Section 1.1

C01S01.001: If $f(x) = \dfrac{1}{x}$, then:

(a) $f(-a) = \dfrac{1}{-a} = -\dfrac{1}{a}$;

(b) $f(a^{-1}) = \dfrac{1}{a^{-1}} = a$;

(c) $f(\sqrt{a}) = \dfrac{1}{\sqrt{a}} = \dfrac{1}{a^{1/2}} = a^{-1/2}$;

(d) $f(a^2) = \dfrac{1}{a^2} = a^{-2}$.

C01S01.002: If $f(x) = x^2 + 5$, then:

(a) $f(-a) = (-a)^2 + 5 = a^2 + 5$;

(b) $f(a^{-1}) = (a^{-1})^2 + 5 = a^{-2} + 5 = \dfrac{1}{a^2} + 5 = \dfrac{1 + 5a^2}{a^2}$;

(c) $f(\sqrt{a}) = (\sqrt{a})^2 + 5 = a + 5$;

(d) $f(a^2) = (a^2)^2 + 5 = a^4 + 5$.

C01S01.003: If $f(x) = \dfrac{1}{x^2 + 5}$, then:

(a) $f(-a) = \dfrac{1}{(-a)^2 + 5} = \dfrac{1}{a^2 + 5}$;

(b) $f(a^{-1}) = \dfrac{1}{(a^{-1})^2 + 5} = \dfrac{1}{a^{-2} + 5} = \dfrac{1 \cdot a^2}{a^{-2} \cdot a^2 + 5 \cdot a^2} = \dfrac{a^2}{1 + 5a^2}$;

(c) $f(\sqrt{a}) = \dfrac{1}{(\sqrt{a})^2 + 5} = \dfrac{1}{a + 5}$;

(d) $f(a^2) = \dfrac{1}{(a^2)^2 + 5} = \dfrac{1}{a^4 + 5}$.

C01S01.004: If $f(x) = \sqrt{1 + x^2 + x^4}$, then:

(a) $f(-a) = \sqrt{1 + (-a)^2 + (-a)^4} = \sqrt{1 + a^2 + a^4}$;

(b) $f(a^{-1}) = \sqrt{1 + (a^{-1})^2 + (a^{-1})^4} = \sqrt{1 + a^{-2} + a^{-4}} = \sqrt{\dfrac{(a^4) \cdot (1 + a^{-2} + a^{-4})}{a^4}}$

$\qquad = \sqrt{\dfrac{a^4 + a^2 + 1}{a^4}} = \dfrac{\sqrt{a^4 + a^2 + 1}}{\sqrt{a^4}} = \dfrac{\sqrt{a^4 + a^2 + 1}}{a^2}$;

(c) $f(\sqrt{a}) = \sqrt{1 + (\sqrt{a})^2 + (\sqrt{a})^4} = \sqrt{1 + a + a^2}$;

(d) $f(a^2) = \sqrt{1 + (a^2)^2 + (a^4)^2} = \sqrt{1 + a^4 + a^8}$.

C01S01.005: If $g(x) = 3x + 4$ and $g(a) = 5$, then $3a + 4 = 5$, so $3a = 1$; therefore $a = \frac{1}{3}$.

C01S01.006: If $g(x) = \dfrac{1}{2x - 1}$ and $g(a) = 5$, then:

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$$\frac{1}{2a - 1} = 5;$$

$$1 = 5 \cdot (2a - 1);$$

$$1 = 10a - 5;$$

$$10a = 6;$$

$$a = \frac{3}{5}.$$

C01S01.007: If $g(x) = \sqrt{x^2 + 16}$ and $g(a) = 5$, then:

$$\sqrt{a^2 + 16} = 5;$$

$$a^2 + 16 = 25;$$

$$a^2 = 9;$$

$$a = 3 \text{ or } a = -3.$$

C01S01.008: If $g(x) = x^3 - 3$ and $g(a) = 5$, then $a^3 - 3 = 5$, so $a^3 = 8$. Hence $a = 2$.

C01S01.009: If $g(x) = \sqrt[3]{x + 25} = (x + 25)^{1/3}$ and $g(a) = 5$, then

$$(a + 25)^{1/3} = 5;$$

$$a + 25 = 5^3 = 125;$$

$$a = 100.$$

C01S01.010: If $g(x) = 2x^2 - x + 4$ and $g(a) = 5$, then:

$$2a^2 - a + 4 = 5;$$

$$2a^2 - a - 1 = 0;$$

$$(2a + 1)(a - 1) = 0;$$

$$2a + 1 = 0 \text{ or } a - 1 = 0;$$

$$a = -\frac{1}{2} \text{ or } a = 1.$$

C01S01.011: If $f(x) = 3x - 2$, then

$$f(a + h) - f(a) = [3(a + h) - 2] - [3a - 2]$$

$$= 3a + 3h - 2 - 3a + 2 = 3h.$$

C01S01.012: If $f(x) = 1 - 2x$, then

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$$f(a + h) - f(a) = [1 - 2(a + h)] - [1 - 2a] = 1 - 2a - 2h - 1 + 2a = -2h.$$

C01S01.013: If $f(x) = x^2$, then

$$f(a + h) - f(a) = (a + h)^2 - a^2$$
$$= a^2 + 2ah + h^2 - a^2 = 2ah + h^2 = h \cdot (2a + h).$$

C01S01.014: If $f(x) = x^2 + 2x$, then

$$f(a + h) - f(a) = [(a + h)^2 + 2(a + h)] - [a^2 + 2a]$$
$$= a^2 + 2ah + h^2 + 2a + 2h - a^2 - 2a = 2ah + h^2 + 2h = h \cdot (2a + h + 2).$$

C01S01.015: If $f(x) = \dfrac{1}{x}$, then

$$f(a + h) - f(a) = \frac{1}{a + h} - \frac{1}{a} = \frac{a}{a(a + h)} - \frac{a + h}{a(a + h)}$$
$$= \frac{a - (a + h)}{a(a + h)} = \frac{-h}{a(a + h)}.$$

C01S01.016: If $f(x) = \dfrac{2}{x + 1}$, then

$$f(a + h) - f(a) = \frac{2}{a + h + 1} - \frac{2}{a + 1} = \frac{2(a + 1)}{(a + h + 1)(a + 1)} - \frac{2(a + h + 1)}{(a + h + 1)(a + 1)}$$
$$= \frac{2a + 2}{(a + h + 1)(a + 1)} - \frac{2a + 2h + 2}{(a + h + 1)(a + 1)} = \frac{(2a + 2) - (2a + 2h + 2)}{(a + h + 1)(a + 1)}$$
$$= \frac{2a + 2 - 2a - 2h - 2}{(a + h + 1)(a + 1)} = \frac{-2h}{(a + h + 1)(a + 1)}.$$

C01S01.017: If $x > 0$ then

$$f(x) = \frac{x}{|x|} = \frac{x}{x} = 1.$$

If $x < 0$ then

$$f(x) = \frac{x}{|x|} = \frac{x}{-x} = -1.$$

We are given $f(0) = 0$, so the range of f is $\{-1, 0, 1\}$. That is, the range of f is the set consisting of the three real numbers -1, 0, and 1.

C01S01.018: Given $f(x) = [\![3x]\!]$, we see that

$$f(x) = 0 \quad \text{if} \quad 0 \leqq x < \tfrac{1}{3},$$

$$f(x) = 1 \quad \text{if} \quad \tfrac{1}{3} \leqq x < \tfrac{2}{3},$$

$$f(2) = 2 \quad \text{if} \quad \tfrac{2}{3} \leqq x < 1;$$

moreover,

$$f(x) = -3 \quad \text{if} \quad -1 \leqq x < -\tfrac{2}{3},$$

$$f(x) = -2 \quad \text{if} \quad -\tfrac{2}{3} \leqq x < -\tfrac{1}{3},$$

$$f(x) = -1 \quad \text{if} \quad -\tfrac{1}{3} \leqq x < 0.$$

In general, if m is any integer, then

$$f(x) = 3m \quad \text{if} \quad m \leqq x < m + \tfrac{1}{3},$$

$$f(x) = 3m + 1 \quad \text{if} \quad m + \tfrac{1}{3} \leqq x < m + \tfrac{2}{3},$$

$$f(x) = 3m + 2 \quad \text{if} \quad m + \tfrac{2}{3} \leqq x < m + 1.$$

Because every integer is equal to $3m$ or to $3m + 1$ or to $3m + 2$ for some integer m, we see that the range of f includes the set $\mathbf{Z}$ of all integers. Because f can assume no values other than integers, we can conclude that the range of f is exactly $\mathbf{Z}$.

