

INSTRUCTOR'S SOLUTIONS MANUAL

SEVENTH EDITION

ANALYTICAL MECHANICS



FOWLES & CASSIDAY

CHAPTER 1

FUNDAMENTAL CONCEPTS: VECTORS

1.1 (a) $\vec{A} + \vec{B} = (\hat{i} + \hat{j}) + (\hat{j} + \hat{k}) = \hat{i} + 2\hat{j} + \hat{k}$

$$|\vec{A} + \vec{B}| = (1 + 4 + 1)^{\frac{1}{2}} = \sqrt{6}$$

(b) $3\vec{A} - 2\vec{B} = 3(\hat{i} + \hat{j}) - 2(\hat{j} + \hat{k}) = 3\hat{i} + \hat{j} - 2\hat{k}$

(c) $\vec{A} \cdot \vec{B} = (1)(0) + (1)(1) + (0)(1) = 1$

(d) $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \hat{i}(1-0) + \hat{j}(0-1) + \hat{k}(1-0) = \hat{i} - \hat{j} + \hat{k}$

$$|\vec{A} \times \vec{B}| = (1 + 1 + 1)^{\frac{1}{2}} = \sqrt{3}$$

1.2 (a) $\vec{A} \cdot (\vec{B} + \vec{C}) = (2\hat{i} + \hat{j}) \cdot (\hat{i} + 4\hat{j} + \hat{k}) = (2)(1) + (1)(4) + (0)(1) = 6$

$$(\vec{A} + \vec{B}) \cdot \vec{C} = (3\hat{i} + \hat{j} + \hat{k}) \cdot 4\hat{j} = (3)(0) + (1)(4) + (1)(0) = 4$$

(b) $\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 4 & 0 \end{vmatrix} = -8$

$$(\vec{A} \times \vec{B}) \cdot \vec{C} = \vec{A} \cdot (\vec{B} \times \vec{C}) = -8$$

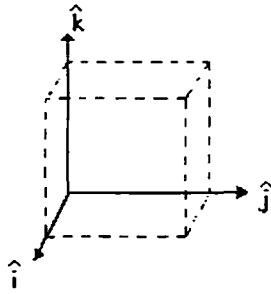
(c) $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C} = 4(\hat{i} + \hat{k}) - 2(4\hat{j}) = 4\hat{i} - 8\hat{j} + 4\hat{k}$

$$\begin{aligned} (\vec{A} \times \vec{B}) \times \vec{C} &= -\vec{C} \times (\vec{A} \times \vec{B}) = -\left[(\vec{C} \cdot \vec{B})\vec{A} - (\vec{C} \cdot \vec{A})\vec{B}\right] \\ &= -\left[0(2\hat{i} + \hat{j}) - 4(\hat{i} + \hat{k})\right] = 4\hat{i} + 4\hat{k} \end{aligned}$$

$$1.3 \quad \cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{(a)(a) + (2a)(2a) + (0)(3a)}{\sqrt{5a^2} \sqrt{14a^2}} = \frac{5a^2}{a^2 \sqrt{5} \sqrt{14}}$$

$$\theta = \cos^{-1} \sqrt{\frac{5}{14}} \approx 53^\circ$$

1.4



(a) $\vec{A} = \hat{i} + \hat{j} + \hat{k}$: body diagonal

$$A = |\vec{A}| = \sqrt{\hat{i} \cdot \hat{i} + \hat{j} \cdot \hat{j} + \hat{k} \cdot \hat{k}} = \sqrt{3}$$

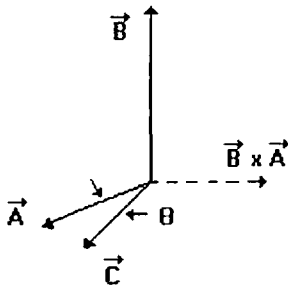
(b) $\vec{B} = \hat{i} + \hat{j}$: face diagonal

$$B = |\vec{B}| = \sqrt{2}$$

$$(c) \quad \vec{C} = \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$(d) \quad \cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{1-1}{\sqrt{3}\sqrt{2}} = 0 \quad \therefore \theta = 90^\circ$$

1.5



$$B = |\vec{B}| = |\vec{A} \times \vec{C}| = AC \sin \theta \quad \therefore C_y = C \sin \theta = \frac{B}{A}$$

$$\vec{A} \cdot \vec{C} = AC \cos \theta = u \quad \therefore C_x = C \cos \theta = \frac{u}{A}$$

$$\begin{aligned} \vec{C} &= \frac{\vec{A}}{A} C_x + \frac{\vec{B} \times \vec{A}}{|\vec{B} \times \vec{A}|} C_y = \frac{u}{A^2} \vec{A} + \frac{\vec{B} \times \vec{A}}{AB} \left(\frac{B}{A} \right) \\ &= \frac{u}{A^2} \vec{A} + \frac{1}{A^2} \vec{B} \times \vec{A} \end{aligned}$$

$$1.6 \quad \frac{d\vec{A}}{dt} = \hat{i} \frac{d}{dt}(\alpha t) + \hat{j} \frac{d}{dt}(\beta t^2) + \hat{k} \frac{d}{dt}(\gamma t^3) = \hat{i}\alpha + \hat{j}2\beta t + \hat{k}3\gamma t^2$$

$$\frac{d^2\vec{A}}{dt^2} = \hat{j}2\beta + \hat{k}6\gamma t$$

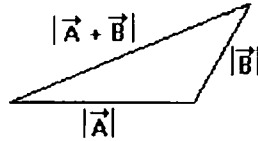
$$1.7 \quad 0 = \vec{A} \cdot \vec{B} = (q)(q) + (3)(-q) + (1)(2) = q^2 - 3q + 2$$

$$(q-2)(q-1) = 0, \quad q = 1 \text{ or } 2$$

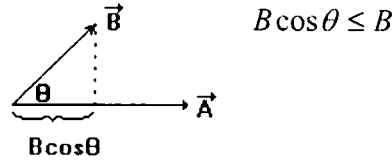
$$1.8 \quad |\vec{A} + \vec{B}|^2 = (\vec{A} + \vec{B}) \cdot (\vec{A} + \vec{B}) = A^2 + B^2 + 2\vec{A} \cdot \vec{B}$$

$$[|\vec{A}| + |\vec{B}|]^2 = A^2 + B^2 + 2AB$$

$$\text{Since } \vec{A} \cdot \vec{B} = AB \cos \theta \leq AB, \quad |\vec{A} + \vec{B}| \leq |\vec{A}| + |\vec{B}|$$



$$|\vec{A} \cdot \vec{B}| = |AB \cos \theta| = |\vec{A}| |\vec{B}| |\cos \theta| \leq |\vec{A}| |\vec{B}|$$



$$1.9 \quad \text{Show } \vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$$

