

92 Chapter 2 Differentiation

$$81. s = (1 + 4t)^{1/2} \Rightarrow v = \frac{ds}{dt} = \frac{1}{2}(1 + 4t)^{-1/2}(4) = 2(1 + 4t)^{-1/2} \Rightarrow v(6) = 2(1 + 4 \cdot 6)^{-1/2} = \frac{2}{5} \text{ m/sec};$$

$$v = 2(1 + 4t)^{-1/2} \Rightarrow a = \frac{dv}{dt} = -\frac{1}{2} \cdot 2(1 + 4t)^{-3/2}(4) = -4(1 + 4t)^{-3/2} \Rightarrow a(6) = -4(1 + 4 \cdot 6)^{-3/2} = -\frac{4}{125} \text{ m/sec}^2$$

$$82. \text{ We need to show } a = \frac{dv}{dt} \text{ is constant: } a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} \text{ and } \frac{dv}{ds} = \frac{d}{ds}(k\sqrt{s}) = \frac{k}{2\sqrt{s}} \Rightarrow a = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v$$

$$= \frac{k}{2\sqrt{s}} \cdot k\sqrt{s} = \frac{k^2}{2} \text{ which is a constant.}$$

$$83. v \text{ proportional to } \frac{1}{\sqrt{s}} \Rightarrow v = \frac{k}{\sqrt{s}} \text{ for some constant } k \Rightarrow \frac{dv}{ds} = -\frac{k}{2s^{3/2}}. \text{ Thus, } a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v$$

$$= -\frac{k}{2s^{3/2}} \cdot \frac{k}{\sqrt{s}} = -\frac{k^2}{2} \left(\frac{1}{s^2}\right) \Rightarrow \text{acceleration is a constant times } \frac{1}{s^2} \text{ so } a \text{ is inversely proportional to } s^2.$$

$$84. \text{ Let } \frac{dx}{dt} = f(x). \text{ Then, } a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot f(x) = \frac{d}{dx} \left(\frac{dx}{dt} \right) \cdot f(x) = \frac{d}{dx} (f(x)) \cdot f(x) = f'(x)f(x), \text{ as required.}$$

85. No. The chain rule says that when g is differentiable at 0 and f is differentiable at $g(0)$, then $f \circ g$ is differentiable at 0. But the chain rule says nothing about what happens when g is not differentiable at 0 so there is no contradiction.

$$86. \text{ The graph of } y = (f \circ g)(x) \text{ has a horizontal tangent at } x = 1 \text{ provided that } (f \circ g)'(1) = 0 \Rightarrow f'(g(1))g'(1) = 0$$

$$\Rightarrow \text{either } f'(g(1)) = 0 \text{ or } g'(1) = 0 \text{ (or both)} \Rightarrow \text{either the graph of } f \text{ has a horizontal tangent at } u = g(1), \text{ or the graph of } g \text{ has a horizontal tangent at } x = 1 \text{ (or both).}$$

$$87. \text{ From the power rule, with } y = x^{1/4}, \text{ we get } \frac{dy}{dx} = \frac{1}{4}x^{-3/4}. \text{ From the chain rule, } y = \sqrt{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{\sqrt{x}}} \cdot \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{\sqrt{x}}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{4}x^{-3/4}, \text{ in agreement.}$$

$$88. \text{ From the power rule, with } y = x^{3/4}, \text{ we get } \frac{dy}{dx} = \frac{3}{4}x^{-1/4}. \text{ From the chain rule, } y = \sqrt{x\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x\sqrt{x}}} \cdot \frac{d}{dx}(x\sqrt{x}) \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x\sqrt{x}}} \cdot \left(x \cdot \frac{1}{2\sqrt{x}} + \sqrt{x}\right) = \frac{1}{2\sqrt{x\sqrt{x}}} \cdot \left(\frac{3}{2}\sqrt{x}\right) = \frac{3\sqrt{x}}{4\sqrt{x\sqrt{x}}}$$

$$= \frac{3\sqrt{x}}{4\sqrt{x}\sqrt{\sqrt{x}}} = \frac{3}{4}x^{-1/4}, \text{ in agreement.}$$

2.8 IMPLICIT DIFFERENTIATION

1. $x^2y + xy^2 = 6$:

$$\text{Step 1: } \left(x^2 \frac{dy}{dx} + y \cdot 2x\right) + \left(x \cdot 2y \frac{dy}{dx} + y^2 \cdot 1\right) = 0$$

$$\text{Step 2: } x^2 \frac{dy}{dx} + 2xy \frac{dy}{dx} = -2xy - y^2$$

$$\text{Step 3: } \frac{dy}{dx}(x^2 + 2xy) = -2xy - y^2$$

$$\text{Step 4: } \frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$$

2. $x^3 + y^3 = 18xy \Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} = 18y + 18x \frac{dy}{dx} \Rightarrow (3y^2 - 18x) \frac{dy}{dx} = 18y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{6y - x^2}{y^2 - 6x}$