C01S01.019: Given $f(x) = (-1)^{[\![x]\!]}$, we first note that the values of the exponent $[\![x]\!]$ consist of all the integers and no other numbers. So all that matters about the exponent is whether it is an even integer or an odd integer, for if even then $f(x) = 1$ and if odd then $f(x) = -1$. No other values of $f(x)$ are possible, so the range of f is the set consisting of the two numbers -1 and 1.

C01S01.020: If $0 < x \leqq 1$, then $f(x) = 34$. If $1 < x \leqq 2$ then $f(x) = 34 + 21 = 55$. If $2 < x \leqq 3$ then $f(x) = 34 + 2 \cdot 21 = 76$. We continue in this way and conclude with the observation that if $11 < x < 12$ then $f(x) = 34 + 11 \cdot 21 = 265$. So the range of f is the set

$$\{34,\ 55,\ 76,\ 97,\ 118,\ 139,\ 160,\ 181,\ 202,\ 223,\ 244,\ 265\}.$$

C01S01.021: Given $f(x) = 10 - x^2$, note that for every real number x, x^2 is defined, and for every such real number x^2, $10 - x^2$ is also defined. Therefore the domain of f is the set $\mathbf{R}$ of all real numbers.

C01S01.022: Given $f(x) = x^3 + 5$, we note that for each real number x, x^3 is defined; moreover, for each such real number x^3, $x^3 + 5$ is also defined. Thus the domain of f is the set $\mathbf{R}$ of all real numbers.

C01S01.023: Given $f(t) = \sqrt{t^2}$, we observe that for every real number t, t^2 is defined and nonnegative, and hence that $\sqrt{t^2}$ is defined as well. Therefore the domain of f is the set $\mathbf{R}$ of all real numbers.

C01S01.024: Given $g(t) = \left(\sqrt{t}\right)^2$, we observe that $\sqrt{t}$ is defined exactly when $t \geqq 0$. In this case, $\left(\sqrt{t}\right)^2$ is also defined, and hence the domain of g is the set $[0, +\infty)$ of all nonnegative real numbers.

C01S01.025: Given $f(x) = \sqrt{3x - 5}$, we note that $3x - 5$ is defined for all real numbers x, but that its square root will be defined when and only when $3x - 5$ is nonnegative; that is, when $3x - 5 \geqq 0$, so that $x \geqq \tfrac{5}{3}$. So the domain of f consists of all those real numbers x in the interval $\left[\tfrac{5}{3}, +\infty\right)$.

C01S01.026: Given $g(t) = \sqrt[3]{t+4} = (t+4)^{1/3}$, we note that $t + 4$ is defined for every real number t and the cube root of $t + 4$ is defined for every possible resulting value of $t + 4$. Therefore the domain of g is the set $\boldsymbol{R}$ of all real numbers.

C01S01.027: Given $f(t) = \sqrt{1 - 2t}$, we observe that $1 - 2t$ is defined for every real number t, but that its square root is defined only when $1 - 2t$ is nonnegative. We solve the inequality $1 - 2t \geq 0$ to find that $f(t)$ is defined exactly when $t \leq \frac{1}{2}$. Hence the domain of f is the interval $\left(-\infty, \frac{1}{2}\right]$.

C01S01.028: Given

$$g(x) = \frac{1}{(x+2)^2},$$

we see that $(x + 2)^2$ is defined for every real number x, but that $g(x)$, its reciprocal, will be defined only when $(x + 2)^2 \neq 0$; that is, when $x + 2 \neq 0$. So the domain of g consists of those real numbers $x \neq -2$.

C01S01.029: Given

$$f(x) = \frac{2}{3 - x},$$

we see that $3 - x$ is defined for all real values of x, but that $f(x)$, double its reciprocal, is defined only when $3 - x \neq 0$. So the domain of f consists of those real numbers $x \neq 3$.

C01S01.030: Given

$$g(t) = \sqrt{\frac{2}{3 - t}},$$

it is necessary that $3 - t$ be both nonzero (so that its reciprocal is defined) and nonnegative (so that the square root is defined). Thus $3 - t > 0$, and therefore the domain of g consists of those real numbers $t < 3$.

C01S01.031: Given $f(x) = \sqrt{x^2 + 9}$, observe that for each real number x, $x^2 + 9$ is defined and, moreover, is positive. So its square root is defined for every real number x. Hence the domain of f is the set $\boldsymbol{R}$ of all real numbers.

C01S01.032: Given

$$h(z) = \frac{1}{\sqrt{4 - z^2}},$$

we note that $4 - z^2$ is defined for every real number z, but that its square root will be defined only if $4 - z^2 \geq 0$. Moreover, the square root cannot be zero, else its reciprocal will be undefined, so we need to solve the inequality $4 - z^2 > 0$; that is, $z^2 < 4$. The solution is $-2 < z < 2$, so the domain of h is the open interval $(-2, 2)$.

C01S01.033: Given $f(x) = \sqrt{4 - \sqrt{x}}$, note first that we require $x \geq 0$ in order that $\sqrt{x}$ be defined. In addition, we require $4 - \sqrt{x} \geq 0$ so that *its* square root will be defined as well. So we solve [simultaneously] $x \geq 0$ and $\sqrt{x} \leq 4$ to find that $0 \leq x \leq 16$. So the domain of f is the closed interval $[0, 16]$.

C01S01.034: Given

$$f(x) = \sqrt{\frac{x + 1}{x - 1}},$$

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we require that $x \neq 1$ so that the fraction is defined. In addition we require that the fraction be nonnegative so that its square root will be defined. These conditions imply that both numerator and denominator be positive or that both be negative; moreover, the numerator may also be zero. But if the denominator is positive then the [larger] numerator will be positive as well; if the numerator is nonpositive then the [smaller] denominator will be negative. So the domain of f consists of those real numbers for which *either* $x - 1 > 0$ or $x + 1 \leqq 0$; that is, either $x > 1$ or $x \leqq -1$. So the domain of f is the union of the two intervals $(-\infty, -1]$ and $(1, +\infty)$. Alternatively, it consists of those real numbers x *not* in the interval $(-1, 1]$.

C01S01.035: Given:

$$g(t) = \frac{t}{|t|}.$$

This fraction will be defined whenever its denominator is nonzero, thus for all real numbers $t \neq 0$. So the domain of g consists of the nonzero real numbers; that is, the union of the two intervals $(-\infty, 0)$ and $(0, +\infty)$.

C01S01.036: If a square has edge length x, then its area A is given by $A = x^2$ and its perimeter P is given by $P = 4x$. To express A in terms of P:

$$x = \tfrac{1}{4}P;$$

$$A = x^2 = \left(\tfrac{1}{4}P\right)^2 = \tfrac{1}{16}P^2.$$

Thus to express A as a function of P, we write

$$A(P) = \tfrac{1}{16}P^2, \quad 0 \leqq P < +\infty.$$

(It will be convenient later in the course to allow the possibility that P, x, and A are zero. If this produces an answer that fails to meet real-world criteria for a solution, then that possibility can simply be eliminated when the answer to the problem is stated.)

C01S01.037: If a circle has radius r, then its circumference C is given by $C = 2\pi r$ and its area A by $A = \pi r^2$. To express C in terms of A, we first express r in terms of A, then substitute in the formula for C:

$$A = \pi r^2; \qquad r = \sqrt{\frac{A}{\pi}};$$

$$C = 2\pi r = 2\pi \sqrt{\frac{A}{\pi}} = 2\sqrt{\frac{\pi^2 A}{\pi}} = 2\sqrt{\pi A}.$$

Therefore to express C as a function of A, we write

$$C(A) = 2\sqrt{\pi A}, \qquad 0 \leqq A < +\infty.$$

It is also permissible simply to write $C(A) = 2\sqrt{\pi A}$ without mentioning the domain, because the "default" domain is correct. In the first displayed equation we do not write $r = \pm\sqrt{A/\pi}$ because we know that r is never negative.

