)(t^2 - 1)}{(t^2 + 1)^4} = -\frac{2t(t^2 - 1)}{(t^2 + 1)^3}.$$

3.2.25: First multiply each term in numerator and denominator by x^4 to obtain

$$g(x) = \frac{x^3 - 2x^2}{2x - 3}.$$

Then apply the quotient rule to obtain

$$g'(x) = \frac{(2x - 3)(3x^2 - 4x) - (x^3 - 2x^2)(2)}{(2x - 3)^2} = \frac{(6x^3 - 17x^2 + 12x) - (2x^3 - 4x^2)}{(2x - 3)^2} = \frac{4x^3 - 13x^2 + 12x}{(2x - 3)^2}.$$

It is usually wise to simplify an expression before differentiating it.

3.2.26: First multiply each term in numerator and denominator by $x^2 + 1$ to obtain

$$f(x) = \frac{x^3(x^2 + 1) - 1}{x^4(x^2 + 1) + 1} = \frac{x^5 + x^3 - 1}{x^6 + x^4 + 1}.$$

Then apply the quotient rule to obtain

$$\begin{aligned}
f'(x) &= \frac{(x^6 + x^4 + 1)(5x^4 + 3x^2) - (x^5 + x^3 - 1)(6x^5 + 4x^3)}{(x^6 + x^4 + 1)^2} \\
&= \frac{(5x^{10} + 8x^8 + 3x^6 + 5x^4 + 3x^2) - (6x^{10} + 10x^8 + 4x^6 - 6x^5 - 4x^3)}{(x^6 + x^4 + 1)^2} \\
&= \frac{5x^{10} + 8x^8 + 3x^6 + 5x^4 + 3x^2 - 6x^{10} - 10x^8 - 4x^6 + 6x^5 + 4x^3}{(x^6 + x^4 + 1)^2} \\
&= \frac{-x^{10} - 2x^8 - x^6 + 6x^5 + 5x^4 + 4x^3 + 3x^2}{(x^6 + x^4 + 1)^2}.
\end{aligned}$$

3.2.27: If $y(x) = x^3 - 6x^5 + \frac{3}{2}x^{-4} + 12$, then the linear combination rule and the power rules yield $h'(x) = 3x^2 - 30x^4 - 6x^{-5}$.

3.2.28: Given:

$$x(t) = \frac{3}{t} - \frac{4}{t^2} - 5 = 3t^{-1} - 4t^{-2} - 5,$$

it follows from the linear combination rule and the power rule for negative integral exponents that

$$x'(t) = -3t^{-2} + 8t^{-3} = \frac{8}{t^3} - \frac{3}{t^2} = \frac{8-3t}{t^3}.$$

3.2.29: Given:

$$y(x) = \frac{5 - 4x^2 + x^5}{x^3} = \frac{5}{x^3} - \frac{4x^2}{x^3} + \frac{x^5}{x^3} = 5x^{-3} - 4x^{-1} + x^2,$$

it follows from the linear combination rule and the power rules that

$$y'(x) = -15x^{-4} + 4x^{-2} + 2x = 2x + \frac{4}{x^2} - \frac{15}{x^4} = \frac{2x^5 + 4x^2 - 15}{x^4}.$$

3.2.30: Given

$$u(x) = \frac{2x - 3x^2 + 2x^4}{5x^2} = \frac{2}{5}x^{-1} - \frac{3}{5} + \frac{2}{5}x^2,$$

it follows from the linear combination rule and the power rules that

$$u'(x) = -\frac{2}{5}x^{-2} + \frac{4}{5}x = \frac{4x^3 - 2}{5x^2}.$$

3.2.31: Because $y(x)$ can be written in the form $y(x) = 3x - \frac{1}{4}x^{-2}$, the linear combination rule and the power rules yield $y'(x) = 3 + \frac{1}{2}x^{-3}$.

3.2.32: We use the reciprocal rule, the linear combination rule, and the power rule for positive integral exponents:

$$f'(z) = -\frac{D_z(z^3 + 2z^2 + 2z)}{z^2(z^2 + 2z + 2)^2} = -\frac{3z^2 + 4z + 2}{z^2(z^2 + 2z + 2)^2}.$$

3.2.33: If we first combine the two fractions, we will need to use the quotient rule only once:

$$y(x) = \frac{x}{x-1} + \frac{x+1}{3x} = \frac{3x^2 + x^2 - 1}{3x(x-1)} = \frac{4x^2 - 1}{3x^2 - 3x},$$

and therefore

$$y'(x) = \frac{(3x^2 - 3x)(8x) - (4x^2 - 1)(6x - 3)}{(3x^2 - 3x)^2} = \frac{24x^3 - 24x^2 - 24x^3 + 12x^2 + 6x - 3}{(3x^2 - 3x)^2} = \frac{-12x^2 + 6x - 3}{(3x^2 - 3x)^2}.$$

3.2.34: First multiply each term in numerator and denominator by t^2 to obtain

$$u(t) = \frac{1}{1 - 4t^{-2}} = \frac{t^2}{t^2 - 4},$$

then apply the quotient rule:

$$u'(t) = \frac{(t^2-4)(2t)-(t^2)(2t)}{(t^2-4)^2} = -\frac{8t}{(t^2-4)^2}.$$

3.2.35: The quotient rule (and other rules, such as the linear combination rule and the power rule) yield

$$\begin{aligned} y'(x) &= \frac{(x^2+9)(3x^2-4)-(x^3-4x+5)(2x)}{(x^2+9)^2} \\ &= \frac{3x^4+23x^2-36-2x^4+8x^2-10x}{(x^2+9)^2} = \frac{x^4+31x^2-10x-36}{(x^2+9)^2}. \end{aligned}$$

3.2.36: Expand $w(z)$ and take advantage of negative exponents:

$$w(z) = z^2\left(2z^3 - \frac{3}{4z^4}\right) = 2z^5 - \frac{3}{4}z^{-2},$$

and so

$$w'(z) = 10z^4 + \frac{3}{2}z^{-3} = 10z^4 + \frac{3}{2z^3} = \frac{20z^7+3}{2z^3}.$$

3.2.37: First multiply each term in numerator and denominator by $5x^4$ to obtain

$$y(x) = \frac{10x^6}{15x^5-4}.$$

Then apply the quotient rule (among others):

$$y'(x) = \frac{(15x^5-4)(60x^5)-(10x^6)(75x^4)}{(15x^5-4)^2} = \frac{900x^{10}-240x^5-750x^{10}}{(15x^5-4)^2} = \frac{150x^{10}-240x^5}{(15x^5-4)^2} = \frac{30x^5(5x^5-8)}{(15x^5-4)^2}.$$

3.2.38: First rewrite

$$z(t) = 4\cdot\frac{1}{t^4-6t^2+9},$$

then apply the linear combination rule and the reciprocal rule to obtain

$$z'(t) = -4\cdot\frac{4t^3-12t}{(t^4-6t^2+9)^2} = \frac{48t-16t^3}{(t^2-3)^4} = -\frac{16t(t^2-3)}{(t^2-3)^4} = -\frac{16t}{(t^2-3)^3}.$$

3.2.39: The quotient rule yields

$$y'(x) = \frac{(x+1)(2x)-(x^2)(1)}{(x+1)^2} = \frac{2x^2+2x-x^2}{(x+1)^2} = \frac{x(x+2)}{(x+1)^2}.$$

3.2.40: Use the quotient rule, or if you prefer write $h(w) = w^{-1} + 10w^{-2}$, so that

$$h'(w) = -w^{-2} - 20w^{-3} = -\left(\frac{1}{w^2} + \frac{20}{w^3}\right) = -\frac{w+20}{w^3}.$$

3.2.41: Given $f(x) = x^3$ and $P(2, 8)$ on its graph, $f'(x) = 3x^2$, so that $f'(2) = 12$ is the slope of the line L tangent to the graph of f at P. So L has equation $y - 8 = 12(x - 2)$; that is, $12x - y = 16$.

3.2.42: Given $f(x) = 3x^2 - 4$ and $P(1, -1)$ on its graph, $f'(x) = 6x$, so that $f'(1) = 6$ is the slope of the line L tangent to the graph of f at P. So L has equation $y + 1 = 6(x - 1)$; that is, $6x - y = 7$.

3.2.43: Given $f(x) = 1/(x - 1)$ and $P(2, 1)$ on its graph,

$$f'(x) = -\frac{D_x(x-1)}{(x-1)^2} = -\frac{1}{(x-1)^2},$$

so that $f'(2) = -1$ is the slope of the line L tangent to the graph of f at P. So L has equation $y-1 = -(x-2)$; that is, $x + y = 3$.

3.2.44: Given $f(x) = 2x - x^{-1}$ and $P(0.5, -1)$ on its graph, $f'(x) = 2 + x^{-2}$, so that $f'(0.5) = 6$ is the slope of the line L tangent to the graph of f at P. So L has equation $y + 1 = 6\left(x - \frac{1}{2}\right)$; that is, $6x - y = 4$.

3.2.45: Given $f(x) = x^3 + 3x^2 - 4x - 5$ and $P(1, -5)$ on its graph, $f'(x) = 3x^2 + 6x - 4$, so that $f'(1) = 5$ is the slope of the line L tangent to the graph of f at P. So L has equation $y + 5 = 5(x - 1)$; that is, $5x - y = 10$.

3.2.46: Given

$$f(x) = \left(\frac{1}{x} - \frac{1}{x^2}\right)^{-1} = \left(\frac{x-1}{x^2}\right)^{-1} = \frac{x^2}{x-1},$$

and $P(2, 4)$ on its graph,

$$f'(x) = \frac{(x-1)(2x) - x^2}{(x-1)^2} = \frac{x^2 - 2x}{(x-1)^2},$$

so that $f'(2) = 0$ is the slope of the line L tangent to the graph of f at P. So L has equation $y-4 = 0\cdot(x-2)$; that is, $y = 4$.

3.2.47: Given $f(x) = 3x^{-2} - 4x^{-3}$ and $P(-1, 7)$ on its graph, $f'(x) = 12x^{-4} - 6x^{-3}$, so that $f'(-1) = 18$ is the slope of the line L tangent to the graph of f at P. So L has equation $y - 7 = 18(x + 1)$; that is, $18x - y = -25$.

3.2.48: Given

$$f(x) = \frac{3x-2}{3x+2}$$

and $P(2, 0.5)$ on its graph,

$$f'(x) = \frac{(3x+2)(3) - (3x-2)(3)}{(3x+2)^2} = \frac{12}{(3x+2)^2},$$

so that $f'(2) = \frac{3}{16}$ is the slope of the line L tangent to the graph of f at P. So L has equation $y-\frac{1}{2} = \frac{3}{16}(x-2)$; that is, $3x - 16y = -2$.

3.2.49: Given

$$f(x) = \frac{3x^2}{x^2 + x + 1}$$

and $P(-1, 3)$ on its graph,

$$f'(x) = \frac{(x^2 + x + 1)(6x) - (3x^2)(2x + 1)}{(x^2 + x + 1)^2} = \frac{3x^2 + 6x}{(x^2 + x + 1)^2},$$

so that $f'(-1) = -3$ is the slope of the line L tangent to the graph of f at P. So an equation of the line L is $y - 3 = -3(x + 1)$; that is, $3x + y = 0$.

3.2.50: Given

$$f(x) = \frac{6}{1 - x^2}$$

and $P(2, -2)$ on its graph,

$$f'(x) = -6 \cdot \frac{-2x}{(1 - x^2)^2} = \frac{12x}{(1 - x^2)^2},$$

so that $f'(2) = \frac{8}{3}$ is the slope of the line L tangent to the graph of f at P. So L has equation $y + 2 = \frac{8}{3}(x - 2)$; that is, $8x - 3y = 22$.

3.2.51: $V = V_0(1 + \alpha T + \beta T^2 + \gamma T^3)$ where $\alpha \approx -0.06427 \times 10^{-3}$, $\beta \approx 8.5053 \times 10^{-6}$, and $\gamma \approx -6.79 \times 10^{-8}$. Now $dV/dt = V_0(\alpha + 2\beta T + 3\gamma T^2)$; $V = V_0 = 1000$ when $T = 0$. Because $V'(0) = \alpha V_0 < 0$, the water contracts when it is first heated. The rate of change of volume at that point is $V'(0) \approx -0.06427$ cm^3 per °C.

3.2.52: $W = \dfrac{2 \times 10^9}{R^2} = (2 \times 10^9)R^{-2}$, so $\dfrac{dW}{dR} = -\dfrac{4 \times 10^9}{R^3}$; when $R = 3960$, $\dfrac{dW}{dR} = -\dfrac{62500}{970299}$ (lb/mi). Thus W decreases initially at about 1.03 ounces per mile.

3.2.53: Draw a cross section of the tank through its axis of symmetry. Let r denote the radius of the (circular) water surface when the height of water in the tank is h. Draw a typical radius, label it r, and label the height h. From similar triangles in your figure, deduce that $h/r = 800/160 = 5$, so $r = h/5$. The volume of water in a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$, so in this case we have $V = V(h) = \frac{1}{75}\pi h^3$. The rate of change of V with respect to h is $dV/dh = \frac{1}{25}\pi h^2$, and therefore when $h = 600$, we have $V'(600) = 14400\pi$; that is, approximately 45239 cm^3 per cm.

3.2.54: Because $y'(x) = 3x^2 + 2x + 1$, the slope of the tangent line at $(1, 3)$ is $y'(1) = 6$. The equation of the tangent line at $(1, 3)$ is $y - 3 = 6(x - 1)$; that is, $y = 6x - 3$. The intercepts of the tangent line are $(0, -3)$ and $\left(\frac{1}{2}, 0\right)$.

3.2.55: The slope of the tangent line can be computed using dy/dx at $x = a$ and also by using the two points known to lie on the line. We thereby find that

$$3a^2 = \frac{a^3 - 5}{a - 1}.$$

This leads to the equation $(a+1)(2a^2-5a+5)=0$. The quadratic factor has negative discriminant, so the only real solution of the cubic equation is $a=-1$. The point of tangency is $(-1, -1)$, the slope there is 3, and the equation of the line in question is $y=3x+2$.

3.2.56: Let (a, a^3) be a point of tangency. The tangent line therefore has slope $3a^2$ and, because it passes through (2, 8), we have

$$3a^2 = \frac{a^3-8}{a-2}; \qquad \text{that is,} \qquad 3a^2(a-2) = a^3-8.$$

This leads to the equation $2a^2-2a-4=0$, so that $a=-1$ or $a=2$. The solution $a=2$ yields the line tangent at (2, 8) with slope 12. The solution $a=-1$ gives the line tangent at $(-1, -1)$ with slope 3. The two lines have equations $y-8=12(x-2)$ and $y+1=3(x+1)$; that is, $y=12x-16$ and $y=3x+2$.

3.2.57: Suppose that some straight line L is tangent to the graph of $f(x)=x^2$ at the points $(a,\, a^2)$ and $(b,\, b^2)$. Our plan is to show that $a=b$, and we may conclude that L cannot be tangent to the graph of f at two *different* points. Because $f'(x)=2x$ and because $(a,\, a^2)$ and $(b,\, b^2)$ both lie on L, the slope of L is equal to both $f'(a)$ and $f'(b)$; that is, $2a=2b$. Hence $a=b$, so that $(a,\, a^2)$ and $(b,\, b^2)$ are the same point. Conclusion: No straight line can be tangent to the graph of $y=x^2$ at two different points.

3.2.58: Let $(a, 1/a)$ be a point of tangency. The slope of the tangent there is $-1/a^2$, so $-1/a^2=-2$. Thus there are two possible values for a: $\pm\frac{1}{2}\sqrt{2}$. These lead to the equations of the two lines: $y=-2x+2\sqrt{2}$ and $y=-2x-2\sqrt{2}$.

3.2.59: Given $f(x)=x^n$, we have $f'(x)=nx^{n-1}$. The line tangent to the graph of f at the point $P(x_0,\, y_0)$ has slope that we compute in two ways and then equate:

$$\frac{y-(x_0)^n}{x-x_0} = n(x_0)^{n-1}.$$

To find the x-intercept of this line, substitute $y=0$ into this equation and solve for x. It follows that the x-intercept is $x=\dfrac{n-1}{n}x_0$.

3.2.60: Because $dy/dx=5x^4+2 \geqq 2>0$ for all x, the curve has no horizontal tangent line. The minimal slope occurs when dy/dx is minimal, and this occurs when $x=0$. So the smallest slope that a line tangent to this graph can have is 2.

3.2.61: $D_x[f(x)]^3 = f'(x)f(x)f(x)+f(x)f'(x)f(x)+f(x)f(x)f'(x) = 3[f(x)]^2f'(x)$.

3.2.62: Suppose that u_1, u_2, u_3, u_4, and u_5 are differentiable functions of x. Let primes denote derivatives with respect to x. Then

$$\begin{aligned} D_x[u_1u_2u_3u_4] &= D_x[(u_1u_2u_3)u_4] = (u_1u_2u_3)'u_4 + (u_1u_2u_3)u_4' \\ &= (u_1'u_2u_3+u_1u_2'u_3+u_1u_2u_3')u_4 + (u_1u_2u_3)u_4' \\ &= u_1'u_2u_3u_4+u_1u_2'u_3u_4+u_1u_2u_3'u_4+u_1u_2u_3u_4'. \end{aligned}$$

Next, using this result,

$$\begin{aligned}D_x[u_1u_2u_3u_4u_5] &= D_x[(u_1u_2u_3u_4)u_5] = (u_1u_2u_3u_4)'u_5 + (u_1u_2u_3u_4)u_5' \\ &= (u_1'u_2u_3u_4 + u_1u_2'u_3u_4 + u_1u_2u_3'u_4 + u_1u_2u_3u_4')u_5 + (u_1u_2u_3u_4)u_5' \\ &= u_1'u_2u_3u_4u_5 + u_1u_2'u_3u_4u_5 + u_1u_2u_3'u_4u_5 + u_1u_2u_3u_4'u_5 + u_1u_2u_3u_4u_5'.\end{aligned}$$

3.2.63: Let $u_1(x) = u_2(x) = u_3(x) = \cdots = u_{n-1}(x) = u_n(x) = f(x)$. Then the left-hand side of Eq. (16) is $D_x[\,(f(x))^n]$ and the right-hand side is

$$f'(x)[f(x)]^{n-1} + f(x)f'(x)[f(x)]^{n-2} + [f(x)]^2 f'(x)[f(x)]^{n-3} + \cdots + [f(x)]^{n-1}f'(x) = n[f(x)]^{n-1} \cdot f'(x).$$

Therefore if n is a positive integer and $f'(x)$ exists, then

$$D_x[\,(f(x))^n] = n(f(x))^{n-1} \cdot f'(x).$$

3.2.64: Substitution of $f(x) = x^2 + x + 1$ and $n = 100$ in the result of Problem 63 yields

$$D_x[\,(x^2 + x + 1)^{100}] = D_x[\,(f(x))^n] = n(f(x))^{n-1} \cdot f'(x) = 100(x^2 + x + 1)^{99} \cdot (2x + 1).$$

3.2.65: Let $f(x) = x^3 - 17x + 35$ and let $n = 17$. Then $g(x) = (f(x))^n$. Hence, by the result in Problem 63,

$$g'(x) = D_x[\,(f(x))^n] = n(f(x))^{n-1} \cdot f'(x) = 17(x^3 - 17x + 35)^{16} \cdot (3x^2 - 17).$$

3.2.66: We begin with $f(x) = ax^3 + bx^2 + cx + d$. Then $f'(x) = 3ax^2 + 2bx + c$. The conditions in the problem require that (simultaneously)

$$\begin{aligned}1 = f(0) = d, &\qquad\qquad 0 = f(1) = a + b + c + d, \\ 0 = f'(0) = c, &\qquad \text{and} \qquad 0 = f'(1) = 3a + 2b + c.\end{aligned}$$

These equations have the unique solution $a = 2$, $b = -3$, $c = 0$, and $d = 1$. Therefore $f(x) = 2x^3 - 3x^2 + 1$ is the only possible solution. It is easy to verify that $f(x)$ satisfies the conditions required in the problem.

3.2.67: If n is a positive integer and

$$f(x) = \frac{x^n}{1 + x^2},$$

then

$$f'(x) = \frac{(1 + x^2)(nx^{n-1}) - (2x)(x^n)}{(1 + x^2)^2} = \frac{nx^{n-1} + nx^{n+1} - 2x^{n+1}}{(1 + x^2)^2} = \frac{x^{n-1}[n + (n-2)x^2]}{(1 + x^2)^2}. \qquad (1)$$

If $n = 0$, then (by the reciprocal rule)

$$f'(x) = -\frac{2x}{(1+x^2)^2}.$$

If $n = 2$, then by Eq. (1)

$$f'(x) = \frac{2x}{(1+x^2)^2}.$$

In each case there can be but one solution of $f'(x) = 0$, so there is only one horizontal tangent line. If $n = 0$ it is tangent to the graph of f at the point $(0, 1)$; if $n = 2$ it is tangent to the graph of f at the point $(0, 0)$.

3.2.68: If $n = 1$, then Eq. (1) of the solution of Problem 67 yields

$$f'(x) = \frac{1-x^2}{(1+x^2)^2}.$$

The equation $f'(x) = 0$ has the two solutions $x = \pm 1$, so there are two points on the graph of f where the tangent line is horizontal: $\left(-1, -\frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$.

3.2.69: If n is a positive integer and $n \geqq 3$, $f'(x) = 0$ only when the numerator is zero in Eq. (1) of the solution of Problem 67; that is, when $x^{n-1}(n + [n-2]x^2) = 0$. But this implies that $x = 0$ (because $n \geqq 3$) or that $n + [n-2]x^2 = 0$. The latter is impossible because $n > 0$ and $[n-2]x^2 \geqq 0$. Therefore the only horizontal tangent to the graph of f is at the point $(0, 0)$.

3.2.70: By Eq. (1) in the solution of Problem 67, if

$$f(x) = \frac{x^3}{1+x^2}, \qquad \text{then} \qquad f'(x) = \frac{x^2(3+x^2)}{(1+x^2)^2}.$$

So $f'(x) = 1$ when

$$\begin{aligned} \frac{x^2(3+x^2)}{(1+x^2)^2} &= 1; \\ x^2(3+x^2) &= (1+x^2)^2; \\ x^4 + 3x^2 &= x^4 + 2x^2 + 1; \\ x^2 &= 1. \end{aligned}$$

Therefore there are two points where the line tangent to the graph of f has slope 1; they are $\left(-1, -\frac{1}{2}\right)$ and $\left(1, \frac{1}{2}\right)$.

3.2.71: If

$$f(x) = \frac{x^3}{1+x^2}, \qquad \text{then} \qquad f'(x) = \frac{x^2(3+x^2)}{(1+x^2)^2} = \frac{x^4+3x^2}{(1+x^2)^2},$$

by Eq. (1) in the solution of Problem 67. A line tangent to the graph of $y = f'(x)$ will be horizontal when the derivative $f''(x)$ of $f'(x)$ is zero. But

$$D_x\,[f'(x)\,] = f''(x) = \frac{(1+x^2)^2(4x^3+6x) - (x^4+3x^2)(4x^3+4x)}{(1+x^2)^4}$$
$$= \frac{(1+x^2)^2(4x^3+6x) - (x^4+3x^2)(4x)(x^2+1)}{(1+x^2)^4} = \frac{(1+x^2)(4x^3+6x) - (x^4+3x^2)(4x)}{(1+x^2)^3}$$
$$= \frac{4x^3+6x+4x^5+6x^3-4x^5-12x^3}{(1+x^2)^3} = \frac{6x-2x^3}{(1+x^2)^3} = \frac{2x(3-x^2)}{(1+x^2)^3}.$$

So $f''(x) = 0$ when $x = 0$ and when $x = \pm\sqrt{3}$. Therefore there are three points on the graph of $y = f'(x)$ at which the tangent line is horizontal: $(0,\,0)$, $\left(-\sqrt{3},\,\frac{9}{8}\right)$, and $\left(\sqrt{3},\,\frac{9}{8}\right)$.

3.2.72: (a) Using the quadratic formula, $V'(T) = 0$ when

$$T \;=\; T_m \;=\; \frac{170100 - 20\sqrt{59243226}}{4074} \;\approx\; 3.96680349529363770572 \quad (\text{in } ^\circ\text{C})$$

and substitution in the formula for $V(T)$ (Example 5) yields

$$V_m \;=\; V(T_m) \;\approx\; 999.87464592037071155281 \quad (\text{cm}^3).$$

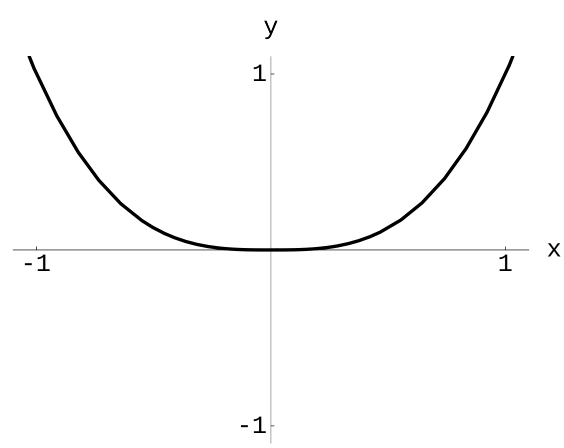
(b) The *Mathematica* command

```
Solve[ V(T) == 1000, T ]
```

yielded three solutions, the only one of which is close to $T = 8$ was

$$T \;=\; \frac{85050 - 10\sqrt{54879293}}{1358} \;\approx\; 8.07764394099814733845 \quad (\text{in } ^\circ\text{C}).$$

3.2.73: The graph of $f(x) = |\,x^3\,|$ is shown next.

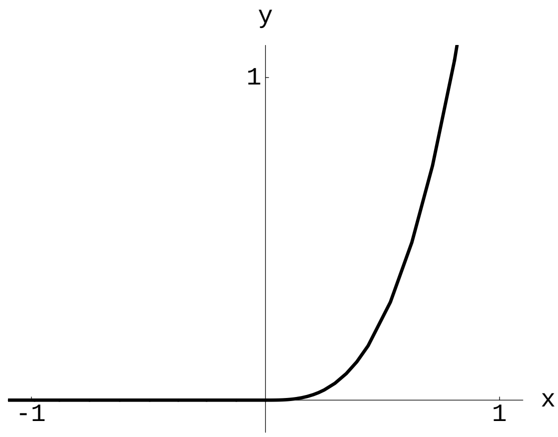


Clearly f is differentiable at x if $x \neq 0$. Moreover,

$$f'_-(0) = \lim_{h \to 0^-} \frac{|(0+h)^3| - 0}{h} = \lim_{h \to 0^-} \frac{-h^3}{h} = 0$$

and $f_+(0) = 0$ by a similar computation. Therefore f is differentiable everywhere.

3.2.74: The graph of $f(x) = x^3 + |x^3|$ is shown next.



Clearly f is differentiable except possibly at $x = 0$. Moreover,

$$f'_-(0) = \lim_{h \to 0^-} \frac{(0+h)^3 + |(0+h)^3|}{h} = \lim_{h \to 0^-} \frac{h^3 - h^3}{h} = 0$$

and

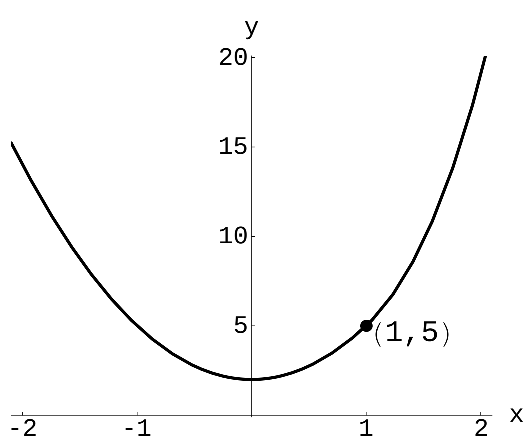
$$f_+(0) = \lim_{h \to 0^+} \frac{(0+h)^3 + |(0+h)^3|}{h} = \lim_{h \to 0^+} \frac{2h^3}{h} = 0.$$

Therefore f is differentiable at x for all x in $\boldsymbol{R}$.

3.2.75: The graph of

$$f(x) = \begin{cases} 2 + 3x^2 & \text{if } x < 1, \\ 3 + 2x^3 & \text{if } x \geqq 1, \end{cases}$$

is shown next.



Clearly f is differentiable except possibly at $x = 1$. But

$$f'_{-}(1) = \lim_{h \to 0^-} \frac{2 + 3(1+h)^2 - 5}{h} = \lim_{h \to 0^-} \frac{6h + 3h^2}{h} = 6$$

and

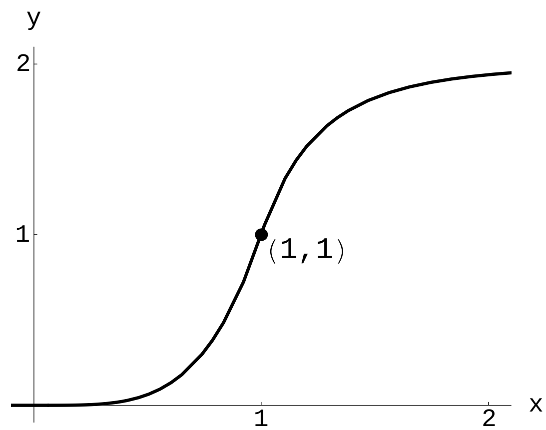
$$f'_{+}(1) = \lim_{h \to 0^+} \frac{3 + 2(1+h)^3 - 5}{h} = \lim_{h \to 0^+} \frac{6h + 6h^2 + 2h^3}{h} = 6.$$

Therefore $f'(1)$ exists (and $f'(1) = 6$), and hence $f'(x)$ exists for every real number x.

3.2.76: The graph of

$$f(x) = \begin{cases} x^4 & \text{if } x < 1, \\ 2 - \dfrac{1}{x^4} & \text{if } x \geqq 1 \end{cases}$$

is shown next.



Clearly f is differentiable except possibly at $x = 1$. But here we have

$$f'_{-}(1) = \lim_{h \to 0^-} \frac{(1+h)^4 - 1}{h} = \lim_{h \to 0^-} \frac{4h + 6h^2 + 4h^3 + h^4}{h} = 4$$

and

$$f'_{+}(1) = \lim_{h \to 0^+} \frac{1}{h} \cdot \left(2 - \frac{1}{(1+h)^4} - 1\right) = \lim_{h \to 0^+} \frac{1}{h} \cdot \left(1 - \frac{1}{(1+h)^4}\right)$$

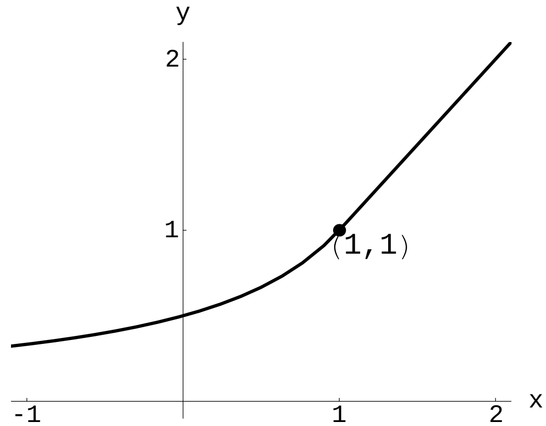
$$= \lim_{h \to 0^+} \frac{4h + 6h^2 + 4h^3 + h^4}{h(1+h)^4} = \lim_{h \to 0^+} \frac{4 + 6h + 4h^2 + h^3}{(1+h)^4} = \frac{4}{1} = 4.$$

Therefore f is differentiable at $x = 1$ as well.

3.2.77: The graph of

$$f(x) = \begin{cases} \dfrac{1}{2 - x} & \text{if } x < 1, \\ x & \text{if } x \geqq 1 \end{cases}$$

is shown next.



Clearly f is differentiable except possibly at $x = 1$. But

$$f'_{-}(1) = \lim_{h\to 0^-} \frac{1}{h} \cdot \left(\frac{1}{2-(1+h)} - 1 \right) = \lim_{h\to 0^+} \frac{1}{h} \cdot \left(\frac{1}{1-h} - 1 \right) = \lim_{h\to 0^-} \frac{h}{h(1-h)} = \lim_{h\to 0^-} \frac{1}{1-h} = 1$$

and

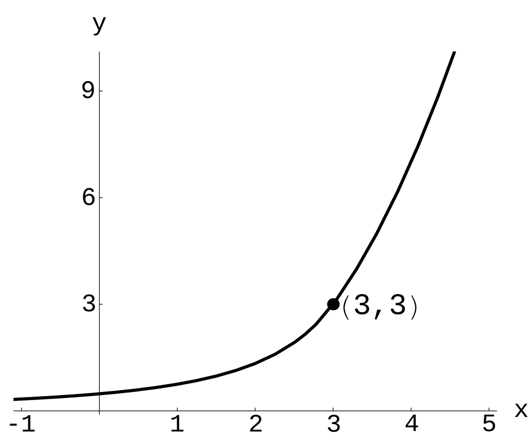
$$f'_{+}(1) = \lim_{h\to 0^+} \frac{1+h-1}{h} = 1.$$

Thus f is differentiable at $x = 1$ as well (and $f'(1) = 1$).

3.2.78: The graph of the function

$$f(x) = \begin{cases} \dfrac{12}{(5-x)^2} & \text{if } x < 3, \\ x^2 - 3x + 3 & \text{if } x \geqq 3 \end{cases}$$

is next.



Clearly f is differentiable except possibly at $x = 3$. But

$$f'_{-}(3) = \lim_{h\to 0^-} \frac{1}{h} \cdot \left(\frac{12}{(5-3-h)^2} - 3 \right)$$

$$= \lim_{h\to 0^-} \frac{12 - 3(2-h)^2}{h(2-h)^2} = \lim_{h\to 0^-} \frac{12h - 3h^2}{h(2-h)^2} = \lim_{h\to 0^-} \frac{12 - 3h}{(2-h)^2} = 3$$

and

$$f'_+(3) = \lim_{h\to 0^+} \frac{(3+h)^2 - 3(3+h) + 3 - 3}{h} = \lim_{h\to 0^+} \frac{6h + h^2 - 3h}{h} = 3.$$

Thus f is differentiable at $x = 3$ as well.

Section 3.3

3.3.1: Given $y = (3x+4)^5$, the chain rule yields

$$\frac{dy}{dx} = 5\cdot(3x+4)^4 \cdot D_x(3x+4) = 5\cdot(3x+4)^4\cdot 3 = 15(3x+4)^4.$$

3.3.2: Given $y = (2-5x)^3$, the chain rule yields

$$\frac{dy}{dx} = 3\cdot(2-5x)^2\cdot D_x(2-5x) = 3\cdot(2-5x)^2\cdot(-5) = -15(2-5x)^2.$$

3.3.3: Rewrite the given function in the form $y = (3x-2)^{-1}$ in order to apply the chain rule. The result is

$$\frac{dy}{dx} = (-1)(3x-2)^{-2}\cdot D_x(3x-2) = (-1)(3x-2)^{-2}\cdot 3 = -3(3x-2)^{-2} = -\frac{3}{(3x-2)^2}.$$

3.3.4: Rewrite the given function in the form $y = (2x+1)^{-3}$ in order to apply the chain rule. The result is

$$\frac{dy}{dx} = (-3)(2x+1)^{-4}\cdot D_x(2x+1) = (-3)(2x+1)^{-4}\cdot 2 = -6(2x+1)^{-4} = -\frac{6}{(2x+1)^4}.$$

3.3.5: Given $y = (x^2+3x+4)^3$, the chain rule yields

$$\frac{dy}{dx} = 3(x^2+3x+4)^2\cdot D_x(x^2+3x+4) = 3(x^2+3x+4)^2(2x+3).$$

3.3.6: $\displaystyle\frac{dy}{dx} = -4\cdot(7-2x^3)^{-5}\cdot D_x(7-2x^3) = -4\cdot(7-2x^3)^{-5}\cdot(-6x^2) = 24x^2(7-2x^3)^{-5} = \frac{24x^2}{(7-2x^3)^5}.$

3.3.7: We use the product rule, and in the process of doing so must use the chain rule twice: Given $y = (2-x)^4(3+x)^7$,

$$\begin{aligned}\frac{dy}{dx} &= (2-x)^4\cdot D_x(3+x)^7 + (3+x)^7\cdot D_x(2-x)^4\\ &= (2-x)^4\cdot 7\cdot(3+x)^6\cdot D_x(3+x) + (3+x)^7\cdot 4(2-x)^3\cdot D_x(2-x)\\ &= (2-x)^4\cdot 7\cdot(3+x)^6\cdot 1 + (3+x)^7\cdot 4(2-x)^3\cdot(-1) = 7(2-x)^4(3+x)^6 - 4(2-x)^3(3+x)^7\\ &= (2-x)^3(3+x)^6(14-7x-12-4x) = (2-x)^3(3+x)^6(2-11x).\end{aligned}$$

The last simplifications would be necessary only if you needed to find where $y'(x)$ is positive, where negative, and where zero.

3.3.8: Given $y = (x+x^2)^5(1+x^3)^2$, the product rule—followed by two applications of the chain rule—yields

$$\begin{aligned}\frac{dy}{dx} &= (x+x^2)^5 \cdot D_x(1+x^3)^2 + (1+x^3)^2 \cdot D_x(x+x^2)^5\\ &= (x+x^2)^5 \cdot 2 \cdot (1+x^3) \cdot D_x(1+x^3) + (1+x^3)^2 \cdot 5 \cdot (x+x^2)^4 \cdot D_x(x+x^2)\\ &= (x+x^2)^5 \cdot 2 \cdot (1+x^3) \cdot 3x^2 + (1+x^3)^2 \cdot 5 \cdot (x+x^2)^4 \cdot (1+2x)\\ &= \cdots = x^4(x+1)^6(x^2-x+1)(16x^3-5x^2+5x+5).\end{aligned}$$

Sometimes you have to factor an expression as much as you can to determine where it is positive, negative, or zero.

3.3.9: We will use the quotient rule, which will require use of the chain rule to find the derivative of the denominator:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(3x-4)^3 D_x(x+2) - (x+2)D_x(3x-4)^3}{[\,(3x-4)^3]^2}\\ &= \frac{(3x-4)^3 \cdot 1 - (x+2) \cdot 3 \cdot (3x-4)^2 \cdot D_x(3x-4)}{(3x-4)^6}\\ &= \frac{(3x-4)^3 - 3(x+2)(3x-4)^2 \cdot 3}{(3x-4)^6} = \frac{(3x-4) - 9(x+2)}{(3x-4)^4} = -\frac{6x+22}{(3x-4)^4}.\end{aligned}$$

3.3.10: We use the quotient rule, and need to use the chain rule twice along the way:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(4+5x+6x^2)^2 \cdot D_x(1-x^2)^3 - (1-x^2)^3 \cdot D_x(4+5x+6x^2)^2}{(4+5x+6x^2)^4}\\ &= \frac{(4+5x+6x^2)^2 \cdot 3 \cdot (1-x^2)^2 \cdot D_x(1-x^2) - (1-x^2)^3 \cdot 2 \cdot (4+5x+6x^2) \cdot D_x(4+5x+6x^2)}{(4+5x+6x^2)^4}\\ &= \frac{(4+5x+6x^2)^2 \cdot 3 \cdot (1-x^2)^2 \cdot (-2x) - (1-x^2)^3 \cdot 2 \cdot (4+5x+6x^2) \cdot (5+12x)}{(4+5x+6x^2)^4}\\ &= \frac{(4+5x+6x^2) \cdot 3 \cdot (1-x^2)^2 \cdot (-2x) - (1-x^2)^3 \cdot 2 \cdot (5+12x)}{(4+5x+6x^2)^3}\\ &= -\frac{2(x^2-1)^2(6x^3+10x^2+24x+5)}{(4+5x+6x^2)^3}.\end{aligned}$$

3.3.11: Here is a problem in which use of the chain rule contains another use of the chain rule. Given $y = [1 + (1 + x)^3]^4$,

$$\frac{dy}{dx} = 4[1 + (1 + x)^3]^3 \cdot D_x[1 + (1 + x)^3] = 4[1 + (1 + x)^3]^3 \cdot [0 + D_x(1 + x)^3]$$
$$= 4[1 + (1 + x)^3]^3 \cdot 3 \cdot (1 + x)^2 \cdot D_x(1 + x) = 12[1 + (1 + x)^3]^3(1 + x)^2.$$

3.3.12: Again a "nested chain rule" problem:

$$\frac{dy}{dx} = -5 \cdot \left[x + (x + x^2)^{-3}\right]^{-6} \cdot D_x \left[x + (x + x^2)^{-3}\right]$$
$$= -5 \cdot \left[x + (x + x^2)^{-3}\right]^{-6} \left[1 + (-3) \cdot (x + x^2)^{-4} \cdot D_x(x + x^2)\right]$$
$$= -5 \cdot \left[x + (x + x^2)^{-3}\right]^{-6} \left[1 + (-3) \cdot (x + x^2)^{-4} \cdot (1 + 2x)\right].$$

3.3.13: Given: $y = (u + 1)^3$ and $u = \dfrac{1}{x^2}$. The chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3(u + 1)^2 \cdot \frac{-2}{x^3} = -\frac{6}{x^3}\left(\frac{1}{x^2} + 1\right)^2 = -\frac{6(x^2 + 1)^2}{x^7}.$$

3.3.14: Write $y = \frac{1}{2}u^{-1} - \frac{1}{3}u^{-2}$. Then, with $u = 2x + 1$, the chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left(-\frac{1}{2}u^{-2} + \frac{2}{3}u^{-3}\right) \cdot 2 = -2\left(\frac{1}{2u^2} - \frac{2}{3u^3}\right)$$
$$= -2\left(\frac{1}{2(2x + 1)^2} - \frac{2}{3(2x + 1)^3}\right) = \cdots = \frac{1 - 6x}{3(1 + 2x)^3}.$$

3.3.15: Given $y = (1 + u^2)^3$ and $u = (4x - 1)^2$, the chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6u(1 + u^2)^2 \cdot 8 \cdot (4x - 1)$$
$$= 48 \cdot (4x - 1)^2(1 + (4x - 1)^4)^2(4x - 1) = 48(4x - 1)^3(1 + (4x - 1)^4)^2.$$

Without the chain rule, our only way to differentiate $y(x)$ would be first to expand it:

$$y(x) = 8 - 192x + 2688x^2 - 25600x^3 + 181248x^4 - 983040x^5 + 4128768x^6$$
$$- 13369344x^7 + 32636928x^8 - 57671680x^9 + 69206016x^{10} - 50331648x^{11} + 16777216x^{12}.$$

Then we could differentiate $y(x)$ using the linear combination and power rules:

$$y'(x) = -192 + 5376x - 76800x^2 + 724992x^3 - 4915200x^4 + 24772608x^5 - 93585408x^6$$
$$+ 261095424x^7 - 519045120x^8 + 692060160x^9 - 553648128x^{10} + 201326592x^{11}.$$

Fortunately, the chain rule is available—and even if not, we still have *Maple*, *Derive*, *Mathematica*, and MATLAB.

3.3.16: If $y = u^5$ and $u = \dfrac{1}{3x-2}$, then the chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 5u^4\cdot\left(-\frac{D_x(3x-2)}{(3x-2)^2}\right) = -\frac{5}{(3x-2)^4}\cdot\frac{3}{(3x-2)^2} = -\frac{15}{(3x-2)^6}.$$

3.3.17: If $y = u(1-u)^3$ and $u = \dfrac{1}{x^4}$, then the chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left[\,(1-u)^3 - 3u(1-u)^2\right]\cdot\left(-4x^{-5}\right) = \left[\,(1-x^{-4})^3 - 3x^{-4}(1-x^{-4})^2\right]\cdot\left(-4x^{-5}\right),$$

which a very patient person can simplify to

$$\frac{dy}{dx} = \frac{16-36x^4+24x^8-4x^{12}}{x^{17}}.$$

3.3.18: If $y = \dfrac{u}{u+1}$ and $u = \dfrac{x}{x+1}$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{(u+1)\cdot 1 - u\cdot 1}{(u+1)^2}\cdot\frac{(x+1)\cdot 1 - x\cdot 1}{(x+1)^2} = \frac{1}{(u+1)^2}\cdot\frac{1}{(x+1)^2}$$

$$= \frac{1}{\left(\dfrac{x}{x+1}+1\right)^2}\cdot\frac{1}{(x+1)^2} = \frac{1}{\left(\dfrac{2x+1}{x+1}\right)^2}\cdot\frac{1}{(x+1)^2} = \frac{1}{(2x+1)^2}.$$

3.3.19: If $y = u^2(u-u^4)^3$ and $u = x^{-2}$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left[2u(u-u^4)^3 + 3u^2(u-u^4)^2(1-4u^3)\,\right]\cdot(-2x^{-3})$$

$$= \left[2x^{-2}(x^{-2}-x^{-8})^3 + 3x^{-4}(x^{-2}-x^{-8})^2(1-4x^{-6})\,\right]\cdot(-2x^{-3}) = \cdots = \frac{28-66x^6+48x^{12}-10x^{18}}{x^{29}}.$$

3.3.20: If $y = \dfrac{u}{(2u+1)^4}$ and $u = x - 2x^{-1}$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{(2u+1)^4 - 8u(2u+1)^3}{(2u+1)^8}\cdot(1+2x^{-2}) = \frac{2u+1-8u}{(2u+1)^5}\cdot(1+2x^{-2})$$

$$= \frac{1-6u}{(2u+1)^5}\cdot\frac{x^2+2}{x^2} = \frac{1-6x+\dfrac{12}{x}}{\left(2x-\dfrac{4}{x}+1\right)^5}\cdot\frac{x^2+2}{x^2} = \frac{x-6x^2+12}{x\left(\dfrac{2x^2-4+x}{x}\right)^5}\cdot\frac{x^2+2}{x^2}$$

$$= \frac{x^4(12+x-6x^2)}{(2x^2+x-4)^5}\cdot\frac{x^2+2}{x^2} = \frac{x^2(12+x-6x^2)(x^2+2)}{(2x^2+x-4)^5}$$

$$= \frac{x^2(12x^2+x^3-6x^4+24+2x-12x^2)}{(2x^2+x-4)^5} = \frac{x^2(24+2x+x^3-6x^4)}{(2x^2+x-4)^5}.$$

3.3.21: Let $u(x) = 2x - x^2$ and $n = 3$. Then $f(x) = u^n$, so that

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = 3u^2 \cdot (2 - 2x) = 3(2x - x^2)^2(2 - 2x).$$

3.3.22: Let $u(x) = 2 + 5x^3$ and $n = -1$. Then $f(x) = u^n$, so that

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = (-1)u^{-2} \cdot 15x^2 = -\frac{15x^2}{(2 + 5x^3)^2}.$$

3.3.23: Let $u(x) = 1 - x^2$ and $n = -4$. Then $f(x) = u^n$, so that

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = (-4)u^{-5} \cdot (-2x) = \frac{8x}{(1 - x^2)^5}.$$

3.3.24: Let $u(x) = x^2 - 4x + 1$ and $n = 3$. Then $f(x) = u^n$, so

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = 3u^2 \cdot (2x - 4) = 3(x^2 - 4x + 1)^2(2x - 4).$$

3.3.25: Let $u(x) = \dfrac{x+1}{x-1}$ and $n = 7$. Then $f(x) = u^n$, and therefore

$$f'(x) = nu^{n-1} \cdot \frac{du}{dx} = 7u^6 \cdot \frac{(x-1) - (x+1)}{(x-1)^2} = 7\left(\frac{x+1}{x-1}\right)^6 \cdot \frac{-2}{(x-1)^2} = -14 \cdot \frac{(x+1)^6}{(x-1)^8}.$$

3.3.26: Let $n = 4$ and $u(x) = \dfrac{x^2 + x + 1}{x + 1}$. Thus

$$\begin{aligned} f'(x) = nu^{n-1} \cdot \frac{du}{dx} &= 4\left(\frac{x^2 + x + 1}{x + 1}\right)^3 \cdot \frac{(x+1)(2x+1) - (x^2 + x + 1)}{(x+1)^2} \\ &= \frac{4(x^2 + x + 1)^3}{(x+1)^3} \cdot \frac{2x^2 + 3x + 1 - x^2 - x - 1}{(x+1)^2} = \frac{4(x^2 + x + 1)^3}{(x+1)^3} \cdot \frac{x^2 + 2x}{(x+1)^2} = \frac{4(x^2 + x + 1)^3(x^2 + 2x)}{(x+1)^5}. \end{aligned}$$

3.3.27: $g'(y) = 1 + 5(2y - 3)^4 \cdot 2 = 1 + 10(2y - 3)^4$.

3.3.28: $h'(z) = 2z(z^2 + 4)^3 + 3z^2(z^2 + 4)^2 \cdot 2z = (2z^3 + 8z + 6z^3)(z^2 + 4)^2 = 8z(z^2 + 1)(z^2 + 4)^2$.