3. $2xy + y^2 = x + y$:

$$\text{Step 1: } \left(2x \frac{dy}{dx} + 2y\right) + 2y \frac{dy}{dx} = 1 + \frac{dy}{dx}$$

$$\text{Step 2: } 2x \frac{dy}{dx} + 2y \frac{dy}{dx} - \frac{dy}{dx} = 1 - 2y$$

$$\text{Step 3: } \frac{dy}{dx}(2x + 2y - 1) = 1 - 2y$$

$$\text{Step 4: } \frac{dy}{dx} = \frac{1 - 2y}{2x + 2y - 1}$$

$$4. \quad x^3 - xy + y^3 = 1 \Rightarrow 3x^2 - y - x \frac{dy}{dx} + 3y^2 \frac{dy}{dx} = 0 \Rightarrow (3y^2 - x) \frac{dy}{dx} = y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{y - 3x^2}{3y^2 - x}$$

$$5. \quad x^2(x - y)^2 = x^2 - y^2:$$

$$\text{Step 1:} \quad x^2 \left[2(x - y) \left(1 - \frac{dy}{dx} \right) \right] + (x - y)^2(2x) = 2x - 2y \frac{dy}{dx}$$

$$\text{Step 2:} \quad -2x^2(x - y) \frac{dy}{dx} + 2y \frac{dy}{dx} = 2x - 2x^2(x - y) - 2x(x - y)^2$$

$$\text{Step 3:} \quad \frac{dy}{dx} [-2x^2(x - y) + 2y] = 2x [1 - x(x - y) - (x - y)^2]$$

$$\begin{aligned} \text{Step 4:} \quad \frac{dy}{dx} &= \frac{2x [1 - x(x - y) - (x - y)^2]}{-2x^2(x - y) + 2y} = \frac{x [1 - x(x - y) - (x - y)^2]}{y - x^2(x - y)} = \frac{x (1 - x^2 + xy - x^2 + 2xy - y^2)}{x^2y - x^3 + y} \\ &= \frac{x - 2x^3 + 3x^2y - xy^2}{x^2y - x^3 + y} \end{aligned}$$

$$\begin{aligned} 6. \quad (3xy + 7)^2 = 6y &\Rightarrow 2(3xy + 7) \cdot \left(3x \frac{dy}{dx} + 3y \right) = 6 \frac{dy}{dx} \Rightarrow 2(3xy + 7)(3x) \frac{dy}{dx} - 6 \frac{dy}{dx} = -6y(3xy + 7) \\ &\Rightarrow \frac{dy}{dx} [6x(3xy + 7) - 6] = -6y(3xy + 7) \Rightarrow \frac{dy}{dx} = -\frac{y(3xy + 7)}{x(3xy + 7) - 1} = \frac{3xy^2 + 7y}{1 - 3x^2y - 7x} \end{aligned}$$

$$7. \quad y^2 = \frac{x-1}{x+1} \Rightarrow 2y \frac{dy}{dx} = \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{y(x+1)^2}$$

$$8. \quad x^2 = \frac{x-y}{x+y} \Rightarrow x^3 + x^2y = x - y \Rightarrow 3x^2 + 2xy + x^2y' = 1 - y' \Rightarrow (x^2 + 1)y' = 1 - 3x^2 - 2xy \Rightarrow y' = \frac{1 - 3x^2 - 2xy}{x^2 + 1}$$

$$9. \quad x = \tan y \Rightarrow 1 = (\sec^2 y) \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$$

$$\begin{aligned} 10. \quad xy = \cot(xy) &\Rightarrow x \frac{dy}{dx} + y = -\csc^2(xy) \left(x \frac{dy}{dx} + y \right) \Rightarrow x \frac{dy}{dx} + x \csc^2(xy) \frac{dy}{dx} = -y \csc^2(xy) - y \\ &\Rightarrow \frac{dy}{dx} [x + x \csc^2(xy)] = -y [\csc^2(xy) + 1] \Rightarrow \frac{dy}{dx} = \frac{-y [\csc^2(xy) + 1]}{x [1 + \csc^2(xy)]} = -\frac{y}{x} \end{aligned}$$

$$\begin{aligned} 11. \quad e^{2x} = \sin(x + 3y) &\Rightarrow 2e^{2x} = (1 + 3y') \cos(x + 3y) \Rightarrow 1 + 3y' = \frac{2e^{2x}}{\cos(x + 3y)} \Rightarrow 3y' = \frac{2e^{2x}}{\cos(x + 3y)} - 1 \\ &\Rightarrow y' = \frac{2e^{2x} - \cos(x + 3y)}{3 \cos(x + 3y)} \end{aligned}$$