C01S01.38: If r denotes the radius of the sphere, then its volume is given by $V = \tfrac{4}{3}\pi r^3$ and its surface area by $S = 4\pi r^2$. Hence

$$r = \frac{1}{2}\sqrt{\frac{S}{\pi}} \ ;$$

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot \frac{1}{8}\left(\frac{S}{\pi}\right)^{3/2} = \frac{1}{6}\pi \left(\frac{S}{\pi}\right)^{3/2}.$$

Answer: $V(S) = \frac{1}{6}\pi \left(\frac{S}{\pi}\right)^{3/2}, \quad 0 \leqq S < +\infty.$

C01S01.039: To avoid decimals, we note that a change of $5°C$ is the same as a change of $9°F$, so when the temperature is $10°C$ it is $32 + 18 = 50°F$; when the temperature is $20°C$ then it is $32 + 2 \cdot 18 = 68°F$. In general we get the Fahrenheit temperature F by adding 32 to the product of $\frac{1}{10}C$ and 18, where C is the Celsius temperature. That is,

$$F = 32 + \frac{9}{5}C,$$

and therefore $C = \frac{5}{9}(F - 32)$. Answer:

$$C(F) = \frac{5}{9}(F - 32), \quad F > -459.67.$$

C01S01.040: Suppose that a rectangle has base length x and perimeter 100. Let h denote the height of such a rectangle. Then $2x + 2h = 100$, so that $h = 50 - x$. Because $x \geq 0$ and $h \geq 0$, we see that $0 \leqq x \leqq 50$. The area A of the rectangle is xh, so that

$$A(x) = x(50 - x), \quad 0 \leqq x \leqq 50.$$

C01S01.041: Let y denote the height of such a rectangle. The rectangle is inscribed in a circle of diameter 4, so the bottom side x and the left side y are the two legs of a right triangle with hypotenuse 4. Consequently $x^2 + y^2 = 16$, so $y = \sqrt{16 - x^2}$ (not $-\sqrt{16 - x^2}$ because $y \geqq 0$). Because $x \geqq 0$ and $y \geqq 0$, we see that $0 \leqq x \leqq 4$. The rectangle has area $A = xy$, so

$$A(x) = x\sqrt{16 - x^2}, \quad 0 \leqq x \leqq 4.$$

C01S042.042: We take the problem to mean that current production is 200 barrels per day per well, that if one new well is drilled then the 21 wells will produce 195 barrels per day per well; in general, that if x new wells are drilled then the $20 + x$ wells will produce $200 - 5x$ barrels per day per well. So total production would be $p = (20 + x)(200 - 5x)$ barrels per day. But because $200 - 5x \geqq 0$, we see that $x \leqq 40$. Because $x \geqq 0$ as well (you don't "undrill" wells), here's the answer:

$$p(x) = 4000 + 100x - 5x^2, \quad 0 \leqq x \leqq 40, \quad x \text{ an integer}.$$

C01S01.043: The square base of the box measures x by x centimeters; let y denote its height (in centimeters). Because the volume of the box is 324 cm^3, we see that $x^2y = 324$. The base of the box costs $2x^2$ cents, each of its four sides costs xy cents, and its top costs x^2 cents. So the total cost of the box is

$$C = 2x^2 + 4xy + x^2 = 3x^2 + 4xy. \tag{1}$$

Because $x > 0$ and $y > 0$ (the box has positive volume), but because y can be arbitrarily close to zero (as well as x), we see also that $0 < x < +\infty$. We use the equation $x^2 y = 324$ to eliminate y from Eq. (1) and thereby find that

$$C(x) = 3x^2 + \frac{1296}{x}, \quad 0 < x < +\infty.$$

C01S01.044: If the rectangle is rotated around its side S of length x to produce a cylinder, then x will also be the height of the cylinder. Let y denote the length of the two sides perpendicular to S; then y will be the radius of the cylinder; moreover, the perimeter of the original rectangle is $2x + 2y = 36$. Hence $y = 18 - x$. Note also that $x \geqq 0$ and that $x \leqq 18$ (because $y \geqq 0$). The volume of the cylinder is $V = \pi y^2 x$, and so

$$V(x) = \pi x (18 - x)^2, \quad 0 \leqq x \leqq 18.$$

C01S01.045: Let h denote the height of the cylinder. Its radius is r, so its volume is $\pi r^2 h = 1000$. The total surface area of the cylinder is

$$A = 2\pi r^2 + 2\pi r h \quad \text{(look inside the front cover of the book);}$$

$$h = \frac{1000}{\pi r^2}, \quad \text{so}$$

$$A = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}.$$

Now r cannot be negative; r cannot be zero, else $\pi r^2 h \neq 1000$. But r can be arbitrarily small positive as well as arbitrarily large positive (by making h sufficiently close to zero). Answer:

$$A(r) = 2\pi r^2 + \frac{2000}{r}, \quad 0 < r < +\infty.$$

C01S01.046: Let y denote the height of the box (in centimeters). Then

$$2x^2 + 4xy = 600, \quad \text{so that} \quad y = \frac{600 - 2x^2}{4x}. \tag{1}$$

The volume of the box is

$$V = x^2 y = \frac{(600 - 2x^2) \cdot x^2}{4x} = \frac{1}{4}(600x - 2x^3) = \frac{1}{2}(300x - x^3)$$

by Eq. (1). Also $x > 0$ by Eq. (1), but the maximum value of x is attained when Eq. (1) forces y to be zero, at which point $x = \sqrt{300} = 10\sqrt{3}$. Answer:

$$V(x) = \frac{300x - x^3}{2}, \quad 0 < x \leqq 10\sqrt{3}.$$

C01S01.047: The base of the box will be a square measuring $50 - 2x$ in. on each side, so the open-topped box will have that square as its base and four rectangular sides each measuring $50 - 2x$ by x (the height of the box). Clearly $0 \leqq x$ and $2x \leqq 50$. So the volume of the box will be

$$V(x) = x(50 - 2x)^2, \quad 0 \leqq x \leqq 25.$$

C01S01.048: Recall that $A(x) = x(50 - x)$, $0 \leqq x \leqq 50$. Here is a table of a few values of the function A at some special numbers in its domain:

x	0	5	10	15	20	25	30	35	40	45	50
A	0	225	400	525	600	625	600	525	400	225	0

It appears that when $x = 25$ (so the rectangle is a square), the rectangle has maximum area 625.

C01S01.049: Recall that the total daily production of the oil field is $p(x) = (20 + x)(200 - 5x)$ if x new wells are drilled (where x is an integer satisfying $0 \leqq x \leqq 40$). Here is a table of *all* of the values of the production function p:

x	0	1	2	3	4	5	6	7
p	4000	4095	4180	4255	4320	4375	4420	4455

x	8	9	10	11	12	13	14	15
p	4480	4495	4500	4495	4480	4455	4420	4375

x	16	17	18	19	20	21	22	23
p	4320	4255	4180	4095	4000	3895	3780	3655

x	24	25	26	27	28	29	30	31
p	3520	3375	3220	3055	2880	2695	2500	2295

x	32	33	34	35	36	37	38	39
p	2080	1855	1620	1375	1120	855	580	295

and, finally, $p(40) = 0$. Answer: Drill ten new wells.

C01S01.050: The surface area A of the box of Example 8 was

$$A(x) = 2x^2 + \frac{500}{x}, \qquad 0 < x < \infty.$$

The restrictions $x \geqq 1$ and $y \geqq 1$ imply that $1 \leqq x \leqq \sqrt{125}$. A small number of values of A, rounded to three places, are given in the following table.

x	1	2	3	4	5	6	7	8	9	10	11
A	502	258	185	157	150	155	169	191	218	250	287

It appears that A is minimized when $x = y = 5$.

C01S01.051: If x is an integer, then $\text{CEILING}(x) = x$ and $-\text{FLOOR}(-x) = -(-x) = x$. If x is not an integer, then choose the integer n so that $n < x < n + 1$. Then $\text{CEILING}(x) = n + 1$, $-(n + 1) < -x < -n$, and

$$-\text{FLOOR}(-x) = -[-(n + 1)] = n + 1.$$

In both cases we see that $\text{CEILING}(x) = -\text{FLOOR}(-x)$.

C01S01.052: The range of $\text{ROUND}(x)$ is the set $\mathbf{Z}$ of all integers. If k is a nonzero constant, then as x varies through all real number values, so does kx. Hence the range of $\text{ROUND}(kx)$ is $\mathbf{Z}$ if $k \neq 0$. If $k = 0$ then the range of $\text{ROUND}(kx)$ consists of the single number zero.

C01S01.053: By the result of Problem 52, the range of $\text{ROUND}(10x)$ is the set of all integers, so the range of $g(x) = \frac{1}{10}\text{ROUND}(10x)$ is the set of all integral multiple of $\frac{1}{10}$.