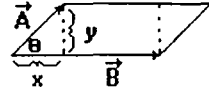
$$\text{or } \vec{A} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (A_x C_x + A_y C_y + A_z C_z) \vec{B} - (A_x B_x + A_y B_y + A_z B_z) \vec{C}$$

$$\begin{aligned} &= (A_x B_x C_x + A_y B_x C_y + A_z B_x C_z - A_x B_y C_x - A_y B_y C_y - A_z B_y C_z) \hat{i} \\ &+ (A_x B_y C_x + A_y B_y C_y + A_z B_y C_z - A_x B_x C_y - A_y B_x C_x - A_z B_x C_z) \hat{j} \\ &+ (A_x B_z C_x + A_y B_z C_y + A_z B_z C_z - A_x B_x C_z - A_y B_x C_z - A_z B_x C_z) \hat{k} \end{aligned}$$

$$\begin{aligned} & (A_y B_x C_y + A_z B_x C_z - A_y B_y C_x - A_z B_z C_x) \hat{i} \\ &= + (A_x B_y C_x + A_z B_y C_z - A_x B_x C_y - A_z B_z C_y) \hat{j} \\ &+ (A_x B_z C_x + A_y B_z C_y - A_x B_x C_z - A_y B_y C_z) \hat{k} \end{aligned}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & B_z C_x - B_x C_z & B_x C_y - B_y C_x \end{vmatrix} = \hat{i}(A_y B_x C_y - A_y B_y C_x - A_z B_z C_x + A_z B_x C_z) \\ + \hat{j}(A_z B_y C_z - A_z B_z C_y - A_x B_x C_y + A_x B_y C_x) \\ + \hat{k}(A_x B_z C_x - A_x B_x C_z - A_y B_y C_z + A_y B_z C_y)$$

1.10



$$y = A \sin \theta$$

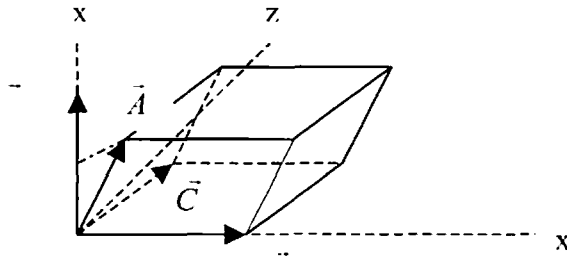
$$A = 2 \left(\frac{1}{2} xy \right) + y(B - x) = xy + yB - xy = AB \sin \theta$$

$$A = |\vec{A} \times \vec{B}|$$

1.11

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{A} \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = - \begin{vmatrix} B_x & B_y & B_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} = -\vec{B} \cdot (\vec{A} \times \vec{C})$$

1.12



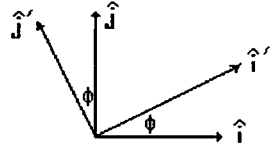
Let $\vec{A} = (A_x, A_y, A_z)$, $\vec{B} = (0, B_y, 0)$ and $\vec{C} = (0, C_y, C_z)$

C_z is the perpendicular distance between the plane $\vec{A}, \vec{B}$ and its opposite. $\vec{u} = \vec{B} \times \vec{C}$ is directed along the x-axis since the vectors $\vec{B}, \vec{C}$ are in the y,z plane. $u_x = |\vec{B} \times \vec{C}| = B_y C_z$

is the area of the parallelogram formed by the vectors $\vec{B}, \vec{C}$. Multiply that area times the height of plane $\vec{A}, \vec{B} = A_x$ to get the volume of the parallelepiped

$$V = A_x u_x = A_x B_y C_z = \vec{A} \cdot (\vec{B} \times \vec{C})$$

1.13 For rotation about the z axis:

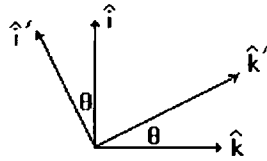


$$\hat{i} \cdot \hat{i}' = \cos \phi = \hat{j} \cdot \hat{j}', \quad \hat{k} \cdot \hat{k}' = 1$$

$$\hat{i} \cdot \hat{j}' = -\sin \phi$$

$$\hat{j} \cdot \hat{i}' = \sin \phi$$

For rotation about the y' axis:



$$\hat{i} \cdot \hat{i}' = \cos \theta = \hat{k} \cdot \hat{k}', \quad \hat{j} \cdot \hat{j}' = 1$$

$$\hat{i} \cdot \hat{k}' = \sin \theta$$

$$\hat{k} \cdot \hat{i}' = -\sin \theta$$

$$\vec{T} = \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \end{pmatrix}$$

1.14

$$\hat{i} \cdot \hat{i}' = \cos 30^\circ = \frac{\sqrt{3}}{2} \quad \hat{j} \cdot \hat{i}' = \sin 30^\circ = \frac{1}{2} \quad \hat{k} \cdot \hat{i}' = 0$$

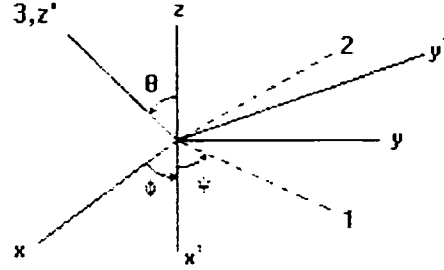
$$\hat{i} \cdot \hat{j}' = -\sin 30^\circ = -\frac{1}{2} \quad \hat{j} \cdot \hat{j}' = \cos 30^\circ = \frac{\sqrt{3}}{2} \quad \hat{k} \cdot \hat{j}' = 0$$

$$\hat{i} \cdot \hat{k}' = 0 \quad \hat{j} \cdot \hat{k}' = 0 \quad \hat{k} \cdot \hat{k}' = 1$$

$$\begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} + \frac{3}{2} \\ \frac{3}{2}\sqrt{3} - 1 \\ -1 \end{bmatrix}$$

$$\vec{A} = 3.232\hat{i}' + 1.598\hat{j}' - \hat{k}'$$

- 1.15
- | | | | |
|----|---------------------------------------|---------------------|------------|
| 1. | Rotate thru ϕ about z-axis | $\phi = 45^\circ$ | R_ϕ |
| 2. | Rotate thru θ about x' -axis | $\theta = 45^\circ$ | R_θ |
| 3. | Rotate thru ψ about z' -axis | $\psi = 45^\circ$ | R_ψ |



$$R_\phi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad R_\psi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R(\psi, \theta, \phi) = R_\psi R_\theta R_\phi = \begin{pmatrix} \frac{1}{2} - \frac{1}{2\sqrt{2}} & \frac{1}{2} + \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2} - \frac{1}{2\sqrt{2}} & -\frac{1}{2} + \frac{1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = R(\psi, \theta, \phi) \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Condition is: $\vec{x}' = R\vec{x}$ where $\vec{x}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

Since $\vec{x} \cdot \vec{x} = 1$ we have: $\psi^2 + \beta^2 + \alpha^2 = 1$

After a lot of algebra: $\alpha = \frac{1}{2} - \frac{\sqrt{2}}{4}$, $\beta = \frac{1}{2} + \frac{\sqrt{2}}{4}$, $\gamma = \frac{1}{2}$