3.3.29: If $F(s) = (s - s^{-2})^3$, then

$$\begin{aligned} F'(s) = 3(s - s^{-2})^2(1 + 2s^{-3}) &= 3\left(s - \frac{1}{s^2}\right)^2 \cdot \left(1 + \frac{2}{s^3}\right) = 3\left(\frac{s^3 - 1}{s^2}\right)^2 \cdot \frac{s^3 + 2}{s^3} \\ &= 3 \cdot \frac{(s^3 - 1)^2(s^3 + 2)}{s^7} = 3 \cdot \frac{(s^6 - 2s^3 + 1)(s^3 + 2)}{s^7} = \frac{3(s^9 - 3s^3 + 2)}{s^7}. \end{aligned}$$

3.3.30: If $G(t) = \left(t^2 + 1 + \dfrac{1}{t}\right)^2$, then

$$G'(t) = 2\left(2t - \frac{1}{t^2}\right)\cdot\left(t^2 + 1 + \frac{1}{t}\right) = 4t^3 + 4t + 2 - \frac{2}{t^2} - \frac{2}{t^3} = \frac{4t^6 + 4t^4 + 2t^3 - 2t - 2}{t^3}.$$

3.3.31: If $f(u) = (1+u)^3(1+u^2)^4$, then

$$f'(u) = 3(1+u)^2(1+u^2)^4 + 8u(1+u)^3(1+u^2)^3 = (1+u)^2(1+u^2)^3(11u^2 + 8u + 3).$$

3.3.32: If $g(w) = (w^2 - 3w + 4)(w+4)^5$, then

$$g'(w) = 5(w+4)^4(w^2 - 3w + 4) + (w+4)^5(2w - 3) = (w+4)^4(7w^2 - 10w + 8).$$

3.3.33: If $h(v) = \left[v - \left(1 - \dfrac{1}{v}\right)^{-1}\right]^{-2}$, then

$$h'(v) = (-2)\left[v - \left(1 - \frac{1}{v}\right)^{-1}\right]^{-3}\left[1 + \left(1 - \frac{1}{v}\right)^{-2}\left(\frac{1}{v^2}\right)\right] = \frac{2(v-1)(v^2 - 2v + 2)}{v^3(2-v)^3}.$$

3.3.34: If $p(t) = \left(\dfrac{1}{t} + \dfrac{1}{t^2} + \dfrac{1}{t^3}\right)^{-4}$, then

$$p'(t) = 4\left(\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3}\right)^{-5}\left(\frac{1}{t^2} + \frac{2}{t^3} + \frac{3}{t^4}\right) = 4\left(\frac{t^2 + t + 1}{t^3}\right)^{-5}\left(\frac{t^2 + 2t + 3}{t^4}\right)$$

$$= 4\left(\frac{t^3}{t^2 + t + 1}\right)^5\left(\frac{t^2 + 2t + 3}{t^4}\right) = \frac{4t^{15}(t^2 + 2t + 3)}{(t^2 + t + 1)^5 t^4} = \frac{4t^{11}(t^2 + 2t + 3)}{(t^2 + t + 1)^5}.$$

3.3.35: If $F(z) = (5z^5 - 4z + 3)^{-10}$, then

$$F'(z) = -10(5z^5 - 4z + 3)^{-11}(25z^4 - 4) = \frac{40 - 250z^4}{(5z^5 - 4z + 3)^{11}}.$$

3.3.36: Given $G(x) = (1 + [x + (x^2 + x^3)^4]^5)^6$,

$$G'(x) = 6(1 + [x + (x^2 + x^3)^4]^5)^5 \cdot 5[x + (x^2 + x^3)^4]^4 \cdot [1 + 4(x^2 + x^3)^3(2x + 3x^2)].$$

When $G'(x)$ is expanded completely (written in polynomial form), it has degree 359 and the term with largest coefficient is $743139421359963600696510590690384174 40x^{287}$.

3.3.37: Chain rule: $\dfrac{dy}{dx} = 4(x^3)^3 \cdot 3x^2$. Power rule: $\dfrac{dy}{dx} = 12x^{11}$.

3.3.38: Chain rule: $\dfrac{dy}{dx} = (-1)\left(\dfrac{1}{x}\right)^{-2}\left(-\dfrac{1}{x^2}\right)$. Power rule: $\dfrac{dy}{dx} = 1$.

3.3.39: Chain rule: $\dfrac{dy}{dx} = 2(x^2 - 1) \cdot 2x$. Without chain rule: $\dfrac{dy}{dx} = 4x^3 - 4x$.

3.3.40: Chain rule: $\dfrac{dy}{dx} = -3(1 - x)^2$. Without chain rule: $\dfrac{dy}{dx} = -3 + 6x - 3x^2$.

3.3.41: Chain rule: $\dfrac{dy}{dx} = 4(x + 1)^3$. Without chain rule: $\dfrac{dy}{dx} = 4x^3 + 12x^2 + 12x + 4$.

3.3.42: Chain rule: $\dfrac{dy}{dx} = -2(x + 1)^{-3}$. Reciprocal rule: $\dfrac{dy}{dx} = -\dfrac{2x + 2}{(x^2 + 2x + 1)^2}$.

3.3.43: Chain rule: $\dfrac{dy}{dx} = -2x(x^2 + 1)^{-2}$. Reciprocal rule: $\dfrac{dy}{dx} = -\dfrac{2x}{(x^2 + 1)^2}$.

3.3.44: Chain rule: $\dfrac{dy}{dx} = 2(x^2 + 1) \cdot 2x$. Product rule: $\dfrac{dy}{dx} = 2x(x^2 + 1) + 2x(x^2 + 1)$.

3.3.45: If $f(x) = \sin(x^3)$, then $f'(x) = \big[\cos(x^3)\big] \cdot D_x(x^3) = 3x^2 \cos(x^3) = 3x^2 \cos x^3$.

3.3.46: If $g(t) = (\sin t)^3$, then $g'(t) = 3(\sin t)^2 \cdot D_t \sin t = (3\sin^2 t)(\cos t) = 3\sin^2 t\, \cos t$.

3.3.47: If $g(z) = (\sin 2z)^3$, then

$$g'(z) = 3(\sin 2z)^2 \cdot D_z(\sin 2z) = 3(\sin 2z)^2(\cos 2z) \cdot D_z(2z) = 6\sin^2 2z\, \cos 2z.$$

3.3.48: If $k(u) = \sin(1 + \sin u)$, then $k'(u) = [\cos(1 + \sin u)] \cdot D_u(1 + \sin u) = [\cos(1 + \sin u)] \cdot \cos u$.

3.3.49: The radius of the circular ripple is $r(t) = 2t$ and its area is $a(t) = \pi(2t)^2$; thus $a'(t) = 8\pi t$. When $r = 10$, $t = 5$, and at that time the rate of change of area with respect to time is $a'(5) = 40\pi$ (in.^{2}/s).

3.3.50: If the circle has area A and radius r, then $A = \pi r^2$, so that $r = \sqrt{A/\pi}$. If t denotes time in seconds, then the rate of change of the radius of the circle is

$$\frac{dr}{dt} = \frac{dr}{dA} \cdot \frac{dA}{dt} = \frac{1}{2\sqrt{\pi A}} \cdot \frac{dA}{dt}. \tag{1}$$

We are given the values $A = 75\pi$ and $dA/dt = -2\pi$; when we substitute these values into the last expression in Eq. (1), we find that $\dfrac{dr}{dt} = -\frac{1}{15}\sqrt{3}$. Hence the radius of the circle is decreasing at the rate of $-\frac{1}{15}\sqrt{3}$ (cm/s) at the time in question.

3.3.51: Let A denote the area of the square and x the length of each edge. Then $A = x^2$, so $dA/dx = 2x$. If t denotes time (in seconds), then

$$\frac{dA}{dt} = \frac{dA}{dx} \cdot \frac{dx}{dt} = 2x\frac{dx}{dt}.$$

All that remains is to substitute the given data $x = 10$ and $dx/dt = 2$ to find that the area of the square is increasing at the rate of 40 in.^{2}/s at the time in question.

3.3.52: Let x denote the length of each side of the triangle. Then its altitude is $\frac{1}{2}x\sqrt{3}$, and so its area is $A = \frac{1}{4}x^2\sqrt{3}$. Therefore the rate of change of its area with respect to time t (in seconds) is

$$\frac{dA}{dt} = \left(\frac{1}{2}x\sqrt{3}\right) \cdot \frac{dx}{dt}.$$

We are given $x = 10$ and $dx/dt = 2$, so at that point the area is increasing at $10\sqrt{3}$ (in.^{2}/s).

3.3.53: The volume of the block is $V = x^3$ where x is the length of each edge. So $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$. We are given $dx/dt = -2$, so when $x = 10$ the volume of the block is decreasing at 600 in.^{3}/h.

3.3.54: By the chain rule, $f'(y) = h'(g(y)) \cdot g'(y)$. Then substitution of the data given in the problem yields $f'(-1) = h'(g(-1)) \cdot g'(-1) = h'(2) \cdot g'(-1) = -1 \cdot 7 = -7$.

3.3.55: $G'(t) = f'(h(t)) \cdot h'(t)$. Now $h(1) = 4$, $h'(1) = -6$, and $f'(4) = 3$, so $G'(1) = 3 \cdot (-6) = -18$.

3.3.56: The derivative of $f(f(f(x)))$ is the product of the three expressions $f'(f(f(x)))$, $f'(f(x))$, and $f'(x)$. When $x = 0$, $f(x) = 0$ and $f'(x) = 1$. Thus when $x = 0$, each of those three expressions has value 1, so the answer is 1.

3.3.57: The volume of the balloon is given by $V = \frac{4}{3}\pi r^3$, so

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2\,\frac{dr}{dt}.$$

Answer: When $r = 10$, $dV/dt = 4\pi \cdot 10^2 \cdot 1 = 400\pi \approx 1256.64$ (cm^3/s).

3.3.58: Let V denote the volume of the balloon and r its radius at time t (in seconds). We are given $dV/dt = 200\pi$. Now

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2\,\frac{dr}{dt}.$$

When $r = 5$, we have $200\pi = 4\pi \cdot 25 \cdot (dr/dt)$, so $dr/dt = 2$. Answer: When $r = 5$ (cm), the radius of the balloon is increasing at 2 cm/s.

3.3.59: Given: $\dfrac{dr}{dt} = -3$. Now $\dfrac{dV}{dt} = -300\pi = 4\pi r^2 \cdot (\dfrac{dr}{dt})$. So $4\pi r^2 = 100\pi$, and thus $r = 5$ (cm) at the time in question.

3.3.60: Let x denote the radius of the hailstone and let V denote its volume. Then

$$V = \frac{4}{3}\pi x^3, \qquad \text{and so} \qquad \frac{dV}{dt} = 4\pi x^2\frac{dx}{dt}.$$

When $x = 2$, $\dfrac{dV}{dt} = -0.1$, and therefore $-\dfrac{1}{10} = 4\pi \cdot 2^2 \cdot \dfrac{dx}{dt}$. So $\dfrac{dx}{dt} = -\dfrac{1}{160\pi}$. Answer: At the time in question, the radius of the hailstone is decreasing at $\dfrac{1}{160\pi}$ cm/s —that is, at about 0.002 cm/s.

3.3.61: Let V denote the volume of the snowball and A its surface area at time t (in hours). Then

$$\frac{dV}{dt} = kA \quad \text{and} \quad A = cV^{2/3}$$

(the latter because A is proportional to r^2, whereas V is proportional to r^3). Therefore

$$\frac{dV}{dt} = \alpha V^{2/3} \quad \text{and thus} \quad \frac{dt}{dV} = \beta V^{-2/3}$$

(α and β are constants). From the last equation we may conclude that $t = \gamma V^{1/3} + \delta$ for some constants γ and δ, so that $V = V(t) = (Pt + Q)^3$ for some constants P and Q. From the information $500 = V(0) = Q^3$ and $250 = V(1) = (P + Q)^3$, we find that $Q = 5\sqrt[3]{4}$ and that $P = -5 \cdot (\sqrt[3]{4} - \sqrt[3]{2})$. Now $V(t) = 0$ when $PT + Q = 0$; it turns out that

$$T = \frac{\sqrt[3]{2}}{\sqrt[3]{2} - 1} \approx 4.8473.$$

Therefore the snowball finishes melting at about 2:50:50 P.M. on the same day.

3.3.62: Let V denote the volume of the block, x the length of each of its edges. Then $V = x^3$. In 8 hours x decreases from 20 to 8, and dx/dt is steady, so t hours after 8:00 A.M. have

$$x = 20 - \frac{3}{2}t.$$

Also

$$\frac{dV}{dt} = \frac{dV}{dx} \cdot \frac{dx}{dt} = 3x^2 \cdot (-\frac{3}{2}) = -\frac{9}{2} \cdot \left(20 - \frac{3}{2}t\right)^2.$$

At 12 noon we have $t = 4$, so at noon $\dfrac{dV}{dt} = -\frac{9}{2}(20 - 6)^2 = -882$. Answer: The volume is decreasing at 882 in.^{3}/h then.

3.3.63: By the chain rule,

$$\frac{dv}{dx} = \frac{dv}{dw} \cdot \frac{dw}{dx}, \quad \text{and therefore} \quad \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx}.$$

3.3.64: Given: n is a fixed integer, f is differentiable, $f(1) = 1$, $F(x) = f(x^n)$, and $G(x) = [\,f(x)\,]^n$. Then

$$F(1) = f(1^n) = f(1) = 1 = 1^n = [\,f(1)\,]^n = G(1).$$

Next,

$$F'(x) = D_x f(x^n) = f'(x^n) \cdot nx^{n-1} \qquad \text{and} \qquad G'(x) = D_x\,[\,f(x)\,]^n = n\,[\,f(x)\,]^{n-1} \cdot f'(x).$$

Therefore

$$F'(1) = f'(1^n) \cdot n \cdot 1 = nf'(1) = n \cdot 1^{n-1} \cdot f'(1) = n \cdot [\, f(1)\,]^{n-1} \cdot f'(1) = G'(1).$$

3.3.65: If $h(x) = \sqrt{x+4}$, then

$$h'(x) = \frac{1}{2\sqrt{x+4}} \cdot D_x(x+4) = \frac{1}{2\sqrt{x+4}} \cdot 1 = \frac{1}{2\sqrt{x+4}}.$$

3.3.66: If $h(x) = x^{3/2} = x \cdot \sqrt{x}$, then

$$h'(x) = 1 \cdot \sqrt{x} + x \cdot D_x\left(\sqrt{x}\right) = \sqrt{x} + \frac{x}{2\sqrt{x}} = \sqrt{x} + \frac{1}{2}\sqrt{x} = \frac{3}{2}\sqrt{x}.$$

3.3.67: If $h(x) = (x^2+4)^{3/2} = (x^2+4)\sqrt{x^2+4}$, then

$$h'(x) = 2x\sqrt{x^2+4} + (x^2+4) \cdot \frac{1}{2\sqrt{x^2+4}} \cdot 2x = 2x\sqrt{x^2+4} + x\sqrt{x^2+4} = 3x\sqrt{x^2+4}.$$

3.3.68: If $h(x) = |x| = \sqrt{x^2}$, then

$$h'(x) = \frac{1}{2\sqrt{x^2}} \cdot D_x\left(x^2\right) = \frac{2x}{2\sqrt{x^2}} = \frac{x}{|x|}.$$

Section 3.4

3.4.1: Write $f(x) = 4x^{5/2} + 2x^{-1/2}$ to find

$$f'(x) = 10x^{3/2} - x^{-3/2} = 10x^{3/2} - \frac{1}{x^{3/2}} = \frac{10x^3 - 1}{x^{3/2}}.$$

3.4.2: Write $g(t) = 9t^{4/3} - 3t^{-1/3}$ to find

$$g'(t) = 12t^{1/3} + t^{-4/3} = 12t^{1/3} + \frac{1}{t^{4/3}} = \frac{12t^{5/3} + 1}{t^{4/3}}.$$

3.4.3: Write $f(x) = (2x+1)^{1/2}$ to find

$$f'(x) = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = \frac{1}{\sqrt{2x+1}}.$$

3.4.4: Write $h(z) = (7-6z)^{-1/3}$ to find that

$$h'(z) = -\frac{1}{3}(7-6z)^{-4/3} \cdot (-6) = \frac{2}{(7-6z)^{4/3}}.$$

3.4.5: Write $f(x) = 6x^{-1/2} - x^{3/2}$ to find that $f'(x) = -3x^{-3/2} - \dfrac{3}{2}x^{1/2} = -\dfrac{3(x^2+2)}{2x^{3/2}}$.

3.4.6: Write $\phi(u) = 7u^{-2/3} + 2u^{1/3} - 3u^{10/3}$ to find that

$$\phi'(u) = -\frac{14}{3}u^{-5/3} + \frac{2}{3}u^{-2/3} - 10u^{7/3} = -\frac{2(15u^4 - u + 7)}{3u^{5/3}}.$$

3.4.7: $D_x(2x+3)^{3/2} = \dfrac{3}{2}(2x+3)^{1/2} \cdot 2 = 3\sqrt{2x+3}$.

3.4.8: $D_x(3x+4)^{4/3} = \dfrac{4}{3}(3x+4)^{1/3} \cdot 3 = 4(3x+4)^{1/3} = 4\sqrt[3]{3x+4}$.

3.4.9: $D_x(3-2x^2)^{-3/2} = -\dfrac{3}{2}(3-2x^2)^{-5/2} \cdot (-4x) = \dfrac{6x}{(3-2x^2)^{5/2}}$.

3.4.10: $D_y(4-3y^3)^{-2/3} = -\dfrac{2}{3}(4-3y^3)^{-5/3} \cdot (-9y^2) = \dfrac{6y^2}{(4-3y^3)^{5/3}}$.

3.4.11: $D_x(x^3+1)^{1/2} = \dfrac{1}{2}(x^3+1)^{-1/2} \cdot (3x^2) = \dfrac{3x^2}{2\sqrt{x^3+1}}$.

3.4.12: $D_z(z^4+3)^{-2} = -2(z^4+3)^{-3} \cdot 4z^3 = -\dfrac{8z^3}{(z^4+3)^3}$.

3.4.13: $D_x(2x^2+1)^{1/2} = \dfrac{1}{2}(2x^2+1)^{-1/2} \cdot 4x = \dfrac{2x}{\sqrt{2x^2+1}}$.

3.4.14: $D_t\left(t(1+t^4)^{-1/2}\right) = (1+t^4)^{-1/2} - \dfrac{1}{2}t(1+t^4)^{-3/2} \cdot 4t^3 = \dfrac{1}{(1+t^4)^{1/2}} - \dfrac{2t^4}{(1+t^4)^{3/2}}$

$= \dfrac{1-t^4}{(1+t^4)^{3/2}}$.

3.4.15: $D_t\left(t^{3/2}\sqrt{2}\right) = \dfrac{3}{2}t^{1/2}\sqrt{2} = \dfrac{3\sqrt{t}}{\sqrt{2}}$.

3.4.16: $D_t\left(\dfrac{1}{\sqrt{3}} \cdot t^{-5/2}\right) = -\dfrac{5}{2\sqrt{3}} \cdot t^{-7/2} = -\dfrac{5}{2t^{7/2}\sqrt{3}}$.

3.4.17: $D_x(2x^2-x+7)^{3/2} = \dfrac{3}{2}(2x^2-x+7)^{1/2} \cdot (4x-1) = \dfrac{3}{2}(4x-1)\sqrt{2x^2-x+7}$.

3.4.18: $D_z(3z^2-4)^{97} = 97(3z^2-4)^{96} \cdot 6z = 582z(3z^2-4)^{96}$.

3.4.19: $D_x(x-2x^3)^{-4/3} = -\dfrac{4}{3}(x-2x^3)^{-7/3} \cdot (1-6x^2) = \dfrac{4(6x^2-1)}{3(x-2x^3)^{7/3}}$.

3.4.20: $D_t\left[\,t^2+(1+t)^4\,\right]^5 = 5\left[\,t^2+(1+t)^4\,\right]^4 \cdot D_t\left[\,t^2+(1+t)^4\,\right]$

$= 5\left[\,t^2+(1+t)^4\,\right]^4 \cdot \left[\,2t+4(1+t)^3 \cdot 1\,\right] = 5\left[\,t^2+(1+t)^4\,\right]^4 \cdot \left[\,2t+4(1+t)^3\,\right]$.

3.4.21: If $f(x) = x(1-x^2)^{1/2}$, then (by the product rule and the chain rule, among others)

$$f'(x) = 1 \cdot (1-x^2)^{1/2} + x \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot D_x(1-x^2)$$
$$= (1-x^2)^{1/2} + x \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = \sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}}.$$

3.4.22: Write $g(x) = \dfrac{(2x+1)^{1/2}}{(x-1)^{1/2}}$ to find

$$g'(x) = \frac{(x-1)^{1/2} \cdot \frac{1}{2}(2x+1)^{-1/2} \cdot 2 - \frac{1}{2}(x-1)^{-1/2} \cdot (2x+1)^{1/2}}{\left[\,(x-1)^{1/2}\,\right]^2}$$
$$= \frac{2(x-1)^{1/2}(2x+1)^{-1/2} - (x-1)^{-1/2}(2x+1)^{1/2}}{2(x-1)}$$
$$= \frac{2(x-1)-(2x+1)}{2(x-1)(x-1)^{1/2}(2x+1)^{1/2}} = -\frac{3}{2(x-1)^{3/2}\sqrt{2x+1}}.$$

3.4.23: If $f(t) = \sqrt{\dfrac{t^2+1}{t^2-1}} = \left(\dfrac{t^2+1}{t^2-1}\right)^{1/2}$, then

$$f'(t) = \frac{1}{2}\left(\frac{t^2+1}{t^2-1}\right)^{-1/2} \cdot \frac{(t^2-1)(2t)-(t^2+1)(2t)}{(t^2-1)^2} = \frac{1}{2}\left(\frac{t^2-1}{t^2+1}\right)^{1/2} \cdot \frac{-4t}{(t^2-1)^2} = -\frac{2t}{(t^2-1)^{3/2}\sqrt{t^2+1}}.$$

3.4.24: If $h(y) = \left(\dfrac{y+1}{y-1}\right)^{17}$, then

$$h'(y) = 17\left(\frac{y+1}{y-1}\right)^{16} \cdot \frac{(y-1)\cdot 1 - (y+1)\cdot 1}{(y-1)^2} = 17\left(\frac{y+1}{y-1}\right)^{16} \cdot \frac{-2}{(y-1)^2} = -\frac{34(y+1)^{16}}{(y-1)^{18}}.$$

3.4.25: $D_x\left(x - \dfrac{1}{x}\right)^3 = 3\left(x - \dfrac{1}{x}\right)^2\left(1 + \dfrac{1}{x^2}\right) = 3\left(\dfrac{x^2-1}{x}\right)^2 \cdot \dfrac{x^2+1}{x^2} = \dfrac{3(x^2-1)^2(x^2+1)}{x^4}.$

3.4.26: Write $g(z) = z^2(1+z^2)^{-1/2}$, then apply the product rule and the chain rule to obtain

$$g'(z) = 2z(1+z^2)^{-1/2} + z^2 \cdot \left(-\frac{1}{2}\right)(1+z^2)^{-3/2} \cdot 2z = \frac{2z}{(1+z^2)^{1/2}} - \frac{z^3}{(1+z^2)^{3/2}} = \frac{z^3+2z}{(1+z^2)^{3/2}}.$$

3.4.27: Write $f(v) = \dfrac{(v+1)^{1/2}}{v}$. Then

$$f'(v) = \frac{v \cdot \frac{1}{2}(v+1)^{-1/2} - 1 \cdot (v+1)^{1/2}}{v^2} = \frac{v \cdot (v+1)^{-1/2} - 2(v+1)^{1/2}}{2v^2} = \frac{v - 2(v+1)}{2v^2(v+1)^{1/2}} = -\frac{v+2}{2v^2(v+1)^{1/2}}.$$

3.4.28: $h'(x) = \dfrac{5}{3}\left(\dfrac{x}{1+x^2}\right)^{2/3} \cdot \dfrac{(1+x^2)\cdot 1 - x \cdot 2x}{(1+x^2)^2} = \dfrac{5}{3}\left(\dfrac{x}{1+x^2}\right)^{2/3} \cdot \dfrac{1-x^2}{(1+x^2)^2}.$

3.4.29: $D_x(1-x^2)^{1/3} = \dfrac{1}{3}(1-x^2)^{-2/3}\cdot(-2x) = -\dfrac{2x}{3(1-x^2)^{2/3}}.$

3.4.30: $D_x(x+x^{1/2})^{1/2} = \dfrac{1}{2}(x+x^{1/2})^{-1/2}\left(1+\dfrac{1}{2}x^{-1/2}\right) = \dfrac{1+2\sqrt{x}}{4\sqrt{x}\,\sqrt{x+\sqrt{x}}}.$

3.4.31: If $f(x) = x(3-4x)^{1/2}$, then (with the aid of the product rule and the chain rule)

$$f'(x) = 1\cdot(3-4x)^{1/2} + x\cdot\frac{1}{2}(3-4x)^{-1/2}\cdot(-4) = (3-4x)^{1/2} - \frac{2x}{(3-4x)^{1/2}} = \frac{3(1-2x)}{\sqrt{3-4x}}.$$

3.4.32: Given $g(t) = \dfrac{t-(1+t^2)^{1/2}}{t^2}$,

$$g'(t) = \frac{t^2\left(1-\frac{1}{2}(1+t^2)^{-1/2}\cdot 2t\right) - 2t\left(t-(1+t^2)^{1/2}\right)}{(t^2)^2} = \frac{t\left(1-t(1+t^2)^{-1/2}\right) - 2\left(t-(1+t^2)^{1/2}\right)}{t^3}$$

$$= \frac{t-t^2(1+t^2)^{-1/2}-2t+2(1+t^2)^{1/2}}{t^3} = \frac{-t(1+t^2)^{1/2}-t^2+2(1+t^2)}{t^3(1+t^2)^{1/2}} = \frac{t^2+2-t(1+t^2)^{1/2}}{t^3(1+t^2)^{1/2}}.$$

3.4.33: If $f(x) = (1-x^2)(2x+4)^{1/3}$, then the product rule (among others) yields

$$\begin{aligned} f'(x) &= -2x(2x+4)^{1/3} + \frac{2}{3}(1-x^2)\cdot(2x+4)^{-2/3} \\ &= \frac{-6x(2x+4)+2(1-x^2)}{3(2x+4)^{2/3}} = \frac{-12x^2-24x+2-2x^2}{3(2x+4)^{2/3}} = \frac{2-24x-14x^2}{3(2x+4)^{2/3}}. \end{aligned}$$

3.4.34: If $f(x) = (1-x)^{1/2}(2-x)^{1/3}$, then

$$f'(x) = \tfrac{1}{2}(1-x)^{-1/2}(-1)\cdot(2-x)^{1/3} + \tfrac{1}{3}(2-x)^{-2/3}(-1)\cdot(1-x)^{1/2} = -\left(\frac{(2-x)^{1/3}}{2(1-x)^{1/2}} + \frac{(1-x)^{1/2}}{3(2-x)^{2/3}}\right)$$

$$= -\frac{3(2-x)+2(1-x)}{6(2-x)^{2/3}(1-x)^{1/2}} = \frac{5x-8}{6(2-x)^{2/3}(1-x)^{1/2}}.$$

3.4.35: If $g(t) = \left(1+\dfrac{1}{t}\right)^2(3t^2+1)^{1/2}$, then

$$g'(t) = \left(1+\frac{1}{t}\right)^2\cdot\frac{1}{2}(3t^2+1)^{-1/2}(6t) + 2\cdot\left(1+\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)(3t^2+1)^{1/2}$$

$$= 3t\cdot\frac{(t+1)^2}{t^2(3t^2+1)^{1/2}} - \frac{2}{t^2}\cdot\frac{t+1}{t}(3t^2+1)^{1/2} = \frac{3t^2(t+1)^2}{t^3(3t^2+1)^{1/2}} - \frac{2(t+1)(3t^2+1)}{t^3(3t^2+1)^{1/2}} = \frac{3t^4-3t^2-2t-2}{t^3\sqrt{3t^2+1}}.$$

3.4.36: If $f(x) = x(1+2x+3x^2)^{10}$, then

$$f'(x) = (1+2x+3x^2)^{10} + 10x(1+2x+3x^2)^9(2+6x) = (3x^2+2x+1)^9(63x^2+22x+1).$$

3.4.37: If $f(x) = \dfrac{2x-1}{(3x+4)^5}$, then

$$f'(x) = \frac{2(3x+4)^5 - (2x-1)\cdot 5(3x+4)^4\cdot 3}{(3x+4)^{10}} = \frac{2(3x+4) - 15(2x-1)}{(3x+4)^6} = \frac{23-24x}{(3x+4)^6}.$$

3.4.38: If $h(z) = (z-1)^4(z+1)^6$, then

$$\begin{aligned} h'(z) &= 4(z-1)^3(z+1)^6 + 6(z+1)^5(z-1)^4 \\ &= (z-1)^3(z+1)^5\big(4(z+1)+6(z-1)\big) = (z-1)^3(z+1)^5(10z-2). \end{aligned}$$

3.4.39: If $f(x) = \dfrac{(2x+1)^{1/2}}{(3x+4)^{1/3}}$, then

$$\begin{aligned} f'(x) &= \frac{(3x+4)^{1/3}(2x+1)^{-1/2} - (2x+1)^{1/2}(3x+4)^{-2/3}}{(3x+4)^{2/3}} \\ &= \frac{(3x+4)-(2x+1)}{(3x+4)^{4/3}(2x+1)^{1/2}} = \frac{x+3}{(3x+4)^{4/3}(2x+1)^{1/2}}. \end{aligned}$$

3.4.40: If $f(x) = (1-3x^4)^5(4-x)^{1/3}$, then

$$\begin{aligned} f'(x) &= 5(1-3x^4)^4(-12x^3)(4-x)^{1/3} + (1-3x^4)^5\cdot\frac{1}{3}(4-x)^{-2/3}(-1) \\ &= -60x^3(1-3x^4)^4(4-x)^{1/3} - \frac{(1-3x^4)^5}{3(4-x)^{2/3}} = \frac{-180x^3(1-3x^4)^4(4-x)}{3(4-x)^{2/3}} - \frac{(1-3x^4)^5}{3(4-x)^{2/3}} \\ &= \frac{\big[(180x^4-720x^3)-(1-3x^4)\big](1-3x^4)^4}{3(4-x)^{2/3}} = \frac{(183x^4-720x^3-1)(1-3x^4)^4}{3(4-x)^{2/3}}. \end{aligned}$$

3.4.41: If $h(y) = \dfrac{(1+y)^{1/2}+(1-y)^{1/2}}{y^{5/3}}$, then

$$\begin{aligned} h'(y) &= \frac{y^{5/3}\big[\frac{1}{2}(1+y)^{-1/2} - \frac{1}{2}(1-y)^{-1/2}\big] - \frac{5}{3}y^{2/3}\big[(1+y)^{1/2}+(1-y)^{1/2}\big]}{y^{10/3}} \\ &= \frac{y\big[\frac{1}{2}(1+y)^{-1/2} - \frac{1}{2}(1-y)^{-1/2}\big] - \frac{5}{3}\big[(1+y)^{1/2}+(1-y)^{1/2}\big]}{y^{8/3}} \\ &= \frac{y\big[3(1+y)^{-1/2} - 3(1-y)^{-1/2}\big] - 10\big[(1+y)^{1/2}+(1-y)^{1/2}\big]}{6y^{8/3}} \\ &= \frac{y\big[3(1-y)^{1/2} - 3(1+y)^{1/2}\big] - 10\big[(1+y)(1-y)^{1/2}+(1-y)(1+y)^{1/2}\big]}{6y^{8/3}(1-y)^{1/2}(1+y)^{1/2}} \\ &= \frac{(7y-10)\sqrt{1+y} - (7y+10)\sqrt{1-y}}{6y^{8/3}\sqrt{1-y}\,\sqrt{1+y}}. \end{aligned}$$

3.4.42: If $f(x) = (1 - x^{1/3})^{1/2}$, then

$$f'(x) = \frac{1}{2}(1 - x^{1/3})^{-1/2}\left(-\frac{1}{3}x^{-2/3}\right) = -\frac{1}{6x^{2/3}\sqrt{1 - x^{1/3}}}.$$

3.4.43: If $g(t) = \left[t + (t + t^{1/2})^{1/2}\right]^{1/2}$, then

$$g'(t) = \frac{1}{2}\left[t + (t + t^{1/2})^{1/2}\right]^{-1/2} \cdot \left[1 + \frac{1}{2}(t + t^{1/2})^{-1/2}\left(1 + \frac{1}{2}t^{-1/2}\right)\right].$$

It is possible to write the derivative without negative exponents. The symbolic algebra program *Mathematica* yields

$$g'(t) = -\frac{(t + (t + t^{1/2})^{1/2})^{1/2}\left[1 - 4t^{3/2} - 4t^2 + 3t^{1/2}\left(1 + (t + t^{1/2})^{1/2}\right) + 2t\left(1 + (t + t^{1/2})^{1/2}\right)\right]}{8t(1 + t^{1/2})(t^{3/2} - t^{1/2} - 1)}.$$

But the first answer that *Mathematica* gives is

$$g'(t) = \frac{1 + \dfrac{1 + \dfrac{1}{2\sqrt{t}}}{2\sqrt{t + \sqrt{t}}}}{2\sqrt{t + \sqrt{t + \sqrt{t}}}}.$$

3.4.44: If $f(x) = x^3\sqrt{1 - \dfrac{1}{x^2 + 1}}$, then

$$f'(x) = 3x^2\sqrt{1 - \frac{1}{x^2 + 1}} + \frac{1}{2}x^3\left(1 - \frac{1}{x^2 + 1}\right)^{-1/2} \cdot \frac{2x}{(x^2 + 1)^2}.$$

The symbolic algebra program *Mathematica* simplifies this to

$$f'(x) = (3x^2 + 4)\left(\frac{x^2}{x^2 + 1}\right)^{3/2}.$$

3.4.45: Because

$$y'(x) = \frac{dy}{dx} = \frac{2}{3x^{1/3}}$$

is never zero, there are no horizontal tangents. Because $y(x)$ is continuous at $x = 0$ and $|y'(x)| \to +\infty$ as $x \to 0$, there is a vertical tangent at $(0, 0)$.

3.4.46: If $f(x) = x\sqrt{4 - x^2}$, then

$$f'(x) = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = \frac{2(2 - x^2)}{\sqrt{4 - x^2}}.$$

Hence there are horizontal tangents at $\left(-\sqrt{2},\ -2\right)$ and at $\left(\sqrt{2},\ 2\right)$. Because f is continuous (from within its interval of definition) at ± 2 and

$$\lim_{x\to -2^+} |f(x)| = +\infty = \lim_{x\to 2^-} |f(x)|,$$

we say that there are vertical tangents at $(-2,\ 0)$ and $(2,\ 0)$.

3.4.47: If $g(x) = x^{1/2} - x^{3/2}$, then

$$g'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2} = \frac{1-3x}{2\sqrt{x}}.$$

Thus there is a horizontal tangent at $\left(\frac{1}{3},\ \frac{2}{9}\sqrt{3}\right)$. Also, because g is continuous at $x = 0$ and

$$\lim_{x\to 0^+} |g'(x)| = \lim_{x\to 0^+} \frac{1-3x}{2\sqrt{x}} = +\infty,$$

the graph of g has a vertical tangent at $(0,\ 0)$.

3.4.48: If $h(x) = (9-x^2)^{-1/2}$, then

$$h'(x) = -\frac{1}{2}(9-x^2)^{-3/2}\cdot(-2x) = \frac{x}{(9-x^2)^{3/2}}.$$

So the graph of h has a horizontal tangent at $\left(0,\ \frac{1}{3}\right)$. There are no vertical tangents because, even though $|h'(x)| \to +\infty$ as $x \to 3^-$ and as $x \to -3^+$, h is not continuous at 3 or at -3, and there are no other values of x at which $|h'(x)| \to +\infty$.

3.4.49: If $y(x) = x(1-x^2)^{-1/2}$, then

$$y'(x) = \frac{dy}{dx} = (1-x^2)^{-1/2} - \frac{1}{2}x(1-x^2)^{-3/2}\cdot(-2x) = \frac{1}{(1-x^2)^{1/2}} + \frac{x^2}{(1-x^2)^{3/2}} = \frac{1}{(1-x^2)^{3/2}}.$$

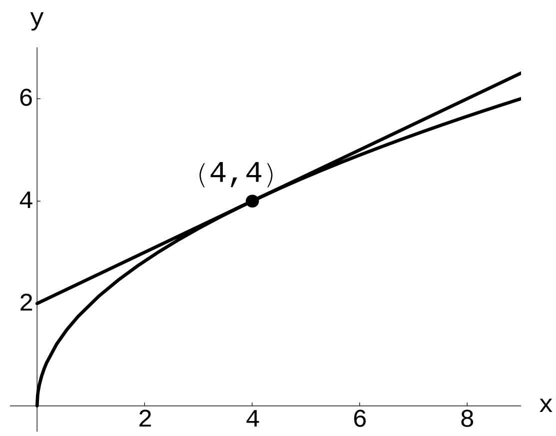
Thus the graph of $y(x)$ has no horizontal tangents because $y'(x)$ is never zero. The only candidates for vertical tangents are at $x = \pm 1$, but there are none because $y(x)$ is not continuous at either of those two values of x.

3.4.50: If $f(x) = \sqrt{(1-x^2)(4-x^2)} = (x^4 - 5x^2 + 4)^{1/2}$, then

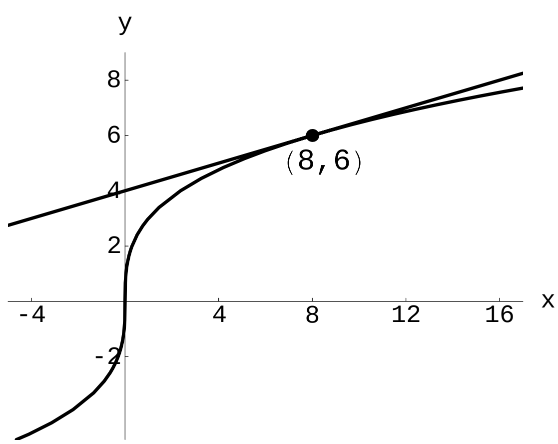
$$f'(x) = \frac{1}{2}(x^4 - 5x^2 + 4)^{-1/2}\cdot(4x^3 - 10x) = \frac{x(2x^2-5)}{\sqrt{(1-x^2)(4-x^2)}}.$$

There are no horizontal tangents where $2x^2 = 5$ because the two corresponding values of x are not in the domain $(-\infty,\ -2] \cup [-1,\ 1] \cup [2,\ +\infty)$ of f. There is a horizontal tangent at $(0,\ 2)$. There are vertical tangents at $(-2,\ 0)$, $(-1,\ 0)$, $(1,\ 0)$, and $(2,\ 0)$ because the appropriate one-sided limits of $|f'(x)|$ are all $+\infty$.

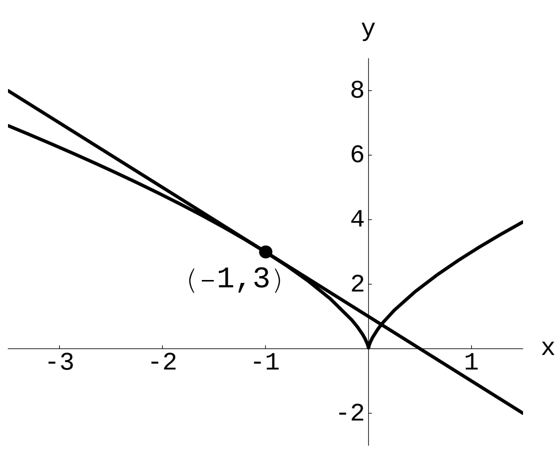
3.4.51: Let $f(x) = 2\sqrt{x}$. Then $f'(x) = x^{-1/2}$, so an equation of the required tangent line is $y - f(4) = f'(4)(x-4)$; that is, $y = \frac{1}{2}(x+4)$. The graph of f and this tangent line are shown next.



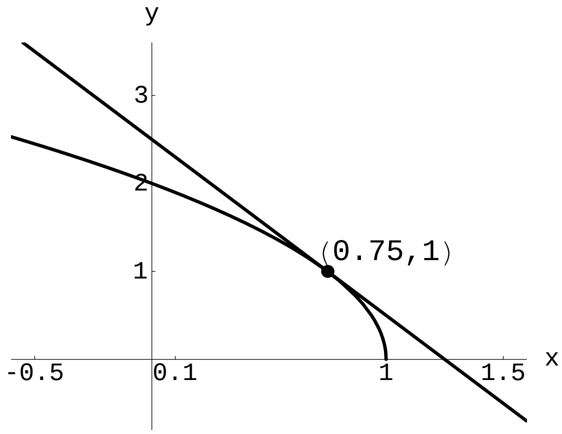
3.4.52: If $f(x) = 3x^{1/3}$ then $f'(x) = x^{-2/3}$, so an equation of the required tangent line is $y - f(8) = f'(8)(x-8)$; that is, $y = \frac{1}{4}(x+16)$. A graph of f and this tangent line are shown next.



3.4.53: If $f(x) = 3x^{2/3}$, then $f'(x) = 2x^{-1/3}$. Therefore an equation of the required tangent line is $y - f(-1) = f'(-1)(x+1)$; that is, $y = -2x+1$. A graph of f and this tangent line are shown next.



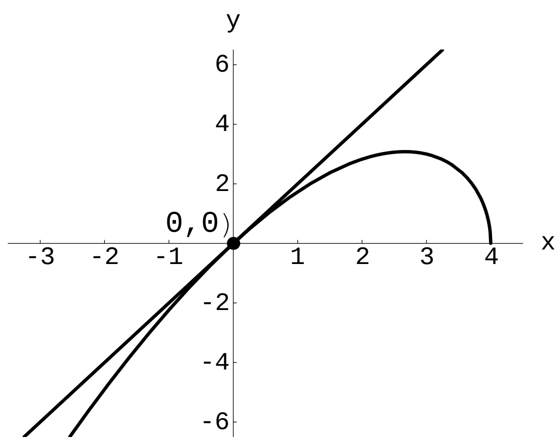
3.4.54: If $f(x) = 2(1-x)^{1/2}$, then $f'(x) = -(1-x)^{-1/2}$, and therefore an equation of the required tangent line is $y - f\left(\frac{3}{4}\right) = f'\left(\frac{3}{4}\right)\left(x - \frac{3}{4}\right)$; that is, $y = -2x + \frac{5}{2}$. The graph of f and this tangent line are shown next.



3.4.55: If $f(x) = x(4-x)^{1/2}$, then

$$f'(x) = (4-x)^{1/2} - \frac{1}{2}x(4-x)^{-1/2} = (4-x)^{1/2} - \frac{x}{2(4-x)^{1/2}} = \frac{8-3x}{2\sqrt{4-x}}.$$

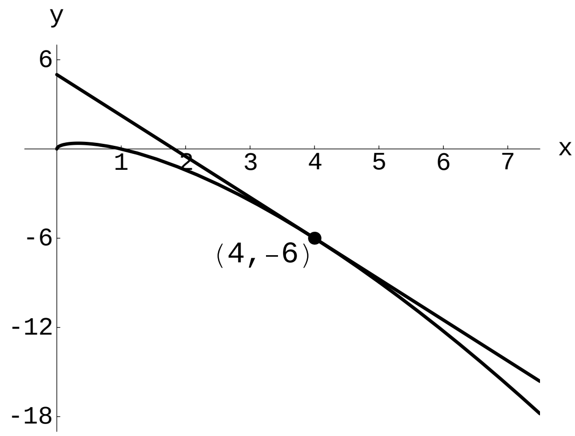
So an equation of the required tangent line is $y - f(0) = f'(0)(x-0)$; that is, $y = 2x$. A graph of f and this tangent line are shown next.



3.4.56: If $f(x) = x^{1/2} - x^{3/2}$, then (as in the solution of Problem 47)

$$f'(x) = \frac{1-3x}{2\sqrt{x}}.$$

Therefore an equation of the required tangent line is $y - f(4) = f'(4)(x-4)$; that is, $y = -\frac{11}{4}x + 5$. A graph of f and this tangent line are shown next.



3.4.57: If $x < 0$ then $f'(x) < 0$; as $x \to 0^-$, $f'(x)$ appears to approach $-\infty$. If $x > 0$ then $f'(x) > 0$; as $x \to 0^+$, $f'(x)$ appears to approach $+\infty$. So the graph of f' must be the one shown in Fig. 3.4.13(d).

3.4.58: If $x \neq 0$, then $f'(x) > 0$; moreover, $f'(x)$ appears to be approaching zero as $|x|$ increases without bound. In contrast, $f'(x)$ appears to approach $+\infty$ as $x \to 0$. Hence the graph of f' must be the one shown in Fig. 3.4.13(f).

3.4.59: Note that $f'(x) > 0$ if $x < 0$ whereas $f'(x) < 0$ if $x > 0$. Moreover, as $x \to 0$, $|f'(x)|$ appears to approach $+\infty$. So the graph of f' must be the one shown in Fig. 3.4.13(b).

3.4.60: We see that $f'(x) > 0$ for $x < 1.4$ (approximately), that $f(x) = 0$ when $x \approx 1.4$, and that $f'(x) < 0$ for $1.4 < x < 2$; moreover, $f'(x) \to -\infty$ as $x \to 2^-$. So the graph of f' must be the one shown in Fig. 3.4.13(a).

3.4.61: We see that $f'(x) < 0$ for $-2 < x < -1.4$ (approximately), that $f'(x) > 0$ for $-1.4 < x < 1.4$ (approximately), and that $f'(x) < 0$ for $1.4 < x < 2$. Also $f'(x) = 0$ when $x \approx \pm 1.4$. Therefore the graph of f must be the one shown in Fig. 3.4.13(e).

3.4.62: Figure 3.4.12 shows a graph whose derivative is negative for $x < -1$, positive for $-1 < x < -0.3$ (approximately), negative for $-0.3 < x < 0$, positive for $0 < x < 0.3$ (approximately), negative for $0.3 < x < 1$, and positive for $1 < x$. Moreover, $f'(x) = 0$ when $x = \pm 1$ and when $x \approx \pm 0.3$. Finally, $f'(x) \to -\infty$ as $x \to 0^-$ whereas $f'(x) \to \infty$ as $x \to 0^+$. Therefore the graph of f' must be the one shown in Fig. 3.4.13(c).

3.4.63: $L = \dfrac{P^2 g}{4\pi^2}$, so $\dfrac{dL}{dP} = \dfrac{Pg}{2\pi^2}$, and hence $\dfrac{dP}{dL} = \dfrac{2\pi^2}{Pg}$. Given $g = 32$ and $P = 2$, we find the value of the latter to be $\frac{1}{32}\pi^2 \approx 0.308$ (seconds per foot).

3.4.64: $dV/dA = \frac{1}{4}\sqrt{A/\pi}$, and $A = 400\pi$ when the radius of the sphere is 10, so the answer is 5 (in appropriate units, such as cubic meters per square meter).

3.4.65: Whether $y = +\sqrt{1-x^2}$ or $y = -\sqrt{1-x^2}$, it follows easily that $dy/dx = -x/y$. The slope of the tangent is -2 when $x = 2y$, so from the equation $x^2 + y^2 = 1$ we see that $x^2 = 4/5$, so that $x = \pm\frac{2}{5}\sqrt{5}$. Because $y = \frac{1}{2}x$, the two points we are to find are $\left(-\frac{2}{5}\sqrt{5},\ -\frac{1}{5}\sqrt{5}\right)$ and $\left(\frac{2}{5}\sqrt{5},\ \frac{1}{5}\sqrt{5}\right)$.

3.4.66: Using some of the results in the preceding solution, we find that the slope of the tangent is 3 when $x = -3y$, so that $y^2 = \frac{1}{10}$. So the two points of tangency are $\left(-\frac{3}{10}\sqrt{10},\ \frac{1}{10}\sqrt{10}\right)$ and $\left(\frac{3}{10}\sqrt{10},\ -\frac{1}{10}\sqrt{10}\right)$.

3.4.67: The line tangent to the parabola $y = x^2$ at the point $Q(a,\ a^2)$ has slope $2a$, so the normal to the parabola at Q has slope $-1/(2a)$. The normal also passes through $P(18,\ 0)$, so we can find its slope another way—by using the two-point formula. Thus

$$-\frac{1}{2a} = \frac{a^2 - 0}{a - 18};$$

$$18 - a = 2a^3;$$

$$2a^3 + a - 18 = 0.$$

By inspection, $a = 2$ is a solution of the last equation. Thus $a - 2$ is a factor of the cubic, and division yields

$$2a^3 + a - 18 = (a-2)(2a^2 + 4a + 9).$$

The quadratic factor has negative discriminant, so $a = 2$ is the only real solution of $2a^3 + a - 18 = 0$. Therefore the normal line has slope $-\frac{1}{4}$ and equation $x + 4y = 18$.