C01S01.054: What works for π will work for every real number; let $\text{ROUND2}(x) = \frac{1}{100}\text{ROUND}(100x)$. To be certain that this is correct (we will verify it only for positive numbers), write the [positive] real number x in the form

$$x = k + \frac{t}{10} + \frac{h}{100} + \frac{m}{1000} + r,$$

where k is a nonnnegative integer, t (the "tenths" digit) is a nonnegative integer between 0 and 9, h (the "hundredths" digit) is a nonnegative integer between 0 and 9, as is m, and $0 \leqq r < 0.001$. Then

$$\text{ROUND2}(x) = \frac{1}{100}\text{FLOOR}(100x + 0.5)$$

$$= \frac{1}{100}\text{FLOOR}(100k + 10t + h + \frac{1}{10}(m + 5) + 100r).$$

If $0 \leqq m \leqq 4$, the last expression becomes

$$\frac{1}{100}(100k + 10t + h) = k + \frac{t}{10} + \frac{h}{100},$$

which is the correct two-digit rounding of x. If $5 \leqq m \leqq 9$, it becomes

$$\frac{1}{100}(100k + 10t + h + 1) = k + \frac{t}{10} + \frac{h+1}{100},$$

also the correct two-digit rounding of x in this case.

C01S01.055: Let $\text{ROUND4}(x) = \frac{1}{10000}\text{ROUND}(10000x)$. To verify that ROUND4 has the desired property for [say] positive values of x, write such a number x in the form

$$x = k + \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \frac{d_4}{10000} + \frac{d_5}{100,000} + r,$$

where k is a nonnegative integer, each d_i is an integer between 0 and 9, and $0 \leqq r < 0.00001$. Application of ROUND4 to x then produces

$$\frac{1}{10000}\text{FLOOR}(10000k + 1000d_1 + 100d_2 + 10d_3 + d_4 + \frac{1}{10}(d_5 + 5) + 10000r).$$

Then consideration of the two cases $0 \leqq d_5 \leqq 4$ and $5 \leqq d_5 \leqq 9$ will show that ROUND4 produces the correct four-place rounding of x in both cases.

C01S01.056: Let $\text{CHOP4}(x) = \frac{1}{10000}\text{FLOOR}(10000x)$. Suppose that $x > 0$. Write x in the form

$$x = k + \frac{d_1}{10} + \frac{d_2}{100} + \frac{d_3}{1000} + \frac{d_4}{10000} + r,$$

where k is a nonnegative integer, each of the d_i is an integer between 0 and 9, and $0 \leqq r < 0.0001$. Then $\text{CHOP4}(x)$ produces

$$\frac{1}{10000}\text{FLOOR}(10000k + 1000d_1 + 100d_2 + 10d_3 + d_4 + 10000r)$$

$$= \frac{1}{10000}(10000k + 1000d_1 + 100d_2 + 10d_3 + d_4)$$

because $10000r < 1$. It follows that CHOP4 has the desired effect.

C01S01.057:

x	0.0	0.2	0.4	0.6	0.8	1.0
y	1.0	0.44	-0.04	-0.44	-0.76	-1.0

The sign change occurs between $x = 0.2$ and $x = 0.4$.

x	0.20	0.25	0.30	0.35	0.40
y	0.44	0.3125	0.19	0.0725	-0.04

The sign change occurs between $x = 0.35$ and $x = 0.40$.

x	0.35	0.36	0.37	0.38	0.39	0.40
y	0.0725	0.0496	0.0269	0.0044	-0.0179	-0.04

From this point on, the data for y will be rounded.

x	0.380	0.382	0.384	0.386	0.388	0.390
y	0.0044	-0.0001	-0.0045	-0.0090	-0.0135	-0.0179

Answer (rounded to two places): 0.38. The quadratic formula yields the two roots $\frac{1}{2}\left(3 \pm \sqrt{5}\right)$; the smaller of these is approximately 0.381966011250105151795.

Problems 58 through 66 are worked in the same way as Problem 57.

C01S01.058: The sign change intervals are $[2, 3]$, $[2.6, 2.8]$, $[2.60, 2.64]$, and $[2.616, 2.624]$. Answer: $\frac{1}{2}\left(3 + \sqrt{5}\right) \approx 2.62$.

C01S01.059: The sign change intervals are $[1, 2]$, $[1.2, 1.4]$, $[1.20, 1.24]$, $[1.232, 1.240]$, and $[1.2352, 1.2368]$. Answer: $-1 + \sqrt{5} \approx 1.24$.

C01S01.060: The sign change intervals are $[-4, -3]$, $[-3.4, -3.2]$, $[-3.24, -3.20]$, $[-3.240, -3.232]$, and $[-3.2368, -3.2352]$. Answer: $-1 - \sqrt{5} \approx -3.24$.

C01S01.061: The sign change intervals are $[0, 1]$, $[0.6, 0.8]$, $[0.68, 0.72]$, $[0.712, 0.720]$, and $[0.7184, 0.7200]$. Answer: $\frac{1}{4}\left(7 - \sqrt{17}\right) \approx 0.72$.

C01S01.062: The sign change intervals are $[2, 3]$, $[2.6, 2.8]$, $[2.76, 2.80]$, $[2.776, 2.784]$, and $[2.7792, 2.7808]$. Answer: $\frac{1}{4}\left(7 + \sqrt{17}\right) \approx 2.78$.

C01S01.063: The sign change intervals are $[3, 4]$, $[3.2, 3.4]$, $[3.20, 3.24]$, $[3.208, 3.216]$, and $[3.2080, 3.2096]$. Answer: $\frac{1}{2}\left(11 - \sqrt{21}\right) \approx 3.21$.

C01S01.064: The sign change intervals are $[7, 8]$, $[7.6, 7.8]$, $[7.76, 7.80]$, $[7.784, 7.792]$, and $[7.7904, 7.7920]$. Answer: $\frac{1}{2}\left(11 + \sqrt{21}\right) \approx 7.79$.

C01S01.065: The sign change intervals are $[1, 2]$, $[1.6, 1.8]$, $[1.60, 1.64]$, $[1.608, 1.616]$, $[1.6144, 1.6160]$, and $[1.61568, 1.61600]$. Answer: $\frac{1}{6}\left(-23 + \sqrt{1069}\right) \approx 1.62$.

C01S01.066: The sign change intervals are $[-10, -9]$, $[-9.4, -9.2]$, $[-9.32, -9.28]$, $[-9.288, -9.280]$, and $[-9.2832, -9.2816]$. Answer: $\frac{1}{6}\left(-23 - \sqrt{1069}\right) \approx -9.28$.

Section 1.2

C01S02.001: The slope of L is $m = (3-0)/(2-0) = \frac{3}{2}$, so L has equation

$$y - 0 = \frac{3}{2}(x - 0); \qquad \text{that is,} \qquad 2y = 3x.$$

C01S02.002: Because L is vertical and $(7, 0)$ lies on L, every point on L has Cartesian coordinates $(7, y)$ for some number y (and every such point lies on L). Hence an equation of L is $x = 7$.

C01S02.003: Because L is horizontal, it has slope zero. Hence an equation of L is

$$y - (-5) = 0 \cdot (x - 3); \qquad \text{that is,} \qquad y = -5.$$

C01S02.004: Because $(2, 0)$ and $(0, -3)$ lie on L, it has slope $(0 + 3)/(2 - 0) = \frac{3}{2}$. Hence an equation of L is

$$y - 0 = \tfrac{3}{2}(x - 2); \qquad \text{that is,} \qquad y = \tfrac{3}{2}x - 3.$$

C01S02.005: The slope of L is $(3 - (-3))/(5 - 2) = 2$, so an equation of L is

$$y - 3 = 2(x - 5); \qquad \text{that is,} \qquad y = 2x - 7.$$

C01S02.006: An equation of L is $y - (-4) = \frac{1}{2}(x - (-1))$; that is, $2y + 7 = x$.

C01S02.007: The slope of L is $\tan(135°) = -1$, so L has equation

$$y - 2 = -1 \cdot (x - 4); \qquad \text{that is,} \qquad x + y = 6.$$

C01S02.008: Equation: $y - 7 = 6(x - 0)$; that is, $y = 6x + 7$.

C01S02.009: The second line's equation can be written in the form $y = -2x + 10$ to show that it has slope -2. Because L is parallel to the second line, L also has slope -2 and thus equation $y - 5 = -2(x - 1)$.

C01S02.010: The equation of the second line can be rewritten as $y = -\frac{1}{2}x + \frac{17}{2}$ to show that it has slope $-\frac{1}{2}$. Because L is perpendicular to the second line, L has slope 2 and thus equation $y - 4 = 2(x + 2)$.

C01S02.011: $x^2 - 4x + 4 + y^2 = 4$: $(x - 2)^2 + (y - 0)^2 = 2^2$. Center $(2, 0)$, radius 2.