1.16 $\vec{v} = v\hat{r} = ct\hat{r}$

$$\vec{a} = \dot{v}\hat{r} + \frac{v^2}{\rho}\hat{n} = c\dot{t}\hat{r} + \frac{c^2 t^2}{b}\hat{n}$$

$$\text{at } t = \sqrt{\frac{b}{c}}, \quad \bar{v} = \hat{r}\sqrt{bc} \quad \text{and} \quad \bar{a} = c\hat{r} + c\hat{n}$$

$$\cos \theta = \frac{\bar{v} \cdot \bar{a}}{va} = \frac{c\sqrt{bc}}{\sqrt{bc}\sqrt{2c^2}} = \frac{1}{\sqrt{2}}$$

$$\theta = 45^\circ$$

$$1.17 \quad \bar{v}(t) = -i b \omega \sin(\omega t) + j 2 b \omega \cos(\omega t)$$

$$|\bar{v}| = (b^2 \omega^2 \sin^2 \omega t + 4b^2 \omega^2 \cos^2 \omega t)^{\frac{1}{2}} = b \omega (1 + 3 \cos^2 \omega t)^{\frac{1}{2}}$$

$$\bar{a}(t) = -i b \omega^2 \cos \omega t - j 2 b \omega^2 \sin \omega t$$

$$|\bar{a}| = b \omega^2 (1 + 3 \sin^2 \omega t)^{\frac{1}{2}}$$

$$\text{at } t = 0, \quad |\bar{v}| = 2b\omega; \quad \text{at } t = \frac{\pi}{2\omega}, \quad |\bar{v}| = b\omega$$

$$1.18 \quad \bar{v}(t) = i b \omega \cos \omega t - j b \omega \sin \omega t + \hat{k} 2 c t$$

$$\bar{a}(t) = -i b \omega^2 \sin \omega t - j b \omega^2 \cos \omega t + \hat{k} 2 c$$

$$|\bar{a}| = (b^2 \omega^4 \sin^2 \omega t + b^2 \omega^4 \cos^2 \omega t + 4c^2)^{\frac{1}{2}} = (b^2 \omega^4 + 4c^2)^{\frac{1}{2}}$$

$$1.19 \quad \bar{v} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta = b k e^{kt} \hat{e}_r + b c e^{kt} \hat{e}_\theta$$

$$\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta = b(k^2 - c^2)e^{kt}\hat{e}_r + 2bck e^{kt}\hat{e}_\theta$$

$$\cos \phi = \frac{\bar{v} \cdot \bar{a}}{va} = \frac{b^2 k (k^2 - c^2) e^{2kt} + 2b^2 c^2 k e^{2kt}}{b e^{kt} (k^2 + c^2)^{\frac{1}{2}} b e^{kt} \left[(k^2 - c^2)^2 + 4c^2 k^2 \right]^{\frac{1}{2}}}$$

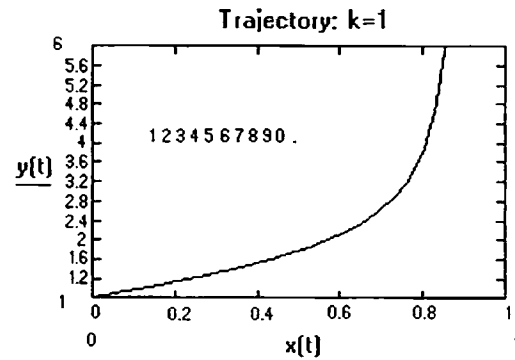
$$\cos \phi = \frac{k(k^2 + c^2)}{(k^2 + c^2)^{\frac{1}{2}}(k^2 + c^2)} = \frac{k}{(k^2 + c^2)^{\frac{1}{2}}}, \quad \text{a constant}$$

$$1.20 \quad \bar{a} = (\ddot{R} - R\phi)\hat{e}_R + (2\dot{R}\phi + R\ddot{\phi})\hat{e}_\phi + \ddot{z}\hat{e}_z$$

$$\bar{a} = -b\omega^2 \hat{e}_R + 2c\hat{e}_z$$

$$|\bar{a}| = (b^2 \omega^4 + 4c^2)^{\frac{1}{2}}$$

$$\begin{aligned}
1.21 \quad \vec{r}(t) &= \hat{i}(1 - e^{-kt}) + \hat{j}e^{kt} \\
\vec{r}(t) &= \hat{i}ke^{-kt} + \hat{j}ke^{kt} \\
\vec{r}(t) &= -\hat{i}k^2 e^{-kt} + \hat{j}k^2 e^{kt}
\end{aligned}$$



$$\begin{aligned}
1.22 \quad \vec{v} &= \hat{e}_r \dot{r} + \hat{e}_\phi r \dot{\phi} \sin \theta + \hat{e}_\theta r \dot{\theta} \\
\vec{v} &= \hat{e}_\phi b \omega \sin \left\{ \frac{\pi}{2} \left[1 + \frac{1}{4} \cos(4\omega t) \right] \right\} - \hat{e}_\theta b \frac{\pi}{2} \omega \sin(4\omega t) \\
\vec{v} &= \hat{e}_\phi b \omega \cos \left[\frac{\pi}{8} \cos(4\omega t) \right] - \hat{e}_\theta b \omega \frac{\pi}{2} \sin(4\omega t) \\
|\vec{v}| &= b \omega \left[\cos^2 \left(\frac{\pi}{8} \cos 4\omega t \right) + \frac{\pi^2}{4} \sin^2 4\omega t \right]^{\frac{1}{2}}
\end{aligned}$$

Path is sinusoidal oscillation about the equator.

$$1.23 \quad \vec{v} \cdot \vec{v} = v^2$$

$$\frac{d\vec{v}}{dt} \cdot \vec{v} + \vec{v} \cdot \frac{d\vec{v}}{dt} = 2v\dot{v}$$

$$2\vec{v} \cdot \vec{a} = 2v\dot{v}$$

$$\vec{v} \cdot \vec{a} = v\dot{v}$$

$$\begin{aligned}
1.24 \quad \frac{d}{dt} [\vec{r} \cdot (\vec{v} \times \vec{a})] &= \frac{d\vec{r}}{dt} \cdot (\vec{v} \times \vec{a}) + \vec{r} \cdot \frac{d}{dt} (\vec{v} \times \vec{a}) \\
&= \vec{v} \cdot (\vec{v} \times \vec{a}) + \vec{r} \cdot \left[\left(\frac{d\vec{v}}{dt} \times \vec{a} \right) + \left(\vec{v} \times \frac{d\vec{a}}{dt} \right) \right] \\
&= 0 + \vec{r} \cdot [0 + (\vec{v} \times \dot{\vec{a}})] \\
\frac{d}{dt} [\vec{r} \cdot (\vec{v} \times \vec{a})] &= \vec{r} \cdot (\vec{v} \times \dot{\vec{a}})
\end{aligned}$$

$$1.25 \quad \vec{v} = v\hat{r} \text{ and } \vec{a} = a_r\hat{r} + a_n\hat{n}$$

$$\vec{v} \cdot \vec{a} = va_r, \text{ so } a_r = \frac{\vec{v} \cdot \vec{a}}{v}$$

$$a^2 = a_r^2 + a_n^2, \text{ so } a_n = (a^2 - a_r^2)^{\frac{1}{2}}$$

$$1.26 \quad \text{For 1.14, } a_r = \frac{-b^2\omega^3 \cos \omega t \cdot \sin \omega t + b^2\omega^3 \sin \omega t \cdot \cos \omega t + 4c^2t}{(b^2\omega^2 \cos^2 \omega t + b^2\omega^2 \sin^2 \omega t + 4c^2t^2)^{\frac{1}{2}}}$$

$$a_r = \frac{4c^2t}{(b^2\omega^2 + 4c^2t^2)^{\frac{1}{2}}}$$

$$a_n = \left(b^2\omega^2 + 4c^2 - \frac{16c^4t^2}{b^2\omega^2 + 4c^2t^2} \right)^{\frac{1}{2}}$$