3.4.68: Let $Q(a,\ a^2)$ be a point on the parabola $y = x^2$ at which some line through $P(3,\ 10)$ is normal to the parabola. Then, as in the solution of Problem 67, we find that

$$\frac{a^2 - 10}{a - 3} = -\frac{1}{2a}.$$

This yields the cubic equation $2a^3 - 19a - 3 = 0$, and after a little computation we find one of its small integral roots to be $r = -3$. So $a + 3$ is a factor of the cubic; by division, the other factor is $2a^2 - 6a - 1$, which is zero when $a = \frac{1}{2}\left(3 \pm \sqrt{11}\right)$. So the three lines have slopes

$$\frac{1}{6}, \qquad -\frac{1}{3-\sqrt{11}}, \qquad \text{and} \qquad -\frac{1}{3+\sqrt{11}}.$$

Their equations are

$$y - 10 = \frac{1}{6}(x-3), \quad y - 10 = -\frac{1}{3-\sqrt{11}}(x-3), \quad \text{and} \quad y - 10 = -\frac{1}{3+\sqrt{11}}(x-3).$$

3.4.69: If a line through $P\left(0,\ \frac{5}{2}\right)$ is normal to $y = x^{2/3}$ at $Q(a,\ a^{2/3})$, then it has slope $-\frac{3}{2}a^{1/3}$. As in the two previous solutions, we find that

$$\frac{a^{2/3} - \frac{5}{2}}{a} = -\frac{3}{2}a^{1/3},$$

which yields $3a^{4/3} + 2a^{2/3} - 5 = 0$. Put $u = a^{2/3}$; we obtain $3u^2 + 2u - 5 = 0$, so that $(3u + 5)(u - 1) = 0$. Because $u = a^{2/3} > 0$, $u = 1$ is the only solution, so $a = 1$ and $a = -1$ yield the two possibilities for the point Q, and therefore the equations of the two lines are

$$y - \frac{5}{2} = -\frac{3}{2}x \qquad \text{and} \qquad y - \frac{5}{2} = \frac{3}{2}x.$$

3.4.70: Suppose that $P = P(u, v)$, so that $u^2 + v^2 = a^2$. Then the slope of the radius OP is $m_r = v/u$ if $u \neq 0$; if $u = 0$ then OP lies on the y-axis. Also, whether $y = +\sqrt{a^2 - x^2}$ or $y = -\sqrt{a^2 - x^2}$, it follows that

$$\frac{dy}{dx} = \pm\frac{-x}{\sqrt{a^2 - x^2}} = \pm\frac{-x}{\pm y} = -\frac{x}{y}. \tag{1}$$

Thus if $u \neq 0$ and $v \neq 0$, then the slope of the line tangent L to the circle at $P(u, v)$ is $m_t = -u/v$. In this case

$$m_r \cdot m_t = \frac{v}{u} \cdot \left(-\frac{u}{v}\right) = -1,$$

so that OP is perpendicular to L if $u \neq 0$ and $v \neq 0$. If $u = 0$ then Eq. (1) shows that L has slope 0, so that L and OP are also perpendicular in this case. Finally, if $v = 0$ then OP lies on the x-axis and L is vertical, so the two are also perpendicular in this case. In every case we see that L and OP are perpendicular.

3.4.71: The equation $x^3 = 3x + 8$ is not an *identity* that says the two functions are equal (and would therefore have equal derivatives). It is merely an equation relating *values* of the two functions x^3 and $3x + 8$ that have nothing in particular to do with one another. So it makes no sense to equate their derivatives.

3.4.72: If $f(x) = x^{1/2}$ and $a > 0$, then

$$f'(a) = \lim_{x \to a} \frac{x^{1/2} - a^{1/2}}{x - a} = \lim_{x \to a} \frac{x^{1/2} - a^{1/2}}{(x^{1/2} - a^{1/2})(x^{1/2} + a^{1/2})} = \lim_{x \to a} \frac{1}{x^{1/2} + a^{1/2}} = \frac{1}{2a^{1/2}}.$$

Therefore $D_x\, x^{1/2} = \dfrac{1}{2}x^{-1/2}$ if $x > 0$.

3.4.73: If $f(x) = x^{1/3}$ and $a > 0$, then

$$f'(a) = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{(x^{1/3} - a^{1/3})(x^{2/3} + x^{1/3}a^{1/3} + a^{2/3})}$$

$$= \lim_{x \to a} \frac{1}{x^{2/3} + x^{1/3}a^{1/3} + a^{2/3}} = \frac{1}{a^{2/3} + a^{2/3} + a^{2/3}} = \frac{1}{3a^{2/3}}.$$

Therefore $D_x\, x^{1/3} = \dfrac{1}{3}x^{-2/3}$ if $x > 0$.

This formula is of course valid for $x < 0$ as well. To show this, observe that the previous argument is valid if $a < 0$, or—if you prefer—you can use the chain rule, laws of exponents, and the preceding result, as follows. Suppose that $x < 0$. Then $-x > 0$; also, $x^{1/3} = -(-x)^{1/3}$. So

$$D_x(x^{1/3}) = D_x\left[-(-x)^{1/3}\right] = -D_x(-x)^{1/3} = -\left[\frac{1}{3}(-x)^{-2/3} \cdot (-1)\right] = \frac{1}{3}(-x)^{2/3} = \frac{1}{3}x^{-2/3}.$$

Therefore $D_x(x^{1/3}) = \dfrac{1}{3}x^{-2/3}$ if $x \neq 0$.

3.4.74: If $f(x) = x^{1/5}$ and $a > 0$, the

$$f'(a) = \lim_{x\to a} \frac{x^{1/5} - a^{1/5}}{x - a} = \lim_{x\to a} \frac{x^{1/5} - a^{1/5}}{(x^{1/5} - a^{1/5})(x^{4/5} + x^{3/5}a^{1/5} + x^{2/5}a^{2/5} + x^{1/5}a^{3/5} + a^{4/5})}$$

$$= \lim_{x\to a} \frac{1}{x^{4/5} + x^{3/5}a^{1/5} + x^{2/5}a^{2/5} + x^{1/5}a^{3/5} + a^{4/5}} = \frac{1}{a^{4/5} + a^{4/5} + a^{4/5} + a^{4/5} + a^{4/5}} = \frac{1}{5a^{4/5}}.$$

Therefore $D_x(x^{1/5}) = \dfrac{1}{5}x^{-4/5}$ if $x > 0$.

As in the concluding paragraph in the previous solution, it is easy to show that this formula holds for all $x \neq 0$.

3.4.75: The preamble to Problems 72 through 75 implies that if q is a positive integer and x and a are positive real numbers, then

$$x - a = (x^{1/q} - a^{1/q})(x^{(q-1)/q} + x^{(q-2)/q}a^{1/q} + x^{(q-3)/q}a^{2/q} + \cdots + x^{1/q}a^{(q-2)/q} + a^{(q-1)/q}).$$

Thus if $f(x) = x^{1/q}$ and $a > 0$, then

$$f'(a) = \lim_{x\to a} \frac{x^{1/q} - a^{1/q}}{x - a}$$

$$= \lim_{x\to a} \frac{x^{1/q} - a^{1/q}}{(x^{1/q} - a^{1/q})(x^{(q-1)/q} + x^{(q-2)/q}a^{1/q} + x^{(q-3)/q}a^{2/q} + \cdots + x^{1/q}a^{(q-2)/q} + a^{(q-1)/q})}$$

$$= \lim_{x\to a} \frac{1}{x^{(q-1)/q} + x^{(q-2)/q}a^{1/q} + x^{(q-3)/q}a^{2/q} + \cdots + a^{(q-1)/q}} \qquad (q \text{ terms in the denominator})$$

$$= \frac{1}{a^{(q-1)/q} + a^{(q-1)/q} + a^{(q-1)/q} + \cdots + a^{(q-1)/q}} \qquad (\text{still } q \text{ terms in the denominator})$$

$$= \frac{1}{qa^{(q-1)/q}} = \frac{1}{q}a^{-(q-1)/q}.$$

Therefore $D_x(x^{1/q}) = \dfrac{1}{q}x^{-(q-1)/q}$ if $x > 0$ and q is a positive integer. This result is easy to extend to the case $x < 0$. Therefore if q is a positive integer and $x \neq 0$, then

$$D_x(x^{1/q}) = \frac{1}{q}x^{(1/q)-1}.$$

Section 3.5

3.5.1: Because $f(x) = 1 - x$ is decreasing everywhere, it can attain a maximum only at a left-hand endpoint of its domain and a minimum only at a right-hand endpoint of its domain. Its domain $[-1, 1)$ has no right-hand endpoint, so f has no minimum value. Its maximum value occurs at -1, is $f(-1) = 2$, and is the global maximum value of f on its domain.

3.5.2: Because $f(x) = 2x + 1$ is increasing everywhere, it can have a minimum only at a left-hand endpoint of its domain and a maximum only at a right-hand endpoint of its domain. But its domain $[-1, 1)$ has no right-hand endpoint, so f has no maximum. It has the global minimum value $f(-1) = -1$ at the left-hand endpoint of its domain.

3.5.3: Because $f(x) = |x|$ is decreasing for $x < 0$ and increasing for $x > 0$, it can have a maximum only at a left-hand or a right-hand endpoint of its domain $(-1, 1)$. But its domain has no endpoints, so f has no maximum value. It has the global minimum value $f(0) = 0$.

3.5.4: Because $g(x) = \sqrt{x}$ is increasing on $(0, 1]$, its reciprocal $f(x) = 1/\sqrt{x}$ is decreasing there. But

$$\lim_{x \to 0^+} \frac{1}{\sqrt{x}} = +\infty,$$

so f has no maximum value. It has the global minimum value $f(1) = 1$ at the right-hand endpoint of its domain.

3.5.5: Given: $f(x) = |x - 2|$ on $(1, 4]$. If $x > 2$ then $f(x) = x - 2$, which is increasing for $x > 2$; if $x < 2$ then $f(x) = 2 - x$, which is decreasing for $x < 2$. So f can have a maximum only at an endpoint of its domain; the only endpoint is at $x = 4$, where $f(x)$ has the maximum value $f(4) = 2$. Because $f(x) \to 1$ as $x \to 1^+$, the extremum at $x = 4$ is in fact a global maximum. Finally, $f(2) = 0$ is the global minimum value of f.

3.5.6: If $f(x) = 5 - x^2$, then $f'(x) = -2x$, so $x = 0$ is the only critical point of f. We note that f is increasing for $x < 0$ and decreasing for $x > 0$, so $f(0) = 5$ is the global maximum value of f. Because $f(-1) = 4$ and $f(x) \to 1$ as $x \to 2^-$, the minimum at $x = -1$ is local but not global.

3.5.7: Given: $f(x) = x^3 + 1$ on $[-1, 1]$. The only critical point of f occurs where $f'(x) = 3x^2$ is zero; that is, at $x = 0$. But $f(x) < 1 = f(0)$ if $x < 0$ whereas $f(x) > 1 = f(0)$ if $x > 0$, so there is no extremum at $x = 0$. By Theorem 1 (page 142), f must have a global maximum and a global minimum. The only possible locations are at the endpoints of the domain of f, and therefore $f(-1) = 0$ is the global minimum value of f and $f(1) = 2$ is its global maximum value.

3.5.8: If

$$f(x) = \frac{1}{x^2 + 1}, \qquad \text{then} \qquad f'(x) = -\frac{2x}{(x^2 + 1)^2},$$

so $x = 0$ is the only critical point of f. Because $g(x) = x^2 + 1$ is increasing for $x > 0$ and decreasing for $x < 0$, we may conclude that f is decreasing for $x > 0$ and increasing for $x < 0$. Therefore f has the global maximum value $f(0) = 1$ at $x = 0$ and no other extrema of any kind.

3.5.9: If

$$f(x) = \frac{1}{x(1-x)}, \qquad \text{then} \qquad f'(x) = \frac{2x-1}{x^2(1-x)^2},$$

which does not exist at $x = 0$ or at $x = 1$ and is zero when $x = \frac{1}{2}$. But none of these points lies in the domain $[2, 3]$ of f, so there are no extrema at those three points. By Theorem 1 f must have a global maximum and a global minimum, which therefore must occur at the endpoints of its domain. Because $f(2) = -\frac{1}{2}$ and $f(3) = -\frac{1}{6}$, the former is the global minimum value of f and the latter is its global maximum value.

3.5.10: If

$$f(x) = \frac{1}{x(1-x)}, \qquad \text{then} \qquad f'(x) = \frac{2x-1}{x^2(1-x)^2},$$

which does not exist at $x = 0$ or at $x = 1$ and is zero when $x = \frac{1}{2}$. But the domain $(0, 1)$ of f includes only the last of these three points. We note that

$$f\left(\tfrac{1}{2}\right) = 4 \qquad \text{and that} \qquad \lim_{x \to 0^+} f(x) = +\infty = \lim_{x \to 1^-} f(x),$$

and therefore f has no global maximum value. The reciprocal of $f(x)$ is

$$g(x) = x - x^2 = -(x^2 - x) = -\left(x^2 - x + \tfrac{1}{4}\right) + \tfrac{1}{4} = \tfrac{1}{4} - \left(x - \tfrac{1}{2}\right)^2,$$

which has the global maximum value $\frac{1}{4}$ at $x = \frac{1}{2}$. Therefore $f(x)$ has the global minimum value 4 at $x = \frac{1}{2}$.

3.5.11: $f'(x) = 3$ is never zero and always exists. Therefore $f(-2) = -8$ is the global minimum value of f and $f(3) = 7$ is its global maximum value.

3.5.12: $f'(x) = -3$ always exists and is never zero. Therefore $f(5) = -11$ is the global minimum value of f and $f(-1) = 7$ is its global maximum value.

3.5.13: $h'(x) = -2x$ always exists and is zero only at $x = 0$, which is not in the domain of h. Therefore $h(1) = 3$ is the global maximum value of h and $h(3) = -5$ is its global minimum value.

3.5.14: $f'(x) = 2x$ always exists and is zero only at $x = 0$, an endpoint of the domain of f. Therefore $f(0) = 3$ is the global minimum value of f and $f(5) = 28$ is its global maximum value.

3.5.15: $g'(x) = 2(x-1)$ always exists and is zero only at $x = 1$. Because $g(-1) = 4$, $g(1) = 0$, and $g(4) = 9$, the global minimum value of g is 0 and the global maximum is 9. If $-1 < x < 0$ then $g(x) = (x-1)^2 < 4$, so the extremum at $x = -1$ is a local maximum.

3.5.16: $h'(x) = 2x + 4$ always exists and is zero only at $x = -2$. Because $h(-3) = 4$, $h(-2) = 3$, and $h(0) = 7$, the global minimum value of h is 3 and its global maximum is 7. Because the graph of h is a parabola opening upward, $h(-3) = 4$ is a local (but not global) maximum value of h.

3.5.17: $f'(x) = 3x^2 - 3 = 3(x+1)(x-1)$ always exists and is zero when $x = -1$ and when $x = 1$. Because $f(-2) = -2$, $f(-1) = 2$, $f(1) = -2$, and $f(4) = 52$, the latter is the global maximum value of f and -2 is its global minimum value—note that the minimum occurs at two different points on the graph. Because f is continuous on $[-2, 2]$, it must have a global maximum there, and our work shows that it occurs at $x = -1$. But because $f(4) = 52 > 2 = f(-1)$, $f(-1) = 2$ is only a local maximum for f on its domain $[-2, 4]$. Summary: Global minimum value -2, local maximum value 2, global maximum value 52.

3.5.18: $g'(x) = 6x^2 - 18x + 12 = 6(x-1)(x-2)$ always exists and is zero when $x = 1$ and when $x = 2$. Because $g(0) = 0$, $g(1) = 5$, $g(2) = 4$, and $g(4) = 32$, the global minimum value of g is $g(0) = 0$ and its global maximum is $g(4) = 32$. Because g is continuous on $[0, 2]$, it must have a global maximum there, so $g(1) = 5$ is a *local* maximum for g on $[0, 4]$. Because g is continuous on $[1, 4]$, it must have a global minimum there, so $g(2) = 4$ is a *local* minimum for g on $[0, 4]$.

3.5.19: If

$$h(x) = x + \frac{4}{x}, \qquad \text{then} \qquad h'(x) = 1 - \frac{4}{x^2} = \frac{x^2-4}{x^2}.$$

Therefore h is continuous on $[1, 4]$ and $x = 2$ is the only critical point of h in its domain. Because $h(1) = 5$, $h(2) = 4$, and $h(4) = 5$, the global maximum value of h is 5 and its global minimum value is 4.

3.5.20: If $f(x) = x^2 + \dfrac{16}{x}$, then $f'(x) = 2x - \dfrac{16}{x^2} = \dfrac{2x^3-16}{x^2} = \dfrac{2(x-2)(x^2+2x+4)}{x^2}$. So f is continuous on its domain $[1, 3]$ and its only critical point is $x = 2$. Because $f(1) = 17$, $f(2) = 12$, and $f(3) = \frac{43}{3} \approx 14.333$, the global maximum value of f is 17 and its global minimum value is 12. Because f is continuous on $[2, 3]$, it must have a global maximum there, and therefore $f(3) = \frac{43}{3}$ is a *local* maximum value of f on $[1, 3]$.

3.5.21: $f'(x) = -2$ always exists and is never zero, so $f(1) = 1$ is the global minimum value of f and $f(-1) = 5$ is its global maximum value.

3.5.22: $f'(x) = 2x - 4$ always exists and is zero when $x = 2$, which is an endpoint of the domain of f. Hence $f(2) = -1$ is the global minimum value of f and $f(0) = 3$ is its global maximum value.

3.5.23: $f'(x) = -12 - 18x$ always exists and is zero when $x = -\frac{2}{3}$. Because $f(-1) = 8$, $f\left(-\frac{2}{3}\right) = 9$, and $f(1) = -16$, the global maximum value of f is 9 and its global minimum value is -16. Consideration of the interval $\left[-1, -\frac{2}{3}\right]$ shows that $f(-1) = 8$ is a local minimum of f.

3.5.24: $f'(x) = 4x - 4$ always exists and is zero when $x = 1$. Because $f(0) = 7$, $f(1) = 5$, and $f(2) = 7$, the global maximum value of f is 7 and its global minimum value is 5.

3.5.25: $f'(x) = 3x^2 - 6x - 9 = 3(x+1)(x-3)$ always exists and is zero when $x = -1$ and when $x = 3$. Because $f(-2) = 3$, $f(-1) = 10$, $f(3) = -22$, and $f(4) = -15$, the global minimum value of f is -22 and its global maximum is 10. Consideration of the interval $[-2, -1]$ shows that $f(-2) = 3$ is a *local* minimum of f; consideration of the interval $[3, 4]$ shows that $f(4) = -15$ is a *local* maximum of f.

3.5.26: $f'(x) = 3x^2 + 1$ always exists and is never zero, so $f(-1) = -2$ is the global minimum value of f and $f(2) = 10$ is its global maximum value.

3.5.27: $f'(x) = 15x^4 - 15x^2 = 15x^2(x+1)(x-1)$ always exists and is zero at $x = -1$, at $x = 0$, and at $x = 1$. We note that $f(-2) = -56$, $f(-1) = 2$, $f(0) = 0$, $f(1) = -2$, and $f(2) = 56$. So the global minimum value of f is -56 and its global maximum value is 56. Consideration of the interval $[-2, 0]$ shows that $f(-1) = 2$ is a *local* maximum of f on its domain $[-2, 2]$. Similarly, $f(1) = -2$ is a *local* minimum of f there. Suppose that x is near, but not equal, to zero. Then $f(x) = x^3(3x^2 - 5)$ is negative if $x > 0$ and positive if $x < 0$. Therefore there is *no extremum* at $x = 0$.

3.5.28: Given: $f(x) = |2x - 3|$ on $[1, 2]$. If $x > \frac{3}{2}$ then $f(x) = 2x - 3$, so that $f'(x) = 2$. If $x < \frac{3}{2}$ then $f(x) = 3 - 2x$, so that $f'(x) = -2$. Therefore $f'(x)$ is never zero. But it fails to exist at $x = \frac{3}{2}$. Because $f(1) = 1$, $f\left(\frac{3}{2}\right) = 0$, and $f(2) = 1$, the global maximum value of f is 1 and its global minimum value is 0.

3.5.29: Given: $f(x) = 5 + |7 - 3x|$ on $[1, 5]$. If $x < \frac{7}{3}$, then $-3x > -7$, so that $7 - 3x > 0$; in this case, $f(x) = 12 - 3x$ and so $f'(x) = -3$. Similarly, if $x > \frac{7}{3}$, then $f(x) = 3x - 2$ and so $f'(x) = 3$. Hence $f'(x)$ is never zero, but it fails to exist at $x = \frac{7}{3}$. Now $f(1) = 9$, $f\left(\frac{7}{3}\right) = 5$, and $f(5) = 13$, so 13 is the global maximum value of f and 5 is its global minimum value. Consideration of the continuous function f on the interval $\left[1, \frac{7}{3}\right]$ shows that $f(1) = 9$ is a *local* maximum of f on its domain.

3.5.30: Given: $f(x) = |x + 1| + |x - 1|$ on $[-2, 2]$. If $x < -1$ then $f(x) = -(x+1) - (x-1) = -2x$, so that $f'(x) = -2$. If $x > 1$ then $f(x) = x + 1 + x - 1 = 2x$, so that $f'(x) = 2$. If $-1 \leqq x \leqq 1$ then $f(x) = x + 1 - (x - 1) = 2$, so that $f'(x) = 0$. But $f'(x)$ does not exist at $x = -1$ or at $x = 1$. We note that $f(-2) = 4$, $f(x) = 2$ for all x such that $-1 \leqq x \leqq 1$, and that $f(2) = 4$. So 4 is the global maximum value of f and 2 is its global minimum value. Observe that f has infinitely many critical points: every number in the interval $[-1, 1]$.

3.5.31: $f'(x) = 150x^2 - 210x + 72 = 6(5x - 3)(5x - 4)$ always exists and is zero at $x = \frac{3}{5}$ and at $x = \frac{4}{5}$. Now $f(0) = 0$, $f\left(\frac{3}{5}\right) = 16.2$, $f\left(\frac{4}{5}\right) = 16$, and $f(1) = 17$. Hence 17 is the global maximum value of f and 0 is its global minimum value. Consideration of the intervals $\left[0, \frac{4}{5}\right]$ and $\left[\frac{3}{5}, 1\right]$ shows that 16.2 is a *local* maximum value of f on $[0, 1]$ and that 16 is a *local* minimum value of f there.

3.5.32: If

$$f(x) = 2x + \frac{1}{2x}, \quad \text{then} \quad f'(x) = 2 - \frac{1}{2x^2} = \frac{4x^2 - 1}{2x^2}.$$

Therefore $f'(x)$ exists for all x in the domain $[1, 4]$ of f and there are no points in the domain of f at which $f'(x) = 0$. Thus the global minimum value of f is $f(1) = 2.5$ and its global maximum value is $f(4) = 8.125$.

3.5.33: If

$$f(x) = \frac{x}{x+1}, \qquad \text{then} \qquad f'(x) = \frac{1}{(x+1)^2},$$

so $f'(x)$ exists for all x in the domain $[0, 3]$ of f and is never zero there. Hence $f(0) = 0$ is the global minimum value of f and $f(3) = \frac{3}{4}$ is its global maximum value.

3.5.34: If

$$f(x) = \frac{x}{x^2+1}, \quad \text{then} \quad f'(x) = \frac{1-x^2}{(1+x^2)^2},$$

so $f'(x)$ exists for all x; the only point in the domain of f at which $f'(x) = 0$ is $x = 1$. Now $f(0) = 0$, $f(1) = \frac{1}{2}$, and $f(3) = \frac{3}{10}$, so 0 is the global minimum value of f and $\frac{1}{2}$ is its global maximum value. By the usual argument, there is a local minimum at $x = 3$.

3.5.35: If

$$f(x) = \frac{1-x}{x^2+3}, \qquad \text{then} \qquad f'(x) = \frac{(x+1)(x-3)}{(x^2+3)^2},$$

so $f'(x)$ always exists and is zero when $x = -1$ and when $x = 3$. Now $f(-2) = \frac{3}{7}$, $f(-1) = \frac{1}{2}$, $f(3) = -\frac{1}{6}$, and $f(5) = -\frac{1}{7}$. So the global minimum value of f is $-\frac{1}{6}$ and its global maximum value is $\frac{1}{2}$. Consideration of the interval $[-2, -1]$ shows that $\frac{3}{7}$ is a *local* minimum value of f; consideration of the interval $[3, 5]$ shows that $-\frac{1}{7}$ is a *local* maximum value of f.

3.5.36: If $f(x) = 2 - x^{1/3}$, then

$$f'(x) = -\frac{1}{3x^{2/3}},$$

so $f'(x)$ is never zero and $f'(x)$ does not exist when $x = 0$. Nevertheless, f is continuous on its domain $[-1, 8]$. And $f(-1) = 3$, $f(0) = 2$, and $f(8) = 0$, so the global maximum value of f is 3 and its global minimum value is 0. Because $g(x) = x^{1/3}$ is an increasing function, $f(x) = 2 - x^{1/3}$ is decreasing on its domain, and therefore there is no extremum at $x = 0$.

3.5.37: Given: $f(x) = x(1-x^2)^{1/2}$ on $[-1, 1]$. First,

$$f'(x) = (1-x^2)^{1/2} + x \cdot \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x) = (1-x^2)^{1/2} - \frac{x^2}{(1-x^2)^{1/2}} = \frac{1-2x^2}{\sqrt{1-x^2}}.$$

Hence $f'(x)$ exists for $-1 < x < 1$ and not otherwise, but we will check the endpoints ± 1 of the domain of f separately. Also $f'(x) = 0$ when $x = \pm\frac{1}{2}\sqrt{2}$. Now $f(-1) = 0$, $f\left(-\frac{1}{2}\sqrt{2}\right) = -\frac{1}{2}$, $f\left(\frac{1}{2}\sqrt{2}\right) = \frac{1}{2}$, and $f(1) = 0$. Therefore the global minimum value of f is $-\frac{1}{2}$ and its global maximum value is $\frac{1}{2}$. Consideration of the interval $\left[-1, -\frac{1}{2}\sqrt{2}\right]$ shows that $f(-1) = 0$ is a *local* maximum value of f on $[-1, 1]$; similarly, $f(1) = 0$ is a *local* minimum value of f there.

3.5.38: Given: $f(x) = x(4-x^2)^{1/2}$ on $[0, 2]$. Then

$$f'(x) = (4-x^2)^{1/2} + x \cdot \frac{1}{2}(4-x^2)^{-1/2} \cdot (-2x) = (4-x^2)^{1/2} - \frac{x^2}{(4-x^2)^{1/2}} = \frac{4-2x^2}{\sqrt{4-x^2}},$$

so $f'(x)$ exists if $0 \leqq x < 2$ and is zero when $x = \sqrt{2}$. Now $f(0) = 0 = f(2)$ and $f(\sqrt{2}) = 2$, so the former is the global minimum value of f on $[0, 2]$ and the latter is its global maximum value there.

3.5.39: Given: $f(x) = x(2-x)^{1/3}$ on $[1, 3]$. Then

$$f'(x) = (2-x)^{1/3} + x \cdot \frac{1}{3}(2-x)^{-2/3} \cdot (-1) = (2-x)^{1/3} - \frac{x}{3(2-x)^{2/3}} = \frac{6-4x}{3(2-x)^{2/3}}.$$

Then $f'(2)$ does not exist and $f'(x) = 0$ when $x = \frac{3}{2}$. Also f is continuous everywhere, and $f(1) = 1$, $f\left(\frac{3}{2}\right) \approx 1.19$, and $f(3) = -3$. Hence the global minimum value of f is -3 and its global maximum value is $f\left(\frac{3}{2}\right) = 3 \cdot 2^{-4/3} \approx 1.190551$. Consideration of the interval $\left[1, \frac{3}{2}\right]$ shows that $f(1) = 1$ is a *local* minimum value of f.

3.5.40: Given: $f(x) = x^{1/2} - x^{3/2}$ on $[0, 4]$. Then

$$f'(x) = \frac{1}{2}x^{-1/2} - \frac{3}{2}x^{1/2} = \frac{1}{2x^{1/2}} - \frac{3x^{1/2}}{2} = \frac{1-3x}{2\sqrt{x}}.$$

Then $f'(x)$ does not exist when $x = 0$, although f is continuous on its domain; also, $f'(x) = 0$ when $x = \frac{1}{3}$. Now $f(0) = 0$, $f\left(\frac{1}{3}\right) = \frac{2}{9}\sqrt{3}$, and $f(4) = -6$. So -6 is the global minimum value of f and its global maximum value is $\frac{2}{9}\sqrt{3}$. Consideration of the interval $\left[0, \frac{1}{3}\right]$ shows that $f(0) = 0$ is a *local* minimum value of f.

3.5.41: If $A \neq 0$, then $f'(x) \equiv A$ is never zero, but because f is continuous it must have global extrema. Therefore they occur at the endpoints. If $A = 0$, then f is a constant function, and its maximum and minimum value B occurs at every point of the interval, including the two endpoints.

3.5.42: The hypotheses imply that f has no critical points in (a, b), but f must have global extrema. Therefore they occur at the endpoints.

3.5.43: $f'(x) = 0$ if x is not an integer; $f'(x)$ does not exist if x is an integer (we saw in Chapter 2 that $f(x) = [\![x]\!]$ is discontinuous at each integer).

3.5.44: If $f(x) = ax^2 + bx + c$ and $a \neq 0$, then $f'(x) = 2ax + b$. Clearly $f'(x)$ exists for all x, and $f'(x) = 0$ has the unique solution $x = -b/(2a)$. Therefore f has exactly one critical point on the real number line.

3.5.45: If $f(x) = ax^3 + bx^2 + cx + d$ and $a \neq 0$, then $f'(x) = 3ax^2 + 2bx + c$ exists for all x, but the quadratic equation $3ax^2 + 2bx + c = 0$ has two solutions if the discriminant $\Delta = 4b^2 - 12ac$ is positive, one solution if $\Delta = 0$, and no [real] solutions if $\Delta < 0$. Therefore f has either no critical points, exactly one critical point, or exactly two. Examples:

$f(x) = x^3 + x$ has no critical points,

$f(x) = x^3$ has exactly one critical point, and

$f(x) = x^3 - 3x$ has exactly two critical points.

3.5.46: A formula for f is

$$f(x) = \min\{x - [\![x]\!],\ 1 + [\![x]\!] - x\}. \tag{1}$$

If you are not comfortable with the idea that "min" is a "function," an equivalent way of defining f is this:

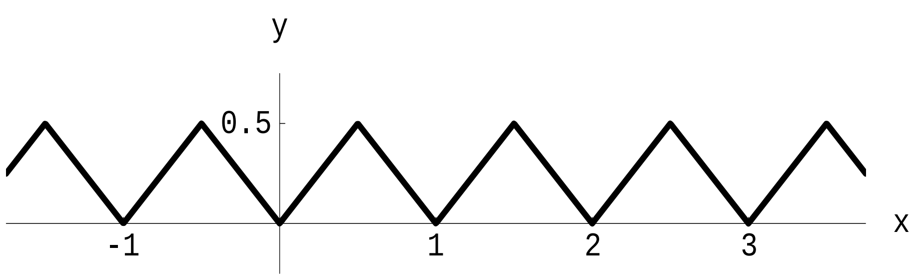
$$f(x) = \frac{1}{2}\left(1 - \left|2x - 1 - 2[\![x]\!]\right|\right).$$

To verify that f performs as advertised, suppose that x is a real number and that $n = [\![x]\!]$, so that $n \leqq x < n + 1$. Case (1): $n \leqq x \leqq n + \frac{1}{2}$. Then

$$x - [\![x]\!] = x - n \leqq \frac{1}{2} \quad \text{and} \quad 1 + [\![x]\!] - x = 1 + n - x = 1 - (x - n) \geqq \frac{1}{2},$$

so that Eq. (1) yields $f(x) = x - n$, which is indeed the distance from x to the nearest integer, because in Case (1) the nearest integer is n. Case (2), in which $n + \frac{1}{2} < x < n + 1$, is handled similarly.

The graph of f is shown next. It should be clear that $f'(x)$ fails to exist at every integral multiple of $\frac{1}{2}$ and that its derivative is either $+1$ or -1 otherwise. Hence its critical points are the integral multiples of $\frac{1}{2}$.



3.5.47: The derivative is positive on $(-\infty, -1.3)$, negative on $(-1.3, 1.3)$, and positive on $(1.3, +\infty)$. So its graph must be the one in Fig. 3.5.15(c). (Numbers with decimal points are approximations.)

3.5.48: The derivative is negative on $(-\infty, -1.0)$, positive on $(-1.0, 1.0)$, negative on $(1.0, 3.0)$, and positive on $(3.0, +\infty)$. So its graph must be the one shown in Fig. 3.5.15(f). (Numbers with decimal points are approximations.)

3.5.49: The derivative is positive on $(-\infty, 0.0)$, negative on $(0.0, 2.0)$, and positive on $(2.0, +\infty)$. So its graph must be the one shown in Fig. 3.5.15(d). (Numbers with decimal points are approximations.)

3.5.50: The derivative is positive on $(-\infty, -2.0)$, negative on $(-2.0, 0.0)$, positive on $(0.0, 2.0)$, and negative on $(2.0, +\infty)$. So its graph must be the one shown in Fig. 3.5.15(b). (Numbers with decimal points are approximations.)

3.5.51: The derivative is negative on $(-\infty, -2.0)$, positive on $(-2.0, 1.0)$, and negative on $(1.0, +\infty)$. Therefore its graph must be the one shown in Fig. 3.5.15(a). (Numbers with decimal points are approximations.)

3.5.52: The derivative is negative on $(-\infty, -2.2)$, positive on $(-2.2, 2.2)$, and negative again on $(2.2, +\infty)$. So its graph must be the one shown in Fig. 3.5.15(e). (Numbers with decimal points are approximations.)

Note: In Problems 53 through 60, we used *Mathematica* and Newton's method (when necessary), carrying 40 decimal digits throughout all computations. Answers are correct or correctly rounded to the number of digits shown. Your answers may differ in the last (or last few) digits beause of differences in hardware or software. Using a graphing calculator or computer to zoom in on solutions has more limited accuracy when using certain machines.

3.5.53: Global maximum value 28 at the left endpoint $x = -2$, global minimum value approximately 6.828387610996 at the critical point where $x = -1 + \frac{1}{3}\sqrt{30} \approx 0.825741858351$, local maximum value 16 at the right endpoint $x = 2$.

3.5.54: Local minimum value 22 at the left endpoint $x = -4$, global maximum value approximately 31.171612389004 at the critical point $x = -1 - \frac{1}{3}\sqrt{30} \approx -2.825741858351$, global minimum value approximately 6.828387610996 at the critical point $x = -1 + \frac{1}{3}\sqrt{30} \approx 0.825741858351$, local maximum value 16 at the right endpoint $x = 2$.

3.5.55: Global maximum value 136 at the left endpoint $x = -3$, global minimum value approximately -8.669500829438 at the critical point $x \approx -0.762212740507$, local maximum value 16 at the right endpont $x = 3$.

3.5.56: Global maximum value 160 at the left endpoint $x = -3$, global minimum value approximately -16.048632589199 at the critical point $x \approx -0.950838582066$, local maximum value approximately 8.976226903748 at the critical point $x \approx 1.323417756580$, local minimum value -8 at the right endpoint $x = 3$.

3.5.57: Global minimum value -5 at the left endpoint $x = 0$, global maximum value approximately 8.976226903748 at the critical point $x \approx 1.323417756580$, local minimum value 5 at the right endpoint $x = 2$.

3.5.58: Local maximum value 3 at the left endpoint $x = -1$, global minimum value approximately -5.767229705222 at the critical point $x \approx -0.460141424682$, global maximum value approximately 21.047667292488 at the critical point $x \approx 0.967947424014$, local minimum value 21 at the right endpoint $x = 1$.

3.5.59: Local minimum value -159 at the left endpoint $x = -3$, global maximum value approximately 30.643243080334 at the critical point $x \approx -1.911336401963$, local minimum value approximately -5.767229705222 at the critical point $x \approx -0.460141424682$, local maximum value approximately

21.047667292488 at the critical point $x \approx 0.967947424014$, global minimum value -345 at the right endpoint $x = 3$.

3.5.60: Local minimum value 0 at the left endpoint $x = 0$, local maximum value approximately 21.047667292488 at the critical point $x \approx 0.967947424014$, global minimum value approximately -1401.923680667600 at the critical point $x \approx 5.403530402632$, global maximum value 36930 at the right endpoint $x = 10$.

Section 3.6

3.6.1: With $x > 0$, $y > 0$, and $x + y = 50$, we are to maximize the product $P = xy$.

$$P = P(x) = x(50 - x) = 50x - x^2, \qquad 0 < x < 50$$

($x < 50$ because $y > 0$.) The product is not maximal if we let $x = 0$ or $x = 50$, so we adjoin the endpoints to the domain of P; thus the continuous function $P(x) = 50x - x^2$ has a global maximum on the closed interval $[0, 50]$, and the maximum does *not* occur at either endpoint. Because f is differentiable, the maximum must occur at a point where $P'(x) = 0$: $50 - 2x = 0$, and so $x = 25$. Because this is the only critical point of P, it follows that $x = 25$ maximizes $P(x)$. When $x = 25$, $y = 50 - 25 = 25$, so the two positive real numbers with sum 50 and maximum possible product are 25 and 25.

3.6.2: If two parallel sides of the rectangle both have length x and the other two sides both have length y, then we are to maximize the area $A = xy$ given that $2x + 2y = 200$. So

$$A = A(x) = x(100 - x), \qquad 0 \leqq x \leqq 100.$$

Clearly the maximum value of A occurs at a critical point of A in the interval (0, 100). But $A'(x) = 100 - 2x$, so $x = 50$ is the location of the maximum. When $x = 50$, also $y = 50$, so the rectangle of maximal area is a square of area $50^2 = 2500$ ft^2.

3.6.3: If the coordinates of the "fourth vertex" are (x, y), then $y = 100 - 2x$ and the area of the rectangle is $A = xy$. So we are to maximize

$$A(x) = x(100 - 2x) \qquad 0 \leqq x \leqq 50.$$

By the usual argument the solution occurs where $A'(x) = 0$, thus where $x = 25$, $y = 50$, and the maximum area is 1250.

3.6.4: If the side of the pen parallel to the wall has length x and the two perpendicular sides both have length y, then we are to maximize area $A = xy$ given $x + 2y = 600$. Thus

$$A = A(y) = y(600 - 2y), \qquad 0 \leqq y \leqq 300.$$

Adjoining the endpoints to the domain is allowed because the maximum we seek occurs at neither endpoint. Therefore the maximum occurs at an interior critical point. We have $A'(y) = 600 - 4y$, so the only critical point of A is $y = 150$. When $y = 150$, we have $x = 300$, so the maximum possible area that can be enclosed is 45000 m^2.

3.6.5: If x is the length of each edge of the base of the box and y denotes the height of the box, then its volume is given by $V = x^2y$. Its total surface area is the sum of the area x^2 of its bottom and four times the area xy of each of its vertical sides, so $x^2 + 4xy = 300$. Thus

$$V = V(x) = x^2 \cdot \frac{300 - x^2}{4x} = \frac{300x - x^3}{4}, \qquad 1 \leqq x \leqq 10\sqrt{3}.$$

Hence

$$V'(x) = \frac{300 - 3x^2}{4},$$

so $V'(x)$ always exists and $V'(x) = 0$ when $x = 10$ (we discard the solution $x = -10$; it's not in the domain of V). Then

$$V(1) = \frac{299}{4} = 74.75, \qquad V(10) = 500, \qquad \text{and} \qquad V\left(10\sqrt{3}\right) = 0,$$

so the maximum possible volume of the box is 500 in.^3.

3.6.6: The excess of the number x over its square is $f(x) = x - x^2$. In this problem we also know that $0 \leqq x \leqq 1$. Then $f(x) = 0$ at the endpoints of its domain, so the maximum value of $f(x)$ must occur at an interior critical point. But $f'(x) = 1 - 2x$, so the only critical point of f is $x = \frac{1}{2}$, which must yield a maximum because f is continuous on $[0, 1]$. So the maximum value of $x - x^2$ for $0 \leqq x \leqq 1$ is $\frac{1}{4}$.

3.6.7: If the two numbers are x and y, then we are to minimize $S = x^2 + y^2$ given $x > 0$, $y > 0$, and $x + y = 48$. So $S(x) = x^2 + (48 - x)^2$, $0 \leqq x \leqq 48$. Here we adjoin the endpoints to the domain of S to ensure the existence of a maximum, but we must test the values of S at these endpoints because it is not immediately clear that neither $S(0)$ nor $S(48)$ yields the maximum value of S. Now $S'(x) = 2x - 2(48 - x)$; the only interior critical point of S is $x = 24$, and when $x = 24$, $y = 24$ as well. Finally, $S(0) = (48)^2 = 2304 = S(48) > 1152 = S(24)$, so the answer is 1152.

3.6.8: Let x be the length of the side around which the rectangle is rotated and let y be the length of each perpendicular side. Then $2x + 2y = 36$. The radius of the cylinder is y and its height is x, so its volume is $V = \pi y^2 x$. So

$$V = V(y) = \pi y^2(18 - y) = \pi(18y^2 - y^3),$$

with natural domain $0 < y < 18$. We adjoin the endpoints to the domain because neither $y = 0$ nor $y = 18$ maximizes $V(y)$, and deduce the existence of a global maximum at an interior critical point. Now

$$V'(y) = \pi(36y - 3y^2) = 3\pi y(12 - y).$$

So $V'(y) = 0$ when $y = 0$ and when $y = 12$. The former value of y *minimizes* $V(y)$, so the maximum possible volume of the cylinder is $V(12) = 864\pi$.

3.6.9: Let x and y be the two numbers. Then $x + y = 10$, $x \geqq 0$, and $y \geqq 0$. We are to minimize the sum of their cubes,

$$S = x^3 + y^3: \qquad S(x) = x^3 + (10 - x)^3, \qquad 0 \leqq x \leqq 10.$$

Now $S'(x) = 3x^2 - 3(10 - x)^2$, so the values of x to be tested are $x = 0$, $x = 5$, and $x = 10$. At the endpoints, $S = 1000$; when $x = 5$, $S = 250$ (the minimum).

3.6.10: Draw a cross section of the cylindrical log—a circle of radius r. Inscribe in this circle a cross section of the beam—a rectangle of width w and height h. Draw a diagonal of the rectangle; the Pythagorean theorem yields $x^2 + h^2 = 4r^2$. The strength S of the beam is given by $S = kwh^2$ where k is a positive constant. Because $h^2 = 4r^2 - w^2$, we have

$$S = S(w) = kw(4r^2 - w^2) = k(4wr^2 - w^3)$$

with natural domain $0 < w < 2r$. We adjoin the endpoints to this domain; this is permissible because $S = 0$ at each, and so is not maximal. Next, $S'(w) = k(4r^2 - 3w^2)$; $S'(w) = 0$ when $3w^2 = 4r^2$, and the corresponding (positive) value of w yields the maximum of S (we know that $S(w)$ must have a maximum on $[0, 2r]$ because of the continuity of S on this interval, and we also know that the maximum does not occur at either endpoint, so there is only one possible location for the maximum). At maximum, $h^2 = 4r^2 - w^2 = 3w^2 - w^2$, so $h = w\sqrt{2}$ describes the shape of the beam of greatest strength.

3.6.11: As in Fig. 3.6.18, let y denote the length of each of the internal dividers and of the two sides parallel to them; let x denote the length of each of the other two sides. The total length of all the fencing is $2x + 4y = 600$ and the area of the corral is $A = xy$. Hence

$$A = A(y) = \frac{600 - 4y}{2} \cdot y = 300y - 2y^2, \qquad 0 \leqq y \leqq 150.$$

Now $A'(y) = 0$ only when $y = 75$, and $A(0) = 0 = A(150)$, and therefore the maximum area of the corral is $A(75) = 11250\ \text{yd}^2$.

3.6.12: Let r denote the radius of the cylinder and h its height. We are to maximize its volume $V = \pi r^2 h$ given the constraint that the total surface area is 150π:

$$2\pi r^2 + 2\pi rh = 150\pi, \qquad \text{so that} \qquad h = \frac{75 - r^2}{r}.$$

Thus

$$V = V(r) = \pi r(75 - r^2) = \pi(75r - r^3), \qquad 0 < r < \sqrt{75}.$$

We may adjoin both endpoints to this domain without creating a spurious maximum, so we use $\left[0,\, 5\sqrt{3}\,\right]$ as the domain of V. Next, $V'(r) = \pi(75 - 3r^2)$. Hence $V'(r)$ always exists and its only zero in the domain of V occurs when $r = 5$ (and $h = 10$). But V is zero at the two endpoints of its domain, so $V(5) = 250\pi$ is the maximum volume of such a cylinder.

3.6.13: If the rectangle has sides x and y, then $x^2 + y^2 = 16^2$ by the Pythagorean theorem. The area of the rectangle is then

$$A(x) = x\sqrt{256 - x^2}\,, \qquad 0 \leqq x \leqq 16.$$

A positive quantity is maximized exactly when its square is maximized, so in place of A we maximize

$$f(x) = (A(x))^2 = 256x^2 - x^4.$$

The only solutions of $f'(x) = 0$ in the domain of A are $x = 0$ and $x = 8\sqrt{2}$. But $A(0) = 0 = A(16)$, so $x = 8\sqrt{2}$ yields the maximum value 128 of A.

3.6.14: If the far side of the rectangle has length $2x$ (this leads to simpler algebra than length x), and the sides perpendicular to the far side have length y, then by the Pythagorean theorem, $x^2 + y^2 = L^2$. The area of the rectangle is $A = 2xy$, so we maximize

$$A(x) = 2x\sqrt{L^2 - x^2}\,, \qquad 0 \leqq x \leqq L$$

by maximizing

$$f(x) = (A(x))^2 = 4(L^2x^2 - x^4).$$

Now $f'(x) = 4(2L^2x - 4x^3) = 8x(L^2 - 2x^2)$ is zero when $x = 0$ (rejected; $A(0) = 0$) and when $x = \frac{1}{2}L\sqrt{2}$. Note also that $A(L) = 0$. By the usual argument, $x = \frac{1}{2}L\sqrt{2}$ maximizes $f(x)$ and thus $A(x)$. The answer is $A\left(\frac{1}{2}L\sqrt{2}\right) = L^2$.

3.6.15: $V'(T) = -0.06426 + (0.0170086)T - (0.0002037)T^2$. The equation $V'(T) = 0$ is quadratic with the two (approximate) solutions $T \approx 79.532$ and $T \approx 3.967$. The formula for $V(T)$ is valid only in the range $0 \leqq T \leqq 30$, so we reject the first solution. Finally, $V(0) = 999.87$, $V(30) \approx 1003.763$, and $V(3.967) \approx 999.71$. Thus the volume is minimized when $T \approx 3.967$, and therefore water has its greatest density at about $3.967°$C.

3.6.16: Let $P(x,\, 0)$ be the lower right-hand corner point of the rectangle. The rectangle then has base $2x$, height $4 - x^2$, and thus area

$$A(x) = 2x(4 - x^2) = 8x - 2x^3, \qquad 0 \leqq x \leqq 2.$$

Now $A'(x) = 8 - 6x^2$; $A'(x) = 0$ when $x = \frac{2}{3}\sqrt{3}$. Because $A(0) = 0$, $A(2) = 0$, and $A\left(\frac{2}{3}\sqrt{3}\right) > 0$, the maximum possible area is $A\left(\frac{2}{3}\sqrt{3}\right) = \frac{32}{9}\sqrt{3}$.

3.6.17: Let x denote the length of each edge of the base and let y denote the height of the box. We are to maximize its volume $V = x^2y$ given the constraint $2x^2 + 4xy = 600$. Solve the latter for y to write

$$V(x) = 150x - \frac{1}{2}x^3, \qquad 1 \leqq x \leqq 10\sqrt{3}.$$

The solution of $V'(x) = 0$ in the domain of V is $x = 10$. Because $V(10) = 1000 > V(1) = 149.5 > V\left(10\sqrt{3}\right) = 0$, this shows that $x = 10$ maximizes V and that the maximum value of V is $1000\ \text{cm}^3$.

3.6.18: Let x denote the radius of the cylinder and y its height. Then its total surface area is $\pi x^2 + 2\pi xy = 300\pi$, so $x^2 + 2xy = 300$. We are to maximize its volume $V = \pi x^2 y$. Because

$$y = \frac{300 - x^2}{2x}, \qquad \text{it follows that} \qquad V = V(x) = \frac{\pi}{2}(300x - x^3), \quad 0 \leqq x \leqq 10\sqrt{3}.$$

It is then easy to show that $x = 10$ maximizes $V(x)$, that $y = x = 10$ as well, and thus that the maximum possible volume of the can is 1000π in.^{3}

3.6.19: Let x be the length of the edge of each of the twelve small squares. Then each of the three cross-shaped pieces will form boxes with base length $1 - 2x$ and height x, so each of the three will have volume $x(1 - 2x)^2$. Both of the two cubical boxes will have edge x and thus volume x^3. So the total volume of all five boxes will be

$$V(x) = 3x(1 - 2x)^2 + 2x^3 = 14x^3 - 12x^2 + 3x, \qquad 0 \leqq x \leqq \frac{1}{2}.$$

Now $V'(x) = 42x^2 - 24x + 3$; $V'(x) = 0$ when $14x^2 - 8x - 1 = 0$. The quadratic formula gives the two solutions $x = \frac{1}{14}\left(4 \pm \sqrt{2}\right)$. These are approximately 0.3867 and 0.1847, and both lie in the domain of V. Finally, $V(0) = 0$, $V(0.1847) \approx 0.2329$, $V(0.3867) \approx 0.1752$, and $V(0.5) = 0.25$. Therefore, to maximize V, one must cut each of the three large squares into four smaller squares of side length $\frac{1}{2}$ each and form the resulting twelve squares into two cubes. At maximum volume there will be only two boxes, not five.

3.6.20: Let x be the length of each edge of the square base of the box and let h denote its height. Then its volume is $V = x^2h$. The total cost of the box is \$144, hence

$$4xh + x^2 + 2x^2 = 144 \qquad \text{and thus} \qquad h = \frac{144 - 3x^2}{4x}.$$

Therefore

$$V = V(x) = \frac{x}{4}\left(144 - 3x^2\right) = 36x - \frac{3}{4}x^3.$$

The natural domain of V is the open interval $\left(0,\ 4\sqrt{3}\,\right)$, but we may adjoin the endpoints as usual to obtain a closed interval. Also

$$V'(x) = 36 - \frac{9}{4}x^2,$$

so $V'(x)$ always exists and is zero only at $x = 4$ (reject the other root $x = -4$). Finally, $V(x) = 0$ at the endpoints of its domain, so $V(4) = 96$ (ft^3) is the maximum volume of such a box. The dimensions of the largest box are 4 ft square on the base by 6 ft high.