C01S02.012: $x^2 + y^2 + 6y + 9 = 9$: $(x - 0)^2 + (y + 3)^2 = 3^2$. Center $(0, -3)$, radius 3.

C01S02.013: $x^2 + 2x + 1 + y^2 + 2y + 1 = 4$: $(x + 1)^2 + (y + 1)^2 = 2^2$. Center $(-1, -1)$, radius 2.

C01S02.014: $x^2 + 10x + 25 + y^2 - 20y + 100 = 25$: $(x + 5)^2 + (y - 10)^2 = 5^2$. Center $(-5, 10)$, radius 5.

C01S02.015: $x^2 + y^2 + x - y = \frac{1}{2}$: $x^2 + x + \frac{1}{4} + y^2 - y + \frac{1}{4} = 1$; $(x + \frac{1}{2})^2 + (y - \frac{1}{2})^2 = 1$. Center: $(-\frac{1}{2}, \frac{1}{2})$, radius 1.

C01S02.016: $x^2 + y^2 - \frac{2}{3}x - \frac{4}{3}y = \frac{11}{9}$: $x^2 - \frac{2}{3}x + \frac{1}{9} + y^2 - \frac{4}{3}y + \frac{4}{9} = \frac{16}{9}$; $(x - \frac{1}{3})^2 + (y - \frac{2}{3})^2 = (\frac{4}{3})^2$. Center $(\frac{1}{3}, \frac{2}{3})$, radius $\frac{4}{3}$.

C01S02.017: $y = (x-3)^2$: Opens upward, vertex at $(3,0)$.

C01S02.018: $y - 16 = -x^2$: Opens downward, vertex at $(0,16)$.

C01S02.019: $y - 3 = (x+1)^2$: Opens upward, vertex at $(-1,3)$.

C01S02.020: $2y = x^2 - 4x + 4 + 4$: $y - 2 = \frac{1}{2}(x-2)^2$. Opens upward, vertex at $(2,2)$.

C01S02.021: $y = 5(x^2 + 4x + 4) + 3 = 5(x+2)^2 + 3$: Opens upward, vertex at $(-2,3)$.

C01S02.022: $y = -(x^2 - x) = -(x^2 - x + \frac{1}{4}) + \frac{1}{4}$: $y - \frac{1}{4} = -(x - \frac{1}{2})^2$. Opens downward, vertex at $(\frac{1}{2}, \frac{1}{4})$.

C01S02.023: $x^2 - 6x + 9 + y^2 + 8y + 16 = 25$: $(x-3)^2 + (y+4)^2 = 5^5$. Circle, center $(3,-4)$, radius 5.

C01S02.024: $(x-1)^2 + (y+1)^2 = 0$: The graph consists of the single point $(1,-1)$.

C01S02.025: $(x+1)^2 + (y+3)^2 = -10$: There are no points on the graph.

C01S02.026: $x^2 + y^2 - x + 3y + 2.5 = 0$: $x^2 - x + 0.25 + y^2 + 3y + 2.25 = 0$: $(x - 0.5)^2 + (y + 1.5)^2 = 0$. The graph consists of the single point $(0.5, -1.5)$.

C01S02.027: The graph is the straight line segment connecting the two points $(-1,7)$ and $(1,-3)$ (including those two points).

C01S02.028: The graph is the straight line segment connecting the two points $(0,2)$ and $(2,-8)$, including the first of these two points but not the second.

C01S02.029: The graph is the parabola that opens downward, symmetric around the y-axis, with vertex at $(0,10)$ and x-intercepts $\pm\sqrt{10}$.

C01S02.030: The graph of $y = 1 + 2x^2$ is a parabola that opens upwards, is symmetric around the y-axis, and has vertex at $(0,1)$.

C01S02.031: The graph of $y = x^3$ can be visualized by modifying the familiar graph of the parabola with equation $y = x^2$: The former may be obtained by multiplying the y-coordinate of the latter's point (x, x^2) by x. Thus both have flat spots at the origin. For $0 < x < 1$, the graph of $y = x^3$ is below that of $y = x^2$. They cross at $(1,1)$, and for $x > 1$ the graph of $y = x^2$ is below that of $y = x^3$, with the difference becoming arbitrarily large as x increases without bound. If the graph of $y = x^3$ for $x \geqq 0$ is rotated $180°$ around the point $(0,0)$, the graph of $y = x^3$ for $x < 0$ is the result.

C01S02.032: The graph of $f(x) = x^4$ can be visualized by first visualizing the graph of $y = x^2$. If the y-coordinate of each point on this graph is replaced with its square (x^4), the result is the graph of f. The effect on the graph of $y = x^2$ is to multiply the y-coordinate by x^2, which is between 0 and 1 for $0 < |x| < 1$ and which is larger than 1 for $|x| > 1$. Thus the graph of f superficially resembles that of $y = x^2$, but is much closer to the x-axis for $|x| < 1$ and much farther away for $|x| > 1$. The two graphs cross at $(0,0)$ (where each has a flat spot) and at $(\pm 1, 1)$, but the graph of f is much steeper at the latter two points.

C01S02.033: To graph $y = f(x) = \sqrt{4 - x^2}$, note that $y \geqq 0$ and that $y^2 = 4 - x^2$; that is, $x^2 + y^2 = 4$. Hence the graph of f is the *upper half* of the circle with center $(0,0)$ and radius 2.

C01S02.034: To graph $y = f(x) = -\sqrt{9 - x^2}$, note that $y \leqq 0$ and that $y^2 = 9 - x^2$; that is, that $x^2 + y^2 = 9$. Hence the graph of f is the *lower half* of the circle with center $(0,0)$ and radius 3.

2

C01S02.035: To graph $f(x) = \sqrt{x^2 - 9}$, note that there is no graph for $-3 < x < 3$, that $f(\pm 3) = 0$, and that $f(x) > 0$ for $x < -3$ and for $x > 3$. If x is large positive, then $\sqrt{x^2 - 9} \approx \sqrt{x^2} = x$, so the graph of f has x-intercept $(3, 0)$ and rises as x increases, nearly coinciding with the graph of $y = x$ for x large positive. The case $x < -3$ is trickier. In this case, if x is a large negative number, then $f(x) = \sqrt{x^2 - 9} \approx \sqrt{x^2} = -x$ (Note the minus sign!). So for $x \leqq -3$, the graph of f has x-intercept $(-3, 0)$ and, for x large negative, almost coincides with the graph of $y = -x$. Later we will see that the graph of f becomes arbitrarily steep as x gets closer and closer to ± 3.

C01S02.036: As x increases without bound—either positively or negatively—$f(x)$ gets arbitrarily close to zero. Moreover, if x is large positive then $f(x)$ is negative and close to zero, so the graph of f lies just below the x-axis for such x. Similarly, the graph of f lies just above the x-axis for x large negative. If x is slightly less than 1 but very close to 1, then $f(x)$ is the reciprocal of a tiny positive number, hence is a large positive number. So the graph of f just to the left of the vertical line $x = 1$ almost coincides with the top half of that line. Similarly, just to the right of the line $x = 1$, then graph of f almost coincides with the bottom half of that line. There is no graph where $x = 1$, so the graph resembles the one in the next figure. The only intercept is the y-intercept $(0, 1)$. The graph correctly shows that the graph of f is increasing for $x < 1$ and for $x > 1$.
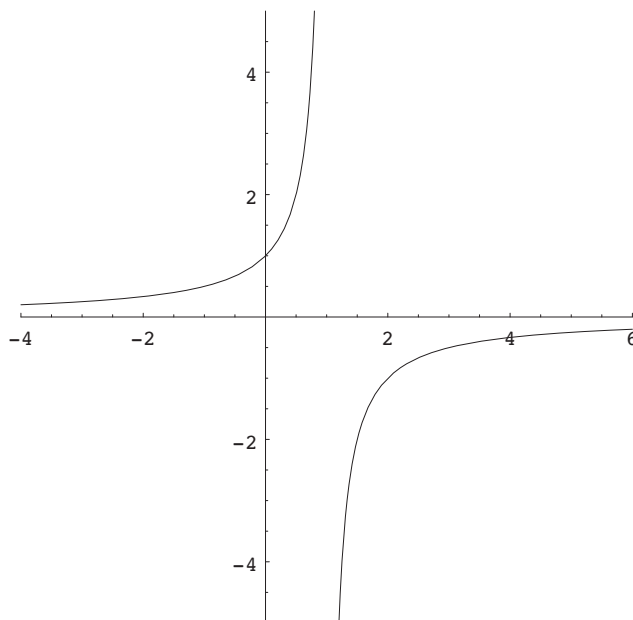
C01S02.037: Note that $f(x)$ is positive and close to zero for x large positive, so that the graph of f is just above the x-axis—and nearly coincides with it—for such x. Similarly, the graph of f is just below the x-axis and nearly coincides with it for x large negative. There is no graph where $x = -2$, but if x is slightly greater than -2 then $f(x)$ is the reciprocal of a very small positive number, so $f(x)$ is large and nearly coincides with the upper half of the vertical line $x = -2$. Similarly, if x is slightly less than -2, then the graph of $f(x)$ is large negative and nearly coincides with the the lower half of the line $x = -3$. The graph of f is decreasing for $x < -2$ and for $x > -2$ and its only intercept is the y-intercept $\left(0, \frac{1}{2}\right)$.