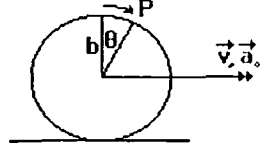
$$\text{For 1.15, } a_r = \frac{b^2k(k^2 - c^2)e^{2kt} + 2b^2c^2ke^{2kt}}{be^{kt}(k^2 + c^2)^{\frac{1}{2}}} = bke^{kt}(k^2 + c^2)^{\frac{1}{2}}$$

$$a_n = \left[b^2e^{2kt}(k^2 + c^2)^2 - b^2k^2e^{2kt}(k^2 + c^2) \right]^{\frac{1}{2}} = bce^{kt}(k^2 + c^2)^{\frac{1}{2}}$$

$$1.27 \quad \vec{v} = v\hat{r}, \quad \vec{a} = \dot{v}\hat{r} + \frac{v^2}{\rho}\hat{n}$$

$$|\vec{v} \times \vec{a}| = v \cdot a_n = v \frac{v^2}{\rho} = \frac{v^3}{\rho}$$

1.28



$$\vec{r}_{p} = \dot{b} \sin \theta + \hat{j} b \cos \theta$$

$$\vec{v}_{rel} = \dot{b} \dot{\theta} \cos \theta - \hat{j} b \dot{\theta} \sin \theta$$

$$\vec{a}_{rel} = \dot{b} (\ddot{\theta} \cos \theta - \dot{\theta}^2 \sin \theta) - \hat{j} b (\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta)$$

at the point $\theta = \frac{\pi}{2}$, $\vec{v}_{rel} = -\vec{v}$

So, $|\vec{v}_{rel}| = b \dot{\theta} = v$

$$\dot{\theta} = \frac{v}{b} \quad \ddot{\theta} = \frac{\dot{v}}{b} = \frac{a_o}{b}$$

Now, $\vec{a}_{rel} = \dot{v}_{rel} \hat{t} + \frac{v_{rel}^2}{\rho} \hat{n} = a_o \hat{t} + \frac{v^2}{b} \hat{n}$

$$|\vec{a}_{rel}| = \left(a_o^2 + \frac{v^4}{b^2} \right)^{\frac{1}{2}}$$

$\vec{v}_p = \vec{v} + \vec{v}_{rel}$ and $\vec{a}_p = \vec{a}_o + \vec{a}_{rel}$

$$\vec{a}_p = \hat{i} \left[a_o + b \left(\frac{a_o}{b} \cos \theta - \frac{v^2}{b^2} \sin \theta \right) \right] - \hat{j} b \left(\frac{a_o}{b} \sin \theta + \frac{v^2}{b^2} \cos \theta \right)$$

$$|\vec{a}_p| = a_o \left(2 + 2 \cos \theta + \frac{v^4}{a_o^2 b^2} - \frac{2v^2}{a_o b} \sin \theta \right)^{\frac{1}{2}}$$

$\vec{a}_p$ is a maximum at $\theta = 0$, i.e., at the top of the wheel.

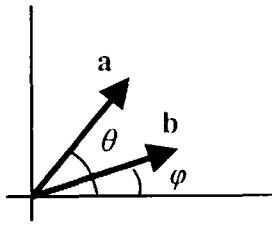
$$-2 \sin \theta - \frac{2v^2}{a_o b} \cos \theta = 0$$

$$\theta = \tan^{-1} \left(-\frac{v^2}{a_o b} \right)$$

1.29 $\tilde{R}R = \begin{pmatrix} x & -x & 0 \\ x & x & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x & x & 0 \\ -x & x & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 2x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Therefore, $x = \frac{1}{\sqrt{2}}$

The transformation represents a rotation of 45° about the z-axis (see Example 1.8.2)

1.30



$$(a) \quad a = \hat{i} \cos \theta + \hat{j} \sin \theta$$

$$b = \hat{i} \cos \varphi + \hat{j} \sin \varphi$$

$$a \cdot b = \cos(\theta - \varphi) = (\hat{i} \cos \theta + \hat{j} \sin \theta) \cdot (\hat{i} \cos \varphi + \hat{j} \sin \varphi)$$

$$\cos(\theta - \varphi) = \cos \theta \cos \varphi + \sin \theta \sin \varphi$$

$$(b) \quad b \times a = |\hat{k}| \sin(\theta - \varphi) = \left| (\hat{i} \cos \theta + \hat{j} \sin \theta) \times (\hat{i} \cos \varphi + \hat{j} \sin \varphi) \right|$$

$$\sin(\theta - \varphi) = \sin \theta \cos \varphi - \cos \theta \sin \varphi$$

CHAPTER 2

NEWTONIAN MECHANICS: RECTILINEAR MOTION OF A PARTICLE

2.1 (a) $\ddot{x} = \frac{1}{m}(F_0 + ct)$

$$\dot{x} = \int_0^t \frac{1}{m}(F_0 + ct) dt = \frac{F_0}{m}t + \frac{c}{2m}t^2$$

$$x = \int_0^t \left(\frac{F_0}{m}t + \frac{c}{2m}t^2 \right) dt = \frac{F_0}{m}t^2 + \frac{c}{6m}t^3$$

(b) $\ddot{x} = \frac{F_0}{m} \sin ct$

$$\dot{x} = \int_0^t \frac{F_0}{m} \sin ct dt = -\frac{F_0}{cm} \cos ct \Big|_0^t = \frac{F_0}{cm}(1 - \cos ct)$$

$$x = \int_0^t \frac{F_0}{cm}(1 - \cos ct) dt = \frac{F_0}{cm} \left(t - \frac{1}{c} \sin ct \right)$$

(c) $\ddot{x} = \frac{F_0}{m} e^{ct}$

$$\dot{x} = \frac{F_0}{cm} e^{ct} \Big|_0^t = \frac{F_0}{cm}(e^{ct} - 1)$$

$$x = \frac{F_0}{cm} \left(\frac{1}{c} e^{ct} - \frac{1}{c} - t \right) = \frac{F_0}{c^2 m} (e^{ct} - 1 - ct)$$