3.6.21: Let x denote the edge length of one square and y that of the other. Then $4x+4y = 80$, so $y = 20-x$. The total area of the two squares is $A = x^2 + y^2$, so

$$A = A(x) = x^2 + (20 - x)^2 = 2x^2 - 40x + 400,$$

with domain $(0, 20)$; adjoin the endpoints as usual. Then $A'(x) = 4x - 40$, which always exists and which vanishes when $x = 10$. Now $A(0) = 400 = A(20)$, whereas $A(10) = 200$. So to minimize the total area of the two squares, make two equal squares. To maximize it, make only one square.

3.6.22: Let r be the radius of the circle and x the edge of the square. We are to maximize total area $A = \pi r^2 + x^2$ given the side condition $2\pi r + 4x = 100$. From the last equation we infer that

$$x = \frac{100 - 2\pi r}{4} = \frac{50 - \pi r}{2}.$$

So

$$A = A(r) = \pi r^2 + \frac{1}{4}(50 - \pi r)^2 = \left(\pi + \frac{1}{4}\pi^2\right)r^2 - 25\pi r + 625$$

for $0 \leqq r \leqq 50/\pi$ (because $x \geqq 0$). Now

$$A'(r) = 2\left(\pi + \frac{1}{4}\pi^2\right)r - 25\pi;$$

$$A'(r) = 0 \quad \text{when} \quad r = \frac{25}{2 + \frac{\pi}{2}} = \frac{50}{\pi + 4};$$

that is, when $r \approx 7$. Finally,

$$A(0) = 625, \quad A\left(\frac{50}{\pi}\right) \approx 795.77 \quad \text{and} \quad A\left(\frac{50}{\pi + 4}\right) \approx 350.06.$$

Results: For minimum area, construct a circle of radius $50/(\pi+4) \approx 7.00124$ (cm) and a square of edge length $100/(\pi + 4) \approx 14.00248$ (cm). For maximum area, bend all the wire into a circle of radius $50/\pi \approx 15.91549$ (cm).

3.6.23: Let x be the length of each segment of fence perpendicular to the wall and let y be the length of each segment parallel to the wall.

Case 1: The internal fence is perpendicular to the wall. Then $y = 600 - 3x$ and the enclosure will have area $A(x) = 600x - 3x^2$, $0 \leqq x \leqq 200$. Then $A'(x) = 0$ when $x = 100$; $A(100) = 30000$ (m^2) is the maximum in Case 1.

Case 2: The internal fence is parallel to the wall. Then $y = 300 - x$, and the area of the enclosure is given by $A(x) = 300x - x^2$, $0 \leqq x \leqq 300$. Then $A'(x) = 0$ when $x = 150$; $A(150) = 22500$ (m^2) is the maximum in Case 2.

Answer: The maximum possible area of the enclosure is 30000 m^2. The divider must be perpendicular to the wall and of length 100 m. The side parallel to the wall is to have length 300 m.

3.6.24: See Fig. 3.6.22 of the text. Suppose that the pen measures x (horizontal) by y (vertical). Then it has area $A = xy$.

Case 1: $x \geqq 10$, $y \geqq 5$. Then

$$x + (x - 10) + y + (y - 5) = 85, \quad \text{so} \quad x + y = 50.$$

Therefore

$$A = A(x) = x(50 - x) = 50x - x^2, \qquad 10 \leqq x \leqq 45.$$

Then $A'(x) = 0$ when $x = 25$; $A(25) = 625$. Note that $A(10) = 400$ and that $A(45) = 225$.

Case 2: $0 \leqq x \leqq 10$, $y \geqq 5$. Then

$$x + y + (y - 5) = 85, \quad \text{so} \quad x + 2y = 90.$$

Therefore

$$A = A(x) = x\frac{90 - x}{2} = \frac{1}{2}(90x - x^2), \qquad 0 \leqq x \leqq 10.$$

In this case, $A'(x) = 0$ when $x = 45$, but 45 doesn't lie in the domain of A. Note that $A(0) = 0$ and that $A(10) = 400$.

Case 3: $x \geqq 10$, $0 \leqq y \leqq 5$. Then

$$x + (x - 10) + y = 85, \quad \text{so} \quad 2x + y = 95.$$

Therefore

$$A = A(x) = x(95 - 2x) = 95x - 2x^2, \qquad 45 \leqq x \leqq 47.5.$$

In this case $A'(x) = 0$ when $x = 23.75$, not in the domain of A. Note that $A(45) = 225$ and that $A(47.5) = 0$.

Conclusion: The area of the pen is maximized when the pen is square, 25 m on each side (the maximum from Case 1).

3.6.25: Let the dimensions of the box be x by x by y. We are to maximize $V = x^2y$ subject to some conditions on x and y. According to the poster on the wall of the Bogart, Georgia Post Office, the *length* of the box is the larger of x and y, and the *girth* is measured around the box in a plane perpendicular to its length.

Case 1: $x < y$. Then the length is y, the girth is $4x$, and the mailing constraint is $4x + y \leqq 100$. It is clear that we take $4x + y = 100$ to maximize V, so that

$$V = V(x) = x^2(100 - 4x) = 100x^2 - 4x^3, \qquad 0 \leqq x \leqq 25.$$

Then $V'(x) = 4x(50 - 3x)$; $V'(x) = 0$ for $x = 0$ and for $x = 50/3$. But $V(0) = 0$, $V(25) = 0$, and $V(50/3) = 250000/27 \approx 9259$ (in.^{3}). The latter is the maximum in Case 1.

Case 2: $x \geqq y$. Then the length is x and the girth is $2x + y$, although you may get some argument from a postal worker who may insist that it's $4x$. So $3x + 2y = 100$, and thus

$$V = V(x) = x^2\left(\frac{100 - 3x}{2}\right) = 50x^2 - \frac{3}{2}x^3, \qquad 0 \leqq x \leqq 100/3.$$

Then $V'(x) = 100x - \frac{9}{2}x^2$; $V'(x) = 0$ when $x = 0$ and when $x = 200/9$. But $V(0) = 0$, $V(100/3) = 0$, and $V(200/9) = 2000000/243 \approx 8230$ (in.^{3}).

Case 3: You lose the argument in Case 2. Then the box has length x and girth $4x$, so $5x = 100$; thus $x = 20$. To maximize the total volume, no calculus is needed—let $y = x$. Then the box of maximum volume will have volume $20^3 = 8000$ (in.^{3}).

Answer: The maximum is $\dfrac{250000}{27}$ in.^{3}

3.6.26: In this problem the girth of the package is its circumference; no one would interpret "girth" in any other way. So suppose that the package has length x and radius r. Then it has volume $V = \pi r^2 x$ where $x + 2\pi r = 100$. We seek to maximize

$$V = V(r) = \pi r^2(100 - 2\pi r) = \pi(100r^2 - 2\pi r^3), \qquad 0 \leqq r \leqq \frac{50}{\pi}.$$

Now

$$V'(r) = \pi(200r - 6\pi r^2) = 2\pi r(100 - 3\pi r);$$

$V'(r) = 0$ when $r = 0$ and when $r = 100/(3\pi)$. But

$$V(0) = 0, \qquad V\left(\frac{50}{\pi}\right) = 0, \qquad \text{and} \qquad V\left(\frac{100}{3\pi}\right) = \frac{1000000}{27\pi} \approx 11789 \ \ (\text{in.}^3),$$

and the latter is clearly the maximum of V.

3.6.27: Suppose that n presses are used, $1 \leqq n \leqq 8$. The total cost of the poster run would then be

$$C(n) = 5n + (10 + 6n)\left(\frac{50000}{3600n}\right) = 5n + \frac{125}{9}\left(\frac{10}{n} + 6\right)$$

dollars. Temporarily assume that n can take on every real number value between 1 and 8. Then

$$C'(n) = 5 - \frac{125}{9} \cdot \frac{10}{n^2};$$

$C'(n) = 0$ when $n = \frac{5}{3}\sqrt{10} \approx 5.27$ presses. But an integral number of presses must be used, so the actual number that will minimize the cost is either 5 or 6, unless the minimum occurs at one of the two endpoints. The values in question are $C(1) \approx 227.2$, $C(5) \approx 136.1$, $C(6) \approx 136.5$, and $C(8) \approx 140.7$. So to minimize cost and thereby maximize the profit, five presses should be used.

3.6.28: Let x denote the number of workers hired. Each worker will pick $900/x$ bushels; each worker will spend $180/x$ hours picking beans. The supervisor cost will be $1800/x$ dollars, and the cost per worker will be $8 + (900/x)$ dollars. Thus the total cost will be

$$C(x) = 8x + 900 + \frac{1800}{x}, \qquad 1 \leqq x.$$

It is clear that large values of x make $C(x)$ large, so the global minimum of $C(x)$ occurs either at $x = 1$ or where $C'(x) = 0$. Assume for the moment that x can take on all real number values in $[1, +\infty)$, not merely integral values, so that C' is defined. Then

$$C'(x) = 8 - \frac{1800}{x^2}; \; C'(x) = 0 \quad \text{when} \quad x^2 = 225.$$

Thus $C'(15) = 0$. Now $C(1) = 2708$ and $C(15) = 1140$, so fifteen workers should be hired; the cost to pick each bushel will be approximately \$1.27.

3.6.29: We are to minimize the total cost C over a ten-year period. This cost is the sum of the initial cost and ten times the annual cost:

$$C(x) = 150x + 10\left(\frac{1000}{2+x}\right), \quad 0 \leqq x \leqq 10.$$

Next,

$$C'(x) = 150 - \frac{10000}{(2+x)^2}; \quad C'(x) = 0 \quad \text{when} \quad 150 = \frac{10000}{(2+x)^2},$$

so that $(2+x)^2 = \frac{200}{3}$. One of the resulting values of x is negative, so we reject it. The other is $x = -2 + \sqrt{200/3} \approx 6.165$ (in.). The problem itself suggests that x must be an integer, so we check $x = 6$ and $x = 7$ along with the endpoints of the domain of C. In dollars, $C(0) = 5000$, $C(6) \approx 2150$, $C(7) \approx 2161$, and $C(10) \approx 2333$. Result: Install six inches of insulation. The annual savings over the situation with no insulation at all then will be one-tenth of $5000 - 2150$, about \$285 per year.

3.6.30: We assume that each one-cent increase in price reduces sales by 50 burritos per night. Let x be the amount, in cents, by which the price is increased. The resulting profit is

$$\begin{aligned}P(x) &= (50 + x)(5000 - 5x) - 25(5000 - 50x) - 100000\\ &= (25 + x)(5000 - 50x) - 100000\\ &= 25000 + 3750x - 50x^2, \qquad -50 \leqq x.\end{aligned}$$

Because $P(x) < 0$ for large values of x and for $x = -50$, P will be maximized where $P'(x) = 0$:

$$P'(x) = 3750 - 100x; \quad P'(x) = 0 \quad \text{when} \quad x = 37.5.$$

Now $P(37) = 953$, $P(37.5) \approx 953.13$, and $P(38) = 953$. Therefore profit is maximized when the selling price is either 87¢ or 88¢, and the maximum profit will be \$953.

3.6.31: Let x be the number of five-cent fare increases. The resulting revenue will be

$$R(x) = (150 + 5x)(600 - 40x), \qquad -30 \leqq x \leqq 15$$

(the revenue is the product of the price and the number of passengers). Now

$$\begin{aligned}&R(x) = 90000 - 3000x - 200x^2;\\ &R'(x) = -3000 - 400x; \qquad R'(x) = 0 \quad \text{when} \quad x = -7.5.\end{aligned}$$

Because the fare must be an integral number of cents, we check $R(-7) = 1012 = R(-8)$ (dollars). Answer: The fare should be either \$1.10 or \$1.15; this is a reduction of 40 or 35 cents, respectively, and each results in the maximum possible revenue of \$1012 per day.

3.6.32: The following figure shows a central cross section of the sphere and inscribed cylinder. The radius of the cylinder is r and its height is h; the radius of the sphere is R. From the Pythagorean theorem we see that $4r^2 + h^2 = 4R^2$. The volume of the cylinder is $V = \pi r^2 h$, and therefore we find that

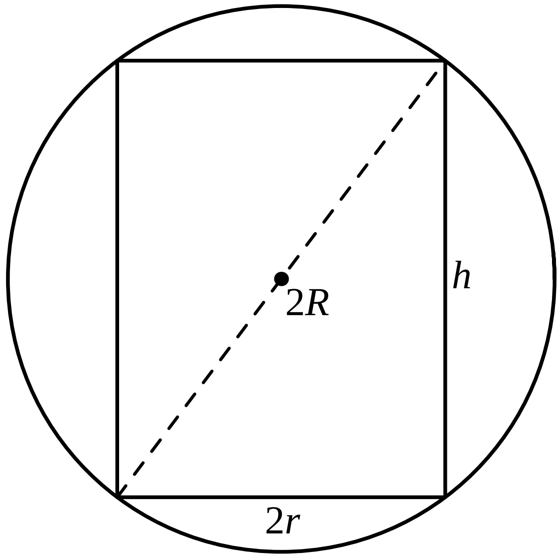
$$\begin{aligned}V = V(h) &= \pi\left(R^2 - \frac{1}{4}h^2\right)h\\ &= \frac{\pi}{4}(4R^2h - h^3), \qquad 0 \leqq h \leqq 2R.\end{aligned}$$

Then

$$V'(h) = \frac{\pi}{4}(4R^2 - 3h^2),$$

so $V(h) = 0$ when $3h^2 = 4R^2$, so that $h = \frac{2}{3}R\sqrt{3}$. This value of h maximizes V because $V(0) = 0$ and $V(2R) = 0$. The corresponding value of r is $\frac{1}{3}R\sqrt{6}$, so the ratio of the height of the cylinder to its radius is

$h/r = \sqrt{2}$. The volume of the maximal cylinder is $\frac{4}{9}\pi R^3\sqrt{3}$ and the volume of the sphere is $\frac{4}{3}\pi R^3$; the ratio of the volume of the sphere to that of the maximal inscribed cylinder is thus $\sqrt{3}$.



3.6.33: The following figure shows a cross section of the cone and inscribed cylinder. Let x be the radius of the cylinder and y its height. By similar triangles in the figure,

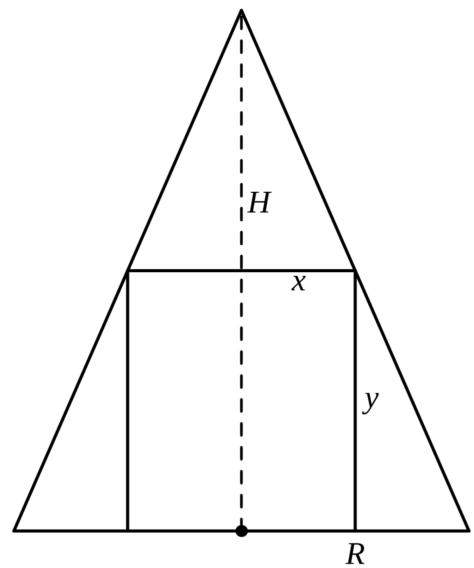
$$\frac{H}{R} = \frac{y}{R-x}, \quad \text{so} \quad y = \frac{H}{R}(R-x).$$

We are to maximize the volume $V = \pi x^2 y$ of the cylinder, so we write

$$V = V(x) = \pi x^2 \frac{H}{R}(R-x)$$

$$= \pi\frac{H}{R}(Rx^2 - x^3), \qquad 0 \leqq x \leqq R.$$

Because $V(0) = 0 = V(r)$, V is maximized when $V'(x) = 0$; this leads to the equation $2xR = 3x^2$ and thus to the results $x = \frac{2}{3}R$ and $y = \frac{1}{3}H$.



3.6.34: Let the circle have equation $x^2 + y^2 = 1$ and let (x, y) denote the coordinates of the upper right-hand vertex of the trapezoid (Fig. 3.6.25). Then the area A of the trapezoid is the product of its altitude y and the average of the lengths of its two bases, so

$$A = \frac{1}{2}y(2x+2) \quad \text{where} \quad y^2 = 1 - x^2.$$

A positive quantity is maximized when its square is maximized, so we maximize instead

$$f(x) = A^2 = (x+1)^2(1-x^2)$$

$$= 1 + 2x - 2x^3 - x^4, \qquad 0 \leqq x \leqq 1.$$

Because $f(0) = 0 = f(1)$, f is maximized when $f'(x) = 0$:

$$0 = 2 - 6x^2 - 4x^3 = 2(1+x)^2(1-2x).$$

But the only solution of $f'(x) = 0$ in the domain of f is $x = \frac{1}{2}$. Finally, $f\left(\frac{1}{2}\right) = \frac{27}{16}$, so the maximum possible area of the trapezoid is $\frac{3}{4}\sqrt{3}$. This is just over 41% of the area of the circle, so the answer meets the test of plausibility.

3.6.35: Draw a circle in the plane with center at the origin and with radius R. Inscribe a rectangle with vertical and horizontal sides and let (x, y) be its vertex in the first quadrant. The base of the rectangle has length $2x$ and its height is $2y$, so the perimeter of the rectangle is $P = 4x + 4y$. Also $x^2 + y^2 = R^2$, so

$$P = P(x) = 4x + 4\sqrt{R^2 - x^2}, \qquad 0 \leqq x \leqq R.$$

$$P'(x) = 4 - \frac{4x}{\sqrt{R^2 - x^2}};$$

$$P'(x) = 0 \quad \text{when} \quad 4\sqrt{R^2 - x^2} = 4x;$$

$$R^2 - x^2 = x^2;$$

$$x^2 = \frac{1}{2}R^2.$$

Because $x > 0$, $x = \frac{1}{2}R\sqrt{2}$. The corresponding value of $P(x)$ is $4R\sqrt{2}$, and $P(0) = 4R = P(R)$. So the former value of x maximizes the perimeter P. Because $y^2 = R^2 - x^2$ and because $R^2 - x^2 = x^2$ at maximum, $y = x$ at maximum. Therefore the rectangle of largest perimeter that can be inscribed in a circle is a square.

3.6.36: Let $(x,\, y)$ be the coordinates of the vertex of the rectangle in the first quadrant. Then, by symmetry, the area of the rectangle is $A = (2x)(2y) = 4xy$. But from the equation of the ellipse we find that

$$y = \tfrac{3}{5}\sqrt{25 - x^2}, \quad \text{so}$$

$$A = A(x) = \tfrac{12}{5}x\sqrt{25 - x^2}\,, \qquad 0 \leqq x \leqq 5.$$

We can simplify the algebra by maximizing instead

$$f(x) = \tfrac{25}{144}A^2 = 25x^2 - x^4;$$

$$f'(x) = 50x - 4x^3;$$

$$f'(x) = 0 \quad \text{when} \quad x = 0 \quad \text{and when} \quad x = \tfrac{5}{2}\sqrt{2}.$$

Now $A(0) = 0 = A(5)$, whereas $A\left(\frac{5}{2}\sqrt{2}\right) = 30$. So the rectangle of maximum area has base $2x = 5\sqrt{2}$ and height $2y = 3\sqrt{2}$.

3.6.37: We are to maximize volume $V = \frac{1}{3}\pi r^2 h$ given $r^2 + h^2 = 100$. The latter relation enables us to write

$$V = V(h) = \frac{1}{3}\pi(100 - h^2)h = \frac{1}{3}\pi(100h - h^3), \qquad 0 \leqq h \leqq 10.$$

Now $V'(h) = \frac{1}{3}\pi(100 - 3h^2)$, so $V'(h) = 0$ when $3h^2 = 100$, thus when $h = \frac{10}{3}\sqrt{3}$. But $V(h) = 0$ at the endpoints of its domain, so the latter value of h maximizes V, and its maximum value is $\frac{2000}{27}\pi\sqrt{3}$.

3.6.38: Put the bases of the poles on the x-axis, one at the origin and the other at $x = 10$. Let the rope touch the ground at the point x. Then the rope reaches straight from $(0,\, 10)$ to $(x,\, 0)$ and straight from $(x,\, 0)$ to $(10,\, 10)$. In terms of x, its length is

$$\begin{aligned} L(x) &= \sqrt{100 + x^2} + \sqrt{100 + (10 - x)^2} \\ &= \sqrt{100 + x^2} + \sqrt{200 - 20x + x^2}, \qquad 0 \leqq x \leqq 10. \end{aligned}$$

So

$$L'(x) = \frac{x}{\sqrt{100 + x^2}} + \frac{x - 10}{\sqrt{200 - 20x + x^2}};$$

$L'(x) = 0$ when

$$x\sqrt{200 - 20x + x^2} = (10 - x)\sqrt{x^2 + 100}\,;$$

$$x^2(x^2 - 20x + 200) = \left(100 - 20x + x^2\right)\left(x^2 + 100\right);$$

$$x^4 - 20x^3 + 200x^2 = x^4 - 20x^3 + 200x^2 - 2000x + 10000;$$

$$2000x = 10000;$$

and thus when $x = 5$. Now $L(0) = L(10) = 10\left(1 + \sqrt{2}\right)$, which exceeds $L(5) = 10\sqrt{5}$. So the latter is the length of the shortest possible rope.

3.6.39: Let x and y be the two numbers. Then $x \geqq 0$, $y \geqq 0$, and $x + y = 16$. We are to find both the maximum and minimum values of $x^{1/3} + y^{1/3}$. Because $y = 16 - x$, we seek the extrema of

$$f(x) = x^{1/3} + (16 - x)^{1/3}, \qquad 0 \leqq x \leqq 16.$$

Now

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-2/3} - \frac{1}{3}(16 - x)^{-2/3} \\ &= \frac{1}{3x^{2/3}} - \frac{1}{3(16 - x)^{2/3}}; \end{aligned}$$

$f'(x) = 0$ when $(16 - x)^{2/3} = x^{2/3}$, so when $16 - x = x$, thus when $x = 8$. Now $f(0) = f(16) = 16^{1/3} \approx 2.52$, so $f(8) = 4$ maximizes f whereas $f(0)$ and $f(16)$ yield its minimum.

3.6.40: If the base of the L has length x, then the vertical part has length $60 - x$. Place the L with its corner at the origin in the xy-plane, its base on the nonnegative x-axis, and the vertical part on the nonnegative y-axis. The two ends of the L have coordinates $(0,\,60 - x)$ and $(x,\,0)$, so they are at distance

$$d = d(x) = \sqrt{x^2 + (60 - x)^2}\,, \qquad 0 \leqq x \leqq 60.$$

A positive quantity is minimized when its square is minimal, so we minimize

$$f(x) = (d(x))^2 = x^2 + (60 - x)^2, \qquad 0 \leqq x \leqq 60.$$

Then $f'(x) = 2x - 2(60 - x) = 4x - 120$; $f'(x) = 0$ when $x = 30$. Now $f(0) = f(60) = 3600$, whereas $f(30) = 1800$. So $x = 30$ minimizes $f(x)$ and thus $d(x)$. The minimum possible distance between the two ends of the wire is therefore $d(30) = 30\sqrt{2}$.

3.6.41: If $(x,\,x^2)$ is a point of the parabola, then its distance from $(0,\,1)$ is

$$d(x) = \sqrt{x^2 + (x^2 - 1)^2}\,.$$

So we minimize

$$f(x) = (d(x))^2 = x^4 - x^2 + 1,$$

where the domain of f is the set of all real numbers. But because $f(x)$ is large positive when $|x|$ is large, we will not exclude a minimum if we restrict the domain of f to be an interval of the form $[-a,\, a]$ where a is a large positive number. On the interval $[-a,\, a]$, f is continuous and thus has a global minimum, which does not occur at $\pm a$ because $f(\pm a)$ is large positive. Because $f'(x)$ exists for all x, the minimum of f occurs at a point where $f'(x) = 0$:

$$4x^3 - 2x = 0; \qquad 2x(2x^2 - 1) = 0.$$

Hence $x = 0$ or $x = \pm\frac{1}{2}\sqrt{2}$. Now $f(0) = 1$ and $f\left(\pm\frac{1}{2}\sqrt{2}\right) = \frac{3}{4}$. So $x = 0$ yields a local maximum value for $f(x)$, and the minimum possible distance is $\sqrt{0.75} = \frac{1}{2}\sqrt{3}$.

3.6.42: It suffices to minimize $x^2 + y^2$ given $y = (3x - 4)^{1/3}$. Let $f(x) = x^2 + (3x - 4)^{2/3}$. Then

$$f'(x) = 2x + 2(3x - 4)^{-1/3}.$$

Then $f'(x) = 0$ when

$$\begin{aligned}
2x + \frac{2}{(3x-4)^{1/3}} &= 0;\\
2x(3x-4)^{1/3} &= -2;\\
x(3x-4)^{1/3} &= -1;\\
x^3(3x-4) &= -1;\\
x^3(3x-4) + 1 &= 0;\\
3x^4 - 4x^3 + 1 &= 0;\\
(x-1)^2(3x^2 + 2x + 1) &= 0.
\end{aligned}$$

Now $x = 1$ is the only real solution of the last equation, $f'(x)$ does not exist when $x = \frac{4}{3}$, and $f(1) = 2 > \frac{16}{9} = f\left(\frac{4}{3}\right)$. So the point closest to the origin is $\left(\frac{4}{3},\, 0\right)$.

3.6.43: Examine the plank on the right on Fig. 3.6.10. Let its height be $2y$ and its width (in the x-direction) be z. The total area of the four small rectangles in the figure is then $A = 4 \cdot z \cdot 2y = 8yz$. The circle has radius 1, and by Problem 35 the large inscribed square has dimensions $\sqrt{2}$ by $\sqrt{2}$. Thus

$$\left(\frac{1}{2}\sqrt{2} + z\right)^2 + y^2 = 1.$$

This implies that

$$y = \sqrt{\frac{1}{2} - z\sqrt{2} - z^2}\,.$$

Therefore

$$A(z) = 8z\sqrt{\frac{1}{2} - z\sqrt{2} - z^2}\,, \qquad 0 \leqq z \leqq 1 - \frac{1}{2}\sqrt{2}.$$

Now $A(z) = 0$ at each endpoint of its domain and

$$A'(z) = \frac{4\sqrt{2}\,\left(1 - 3z\sqrt{z} - 4z^2\right)}{\sqrt{1 - 2z\sqrt{z} - 2z^2}}.$$

So $A'(z) = 0$ when $z = \frac{1}{8}\left(-3\sqrt{2} \pm \sqrt{34}\right)$; we discard the negative solution, and find that when $A(z)$ is maximized

$$z = \frac{-3\sqrt{2} + \sqrt{34}}{8} \approx 0.198539,$$

$$2y = \frac{\sqrt{7 - \sqrt{17}}}{2} \approx 0.848071, \quad \text{and}$$

$$A(z) = \frac{\sqrt{142 + 34\sqrt{17}}}{2} \approx 0.673500.$$

The four small planks use just under 59% of the wood that remains after the large plank is cut, a very efficient use of what might be scrap lumber.

3.6.44: Place the base of the triangle on the x-axis and its upper vertex on the y-axis. Then its lower right vertex is at the point $\left(\frac{1}{2},\, 0\right)$ and its upper vertex is at $\left(0,\, \frac{1}{2}\sqrt{3}\right)$. It follows that the slope of the side of the triangle joining these two vertices is $-\sqrt{3}$. So this side lies on the straight line with equation

$$y = \sqrt{3}\left(\frac{1}{2} - x\right).$$

Let $(x,\, y)$ be the coordinates of the upper right-hand vertex of the rectangle. Then the rectangle has area $A = 2xy$, so

$$A(x) = \sqrt{3}\,(x - 2x^2), \qquad 0 \leqq x \leqq \frac{1}{2}.$$

Now $A'(x) = 0$ when $x = \frac{1}{4}$, and because $A(x) = 0$ at the endpoints of its domain, it follows that the maximum area of such a rectangle is $A\left(\frac{1}{4}\right) = \frac{1}{8}\sqrt{3}$.

3.6.45: Set up a coordinate system in which the island is located at $(0,\, 2)$ and the village at $(6,\, 0)$, and let $(x,\, 0)$ be the point at which the boat lands. It is clear that $0 \leqq x \leqq 6$. The trip involves the land distance $6 - x$ traveled at 20 km/h and the water distance $(4 + x^2)^{1/2}$ traveled at 10 km/h. The total time of the trip is then given by

$$T(x) = \frac{1}{10}\sqrt{4 + x^2} + \frac{1}{20}(6 - x), \qquad 0 \leqq x \leqq 6.$$

Now

$$T'(x) = \frac{x}{10\sqrt{4+x^2}} - \frac{1}{20}.$$

Thus $T'(x) = 0$ when $3x^2 = 4$; because $x \geqq 0$, we find that $x = \frac{2}{3}\sqrt{3}$. The value of T there is

$$\frac{1}{10}\left(3+\sqrt{3}\right) \approx 0.473,$$

whereas $T(0) = 0.5$ and $T(6) \approx 0.632$. Therefore the boater should make landfall at $\frac{2}{3}\sqrt{3} \approx 1.155$ km from the point on the shore closest to the island.

3.6.46: Set up a coordinate system in which the factory is located at the origin and the power station at (L, W) in the xy-plane—$L = 4500$, $W = 2000$. Part of the path of the power cable will be straight along the river bank and part will be a diagonal running under water. It makes no difference whether the straight part is adjacent to the factory or to the power station, so we assume the former. Thus we suppose that the power cable runs straight from (0, 0) to $(x, 0)$, then straight from $(x, 0)$ to (L, W), where $0 \leqq x \leqq L$. Let y be the length of the diagonal stretch of the cable. Then by the Pythagorean theorem,

$$W^2 + (L-x)^2 = y^2, \qquad \text{so} \qquad y = \sqrt{W^2 + (L-x)^2}\,.$$

The cost C of the cable is $C = kx + 3ky$ where k is the cost per unit distance of over-the-ground cable. Therefore the total cost of the cable is

$$C(x) = kx + 3k\sqrt{W^2 + (L-x)^2}\,, \qquad 0 \leqq x \leqq L.$$

It will not change the solution if we assume that $k = 1$, and in this case we have

$$C'(x) = 1 - \frac{3(L-x)}{\sqrt{W^2 + (L-x)^2}}.$$

Next, $C'(x) = 0$ when $x^2 + (L-x)^2 = 9(L-x)^2$, and this leads to the solution

$$x = L - \frac{1}{4}W\sqrt{2} \text{ and } y = \frac{3}{4}W\sqrt{2}.$$

It is not difficult to verify that the latter value of x yields a value of C smaller than either $C(0)$ or $C(L)$. Answer: Lay the cable $x = 4500 - 500\sqrt{2} \approx 3793$ meters along the bank and $y = 1500\sqrt{2} \approx 2121$ meters diagonally across the river.

3.6.47: The distances involved are $|AP| = |BP| = \sqrt{x^2+1}$ and $|CP| = 3-x$. Therefore we are to minimize

$$f(x) = 2\sqrt{x^2+1} + 3 - x, \qquad 0 \leqq x \leqq 3.$$

Now

$$f'(x) = \frac{2x}{\sqrt{x^2+1}} - 1; \quad f'(x) = 0 \quad \text{when} \quad \frac{2x}{\sqrt{x^2+1}} = 1.$$

This leads to the equation $3x^2 = 1$, so $x = \frac{1}{3}\sqrt{3}$. Now $f(0) = 5$, $f(3) \approx 6.32$, and at the critical point, $f(x) = 3 + \sqrt{3} \approx 4.732$. Answer: The distribution center should be located at the point $P(\frac{1}{3}\sqrt{3}, 0)$.

3.6.48: **(a)** $T = \frac{1}{c}\sqrt{a^2+x^2} + \frac{1}{v}\sqrt{(s-x)^2+b^2}$.

(b) $T'(x) = \frac{x}{c\sqrt{a^2+x^2}} - \frac{s-x}{v\sqrt{(s-x)^2+b^2}}$.

$$T'(x) = 0 \quad \text{when} \quad \frac{x}{c\sqrt{a^2+x^2}} = \frac{s-x}{v\sqrt{(s-x)^2+b^2}};$$

$$\frac{x}{\sqrt{a^2+x^2}} \cdot \frac{\sqrt{(x-s)^2+b^2}}{s-x} = \frac{c}{v};$$

$$\sin\alpha \csc\beta = \frac{c}{v}$$

$$\frac{\sin\alpha}{\sin\beta} = \frac{c}{v} = n.$$

3.6.49: We are to minimize total cost

$$C = c_1\sqrt{a^2+x^2} + c_2\sqrt{(L-x)^2+b^2}.$$

$$C'(x) = \frac{c_1 x}{\sqrt{a^2+x^2}} - \frac{c_2(L-x)}{\sqrt{(L-x)^2+b^2}};$$

$$C'(x) = 0 \text{ when } \frac{c_1 x}{\sqrt{a^2+x^2}} = \frac{c_2(L-x)}{\sqrt{(L-x)^2+b^2}}.$$

The result in Part (a) is equivalent to the last equation. For Part (b), assume that $a = b = c_1 = 1$, $c_2 = 2$, and $L = 4$. Then we obtain

$$\frac{x}{\sqrt{1+x^2}} = \frac{2(4-x)}{\sqrt{(4-x)^2+1}};$$

$$\frac{x^2}{1+x^2} = \frac{4(16-8x+x^2)}{16-8x+x^2+1};$$

$$x^2(17-8x+x^2) = (4+4x^2)(16-8x+x^2);$$

$$17x^2 - 8x^3 + x^4 = 64 - 32x + 68x^2 - 32x^3 + 4x^4.$$

Therefore we wish to solve $f(x) = 0$ where

$$f(x) = 3x^4 - 24x^4 + 51x^2 - 32x + 64.$$

Now $f(0) = 64$, $f(1) = 62$, $f(2) = 60$, $f(3) = 22$, and $f(4) = -16$. Because $f(3) > 0 > f(4)$, we interpolate to estimate the zero of $f(x)$ between 3 and 4; it turns out that interpolation gives $x \approx 3.58$. Subsequent interpolation yields the more accurate estimate $x \approx 3.45$. (The equation $f(x) = 0$ has exactly two solutions, $x \approx 3.452462314$ and $x \approx 4.559682567$.)

3.6.50: Because $x^3 + y^3 = 2000$, $y = (2000 - x^3)^{1/3}$. We want to maximize and minimize total surface area $A = 6x^2 + 6y^2$;

$$A = A(x) = 6x^2 + 6(2000 - x^3)^{2/3}, \qquad 0 \leqq x \leqq 10\sqrt[3]{2}.$$

$$A'(x) = \frac{-12[x^2 - x(2000 - x^3)^{1/3}]}{(2000 - x^3)^{1/3}}.$$

Now $A'(x) = 0$ at $x = 0$ and at $x = 10$; $A'(x)$ does not exist at $x = 10\sqrt[3]{2}$, the right-hand endpoint of the domain of A (at that point, the graph of A has a vertical tangent). Also $A(0) = 600 \cdot 2^{2/3} \approx 952.441$ and $A\left(10\sqrt[3]{2}\right)$ is the same; $A(10) = 1200$. So the maximum surface area is attained when each cube has edge length 10 and the minimum is attained when there is only one cube, of edge length $10\sqrt[3]{2} \approx 12.5992$.

3.6.51: Let r be the radius of the sphere and x the edge length of the cube. We are to maximize and minimize total volume

$$V = \frac{4}{3}\pi r^3 + x^3 \quad \text{given} \quad 4\pi r^2 + 6x^2 = 1000.$$

The latter equation yields

$$x = \sqrt{\frac{1000 - 4\pi r^2}{6}},$$

so

$$V = V(r) = \frac{4}{3}\pi r^3 + \left(\frac{500 - 2\pi r^2}{3}\right)^{3/2}, \qquad 0 \leqq r \leqq r_1 = 5\sqrt{\frac{10}{\pi}}.$$

Next,

$$V'(r) = 4\pi r^2 - 2\pi r\sqrt{\frac{500 - 2\pi r^2}{3}},$$

and $V'(r) = 0$ when

$$4\pi r^2 = 2\pi r\sqrt{\frac{500 - 2\pi r^2}{3}}.$$

So $r = 0$ or

$$2r = \sqrt{\frac{500 - 2\pi r^2}{3}}.$$

The latter equation leads to

$$r = r_2 = 5\sqrt{\frac{10}{\pi + 6}}.$$

Now $V(0) \approx 2151.66$, $V(r_1) \approx 2973.54$, and $V(r_2) \approx 1743.16$. Therefore, to minimize the sum of the volumes, choose $r = r_2 \approx 5.229$ in. and $x = 2r_2 \approx 10.459$ in. To maximize the sum of their volumes, take $r = r_1 \approx 8.921$ in. and $x = 0$ in.

3.6.52: Let the horizontal piece of wood have length $2x$ and the vertical piece have length $y + z$ where y is the length of the part above the horizontal piece and z the length of the part below it. Then

$$y = \sqrt{4 - x^2} \quad \text{and} \quad z = \sqrt{16 - x^2}.$$

Also the kite area is $A = x(y + z)$; $\dfrac{dA}{dx} = 0$ implies that

$$y + z = \frac{x^2}{y} + \frac{x^2}{z}.$$

Multiply each side of the last equation by yz to obtain

$$y^2z + yz^2 = x^2z + x^2y,$$

so that

$$yz(y + z) = x^2(y + z);$$

$$x^2 = yz;$$

$$x^4 = y^2z^2 = (4 - x^2)(16 - x^2);$$

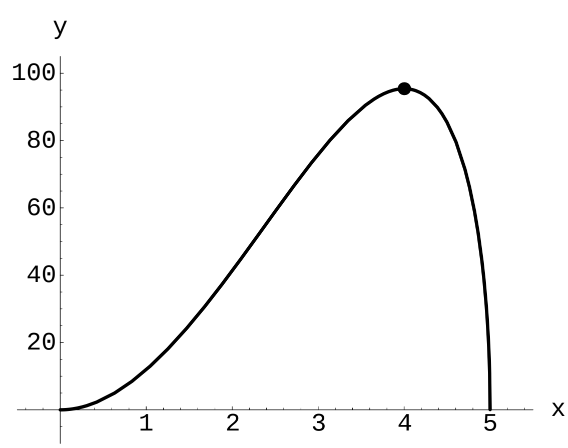
$$x^4 = 64 - 20x^2 + x^4;$$

$$20x^2 = 64;$$

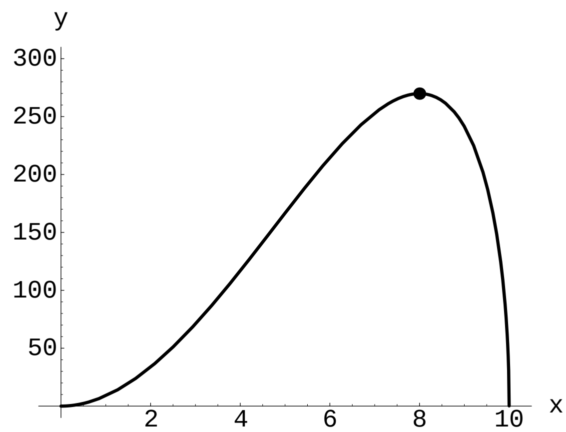
$$x = \frac{4}{5}\sqrt{5}, \;\; y = \frac{2}{5}\sqrt{5}, \;\; z = \frac{8}{5}\sqrt{5}.$$

Therefore $L_1 = \frac{8}{5}\sqrt{5} \approx 3.5777$ and $L_2 = 2\sqrt{5} \approx 4.47214$ for maximum area.

3.6.53: The graph of $V(x)$ is shown next. The maximum volume seems to occur near the point $(4, V(4)) \approx (4, 95.406)$, so the maximum volume is approximately 95.406 cubic feet.



3.6.54: The graph of $V(x)$ is shown next. The maximum volume seems to occur near the point $(8, V(8)) \approx (8, 269.848)$, so the maximum volume is approximately 269.848 cubic feet.



3.6.55: Let V_1 and V_2 be the volume functions of problems 53 and 54, respectively. Then

$$V_1'(x) = \frac{20\sqrt{5}\,(4x - x^2)}{3\sqrt{5 - x}},$$

which is zero at $x = 0$ and at $x = 4$, and

$$V_2'(x) = \frac{10\sqrt{5}\,(8x - x^2)}{3\sqrt{10 - x}},$$

which is zero at $x = 0$ and at $x = 8$, as expected. Finally, $\dfrac{V_2(8)}{V_1(4)} = 2\sqrt{2}$.

3.6.56: Let x denote the length of each edge of the base of the box; let y denote its height. If the box has total surface area A, then $2x^2 + 4xy = A$, and hence

$$y = \frac{A - 2x^2}{4x}. \qquad (1)$$

The box has volume $V = x^2 y$, so its volume can be expressed as a function of x alone:

$$V(x) = \frac{Ax - 2x^3}{4}, \quad 0 \leqq x \leqq \sqrt{A/2}\,.$$

Then

$$V'(x) = \frac{A - 6x^2}{4}; \qquad V'(x) = 0 \quad \text{when} \quad x = \sqrt{A/6}\,.$$

This critical point clearly lies in the interior of the domain of V, and (almost as clearly) $V'(x)$ is increasing to its left and decreasing to its right. Hence this critical point yields the box of maximal volume. Moreover, when $x = \sqrt{A/6}$, we have—by Eq. (1)—

$$y == \frac{A-(A/3)}{4\sqrt{A/6}} = \frac{2A}{3}\cdot\frac{\sqrt{6}}{4\sqrt{A}} = \frac{\sqrt{6}}{6}\cdot\sqrt{A} = \frac{\sqrt{A}}{\sqrt{6}}.$$

Therefore the closed box with square base, fixed surface area, and maximal volume is a cube.

3.6.57: Let x denote the length of each edge of the square base of the box and let y denote its height. Given total surface area A, we have $x^2 + 4xy = A$, and hence

$$y = \frac{A-x^2}{4x}. \tag{1}$$

The volume of the box is $V = x^2y$, and therefore

$$V(x) = \frac{Ax-x^2}{4}, \quad 0 \leqq x \leqq \sqrt{A}.$$

Next,

$$V'(x) = \frac{A-3x^2}{4}; \qquad V'(x) = 0 \quad \text{when} \quad x = \sqrt{A/3}.$$

Because $V'(x) > 0$ to the left of this critical point and $V'(x) < 0$ to the right, it yields the global maximum value of $V(x)$. By Eq. (1), the corresponding height of the box is $\frac{1}{2}\sqrt{A/3}$. Therefore the open box with square base and maximal volume has height equal to half the length of the edge of its base.

3.6.58: Let r denote the radius of the base of the closed cylindrical can, h its height, and A its total surface area. Then

$$2\pi r^2 + 2\pi rh = A, \quad \text{and hence} \quad h = \frac{A-2\pi r^2}{2\pi r}. \tag{1}$$

The volume of the can is $V = \pi r^2 h$, and thus

$$V(r) = \frac{Ar-2\pi r^3}{2}, \quad 0 \leqq r \leqq \left(\frac{A}{2\pi}\right)^{1/2}.$$

Next,

$$V'(r) = \frac{A-6\pi r^2}{2}; \qquad V'(r) = 0 \quad \text{when} \quad r = \left(\frac{A}{6\pi}\right)^{1/2}.$$

Because $V'(r) > 0$ to the left of this critical point and $V'(r) < 0$ to the right, it determines the global maximum value of $V(r)$. By Eq. (1), it follows that the can of maximum volume has equal height and diameter.

3.6.59: Let r denote the radius of the base of the open cylindrical can and let h denote its height. Its total surface area A then satisfies the equation $\pi r^2 + 2\pi rh = A$, and therefore

$$h = \frac{A-\pi r^2}{2\pi r}. \tag{1}$$

Thus the volume of the can is given by

$$V(r) = \frac{Ar - \pi r^3}{2}, \quad 0 \leqq r \leqq \left(\frac{A}{\pi}\right)^{1/2}.$$

Next,

$$V'(r) = \frac{A - 3\pi r^2}{2}; \qquad V'(r) = 0 \quad \text{when} \quad r = \left(\frac{A}{3\pi}\right)^{1/2}.$$

Clearly $V'(r) > 0$ to the left of this critical point and $V'(r) < 0$ to the right, so it determines the global maximum value of $V(r)$. By Eq. (1) the corresponding value of h is the same, so the open cylindrical can of maximum volume has height equal to its radius.

3.6.60: Let r denote the interior radius of the cylindrical can and h its interior height. Because the thickness t of the material of the can will be very small in comparison with r and h, the total amount of material M used to make the can will be very accurately approximated by multiplying the thickness of the bottom by its area, the thickness of the curved side by its area, and the thickness of the top by its area. That is,

$$\pi r^2 t + 2\pi rht + 3\pi r^2 t = M, \quad \text{so that} \quad h = \frac{M - 4\pi r^2 t}{2\pi rt}. \tag{1}$$

Thus the volume of the can will be given by (the very accurate approximation)

$$V(r) = \frac{Mr - 4\pi r^3 t}{2t}, \quad 0 \leqq r \leqq \left(\frac{M}{4\pi t}\right)^{1/2}.$$

Next,

$$V'(r) = \frac{M - 12\pi r^2 t}{2t}; \qquad V'(r) = 0 \quad \text{when} \quad r = \frac{1}{2} \cdot \left(\frac{M}{3\pi t}\right)^{1/2}.$$

Because $V'(r) > 0$ to the left of this critical point and $V'(r) < 0$ to the right, it yields the global maximum value of $V(r)$. By (1), the height of the corresponding can is four times as great, so the can of maximum volume has height twice its diameter (approximately, but quite accurately).

To solve this problem exactly, first establish that

$$4\pi t(r+t)^2 + \pi ht^2(2r+t) = M,$$

then that

$$V(r) = \frac{Mr^2 + 4\pi tr^2(r+t)^2}{t^2(2r+t)}.$$

Then show that $V'(r) = 0$ when

$$12\pi tr^4 + 24\pi t^2 r^3 + (16\pi t^3 - M)r^2 + (4\pi t^4 - Mt)r = 0,$$

and that the relevant critical point is

$$r = \frac{-3\pi t^2 + \sqrt{3\pi t(M - \pi t^3)}}{6\pi t}.$$

3.6.61: Let

$$f(t) = \frac{1}{1+t^2}, \qquad \text{so that} \qquad f'(t) = -\frac{2t}{(1+t^2)^2}.$$

The line tangent to the graph of $y = f(t)$ at the point $(t,\, f(t))$ then has x-intercept and y-intercept

$$\frac{1+3t^2}{2t} \qquad \text{and} \qquad \frac{1+3t^2}{(1+t^2)^2},$$

respectively. The area of the triangle bounded by the part of the tangent line in the first quadrant and the coordinate axes is

$$A(t) = \frac{1}{2} \cdot \frac{1+3t^2}{2t} \cdot \frac{1+3t^2}{(1+t^2)^2}, \tag{1}$$

and

$$A'(t) = \frac{-9t^6 + 9t^4 + t^2 - 1}{4t^2(1+t^2)^3}.$$

Next, $A'(t) = 0$ when

$$(t-1)(t+1)(3t^2-1)(3t^2+1) = 0,$$

and the only two critical points of A in the interval $[0.5,\, 2]$ are

$$t_1 = 1 \qquad \text{and} \qquad t_2 = \frac{\sqrt{3}}{3} \approx 0.57735.$$

Significant values of $A(t)$ are then

$$A(0.5) = 0.98, \quad A(0.57735) \approx 0.97428, \quad A(1) = 1, \quad \text{and} \quad A(2) = 0.845.$$

Therefore $A(t)$ has a local maximum at $t = 0.5$, a local minimum at t_2, its global maximum at $t = 1$, and its global minimum at $t = 2$.

To answer the first question in Problem 61, Eq. (1) makes it clear that $A(t) \to +\infty$ as $t \to 0^+$ and, in addition, that $A(t) \to 0$ as $t \to +\infty$.

3.6.62: If $0 \leqq x < 1$, then the cost of the power line will be

$$C(x) = 40x + 100\sqrt{1 + (1-x)^2}$$

(in thousand of dollars). If $x = 1$, then the cost will be 80 thousand dollars because there is no need to use underground cable. Next,

$$C'(x) = \frac{100x - 100 + 40\sqrt{x^2 - 2x + 2}}{\sqrt{x^2 - 2x + 2}},$$

and $C'(x) = 0$ when

$$x = x_0 = \frac{21 - 2\sqrt{21}}{21} \approx 0.563564219528.$$

The graph of $C(x)$ (using a computer algebra system) establishes that x_0 determines the global minimum for $C(x)$ on the interval $[0, 1)$, yielding the corresponding value $C(x_0) \approx 131.651513899117$. Hence the global minimum for $C(x)$ on $[0, 1]$ is $C(1) = 80$ (thousand dollars). It is neither necessary to cross the park nor to use underground cable.

Section 3.7

3.7.1: If $f(x) = 3\sin^2 x = 3(\sin x)^2$, then $f'(x) = 6\sin x \cos x$.

3.7.2: If $f(x) = 2\cos^4 x = 2(\cos x)^4$, then $f'(x) = 8(\cos x)^3(-\sin x) = -8\cos^3 x \sin x$.

3.7.3: If $f(x) = x\cos x$, then $f'(x) = 1 \cdot \cos x + x \cdot (-\sin x) = \cos x - x\sin x$.