C01S02.038: Note that $f(x)$ is very small but positive if x is either large positive or large negative. There is no graph for $x = 0$, but if x is very close to zero, then $f(x)$ is the reciprocal of a very small positive number, and hence is large positive. So the graph of f is just above the x-axis and almost coincides with it if $|x|$ is large, whereas the graph of f almost coincides with the positive y-axis for x near zero. There are no intercepts; the graph of f is increasing for $x < 0$ and is decreasing for $x > 0$.

C01S02.039: Note that $f(x) > 0$ for all x other than $x = 1$, where f is not defined. If $|x|$ is large, then $f(x)$ is near zero, so the graph of f almost coincides with the x-axis for such x. If x is very close to 1, then $f(x)$ is the reciprocal of a very small positive number, hence $f(x)$ is large positive. So for such x, the graph of $f(x)$ almost coincides with the upper half of the vertical line $x = 1$. The only intercept is $(0, 1)$.

C01S02.040: Note first that $f(x)$ is undefined at $x = 0$. To handle the absolute value symbol, we look at two cases: If $x > 0$, then $f(x) = 1$; if $x < 0$, then $f(x) = -1$. So the graph of f consists of the part of the horizontal line $y = 1$ for which $x > 0$, together with the part of the horizontal line $y = -1$ for which $x < 0$.

C01S02.041: Note that $f(x)$ is undefined when $2x + 3 = 0$; that is, when $x = -\frac{3}{2}$. If x is large positive, then $f(x)$ is positive and close to zero, so the graph of f is slightly above the x-axis and almost coincides with the x-axis. If x is large negative, then $f(x)$ is negative and close to zero, so the graph of f is slightly below the x-axis and almost coincides with the x-axis. If x is slightly greater than $-\frac{3}{2}$ then $f(x)$ is very large positive, so the graph of f almost coincides with the upper half of the vertical line $x = -\frac{3}{2}$. If x is slightly less than $-\frac{3}{2}$ then $f(x)$ is very large negative, so the graph of f almost coincides with the lower half of that vertical line. The graph of f is decreasing for $x < -\frac{3}{2}$ and also decreasing for $x > -\frac{3}{2}$. The only intercept is at $\left(0, \frac{1}{3}\right)$.

C01S02.042: Note that $f(x)$ is undefined when $2x + 3 = 0$; that is, when $x = -\frac{3}{2}$. If x is large positive or large negative, then $f(x)$ is positive and close to zero, so the graph of f is slightly above the x-axis and almost coincides with the x-axis for $|x|$ large. If x is close to $-\frac{3}{2}$ then $f(x)$ is very large positive, so the graph of f almost coincides with the upper half of the vertical line $x = -\frac{3}{2}$. The graph of f is increasing for $x < -\frac{3}{2}$ and decreasing for $x > -\frac{3}{2}$. The only intercept is at $\left(0, \frac{1}{9}\right)$.

C01S02.043: Given $y = f(x) = \sqrt{1 - x}$, note that $y \geqq 0$ and that $y^2 = 1 - x$; that is, $x = 1 - y^2$. So the graph is the part of the parabola $x = 1 - y^2$ for which $y \geqq 0$. This parabola has horizontal axis of symmetry the y-axis, opens to the left (because the coefficient of y^2 is negative), and has vertex $(1, 0)$. Therefore the graph of f is the upper half of this parabola.

C01S02.044: Note that the interval $x < 1$ is the domain of f, so there is no graph for $x \geqq 1$. If x is a large negative number, then the denominator is large positive, so that its reciprocal $f(x)$ is very small positive. As x gets closer and closer to 1 (while $x < 1$), the denominator approaches zero, so its reciprocal $f(x)$ takes on arbitrarily large positive values. So the graph of f is slightly above the x-axis and almost coincides with that axis for x large negative; the graph of f almost coincides with the upper half of the vertical line $x = 1$ for x near (and less than) 1. The graph of f is increasing for all $x < 1$ and $(0, 1)$ is the only intercept.

C01S02.045: Note that $f(x)$ is defined only if $2x + 3 > 0$; that is, if $x > -\frac{3}{2}$. Note also that $f(x) > 0$ for all such x. If x is large positive, then $f(x)$ is positive but near zero, so the graph of f is just above the x-axis and almost coincides with it. If x is very close to $-\frac{3}{2}$ (but larger), then the denominator in $f(x)$ is very tiny positive, so the graph of f almost coincides with the upper half of the vertical line $x = -\frac{3}{2}$ for such x. The graph of f is decreasing for all $x > -\frac{3}{2}$.

C01S02.046: Given: $f(x) = |2x - 2|$. Case 1: $x \geqq 1$. Then $2x - 2 \geqq 0$, so that $f(x) = 2x - 2$. Because $f(1) = 0$, the graph of f for $x \geqq 1$ consists of the part of the straight line through $(1, 0)$ with slope 2. Case 2: $x < 1$. Then $2x - 2 < 0$, so that $f(x) = -2x + 2$. The line $y = -2x + 2$ passes through $(1, 0)$, so the graph of f for $x < 1$ consists of the part of the straight line through $(1, 0)$ with slope -2.

C01S02.047: Given: $f(x) = |x| + x$. If $x \geqq 0$ then $f(x) = x + x = 2x$, so if $x \geqq 0$ then the graph of f is the part of the straight line through $(0, 0)$ with slope 2 for which $x \geqq 0$. If $x < 0$ then $f(x) = -x + x = 0$, so the rest of the graph of f coincides with the negative x-axis.

C01S02.048: Given: $f(x) = |x - 3|$. If $x \geqq 0$ then $f(x) = x - 3$, so the graph of f consists of the straight line through $(3, 0)$ with slope 1 for $x \geqq 3$. If $x < 0$ then $f(x) = -x + 3$, so the graph of f consists of that part of the straight line with slope -3 and y-intercept $(0, 3)$. These two line segments fit together perfectly at the point $(3, 0)$; there is no break or gap or discontinuity in the graph of f.

C01S02.049: Given: $f(x) = |2x + 5|$. The two cases are determined by the point where $2x + 5$ changes sign, which is where $x = -\frac{5}{2}$. If $x \geqq -\frac{5}{2}$, then $f(x) = 2x + 5$, so the graph of f consists of the part of the line with slope 2 and y-intercept 5 for which $x \geqq -\frac{5}{2}$. If $x < -\frac{5}{2}$, then the graph of f is the part of the straight line $y = -2x - 5$ for which $x < -\frac{5}{2}$.

C01S02.050: The graph consists of the part of the line $y = -x$ for which $x < 0$ together with the part of the parabola $y = x^2$ for which $x \geqq 0$. The two graphs fit together perfectly at the point $(0, 0)$; there is no break, gap, jump, or discontinuity there. The graph is shown next.
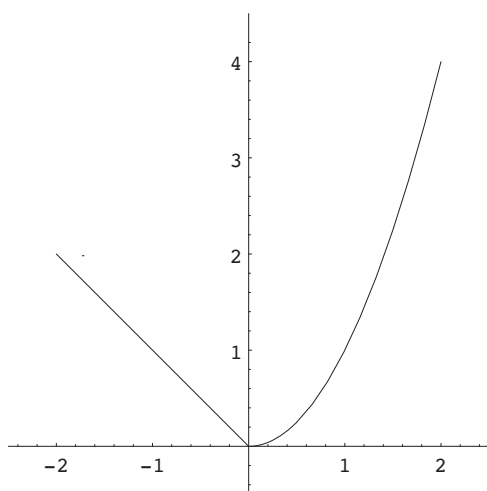
C01S02.051: The graph consists of the horizontal line $y = 0$ for $x < 0$ together with the horizontal line $y = 1$ for $x \geqq 0$. As x moves from left to right through the value zero, there is an abrupt and unavoidable "jump" in the value of f from 0 to 1. That is, f is discontinuous at $x = 0$. To see part of the graph of f, enter the *Mathematica* commands

f[x_] := If[x < 0, 0, 1]
Plot[f[x], {x, −3.5, 3.5 }, AspectRatio −> Automatic, PlotRange −> {{ −3.5, 3.5 }, {−1.5, 2.5 }}];

C01S02.052: The graph of f consists of the open intervals $\ldots$, $(-2, -1)$, $(-1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 3)$, $\ldots$ on the x-axis together with the isolated points $\ldots$, $(-1, 1)$, $(0, 1)$, $(1, 1)$, $(2, 1)$, $(3, 1)$, $\ldots$. There is a discontinuity at every integral value of x. A *Mathematica* plot of

f[x_] := If[IntegerQ[x], 1, 0]

will produce a graph that's completely different because *Mathematica*, like most plotting programs, "connects the dots," in effect assuming that every function is continuous at every point in its domain.