2.2 (a) $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \cdot \frac{dx}{dt} = \dot{x} \frac{d\dot{x}}{dx}$

$$\dot{x} \frac{d\dot{x}}{dx} = \frac{1}{m}(F_0 + cx)$$

$$\dot{x} d\dot{x} = \frac{1}{m}(F_0 + cx) dx$$

$$\frac{1}{2} \dot{x}^2 = \frac{1}{m} \left(F_0 x + \frac{cx^2}{2} \right)$$

$$\dot{x} = \left[\frac{x}{m} (2F_0 + cx) \right]^{\frac{1}{2}}$$

(b) $\ddot{x} = \dot{x} \frac{d\dot{x}}{dx} = \frac{1}{m} F_0 e^{-cx}$

$$\dot{x}d\dot{x} = \frac{1}{m} F_0 e^{-cx} dx$$

$$\frac{1}{2} \dot{x}^2 = -\frac{F_0}{cm} (e^{-cx} - 1) = \frac{F_0}{cm} (1 - e^{-cx})$$

$$\dot{x} = \left[\frac{2F_0}{cm} (1 - e^{-cx}) \right]^{\frac{1}{2}}$$

$$(c) \quad \ddot{x} = \dot{x} \frac{d\dot{x}}{dx} = \frac{1}{m} (F_0 \cos cx)$$

$$\dot{x}d\dot{x} = \frac{F_0}{m} \cos cx dx$$

$$\frac{1}{2} \dot{x}^2 = \frac{F_0}{cm} \sin cx$$

$$\dot{x} = \left(\frac{2F_0}{cm} \sin cx \right)^{\frac{1}{2}}$$

$$2.3 \quad (a) \quad V(x) = -\int_x^x (F_0 + cx) dx = -F_0 x - \frac{cx^2}{2} + C$$

$$(b) \quad V(x) = -\int_x^x F_0 e^{-cx} dx = \frac{F_0}{c} e^{-cx} + C$$

$$(c) \quad V(x) = -\int_x^x F_0 \cos cx dx = -\frac{F_0}{c} \sin cx + C$$

$$2.4 \quad (a) \quad F(x) = -\frac{dV(x)}{dx} = -kx$$

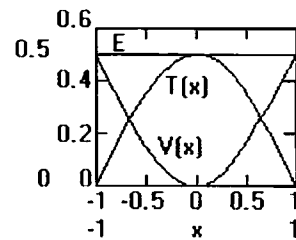
$$V(x) = \int_0^x kx dx = \frac{1}{2} kx^2$$

$$(b) \quad T_0 = T(x) + V(x)$$

$$T(x) = T_0 - V(x) = \frac{1}{2} k(A^2 - x^2)$$

$$(c) \quad E = T_0 = \frac{1}{2} kA^2$$

$$(d) \quad \text{turning points @ } T(x_1) \rightarrow 0 \quad \therefore x_1 = \pm A$$



$$2.5 \quad (a) \quad F(x) = -kx + \frac{kx^3}{A^2} \quad \text{so} \quad V(x) = \int_0^x \left(kx - \frac{kx^3}{A^2} \right) dx = \frac{1}{2} kx^2 - \frac{1}{4} \frac{kx^4}{A^2}$$

$$(b) \quad T(x) = T_0 - V(x) = T_0 - \frac{1}{2} kx^2 + \frac{1}{4} \frac{kx^4}{A^2}$$

$$(c) \quad E = T_0$$

(d) $V(x)$ has maximum at $|F(x_m)| \rightarrow 0$

$$kx_m - \frac{kx_m^3}{A^2} = 0 \quad x_m = \pm A$$

$$V(x_m) = \frac{1}{2}kA^2 - \frac{1}{4}\frac{kA^4}{A^2} = \frac{1}{4}kA^2$$

If $E < V(x_m)$ turning points exist.

Turning points @ $T(x_1) \rightarrow 0$ let $u = x_1^2$

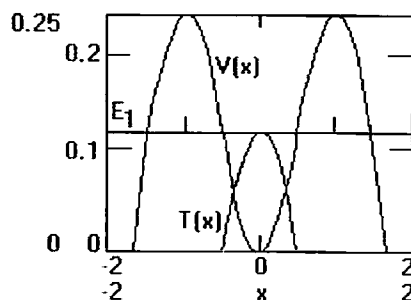
$$E - \frac{1}{2}ku + \frac{1}{4}\frac{ku^2}{A^2} = 0$$

solving for u , we obtain

$$u = A^2 \left[1 \pm \left(1 - \frac{4E}{kA^2} \right)^{\frac{1}{2}} \right]$$

or

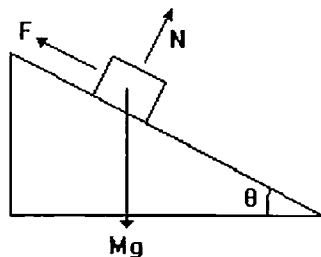
$$x_1 = \pm A \left[1 - \sqrt{\left(1 - \frac{4E}{kA^2} \right)} \right]^{\frac{1}{2}}$$



$$2.6 \quad \dot{x} = v(x) = \frac{\alpha}{x} \quad \ddot{x} = -\frac{\alpha}{x^2} \dot{x} = -\frac{\alpha^2}{x^3}$$

$$F(x) = m\ddot{x} = -\frac{m\alpha^2}{x^3}$$

2.7



$$F \geq Mg \sin \theta$$

$$2.8 \quad F = m\ddot{x} = m\dot{x} \frac{d\dot{x}}{dx}$$

$$\dot{x} = bx^{-3}$$

$$\frac{d\dot{x}}{dx} = -3bx^{-4}$$

$$F = m(bx^{-3})(-3bx^{-4})$$

$$F = -3mb^2x^{-7}$$

$$2.9 \quad (a) \quad V = mgx = (.145 \text{ kg}) \left(9.8 \frac{\text{m}}{\text{s}^2} \right) (1250 \text{ ft}) \left(.3048 \frac{\text{m}}{\text{ft}} \right) = 541 \text{ J}$$

$$(b) \quad T = \frac{1}{2}mv^2 = \frac{1}{2}mv_t^2 = \frac{1}{2}m\left(\frac{mg}{c_2}\right) = \frac{1}{2}\frac{m^2g}{.22D^2}$$

$$T = \frac{(.145kg)^2\left(9.8\frac{m}{s^2}\right)}{(2)(.22)\left[(2)(.0366)\right]^2\frac{kg}{m}} = 87J$$

$$\begin{aligned}\int Fdx &= \int -cv^2dx = -c \int v^3dt = -c \int \left(-v_t \tanh\left(\frac{t}{\tau}\right)\right)^3 dt \\ &= cv_t^3\tau \left[-\frac{1}{2}\tanh^2\left(\frac{t}{\tau}\right) + \int \tanh\left(\frac{t}{\tau}\right)d\left(\frac{t}{\tau}\right)\right] \\ &= cv_t^3\tau \left[-\frac{1}{2}\tanh^2\left(\frac{t}{\tau}\right) + \ln \cosh\left(\frac{t}{\tau}\right)\right]\end{aligned}$$

$$\text{Now } \tanh^2\left(\frac{t}{\tau}\right) \cong 1 \text{ for } t \ll \tau$$

$$\text{Meanwhile } x = \int vdt = \int \left(-v_t \tanh\left(\frac{t}{\tau}\right)\right)dt = v_t\tau \ln \cosh\left(\frac{t}{\tau}\right)$$

$$\ln \cosh\left(\frac{t}{\tau}\right) = \frac{x}{v_t\tau}$$

$$x = (1250 ft)\left(.3048\frac{m}{ft}\right) = 381m$$

$$v_t = \left(\frac{mg}{c_2}\right)^{\frac{1}{2}} = \left[\frac{(.145kg)\left(9.8\frac{m}{s^2}\right)}{(.22)(.0732)^2\frac{kg}{m}}\right]^{\frac{1}{2}} = 34.72\frac{m}{s}$$