3.7.4: If $f(x) = x^{1/2}\sin x$, then $f'(x) = \frac{1}{2}x^{-1/2}\sin x + x^{1/2}\cos x = \dfrac{\sin x + 2x\cos x}{2\sqrt{x}}$.

3.7.5: If $f(x) = \dfrac{\sin x}{x}$, then $f'(x) = \dfrac{x\cos x - \sin x}{x^2}$.

3.7.6: If $f(x) = \dfrac{\cos x}{x^{1/2}}$, then $f'(x) = \dfrac{x^{1/2}(-\sin x) - \frac{1}{2}x^{-1/2}\cos x}{x} = -\dfrac{2x\sin x + \cos x}{2x\sqrt{x}}$.

3.7.7: If $f(x) = \sin x \cos^2 x$, then

$$f'(x) = \cos x\cos^2 x + (\sin x)(2\cos x)(-\sin x) = \cos^3 x - 2\sin^2 x\cos x.$$

3.7.8: If $f(x) = \cos^3 x \sin^2 x$, then

$$f'(x) = (3\cos^2 x)(-\sin x)(\sin^2 x) + (2\sin x\cos x)(\cos^3 x) = -3\cos^2 x\sin^3 x + 2\sin x\cos^4 x.$$

3.7.9: If $g(t) = (1 + \sin t)^4$, then $g'(t) = 4(1 + \sin t)^3 \cdot \cos t$.

3.7.10: If $g(t) = (2 - \cos^2 t)^3$, then $g'(t) = 3(2 - \cos^2 t)^2 \cdot (2\cos t\sin t) = 6(2 - \cos^2 t)^2(\sin t\cos t)$.

3.7.11: If $g(t) = \dfrac{1}{\sin t + \cos t}$, then (by the reciprocal rule) $g'(t) = -\dfrac{\cos t - \sin t}{(\sin t + \cos t)^2} = \dfrac{\sin t - \cos t}{(\sin t + \cos t)^2}$.

3.7.12: If $g(t) = \dfrac{\sin t}{1 + \cos t}$, then (by the quotient rule)

$$g'(t) = \frac{(1 + \cos t)(\cos t) - (\sin t)(-\sin t)}{(1 + \cos t)^2} = \frac{\sin^2 t + \cos^2 t + \cos t}{(1 + \cos t)^2} = \frac{1 + \cos t}{(1 + \cos t)^2} = \frac{1}{1 + \cos t}.$$

3.7.13: If $f(x) = 2x \sin x - 3x^2 \cos x$, then (by the product rule)

$$f'(x) = 2 \sin x + 2x \cos x - 6x \cos x + 3x^2 \sin x = 3x^2 \sin x - 4x \cos x + 2 \sin x.$$

3.7.14: If $f(x) = x^{1/2} \cos x - x^{-1/2} \sin x$,

$$f'(x) = \frac{1}{2}x^{-1/2} \cos x - x^{1/2} \sin x + \frac{1}{2}x^{-3/2} \sin x - x^{-1/2} \cos x = \frac{(1 - 2x^2) \sin x - x \cos x}{2x\sqrt{x}}.$$

3.7.15: If $f(x) = \cos 2x \sin 3x$, then $f'(x) = -2 \sin 2x \sin 3x + 3 \cos 2x \cos 3x$.

3.7.16: If $f(x) = \cos 5x \sin 7x$, then $f'(x) = -5 \sin 5x \sin 7x + 7 \cos 5x \cos 7x$.

3.7.17: If $g(t) = t^3 \sin^2 2t = t^3 (\sin 2t)^2$, then

$$g'(t) = 3t^2 (\sin 2t)^2 + t^3 \cdot (2 \sin 2t) \cdot (\cos 2t) \cdot 2 = 3t^2 \sin^2 2t + 4t^3 \sin 2t \cos 2t.$$

3.7.18: If $g(t) = \sqrt{t}\, \cos^3 3t = t^{1/2} (\cos 3t)^3$, then

$$g'(t) = \frac{1}{2}t^{-1/2} (\cos 3t)^3 + t^{1/2} \cdot 3(\cos 3t)^2 \cdot (-3 \sin 3t) = \frac{\cos^3 3t}{2\sqrt{t}} - 9\sqrt{t}\, \cos^2 3t\, \sin 3t.$$

3.7.19: If $g(t) = (\cos 3t + \cos 5t)^{5/2}$, then $g'(t) = \frac{5}{2}(\cos 3t + \cos 5t)^{3/2}(-3 \sin 3t - 5 \sin 5t)$.

3.7.20: If $g(t) = \dfrac{1}{\sqrt{\sin^2 t + \sin^2 3t}} = (\sin^2 t + \sin^2 3t)^{-1/2}$, then

$$g'(t) = -\frac{1}{2}(\sin^2 t + \sin^2 3t)^{-3/2}(2 \sin t \cos t + 6 \sin 3t \cos 3t) = -\frac{\sin t \cos t + 3 \sin 3t \cos 3t}{(\sin^2 2t + \sin^2 3t)^{3/2}}.$$

3.7.21: If $y = y(x) = \sin^2 \sqrt{x} = (\sin x^{1/2})^2$, then

$$\frac{dy}{dx} = 2(\sin x^{1/2})(\cos x^{1/2}) \cdot \frac{1}{2}x^{-1/2} = \frac{\sin \sqrt{x}\, \cos \sqrt{x}}{\sqrt{x}}.$$

3.7.22: If $y = y(x) = \dfrac{\cos 2x}{x}$, then $\dfrac{dy}{dx} = \dfrac{x \cdot (-2 \sin 2x) - 1 \cdot \cos 2x}{x^2} = -\dfrac{2x \sin 2x + \cos 2x}{x^2}$.

3.7.23: If $y = y(x) = x^2 \cos(3x^2 - 1)$, then

$$\frac{dy}{dx} = 2x \cos(3x^2 - 1) - x^2 \cdot 6x \cdot \sin(3x^2 - 1) = 2x \cos(3x^2 - 1) - 6x^3 \sin(3x^2 - 1).$$

3.7.24: If $y = y(x) = \sin^3 x^4 = (\sin x^4)^3$, then

$$\frac{dy}{dx} = 3(\sin x^4)^2 \cdot D_x(\sin x^4) = 3(\sin x^4)^2 \cdot (\cos x^4) \cdot D_x(x^4) = 12x^3 \sin^2 x^4\, \cos x^4.$$

3.7.25: If $y = y(x) = \sin 2x \cos 3x$, then

$$\frac{dy}{dx} = (\sin 2x) \cdot D_x(\cos 3x) + (\cos 3x) \cdot D_x(\sin 2x) = -3\sin 2x \sin 3x + 2\cos 3x \cos 2x.$$

3.7.26: If $y = y(x) = \dfrac{x}{\sin 3x}$, then $\dfrac{dy}{dx} = \dfrac{(\sin 3x) \cdot 1 - x \cdot D_x(\sin 3x)}{(\sin 3x)^2} = \dfrac{\sin 3x - 3x\cos 3x}{\sin^2 3x}$.

3.7.27: If $y = y(x) = \dfrac{\cos 3x}{\sin 5x}$, then

$$\frac{dy}{dx} = \frac{(\sin 5x)(-3\sin 3x) - (\cos 3x)(5\cos 5x)}{(\sin 5x)^2} = -\frac{3\sin 3x \sin 5x + 5\cos 3x \cos 5x}{\sin^2 5x}.$$

3.7.28: If $y = y(x) = \sqrt{\cos\sqrt{x}} = (\cos x^{1/2})^{1/2}$, then

$$\frac{dy}{dx} = \frac{1}{2}(\cos x^{1/2})^{-1/2}(-\sin x^{1/2}) \cdot \frac{1}{2}x^{-1/2} = -\frac{\sin\sqrt{x}}{4\sqrt{x}\,\sqrt{\cos\sqrt{x}}}.$$

3.7.29: If $y = y(x) = \sin^2 x^2 = (\sin x^2)^2$, then

$$\frac{dy}{dx} = 2(\sin x^2) \cdot D_x(\sin x^2) = 2(\sin x^2) \cdot (\cos x^2) \cdot D_x(x^2) = 4x \sin x^2 \ \cos x^2.$$

3.7.30: If $y = y(x) = \cos^3 x^3 = (\cos x^3)^3$, then

$$\frac{dy}{dx} = 3(\cos x^3)^2 \cdot D_x(\cos x^3) = 3(\cos x^3)^2 \cdot (-\sin x^3) \cdot D_x(x^3) = -9x^2 \cos^2 x^3 \ \sin x^3.$$

3.7.31: If $y = y(x) = \sin 2\sqrt{x} = \sin(2x^{1/2})$, then

$$\frac{dy}{dx} = \Big[\cos(2x^{1/2})\Big] \cdot D_x(2x^{1/2}) = x^{-1/2}\cos(2x^{1/2}) = \frac{\cos 2\sqrt{x}}{\sqrt{x}}.$$

3.7.32: If $y = y(x) = \cos 3\sqrt[3]{x} = \cos(3x^{1/3})$, then

$$\frac{dy}{dx} = \Big[-\sin(3x^{1/3})\Big] \cdot D_x(3x^{1/3}) = (-\sin 3x^{1/3}) \cdot (x^{-2/3}) = -\frac{\sin 3\sqrt[3]{x}}{\sqrt[3]{x^2}}.$$

3.7.33: If $y = y(x) = x\sin x^2$, then $\dfrac{dy}{dx} = 1 \cdot \sin x^2 + x \cdot (\cos x^2) \cdot 2x = \sin x^2 + 2x^2 \cos x^2$.

3.7.34: If $y = y(x) = x^2 \cos\left(\dfrac{1}{x}\right)$, then

$$\frac{dy}{dx} = 2x\cos\left(\frac{1}{x}\right) + x^2\left[-\sin\left(\frac{1}{x}\right)\right] \cdot D_x\left(\frac{1}{x}\right)$$

$$= 2x\cos\left(\frac{1}{x}\right) - x^2\left[\sin\left(\frac{1}{x}\right)\right] \cdot \left(-\frac{1}{x^2}\right) = 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right).$$

3.7.35: If $y = y(x) = \sqrt{x}\,\sin\sqrt{x} = x^{1/2}\sin x^{1/2}$, then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}\sin x^{1/2} + x^{1/2}(\cos x^{1/2})\cdot\frac{1}{2}x^{-1/2} = \frac{\sin\sqrt{x}}{2\sqrt{x}} + \frac{\cos\sqrt{x}}{2} = \frac{\sin\sqrt{x} + \sqrt{x}\,\cos\sqrt{x}}{2\sqrt{x}}.$$

3.7.36: If $y = y(x) = (\sin x - \cos x)^2$, then

$$\frac{dy}{dx} = 2(\sin x - \cos x)(\cos x + \sin x) = 2(\sin^2 x - \cos^2 x) = -2\cos 2x.$$

3.7.37: If $y = y(x) = \sqrt{x}\,(x - \cos x)^3 = x^{1/2}(x - \cos x)^3$, then

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}x^{-1/2}(x - \cos x)^3 + 3x^{1/2}(x - \cos x)^2(1 + \sin x)\\ &= \frac{(x - \cos x)^3}{2\sqrt{x}} + 3\sqrt{x}\,(x - \cos x)^2(1 + \sin x) = \frac{(x - \cos x)^3 + 6x(x - \cos x)^2(1 + \sin x)}{2\sqrt{x}}.\end{aligned}$$

3.7.38: If $y = y(x) = \sqrt{x}\,\sin\sqrt{x + \sqrt{x}} = x^{1/2}\sin(x + x^{1/2})^{1/2}$, then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}\sin(x + x^{1/2})^{1/2} + x^{1/2}\left[\cos(x + x^{1/2})^{1/2}\right]\cdot\frac{1}{2}(x + x^{1/2})^{-1/2}\left(1 + \frac{1}{2}x^{-1/2}\right).$$

The symbolic algebra program *Mathematica* simplifies this to

$$\frac{dy}{dx} = \frac{(2x + \sqrt{x}\,)\cos\sqrt{x + \sqrt{x}} + 2\left(\sqrt{x + \sqrt{x}}\right)\sin\sqrt{x + \sqrt{x}}}{4\sqrt{x}\,\sqrt{x + \sqrt{x}}}.$$

3.7.39: If $y = y(x) = \cos(\sin x^2)$, then $\dfrac{dy}{dx} = \left[-\sin(\sin x^2)\right]\cdot(\cos x^2)\cdot 2x = -2x\left[\sin(\sin x^2)\right]\cos x^2$.

3.7.40: If $y = y(x) = \sin\left(1 + \sqrt{\sin x}\right)$, then

$$\frac{dy}{dx} = \left[\cos\left(1 + \sqrt{\sin x}\right)\right]\cdot\frac{1}{2}(\sin x)^{-1/2}\cdot\cos x = \frac{(\cos x)\cos\left(1 + \sqrt{\sin x}\right)}{2\sqrt{\sin x}}.$$

3.7.41: If $x = x(t) = \tan t^7 = \tan(t^7)$, then $\dfrac{dx}{dt} = \left(\sec t^7\right)^2\cdot D_t(t^7) = 7t^6\sec^2 t^7$.

3.7.42: If $x = x(t) = \sec t^7 = \sec(t^7)$, then $\dfrac{dx}{dt} = (\sec t^7\,\tan t^7)\cdot D_t(t^7) = 7t^6\sec t^7\,\tan t^7$.

3.7.43: If $x = x(t) = (\tan t)^7 = \tan^7 t$, then

$$\frac{dx}{dt} = 7(\tan t)^6\cdot D_t\tan t = 7(\tan t)^6\,\sec^2 t = 7\tan^6 t\,\sec^2 t.$$

3.7.44: If $x = x(t) = (\sec 2t)^7 = \sec^7 2t$, then

$$\frac{dx}{dt} = 7(\sec 2t)^6 \cdot D_t(\sec 2t) = 7(\sec 2t)^6(\sec 2t \tan 2t) \cdot D_t(2t) = 14 \sec^7 2t\, \tan 2t.$$

3.7.45: If $x = x(t) = t^7 \tan 5t$, then $\dfrac{dx}{dt} = 7t^6 \tan 5t + 5t^7 \sec^2 5t$.

3.7.46: If $x = x(t) = \dfrac{\sec t^5}{t}$, then

$$\frac{dx}{dt} = \frac{t \cdot (\sec t^5\, \tan t^5) \cdot 5t^4 - \sec t^5}{t^2} = \frac{5t^5 \sec t^5\, \tan t^5 - \sec t^5}{t^2}.$$

3.7.47: If $x = x(t) = \sqrt{t} \sec \sqrt{t} = t^{1/2} \sec(t^{1/2})$, then

$$\frac{dx}{dt} = \frac{1}{2}t^{-1/2} \sec(t^{1/2}) + t^{1/2}\left[\sec(t^{1/2}) \tan(t^{1/2})\right] \cdot \frac{1}{2}t^{-1/2} = \frac{\sec \sqrt{t} + \sqrt{t}\, \sec \sqrt{t}\, \tan \sqrt{t}}{2\sqrt{t}}.$$

3.7.48: If $x = x(t) = \sec \sqrt{t}\, \tan \sqrt{t} = \sec t^{1/2}\, \tan t^{1/2}$, then

$$\frac{dx}{dt} = \left(\sec t^{1/2}\right)\left(\frac{1}{2}t^{-1/2} \sec^2 t^{1/2}\right) + \left(\frac{1}{2}t^{-1/2} \sec t^{1/2}\, \tan t^{1/2}\right)\left(\tan t^{1/2}\right) = \frac{\sec^3 \sqrt{t} + \sec \sqrt{t}\, \tan^2 \sqrt{t}}{2\sqrt{t}}.$$

3.7.49: If $x = x(t) = \csc\left(\dfrac{1}{t^2}\right)$, then

$$\frac{dx}{dt} = \left[-\csc\left(\frac{1}{t^2}\right) \cot\left(\frac{1}{t^2}\right)\right] \cdot \left(-\frac{2}{t^3}\right) = \frac{2}{t^3} \csc\left(\frac{1}{t^2}\right) \cot\left(\frac{1}{t^2}\right).$$

3.7.50: If $x = x(t) = \cot\left(\frac{1}{\sqrt{t}}\right) = \cot t^{-1/2}$, then

$$\frac{dx}{dt} = -\left(\csc t^{-1/2}\right)^2 \cdot D_t\left(t^{-1/2}\right) = \frac{1}{2}t^{-3/2} \csc^2 t^{-1/2} = \frac{2}{t\sqrt{t}} \csc^2\left(\frac{1}{\sqrt{t}}\right).$$

3.7.51: If $x = x(t) = \dfrac{\sec 5t}{\tan 3t}$, then

$$\frac{dx}{dt} = \frac{5 \tan 3t \sec 5t \tan 5t - 3 \sec 5t \sec^2 3t}{(\tan 3t)^2} = 5 \cot 3t \sec 5t \tan 5t - 3 \csc^2 3t \sec 5t.$$

3.7.52: If $x = x(t) = \sec^2 t - \tan^2 t$, then $\dfrac{dx}{dt} = (2 \sec t)(\sec t \tan t) - (2 \tan t)(\sec^2 t) \equiv 0$.

3.7.53: If $x = x(t) = t \sec t \csc t$, then

$$\frac{dx}{dt} = \sec t\, \csc t + t \sec t\, \tan t\, \csc t - t \sec t\, \csc t\, \cot t = t \sec^2 t + \sec t\, \csc t - t \csc^2 t.$$

3.7.54: If $x = x(t) = t^3 \tan^3 t^3 = t^3(\tan t^3)^3$, then

$$\frac{dx}{dt} = 3t^2(\tan t^3)^3 + t^3 \cdot 3(\tan t^3)^2(\sec t^3)^2 \cdot 3t^2 = 3t^2 \tan^3 t^3 + 9t^5 \sec^2 t^3 \tan^2 t^3.$$

3.7.55: If $x = x(t) = \sec(\sin t)$, then $\dfrac{dx}{dt} = [\sec(\sin t)\,\tan(\sin t)] \cdot \cos t$.

3.7.56: If $x = x(t) = \cot(\sec 7t)$, then $\dfrac{dx}{dt} = \left[-\csc^2(\sec 7t)\right] \cdot 7 \sec 7t \tan 7t$.

3.7.57: If $x = x(t) = \dfrac{\sin t}{\sec t} = \sin t \cos t$, then $\dfrac{dx}{dt} = \cos^2 t - \sin^2 t = \cos 2t$.

3.7.58: If $x = x(t) = \dfrac{\sec t}{1 + \tan t}$, then

$$\frac{dx}{dt} = \frac{(1+\tan t)\sec t \tan t - \sec t \sec^2 t}{(1+\tan t)^2} = \frac{\sec t \tan t + \sec t \tan^2 t - (1+\tan^2 t)\sec t}{(1+\tan t)^2} = \frac{\sec t \tan t - \sec t}{(1+\tan t)^2}.$$

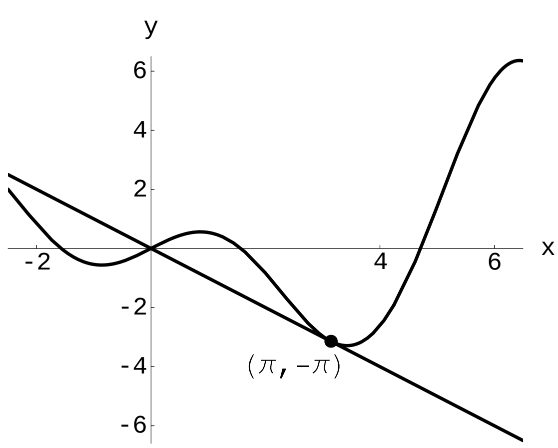
3.7.59: If $x = x(t) = \sqrt{1 + \cot 5t} = (1 + \cot 5t)^{1/2}$, then

$$\frac{dx}{dt} = \frac{1}{2}(1+\cot 5t)^{-1/2}(-5\csc^2 5t) = -\frac{5\csc^2 5t}{2\sqrt{1+\cot 5t}}.$$

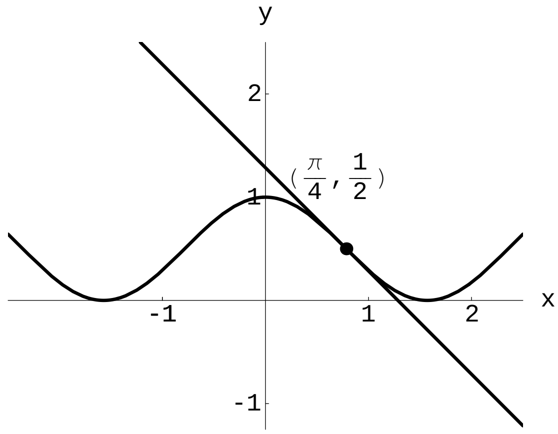
3.7.60: If $x = x(t) = \sqrt{\csc\sqrt{t}} = (\csc t^{1/2})^{1/2}$, then

$$\frac{dx}{dt} = \frac{1}{2}(\csc t^{1/2})^{-1/2}(-\csc t^{1/2}\cot t^{1/2}) \cdot \frac{1}{2}t^{-1/2} = -\frac{\left(\cot\sqrt{t}\right)\sqrt{\csc\sqrt{t}}}{4\sqrt{t}} = -\frac{\left(\csc\sqrt{t}\right)^{3/2}\cos\sqrt{t}}{4\sqrt{t}}.$$

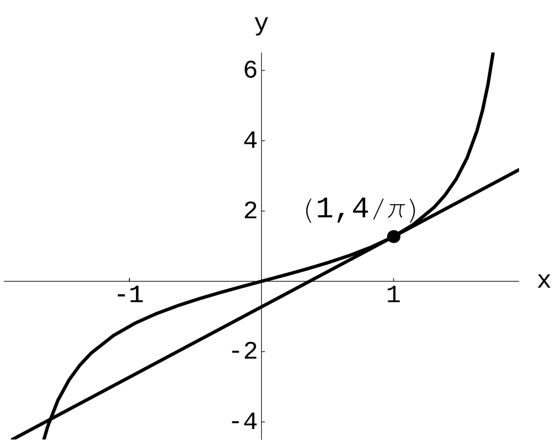
3.7.61: If $f(x) = x\cos x$, then $f'(x) = -x\sin x + \cos x$, so the slope of the tangent at $x = \pi$ is $f'(\pi) = -\pi \sin\pi + \cos\pi = -1$. Because $f(\pi) = -\pi$, an equation of the tangent line is $y + \pi = -(x - \pi)$; that is, $y = -x$. The graph of f and this tangent line are shown next.



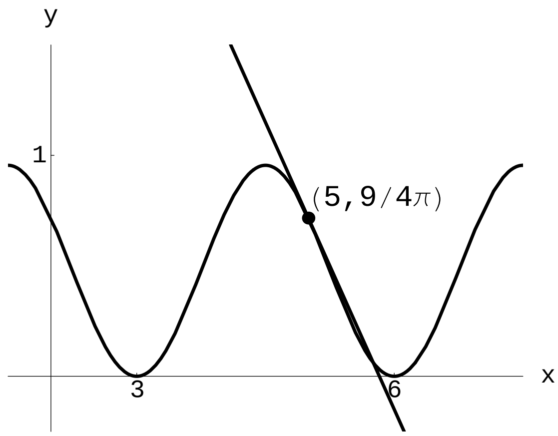
3.7.62: If $f(x) = \cos^2 x$ then $f'(x) = -2\cos x \sin x$, so the slope of the tangent at $x = \pi/4$ is $f'(\pi/4) = -2\cos(\pi/4)\sin(\pi/4) = -1$. Because $f(\pi/4) = \frac{1}{2}$, an equation of the tangent line is $y - \frac{1}{2} = -(x - \pi/4)$; that is, $4y = -4x + 2 + \pi$. The graph of f and this line are shown next.



3.7.63: If $f(x) = \dfrac{4}{\pi}\tan\left(\dfrac{\pi x}{4}\right)$, then $f'(x) = \sec^2\left(\dfrac{\pi x}{4}\right)$, so the slope of the tangent at $x = 1$ is $f'(1) = \sec^2\left(\dfrac{\pi}{4}\right) = 2$. Because $f(1) = \dfrac{4}{\pi}$, an equation of the tangent line is $y - \dfrac{4}{\pi} = 2\,(x-1)$; that is, $y = 2x - 2 + \dfrac{4}{\pi}$. The graph of f and this tangent line are shown next.



3.7.64: If $f(x) = \dfrac{3}{\pi}\sin^2\left(\dfrac{\pi x}{3}\right)$, then $f'(x) = 2\sin\left(\dfrac{\pi x}{3}\right)\cos\left(\dfrac{\pi x}{3}\right)$, so the slope of the tangent at $x = 5$ is $f'(5) = 2\sin\dfrac{5\pi}{3}\cos\dfrac{5\pi}{3} = -\frac{1}{2}\sqrt{3}$. Because $f(5) = \dfrac{9}{4\pi}$, an equation of the tangent line is $y - \dfrac{9}{4\pi} = -\frac{1}{2}\sqrt{3}\,(x-5)$; that is, $y = -\dfrac{x\sqrt{3}}{2} + \dfrac{9 + 10\pi\sqrt{3}}{4\pi}$. The graph of f and this tangent line are shown next.



3.7.65: $\dfrac{dy}{dx} = -2\sin 2x$. This derivative is zero at all values of x for which $\sin 2x = 0$; i.e., values of x for which $2x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Therefore the tangent line is horizontal at points with x-coordinate an integral multiple of $\frac{1}{2}\pi$. These are points of the form $(n\pi, 1)$ for any integer n and $\left(\frac{1}{2}m\pi, -1\right)$ for any odd integer m.

3.7.66: $\dfrac{dy}{dx} = 1 - 2\cos x$, which is zero for $x = \left(\frac{1}{3}\pi\right) + 2k\pi$ and for $x = -\left(\frac{1}{3}\pi\right) + 2k\pi$ for any integer k. The tangent line is horizontal at all points of the form $\Big(\pm\frac{1}{3}\pi + 2k\pi, y\left(\pm\frac{1}{3}\pi + 2k\pi\right)\Big)$ where k is an integer.

3.7.67: If $f(x) = \sin x \cos x$, then $f'(x) = \cos^2 x - \sin^2 x$. This derivative is zero at $x = \frac{1}{4}\pi + n\pi$ and at $x = \frac{3}{4}\pi + n\pi$ for any integer n. The tangent line is horizontal at all points of the form $\left(n\pi + \frac{1}{4}\pi, \frac{1}{2}\right)$ and at all points of the form $\left(n\pi + \frac{3}{4}\pi, -\frac{1}{2}\right)$ where n is an integer.

3.7.68: If

$$f(x) = \frac{1}{3\sin^2 x + 2\cos^2 x}, \qquad \text{then} \qquad f'(x) = -\frac{\sin 2x}{\left(2 + \sin^2 x\right)^2}.$$

This derivative is zero at all values of x for which $\sin 2x = 0$; i.e., values of x for which $2x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Therefore the tangent line is horizontal at points with x-coordinate an integral multiple of $\frac{1}{2}\pi$. These are points of the form $\left(n\pi, \frac{1}{2}\right)$ for any integer n and $\left(\frac{1}{2}m\pi, \frac{1}{3}\right)$ for any odd integer m.

3.7.69: Let $f(x) = x - 2\cos x$. Then $f'(x) = 1 + 2\sin x$, so $f'(x) = 1$ when $2\sin x = 0$; that is, when $x = n\pi$ for some integer n. Moreover, if n is an integer then $f(n\pi) = n\pi - 2\cos n\pi$, so $f(n\pi) = n\pi + 2$ if n is even and $f(n\pi) = n\pi - 2$ if n is odd. In particular, $f(0) = 2$ and $f(\pi) = \pi - 2$. So the two lines have equations $y = x + 2$ and $y = x - 2$, respectively.

3.7.70: If

$$f(x) = \frac{16 + \sin x}{3 + \sin x}, \qquad \text{then} \qquad f'(x) = -\frac{13\cos x}{(3 + \sin x)^2},$$

so that $f'(x) = 0$ when $\cos x = 0$; that is, when x is an odd integral multiple of $\frac{1}{2}\pi$. In particular,

$$f\left(\tfrac{1}{2}\pi\right) = \frac{16 + 1}{3 + 1} = \frac{17}{4};$$

similarly, $f\left(\frac{3}{4}\pi\right) = \frac{15}{2}$. Hence equations of the two lines are $y \equiv \frac{17}{4}$ and $y \equiv \frac{15}{2}$.

3.7.71: To derive the formulas for the derivatives of the cotangent, secant, and cosecant functions, express each in terms of sines and cosines and apply the quotient rule (or the reciprocal rule) and various trigonometric identities (see Appendix C). Thus

$$D_x \cot x = D_x \frac{\cos x}{\sin x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} - -\frac{1}{\sin^2 x} = -\csc^2 x,$$

$$D_x \sec x = D_x \frac{1}{\cos x} = -\frac{-\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \sec x \, \tan x, \qquad \text{and}$$

$$D_x \csc x = D_x \frac{1}{\sin x} = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x} = -\csc x \, \cot x.$$

3.7.72: If $g(x) = \cos x$, then

$$\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{\cos x \, \cos h - \sin x \, \sin h - \cos x}{h} \\
&= \lim_{h \to 0} \frac{1 - \cos h}{h}(-\cos x) - \lim_{h \to 0} \frac{\sin h}{h}(\sin x) \\
&= 0 \cdot (-\cos x) - 1 \cdot \sin x = -\sin x.
\end{aligned}$$

3.7.73: Write $R = R(\alpha) = \dfrac{1}{32}v^2 \sin 2\alpha$. Then

$$R'(\alpha) = \frac{1}{16}v^2 \cos 2\alpha,$$

which is zero when $\alpha = \pi/4$ (we assume $0 \leqq \alpha \leqq \pi/2$). Because R is zero at the endpoints of its domain, we conclude that $\alpha = \pi/4$ maximizes the range R.

3.7.74: Let h be the altitude of the balloon (in feet) at time t (in seconds) and let θ be its angle of elevation with respect to the observer. From the obvious figure, $h = 300 \tan \theta$, so

$$\frac{dh}{dt} = (300 \sec^2 \theta) \frac{d\theta}{dt}.$$

When $\theta = \pi/4$ and $\dfrac{d\theta}{dt} = \pi/180$, we have

$$\frac{dh}{dt} = 300 \cdot 2 \cdot \frac{\pi}{180} = \frac{10\pi}{3} \approx 10.47 \text{ (ft/s)}$$

as the rate of the balloon's ascent then.

3.7.75: Let h be the altitude of the rocket (in miles) at time t (in seconds) and let α be its angle of elevation then. From the obvious figure, $h = 2\tan\alpha$, so

$$\frac{dh}{dt} = (2\sec^2\alpha)\frac{d\alpha}{dt}.$$

When $\alpha = 5\pi/18$ and $d\alpha/dt = 5\pi/180$, we have $dh/dt \approx 0.4224$ (mi/s; about 1521 mi/h).

3.7.76: Draw a figure in which the airplane is located at (0, 25000) and the fixed point on the ground is located at $(x, 0)$. A line connecting the two produces a triangle with angle θ at $(x, 0)$. This angle is also the angle of depression of the pilot's line of sight, and when $\theta = 65°$, $d\theta/dt = 1.5°/\text{s}$. Now

$$\tan\theta = \frac{25000}{x}, \text{ so } x = 25000\frac{\cos\theta}{\sin\theta},$$

thus

$$\frac{dx}{d\theta} = -\frac{25000}{\sin^2\theta}.$$

The speed of the airplane is

$$-\frac{dx}{dt} = \frac{25000}{\sin^2\theta}\cdot\frac{d\theta}{dt}.$$

When $\theta = \dfrac{13}{36}\pi$, $\dfrac{d\theta}{dt} = \dfrac{\pi}{120}$. So the ground speed of the airplane is

$$\frac{25000}{\sin^2\left(\dfrac{13\pi}{36}\right)}\cdot\frac{\pi}{120} \approx 796.81 \text{ (ft/s)}.$$

Answer: About 543.28 mi/h.

3.7.77: Draw a figure in which the observer is located at the origin, the x-axis corresponds to the ground, and the airplane is located at $(x, 20000)$. The observer's line of sight corrects the origin to the point $(x, 20000)$ and makes an angle θ with the ground. Then

$$\tan\theta = \frac{20000}{x},$$

so that $x = 20000\cot\theta$. Thus

$$\frac{dx}{dt} = (-20000\csc^2\theta)\frac{d\theta}{dt}.$$

When $\theta = 60°$, we are given $\dfrac{d\theta}{dt} = 0.5°/\text{s}$; that is, $\dfrac{d\theta}{dt} = \pi/360$ radians per second when $\theta = \pi/3$. We evaluate dx/dt at this time with these values to obtain

$$\frac{dx}{dt} = (-20000)\frac{1}{\sin^2\left(\dfrac{\pi}{3}\right)}\cdot\frac{\pi}{360} = -\frac{2000\pi}{27},$$

approximately -232.71 ft/s. Answer: About 158.67 mi/h.

3.7.78: The area of the rectangle is $A = 4xy$, but $x = \cos\theta$ and $y = \sin\theta$, so

$$A = A(\theta) = 4\sin\theta\,\cos\theta, \qquad 0 \leqq \theta \leqq \pi/2.$$

Now $A'(\theta) = 4(\cos^2\theta - \sin^2\theta) = 4\cos 2\theta$, so $A'(\theta) = 0$ when $\cos 2\theta = 0$. Because $0 \leqq 2\theta \leqq \pi$, it follows that $2\theta = \pi/2$, so $\theta = \pi/4$. But $A(0) = 0 = A(\pi/2)$ and $A(\pi/4) = 2$, so the latter is the largest possible area of a rectangle inscribed in the unit circle.

3.7.79: The cross section of the trough is a trapezoid with short base 2, long base $2 + 4\cos\theta$, and height $2\sin\theta$. Thus its cross-sectional area is

$$\begin{aligned} A(\theta) &= \frac{2 + (2 + 4\cos\theta)}{2} \cdot 2\sin\theta \\ &= 4(\sin\theta + \sin\theta\cos\theta), \qquad 0 \leqq \theta \leqq \pi/2 \end{aligned}$$

(the real upper bound on θ is $2\pi/3$, but the maximum value of A clearly occurs in the interval $[0,\,\pi/2]$).

$$\begin{aligned} A'(\theta) &= 4(\cos\theta + \cos^2\theta - \sin^2\theta) \\ &= 4(2\cos^2\theta + \cos\theta - 1) \\ &= 4(2\cos\theta - 1)(\cos\theta + 1). \end{aligned}$$

The only solution of $A'(\theta) = 0$ in the given domain occurs when $\cos\theta = \frac{1}{2}$, so that $\theta = \frac{1}{3}\pi$. It is easy to verify that this value of θ maximizes the function A.

3.7.80: In the situation described in the problem, we have $D = 20\sec\theta$. The illumination of the walkway is then

$$\begin{aligned} I = I(\theta) &= \frac{k}{400}\sin\theta\,\cos^2\theta, \qquad 0 \leqq \theta \leqq \pi/2. \\ \frac{dI}{d\theta} &= \frac{k\cos\theta}{400}(\cos^2\theta - 2\sin^2\theta); \end{aligned}$$

$dI/d\theta = 0$ when $\theta = \pi/2$ and when $\cos^2\theta = 2\sin^2\theta$. The solution θ in the domain of I of the latter equation has the property that $\sin\theta = \sqrt{3}/3$ and $\cos\theta = \sqrt{6}/3$. But $I(0) = 0$ and $I(\theta) \to 0$ as $\theta \to (\pi/2)^-$, so the optimal height of the lamp post occurs when $\sin\theta = \sqrt{3}/3$. This implies that the optimal height is $10\sqrt{2} \approx 14.14$ m.

3.7.81: The following figure shows a cross section of the sphere-with-cone through the axis of the cone and a diameter of the sphere. Note that $h = r\tan\theta$ and that

$$\cos\theta = \frac{R}{h - R}.$$

Therefore

$$h = R + R\sec\theta, \qquad \text{and thus} \qquad r = \frac{R + R\sec\theta}{\tan\theta}.$$

Now $V = \frac{1}{3}\pi r^2 h$, so for θ in the interval $(0,\ \pi/2)$, we have

$$V = V(\theta) = \frac{1}{3}\pi R^3 \cdot \frac{(1+\sec\theta)^3}{\tan^2\theta}.$$

Therefore

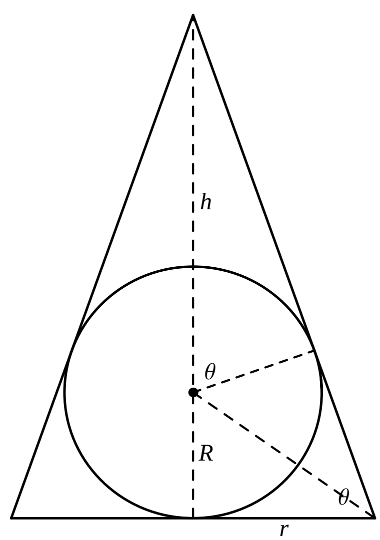
$$V'(\theta) = \frac{\pi R^3}{3\tan^4\theta}\left[3(\tan^2\theta)(1+\sec\theta)^2\sec\theta\ \tan\theta - (1+\sec\theta)^3(2\tan\theta\ \sec^2\theta)\right].$$

If $V'(\theta) = 0$ then either $\sec\theta = -1$ (so $\theta = \pi$, which we reject), or $\sec\theta = 0$ (which has no solutions), or $\tan\theta = 0$ (so either $\theta = 0$ or $\theta = \pi$, which we also reject), or (after replacement of $\tan^2\theta$ with $\sec^2\theta - 1$)

$$3(\sec^2\theta - 1) - 2(\sec\theta)(1+\sec\theta) = 0;$$

$$\sec^2\theta - 2\sec\theta - 3 = 0.$$

It follows that $\sec\theta = 3$ or $\sec\theta = -1$. We reject the latter as before, and find that $\sec\theta = 3$, so $\theta \approx 1.23095$ (radians). The resulting minimum volume of the cone is $\frac{8}{3}\pi R^3$, twice the volume of the sphere!



3.7.82: Let L be the length of the crease. Then the right triangle of which L is the hypotenuse has sides $L\cos\theta$ and $L\sin\theta$. Now $20 = L\sin\theta + L\sin\theta\ \cos 2\theta$, so

$$L = L(\theta) = \frac{20}{(\sin\theta)(1+\cos 2\theta)}, \qquad 0 < \theta \leqq \frac{\pi}{4}.$$

Next, $\dfrac{dL}{d\theta} = 0$ when

$$(\cos\theta)(1+\cos 2\theta) = (\sin\theta)(2\sin 2\theta);$$

$$(\cos\theta)(2\cos^2\theta) = 4\sin^2\theta\ \cos\theta;$$

so $\cos\theta = 0$ (which is impossible given the domain of L) or

$$\cos^2\theta = 2\sin^2\theta = 2 - 2\cos^2\theta; \qquad \cos^2\theta = \frac{2}{3}.$$

This implies that $\cos\theta = \frac{1}{3}\sqrt{6}$ and $\sin\theta = \frac{1}{3}\sqrt{3}$. Because $L \to +\infty$ as $\theta \to 0^+$, we have a minimum either at the horizontal tangent just found or at the endpoint $\theta = \frac{1}{4}\pi$. The value of L at $\frac{1}{4}\pi$ is $20\sqrt{2} \approx 28.28$ and at the horizontal tangent we have $L = 15\sqrt{3} \approx 25.98$. So the shortest crease is obtained when $\cos\theta = \frac{1}{3}\sqrt{6}$; that is, for θ approximately $35^\circ 15'\ 52''$. The bottom of the crease should be one-quarter of the way across the page from the lower left-hand corner.

3.7.83: Set up coordinates so the diameter is on the x-axis and the equation of the circle is $x^2 + y^2 = 1$; let $(x\,, y)$ denote the northwest corner of the trapezoid. The chord from $(1,\ 0)$ to $(x,\ y)$ forms a right triangle with hypotenuse 2, side z opposite angle θ, and side w; moreover, $z = 2\sin\theta$ and $w = 2\cos\theta$. It follows that

$$y = w\sin\theta = 2\sin\theta\,\cos\theta \qquad \text{and}$$

$$-x = 1 - w\cos\theta = -\cos 2\theta.$$

Now

$$A = y(1-x) = (2\sin\theta\cos\theta)(1-\cos 2\theta) = 4\sin\theta\cos\theta\sin^2\theta,$$

and therefore

$$A = A(\theta) = 4\sin^3\theta\,\cos\theta, \qquad \frac{\pi}{4} \leqq \theta \leqq \frac{\pi}{2}.$$

$$A'(\theta) = 12\sin^2\theta\,\cos^2\theta - 4\sin^4\theta$$

$$= (4\sin^2\theta)(3\cos^2\theta - \sin^2\theta).$$

To solve $A'(\theta) = 0$, we note that $\sin\theta \neq 0$, so we must have $3\cos^2\theta = \sin^2\theta$; that is $\tan^2\theta = 3$. It follows that $\theta = \frac{1}{3}\pi$. The value of A here exceeds its value at the endpoints, so we have found the maximum value of the area—it is $\frac{3}{4}\sqrt{3}$.

3.7.84: Let $\theta = \alpha/2$ (see Fig. 3.7.18 of the text) and denote the radius of the circular log by r. Using the technique of the solution of Problem 82, we find that the area of the hexagon is

$$A = A(\theta) = 8r^2\sin^3\theta\,\cos\theta, \qquad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

After some simplifications we also find that

$$\frac{dA}{d\theta} = 8r^2(\sin^2\theta)(4\cos^2\theta - 1).$$

Now $dA/d\theta = 0$ when $\sin\theta = 0$ and when $\cos\theta = \frac{1}{2}$. When $\sin\theta = 0$, $A = 0$; also, $A(0) = 0 = A\left(\frac{1}{2}\pi\right)$. Therefore A is maximal when $\cos\theta = \frac{1}{2}$: $\theta = \frac{1}{3}\pi$. When this happens, we find that $\alpha = \frac{2}{3}\pi$ and that $\beta = \pi - \theta = \frac{2}{3}\pi$. Therefore the figure of maximal area is a regular hexagon.

3.7.85: The area in question is the area of the sector minus the area of the triangle in Fig. 3.7.19 and turns out to be

$$\begin{aligned} A &= \frac{1}{2}r^2\theta - r^2\cos\frac{\theta}{2}\,\sin\frac{\theta}{2} \\ &= \frac{1}{2}r^2(\theta - \sin\theta) = \frac{s^2(\theta - \sin\theta)}{2\theta^2} \end{aligned}$$

because $s = r\theta$. Now

$$\frac{dA}{d\theta} = \frac{s^2(2\sin\theta - \theta\cos\theta - \theta)}{2\theta^3},$$

so $dA/d\theta = 0$ when $\theta(1 + \cos\theta) = 2\sin\theta$. Let $\theta = 2x$; note that $0 < x \leqq \pi$ because $0 < \theta \leqq 2\pi$. So the condition that $dA/d\theta = 0$ becomes

$$x = \frac{\sin\theta}{1 + \cos\theta} = \tan x.$$

But this equation has no solution in the interval $(0, \pi]$. So the only possible maximum of A must occur at an endpoint of its domain, or where x is undefined because the denominator $1 + \cos\theta$ is zero—and this occurs when $\theta = \pi$. Finally,

$$A(2\pi) = \frac{s^2}{4\pi} \qquad \text{and} \qquad A(\pi) = \frac{s^2}{2\pi},$$

so the maximum area is attained when the arc is a semicircle.

3.7.86: The length of the forest path is $2\csc\theta$. So the length of the part of the trip along the road is $3 - 2\csc\theta\,\cos\theta$. Thus the total time for the trip is given by

$$T = T(\theta) = \frac{2}{3\sin\theta} + \frac{3 - \dfrac{2\cos\theta}{\sin\theta}}{8}.$$

Note that the range of values of θ is determined by the condition

$$\frac{3\sqrt{13}}{13} \geqq \cos\theta \geqq 0.$$

After simplifications, we find that

$$T'(\theta) = \frac{3 - 8\cos\theta}{12\sin^2\theta}.$$

Now $T'(\theta) = 0$ when $\cos\theta = \frac{3}{8}$; that is, when θ is approximately $67°58'\,32''$. For this value of θ, we find that $\sin\theta = \frac{1}{8}\sqrt{55}$. There's no problem in verifying that we have found the minimum. Answer: The distance to walk down the road is

$$\left(3 - 2\frac{\cos\theta}{\sin\theta}\right)\Bigg|_{\sin\theta = \frac{1}{8}\sqrt{55}} = 3 - \frac{6\sqrt{55}}{55} \approx 2.19096 \text{ (km)}.$$

3.7.87: Following the *Suggestion*, we note that if n is a positive integer and

$$h = \frac{2}{(4n+1)\pi},$$

then

$$\frac{f(h) - f(0)}{h} = \frac{(4n+1)\pi \sin \frac{1}{2}(4n+1)\pi}{(4n+1)\pi} = 1,$$

and that if

$$h = \frac{2}{(4n-1)\pi},$$

then

$$\frac{f(h) - f(0)}{h} = \frac{(4n-1)\pi \sin \frac{1}{2}(4n-1)\pi}{(4n-1)\pi} = -1.$$

Therefore there are values of h arbitrarily close to zero for which

$$\frac{f(0+h) - f(0)}{h} = +1$$

and values of h arbitrarily close to zero for which

$$\frac{f(0+h) - f(0)}{h} = -1.$$

It follows that

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \quad \text{does not exist;}$$

that is, $f'(0)$ does not exist, and so f is not differentiable at $x = 0$.

3.7.88: $f'(0) = \lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \dfrac{h^2 \sin \dfrac{1}{h}}{h} = \lim_{h \to 0} h \sin \dfrac{1}{h}.$

It now follows from the Squeeze Law of Section 2.3 that

$$f'(0) = \lim_{h \to 0} h \sin \frac{1}{h} = 0$$

because $-|h| \leqq h \sin \dfrac{1}{h} \leqq |h|$ if $h \neq 0$.

Section 3.8

3.8.1: If $f(x) = e^{2x}$, then $f'(x) = e^{2x} \cdot D_x(2x) = 2e^{2x}$.

3.8.2: If $f(x) = e^{3x-1}$, then $f'(x) = e^{3x-1} \cdot D_x(3x-1) = 3e^{3x-1}$.

3.8.3: If $f(x) = \exp(x^2)$, then $f'(x) = \left[\exp(x^2)\right] \cdot D_x(x^2) = 2x\exp(x^2)$.

3.8.4: If $f(x) = e^{4-x^3}$, then $f'(x) = e^{4-x^3} \cdot D_x(4-x^3) = -3x^2e^{4-x^3}$.

3.8.5: If $f(x) = e^{1/x^2}$, then $f'(x) = e^{1/x^2} \cdot D_x(1/x^2) = -\dfrac{2}{x^3}\,e^{1/x^2}$.

3.8.6: If $f(x) = x^2\exp(x^3)$, then $f'(x) = 2x\exp(x^3) + x^2 \cdot 3x^2\exp(x^3) = (2x + 3x^4)\exp(x^3)$.

3.8.7: If $g(t) = t\exp(t^{1/2})$, then $g'(t) = \exp(t^{1/2}) + t \cdot \dfrac{1}{2}t^{-1/2}\exp(t^{1/2}) = \dfrac{2+\sqrt{t}}{2}\exp(t^{1/2})$.

3.8.8: If $g(t) = (e^{2t} + e^{3t})^7$, then $g'(t) = 7(e^{2t} + e^{3t})^6(2e^{2t} + 3e^{3t})$.

3.8.9: If $g(t) = (t^2-1)e^{-t}$, then $g'(t) = 2te^{-t} - (t^2-1)e^{-t} = (1 + 2t - t^2)e^{-t}$.

3.8.10: If $g(t) = (e^t - e^{-t})^{1/2}$, then $g'(t) = \dfrac{1}{2}(e^t - e^{-t})^{-1/2}(e^t + e^{-t})$.

3.8.11: If $g(t) = e^{\cos t} = \exp(\cos t)$, then $g'(t) = (-\sin t)\exp(\cos t)$.

3.8.12: If $f(x) = xe^{\sin x} = x\exp(\sin x)$, then

$$f'(x) = \exp(\sin x) + (x\cos x)\exp(\sin x) = e^{\sin x}(1 + x\cos x).$$

3.8.13: If $g(t) = \dfrac{1-e^{-t}}{t}$, then $g'(t) = \dfrac{te^{-t} - (1-e^{-t})}{t^2} = \dfrac{te^{-t} + e^{-t} - 1}{t^2}$.

3.8.14: If $f(x) = e^{-1/x}$, then $f'(x) = \dfrac{1}{x^2}\,e^{-1/x}$.

3.8.15: If $f(x) = \dfrac{1-x}{e^x}$, then

$$f'(x) = \frac{(-1)e^x - (1-x)e^x}{(e^x)^2} = \frac{-1-1+x}{e^x} = \frac{x-2}{e^x}.$$

3.8.16: If $f(x) = \exp\left(\sqrt{x}\,\right) + \exp\left(-\sqrt{x}\,\right)$, then

$$f'(x) = \frac{1}{2}x^{-1/2}\exp\left(\sqrt{x}\,\right) - \frac{1}{2}x^{-1/2}\exp\left(-\sqrt{x}\,\right) = \frac{\exp\left(\sqrt{x}\,\right) - \exp\left(-\sqrt{x}\,\right)}{2\sqrt{x}}.$$

3.8.17: If $f(x) = \exp(e^x)$, then $f'(x) = e^x\exp(e^x)$.