C01S02.053: Because the graph of the greatest integer function changes at each integral value of x, the graph of $f(x) = [\![2x]\!]$ changes twice as often—at each integral multiple of $\frac{1}{2}$. So as x moves from left to right through such points, the graph jumps upward one unit. Thus there is a discontinuity at each integral multiple of $\frac{1}{2}$. Because f is constant otherwise, these are the only discontinuities. To see something like the graph of f, enter the *Mathematica* commands

f[x_] := Floor[2*x];

Plot[f[x], {x, −3.5, 3.5 }, AspectRatio −> Automatic, PlotRange −> {{ −3.5, 3.5 }, {−4.5, 4.5 }}];

Mathematica will draw vertical lines connecting points that it shouldn't, making the graph look like treads and risers of a staircase, whereas only the treads are on the graph.

C01S02.054: The function f is undefined at $x = 1$. The graph consists of the horizontal line $y = 1$ for $x > 1$ together with the horizontal line $y = -1$ for $x < 1$. There is a discontinuity at $x = 1$.

C01S02.055: Given: $f(x) = [\![x]\!]$. If n is an integer and $n \leq x < n + 1$, then express x as $x = n + (\!(x)\!)$ where $(\!(x)\!) = x - [\![x]\!]$ is the *fractional part* of x. Then $f(x) = n - x = n - [n + (\!(x)\!)] = -(\!(x)\!)$. So $f(x)$ is the negative of the fractional part of x. So as x ranges from n up to (but not including) $n + 1$, $f(x)$ begins at 0 and drops linearly down not quite to -1. That is, on the interval $(n, n + 1)$, the graph of f is the straight line segment connecting the two points $(n, 0)$ and $(n + 1, -1)$ with the first of these points included and the second excluded. There is a discontinuity at each integral value of x.

C01S02.056: Given: $f(x) = [\![x]\!] + [\![-x]\!] + 1$. If x is an integer, then $f(x) = x + (-x) + 1 = 1$. If x is not an integer, then choose the integer n such that $n < x < n + 1$. Then $-(n+1) < -x < -n$, so

$$f(x) = [\![x]\!] + [\![-x]\!] + 1 = n - (n + 1) + 1 = 0.$$

So f is the same function as the one defined in Problem 52 and has the same discontinuities: one at each integral value of x.

C01S02.057: Because $y = 2x^2 - 6x + 7 = 2(x^2 - 3x + 3.5) = 2(x^2 - 3x + 2.25 + 1.25) = 2(x - 1.5)^2 + 2.5$, the vertex of the parabola is at $(1.5, 2.5)$.

C01S02.058: Because $y = 2x^2 - 10x + 11 = 2(x^2 - 5x + 5.5) = 2(x^2 - 5x + 6.25 - 0.75) = 2(x - 2.5)^2 - 1.5$, the vertex of the parabola is at $(2.5, -1.5)$.

C01S02.059: Because $y = 4x^2 - 18x + 22 = 4(x^2 - (4.5)x + 5.5) = 4(x^2 - (4.5)x + 5.0625 + 0.4375)$ $= 4(x - 2.25)^2 + 1.75$, the vertex of the parabola is at $(2.25, 1.75)$.

C01S02.060: Because $y = 5x^2 - 32x + 49 = 5(x^2 - (6.4)x + 9.8) = 5(x^2 - (6.4)x + 10.24 - 0.44)$ $= 5(x - 3.2)^2 - 2.2$, the vertex of the parabola is at $(3.2, -2.2)$.

C01S02.061: Because $y = -8x^2 + 36x - 32 = -8(x^2 - (4.5)x + 4) = -8(x^2 - (4.5)x + 5.0625 - 1.0625)$ $= -8(x - 2.25)^2 + 8.5$, the vertex of the parabola is at $(2.25, 8.5)$.

C01S02.062: Because $y = -5x^2 - 34x - 53 = -5(x^2 + (6.8)x + 10.6) = -5(x^2 + (6.8)x + 11.56 - 0.96)$ $= -5(x + 3.4)^2 + 4.8$, the vertex of the parabola is at $(-3.4, 4.8)$.

C01S02.063: Because $y = -3x^2 - 8x + 3 = -3\left(x^2 + \frac{8}{3}x - 1\right) = -3\left(x^2 + \frac{8}{3}x + \frac{16}{9} - \frac{25}{9}\right) = -3\left(x + \frac{4}{3}\right)^2 + \frac{25}{3}$, the vertex of the parabola is at $\left(-\frac{4}{3}, \frac{25}{3}\right)$.

C01S02.064: Because $y = -9x^2 + 34x - 28 = -9\left(x^2 - \frac{34}{9}x + \frac{28}{9}\right) = -9\left(x^2 - \frac{34}{9}x + \frac{289}{81} - \frac{37}{81}\right)$ $= -9\left(x - \frac{17}{9}\right)^2 + \frac{37}{9}$, the vertex of the parabola is at $\left(\frac{17}{9}, \frac{37}{9}\right)$.

C01S02.065: To find the maximum height $y = -16t^2 + 96t$ of the ball, we find the vertex of the parabola: $y = -16(t^2 - 6t) = -16(t^2 - 6t + 9 - 9) = -16(t - 3)^2 + 144$. The vertex of the parabola is at $(3, 144)$ and therefore the maximum height of the ball is 144 ft.

C01S02.066: Recall that the area of the rectangle is given by $y = A(x) = x(50-x)$. To maximize $A(x)$ we find the vertex of the parabola: $y = 50x - x^2 = -(x^2 - 50x) = -(x^2 - 50x + 625 - 625) = -(x-25)^2 + 625$. Because the vertex of the parabola is at $(25, 625)$ and $x = 25$ is in the domain of the function A, the maximum value of $A(x)$ occurs at $x = 25$ and is $A(25) = 625$ (ft^2).

C01S02.067: If two positive numbers x and y have sum 50, then $y = 50 - x$ and $x < 50$ (because $y > 0$). To maximize their product $p(x)$ we find the vertex of the parabola

$$y = p(x) = x(50-x) = -(x^2 - 50x)$$

$$= -(x^2 - 50x + 625 - 625) = -(x-25)^2 + 625,$$

which is at $(25, 625)$. Because $0 < 25 < 50$, $x = 25$ is in the domain of the product function $p(x) = x(50-x)$, and hence the maximum value of the product of x and y is $p(25) = 625$.

C01S02.068: Recall that if x new wells are drilled, then the resulting total production p is given by $p(x) = 4000 + 100x - 5x^2$. To maximize $p(x)$ we find the vertex of the parabola

$$y = p(x) = -5x^2 + 100x + 4000 = -5(x^2 - 20x - 800)$$

$$= -5(x^2 - 20x + 100 - 900) = -5(x-10)^2 + 4500.$$

The vertex of the parabola $y = p(x)$ is therefore at $(10, 4500)$. Because $x = 10$ is in the domain of p (it is an integer between 0 and 40) and because the parabola opens downward (the coefficient of x^2 is negative), $x = 10$ indeed maximizes $p(x)$.

C01S02.069: The graph looks like the graph of $y = |x|$ because the slope of the left-hand part is -1 and that of the right-hand part is 1; but the vertex is shifted to $(-1, 0)$, so—using the translation principle—the graph in Fig. 1.2.29 must be the graph of $f(x) = |x+1|$, $-2 \leqq x \leqq 2$.