$$\tau = \left(\frac{m}{c_2g}\right)^{\frac{1}{2}} = \left[\frac{(.145kg)}{(.22)(.0732)^2\frac{kg}{m}\left(9.8\frac{m}{s^2}\right)}\right]^{\frac{1}{2}} = 3.543s$$

$$\int Fdx = (.22)(.0732)^2(34.72)^3(3.543)\left[-.5 + \frac{3.81}{(34.72)(3.54)}\right] = 454J$$

$$V - T = 541J - 87J = 454J$$

2.10 For $0 \leq t \leq t_1$: $v = \frac{F_o}{m}t$, $x = \frac{1}{2}\frac{F_o}{m}t^2$

For $t_1 \leq t \leq 2t_1$: $v_o = \frac{F_o}{m}t_1$, $x_o = \frac{F_o}{2m}t_1^2$, $t_o = t_1$

$$x = \frac{F_o}{2m} t_1^2 + \frac{F_o}{m} t_1 (t - t_1) + \frac{1}{2} \frac{2F_o}{m} (t - t_1)^2$$

$$\text{At } t = 2t_1: x = \frac{F_o}{2m} t_1^2 + \frac{F_o}{m} t_1^2 + \frac{F_o}{m} t_1^2 = \frac{5F_o}{2m} t_1^2$$

$$2.11 \quad a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx} = -\frac{c}{m} v^{\frac{3}{2}}$$

$$v^{-\frac{1}{2}} dv = -\frac{c}{m} dx$$

$$\int_v^0 v^{-\frac{1}{2}} dv = \int_0^{x_{\max}} -\frac{c}{m} dx$$

$$-2v_o^{\frac{1}{2}} = -\frac{c}{m} x_{\max}$$

$$x_{\max} = \frac{2mv_o^{\frac{1}{2}}}{c}$$

$$2.12 \quad \text{Going up: } F_x = -mg \sin 30^\circ - \mu mg \cos 30^\circ$$

$$\ddot{x} = -g(\sin 30^\circ + 0.1 \cos 30^\circ) = -5.749 \frac{m}{s^2}$$

$$v = v_o + at$$

$$\text{at the highest point } v = 0 \text{ so } t_{up} = -\frac{v_o}{a} = 0.174 v_o s$$

$$x_{up} = v_o t_{up} + \frac{1}{2} a t_{up}^2 = 0.174 v_o^2 - 0.087 v_o^2 = 0.087 v_o^2 m$$

$$\text{Going down: } x_o' = 0.087 v_o^2, \quad v_o' = 0, \quad a' = -9.8(0.5 - 0.0866)$$

$$x_{down} = 0 = 0.087 v_o^2 - \frac{1}{2} 4.0513 t_{down}^2$$

$$t_{down} = 0.207 v_o s$$

$$t_{total} = t_{up} + t_{down} = 0.381 v_o s$$

$$2.13 \quad \text{At the top } v = 0 \text{ so } e^{-2kx_{\max}} = \frac{\frac{g}{k}}{\frac{g}{k} + v_o^2}$$

$$\text{Coming down } x_o = x_{\max} \text{ and at the bottom } x = 0$$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k} \right)^2 \frac{1}{\left(\frac{g}{k} + v_o^2 \right)} (1) = \frac{\left(\frac{g}{k} \right) v_o^2}{\frac{g}{k} + v_o^2}$$

$$v = \frac{v_t v_o}{(v_t^2 + v_o^2)^{\frac{1}{2}}}, \quad v_t = \sqrt{\frac{g}{k}} = \sqrt{\frac{mg}{c_2}}$$

2.14 Going up: $F_x = -mg - c_2 v^2$

$$a = v \frac{dv}{dx} = -g - kv^2, \quad k = \frac{c_2}{m}$$

$$\int_x^v \frac{v dv}{-g - kv^2} = \int_0^x dx$$

$$-\frac{1}{2k} \ln(-g - kv^2) \Big|_v^v = x$$

$$\frac{g + kv^2}{g + kv_o^2} = e^{-2kx}$$

$$v^2 = \left(\frac{g}{k} + v_o^2 \right) e^{-2kx} - \frac{g}{k}$$

Going down: $F_x = -mg + c_2 v^2$

$$v \frac{dv}{dx} = -g + kv^2$$

$$\int_0^v \frac{v dv}{-g + kv^2} = \int_0^x dx$$

$$\frac{1}{2k} \ln(-g + kv^2) \Big|_0^v = x - x_o$$

$$1 - \frac{k}{g} v^2 = e^{2kx} e^{-2kx_o}$$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k} e^{-2kx_o} \right) e^{2kx}$$

2.15 $m \frac{dv}{dt} = mg - c_1 v - c_2 v^2$

$$\int_0^t \frac{dt}{m} = \int_0^v \frac{dv}{mg - c_1 v - c_2 v^2}$$

Using $\int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{b^2 - 4ac}} \ln \frac{2cx + b + \sqrt{b^2 - 4ac}}{2cx + b - \sqrt{b^2 - 4ac}},$

$$\frac{t}{m} = \frac{1}{\sqrt{c_1^2 + 4mgc_2}} \ln \left. \frac{-2c_2v - c_1 - \sqrt{c_1^2 + 4mgc_2}}{-2c_2v - c_1 + \sqrt{c_1^2 + 4mgc_2}} \right|_0^v$$

$$\frac{t}{m} (c_1^2 + 4mgc_2)^{\frac{1}{2}} = \ln \frac{(2c_2v + c_1 + \sqrt{c_1^2 + 4mgc_2})(c_1 - \sqrt{c_1^2 + 4mgc_2})}{(2c_2v + c_1 - \sqrt{c_1^2 + 4mgc_2})(c_1 + \sqrt{c_1^2 + 4mgc_2})}$$

as $t \rightarrow \infty$, $2c_2v_t + c_1 - \sqrt{c_1^2 + 4mgc_2} = 0$

$$v_t = -\frac{c_1}{2c_2} + \left[\left(\frac{c_1}{2c_2} \right)^2 + \frac{mg}{c_2} \right]^{\frac{1}{2}}$$

Alternatively, when $v = v_t$,

$$m \frac{dv}{dt} = 0 = mg - c_1v_t - c_2v_t^2$$

$$v_t = -\frac{c_1}{2c_2} + \left[\left(\frac{c_1}{2c_2} \right)^2 + \frac{mg}{c_2} \right]^{\frac{1}{2}}$$

2.16 $a = v \frac{dv}{dx} = -\frac{k}{m} x^{-2}$

$$\int_0^x v dv = \int_b^x -\frac{k dx}{mx^2}$$

$$\frac{1}{2} v^2 = \frac{k}{m} \left(\frac{1}{x} - \frac{1}{b} \right)$$

$$v = \frac{dx}{dt} = \left[\frac{2k}{m} \left(\frac{1}{x} - \frac{1}{b} \right) \right]^{\frac{1}{2}} = \left[\frac{2k}{mb} \left(\frac{b-x}{x} \right) \right]^{\frac{1}{2}}$$

$$\int_0^t dt = \int_b^0 \left[\frac{mb}{2k} \left(\frac{x}{b-x} \right) \right]^{\frac{1}{2}} dx = \left(\frac{mb^3}{2k} \right)^{\frac{1}{2}} \int_1^0 \left(\frac{\frac{x}{b}}{1 - \frac{x}{b}} \right)^{\frac{1}{2}} d\left(\frac{x}{b} \right)$$