3.8.18: If $f(x) = \left(e^{2x} + e^{-2x}\right)^{1/2}$, then

$$f'(x) = \frac{1}{2}\left(e^{2x} + e^{-2x}\right)^{-1/2}\left(2e^{2x} - 2e^{-2x}\right) = \frac{e^{2x} - e^{-2x}}{\sqrt{e^{2x} + e^{-2x}}}.$$

3.8.19: If $f(x) = \sin(2e^x)$, then $f'(x) = 2e^x \cos(2e^x)$.

3.8.20: If $f(x) = \cos(e^x + e^{-x})$, then $f'(x) = (e^{-x} - e^x)\sin(e^x + e^{-x})$.

3.8.21: If $f(x) = \ln(3x-1)$, then $f'(x) = \dfrac{1}{3x-1} \cdot D_x(3x-1) = \dfrac{3}{3x-1}$.

3.8.22: If $f(x) = \ln(4-x^2)$, then $f'(x) = \dfrac{2x}{x^2-4}$.

3.8.23: If $f(x) = \ln\big[(1+2x)^{1/2}\big]$, then $f'(x) = \dfrac{\frac{1}{2}\cdot 2(1+2x)^{-1/2}}{(1+2x)^{1/2}} = \dfrac{1}{1+2x}$.

3.8.24: If $f(x) = \ln\big[(1+x)^2\big]$, then $f'(x) = \dfrac{2(1+x)}{(1+x)^2} = \dfrac{2}{1+x}$.

3.8.25: If $f(x) = \ln\big[(x^3-x)^{1/3}\big] = \dfrac{1}{3}\ln(x^3-x)$, then $f'(x) = \dfrac{3x^2-1}{3(x^3-x)}$.

3.8.26: If $f(x) = \ln\big[(\sin x)^2\big] = 2\ln(\sin x)$, then $f'(x) = \dfrac{2\cos x}{\sin x} = 2\cot x$.

3.8.27: If $f(x) = \cos(\ln x)$, then $f'(x) = -\dfrac{\sin(\ln x)}{x}$.

3.8.28: If $f(x) = (\ln x)^3$, then $f'(x) = \dfrac{3(\ln x)^2}{x}$.

3.8.29: If $f(x) = \dfrac{1}{\ln x}$, then (by the reciprocal rule) $f'(x) = -\dfrac{1}{x(\ln x)^2}$.

3.8.30: If $f(x) = \ln(\ln x)$, then $f'(x) = \dfrac{1}{x\ln x}$.

3.8.31: If $f(x) = \ln\big[x(x^2+1)^{1/2}\big]$, then

$$f'(x) = \frac{(x^2+1)^{1/2} + x^2(x^2+1)^{-1/2}}{x(x^2+1)^{1/2}} = \frac{2x^2+1}{x(x^2+1)}.$$

3.8.32: If $g(t) = t^{3/2}\ln(t+1)$, then

$$g'(t) = \frac{3}{2}t^{1/2}\ln(t+1) + \frac{t^{3/2}}{t+1} = \frac{t^{1/2}\,[\,2t + 3\ln(t+1) + 3t\ln(t+1)\,]}{2(t+1)}.$$

3.8.33: If $f(x) = \ln\cos x$, then $f'(x) = \dfrac{-\sin x}{\cos x} = -\tan x$.

3.8.34: If $f(x) = \ln(2\sin x) = (\ln 2) + \ln(\sin x)$, then $f'(x) = \dfrac{\cos x}{\sin x} = \cot x$.

3.8.35: If $f(t) = t^2\ln(\cos t)$, then $f'(t) = 2t\ln(\cos t) - \dfrac{t^2\sin t}{\cos t} = t\,[\,2\ln(\cos t) - t\tan t\,]$.

3.8.36: If $f(x) = \sin(\ln 2x)$, then $f'(x) = [\cos(\ln 2x)] \cdot \dfrac{2}{2x} = \dfrac{\cos(\ln 2x)}{x}$.

3.8.37: If $g(t) = t(\ln t)^2$, then

$$g'(t) = (\ln t)^2 + t \cdot \frac{2\ln t}{t} = (2 + \ln t)\ln t.$$

3.8.38: If $g(t) = t^{1/2}\,[\cos(\ln t)]^2$, then

$$g'(t) = \frac{1}{2}t^{-1/2}\,[\cos(\ln t)]^2 + 2t^{1/2}\,[\cos(\ln t)] \cdot \frac{-\sin(\ln t)}{t} = \frac{[\cos(\ln t)]\,[\cos(\ln t) - 4\sin(\ln t)]}{2t^{1/2}}.$$

3.8.39: Because $f(x) = 3\ln(2x+1) + 4\ln(x^2-4)$, we have

$$f'(x) = \frac{6}{2x+1} + \frac{8x}{x^2-4} = \frac{22x^2+8x-24}{(2x+1)(x^2-4)}.$$

3.8.40: If

$$f(x) = \ln\left(\frac{1-x}{1+x}\right)^{1/2} = \frac{1}{2}\ln(1-x) - \frac{1}{2}\ln(1+x),$$

then

$$f'(x) = -\frac{1}{2(1-x)} - \frac{1}{2(1+x)} = \frac{1}{(x+1)(x-1)}.$$

3.8.41: If

$$f(x) = \ln\left(\frac{4-x^2}{9+x^2}\right)^{1/2} = \frac{1}{2}\ln(4-x^2) - \frac{1}{2}\ln(9+x^2),$$

then

$$f'(x) = -\frac{x}{4-x^2} - \frac{x}{9+x^2} = \frac{13x}{(x^2-4)(x^2+9)}.$$

3.8.42: If

$$f(x) = \ln\frac{\sqrt{4x-7}}{(3x-2)^3} = \frac{1}{2}\ln(4x-7) - 3\ln(3x-2),$$

then

$$f'(x) = \frac{2}{4x-7} - \frac{9}{3x-2} = \frac{59-30x}{(3x-2)(4x-7)}.$$

3.8.43: If

$$f(x) = \ln\frac{x+1}{x-1} = \ln(x+1) - \ln(x-1), \quad \text{then} \quad f'(x) = \frac{1}{x+1} - \frac{1}{x-1} = -\frac{2}{(x-1)(x+1)}.$$

3.8.44: If

$$f(x) = x^2\ln\frac{1}{2x+1} = -x^2\ln(2x+1), \quad \text{then} \quad f'(x) = -\frac{2x^2}{2x+1} - 2x\ln(2x+1).$$

3.8.45: If

$$g(t) = \ln \frac{t^2}{t^2+1} = 2\ln t - \ln(t^2+1), \quad \text{then} \quad g'(t) = \frac{2}{t} - \frac{2t}{t^2+1} = \frac{2}{t(t^2+1)}.$$

3.8.46: If

$$f(x) = \ln \frac{\sqrt{x+1}}{(x-1)^3} = \frac{1}{2}\ln(x+1) - 3\ln(x-1), \quad \text{then} \quad f'(x) = \frac{1}{2(x+1)} - \frac{3}{x-1} = -\frac{5x+7}{2(x-1)(x+1)}.$$

3.8.47: Given: $y = 2^x$. Then

$$\ln y = \ln\left(2^x\right) = x\ln 2;$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \ln 2;$$

$$\frac{dy}{dx} = y(x)\cdot\ln 2 = 2^x\ln 2.$$

3.8.48: Given: $y = x^x$. Then

$$\ln y = \ln\left(x^x\right) = x\ln x;$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1 + \ln x;$$

$$\frac{dy}{dx} = y(x)\cdot(1+\ln x) = x^x(1+\ln x).$$

3.8.49: Given: $y = x^{\ln x}$. Then

$$\ln y = \ln\left(x^{\ln x}\right) = (\ln x)\cdot(\ln x) = (\ln x)^2;$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{2\ln x}{x};$$

$$\frac{dy}{dx} = y(x)\cdot\frac{2\ln x}{x} = \frac{2x^{\ln x}\ln x}{x}.$$

3.8.50: Given: $y = (1+x)^{1/x}$. Then

$$\ln y = \ln(1+x)^{1/x} = \frac{1}{x}\ln(1+x);$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x(1+x)} - \frac{\ln(1+x)}{x^2} = \frac{x - \ln(1+x) - x\ln(1+x)}{x^2(1+x)};$$

$$\frac{dy}{dx} = y(x)\cdot\frac{x - \ln(1+x) - x\ln(1+x)}{x^2(1+x)} = \frac{[\,x - \ln(1+x) - x\ln(1+x)\,]\cdot(1+x)^{1/x}}{x^2(1+x)}.$$

3.8.51: Given: $y = (\ln x)^{\sqrt{x}}$. Then

$$\ln y = \ln (\ln x)^{\sqrt{x}} = x^{1/2} \ln (\ln x);$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2} x^{-1/2} \ln (\ln x) + \frac{x^{1/2}}{x \ln x};$$

$$\frac{dy}{dx} = y(x) \cdot \left[\frac{\ln (\ln x)}{2x^{1/2}} + \frac{1}{x^{1/2} \ln x} \right];$$

$$\frac{dy}{dx} = \frac{2 + (\ln x) \ln (\ln x)}{2x^{1/2} \ln x} \cdot (\ln x)^{\sqrt{x}}.$$

3.8.52: Given: $y = (3 + 2^x)^x$. Then

$$\ln y = \ln (3 + 2^x)^x = x \ln (3 + 2^x);$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x \cdot 2^x \cdot \ln 2}{3 + 2^x} + \ln (3 + 2^x);$$

$$\frac{dy}{dx} = y(x) \cdot \frac{x \cdot 2^x \cdot (\ln 2) + 3 \ln (3 + 2^x) + 2^x \ln (3 + 2^x)}{3 + 2^x};$$

$$\frac{dy}{dx} = \frac{x \cdot 2^x \cdot (\ln 2) + 3 \ln (3 + 2^x) + 2^x \ln (3 + 2^x)}{3 + 2^x} \cdot (3 + 2^x)^x.$$

3.8.53: If $y = (1 + x^2)^{3/2}(1 + x^3)^{-4/3}$, then

$$\ln y = \frac{3}{2} \ln(1 + x^2) - \frac{4}{3} \ln(1 + x^3);$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{3x}{1 + x^2} - \frac{4x^2}{1 + x^3} = \frac{3x - 4x^2 - x^4}{(1 + x^2)(1 + x^3)};$$

$$\frac{dy}{dx} = y(x) \cdot \frac{3x - 4x^2 - x^4}{(1 + x^2)(1 + x^3)} = \frac{3x - 4x^2 - x^4}{(1 + x^2)(1 + x^3)} \cdot \frac{(1 + x^2)^{3/2}}{(1 + x^3)^{4/3}};$$

$$\frac{dy}{dx} = \frac{(3x - 4x^2 - x^4)(1 + x^2)^{1/2}}{(1 + x^3)^{7/3}}.$$

3.8.54: If $y = (x + 1)^x$, then

$$\ln y = \ln(x + 1)^x = x \ln(x + 1);$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{x + 1} + \ln(x + 1);$$

$$\frac{dy}{dx} = \left[\frac{x}{x + 1} + \ln(x + 1) \right] \cdot (x + 1)^x.$$

3.8.55: If $y = (x^2+1)^{x^2}$, then

$$\ln y = \ln(x^2+1)^{x^2} = x^2 \ln(x^2+1);$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{2x^3}{x^2+1} + 2x\ln(x^2+1);$$

$$\frac{dy}{dx} = y(x) \cdot \left[\frac{2x^3}{x^2+1} + 2x\ln(x^2+1)\right] = \left[\frac{2x^3}{x^2+1} + 2x\ln(x^2+1)\right] \cdot (x^2+1)^{x^2}.$$

3.8.56: If $y = \left(1 + \dfrac{1}{x}\right)^x$, then

$$\ln y = \ln\left(1+\frac{1}{x}\right)^x = x\ln\left(1+\frac{1}{x}\right) = x\ln(x+1) - x\ln x;$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{x+1} + \ln(x+1) - 1 - \ln x;$$

$$\frac{dy}{dx} = \left[\frac{x}{x+1} + \ln(x+1) - 1 - \ln x\right] \cdot \left(1+\frac{1}{x}\right)^x.$$

3.8.57: Given: $y = (\sqrt{x}\,)^{\sqrt{x}}$. Then

$$\ln y = \ln\left(\sqrt{x}\,\right)^{\sqrt{x}} = x^{1/2}\ln\left(x^{1/2}\right) = \frac{1}{2}x^{1/2}\ln x;$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{2x^{1/2}} + \frac{\ln x}{4x^{1/2}} = \frac{2+\ln x}{4\sqrt{x}};$$

$$\frac{dy}{dx} = \frac{(2+\ln x)\left(\sqrt{x}\,\right)^{\sqrt{x}}}{4\sqrt{x}}.$$

3.8.58: If $y = x^{\sin x}$, then

$$\ln y = (\sin x)\ln x;$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{\sin x}{x} + (\cos x)\ln x;$$

$$\frac{dy}{dx} = \frac{\sin x + x(\cos x)\ln x}{x} \cdot \left(x^{\sin x}\right).$$

3.8.59: If $f(x) = xe^{2x}$, then $f'(x) = e^{2x} + 2xe^{2x}$, so the slope of the graph of $y = f(x)$ at $(1,\, e^2)$ is $f'(1) = 3e^2$. Hence an equation of the line tangent to the graph at that point is $y - e^2 = 3e^2(x-1)$; that is, $y = 3e^2x - 2e^2$.

3.8.60: If $f(x) = e^{2x}\cos x$, then $f'(x) = 2e^{2x}\cos x - e^{2x}\sin x$, so the slope of the graph of $y = f(x)$ at the point $(0,\, 1)$ is $f'(0) = 2$. So an equation of the line tangent to the graph at that point is $y - 1 = 2(x-0)$; that is, $y = 2x + 1$.

3.8.61: If $f(x) = x^3 \ln x$, then $f'(x) = x^2 + 3x^2 \ln x$, so the slope of the graph of $y = f(x)$ at the point $(1,\, 0)$ is $f'(1) = 1$. Hence an equation of the line tangent to the graph at that point is $y - 0 = 1 \cdot (x-1)$; that is, $y = x - 1$.

3.8.62: If

$$f(x) = \frac{\ln x}{x^2}, \qquad \text{then} \qquad f'(x) = \frac{x - 2x\ln x}{x^4} = \frac{1 - 2\ln x}{x^3}.$$

Hence the slope of the graph of $y = f(x)$ at the point $\left(e,\, e^{-2}\right)$ is $f'(e) = -1/e^3$. Therefore an equation of the line tangent to the graph at that point is

$$y - \frac{1}{e^2} = -\frac{1}{e^3}(x - e); \qquad \text{that is,} \qquad y = \frac{2e - x}{e^3}.$$

3.8.63: If $f(x) = e^{2x}$, then

$$f'(x) = 2e^{2x}, \quad f''(x) = 4e^{2x}, \quad f'''(x) = 8e^{2x}, \quad f^{(4)}(x) = 16e^{2x}, \quad \text{and} \quad f^{(5)}(x) = 32e^{2x}.$$

It appears that $f^{(n)}(x) = 2^n e^{2x}$.

3.8.64: If $f(x) = xe^x$, then

$$f'(x) = (x+1)e^x, \quad f''(x) = (x+2)e^x, \quad f'''(x) = (x+3)e^x, \quad f^{(4)}(x) = (x+4)e^x, \quad \text{and} \quad f^{(5)}(x) = (x+5)e^x.$$

It appears that $f^{(n)}(x) = (x+n)e^x$.

3.8.65: If $f(x) = e^{-x/6}\sin x$, then

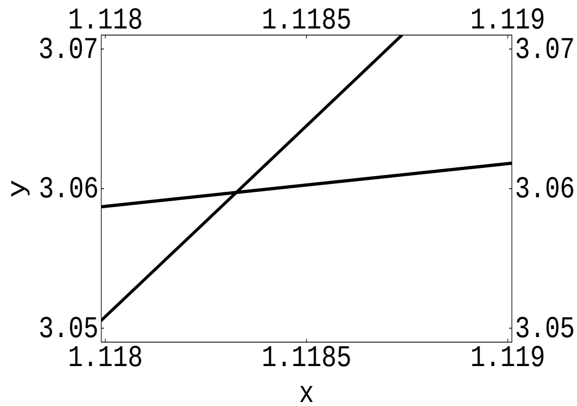
$$f'(x) = -\frac{1}{6}e^{-x/6}\sin x + e^{-x/6}\cos x = \frac{6\cos x - \sin x}{6e^{x/6}}.$$

Hence the first local maximum point for $x > 0$ occurs when $x = \arctan 6$ and the first local minimum point occurs when $x = \pi + \arctan 6$. The corresponding y-coordinates are, respectively,

$$\frac{6}{e^{(\arctan 6)/6}\sqrt{37}} \qquad \text{and} \qquad -\frac{6}{e^{(\pi + \arctan 6)/6}\sqrt{37}}.$$

3.8.66: Given $f(x) = e^{-x/6}\sin x$, let $g(x) = e^{-x/6}$ and $h(x) = -e^{-x/6}$. We solve the equation $f(x) = g(x)$ by hand; the x-coordinate of the first point of tangency is $\pi/2$. Similarly, the x-coordinate of the second point of tangency is $3\pi/2$. These are *not* the same as $\arctan 6$ and $\pi + \arctan 6$.

3.8.67: The figure below shows the intersection of the two graphs in the viewing window $1.118 \leqq x \leqq 1.119$, $3.05 \leqq y \leqq 3.07$. We see that, accurate to three decimal places, the indicated solution of $e^x = x^{10}$ is $x = 1.118$.



3.8.68: The viewing rectangle with $35.771515 \leqq x \leqq 35.771525$ reveals a solution of $e^x = x^{10}$ near $x = 35.75152$. Therefore this solution is approximately 3.58×10^1. Newton's method applied to $f(x) = e^x - x^{10}$ reveals the more accurate approximation 35.7715206396.

3.8.69: We first let

$$f(k) = \left(1 + \frac{1}{10^k}\right)^{10^k}.$$

Then *Mathematica* yields the following approximations:

k	$f(k)$ (rounded)
1	2.593742460100
2	2.704813829422
3	2.716923932236
4	2.718145926825
5	2.718268237174
6	2.718280469319
7	2.718281692545
8	2.718281814868
9	2.718281827100
10	2.718281828323
11	2.718281828445
12	2.718281828458
13	2.718281828459
14	2.718281828459

15	2.718281828459
16	2.718281828459045099
17	2.719291929459045222
18	2.718281828459045234
19	2.718281828459045235
20	2.718281828459045235
21	2.718281828459045235

3.8.70: If $y = u^v$ where all are differentiable functions of x, then $\ln y = v \ln u$. With $u'(x)$ denoted simply by u', etc., we now have

$$\frac{1}{y}y' = v' \ln u + \frac{vu'}{u}.$$

Thus $y' = u^v\, v' \ln u + \dfrac{u^v vu'}{u} = vu^{v-1}u' + u^v(\ln u)v'$.

(a) If u is constant, this implies that $\dfrac{dy}{dx} = u^{v(x)}(\ln u)v'(x)$.

(b) If v is constant, this implies that $\dfrac{dy}{dx} = v\,(u(x))^{v-1}\,u'(x)$.

3.8.71: Solution:

$$\ln y = \ln u + \ln v + \ln w - \ln p - \ln q - \ln r;$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\cdot\frac{dv}{dx} + \frac{1}{w}\cdot\frac{dw}{dx} - \frac{1}{p}\cdot\frac{dp}{dx} - \frac{1}{q}\cdot\frac{dq}{dx} - \frac{1}{r}\cdot\frac{dr}{dx};$$

$$\frac{dy}{dx} = y\cdot\left(\frac{1}{u}\cdot\frac{du}{dx} + \frac{1}{v}\cdot\frac{dv}{dx} + \frac{1}{w}\cdot\frac{dw}{dx} - \frac{1}{p}\cdot\frac{dp}{dx} - \frac{1}{q}\cdot\frac{dq}{dx} - \frac{1}{r}\cdot\frac{dr}{dx}\right).$$

The solution makes the generalization obvious.

3.8.72: Suppose by way of contradiction that $\log_2 3$ is a rational number. Then $\log_2 3 = p/q$ where p and q are positive integers (both positive because $\log_2 3 > 0$). Thus $2^{p/q} = 3$, so that $2^p = 3^q$. But if p and q are *positive* integers, then 2^p is even and 3^q is odd, so they cannot be equal. Therefore the assumption that $\log_2 3$ is rational leads to a contradiction, and thus $\log_2 3$ is irrational.

3.8.73: (a): If $f(x) = \log_{10} x$, then the definition of the derivative yields

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{h\to 0}\frac{1}{h}\log_{10}(1+h) = \lim_{h\to 0}\log_{10}(1+h)^{1/h}.$$

(b): When $h = 0.1$ the value of $\log_{10}(1+h)^{1/h}$ is approximately 0.4139. With $h = 0.01$ we get 0.4321, with $h = 0.001$ we get 0.4341, and with $h = \pm 0.0001$ we get 0.4343.

3.8.74: Because $\exp(\ln x) = x$, we see first that

$$10^x = \exp(\ln 10^x) = \exp(x \ln 10) = e^{x \ln 10}.$$

Hence

$$D_x\, 10^x = D_x\left(e^{x \ln 10}\right) = e^{x \ln 10} \ln 10 = 10^x \ln 10.$$

Thus, by the chain rule, if u is a differentiable function of x, then

$$D_x 10^u = (10^u \ln 10)\,\frac{du}{dx}.$$

Finally, if $u(x) = \log_{10} x$, so that $10^u \equiv x$, then differentiation of this last identity yields

$$(10^u \ln 10)\frac{du}{dx} \equiv 1, \qquad \text{so that} \qquad \frac{du}{dx} = D_x \log_{10} x = \frac{1}{x \ln 10} \approx \frac{0.4343}{x}.$$

Section 3.9

3.9.1: $2x - 2y\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = \dfrac{x}{y}$. Also, $y = \pm\sqrt{x^2-1}$, so $\dfrac{dy}{dx} = \pm\dfrac{x}{\sqrt{x^2-1}} = \dfrac{x}{\pm\sqrt{x^2-1}} = \dfrac{x}{y}$.

3.9.2: $x\dfrac{dy}{dx} + y = 0$, so $\dfrac{dy}{dx} = -\dfrac{y}{x}$. By substituting $y = x^{-1}$ here, we get $\dfrac{dy}{dx} = -\dfrac{x^{-1}}{x} = -x^{-2}$, which is the result obtained by explicit differentiation.

3.9.3: $32x + 50y\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{16x}{25y}$. Substituting $y = \pm\frac{1}{5}\sqrt{400-16x^2}$ into the derivative, we get $\dfrac{dy}{dx} = \mp\dfrac{16x}{5\sqrt{400-16x^2}}$, which is the result obtained by explicit differentiation.

3.9.4: $3x^2 + 3y^2\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{x^2}{y^2}$. $y = \sqrt[3]{1-x^3}$, so substitution results in $\dfrac{dy}{dx} = -\dfrac{x^2}{(1-x^3)^{2/3}}$. Explicit differentiation yields the same answer.

3.9.5: $\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$.

3.9.6: $4x^3 + 2x^2y\dfrac{dy}{dx} + 2xy^2 + 4y^3\dfrac{dy}{dx} = 0$: $\quad (2x^2y + 4y^3)\dfrac{dy}{dx} = -(4x^3 + 2xy^2)$; $\quad \dfrac{dy}{dx} = -\dfrac{4x^3 + 2xy^2}{2x^2y + 4y^3}$.

3.9.7: $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\left(\dfrac{x}{y}\right)^{-1/3} = -\left(\dfrac{y}{x}\right)^{1/3}$.

3.9.8: $y^2 + 2(x-1)y\dfrac{dy}{dx} = 1$, so $\dfrac{dy}{dx} = \dfrac{1-y^2}{2y(x-1)}$.

3.9.9: Given: $x^3 - x^2y = xy^2 + y^3$:

$$3x^2 - x^2\frac{dy}{dx} - 2xy = y^2 + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx};$$

$$3x^2 - 2xy - y^2 = (2xy + 3y^2 + x^2)\frac{dy}{dx};$$

$$\frac{dy}{dx} = \frac{3x^2 - 2xy - y^2}{3y^2 + 2xy + x^2}.$$

3.9.10: Given: $x^5 + y^5 = 5x^2y^2$:

$$5x^4 + 5y^4\frac{dy}{dx} = 10x^2y\frac{dy}{dx} + 10xy^2;$$

$$\frac{dy}{dx} = \frac{10xy^2 - 5x^4}{5y^4 - 10x^2y}.$$

3.9.11: Given: $x\sin y + y\sin x = 1$:

$$x\cos y\frac{dy}{dx} + \sin y + y\cos x + \sin x\frac{dy}{dx} = 0;$$

$$\frac{dy}{dx} = -\frac{\sin y + y\cos x}{x\cos y + \sin x}.$$

3.9.12: Given: $\cos(x+y) = \sin x\ \sin y$:

$$-\sin(x+y)(1 + \frac{dy}{dx}) = \sin x\ \cos y\ \frac{dy}{dx} + \sin y\ \cos x;$$

$$\frac{dy}{dx} = -\frac{\sin y\ \cos x + \sin(x+y)}{\sin(x+y) + \sin x\ \cos y}.$$

3.9.13: Given: $2x + 3e^y = e^{x+y}$. Differentiation of both sides of this equation (actually, an *identity*) with respect to x yields

$$2 + 3e^y\,\frac{dy}{dx} = e^{x+y}\left(1 + \frac{dy}{dx}\right), \qquad \text{and so} \qquad \frac{dy}{dx} = \frac{e^{x+y} - 2}{3e^y - e^{x+y}} = \frac{3e^y + 2x - 2}{(3 - e^x)e^y}.$$

3.9.14: Given: $xy = e^{-xy}$. Differentiation of both sides with respect to x yields

$$x\,\frac{dy}{dx} + y = -e^{xy}\left(x\,\frac{dy}{dx} + y\right), \qquad \text{and so} \qquad (1 + e^{-xy})x\,\frac{dy}{dx} = -(1 + e^{-xy})y.$$

Because $1 + e^{-xy} > 0$ for all x and y, it follows that

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Another way to solve this problem is to observe that the equation $e^{-z} = z$ has exactly one real solution $a \approx 0.5671432904$. Hence if $e^{-xy} = xy$, then $xy = a$, so that $y = a/x$. Hence

$$\frac{dy}{dx} = -\frac{a}{x^2} = -\frac{xy}{x^2} = -\frac{y}{x}.$$

3.9.15: $2x+2y\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{x}{y}$. At $(3, -4)$ the tangent has slope $\frac{3}{4}$ and thus equation $y+4 = \frac{3}{4}(x-3)$.

3.9.16: $x\dfrac{dy}{dx} + y = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{y}{x}$. At $(4, -2)$ the tangent has slope $\frac{1}{2}$ and thus equation $y+2 = \frac{1}{2}(x-4)$.

3.9.17: $x^2\dfrac{dy}{dx} + 2xy = 1$, so $\dfrac{dy}{dx} = \dfrac{1-2xy}{x^2}$. At $(2, 1)$ the tangent has slope $-\frac{3}{4}$ and thus equation $3x+4y = 10$.

3.9.18: $\frac{1}{4}x^{-3/4} + \frac{1}{4}y^{-3/4}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -(y/x)^{3/4}$. At $(16, 16)$ the tangent has slope -1 and thus equation $x + y = 32$.

3.9.19: $y^2 + 2xy\dfrac{dy}{dx} + 2xy + x^2\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{2xy+y^2}{2xy+x^2}$. At $(1, -2)$ the slope is zero, so an equation of the tangent there is $y = -2$.

3.9.20: $-\dfrac{1}{(x+1)^2} - \dfrac{1}{(y+1)^2}\cdot\dfrac{dy}{dx} = 0$, so $\dfrac{dy}{dx} = -\dfrac{(y+1)^2}{(x+1)^2}$. At $(1, 1)$ the tangent line has slope -1 and thus equation $y - 1 = -(x-1)$.

3.9.21: $24x + 24y\dfrac{dy}{dx} = 25y + 25x\dfrac{dy}{dx}$: $\quad \dfrac{dy}{dx} = \dfrac{25y-24x}{24y-25x}$. At $(3, 4)$ the tangent line has slope $\frac{4}{3}$ and thus equation $4x = 3y$.

3.9.22: $2x + y + x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{2x+y}{x+2y}$. At $(3, -2)$ the tangent line has slope 4 and thus equation $y + 2 = 4(x-3)$.

3.9.23: $\dfrac{dy}{dx} = \dfrac{3e^{2x}+2e^{y}}{3e^{2x}+e^{x+2y}}$, so the tangent line at $(0, 0)$ has slope $\frac{5}{4}$ and equation $4y = 5x$.

3.9.24: $\dfrac{dy}{dx} = \dfrac{12e^{2x}-ye^{3y}}{18e^{2x}+xe^{3y}}$, so the tangent line at $(3, 2)$ has slope $\frac{10}{21}$ and equation $10x + 12 = 21y$.

3.9.25: $\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\dfrac{dy}{dx} = 0$: $\quad \dfrac{dy}{dx} = -\dfrac{y^{1/3}}{x^{1/3}}$. At $(8, 1)$ the tangent line has slope $-\frac{1}{2}$ and thus equation $y - 1 = -\frac{1}{2}(x-8)$; that is, $x + 2y = 10$.

3.9.26: $2x - x\dfrac{dy}{dx} - y + 2y\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = \dfrac{y-2x}{2y-x}$. At $(3, -2)$ the tangent line has slope $\frac{8}{7}$ and thus equation $y + 2 = \frac{8}{7}(x-3)$; that is, $7y = 8x - 38$.

3.9.27: $2\left(x^2+y^2\right)\left(2x + 2y\dfrac{dy}{dx}\right) = 50x\dfrac{dy}{dx} + 50y$:

$$\frac{dy}{dx} = -\frac{2x^3 - 25y + 2xy^2}{-25x + 2x^2y + 2y^3}.$$

At (2, 4) the tangent line has slope $\frac{2}{11}$ and thus equation $y - 4 = \frac{2}{11}(x - 2)$; that is, $11y = 2x + 40$.

3.9.28: $2y\dfrac{dy}{dx} = 3x^2 + 14x$: $\quad \dfrac{dy}{dx} = \dfrac{3x^2 + 14x}{2y}$. At $(-3, 6)$ the tangent line has slope $-\frac{5}{4}$ and thus equation $y - 6 = -\frac{5}{4}(x + 3)$; alternatively, $4y = 9 - 5x$.

3.9.29: $3x^2 + 3y^2\dfrac{dy}{dx} = 9x\dfrac{dy}{dx} + 9y$: $\quad \dfrac{dy}{dx} = \dfrac{3y - x^2}{y^2 - 3x}$.

(a): At (2, 4) the tangent line has slope $\frac{4}{5}$ and thus equation $y - 4 = \frac{4}{5}(x - 2)$; that is, $5y = 4x + 12$.

(b): At a point on the curve at which $\dfrac{dy}{dx} = -1$, $3y - x^2 = -y^2 - 3x$ and $x^3 + y^3 = 9xy$. This pair of simultaneous equations has solutions $x = 0$, $y = 0$ and $x = \frac{9}{2}$, $y = \frac{9}{2}$, but the derivative does not exist at the point (0, 0). Therefore the tangent line with slope -1 has equation $y - \frac{9}{2} = -\left(x - \frac{9}{2}\right)$.

3.9.30: First, $2x^2 - 5xy + 2y^2 = (y - 2x)(2y - x)$.

(a): Hence if $2x^2 - 5xy + 2y^2 = 0$, then $y - 2x = 0$ or $2y - x = 0$. This is a pair of lines through the origin; the first has slope 2 and the second has slope $\frac{1}{2}$.

(b): Differentiating implicitly, we obtain $4x - 5x\dfrac{dy}{dx} - 5y + 4y\dfrac{dy}{dx} = 0$, which gives $\dfrac{dy}{dx} = \dfrac{5y - 4x}{4y - 5x}$, which is 2 if $y = 2x$ and $-\frac{1}{2}$ if $y = -\frac{1}{2}x$.

3.9.31: Here $\dfrac{dy}{dx} = \dfrac{2 - x}{y - 2}$, so horizontal tangents can occur only if $x = 2$ and $y \neq 2$. When $x = 2$, the original equation yields $y^2 - 4y - 4 = 0$, so that $y = 2 \pm \sqrt{8}$. Thus there are two points at which the tangent line is horizontal: $\left(2,\ 2 - \sqrt{8}\right)$ and $\left(2,\ 2 + \sqrt{8}\right)$.

3.9.32: First, $\dfrac{dy}{dx} = \dfrac{y - x^2}{y^2 - x}$ and $\dfrac{dx}{dy} = \dfrac{y^2 - x}{y - x^2}$. Horizontal tangents require $y = x^2$, and the equation $x^3 + y^3 = 3xy$ of the folium yields $x^3(x^3 - 2) = 0$, so either $x = 0$ or $x = \sqrt[3]{2}$. But dy/dx is not defined at (0, 0), so only at $\left(\sqrt[3]{2},\ \sqrt[3]{4}\right)$ is there a horizontal tangent. By symmetry or by a similar argument, there is a vertical tangent at $\left(\sqrt[3]{4},\ \sqrt[3]{2}\right)$ and nowhere else.

3.9.33: By direct differentiation, $dx/dy = (1 + y)e^y$. By implicit differentiation, $\dfrac{dy}{dx} = \dfrac{1}{(1 + y)e^y}$, and the results are equivalent.

(a): At $(0, 0)$, $dy/dx = 1$, so an equation of the line tangent to the curve at (0, 0) is $y = x$.

(b): At $(e,\ 1)$, $dy/dx = 1/(2e)$, so an equation of the line tangent to the curve at $(e,\ 1)$ is $x + e = 2ey$.

3.9.34: (a): By direct differentiation, $dx/dy = (1 + y)e^y$, so there is only one point on the curve where the tangent line is vertical ($dx/dy = 0$): $(-1/e,\ -1)$.

(b): Because $\dfrac{dy}{dx} = \dfrac{1}{(1 + y)e^y}$ is never zero, the graph has no horizontal tangents.

3.9.35: From $2(x^2 + y^2)\left(2x + 2y\dfrac{dy}{dx}\right) = 2x - 2y\dfrac{dy}{dx}$ it follows that

$$\frac{dy}{dx} = \frac{x[1 - 2(x^2 + y^2)]}{y[1 + 2(x^2 + y^2)]}.$$

So $dy/dx = 0$ when $x^2 + y^2 = \frac{1}{2}$, but is undefined when $x = 0$, for then $y = 0$ as well. If $x^2 + y^2 = \frac{1}{2}$, then $x^2 - y^2 = \frac{1}{4}$, so that $x^2 = \frac{3}{8}$, and it follows that $y^2 = \frac{1}{8}$. Consequently there are horizontal tangents at all four points where $|x| = \frac{1}{4}\sqrt{6}$ and $|y| = \frac{1}{4}\sqrt{2}$.

Also $dx/dy = 0$ only when $y = 0$, and if so, then $x^4 = x^2$, so that $x = \pm 1$ (dx/dy is undefined when $x = 0$). So there are vertical tangents at the two points $(-1, 0)$ and $(1, 0)$.

3.9.36: Base edge of block: x. Height: y. Volume: $V = x^2 y$. We are given $dx/dt = -2$ and $dy/dt = -3$. Implicit differentiation yields

$$\frac{dV}{dt} = x^2 \frac{dy}{dt} + 2xy\frac{dx}{dt}.$$

When $x = 20$ and $y = 15$, $dV/dt = (400)(-3) + (600)(-2) = -2400$. So the rate of flow at the time given is 2400 in.^{3}/h.

3.9.37: Suppose that the pile has height $h = h(t)$ at time t (seconds) and radius $r = r(t)$ then. We are given $h = 2r$ and we know that the volume of the pile at time t is

$$V = V(t) = \frac{\pi}{3}r^2 h = \frac{2}{3}\pi r^3. \quad \text{Now} \quad \frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}, \quad \text{so} \quad 10 = 2\pi r^2 \frac{dr}{dt}.$$

When $h = 5$, $r = 2.5$; at that time $\dfrac{dr}{dt} = \dfrac{10}{2\pi(2.5)^2} = \dfrac{4}{5\pi} \approx 0.25645$ (ft/s).

3.9.38: Draw a vertical cross section through the center of the tank. Let r denote the radius of the (circular) water surface when the depth of water in the tank is y. From the drawing and the Pythagorean theorem derive the relationship $r^2 + (10 - y)^2 = 100$. Therefore

$$2r\frac{dr}{dt} - 2(10 - y)\frac{dy}{dt} = 0, \quad \text{and so} \quad r\frac{dr}{dt} = (10 - y)\frac{dy}{dt}.$$

We are to find dr/dt when $y = 5$, given $dy/dt = -3$. At that time, $r^2 = 100 - 25$, so $r = 5\sqrt{3}$. Thus

$$\left.\frac{dr}{dt}\right|_{y=5} = \frac{10 - y}{r} \cdot \left.\frac{dy}{dt}\right|_{y=5} = \frac{5}{5\sqrt{3}}(-3) = -\sqrt{3}.$$

Answer: The radius of the top surface is decreasing at $\sqrt{3}$ ft/s then.

3.9.39: We assume that the oil slick forms a solid right circular cylinder of height (thickness) h and radius r. Then its volume is $V = \pi r^2 h$, and we are given $V = 1$ (constant) and $dh/dt = -0.001$. Therefore $0 = \pi r^2 \dfrac{dh}{dt} + 2\pi rh\dfrac{dr}{dt}$. Consequently $2h\dfrac{dr}{dt} = \dfrac{r}{1000}$, and so $\dfrac{dr}{dt} = \dfrac{r}{2000h}$. When $r = 8$, $h = \dfrac{1}{\pi r^2} = \dfrac{1}{64\pi}$. At that time, $\dfrac{dr}{dt} = \dfrac{8 \cdot 64\pi}{2000} = \dfrac{32\pi}{125} \approx 0.80425$ (m/h).

3.9.40: Let x be the distance from the ostrich to the street light and u the distance from the base of the light pole to the tip of the ostrich's shadow. Draw a figure and so label it; by similar triangles you find that

$\dfrac{u}{10} = \dfrac{u-x}{5}$, and it follows that $u = 2x$. We are to find du/dt and $D_t(u-x) = du/dt - dx/dt$. But $u = 2x$, so

$$\frac{du}{dt} = 2\frac{dx}{dt} = (2)(-4) = -8; \qquad \frac{du}{dt} - \frac{dx}{dt} = -8 - (-4) = -4.$$

Answers: (a): +8 ft/s; (b): +4 ft/s.

3.9.41: Let x denote the width of the rectangle; then its length is $2x$ and its area is $A = 2x^2$. Thus $\dfrac{dA}{dt} = 4x\dfrac{dx}{dt}$. When $x = 10$ and $dx/dt = 0.5$, we have

$$\left.\frac{dA}{dt}\right|_{x=10} = (4)(10)(0.5) = 20 \text{ (cm}^2\text{/s)}.$$

3.9.42: Let x denote the length of each edge of the triangle. Then the triangle's area is $A(x) = \left(\frac{1}{4}\sqrt{3}\right)x^2$, and therefore $\dfrac{dA}{dt} = \left(\frac{1}{2}\sqrt{3}\right)x\dfrac{dx}{dt}$. Given $x = 10$ and $\dfrac{dx}{dt} = 0.5$, we find that

$$\left.\frac{dA}{dt}\right|_{x=10} = \frac{\sqrt{3}}{2}\cdot 10 \cdot (0.5) = \frac{5\sqrt{3}}{2} \text{ (cm}^2\text{/s)}.$$

3.9.43: Let r denote the radius of the balloon and V its volume at time t (in seconds). Then

$$V = \frac{4}{3}\pi r^3, \quad \text{so} \quad \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}.$$

We are to find dr/dt when $r = 10$, and we are given the information that $dV/dt = 100\pi$. Therefore

$$100\pi = 4\pi(10)^2\left.\frac{dr}{dt}\right|_{r=10},$$

and so at the time in question the radius is increasing at the rate of $dr/dt = \frac{1}{4} = 0.25$ (cm/s).

3.9.44: Because $pV = 1000$, $V = 10$ when $p = 100$. Moreover, $p\dfrac{dV}{dt} + V\dfrac{dp}{dt} = 0$. With $p = 100$, $V = 10$, and $dp/dt = 2$, we find that

$$\left.\frac{dV}{dt}\right|_{p=100} = -\frac{V}{p}\cdot\left.\frac{dp}{dt}\right|_{p=100} = -\frac{10}{100}\cdot 2 = -\frac{1}{5}.$$

Therefore the volume is decreasing at 0.2 in.^{3}/s.

3.9.45: Place the person at the origin and the kite in the first quadrant at $(x, 400)$ at time t, where $x = x(t)$ and we are given $dx/dt = 10$. Then the length $L = L(t)$ of the string satisfies the equation $L^2 = x^2 + 160000$, and therefore $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$. Moreover, when $L = 500$, $x = 300$. So

$$1000\left.\frac{dL}{dt}\right|_{L=500} = 600 \cdot 10,$$

which implies that the string is being let out at 6 ft/s.

3.9.46: Locate the observer at the origin and the balloon in the first quadrant at (300, y), where $y = y(t)$ is the balloon's altitude at time t. Let θ be the angle of elevation of the balloon (in radians) from the observer's point of view. Then $\tan\theta = y/300$. We are given $d\theta/dt = \pi/180$ rad/s. Hence we are to find dy/dt when $\theta = \pi/4$. But $y = 300\tan\theta$, so

$$\frac{dy}{dt} = (300\sec^2\theta)\frac{d\theta}{dt}.$$

Substitution of the given values of θ and $d\theta/dt$ yields the answer

$$\left.\frac{dy}{dt}\right|_{\theta=45^\circ} = 300 \cdot 2 \cdot \frac{\pi}{180} = \frac{10\pi}{3} \approx 10.472 \text{ (ft/s)}.$$

3.9.47: Locate the observer at the origin and the airplane at (x, 3), with $x > 0$. We are given dx/dt where the units are in miles, hours, and miles per hour. The distance z between the observer and the airplane satisfies the identity $z^2 = x^2 + 9$, and because the airplane is traveling at 8 mi/min, we find that $x = 4$, and therefore that $z = 5$, at the time 30 seconds after the airplane has passed over the observer. Also $2z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt}$, so at the time in question, $10\dfrac{dz}{dt} = 8 \cdot 480$. Therefore the distance between the airplane and the observer is increasing at 384 mi/h at the time in question.

3.9.48: In this problem we have $V = \frac{1}{3}\pi y^2(15 - y)$ and $(-100)(0.1337) = \dfrac{dV}{dt} = \pi(10y - y^2)\dfrac{dy}{dt}$. Therefore $\dfrac{dy}{dt} = -\dfrac{13 \cdot 37}{\pi y(10 - y)}$. Answers: (a): Approximately 0.2027 ft/min; (b): The same.

3.9.49: We use $a = 10$ in the formula given in Problem 42. Then

$$V = \frac{1}{3}\pi y^2(30 - y).$$

Hence $(-100)(0.1337) = \dfrac{dV}{dt} = \pi(20y - y^2)\dfrac{dy}{dt}$. Thus $\dfrac{dy}{dt} = -\dfrac{13 \cdot 37}{\pi y(20 - y)}$. Substitution of $y = 7$ and $y = 3$ now yields the two answers:

$$\text{(a):} \quad -\frac{191}{1300\pi} \approx -0.047 \ \text{(ft/min)}; \qquad \text{(b):} \quad -\frac{1337}{5100\pi} \approx -0.083 \ \text{(ft/min)}.$$

3.9.50: When the height of the water at the deep end of the pool is 10 ft, the length of the water surface is 50 ft. So by similar triangles, if the height of the water at the deep end is y feet ($y \geqq 10$), then the length of the water surface is $x = 5y$ feet. A cross section of the water perpendicular to the width of the pool thus forms a right triangle of area $5y^2/2$. Hence the volume of the pool is $V(y) = 50y^2$. Now $133.7 = \dfrac{dV}{dt} = 100y\dfrac{dy}{dt}$, so when $y = 6$ we have

$$\left.\frac{dy}{dt}\right|_{y=6} = \frac{133.7}{600} \approx 0.2228 \text{ (ft/min)}.$$

3.9.51: Let the positive y-axis represent the wall and the positive x-axis the ground, with the top of the ladder at $(0, y)$ and its lower end at $(x, 0)$ at time t. Given: $dx/dt = 4$, with units in feet, seconds, and feet per second. Also $x^2 + y^2 = 41^2$, and it follows that $y\dfrac{dy}{dt} = -x\dfrac{dx}{dt}$. Finally, when $y = 9$, we have $x = 40$, so at that time $9\dfrac{dy}{dt} = -40 \cdot 4$. Therefore the top of the ladder is moving downward at $\frac{160}{9} \approx 17.78$ ft/s.

3.9.52: Let x be the length of the base of the rectangle and y its height. We are given $dx/dt = +4$ and $dy/dt = -3$, with units in centimeters and seconds. The area of the rectangle is $A = xy$, so

$$\frac{dA}{dt} = x\frac{dy}{dt} + y\frac{dx}{dt} = -3x + 4y.$$

Therefore when $x = 20$ and $y = 12$, we have $dA/dt = -12$, so the area of the rectangle is decreasing at the rate of 12 cm^2/s then.

3.9.53: Let r be the radius of the cone, h its height. We are given $dh/dt = -3$ and $dr/dt = +2$, with units in centimeters and seconds. The volume of the cone at time t is $V = \frac{1}{3}\pi r^2 h$, so

$$\frac{dV}{dt} = \frac{2}{3}\pi r h\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt}.$$

When $r = 4$ and $h = 6$, $\dfrac{dV}{dt} = \dfrac{2}{3} \cdot 24\pi \cdot 2 + \dfrac{1}{3} \cdot 16\pi \cdot (-3) = 16\pi$, so the volume of the cone is increasing at the rate of 16π cm^3/s then.

3.9.54: Let x be the edge length of the square and $A = x^2$ its area. Given: $\dfrac{dA}{dt} = 120$ when $x = 10$. But $dA/dt = 2x(dx/dt)$, so $dx/dt = 6$ when $x = 10$. Answer: At 6 in./s.

3.9.55: Locate the radar station at the origin and the rocket at $(4,\ y)$ in the first quadrant at time t, with y in miles and t in hours. The distance z between the station and the rocket satisfies the equation $y^2 + 16 = z^2$, so $2y\dfrac{dy}{dt} = 2z\dfrac{dz}{dt}$. When $z = 5$, we have $y = 3$, and because $dz/dt = 3600$ it follows that $dy/dt = 6000$ mi/h.

3.9.56: Locate the car at $(x,\ 0)$, the truck at $(0,\ y)$ $\ (x, y > 0)$. Then at 1 P.M. we have $x = 90$ and $y = 80$. We are given that data $dx/dt = 30$ and $dy/dt = 40$, with units in miles, hours, and miles per hour. The distance z between the vehicles satisfies the equation $z^2 = x^2 + y^2$, so

$$z\frac{dz}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}.$$

Finally, at 1 P.M. $z^2 = 8100 + 6400 = 14500$, so $z = 10\sqrt{145}$ then. So at 1 P.M.

$$\frac{dz}{dt} = \frac{2700 + 3200}{10\sqrt{145}} = \frac{590}{\sqrt{145}}$$

mi/h—approximately 49 mi/h.

3.9.57: Put the floor on the nonnegative x-axis and the wall on the nonnegative y-axis. Let x denote the distance from the wall to the foot of the ladder (measured along the floor) and let y be the distance from the floor to the top of the ladder (measured along the wall). By the Pythagorean theorem, $x^2 + y^2 = 100$, and we are given $dx/dt = \frac{22}{15}$ (because we will use units of feet and seconds rather than miles and hours). From the Pythagorean relation we find that

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0,$$

so that $\dfrac{dy}{dt} = -\dfrac{x}{y} \cdot \dfrac{dx}{dt} = -\dfrac{22x}{15y}$.

(a): If $y = 4$, then $x = \sqrt{84} = 2\sqrt{21}$. Hence when the top of the ladder is 4 feet above the ground, it is moving a a rate of

$$\left.\frac{dy}{dt}\right|_{y=4} = -\frac{44\sqrt{21}}{60} = -\frac{11\sqrt{21}}{15} \approx -3.36$$

feet per second, about 2.29 miles per hour downward.

(b): If $y = \frac{1}{12}$ (one inch), then

$$x^2 = 100 - \frac{1}{144} = \frac{14399}{144}, \qquad \text{so that} \qquad x \approx 9.99965.$$

In this case,

$$\left.\frac{dy}{dt}\right|_{y=1/12} = -\frac{22 \cdot (9.99965)}{15 \cdot \frac{1}{12}} = -\frac{88}{5} \cdot (9.99965) \approx -176$$

feet per second, about 120 miles per hour downward.

(c): If $y = 1$ mm, then $x \approx 10$ (ft), and so

$$\frac{dy}{dt} \approx -\frac{22}{15} \cdot (3048) \approx 4470$$

feet per second, about 3048 miles per hour.

The results in parts (b) and (c) are not plausible. This shows that the assumption that the top of the ladder never leaves the wall is invalid

3.9.58: Let x be the distance between the *Pinta* and the island at time t and y the distance between the *Niña* and the island then. We know that $x^2 + y^2 = z^2$ where $z = z(t)$ is the distance between the two ships, so

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}. \tag{1}$$

When $x = 30$ and $y = 40$, $z = 50$. It follows from Eq. (1) that $dz/dt = -25$ then. Answer: They are drawing closer at 25 mi/h at the time in question.