C01S02.070: Because the graph in Fig. 1.2.30 is composed of three straight-line segments, it can be described most easily using a "three-part" function:

$$f(x) = \begin{cases} 2x + 6 & \text{if } -3 \leqq x < -2; \\ 2 & \text{if } -2 \leqq x < 2; \\ \frac{1}{3}(10 - 2x) & \text{if } \quad 2 \leqq x \leqq 5. \end{cases}$$

C01S02.071: The graph in Fig. 1.2.31 is much like the graph of the greatest integer function—it takes on only integral values—but the "jumps" occur twice as often, so this must be very like—indeed, it is exactly—the graph of $f(x) = [\![2x]\!]$, $-1 \leqq x < 2$.

C01S02.072: The graph in Fig. 1.2.32 resembles the graph of the greatest integer function in that it takes on all integral values and only those, but it is decreasing rather than increasing and the "jumps" occur only at the even integers. Thus it must be the graph of something similar to $f(x) = -[\![\frac{1}{2}x]\!]$, $-4 \leqq x < 4$. Comparing values of f at $x = -4$, -3, -2.1, -2, -1, -0.1, 0, 1, 1.9, 2, 3, and 3.9 with points on the graph is sufficient evidence that the graph of f is indeed that shown in the figure.

C01S02.073: Clearly $x(t) = 45t$ for the first hour; that is, for $0 \leqq t \leqq 1$. In the second hour the graph of $x(t)$ must be a straight line (because of constant speed) of slope 75, thus with equation $x(t) = 75t + C$ for some constant C. The constant C is determined by the fact that $45t$ and $75t + C$ must be equal at time

$t = 1$, as the automobile cannot suddenly jump from one position to a completely different position in an instant. Hence $45 = 75 + C$, so that $C = -30$. Therefore

$$x(t) = \begin{cases} 45t & \text{if } 0 \leqq t \leqq 1; \\ 75t - 30 & \text{if } 1 < t \leqq 2. \end{cases}$$

To see the graph of $x(t)$, plot in *Mathematica*

$\quad$ x[t_] := If[t < 1, 45*t, 75*t − 30]

on the interval $0 \leqq t \leqq 2$.

C01S02.074: The graph of $x(t)$ will consist of three straight-line segments (because of the constant speeds), the first of slope 60 for $0 \leqq t \leqq 1$, the second of slope zero for $1 \leqq t \leqq 1.5$, and the third of slope 60 for $1.5 \leqq t \leqq 2.5$. The first pair must coincide when $t = 1$ and the second pair must coincide when $t = 1.5$ because the graph of $x(t)$ can have no discontinuities. So if we write $x(t) = 60$ for $0 \leqq t \leqq 1$, we must have $x(t) = 60$ for $1 \leqq t \leqq 1.5$. Finally, $x(t) = 60t + C$ for some constant C if $1.5 \leqq t \leqq 2.5$, but the latter must equal 60 when $t = 1.5$, so that $C = -30$. Hence

$$x(t) = \begin{cases} 60t & \text{if } 0 \leqq t \leqq 1, \\ 60 & \text{if } 1 < t \leqq 1.5, \\ 60t - 30 & \text{if } 1.5 < t \leqq 2.5. \end{cases}$$

The graph of $x(t)$ is shown next.
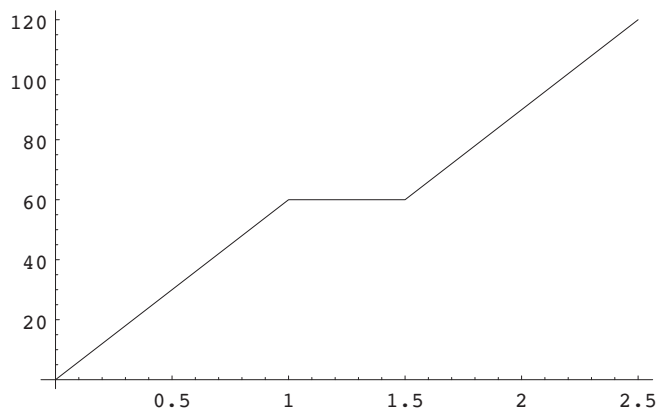
C01S02.075: The graph must consist of two straight-line segments (because of the constant speeds). The first must have slope 60, so we have $x(t) = 60t$ for $0 \leqq t \leqq 1$. The second must have slope -30, negative because you're driving in the reverse direction, so $x(t) = -30t + C$ for some constant C if $1 \leqq t \leqq 3$. The two segments must coincide when $t = 1$, so that $60 = -30 + C$. Thus $C = 90$ and thus a formula for $x(t)$ is

$$x(t) = \begin{cases} 60t & \text{if } 0 \leqq t \leqq 1, \\ 90 - 30t & \text{if } 1 < t \leqq 3. \end{cases}$$

C01S02.076: We need three straight line segments, the first of slope 60 for $0 \leqq t \leqq 0.5$, the second of slope -60 for $0.5 \leqq t \leqq 1$, and the third of slope 60 for $1 \leqq t \leqq 3$. Clearly the first must be $x(t) = 60t$ for $0 \leqq t \leqq 0.5$. The second must have the form $x(t) = -60t + C$ for some constant C, and the first and second must coincide when $t = 0.5$, so that $30 = -30 + C$, and thus $C = 60$. The third segment must have the form $x(t) = 60t + K$ for some constant K, and the second and third must coincide when $t = 1$, so that $0 = 60 + K$, and so $K = -60$. Therefore a formula for $x(t)$ is

$$x(t) = \begin{cases} 60t & \text{if } 0 \leqq t \leqq 0.5, \\ 60 - 60t & \text{if } 0.5 < t \leqq 1, \\ 60t - 60 & \text{if } 1 < t \leqq 3. \end{cases}$$

The graph of $x(t)$ is shown next.
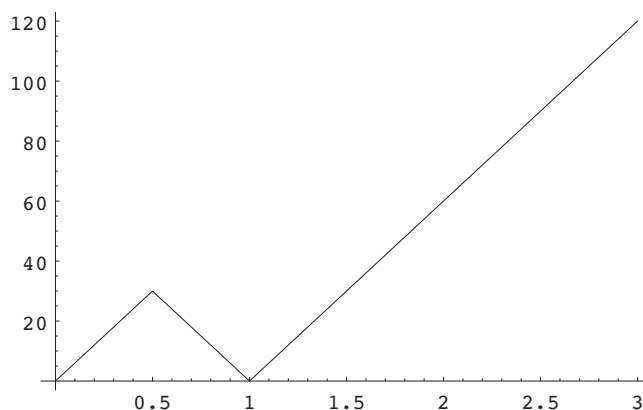
C01S02.077: Initially we work in units of pages and cents (to avoid decimals and fractions). The graph of C, as a function of p, must be a straight line segment, and its slope is (by information given)

$$\frac{C(79) - C(34)}{79 - 34} = \frac{305 - 170}{79 - 34} = \frac{135}{45} = 3.$$

Thus $C(p) = 3p + K$ for some constant K. So $3 \cdot 34 + K = 170$, and it follows that $K = 68$. So $C(p) = 3p + 68$, $1 \leqq p \leqq 100$, if C is to be expressed in cents. If C is to be expressed in dollars, we have

$$C(p) = (0.03)p + 0.68, \quad 1 \leqq p \leqq 100.$$

The "fixed cost" is incurred regardless of the number of pamphlets printed; it is \$0.68. The "marginal cost" of printing each additional page of the pamphlet is the coefficient \$0.03 of p.

C01S02.078: We are given $C(x) = a + bx$ where a and b are constants; we are also given

$$99.45 = C(207) = a + 207b \qquad \text{and}$$

$$79.15 = C(149) = a + 149b.$$

Subtraction of the second equation from the first yields $20.3 = 58b$, so that $b = 0.35$. Substitution of this datum in the first of the preceding equations then yields

$$99.45 = a + 207 \cdot 0.35 = a + 72.45, \quad \text{so that} \quad a = 27.$$

Therefore $C(x) = 27 + (0.35)x, \; 0 \leqq x < +\infty$. Thus if you drive 175 miles on the third day, the cost for that day will be $C(175) = 88.25$ (in dollars). The slope $b = 0.35$ represents a cost of \$0.35 per mile. The C-intercept $a = 27$ represents the daily base cost of renting the car. In civil engineering and in some branches of applied mathematics, the intercept $a = 27$ is sometimes called the *offset*, representing the vertical amount by which $C(0)$ is "offset" from zero.

C01S02.079: Suppose that the letter weighs x ounces, $0 < x \leqq 16$. If $x \leqq 8$, then the cost is simply 8 (dollars). If $8 < x \leqq 9$, add \$0.80; if $9 < x \leqq 10$, add \$1.60, and so on. Very roughly, one adds \$0.80 if

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