Since $x \leq b$, say $\frac{x}{b} = \sin^2 \theta$

$$t = \left(\frac{mb^3}{2k} \right)^{\frac{1}{2}} \int_{\frac{\pi}{2}}^0 \frac{\sin \theta (2 \sin \theta \cos \theta d\theta)}{\cos \theta} = \left(\frac{2mb^3}{k} \right)^{\frac{1}{2}} \int_{\frac{\pi}{2}}^0 \sin^2 \theta d\theta$$

$$t = \left(\frac{mb^3}{8k} \right)^{\frac{1}{2}} \pi$$

$$2.17 \quad m \frac{dv}{dt} = mv \frac{dv}{dx} = f(x) \cdot g(v)$$

$$\frac{mv dv}{g(v)} = f(x) dx$$

By integration, get $v = v(x) = \frac{dx}{dt}$

If $F(x, t) = f(x) \cdot g(t)$:

$$m \frac{d^2 x}{dt^2} = m \frac{d}{dt} \left(\frac{dx}{dt} \right) = f(x) \cdot g(t)$$

This cannot, in general, be solved by integration.

If $F(v, t) = f(v) \cdot g(t)$:

$$m \frac{dv}{dt} = f(v) \cdot g(t)$$

$$\frac{mdv}{f(v)} = g(t) dt$$

Integration gives $v = v(t)$

$$\frac{dx}{dt} = v(t)$$

$$dx = v(t) dt$$

A second integration gives $x = x(t)$

2.18

$$c_1 = (1.55 \times 10^{-4})(10^{-2}) = 1.55 \times 10^{-6} \frac{kg}{s}$$

$$c_2 = (0.22)(10^{-2})^2 = 2.2 \times 10^{-5} \frac{kg}{s}$$

$$v_t = -\frac{1.55 \times 10^{-6}}{2 \times 2.2 \times 10^{-5}} + \left[\left(\frac{1.55 \times 10^{-6}}{2 \times 2.2 \times 10^{-5}} \right)^2 + \frac{(10^{-7})(9.8)}{2.2 \times 10^{-5}} \right]^{\frac{1}{2}}$$

$$v_t = 0.179 \frac{m}{s}$$

$$\text{Using equation 2.29, } v_t = \sqrt{\frac{(10^{-7})(9.8)}{2.2 \times 10^{-5}}} = 0.211 \frac{m}{s}$$

2.19

$$F(x) = -Ae^{\alpha x} = m\ddot{x} \quad \text{or} \quad F(v) = -Ae^{\alpha v} = m\dot{v} \quad \frac{dv}{e^{\alpha v}} = -\frac{A}{m} dt$$

$$\text{Let } u = e^{\alpha v} \quad du = \alpha e^{\alpha v} dv \quad dv = \frac{du}{\alpha e^{\alpha v}} = \frac{du}{\alpha u} \quad \therefore \frac{du}{u^2} = -\frac{\alpha A}{m} dt$$

Integrating

$$\frac{1}{u} - \frac{1}{u_s} = \frac{A}{m} \alpha t \quad \text{and substituting } e^{\alpha v} = u$$

$$(a) \quad v = v_s - \frac{1}{\alpha} \ln \left[1 + \frac{A}{m} e^{\alpha v} \alpha t \right]$$

$$(b) \quad t = T @ v = 0$$

$$\alpha v_s = \ln \left[1 + \frac{A}{m} e^{\alpha v} \alpha T \right]$$

$$e^{\alpha v} = 1 + \frac{A}{m} e^{\alpha v} \alpha T \quad T = \frac{m}{\alpha A} [1 - e^{-\alpha v}]$$

$$(c) \quad v \frac{dv}{dx} = v = -\frac{A}{m} e^{\alpha v} \quad \frac{v dv}{e^{\alpha v}} = -\frac{A}{m} dx$$

$$\text{again, let } u = e^{\alpha v} \quad du = \alpha u dv \quad \text{or} \quad dv = \frac{du}{\alpha u} \quad v = \frac{1}{\alpha} \ln u$$

$$\frac{\left[\frac{1}{\alpha} \ln u \right] \frac{du}{\alpha u}}{u} = -\frac{A}{m} dx \quad \text{Integrating and solving}$$

$$x = \frac{m}{\alpha^2 A} [1 - (1 + \alpha v_s) e^{-\alpha v}]$$

2.20

$$F = \frac{d(mv)}{dt} = mv + vm = mg$$

$$\text{but } m = \rho_s \frac{4}{3} \pi r^3 \quad m = \rho_1 \pi r^2 v$$

$$\text{so (1) } \frac{4}{3} \pi \rho_s r^3 v + \pi \rho_1 r^2 v^2 = \frac{4}{3} \pi^2 = \frac{4}{3} \pi \rho_s r^3 g$$

$$\text{Now } \frac{\rho_1}{\rho_s} \approx 10^{-3} \quad \text{so, second term is negligible-small}$$

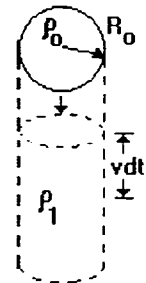
$$\text{hence } v \approx g \quad \text{and } \boxed{v \approx gt} \quad \text{speed} \propto t \quad \text{but}$$

$$\dot{m} = \rho_s 4 \pi r^2 \dot{r} = \rho_1 \pi r^2 v \quad \text{or} \quad \dot{r} \cong \frac{1}{4} \frac{\rho_1}{\rho_s} v \quad \text{Hence } \boxed{r \approx \frac{1}{4} \frac{\rho_1}{\rho_s} gt} \quad \text{and rate of}$$

growth $\propto t$

The exact differential equation from (1) above is:

$$\frac{4}{3} \pi \rho_s r \left| \frac{4 \rho_s}{\rho_1} \dot{r} \right| + \pi \rho_1 \left| \frac{4 \rho_s \dot{r}}{\rho_1} \right|^2 = \frac{4}{3} \pi \rho_s r g$$



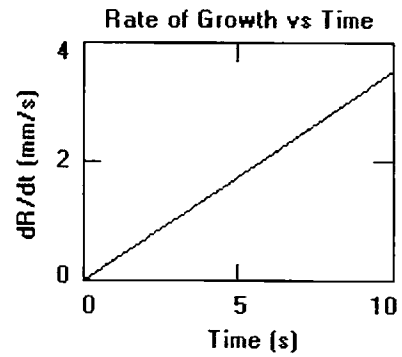
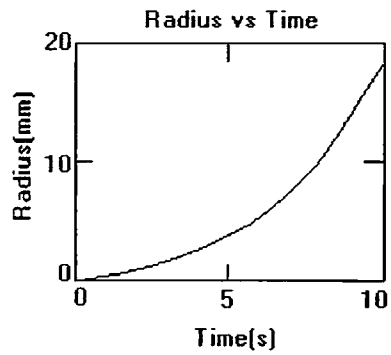
which reduces to: $\ddot{r} + \frac{3\dot{r}^2}{r} = \frac{\rho_l}{4\rho_v} g$