3.9.59: Locate the military jet at $(x,\,0)$ with $x < 0$ and the other aircraft at $(0,\,y)$ with $y \geqq 0$. With units in miles, minutes, and miles per minute, we are given $dx/dt = +12$, $dy/dt = +8$, and when $t = 0$, $x = -208$ and $y = 0$. The distance z between the aircraft satisfies the equation $x^2 + y^2 = z^2$, so

$$\frac{dz}{dt} = \frac{1}{\sqrt{x^2+y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = \frac{12x+8y}{\sqrt{x^2+y^2}}.$$

The closest approach will occur when $dz/dt = 0$: $y = -3x/2$. Now $x(t) = 12t - 208$ and $y(t) = 8t$. So at closest approach we have

$$8t = y(t) = -\frac{3}{2}x(t) = -\frac{3}{2}(12t - 208).$$

Hence at closest approach, $16t = 624 - 36t$, and thus $t = 12$. At this time, $x = -64$, $y = 96$, and $z = 32\sqrt{13} \approx 115.38$ (mi).

3.9.60: Let x be the distance from the anchor to the point on the seabed directly beneath the hawsehole; let L be the amount of anchor chain out. We must find dx/dt when $L = 13$ (fathoms), given $dL/dt = -10$. Now $x^2 + 144 = L^2$, so $2L\dfrac{dL}{dt} = 2x\dfrac{dx}{dt}$. Consequently, $\dfrac{dx}{dt} = \dfrac{L}{x}\cdot\dfrac{dL}{dt}$. At the time in question in the problem, $x^2 = 13^2 - 12^2$, so $x = 5$. It follows that $dx/dt = -26$ then. Thus the ship is moving at 26 fathoms per minute—about 1.77 mi/h.

3.9.61: Let x be the radius of the water surface at time t and y the height of the water remaining at time t. If Q is the amount of water remaining in the tank at time t, then (because the water forms a cone) $Q = Q(t) = \frac{1}{3}\pi x^2 y$. But by similar triangles, $\dfrac{x}{y} = \dfrac{3}{5}$, so $x = \dfrac{3y}{5}$. So

$$Q(t) = \frac{1}{3}\pi\frac{9}{25}y^3 = \frac{3}{25}\pi y^3.$$

We are given $dQ/dt = -2$ when $y = 3$. This implies that when $y = 3$, $-2 = \dfrac{dQ}{dt} = \dfrac{9}{25}\pi y^2\dfrac{dy}{dt}$. So at the time in question,

$$\left.\frac{dy}{dt}\right|_{y=3} = -\frac{50}{81\pi} \approx -0.1965 \text{ (ft/s)}.$$

3.9.62: Given $V = \frac{1}{3}\pi(30y^2 - y^3)$, find dy/dt given V, y, and dy/dt. First,

$$\frac{dV}{dt} = \frac{1}{3}\pi(60y - 3y^2)\frac{dy}{dt} = \pi(20y - y^2)\frac{dy}{dt}.$$

So $\dfrac{dy}{dt} = \dfrac{1}{\pi(20y - y^2)}\cdot\dfrac{dV}{dt}$. Therefore, when $y = 5$, we have

$$\left.\frac{dy}{dt}\right|_{y=5} = \frac{(200)(0.1337)}{\pi(100-25)} \approx 0.113488 \text{ (ft/min)}.$$

3.9.63: Let r be the radius of the water surface at time t, h the depth of water in the bucket then. By similar triangles we find that

$$\frac{r-6}{h} = \frac{1}{4}, \text{ so } r = 6 + \frac{h}{4}.$$

The volume of water in the bucket then is

$$\begin{aligned} V &= \frac{1}{3}\pi h(36 + 6r + r^2) \\ &= \frac{1}{3}\pi\left(36 + 36 + \frac{3}{2}h + 36 + 3h + \frac{1}{16}h^2\right) \\ &= \frac{1}{3}\pi h\left(108 + \frac{9}{2}h + \frac{1}{16}h^2\right). \end{aligned}$$

Now $\dfrac{dV}{dt} = -10$; we are to find dh/dt when $h = 12$.

$$\frac{dV}{dt} = \frac{1}{3}\pi(108 + 9h + \frac{3}{16}h^2)\frac{dh}{dt}.$$

Therefore $\left.\dfrac{dh}{dt}\right|_{h=12} = \dfrac{3}{\pi} \cdot \dfrac{-10}{108 + 9 \cdot 12 + \dfrac{3 \cdot 12^2}{16}} = -\dfrac{10}{81\pi} \approx -0.0393$ (in./min).

3.9.64: Let x denote the distance between the ship and A, y the distance between the ship and B, h the perpendicular distance from the position of the ship to the line AB, u the distance from A to the foot of this perpendicular, and v the distance from B to the foot of the perpendicular. At the time in question, we know that $x = 10.4$, $dx/dt = 19.2$, $y = 5$, and $dy/dt = -0.6$. From the right triangles involved, we see that $u^2 + h^2 = x^2$ and $(12.6 - u)^2 + h^2 = y^2$. Therefore

$$x^2 - u^2 = y^2 - (12.6 - u)^2. \tag{1}$$

We take $x = 10.4$ and $y = 5$ in Eq. (1); it follows that $u = 9.6$ and that $v = 12.6 - u = 3$. From Eq. (1), we know that

$$x\frac{dx}{dt} - u\frac{du}{dt} = y\frac{dy}{dt} + (12.6 - u)\frac{du}{dt},$$

so

$$\frac{du}{dt} = \frac{1}{12.6}\left(x\frac{dx}{dt} - y\frac{dy}{dt}\right).$$

From the data given, $du/dt \approx 16.0857$. Also, because $h = \sqrt{x^2 - u^2}$, $h = 4$ when $x = 10.4$ and $y = 9.6$. Moreover, $h\dfrac{dh}{dt} = x\dfrac{dx}{dt} - u\dfrac{du}{dt}$, and therefore

$$\left.\frac{dh}{dt}\right|_{h=4} \approx \frac{1}{4}\left[(10.4)(19.2) - (9.6)(16.0857)\right] \approx 11.3143.$$

Finally, $\dfrac{dh/dt}{du/dt} \approx 0.7034$, so the ship is sailing a course about $35°7'$ north *or* south of east at a speed of $\sqrt{(du/dt)^2 + (dh/dt)^2} \approx 19.67$ mi/h. It is located 9.6 miles east and 4 miles north *or* south of A, or 10.4 miles from A at a bearing of either $67°22'\,48''$ or $112°37'\,12''$.

3.9.65: Set up a coordinate system in which the radar station is at the origin, the plane passes over it at the point (0, 1) (so units on the axes are in miles), and the plane is moving along the graph of the equation $y = x + 1$. Let s be the distance from (0, 1) to the plane and let u be the distance from the radar station to the plane. We are given $du/dt = +7$ mi/min. We may deduce from the law of cosines that $u^2 = s^2 + 1 + s\sqrt{2}$. Let v denote the speed of the plane, so that $v = ds/dt$. Then

$$2u\frac{du}{dt} = 2sv + v\sqrt{2} = v\left(2s + \sqrt{2}\right), \quad \text{and so} \quad v = \frac{2u}{2s + \sqrt{2}} \cdot \frac{du}{dt}.$$

When $u = 5$, $s^2 + s\sqrt{2} - 24 = 0$. The quadratic formula yields the solution $s = 3\sqrt{2}$, and it follows that $v = 5\sqrt{2}$ mi/min; alternatively, $v \approx 424.26$ mi/h.

3.9.66: $V(y) = \frac{1}{3}\pi(30y^2 - y^3)$ where the depth is y. Now $\dfrac{dV}{dt} = -k\sqrt{y} = \dfrac{dV}{dy} \cdot \dfrac{dy}{dt}$, and therefore

$$\frac{dy}{dt} = -\frac{k\sqrt{y}}{\dfrac{dV}{dy}} = -\frac{k\sqrt{y}}{\pi(20y - y^2)}.$$

To minimize dy/dt, write $F(y) = dy/dt$. It turns out (after simplifications) that

$$F'(y) = \frac{k}{2\pi} \cdot \frac{20y - 3y^2}{(20y - y^2)^2\sqrt{y}}.$$

So $F'(y) = 0$ when $y = 0$ and when $y = \frac{20}{3}$. When y is near 20, $F(y)$ is very large; the same is true for y near zero. So $y = \frac{20}{3}$ minimizes dy/dt, and therefore the answer to part (b) is 6 ft 8 in.

3.9.67: Place the pole at the origin in the plane, and let the horizontal strip $0 \leqq y \leqq 30$ represent the road. Suppose that the person is located at $(x, 30)$ with $x > 0$ and is walking to the right, so $dx/dt = +5$. Then the distance from the pole to the person will be $\sqrt{x^2 + 900}$. Let z be the length of the person's shadow. By similar triangles it follows that $2z = \sqrt{x^2 + 900}$, so $4z^2 = x^2 + 900$, and thus $8z\dfrac{dz}{dt} = 2x\dfrac{dx}{dt}$. When $x = 40$, we find that $z = 25$, and therefore that

$$100\frac{dz}{dt}\bigg|_{z=25} = 40 \cdot 5 = 200.$$

Therefore the person's shadow is lengthening at 2 ft/s at the time in question.

3.9.68: Set up a coordinate system in which the officer is at the origin and the van is moving in the positive direction along the line $y = 200$ (so units on the coordinate axes are in feet). When the van is at position

$(x,\,200)$, the distance from the officer to the van is z, where $x^2 + 200^2 = z^2$, so that $x\dfrac{dx}{dt} = z\dfrac{dz}{dt}$. When the van reaches the call box, $x = 200$, $z = 200\sqrt{2}$, and $dz/dt = 66$. It follows that

$$\left.\frac{dx}{dt}\right|_{x=200} = 66\sqrt{2},$$

which translates to about 63.6 mi/h.

Section 3.10

Note: Your answers may differ from ours in the last one or two decimal places because of differences in hardware or in the way the problem was solved. We used *Mathematica* and carried 40 decimal digits throughout all calculations, and our answers are correct or correctly rounded to the number of digits shown here. In most of the first 20 problems the initial guess x_0 was obtained by linear interpolation. Finally, the equals mark is used in this section to mean "equal or approximately equal."

3.10.1: With $f(x) = x^2 - 5$, $a = 2$, $b = 3$, and

$$x_0 = a - \frac{(b-a)f(a)}{f(b) - f(a)} = 2.2,$$

we used the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{for} \quad n \geqq 0.$$

Thus we obtained $x_1 = 2.236363636$, $x_2 = 2.236067997$, and $x_3 = x_4 = 2.236067977$. Answer: 2.2361.

3.10.2: $x_0 = 1.142857143$; we use $f(x) = x^3 - 2$. Then $x_1 = 1.272321429$, $x_2 = 1.260041515$, $x_3 = 1.259921061$, and $x_4 = x_5 = 1.259921050$. Answer: 1.2599.

3.10.3: $x_0 = 2.322274882$; we use $f(x) = x^5 - 100$. Then $x_1 = 2.545482651$, $x_2 = 2.512761634$, $x_3 = 2.511887041$, and $x_4 = x_5 = 2.511886432$. Answer: 2.5119.

3.10.4: Let $f(x) = x^{3/2} - 10$. Then $x_0 = 4.628863603$. From the iterative formula

$$x - \frac{x^{3/2} - 10}{\frac{3}{2}x^{1/2}} \longrightarrow x$$

we obtain $x_1 = 4.641597575$, $x_2 = 4.641588834 = x_3$. Answer: 4.6416.

3.10.5: 0.25, 0.3035714286, 0.3027758133, 0.3027756377. Answer: 0.3028.

3.10.6: 0.2, 0.2466019417, 0.2462661921, 0.2462661722. Answer: 0.2463.

3.10.7: $x_0 = -0.5$, $x_1 = -0.8108695652$, $x_2 = -0.7449619516$, $x_3 = -0.7402438226$, $x_4 = -0.7402217826 = x_5$. Answer: 0.7402.

3.10.8: Let $f(x) = x^3 + 2x^2 + 2x - 10$. With initial guess $x_0 = 1.5$ (the midpoint of the interval), we obtain $x_1 = 1.323943661972$, $x_2 = 1.309010783652$, $x_3 = 1.308907324710$, $x_4 = 1.308907319765$, and $x_5 = 1.308907319765$. Answer: 1.3089.

3.10.9: With $f(x) = x - \cos x$, $f'(x) = 1 + \sin x$, and calculator set in *radian* mode, we obtain $x_0 = 0.5854549279$, $x_1 = 0.7451929664$, $x_2 = 0.7390933178$, $x_3 = 0.7390851332$, and $x_4 = x_3$. Answer: 0.7391.

3.10.10: Let $f(x) = x^2 - \sin x$. Then $f'(x) = 2x - \cos x$. The linear interpolation formula yields $x_0 = 0.7956861008$, and the iterative formula

$$x - \frac{x^2 - \sin x}{2x - \cos x} \longrightarrow x$$

(with calculator in *radian* mode) yields the following results: $x_1 = 0.8867915207$, $x_2 = 0.8768492470$, $x_3 = 0.8767262342$, and $x_4 = 0.8767262154 = x_5$. Answer: 0.8767.

3.10.11: With $f(x) = 4x - \sin x - 4$ and calculator in *radian* mode, we get the following results: $x_0 = 1.213996400$, $x_1 = 1.236193029$, $x_2 = 1.236129989 = x_3$. Answer: 1.2361.

3.10.12: $x_0 = 0.8809986055$, $x_1 = 0.8712142932$, $x_3 = 0.8712215145$, and $x_4 = x_3$. Answer: 0.8712.

3.10.13: With $x_0 = 2.188405797$ and the iterative formula

$$x - \frac{x^4(x+1) - 100}{x^3(5x+4)} \longrightarrow x,$$

we obtain $x_1 = 2.360000254$, $x_2 = 2.339638357$, $x_3 = 2.339301099$, and $x_4 = 2.339301008 = x_5$. Answer: 2.3393.

3.10.14: $x_0 = 0.7142857143$, $x_1 = 0.8890827860$, $x_2 = 0.8607185590$, $x_3 = 0.8596255544$, and $x_4 = 0.8596240119 = x_5$. Answer: 0.8596.

3.10.15: The nearest discontinuities of $f(x) = x - \tan x$ are at $\pi/2$ and at $3\pi/2$, approximately 1.571 and 4.712. Therefore the function $f(x) = x - \tan x$ has the intermediate value property on the interval $[2, 3]$. Results: $x_0 = 2.060818495$, $x_1 = 2.027969226$, $x_2 = 2.028752991$, and $x_3 = 2.028757838 = x_4$. Answer: 2.0288.

3.10.16: As $\frac{7}{2}\pi \approx 10.9956$ and $\frac{9}{2}pi \approx 14.1372$ are the nearest discontinuities of $f(x) = x - \tan x$, this function has the intermediate value property on the interval $[11, 12]$. Because $f(11) \approx -214.95$ and $f(12) \approx 11.364$, the equation $f(x) = 0$ has a solution in $[11, 12]$. We obtain $x_0 = 11.94978618$ by interpolation, and the iteration

$$x - \frac{x + \tan x}{1 + \sec^2 x} \longrightarrow x$$

of Newton's method yields the successive approximations

$$x_1 = 7.457596948,\ x_2 = 6.180210620,\ x_3 = 3.157913273,\ x_4 = 1.571006986;$$

after many more iterations we arrive at the answer 2.028757838 of Problem 15. The difficulty is caused by the fact that $f(x)$ is generally a very large number, so the iteration of Newton's method tends to alter the value of x excessively. A little experimentation yields the fact that $f(11.08) \approx -0.736577$ and $f(11.09) \approx 0.531158$. We begin anew on the better interval $[11.08, 11.09]$ and obtain $x_0 = 11.08581018$, $x_1 = 11.08553759$, $x_2 = 11.08553841$, and $x_3 = x_2$. Answer: 11.0855.

3.10.17: $x - e^{-x} = 0$; $[0, 1]$: $x_0 = 0.5$, $x_1 \approx 0.5663$, $x_2 \approx 0.5671$, $x_3 \approx 0.5671$.

3.10.18: $x_0 = 2.058823529$, $x_1 = 2.095291459$, $x_2 = 2.094551790$, $x_3 = 2.094551482 = x_4$. Answer: 2.0946.

3.10.19: $e^x + x - 2 = 0$; $[0, 1]$: $x_0 = 0.5$, $x_1 \approx 0.4439$, $x_2 \approx 0.4429$, $x_3 \approx 0.4429$.

3.10.20: $e^{-x} - \ln x = 0$; $[1, 2]$: $x_0 = 1.5$, $x_1 \approx 1.2951$, $x_2 \approx 1.3097$, $x_3 \approx 1.3098 \approx x_4$.

3.10.21: Let $f(x) = x^3 - a$. Then the iteration of Newton's method in Eq. (6) takes the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n)^3 - a}{3(x_n)^2} = \frac{2(x_n)^3 + a}{3(x_n)^2} = \frac{1}{3}\left(2x_n + \frac{a}{(x_n)^2}\right).$$

Because $1 < \sqrt[3]{2} < 2$, we begin with $x_0 = 1.5$ and apply this formula with $a = 2$ to obtain $x_1 = 1.296296296$, $x_2 = 1.260932225$, $x_3 = 1.259921861$, and $x_4 = 1.259921050 = x_5$. Answer: 1.25992.

3.10.22: The formula in Eq. (6) of the text, with $f(x) = x^k - a$, takes the form

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n)^k - a}{k(x_n)^{k-1}} = \frac{(k-1)(x_n)^k + a}{k(x_n)^{k-1}} = \frac{1}{k}\left[(k-1)x_n + \frac{a}{(x_n)^{k-1}}\right].$$

We take $a = 100$, $k = 10$, and $x_0 = 1.5$ and obtain $x_1 = 1.610122949$, $x_2 = 1.586599871$, $x_3 = 1.584901430$, $x_4 = 1.584893193$, and $x_5 = 1.584893192 = x_6$. Answer: 1.58489.

3.10.23: We get $x_0 = 0.5$, $x_1 = 0.4387912809$, $x_2 = 0.4526329217$, $x_3 = 0.4496493762$, $\dots$, $x_{14} = 0.4501836113 = x_{15}$. The method of repeated substitution tends to converge much more slowly than Newton's method, has the advantage of not requiring that you compute a derivative or even that the functions involved are differentiable, and has the disadvantage of more frequent failure than Newton's method when both are applicable (see Problems 24 and 25).

3.10.24: Our results using the first formula: $x_0 = 1.5$, $x_1 = 1.257433430$, $x_2 = 1.225755182$, $x_3 = 1.221432153, \ldots,$ $x_{10} = 1.220745085 = x_{11}$. When we use the second formula, we obtain $x_1 = 4.0625$, $x_2 = 271.3789215$, $x_3 = 5423829645$, and x_4 has 39 digits to the left of the decimal point. It frequently requires some ingenuity to find a suitable way to put the equation $f(x) = 0$ into the form $x = G(x)$.

3.10.25: Beginning with $x_0 = 0.5$, the first formula yields $x_0 = 0.5$, $x_1 = -1$, $x_2 = 2$, $x_3 = 2.75$, $x_4 = 2.867768595, \ldots,$ $x_{12} = 2.879385242 = x_{13}$. Wrong root! At least the method converged. If your calculator or computer balks at computing the cube root of a negative number, then you can rewrite the second formula in Problem 25 in the form

$$x = \operatorname{Sgn}(3x^2 - 1) \cdot |3x^2 - 1|^{1/3}.$$

The results, again with $x_0 = 0.5$, are $x_1 = -0.629960525$, $x_2 = 0.575444686$, $x_3 = -0.187485243$, $x_4 = -0.963535808$, $\ldots,$ $x_{25} = 2.877296053$, $x_{26} = 2.877933902$, $\ldots,$ and $x_{62} = 2.879385240 = x_{63}$. Not only is convergence extremely slow, the method of repeated substitution again leads to the wrong root. Finally, the given equation can also be written in the form

$$x = \frac{1}{\sqrt{3-x}},$$

and in this case, again with $x_0 = 0.5$, we obtain $x_1 = 0.632455532$, $x_2 = 0.649906570$, $x_3 = 0.652315106$, $x_4 = 0.652649632$, $\ldots,$ and $x_{12} = 0.652703645 = x_{13}$.

3.10.26: If $f(x) = \dfrac{1}{x} - a$, then Newton's method uses the iteration

$$x - \frac{\dfrac{1}{x} - a}{-\dfrac{1}{x^2}} = x + x^2\left(\frac{1}{x} - a\right) = 2x - ax^2 \longrightarrow x.$$

3.10.27: Let $f(x) = x^5 + x - 1$. Then $f(x)$ is a polynomial, thus is continuous everywhere, and thus has the intermediate value property on every interval. Also $f(0) = -1$ and $f(1) = 1$, so $f(x)$ must assume the intermediate value 0 somewhere in the interval $[0, 1]$. Thus the equation $f(x) = 0$ has *at least* one solution. Next, $f'(x) = 5x^4 + 1$ is positive for all x, so f is an increasing function. Because f is continuous, its graph can therefore cross the x-axis at most once, and so the equation $f(x) = 0$ has *at most* one solution. Thus it has exactly one solution. Incidentally, Newton's method yields the approximate solution 0.75487766624669276. To four places, 0.7549.

3.10.28: Let $f(x) = x^2 - \cos x$. The graph of f on $[-1, 1]$ shows that there are two solutions, one near -0.8 and the other near 0.8. With $x_0 = 0.8$, Newton's method yields $x_1 = 0.824470434$, $x_2 = 0.824132377$, and $x_3 = 0.824132312 = x_4$. Because $f(-x) = f(x)$, the other solution is -0.824132312. Answer: ± 0.8241.

3.10.29: Let $f(x) = x - 2\sin x$. The graph of f on $[-2, 2]$ shows that there are exactly three solutions, the largest of which is approximately $x_0 = 1.9$. With Newton's method we obtain $x_1 = 1.895505940$, and $x_2 = 1.895494267 = x_3$. Because $f(-x) = -f(x)$, the other two solutions are 0 and -1.895494267. Answer: ± 1.8955 and 0.

3.10.30: Let $f(x) = x + 5\cos x$. The graph of f on the interval $[-5, 5]$ shows that there are exactly three solutions, approximately -1.3, 2.0, and 3.9. Newton's method then yields

n	First x_n	Second x_n	Third x_n
1	-1.306444739	1.977235450	3.839096917
2	-1.306440008	1.977383023	3.837468316
3	-1.306440008	1.977383029	3.837467106
4	-1.306440008	1.977383029	3.837467106

Answers: -1.3064, 1.9774, and 3.8375.

3.10.31: Let $f(x) = x^7 - 3x^3 + 1$. Then $f(x)$ is a polynomial, so f is continuous on every interval of real numbers, including the intervals $[-2, -1]$, $[0, 1]$, and $[1, 2]$. Also $f(-2) = -103 < 0 < 3 = f(-1)$, $f(0) = 1 > 0 > -1 = f(1)$, and $f(1) = -1 < 0 < 105 = f(2)$. Therefore the equation $f(x) = 0$ has one solution in $(-2, -1)$, another in $(0, 1)$, and a third in $(1, 2)$. (It has no other real solutions.) The graph of f shows that the first solution is near -1.4, the second is near 0.7, and the third is near 1.2. Then Newton's method yields

n	First x_n	Second x_n	Third x_n
1	-1.362661201	0.714876604	1.275651936
2	-1.357920265	0.714714327	1.258289744
3	-1.357849569	0.714714308	1.256999591
4	-1.357849553	0.714714308	1.256992779
5	-1.357849553	0.714714308	1.256992779

Answers: -1.3578, 0.7147, and 1.2570.

3.10.32: Let $f(x) = x^3 - 5$. Use the iteration

$$x - \frac{x^3 - 5}{3x^2} \longrightarrow x.$$

With $x_0 = 2$, we obtain the sequence of approximations 1.75, 1.710884354, 1.709976429, 1.709975947, and 1.709975947. Answer: 1.7100.

3.10.33: There is only one solution of $x^3 = \cos x$ for the following reasons: $x^3 < -1 \leqq \cos x$ if $x < -1$, $x^3 < 0 < \cos x$ if $-1 < x < 0$, x^3 is increasing on $[0, 1]$ whereas $\cos x$ is decreasing there (and their graphs cross in this interval as a consequence of the intermediate value property of continuous functions), and $x^3 > 1 \geqq \cos x$ for $x > 1$. The graph of $f(x) = x^3 - \cos x$ crosses the x-axis near $x_0 = 0.9$, and Newton's method yields $x_1 = 0.866579799$, $x_2 = 0.865475218$, and $x_3 = 0.865474033 = x_4$. Answer: Approximately 0.8654740331016145.

3.10.34: The graphs of $y = x$ and $y = \tan x$ show that the smallest positive solution of the equation $f(x) = x - \tan x = 0$ is between π and $3\pi/2$. With initial guess $x = 4.5$ we obtain 4.493613903, 4.493409655, 4.493409458, and 4.493409458. Answer: Approximately 4.493409457909064.

3.10.35: With $x_0 = 3.5$, we obtain the sequence $x_1 = 3.451450588$, $x_2 = 3.452461938$, and finally $x_3 = 3.452462314 = x_4$. Answer: Approximately 3.452462314057969.

3.10.36: To find a zero of $f(\theta) = \theta - \frac{1}{2}\sin\theta - \frac{17}{50}\pi$, we use the iteration

$$\theta - \frac{\theta - \frac{1}{2}\sin\theta - \frac{17}{50}\pi}{1 - \frac{1}{2}\cos\theta} \longrightarrow \theta.$$

The results, with $\theta_0 = 1.5$ $(86°56'\,37'')$, are: $\theta_1 = 1.569342$ $(89°55'\,00'')$, $\theta_2 = 1.568140$ $(89°50'\,52'')$, $\theta_3 = 1.568140$.

3.10.37: If the plane cuts the sphere at distance x from its center, then the smaller spherical segment has height $h = a - x = 1 - x$ and the larger has height $h = a + x = 1 + x$. So the smaller has volume

$$V_1 = \frac{1}{3}\pi h^2(3a - h) = \frac{1}{3}\pi(1 - x)^2(2 + x)$$

and the larger has volume

$$V_2 = \frac{1}{3}\pi h^2(3a - h) = \frac{1}{3}\pi(1 + x)^2(2 - x) = 2V_1.$$

These equations leads to

$$(1 + x)^2(2 - x) = 2(1 - x)^2(2 + x);$$

$$(x^2 + 2x + 1)(x - 2) + 2(x^2 - 2x + 1)(x + 2) = 0;$$

$$x^3 - 3x - 2 + 2x^3 - 6x + 4 = 0;$$

$$3x^3 - 9x + 2 = 0.$$

The last of these equations has three solutions, one near -1.83 (out of range), one near 1.61 (also out of range), and one near $x_0 = 0.2$. Newton's method yields $x_1 = 0.225925926$, $x_2 = 0.226073709$, and $x_3 = 0.226073714 = x_4$. Answer: 0.2261.

3.10.38: This table shows that the equation $f(x) = 0$ has solutions in each of the intervals $(-3, -2)$, $(0, 1)$, and $(1, 2)$.

x	-3	-2	-1	0	1	2	3
$f(x)$	-14	1	4	1	-2	1	16

The next table shows the results of the iteration of Newton's method:

n	x_n	x_n	x_n
0	1.5	0.5	-2.5
1	2.090909091	0.2307692308	-2.186440678
2	1.895903734	0.2540002371	-2.118117688
3	1.861832371	0.2541016863	-2.114914461
4	1.860806773	0.2541016884	-2.114907542
5	1.860805853	0.2541016884	-2.114907541
6	1.860805853		-2.114907541

Answer: -2.1149, 0.2541, and 1.8608.

3.10.39: We iterate using the formula

$$x - \frac{x + \tan x}{1 + \sec^2 x} \longrightarrow x.$$

Here is a sequence of simple *Mathematica* commands to find approximations to the four least positive solutions of the given equation, together with the results. (The command **list=g[list]** was executed repeatedly, but deleted from the output to save space.)

```
list={2.0,  5.0,  8.0,  11.0}
f[x_]:=x+Tan[x]
g[x_]:=N[x−f[x]/f′[x],  10]
list=g[list]
2.027314579, 4.879393859, 7.975116372, 11.00421012
2.028754298, 4.907699753, 7.978566616, 11.01202429
2.028757838, 4.913038110, 7.978665635, 11.02548807
2.028757838, 4.913180344, 7.978665712, 11.04550306
2.028757838, 4.913180439, 7.978665712, 11.06778114
2.028757838, 4.913180439, 7.978665712, 11.08205766
2.028757838, 4.913180439, 7.978665712, 11.08540507
2.028757838, 4.913180439, 7.978665712, 11.08553821
2.028757838, 4.913180439, 7.978665712, 11.08553841
```

Answer: 2.029 and 4.913.

3.10.40: Plot the graph of $f(x) = 4x^3 - 42x^2 - 19x - 28$ on $[-3, 12]$ to see that the equation $f(x) = 0$ has exactly one real solution neear $x = 11$. The initial guess $x_0 = 0$ yields the solution $x = 10.9902$ after 20 iterations. The initial guess $x_0 = 10$ yields the solution after three iterations. The initial guess $x_0 = 100$ yields the solution after ten iterations.

3.10.41: Similar triangles show that

$$\frac{x}{u+v} = \frac{5}{v} \qquad \text{and} \qquad \frac{y}{u+v} = \frac{5}{u},$$

so that

$$x = 5 \cdot \frac{u+v}{v} = 5(1+t) \qquad \text{and} \qquad y = 5 \cdot \frac{u+v}{u} = 5\left(1 + \frac{1}{t}\right).$$

Next, $w^2 + y^2 = 400$ and $w^2 + x^2 = 225$, so that:

$$400 - y^2 = 225 - x^2;$$

$$175 + x^2 = y^2;$$

$$175 + 25(1+t)^2 = 25\left(1 + \frac{1}{t}\right)^2;$$

$$175t^2 + 25t^2(1+t)^2 = 25(1+t)^2;$$

$$7t^2 + t^4 + 2t^3 + t^2 = t^2 + 2t + 1;$$

$$t^4 + 2t^3 + 7t^2 - 2t - 1 = 0.$$

The graph of $f(t) = t^4 + 2t^3 + 7t^2 - 2t - 1$ shows a solution of $f(t) = 0$ near $x_0 = 0.5$. Newton's method yields $x_1 = 0.491071429$, $x_2 = 0.490936940$, and $x_3 = 0.490936909 = x_4$. It now follows that $x = 7.454684547$, that $y = 15.184608052$, that $w = 13.016438772$, that $u = 4.286063469$, and that $v = 8.730375303$. Answers: $t = 0.4909$ and $w = 13.0164$.

3.10.42: We let $f(x) = 3\sin x - \ln x$. The graph of f on the interval $[1, 22]$ does not make it clear whether there are no solutions of $f(x) = 0$ between 20 and 22, or one solution, or two. But the graph on $[20, 21]$ makes it quite clear that there is no solution there: The maximum value of $f(x)$ there is approximately -0.005 and occurs close to $x = 20.4$. The iteration of Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

beginning with the initial values $x_0 = 7$, $x_0 = 9$, $x_0 = 13.5$, and $x_0 = 14.5$, yielded the following (rounded) results:

n	x_n	x_n	x_n	x_n
1	6.9881777136	8.6622012723	13.6118476398	14.6151381365
2	6.9882410659	8.6242919485	13.6226435579	14.6025252483
3	6.9882410677	8.6236121268	13.6227513693	14.6023754151
4	6.9882410677	8.6236119024	13.6227513801	14.6023753939
5	6.9882410677	8.6236119024	13.6227513801	14.6023753939

The last line in the table gives the other four solutions to ten-place accuracy.

3.10.43: Let $f(\theta) = (100+\theta)\cos\theta - 100$. The iterative formula of Newton's method is

$$\theta_{i+1} = \theta_i - \frac{f(\theta_i)}{f'(\theta_i)} \tag{1}$$

where, of course, $f'(\theta) = \cos\theta - (100+\theta)\sin\theta$. Beginning with $\theta_0 = 1$, iteration of the formula in (1) yields

0.4620438212,	0.2325523723,	0.1211226155,	0.0659741863,
0.0388772442,	0.0261688780,	0.0211747166,	0.0200587600,
0.0199968594,	0.0199966678,	0.0199966678,	0.0199966678.

We take the last value of θ_i to be sufficiently accurate. The corresponding radius of the asteriod is thus approximately $1000/\theta_{12} \approx 50008.3319$ ft, about 9.47 mi.

3.10.44: The length of the circular arc is $2R\theta = 5281$; the length of its chord is $2R\sin\theta = 5280$ (units are radians and feet). Division of the second of these equations by the first yields

$$\frac{\sin\theta}{\theta} = \frac{5280}{5281}.$$

To solve for θ by means of Newton's method, we let $f(\theta) = 5281\sin\theta - 5280\theta$. The iterative formula of Newton's method is

$$\theta_{i+1} = \theta_i - \frac{5281\sin\theta - 5280\theta}{5281\cos\theta - 5280}. \tag{1}$$

Beginning with the [poor] initial guess $\theta_0 = 1$, iteration of the formula in (1) yields these results:

0.655415,	0.434163,	0.289117,	0.193357,	0.130147,
0.0887267,	0.0621344,	0.0459270,	0.0373185,	0.0341721,
0.0337171,	0.0337078,	0.0337078,	0.0337078,	0.0337078.

Hence the radius of the circular arc is

$$R \approx \frac{5281}{2\theta_{15}} \approx 78335.1,$$

and its height at its center is

$$x = R(1-\cos\theta) \approx 44.4985.$$

That is, the maximum height is about 44.5 feet! Surprising to almost everyone.

Chapter 3 Miscellaneous Problems

3.M.1: If $y = y(x) = x^2 + 3x^{-2}$, then $\dfrac{dy}{dx} = 2x - 6x^{-3} = 2x - \dfrac{6}{x^3}$.

3.M.2: Given $y^2 = x^2$, implicit differentiation with respect to x yields

$$2y\,\frac{dy}{dx} = 2x, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{x}{y}.$$

3.M.3: If $y = y(x) = \sqrt{x} + \dfrac{1}{\sqrt[3]{x}} = x^{1/2} + x^{-1/3}$, then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - \frac{1}{3}x^{-4/3} = \frac{1}{2x^{1/2}} - \frac{1}{3x^{4/3}} = \frac{3x^{5/6} - 2}{6x^{4/3}}.$$

3.M.4: Given $y = y(x) = (x^2 + 4x)^{5/2}$, the chain rule yields $\dfrac{dy}{dx} = \dfrac{5}{2}(x^2 + 4x)^{3/2}(2x + 4)$.

3.M.5: Given $y = y(x) = (x - 1)^7(3x + 2)^9$, the product rule and the chain rule yield

$$\frac{dy}{dx} = 7(x-1)^6(3x+2)^9 + 27(x-1)^7(3x+2)^8 = (x-1)^6(3x+2)^8(48x - 13).$$

3.M.6: Given $y = y(x) = \dfrac{x^4 + x^2}{x^2 + x + 1}$, the quotient rule yields

$$\frac{dy}{dx} = \frac{(x^2+x+1)(4x^3+2x) - (x^4+x^2)(2x+1)}{(x^2+x+1)^2} = \frac{2x^5 + 3x^4 + 4x^3 + x^2 + 2x}{(x^2+x+1)^2}.$$

3.M.7: If $y = y(x) = \left(3x - \dfrac{1}{2x^2}\right)^4 = \left(3x - \frac{1}{2}x^{-2}\right)^4$, then

$$\frac{dy}{dx} = 4\left(3x - \tfrac{1}{2}x^{-2}\right)^3 \cdot \left(3 + x^{-3}\right) = 4\left(3x - \frac{1}{2x^2}\right)^3 \cdot \left(3 + \frac{1}{x^3}\right).$$

3.M.8: Given $y = y(x) = x^{10}\sin 10x$, the product rule and the chain rule yield

$$\frac{dy}{dx} = 10x^9 \sin 10x + 10x^{10}\cos 10x = 10x^9(\sin 10x + x\cos 10x).$$

3.M.9: Given $xy = 9$, implicit differentiation with respect to x yields

$$x\,\frac{dy}{dx} + y = 0, \qquad \text{so that} \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Alternatively, $y = y(x) = \dfrac{9}{x}$, so that $\dfrac{dy}{dx} = -\dfrac{9}{x^2}$.

3.M.10: $y = y(x) = (5x^6)^{-1/2}$: $\dfrac{dy}{dx} = -\dfrac{1}{2}(5x^6)^{-3/2}(30x^5) = -\dfrac{3}{x\sqrt{5x^6}} = -\dfrac{3\sqrt{5}}{5x^4}$.

3.M.11: Given $y = y(x) = \dfrac{1}{\sqrt{(x^3-x)^3}} = (x^3-x)^{-3/2}$,

$$\frac{dy}{dx} = -\frac{3}{2}(x^3-x)^{-5/2}(3x^2-1) = -\frac{3(3x^2-1)}{2(x^3-x)^{5/2}}.$$

3.M.12: Given $y = y(x) = (2x+1)^{1/3}(3x-2)^{1/5}$,

$$\begin{aligned}\frac{dy}{dx} &= \frac{2}{3}(2x+1)^{-2/3}(3x-2)^{1/5} + \frac{3}{5}(3x-2)^{-4/5}(2x+1)^{1/3}\\ &= \frac{10(3x-2)+9(2x+1)}{15(2x+1)^{2/3}(3x-2)^{4/5}} = \frac{48x-11}{15(2x+1)^{2/3}(3x-2)^{4/5}}.\end{aligned}$$

3.M.13: $\dfrac{dy}{dx} = \dfrac{dy}{du}\cdot\dfrac{du}{dx} = \dfrac{-2u}{(1+u^2)^2}\cdot\dfrac{-2x}{(1+x^2)^2}$. Now $1+u^2 = 1+\dfrac{1}{(1+x^2)^2} = \dfrac{x^4+2x^2+2}{(1+x^2)^2}$.

So $\dfrac{dy}{du} = \dfrac{-2u}{(1+u^2)^2} = \dfrac{-2}{1+x^2}\cdot\dfrac{(1+x^2)^4}{(x^4+2x^2+2)^2} = \dfrac{-2(1+x^2)^3}{(x^4+2x^2+2)^2}$.

Therefore $\dfrac{dy}{dx} = \dfrac{-2(1+x^2)^3}{(x^4+2x^2+2)^2}\cdot\dfrac{-2x}{(1+x^2)^2} = \dfrac{4x(1+x^2)}{(x^4+2x^2+2)^2}$.

3.M.14: $3x^2 = 2\dfrac{dy}{dx}\sin y\,\cos y$, so $\dfrac{dy}{dx} = \dfrac{3x^2}{2\sin y\,\cos y}$.

3.M.15: Given $y = y(x) = \left(x^{1/2}+2^{1/3}x^{1/3}\right)^{7/3}$,

$$\frac{dy}{dx} = \frac{7}{3}\left(x^{1/2}+2^{1/3}x^{1/3}\right)^{4/3}\cdot\left(\frac{1}{2}x^{-1/2}+\frac{2^{1/3}}{3}x^{-2/3}\right).$$

3.M.16: Given $y = y(x) = \sqrt{3x^5-4x^2} = (3x^5-4x^2)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}(3x^5-4x^2)^{-1/2}\cdot(15x^4-8x) = \frac{15x^4-8x}{2\sqrt{3x^5-4x^2}}.$$

3.M.17: If $y = \dfrac{u+1}{u-1}$ and $u = (x+1)^{1/2}$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{(u-1)-(u+1)}{(u-1)^2}\cdot\frac{1}{2}(x+1)^{-1/2} = -\frac{2}{(u-1)^2}\cdot\frac{1}{2\sqrt{x+1}} = -\frac{1}{\left(\sqrt{x+1}-1\right)^2\sqrt{x+1}}.$$

3.M.18: Given $y = y(x) = \sin(2\cos 3x)$,

$$\frac{dy}{dx} = [\cos(2\cos 3x)]\cdot(-6\sin 3x) = -6(\sin 3x)\,\cos(2\cos 3x).$$

3.M.19: Given $x^2y^2 = x + y$, we differentiate (both sides) implicitly with respect to x and obtain

$$2xy^2 + 2x^2y\,\frac{dy}{dx} = 1 + \frac{dy}{dx};$$

$$2x^2y\,\frac{dy}{dx} - \frac{dy}{dx} = 1 - 2xy^2;$$

$$\frac{dy}{dx} = \frac{1-2xy^2}{2x^2y-1};$$

$$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{x-2x^2y^2}{2x^2y^2-y};$$

$$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{x-2(x+y)}{2(x+y)-y};$$

$$\frac{dy}{dx} = \frac{y}{x}\cdot\frac{-x-2y}{2x+y} = -\frac{(x+2y)y}{(2x+y)x}.$$

Of course you may stop with the third line, but to find horizontal and vertical lines tangent to the graph of $x^2y^2 = x + y$, the last line is probably the most convenient.

3.M.20: Given $y = y(x) = \left(1 + \sin x^{1/2}\right)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}\left(1+\sin x^{1/2}\right)^{-1/2}\left(\cos x^{1/2}\right)\cdot\frac{1}{2}x^{-1/2} = \frac{\cos\sqrt{x}}{4\sqrt{x}\,\sqrt{1+\sin\sqrt{x}}}.$$

3.M.21: Given $y = y(x) = \sqrt{x + \sqrt{2x+\sqrt{3x}}} = \left(x + \left[2x + (3x)^{1/2}\right]^{1/2}\right)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}\left(x + \left[2x+(3x)^{1/2}\right]^{1/2}\right)^{-1/2}\cdot\left(1 + \frac{1}{2}\left[2x+(3x)^{1/2}\right]^{-1/2}\cdot\left[2+\frac{3}{2}(3x)^{-1/2}\right]\right).$$

The symbolic algebra program *Mathematica* writes this answer without exponents as follows:

$$\frac{dy}{dx} = \frac{1 + \dfrac{2+\dfrac{\sqrt{3}}{2\sqrt{x}}}{2\sqrt{2x+\sqrt{3x}}}}{2\sqrt{x+\sqrt{2x+\sqrt{3x}}}}.$$

3.M.22: $\displaystyle\frac{dy}{dx} = \frac{(x^2+\cos x)(1+\cos x)-(x+\sin x)(2x-\sin x)}{(x^2+\cos x)^2} = \frac{1-x^2-x\sin x+\cos x+x^2\cos x}{(x^2+\cos x)^2}.$

3.M.23: Given $x^{1/3} + y^{1/3} = 4$, differentiate both sides with respect to x:

$$\frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0, \qquad \text{so} \qquad \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{2/3}.$$

3.M.24: Given $x^3 + y^3 = xy$, differentiate both sides with respect to x to obtain

$$3x^2 + 3y^2 \frac{dy}{dx} = x\frac{dy}{dx} + y, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}.$$

3.M.25: Given $y = (1+2u)^3$ where $u = (1+x)^{-3}$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 6(1+2u)^2 \cdot (-3)(1+x)^{-4} = -\frac{18(1+2u)^2}{(1+x)^4} = -\frac{18(1+2(1+x)^{-3})^2}{(1+x)^4}$$

$$= -\frac{18(1+x)^6(1+2(1+x)^{-3})^2}{(1+x)^{10}} = -\frac{18((1+x)^3+2)^2}{(1+x)^{10}} = -18 \cdot \frac{(x^3+3x^2+3x+3)^2}{(x+1)^{10}}.$$

3.M.26: $\dfrac{dy}{dx} = \big(-2\cos(\sin^2 x)\sin(\sin^2 x)\big) \cdot \big(2\sin x\,\cos x\big).$

3.M.27: Given $y = y(x) = \left(\dfrac{\sin^2 x}{1+\cos x}\right)^{1/2}$,

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{\sin^2 x}{1+\cos x}\right)^{-1/2} \cdot \frac{(1+\cos x)(2\sin x\,\cos x) + \sin^3 x}{(1+\cos x)^2}$$

$$= \left(\frac{1+\cos x}{\sin^2 x}\right)^{1/2} \cdot \frac{2\sin x\,\cos x + 2\sin x\,\cos^2 x + \sin^3 x}{2(1+\cos x)^2}.$$

3.M.28: $\dfrac{dy}{dx} = \dfrac{3\,(1+\sqrt{x}\,)^2}{2\sqrt{x}}\,(1-2\sqrt[3]{x}\,)^4 + 4\,(1-2\sqrt[3]{x}\,)^3\,\big(-\tfrac{2}{3}x^{-2/3}\big)\,(1+\sqrt{x}\,)^3.$

3.M.29: Given: $y = y(x) = \dfrac{\cos 2x}{\sqrt{\sin 3x}} = (\cos 2x)(\sin 3x)^{-1/2}$,

$$\frac{dy}{dx} = (-2\sin 2x)(\sin 3x)^{-1/2} + (\cos 2x)\left(-\frac{1}{2}(\sin 3x)^{-3/2}\right)(3\cos 3x)$$

$$= -\frac{2\sin 2x}{\sqrt{\sin 3x}} - \frac{3\cos 2x\,\cos 3x}{2(\sin 3x)^{3/2}} = -\frac{4\sin 2x\,\sin 3x + 3\cos 2x\,\cos 3x}{2(\sin 3x)^{3/2}}.$$

3.M.30: $3x^2 - x^2\dfrac{dy}{dx} - 2xy + y^2 + 2xy\dfrac{dy}{dx} - 3y^2\dfrac{dy}{dx} = 0$: $\dfrac{dy}{dx} = \dfrac{3x^2 - 2xy + y^2}{3y^2 - 2xy + x^2}.$

3.M.31: $\dfrac{dy}{dx} = (e^x) \cdot \cos x + e^x \cdot (-\sin x) = e^x(\cos x - \sin x).$

3.M.32: $\dfrac{dy}{dx} = e^{-2x}(3\cos 3x - 2\sin 3x).$

3.M.33: $\dfrac{dy}{dx} = -\dfrac{3e^x}{(2+3e^x)^{5/2}\left[1 + (2+3e^x)^{-3/2}\right]^{1/3}}.$

3.M.34: $\dfrac{dy}{dx} = 5(e^x + e^{-x})^4(e^x - e^{-x})$.

3.M.35: $\dfrac{dy}{dx} = -\dfrac{\cos^2\left([1+\ln x]^{1/3}\right)\sin\left([1+\ln x]^{1/3}\right)}{x[1+\ln x]^{2/3}}$.

3.M.36: If $f(x) = \cos(1 - e^{-x})$, then $f'(x) = -e^{-x}\sin(1 - e^{-x})$.

3.M.37: If $f(x) = \sin^2(e^{-x}) = [\sin(e^{-x})]^2$, then $f'(x) = -2e^{-x}\sin(e^{-x})\,\cos(e^{-x})$.

3.M.38: If $f(x) = \ln(x + e^{-x})$, then $f'(x) = \dfrac{1 - e^{-x}}{x + e^{-x}}$.

3.M.39: If $f(x) = e^x\cos 2x$, then $f'(x) = e^x\cos 2x - 2e^x\sin 2x$.

3.M.40: If $f(x) = e^{-2x}\sin 3x$, then $f'(x) = 3e^{-2x}\cos 3x - 2e^{-2x}\sin 3x$.

3.M.41: If $g(t) = \ln(te^{t^2}) = (\ln t) + t^2\ln e = (\ln t) + t^2$, then

$$g'(t) = \frac{1}{t} + 2t = \frac{1 + 2t^2}{t}.$$

3.M.42: If $g(t) = 3(e^t - \ln t)^5$, then $g'(t) = 15(e^t - \ln t)^4\left(e^t - \dfrac{1}{t}\right)$.

3.M.43: If $g(t) = \sin(e^t)\,\cos(e^{-t})$, then

$$g'(t) = e^t\cos(e^t)\,\cos(e^{-t}) + e^{-t}\sin(e^t)\,\sin(e^{-t}).$$

3.M.44: If $f(x) = \dfrac{2 + 3x}{e^{4x}}$, then

$$f'(x) = \frac{3e^{4x} - 4(2 + 3x)e^{4x}}{(e^{4x})^2} = \frac{3 - 8 - 12x}{e^{4x}} = -\frac{12x + 5}{e^{4x}}.$$

3.M.45: If $g(t) = \dfrac{1 + e^t}{1 - e^t}$, then $g'(t) = \dfrac{(1 - e^t)e^t + (1 + e^t)e^t}{(1 - e^t)^2} = \dfrac{2e^t}{(1 - e^t)^2}$.

3.M.46: Given $xe^y = y$, we apply D_x to both sides and find that

$$e^y + xe^y\frac{dy}{dx} = \frac{dy}{dx}; \qquad (1 - xe^y)\frac{dy}{dx} = e^y;$$

$$\frac{dy}{dx} = \frac{e^y}{1 - xe^y}; \qquad \frac{dy}{dx} = \frac{e^y}{1 - y}.$$

In the last step we used the fact that $xe^y = y$ to simplify the denominator.

3.M.47: Given $\sin(e^{xy}) = x$, we apply D_x to both sides and find that

$$[\cos(e^{xy})] \cdot e^{xy} \cdot \left(y + x\frac{dy}{dx}\right) = 1; \qquad xe^{xy}[\cos(e^{xy})] \cdot \frac{dy}{dx} = 1 - ye^{xy}\cos(e^{xy});$$

$$\frac{dy}{dx} = \frac{1 - ye^{xy}\cos(e^{xy})}{xe^{xy}\cos(e^{xy})}.$$

3.M.48: Given $e^x + e^y = e^{xy}$, we apply D_x to both sides and find that

$$e^x + e^y\frac{dy}{dx} = e^{xy}\left(y + x\frac{dy}{dx}\right); \qquad (e^y - xe^{xy})\frac{dy}{dx} = ye^{xy} - e^x;$$

$$\frac{dy}{dx} = \frac{ye^{xy} - e^x}{e^y - xe^{xy}}.$$

3.M.49: Given $x = ye^y$, we apply D_x to both sides and obtain

$$1 = e^y\frac{dy}{dx} + ye^y\frac{dy}{dx}; \qquad \frac{dy}{dx} = \frac{1}{e^y + ye^y} = \frac{y}{ye^y + y^2e^y} = \frac{y}{x + xy}.$$

We used the fact that $ye^y = x$ in the simplification in the last step.

Here is an alternative approach to finding dy/dx. Beginning with $x = ye^y$, we differentiate with respect to y and find that

$$\frac{dx}{dy} = e^y + ye^y, \qquad \text{so that} \qquad \frac{dy}{dx} = \frac{1}{e^y + ye^y}.$$

3.M.50: Given $e^{x-y} = xy$, we apply D_x to both sides and find that

$$e^{x-y}\left(1 - \frac{dy}{dx}\right) = y + x\frac{dy}{dx}; \qquad \left(x + e^{x-y}\right)\frac{dy}{dx} = e^{x-y} - y;$$

$$\frac{dy}{dx} = \frac{e^{x-y} - y}{e^{x-y} + x}; \qquad \frac{dy}{dx} = \frac{xy - y}{xy + x} = \frac{(x-1)y}{(y+1)x}.$$

We used the fact that $e^{x-y} = xy$ to make the simplification in the last step.

3.M.51: Given $x\ln y = x + y$, we apply D_x to both sides and find that

$$\ln y + \frac{x}{y} \cdot \frac{dy}{dx} = 1 + \frac{dy}{dx};$$

$$\left(\frac{x}{y} - 1\right) \cdot \frac{dy}{dx} = 1 - \ln y;$$

$$\frac{x - y}{y} \cdot \frac{dy}{dx} = 1 - \ln y;$$

$$\frac{dy}{dx} = \frac{y(1 - \ln y)}{x - y}.$$

3.M.52: Given: $y = \sqrt{(x^2-4)\sqrt{2x+1}}$. Thus

$$\ln y = \ln\left[(x^2-4)(2x+1)^{1/2}\right]^{1/2} = \frac{1}{2}\ln\left[(x^2-4)(2x+1)^{1/2}\right] = \frac{1}{2}\left[\ln(x^2-4) + \frac{1}{2}\ln(2x+1)\right].$$

Therefore

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{x}{x^2-4} + \frac{1}{2(2x+1)} = \frac{5x^2+2x-4}{2(x^2-4)(2x+1)},$$

and so

$$\frac{dy}{dx} = y(x)\cdot\frac{5x^2+2x-4}{2(x^2-4)(2x+1)} = \frac{(5x^2+2x-4)\sqrt{(x^2-4)\sqrt{2x+1}}}{2(x^2-4)(2x+1)}.$$

3.M.53: Given: $y = (3-x^2)^{1/2}(x^4+1)^{-1/4}$. Thus

$$\ln y = \frac{1}{2}\ln(3-x^2) - \frac{1}{4}\ln(x^4+1),$$

and therefore

$$\frac{1}{y}\cdot\frac{dy}{dx} = -\frac{x}{3-x^2} - \frac{x^3}{x^4+1} = \frac{x(3x^2+1)}{(x^2-3)(x^4+1)}.$$

Thus

$$\frac{dy}{dx} = y(x)\cdot\frac{x(3x^2+1)}{(x^2-3)(x^4+1)} = -\frac{x(3x^2+1)(3-x^2)^{1/2}}{(3-x^2)(x^4+1)(x^4+1)^{1/4}} = -\frac{x(3x^2+1)}{(3-x^2)^{1/2}(x^4+1)^{5/4}}.$$

3.M.54: Given: $y = \left[\dfrac{(x+1)(x+2)}{(x^2+1)(x^2+2)}\right]^{1/3}$. Then

$$\ln y = \frac{1}{3}\left[\ln(x+1) + \ln(x+2) - \ln(x^2+1) - \ln(x^2+2)\right];$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{3}\left(\frac{1}{x+1} + \frac{1}{x+2} - \frac{2x}{x^2+1} - \frac{2x}{x^2+2}\right);$$

$$\frac{dy}{dx} = y(x)\cdot\frac{6-8x-9x^2-8x^3-9x^4-2x^5}{3(x+1)(x+2)(x^2+1)(x^2+2)};$$

$$\frac{dy}{dx} = \frac{6-8x-9x^2-8x^3-9x^4-2x^5}{3(x+1)(x+2)(x^2+1)(x^2+2)}\cdot\left[\frac{(x+1)(x+2)}{(x^2+1)(x^2+2)}\right]^{1/3};$$

$$\frac{dy}{dx} = \frac{6-8x-9x^2-8x^3-9x^4-2x^5}{3(x+1)^{2/3}(x+2)^{2/3}(x^2+1)^{4/3}(x^2+2)^{4/3}}.$$

3.M.55: If $y = (x+1)^{1/2}(x+2)^{1/3}(x+3)^{1/4}$, then

$$\ln y = \frac{1}{2}\ln(x+1) + \frac{1}{3}\ln(x+2) + \frac{1}{4}\ln(x+3);$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{2(x+1)} + \frac{1}{3(x+2)} + \frac{1}{4(x+3)};$$

$$\frac{dy}{dx} = y(x)\cdot\frac{13x^2+55x+54}{12(x+1)(x+2)(x+3)} = \frac{13x^2+55x+54}{12(x+1)^{1/2}(x+2)^{2/3}(x+3)^{3/4}}.$$

3.M.56: If $y = x^{(e^x)}$, then

$$\ln y = e^x \ln x; \qquad \frac{1}{y}\cdot\frac{dy}{dx} = \frac{e^x}{x} + e^x \ln x;$$

$$\frac{dy}{dx} = y(x)\cdot\frac{(1 + x\ln x)e^x}{x}; \qquad \frac{dy}{dx} = \frac{(1 + x\ln x)e^x}{x}\cdot\left(x^{(e^x)}\right).$$

3.M.57: Given: $y = (\ln x)^{\ln x}$, $x > 1$. Then

$$\ln y = (\ln x)\ln(\ln x);$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{x}\ln(\ln x) + \frac{\ln x}{x\ln x} = \frac{1 + \ln(\ln x)}{x};$$

$$\frac{dy}{dx} = \frac{1 + \ln(\ln x)}{x}\cdot(\ln x)^{\ln x}.$$

3.M.58: $\dfrac{dy}{dx} = \dfrac{(x-1)-(x+1)}{(x-1)^2} = -\dfrac{2}{(x-1)^2}$; the slope of the line tangent at $(0, -1)$ is -2; an equation of the tangent line is $y + 1 = -2x$; that is, $2x + y + 1 = 0$.

3.M.59: $1 = (2\cos 2y)\dfrac{dy}{dx}$, so $\dfrac{dy}{dx} = \dfrac{1}{2\cos 2y}$. Because $\dfrac{dy}{dx}$ is undefined at $(1, \pi/4)$, there may well be a vertical tangent at that point. And indeed there is: $\dfrac{dx}{dy} = 0$ at $(1, \pi/4)$. So an equation of the tangent line is $x = 1$.

3.M.60: $\dfrac{dy}{dx} = \dfrac{3y - 2x}{4y - 3x}$; at $(2, 1)$ the slope is $\frac{1}{2}$. So an equation of the tangent is $y - 1 = \frac{1}{2}(x - 2)$; that is, $x = 2y$.

3.M.61: $\dfrac{dy}{dx} = \dfrac{2x + 1}{3y^2}$; at $(0, 0)$, $\dfrac{dx}{dy} = 0$, so the tangent line is vertical. Its equation is $x = 0$.

3.M.62: $V(x) = \frac{1}{3}\pi(36x^2 - x^3)$: $V'(x) = \pi x(24 - x)$. Now $\dfrac{dV}{dt} = \dfrac{dV}{dx}\cdot\dfrac{dx}{dt}$; when $x = 6$, $36\pi = -108\pi\,\dfrac{dx}{dt}$, so $\dfrac{dx}{dt} = -\frac{1}{3}$ (in./s) when $x = 6$.

3.M.63: Let r be the radius of the sandpile, h its height, each a function of time t. We know that $2r = h$, so the volume of the sandpile at time t is

$$V = \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^3.$$

$$\text{So}\quad 25\pi = \frac{dV}{dt} = 2\pi r^2\frac{dr}{dt};$$

substitution of $r = 5$ yields the answer: $dr/dt = \frac{1}{2}$ (ft/min) when $r = 5$ (ft).

3.M.64: Divide each term in the numerator and denominator by $\sin x$ to obtain

$$\lim_{x\to 0} \frac{x}{\sin x} - \lim_{x\to 0} \frac{1}{\cos x} = 1 - 1 = 0.$$

3.M.65: $x \cot 3x = \dfrac{1}{3} \cdot \dfrac{3x}{\sin 3x} \to \dfrac{1}{3} \cdot 1 = \dfrac{1}{3}$ as $x \to 0$.

3.M.66: $\dfrac{\sin 2x}{\sin 5x} = \dfrac{2}{5} \cdot \dfrac{\sin 2x}{2x} \cdot \dfrac{5x}{\sin 5x} \to \dfrac{2}{5}$ as $x \to 0$.

3.M.67: $x^2 \csc 2x \cot 2x = \dfrac{1}{4} \cdot \dfrac{2x}{\sin 2x} \cdot \dfrac{2x}{\sin 2x} \cdot \cos 2x \to \dfrac{1}{4} \cdot 1 \cdot 1 \cdot 1 = \dfrac{1}{4}$ as $x \to 0$.

3.M.68: $-1 \leqq \sin u \leqq 1$ for all u. So

$$-x^2 \leqq x^2 \sin \frac{1}{x^2} \leqq x^2$$

for all $x \neq 0$. But $x^2 \to 0$ as $x \to 0$, so the limit of the expression caught in the squeeze is also zero.

3.M.69: $-1 \leqq \sin u \leqq 1$ for all u. So

$$-\sqrt{x} \leqq \sqrt{x} \sin \frac{1}{x} \leqq \sqrt{x}$$

for all $x > 0$. But $\sqrt{x} \to 0$ as $x \to 0^+$, so the limit is zero.

3.M.70: $h(x) = (x + x^4)^{1/3} = f(g(x))$ where $f(x) = x^{1/3}$ and $g(x) = x + x^4$. Therefore $h'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{3}(x + x^4)^{-2/3} \cdot (1 + 4x^3)$.

3.M.71: $h(x) = (x^2 + 25)^{-1/2} = f(g(x))$ where $f(x) = x^{-1/2}$ and $g(x) = x^2 + 25$. Therefore $h'(x) = f'(g(x)) \cdot g'(x) = -\frac{1}{2}(x^2 + 25)^{-3/2} \cdot 2x$.

3.M.72: First,

$$h(x) = \sqrt{\frac{x}{x^2+1}} = \left(\frac{x}{x^2+1}\right)^{1/2} = f(g(x)) \quad \text{where} \quad f(x) = x^{1/2} \quad \text{and} \quad g(x) = \frac{x}{x^2+1}.$$

Therefore

$$h'(x) = \frac{1}{2}\left(\frac{x}{x^2+1}\right)^{-1/2} \cdot \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \left(\frac{x^2+1}{x}\right)^{1/2} \cdot \frac{1 - x^2}{2(x^2+1)^2} = \frac{1 - x^2}{2x^{1/2}(x^2+1)^{3/2}}.$$

3.M.73: One solution: $h(x) = (x - 1)^{5/3} = f(g(x))$ where $f(x) = x^{5/3}$ and $g(x) = x - 1$. Therefore $h'(x) = f'(g(x)) \cdot g'(x) = \frac{5}{3}(x - 1)^{2/3} \cdot 1 = \frac{5}{3}(x - 1)^{2/3}$. You might alternatively choose $f(x) = x^{1/3}$ and $g(x) = (x - 1)^5$.

3.M.74: If

$$h(x) = \frac{(x+1)^{10}}{(x-1)^{10}}, \quad \text{then} \quad h(x) = f(g(x)) \quad \text{where} \quad f(x) = x^{10} \quad \text{and} \quad g(x) = \frac{x+1}{x-1}.$$

Hence

$$h'(x) = f'(g(x)) \cdot g'(x) = 10\left(\frac{x+1}{x-1}\right)^9 \cdot \frac{(x+1)-(x-1)}{(x-1)^2} = 10\left(\frac{x+1}{x-1}\right)^9 \cdot \frac{2}{(x-1)^2} = \frac{20(x+1)^9}{(x-1)^{11}}.$$

3.M.75: $h(x) = \cos(x^2+1) = f(g(x))$ where $f(x) = \cos x$ and $g(x) = x^2+1$. Therefore $h'(x) = f'(g(x)) \cdot g'(x) = -2x\sin(x^2+1)$.

3.M.76: $T = 2\pi\sqrt{\dfrac{L}{32}}$; $\dfrac{dT}{dL} = \dfrac{\pi}{32}\sqrt{\dfrac{32}{L}}$. So $\left.\dfrac{dT}{dL}\right|_{L=4} = \dfrac{\pi\sqrt{2}}{16}$. Hence when $L = 4$, T is changing at approximately 0.27768 seconds per foot.

3.M.77: Of course r denotes the radius of the sphere. First, $\dfrac{dV}{dA} \cdot \dfrac{dA}{dr} = \dfrac{dV}{dr}$. Now $V = \dfrac{4}{3}\pi r^3$ and $A = 4\pi r^2$, so $\dfrac{dV}{dA} \cdot 8\pi r = 4\pi r^2$, and therefore $\dfrac{dV}{dA} = \dfrac{r}{2} = \dfrac{1}{4}\sqrt{\dfrac{A}{\pi}}$.

3.M.78: Let $(a,\, b)$ denote the point of tangency; note that

$$b = a + \frac{1}{a}, \quad a > 0, \quad \text{and} \quad h'(x) = 1 - \frac{1}{x^2}.$$

The slope of the tangent line can be computed using the two-point formula for slope and by using the derivative. We equate the results to obtain

$$\frac{a + \dfrac{1}{a} - 0}{a-1} = 1 - \frac{1}{a^2} = \frac{a^2-1}{a^2}.$$

It follows that $a^3 + a = (a-1)(a^2-1) = a^3 - a^2 - a + 1$. Thus $a^2 + 2a - 1 = 0$, and so $a = -1 + \sqrt{2}$ (the positive root because $a > 0$). Consequently the tangent line has slope $-2\left(1+\sqrt{2}\right)$ and thus equation

$$y = -2\left(1+\sqrt{2}\right)(x-1).$$

3.M.79: Let $y = y(t)$ denote the altitude of the rocket at time t; let $u = u(t)$ denote the angle of elevation of the observer's line of sight at time t. Then $\tan u = y/3$, so that $y = 3\tan u$ and, therefore,

$$\frac{dy}{dt} = (3\sec^2 u)\frac{du}{dt}.$$

When $u = 60°$, we take $du/dt = \dfrac{\pi}{3}$ and find that the speed of the rocket is

$$\left.\frac{dy}{dt}\right|_{u=60°} = \frac{3}{\cos^2(\pi/3)} \cdot \frac{\pi}{30} = \frac{2}{5}\pi \approx 1.2566 \text{ (mi/s)},$$

about 4524 mi/h, or about 6635 ft/s.

3.M.80: Current production per well: 200 (bbl/day). Number of new wells: x ($x \geqq 0$). Production per well: $200 - 5x$. Total production:

$$T = T(x) = (20 + x)(200 - 5x), \qquad 0 \leqq x \leqq 40.$$

Now $T(x) = 4000 + 100x - 5x^2$, so $T'(x) = 100 - 10x$. $T'(x) = 0$ when $x = 10$. $T(0) = 4000$, $T(40) = 0$, and $T(10) = 4500$. So $x = 10$ maximizes $T(x)$. Answer: Ten new wells should be drilled, thereby increasing total production from 4000 bbl/day to 4500 bbl/day.

3.M.81: Let the circle be the one with equation $x^2 + y^2 = R^2$ and let the base of the triangle lie on the x-axis; denote the opposite vertex of the triangle by (x, y). The area of the triangle $A = Ry$ is clearly maximal when y is maximal; that is, when $y = R$. To solve this problem using calculus, let θ be the angle of the triangle at $(-R, 0)$. Because the triangle has a right angle at (x, y), its two short sides are $2R\cos\theta$ and $2R\sin\theta$, so its area is

$$A(\theta) = 2R^2 \sin\theta\, \cos\theta = R^2 \sin 2\theta, \qquad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

Then $A'(\theta) = 2R^2 \cos 2\theta$; $A'(\theta) = 0$ when $\cos 2\theta = 0$; because θ lies in the first quadrant, $\theta = \frac{1}{4}\pi$. Finally, $A(0) = 0 = A(\pi/2)$, but $A(\pi/4) = R^2 > 0$. Hence the maximum possible area of such a triangle is R^2.

3.M.82: Let x be the length of the edges of each of the 20 small squares. The first five boxes measure $210 - 2x$ by $336 - 2x$ by x. The total volume is then

$$V(x) = 5x(210 - 2x)(336 - 2x) + 8x^3, \qquad 0 \leqq x \leqq 105.$$

Thus $V(x) = 28x^3 - 5460x^2 + 352800x$, and so

$$V'(x) = 84x^2 - 10290x + 352800 = 84(x^2 - 130x + 4200) = 84(x - 60)(x - 70).$$

So $V'(x) = 0$ when $x = 60$ and when $x = 70$. But $V(0) = 0$, $V(60) = 7{,}560{,}000$, $V(70) = 7{,}546{,}000$, and $V(105) = 9{,}261{,}000$. Answer: For maximal volume, make x as large as possible: 105 cm. This yields the maximum volume, 9,261,000 cm^3. Note that it is attained by constructing one large cubical box and that some material is wasted.

3.M.83: Let one sphere have radius r; the other, s. We seek the extrema of $A = 4\pi(r^2 + s^2)$ given $\frac{4}{3}\pi(r^3 + s^3) = V$, a constant. We illustrate here the **method of auxiliary variables:**

$$\frac{dA}{dr} = 4\pi\left(2r + 2s\frac{ds}{dr}\right);$$

the condition $dA/dr = 0$ yields $ds/dr = -r/s$. But we also know that $\frac{4}{3}\pi(r^3 + s^3) = V$; differentiation of both sides of this *identity* with respect to r yields

$$\frac{4}{3}\pi\left(3r^2 + 3s^2\frac{ds}{dr}\right) = 0, \quad \text{and so}$$
$$3r^2 + 3s^2\left(-\frac{r}{s}\right) = 0;$$
$$r^2 - rs = 0.$$

Therefore $r = 0$ or $r = s$. Also, ds/dr is undefined when $s = 0$. So we test these three critical points. If $r = 0$ or if $s = 0$, there is only one sphere, with radius $(3V/4\pi)^{1/3}$ and surface area $(36\pi V^2)^{1/3}$. If $r = s$, then there are two spheres of equal size, both with radius $\frac{1}{2}(3V/\pi)^{1/3}$ and surface area $(72\pi V^2)^{1/3}$. Therefore, for maximum surface area, make two equal spheres. For minimum surface area, make only one sphere.

3.M.84: Let x be the length of the edge of the rectangle on the side of length 4 and y the length of the adjacent edges. By similar triangles, $3/4 = (3 - y)/x$, so $x = 4 - \frac{4}{3}y$. We are to maximize $A = xy$; that is,

$$A = A(y) = 4y - \frac{4}{3}y^2, \qquad 0 \leqq y \leqq 3.$$

Now $dA/dy = 4 - \frac{8}{3}y$; $dA/dy = 0$ when $y = \frac{3}{2}$. Because $A(0) = A(3) = 0$, the maximum is $A(2) = 3$ (m^2).

3.M.85: Let r be the radius of the cone; let its height be $h = R + y$ where $0 \leqq y \leqq R$. (Actually, $-R \leqq y \leqq R$, but the cone will have maximal volume if $y \geqq 0$.) A central vertical cross section of the figure (*draw it!*) shows a right triangle from which we read the relation $y^2 = R^2 - r^2$. We are to maximize $V = \frac{1}{3}\pi r^2 h$, so we write

$$V = V(r) = \frac{1}{3}\pi\left[r^2\left(R + \sqrt{R^2 - r^2}\right)\right], \qquad 0 \leqq r \leqq R.$$

The condition $V'(r) = 0$ leads to the equation $r\left(2R^2 - 3r^2 + 2R\sqrt{R^2 - r^2}\right) = 0$, which has the two solutions $r = 0$ and $r = \frac{2}{3}R\sqrt{2}$. Now $V(0) = 0$, $V(R) = \frac{1}{3}\pi R^3$ (which is one-fourth the volume of the sphere), and $V\left(\frac{2}{3}R\sqrt{2}\right) = \frac{32}{81}\pi R^3$ (which is 8/27 of the volume of the sphere). Answer: The maximum volume is $\frac{32}{81}\pi R^3$.

3.M.86: Let x denote the length of the two sides of the corral that are perpendicular to the wall. There are two cases to consider.

Case 1: Part of the wall is used. Let y be the length of the side of the corral parallel to the wall. Then $y = 400 - 2x$, and we are to maximize the area

$$A = xy = x(400 - 2x), \qquad 150 \leqq x \leqq 200.$$

Then $A'(x) = 400 - 4x$; $A'(x) = 0$ when $x = 100$, but that value of x is not in the domain of A. Note that $A(150) = 15000$ and that $A(200) = 0$.

Case 2: All of the wall is used. Let y be the length of fence added to one end of the wall, so that the side parallel to the wall has length $100 + y$. Then $100 + 2y + 2x = 400$, so $y = 150 - x$. We are to maximize the area

$$A = x(100 + y) = x(250 - x), \qquad 0 \leqq x \leqq 150.$$

In this case $A'(x) = 0$ when $x = 125$. And in this case $A(150) = 15000$, $A(0) = 0$, and $A(125) = 15625$.

Answer: The maximum area is 15625 ft^2; to attain it, use all the existing wall and build a square corral.

3.M.87: First, $R'(x) = kM - 2kx$; because $k \neq 0$, $R'(x) = 0$ when $x = M/2$. Moreover, because $R(0) = 0 = R(M)$ and $R(M/2) > 0$, the latter is the maximum value of $R(x)$. Therefore the incidence of the disease is the highest when half the susceptible individuals are infected.

3.M.88: The trapezoid is shown next. It has altitude $h = L\cos\theta$ and the length of its longer base is $L + 2L\sin\theta$, so its area is

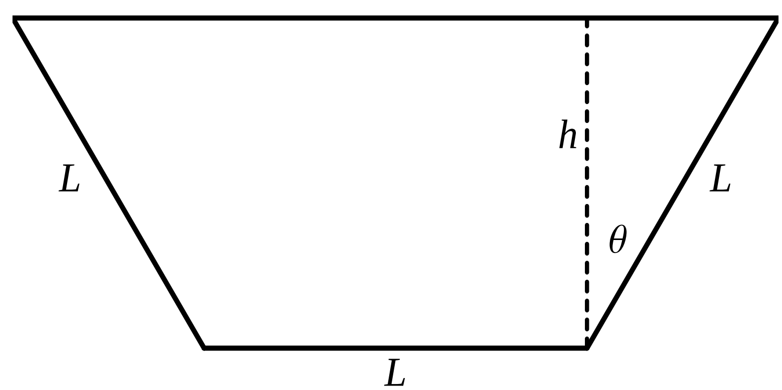
$$A(\theta) = L^2(1 + \sin\theta)\cos\theta, \qquad -\frac{\pi}{6} \leqq \theta \leqq \frac{\pi}{2}.$$

Now $dA/d\theta = 0$ when

$$1 - \sin\theta - 2\sin^2\theta = 0;$$

$$(2\sin\theta - 1)(\sin\theta + 1) = 0;$$

the only solution is $\theta = \pi/6$ because $\sin\theta$ cannot equal -1 in the range of A. Finally, $A(\pi/2) = 0$, $A(-\pi/6) = \frac{1}{4}L^2\sqrt{3}$, and $A(\pi/6) = \frac{3}{4}L^2\sqrt{3}$. The latter maximizes $A(\theta)$, and the fourth side of the trapezoid then has length $2L$.



3.M.89: Let x be the width of the base of the box, so that the base has length $2x$; let y be the height of the box. Then the volume of the box is $V = 2x^2y$, and for its total surface area to be 54 ft^2, we require $2x^2 + 6xy = 54$. Therefore the volume of the box is given by

$$V = V(x) = 2x^2\left(\frac{27 - x^2}{3x}\right) = \frac{2}{3}(27x - x^3), \qquad 0 < x \leqq 3\sqrt{3}.$$

Now $V'(x) = 0$ when $x^2 = 9$, so that $x = 3$. Also $V(x) \to 0$ as $x \to 0^+$ and $V(3\sqrt{3}) = 0$, so $V(3) = 36$ (ft^3) is the maximum possible volume of the box.

3.M.90: Suppose that the small cone has radius x and height y. Similar triangles that appear in a vertical cross section of the cones (*draw it!*) show that $\dfrac{x}{H-y} = \dfrac{R}{H}$. Hence $y = H - \dfrac{H}{R}x$, and we seek to maximize the volume $V = \frac{1}{3}\pi x^2 y$. Now

$$V = V(x) = \frac{\pi H}{3R}(Rx^2 - x^3), \qquad 0 \leqq x \leqq R.$$

So $V'(x) = \dfrac{\pi H}{3R}x(2R - 3x)$. $V'(x) = 0$ when $x = 0$ and when $x = \frac{2}{3}R$ (in this case, $y = H/3$). But $V(0) = 0$ and $V(R) = 0$, so $x = \frac{2}{3}R$ maximizes V. Finally, it is easy to find that $V_{\text{max}} = \dfrac{4}{27} \cdot \dfrac{\pi}{3}R^2H$, so the largest fraction of the large cone that the small cone can occupy is $4/27$.

3.M.91: Let $(x,\, y)$ be the coordinates of the vertex of the trapezoid lying properly in the first quadrant and let θ be the angle that the radius of the circle to $(x,\, y)$ makes with the x-axis. The bases of the trapezoid have lengths 4 and $4\cos\theta$ and its altitude is $2\sin\theta$, so its area is

$$A(\theta) = \frac{1}{2}(4 + 4\cos\theta)(2\sin\theta) = 4(1+\cos\theta)\sin\theta, \quad 0 \leqq \theta \leqq \frac{\pi}{2}.$$

Now

$$\begin{aligned} A'(\theta) &= 4(\cos\theta + \cos^2\theta - \sin^2\theta) \\ &= 4(2\cos^2\theta + \cos\theta - 1) \\ &= 4(2\cos\theta - 1)(\cos\theta + 1). \end{aligned}$$

The only zero of A' in its domain occurs at $\theta = \pi/3$. At the endpoints, we have $A(0) = 0$ and $A(\pi/2) = 4$. But $A(\pi/3) = 3\sqrt{3} \approx 5.196$, so the latter is the maximum possible area of such a trapezoid.

3.M.92: The square of the length of PQ is a function of x, $G(x) = (x - x_0)^2 + (y - y_0)^2$, which we are to maximize given the constraint $C(x) = y - f(x) = 0$. Now

$$\frac{dG}{dx} = 2(x - x_0) + 2(y - y_0)\frac{dy}{dx} \quad \text{and} \quad \frac{dC}{dx} = \frac{dy}{dx} - f'(x).$$

When both vanish, $f'(x) = \dfrac{dy}{dx} = -\dfrac{x - x_0}{y - y_0}$. The line containing P and Q has slope

$$\frac{y - y_0}{x - x_0} = -\frac{1}{f'(x)},$$

and therefore this line is normal to the graph at Q.

C03S0M.093: If $Ax + By + C = 0$ is an equation of a straight line L, then not both A and B can be zero.

Case 1: $A = 0$ and $B \neq 0$. Then L has equation $y = -C/B$ and thus is a horizontal line. So the shortest segment from $P(x_0, y_0)$ to Q on L is a vertical segment that therefore meets L in the point $Q(x_0, -C/B)$. Therefore, because $A = 0$, the distance from P to Q is

$$\left| y_0 + \frac{C}{B} \right| = \frac{|By_0 + C|}{|B|} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

Case 2: $A \neq 0$ and $B = 0$. Then L has equation $x = -C/A$ and thus is a vertical line. So the shortest segment from $P(x_0, y_0)$ to Q on L is a horizontal segment that therefore meets L in the point $Q(-C/A, y_0)$. Therefore, because $B = 0$, the distance from P to Q is

$$\left| x_0 + \frac{C}{A} \right| = \frac{|Ax_0 + C|}{|A|} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.$$

Case 3: $A \neq 0$ and $B \neq 0$. Then L is neither horizontal nor vertical, and the segment joining $P(x_0, y_0)$ to the nearest point $Q(u, v)$ on L is also neither horizontal nor vertical. The equation of L may be written in the form

$$y = -\frac{A}{B} - \frac{C}{B},$$

so L has slope $-A/B$. Thus the slope of PQ is B/A (by the result in Problem 70), and therefore PQ lies on the line K with equation

$$y - y_0 = \frac{B}{A}(x - x_0).$$

Consequently $A(v - y_0) = B(u - x_0)$. But $Q(u, v)$ also lies on L, and so $Au + Bv = -C$. Thus we have the simultaneous equations

$$Au + Bv = -C;$$

$$Bu - Av = Bx_0 - Ay_0.$$

These equations may be solved for

$$u = \frac{-AC + B^2x_0 - ABy_0}{A^2 + B^2} \qquad \text{and} \qquad v = \frac{-BC - ABx_0 + A^2y_0}{A^2 + B^2},$$

and it follows that

$$u - x_0 = \frac{A(-C - Ax_0 - By_0)}{A^2 + B^2} \qquad \text{and} \qquad v - y_0 = \frac{B(-C - Ax_0 - By_0)}{A^2 + B^2}.$$

Therefore

$$(u - x_0)^2 + (v - y_0)^2 = \frac{A^2(-C - Ax_0 - By_0)^2}{(A^2 + B^2)^2} + \frac{B^2(-C - Ax_0 - By_0)^2}{(A^2 + B^2)^2}$$

$$= \frac{(A^2 + B^2)(-C - Ax_0 - By_0)^2}{(A^2 + B^2)^2} = \frac{(Ax_0 + By_0 + C)^2}{A^2 + B^2}.$$

The square root of this expression then gives the distance from P to Q as

$$\frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}},$$

and the proof is complete.

3.M.94: Let r be the radius of each semicircle and x the length of the straightaway. We wish to maximize $A = 2rx$ given $C = 2\pi r + 2x - 4 = 0$. We use the method of auxiliary variables (as in the solution of Problem 83):

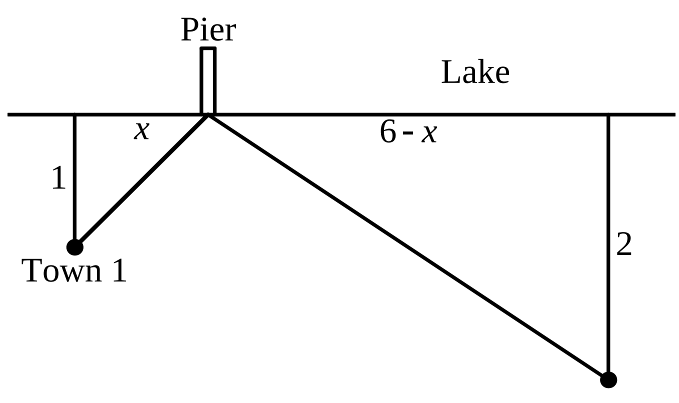
$$\frac{dA}{dx} = 2r + 2x\frac{dr}{dx} \quad \text{and} \quad \frac{dC}{dx} = 2\pi\frac{dr}{dx} + 2.$$

When both derivatives are zero, $-r/x = dr/dx = -1/\pi$, and so $x = \pi r$. Also $2\pi r + 2x = 4$, and it follows that $r = \dfrac{1}{\pi}$ and that $x = 1$. Answer: Design the straightaway 1 km long with semicircles of radius $\dfrac{1}{\pi}$ at each end.

3.M.95: As the following diagram suggests, we are to minimize the sum of the lengths of the two diagonals. Fermat's principle of least time may be used here, so we know that the angles at which the roads meet the shore are equal, and thus so are the tangents of those angles: $\dfrac{x}{1} = \dfrac{6-x}{2}$. It follows that the pier should be built two miles from the point on the shore nearest the first town. To be sure that we have found a minimum, consider the function that gives the total length of the two diagonals:

$$f(x) = \sqrt{x^2 + 1} + \sqrt{(6-x)^2 + 4}, \qquad 0 \leqq x \leqq 6.$$

(The domain certainly contains the global minimum value of f.) Moreover, $f(0) = 1 + \sqrt{40} \approx 7.32$, $f(6) = 2 + \sqrt{37} \approx 8.08$, and $f(2) = \sqrt{5} + \sqrt{20} \approx 6.71$. This establishes that $x = 2$ yields the global minimum of $f(x)$.



3.M.96: The length of each angled path is $\dfrac{2}{\sin\theta}$. The length of the roadway path is $10 - \dfrac{4\cos\theta}{\sin\theta}$. So the total time of the trip will be

$$T = T(\theta) = \frac{5}{4} + \frac{32 - 12\cos\theta}{24\sin\theta}.$$

Note that $\cos\theta$ varies in the range $0 \leqq \cos\theta \leqq \frac{5}{29}\sqrt{29}$, so $21.80° \leqq \theta° \leqq 90°$. After simplifications,

$$T'(\theta) = \frac{12 - 32\cos\theta}{24\sin^2\theta};$$

$T'(\theta) = 0$ when $\cos\theta = \frac{3}{8}$, so $\theta° \approx 67.98°$. With this value of θ, we find that the time of the trip is

$$T = \frac{2\sqrt{55} + 15}{12} \approx 2.486 \text{ (hours)}.$$

Because $T \approx 3.590$ with $\theta° \approx 21.80°$ and $T \approx 2.583$ when $\theta° \approx 90°$, the value $\theta° \approx 69.98°$ minimizes T, and the time saved is about 50.8 minutes.

3.M.97: Denote the initial velocity of the arrow by v. First, we have

$$\frac{dy}{dx} = m - \frac{32x}{v^2}(m^2 + 1);$$

$dy/dx = 0$ when $mv^2 = 32x(m^2+1)$, so that $x = \dfrac{mv^2}{32(m^2+1)}$. Substitution of this value of x in the formula given for y in the problem yields the maximum height

$$y_{\max} = \frac{m^2v^2}{64(m^2+1)}.$$

For part (b), we set $y = 0$ and solve for x to obtain the range

$$R = \frac{mv^2}{16(m^2+1)}.$$

Now R is a continuous function of the slope m of the arrow's path at time $t = 0$, with domain $0 \leqq m < +\infty$. Because $R(m) = 0$ and $R(m) \to 0$ as $m \to +\infty$, the function R has a global maximum; because R is differentiable, this maximum occurs at a point where $R'(m) = 0$. But

$$\frac{dR}{dm} = \frac{v^2}{16} \cdot \frac{(m^2+1) - 2m^2}{(m^2+1)^2},$$

so $dR/dm = 0$ when $m = 1$ and only then. So the maximum range occurs when $\tan\alpha = 1$; that is, when $\alpha = \frac{1}{4}\pi$.

3.M.98: Here we have

$$R = R(\theta) = \frac{v^2\sqrt{2}}{16}(\cos\theta\,\sin\theta - \cos^2\theta) \quad \text{for} \quad \tfrac{1}{4}\pi \leqq \theta \leqq \tfrac{1}{2}\pi.$$

Now

$$R'(\theta) = \frac{v^2\sqrt{2}}{16}(\cos^2\theta - \sin^2\theta + 2\sin\theta\,\cos\theta);$$

$R'(\theta) = 0$ when $\cos 2\theta + \sin 2\theta = 0$, so that $\tan 2\theta = -1$. It follows that $\theta = 3\pi/8$ (67.5°). This yields the maximum range because $R(\pi/4) = 0 = R(\pi/2)$.

3.M.99: With initial guess $x_0 = 2.5$ (the midpoint of the given interval [2, 3]), the iteration

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{(x_n)^2 - 7}{2x_n}$$

of Newton's method yields $x_1 = 2.65$, $x_2 = 2.645754717$, and $x_3 = 2.645751311$. Answer: 2.6458.

3.M.100: We get $x_0 = 1.5$, $x_1 = 1.444444444$, $x_2 = 1.442252904$, and $x_3 = 1.442249570$. Answer: 1.4422.

3.M.101: With $x_0 = 2.5$, we obtain $x_1 = 2.384$, $x_2 = 2.371572245$, $x_3 = 2.371440624$, and $x_4 = 2.371440610$. Answer: 2.3714.

3.M.102: With $x_0 = 5.5$ we get $x_1 = 5.623872512$, $x_2 = 5.623413258$, and $x_3 = 5.623413252$. Answer: 5.6234. If your calculator won't raise numbers to fractional powers, you could solve instead the equation $x^4 - 1000 = 0$. With $x_0 = 5.5$ the results should be $x_1 = 5.627629602$, $x_2 = 5.623417988$, and $x_3 = 5.623413252$.

3.M.103: With $x_0 = -0.5$ we obtain $x_1 = -0.333333333$, $x_2 = -0.347222222$, $x_3 = -0.347296353$, and $x_4 = -0.347296355$. Answer: -0.3473.

3.M.104: With $x_0 = -0.5$ we obtain $x_1 = -0.230769231$, $x_2 = -0.254000237$, $x_3 = -0.254101686$, and $x_4 = -0.254101688$. Answer: -0.2541.

3.M.105: With $f(x) = e^{-x} - \sin x$ and initial guess $x_0 = 0.6$, five iterations of the formula of Newton's method yields the approximate solution 0.588532744 of the equation $f(x) = 0$.

3.M.106: With $f(x) = \cos x - \ln x$ and initial guess $x_0 = 1.3$, four iterations of the formula of Newton's method yield the approximate solution 1.302964001 of the equation $f(x) = 0$.

3.M.107: With $x_0 = -1.0$ we obtain $x_1 = -0.750363868$, $x_2 = -0.739112891$, $x_3 = -0.739085133$, and $x_4 = -0.739085133$. Answer: -0.7391.

3.M.108: With $x_0 = -0.75$, we obtain $x_1 = -0.905065774$, $x_2 = -0.877662556$, $x_3 = -0.876727303$, $x_4 = -0.876726215$, and $x_5 = -0.876726215$. Answer: -0.8767.

3.M.109: With $x_0 = -1.5$, we obtain $x_1 = -1.244861806$, $x_2 = -1.236139793$, $x_3 = -1.236129989$, and $x_4 = -1.236129989$. Answer: -1.2361.

3.M.110: With $x_0 = -0.5$ we obtain $x_1 = -0.858896298$, $x_2 = -0.871209876$, $x_3 = -0.871221514$, and $x_4 = -0.871221514$. Answer: -0.8712.

3.M.111: The volume of a spherical segment of height h is

$$V = \frac{1}{3}\pi h^2(3r - h)$$

if the sphere has radius r. If ρ is the density of water and the ball sinks to the depth h, then the weight of the water that the ball displaces is equal to the total weight of the ball, so

$$\frac{1}{3}\pi\rho h^2(3r - h) = \frac{4}{3^2}\pi\rho r^3.$$

Because $r = 2$, this leads to the equation $p(h) = 3h^3 - 18h^2 + 32 = 0$. This equation has at most three [real] solutions because $p(h)$ is a polynomial of degree 3, and it turns out to have exactly three solutions because $p(-2) = -64$, $p(-1) = 11$, $p(2) = -16$, and $p(6) = 32$. Newton's method yields the three approximate solutions $h = -1.215825766$, $h = 1.547852572$, and $h = 5.667973193$. Only one is plausible, so the answer is that the ball sinks to a depth of approximately 1.54785 ft, about 39% of the way up a diameter.

3.M.112: The iteration is

$$x - \frac{x^2 + 1}{2x} = \frac{x^2 - 1}{2x} \longrightarrow x.$$

With $x_0 = 2.0$, the sequence obtained by iteration of Newton's method is $0.75,\ -0.2917,\ 1.5685,\ 0.4654,\ -0.8415,\ 0.1734,\ -2.7970,\ -1.2197,\ -0.1999,\ 2.4009,\ 0.9922,\ -0.0078,\ 63.7100, \dots$.

3.M.113: Let $f(x) = x^5 - 3x^3 + x^2 - 23x + 19$. Then $f(-3) = -65$, $f(0) = 19$, $f(1) = -5$, and $f(3) = 121$. So there are at least three, and at most five, real solutions. Newton's method produces three real solutions, specifically $r_1 = -2.722493355$, $r_2 = 0.8012614801$, and $r_3 = 2.309976541$. If one divides the polynomial $f(x)$ by $(x - r_1)(x - r_2)(x - r_3)$, one obtains the quotient polynomial $x^2 + (0.38874466)x + 3.770552031$, which has no real roots—the quadratic formula yields the two complex roots $-0.194372333 \pm (1.932038153)i$. Consequently we have found all three real solutions.

3.M.114: Let $f(x) = \tan x - \dfrac{1}{x}$. We iterate

$$x - \frac{\tan x - \dfrac{1}{x}}{\sec^2 x + \dfrac{1}{x^2}} \longrightarrow x.$$

The results are shown in the following table. The instability in the last one or two digits is caused by machine rounding and is common. Answers: To three places, $\alpha_1 = 0.860$ and $\alpha_2 = 3.426$.

```
f[x_]:=Tan[x]−1/x
g[x_]:=N[x−f[x]/f'[x], 20]
list={1.0,4.0};
g[list]
0.8740469203219249386,    3.622221245370322529
0.8604001629909660496,    3.440232462677783381
0.8603335904117901655,    3.425673797668214504
0.8603335890193797612,    3.425618460245614115
0.8603335890193797636,    3.425618459481728148
0.8603335890193797608,    3.425618459481728146
0.8603335890193797634,    3.425618459481728148
```

3.M.115: The number of summands on the right is variable, and we have no formula for finding its derivative. One thing is certain: Its derivative is *not* $2x^2$.

3.M.116: We factor:

$$z^{3/2} - x^{3/2} = (z^{1/2})^3 - (x^{1/2})^3 = (z^{1/2} - x^{1/2})(z + z^{1/2}x^{1/2} + x)$$

and $z - x = (z^{1/2})^2 - (x^{1/2})^2 = (z^{1/2} - x^{1/2})(z^{1/2} + x^{1/2})$. Therefore

$$\frac{z^{3/2} - x^{3/2}}{z - x} = \frac{z + z^{1/2}x^{1/2} + x}{z^{1/2} + x^{1/2}} \to \frac{3x}{2x^{1/2}} = \frac{3}{2}x^{1/2} \quad \text{as} \quad z \to x.$$

3.M.117: We factor:

$$z^{2/3} - x^{2/3} = (z^{1/3})^2 - (x^{1/3})^2 = (z^{1/3} - x^{1/3})(z^{1/3} + x^{1/3}) \qquad \text{and}$$

$$z - x = (z^{1/3})^3 - (x^{1/3})^3 = (z^{1/3} - x^{1/3})(z^{2/3} + z^{1/3}x^{1/3} + x^{2/3}).$$

Therefore

$$\frac{z^{2/3} - x^{2/3}}{z - x} = \frac{z^{1/3} + x^{1/3}}{z^{2/3} + z^{1/3}x^{1/3} + z^{2/3}} \to \frac{2x^{1/3}}{3x^{2/3}} = \frac{2}{3}x^{-1/3} \quad \text{as} \quad x \to x.$$

3.M.118: The volume of the block is $V = x^2y$, and V is constant while x and y are functions of time t (in minutes). So

$$0 = \frac{dV}{dt} = 2xy\frac{dx}{dt} + x^2\frac{dy}{dt}. \tag{1}$$

We are given $dy/dt = -2$, $x = 30$, and $y = 20$, so by Eq. (1) $dx/dt = \frac{3}{2}$. Answer: At the time in question the edge of the base is increasing at 1.5 cm/min.

3.M.119: The balloon has volume $V = \frac{4}{3}\pi r^3$ and surface area $A = 4\pi r^2$ where r is its radius and V, A, and r are all functions of time t. We are given $dV/dt = +10$, and we are to find dA/dt when $r = 5$.

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}, \quad \text{so} \quad 10 = 4\pi \cdot 25 \cdot \frac{dr}{dt}.$$

$$\text{Thus} \quad \frac{dr}{dt} = \frac{10}{100\pi} = \frac{1}{10\pi}.$$

$$\text{Also} \quad \frac{dA}{dt} = 8\pi r \frac{dr}{dt}, \quad \text{and therefore}$$

$$\left.\frac{dA}{dt}\right|_{r=5} = 8\pi \cdot 5 \cdot \frac{1}{10\pi} = 4.$$

Answer: At 4 in.^{2}/s.

3.M.120: Let the nonnegative x-axis represent the ground and the nonnegative y-axis the wall. Let x be the distance from the base of the wall to the foot of the ladder; let y be the height of the top of the ladder above the ground. From the Pythagorean theorem we obtain $x^2 + y^2 = 100$, so

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0.$$

Thus $\dfrac{dy}{dt} = -\dfrac{x}{y} \cdot \dfrac{dx}{dt}$. We are given $\dfrac{dx}{dt} = \dfrac{5280}{3600} = \dfrac{22}{15}$ ft/s, and at the time when $y = 1$, we have

$$x = \sqrt{100 - (0.01)^2} = \sqrt{99.9999}.$$

At that time,

$$\left.\frac{dy}{dt}\right|_{y=0.01} = -\frac{\sqrt{99.9999}}{0.01} \cdot \frac{22}{15} \approx -1466.666 \text{ (ft/s)},$$

almost exactly 1000 mi/h. This shows that in reality, the top of the ladder cannot remain in contact with the wall. If it is forced to do so by some latching mechanism, then a downward force much greater than that caused by gravity will be needed to keep the bottom of the latter moving at the constant rate of 1 mi/h.

3.M.121: Let Q be the amount of water in the cone at time t, r the radius of its upper surface, and h its height. From similar triangles we find that $h = 2r$, so

$$Q = \frac{1}{3}\pi r^2 h = \frac{2}{3}\pi r^3 = \frac{1}{12}\pi h^3.$$

Now

$$-50 = \frac{dQ}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}, \text{ so } \frac{dh}{dt} = -\frac{200}{36\pi}.$$

Therefore $\dfrac{dh}{dt} = -\dfrac{50}{9\pi} \approx -1.7684$ (ft/min).

3.M.122: Let x denote the distance from plane A to the airport, y the distance from plane B to the airport, and z the distance between the two aircraft. Then

$$z^2 = x^2 + y^2 + (3-2)^2 = x^2 + y^2 + 1$$

and $dx/dt = -500$. Now

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt},$$

and when $x = 2$, $y = 2$. Therefore $z = 3$ at that time. Therefore,

$$3 \cdot (-600) = 2 \cdot (-500) + 2 \cdot \left.\frac{dy}{dt}\right|_{x=2},$$

and thus $\left.\dfrac{dy}{dt}\right|_{x=2} = -400$. Answer: Its speed is 400 mi/h.

3.M.123: $\dfrac{dV}{dt} = 3\sqrt{y}\,\dfrac{dy}{dt} = -3\sqrt{y}$, so $\dfrac{dy}{dt} = -1$. Answer: At 1 in./min—a constant rate. The tank is a *clock!*

3.M.124: As in the solution of Problem 121, we find that when the height of water in the tank is y, its volume is $V = \frac{1}{12}\pi y^3$. For part (a), we have

$$+50 - 10\sqrt{y} = \frac{dV}{dt} = \frac{1}{4}\pi y^2\frac{dy}{dt}.$$

So when $y = 5$,

$$50 - 10\sqrt{5} = \frac{25}{4}\pi\left.\frac{dy}{dt}\right|_{y=5},$$

and therefore $\left.\dfrac{dy}{dt}\right|_{y=5} = \dfrac{1}{5\pi}(40 - 8\sqrt{5}) \approx 1.40766$ (ft/min). In part (b),

$$\frac{dV}{dt} = 25 - 10\sqrt{y} = \frac{1}{4}\pi y^2\frac{dy}{dt}; \tag{1}$$

$dy/dt = 0$ when $25 = 10\sqrt{y}$, so that $y = 6.25$ (ft) would seem to be the maximum height ever attained by the water. What actually happens is that the water level rises more and more slowly as time passes, approaching the limiting height of 6.25 ft as a right-hand limit, but never reaching it. This is not obvious; you must solve the differential equation in (1) (use the substitution $y = u^2$) and analyze the solution to establish this conclusion.

3.M.125: The straight line through $P(x_0, y_0)$ and $Q(a, a^2)$ has slope $\dfrac{a^2 - y_0}{a - x_0} = 2a$, a consequence of the two-point formula for slope and the fact that the line is tangent to the parabola at Q. Hence $a^2 - 2ax_0 + y_0 = 0$. Think of this as a quadratic equation in the unknown a. It has two real solutions when the discriminant is positive: $(x_0)^2 - y_0 > 0$, and this establishes the conclusion in part (b). There are no real solutions when $(x_0)^2 - y_0 < 0$, and this establishes the conclusion in part (c). What if $(x_0)^2 - y_0 = 0$?