Using Mathcad, solve the above non-linear d.e. letting

$\frac{\rho_l}{\rho_v} \approx 10^{-3}$ and $R_0 \approx 0.01mm$ (small raindrop). Graphs

show that

$$v \propto \dot{r} \propto t \text{ and } r \propto t^2$$



CHAPTER 3 OSCILLATIONS

- 3.1** $x = 0.002 \sin \left[2\pi (512 \text{ s}^{-1}) t \right] [m]$
 $\dot{x}_{\max} = (0.002)(2\pi)(512) \left[\frac{m}{s} \right] = 6.43 \left[\frac{m}{s} \right]$
 $\ddot{x}_{\max} = (0.002)(2\pi)^2 (512)^2 \left[\frac{m}{s^2} \right] = 2.07 \times 10^4 \left[\frac{m}{s^2} \right]$
- 3.2** $x = 0.1 \sin \omega_o t [m]$ $\dot{x} = 0.1 \omega_o \cos \omega_o t \left[\frac{m}{s} \right]$
 When $t = 0$, $x = 0$ and $\dot{x} = 0.5 \left[\frac{m}{s} \right] = 0.1 \omega_o$
 $\omega_o = 5 \text{ s}^{-1}$ $T = \frac{2\pi}{\omega_o} = 1.26 \text{ s}$
- 3.3** $x(t) = x_o \cos \omega_o t + \frac{\dot{x}_o}{\omega_o} \sin \omega_o t$ and $\omega_o = 2\pi f$
 $x = 0.25 \cos(20\pi t) + 0.00159 \sin(20\pi t) [m]$
- 3.4** $\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$
 $x = A \cos(\omega_o t - \phi) = A \cos \phi \cos \omega_o t + A \sin \phi \sin \omega_o t$
 $x = A \cos \omega_o t + B \sin \omega_o t$, $A = A \cos \phi$, $B = A \sin \phi$
- 3.5** $\frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} k x_1^2 = \frac{1}{2} m \dot{x}_2^2 + \frac{1}{2} k x_2^2$
 $k(x_1^2 - x_2^2) = m(\dot{x}_2^2 - \dot{x}_1^2)$
 $\omega_o = \sqrt{\frac{k}{m}} = \left(\frac{\dot{x}_2^2 - \dot{x}_1^2}{x_1^2 - x_2^2} \right)^{\frac{1}{2}}$
 $\frac{1}{2} k A^2 = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} k x_1^2$
 $A^2 = \frac{m}{k} \dot{x}_1^2 + x_1^2 = \frac{x_1^2 \dot{x}_1^2 - x_2^2 \dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2} + x_1^2$
 $A = \left(\frac{x_1^2 \dot{x}_2^2 - x_2^2 \dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2} \right)^{\frac{1}{2}}$

$$3.6 \quad \frac{1}{2}T_0 = \pi \sqrt{\frac{l}{g}} = \pi \sqrt{\frac{1}{\frac{9.8}{6}}} s \approx 2.5 s$$

3.7 For springs tied in parallel:

$$F_s(x) = -k_1 x - k_2 x = -(k_1 + k_2)x$$

$$\omega = \left[\frac{(k_1 + k_2)}{m} \right]^{\frac{1}{2}}$$

For springs tied in series:

The upward force m is $k_{eq} x$.

Therefore, the downward force on spring k_2 is $k_{eq} x$.

The upward force on the spring k_2 is $k_1 x'$ where x' is the displacement of P, the point at which the springs are tied.

Since the spring k_2 is in equilibrium, $k_1 x' = k_{eq} x$.

Meanwhile,

The upward force at P is $k_1 x'$.

The downward force at P is $k_2 (x - x')$.

Therefore, $k_1 x' = k_2 (x - x')$

$$x' = \frac{k_2 x}{k_1 + k_2}$$

$$\text{And } k_{eq} x = k_1 \left(\frac{k_2 x}{k_1 + k_2} \right)$$

$$\omega = \sqrt{\frac{k_{eq}}{m}} = \left[\frac{k_1 k_2}{(k_1 + k_2) m} \right]^{\frac{1}{2}}$$

3.8 For the system $(M + m)$, $-kX = (M + m)\ddot{X}$

The position and acceleration of m are the same as for $(M + m)$:

$$\ddot{x}_m = -\frac{k}{M + m} x_m$$

$$x_m = A \cos \left(\sqrt{\frac{k}{M + m}} t + \delta \right) = d \cos \sqrt{\frac{k}{M + m}} t$$

The total force on m . $F_m = m\ddot{x}_m = mg - F_r$

$$F_r = mg + \frac{mk}{M+m} x_m = mg + \frac{mkd}{M+m} \cos \sqrt{\frac{k}{M+m}} t$$

For the block to just begin to leave the bottom of the box at the top of the vertical oscillations. $F_r = 0$ at $x_m = -d$:

$$0 = mg - \frac{mkd}{M+m}$$

$$d = \frac{g(M+m)}{k}$$

3.9 $x = e^{-\gamma t} A \cos(\omega_d t - \phi)$

$$\frac{dx}{dt} = -e^{-\gamma t} A \omega_d \sin(\omega_d t - \phi) - \gamma e^{-\gamma t} A \cos(\omega_d t - \phi)$$

$$\text{maxima at } \frac{dx}{dt} = 0 = \omega_d \sin(\omega_d t - \phi) + \gamma \cos(\omega_d t - \phi)$$

$$\tan(\omega_d t - \phi) = -\frac{\gamma}{\omega_d}$$

thus the condition of relative maximum occurs every time that t increases by $\frac{2\pi}{\omega_d}$:

$$t_{i+1} = t_i + \frac{2\pi}{\omega_d}$$

$$\text{For the } i \text{ th maximum: } x_i = e^{-\gamma t_i} A \cos(\omega_d t_i - \phi)$$

$$x_{i+1} = e^{-\gamma t_{i+1}} A \cos(\omega_d t_{i+1} - \phi) = e^{-\gamma \frac{2\pi}{\omega_d}} x_i$$

$$\frac{x_i}{x_{i+1}} = e^{-\gamma \frac{2\pi}{\omega_d}} = e^{\gamma T_d}$$

3.10 (a) $\gamma = \frac{c}{2m} = 3 \text{ s}^{-1}$ $\omega_o^2 = \frac{k}{m} = 25 \text{ s}^{-2}$

$$\omega_d^2 = \omega_o^2 - \gamma^2 = 16 \text{ s}^{-2}$$

$$\omega_r^2 = \omega_d^2 - \gamma^2 = 7 \text{ s}^{-2}$$

$$\therefore \omega_r = \sqrt{7} \text{ s}^{-1}$$

(b) $A_{\max} = \frac{F_o}{C\omega_d} = \frac{48}{60.4} m = 0.2 \text{ m}$

(c) $\tan \phi = \frac{2\gamma\omega_r}{(\omega_o^2 - \omega_r^2)} = \frac{2\gamma\omega_r}{2\gamma^2} = \frac{\omega_r}{\gamma} = \frac{\sqrt{7}}{3}$ $\therefore \phi \approx 41.4^\circ$