$$12. \quad x + \sin y = xy \Rightarrow 1 + (\cos y) \frac{dy}{dx} = y + x \frac{dy}{dx} \Rightarrow (\cos y - x) \frac{dy}{dx} = y - 1 \Rightarrow \frac{dy}{dx} = \frac{y-1}{\cos y - x}$$

$$\begin{aligned} 13. \quad y \sin\left(\frac{1}{y}\right) = 1 - xy &\Rightarrow y \left[\cos\left(\frac{1}{y}\right) \cdot \left(-1\right) \frac{1}{y^2} \cdot \frac{dy}{dx} \right] + \sin\left(\frac{1}{y}\right) \cdot \frac{dy}{dx} = -x \frac{dy}{dx} - y \Rightarrow \\ \frac{dy}{dx} \left[-\frac{1}{y} \cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x \right] &= -y \Rightarrow \frac{dy}{dx} = \frac{-y}{-\frac{1}{y} \cos\left(\frac{1}{y}\right) + \sin\left(\frac{1}{y}\right) + x} = \frac{-y^2}{y \sin\left(\frac{1}{y}\right) - \cos\left(\frac{1}{y}\right) + xy} \end{aligned}$$

$$\begin{aligned} 14. \quad e^{x^2y} = 2x + 2y &\Rightarrow e^{x^2y}(x^2y' + 2xy) = 2 + 2y' \Rightarrow x^2e^{x^2y}y' + 2xye^{x^2y} = 2 + 2y' \Rightarrow x^2e^{x^2y}y' - 2y' = 2 - 2xye^{x^2y} \\ &\Rightarrow y' = \frac{2 - 2xye^{x^2y}}{x^2e^{x^2y} - 2} \end{aligned}$$

$$15. \quad \theta^{1/2} + r^{1/2} = 1 \Rightarrow \frac{1}{2} \theta^{-1/2} + \frac{1}{2} r^{-1/2} \cdot \frac{dr}{d\theta} = 0 \Rightarrow \frac{dr}{d\theta} \left[\frac{1}{2\sqrt{r}} \right] = \frac{-1}{2\sqrt{\theta}} \Rightarrow \frac{dr}{d\theta} = -\frac{2\sqrt{r}}{2\sqrt{\theta}} = -\frac{\sqrt{r}}{\sqrt{\theta}}$$

$$16. \quad r - 2\sqrt{\theta} = \frac{3}{2} \theta^{2/3} + \frac{4}{3} \theta^{3/4} \Rightarrow \frac{dr}{d\theta} - \theta^{-1/2} = \theta^{-1/3} + \theta^{-1/4} \Rightarrow \frac{dr}{d\theta} = \theta^{-1/2} + \theta^{-1/3} + \theta^{-1/4}$$

$$\begin{aligned} 17. \quad \sin(r\theta) = \frac{1}{2} &\Rightarrow [\cos(r\theta)] \left(r + \theta \frac{dr}{d\theta} \right) = 0 \Rightarrow \frac{dr}{d\theta} [\theta \cos(r\theta)] = -r \cos(r\theta) \Rightarrow \frac{dr}{d\theta} = \frac{-r \cos(r\theta)}{\theta \cos(r\theta)} = -\frac{r}{\theta}, \\ \cos(r\theta) &\neq 0 \end{aligned}$$

$$18. \cos r + \cot \theta = e^{r\theta} \Rightarrow -\sin r \cdot \frac{dr}{d\theta} - \csc^2 \theta = e^{r\theta} \left(r + \theta \frac{dr}{d\theta} \right) \Rightarrow -\sin r \cdot \frac{dr}{d\theta} - \csc^2 \theta = r e^{r\theta} + \theta e^{r\theta} \frac{dr}{d\theta} \\ \Rightarrow -\sin r \frac{dr}{d\theta} - \theta e^{r\theta} \frac{dr}{d\theta} = r e^{r\theta} + \csc^2 \theta \Rightarrow \frac{dr}{d\theta} = -\frac{r e^{r\theta} + \csc^2 \theta}{\theta e^{r\theta} + \sin r}$$

$$19. x^2 + y^2 = 1 \Rightarrow 2x + 2yy' = 0 \Rightarrow 2yy' = -2x \Rightarrow \frac{dy}{dx} = y' = -\frac{x}{y}; \text{ now to find } \frac{d^2y}{dx^2}, \frac{d}{dx} \left(-\frac{x}{y} \right) \\ \Rightarrow y'' = \frac{y(-1) + xy'}{y^2} = \frac{-y + x \left(-\frac{x}{y} \right)}{y^2} \text{ since } y' = -\frac{x}{y} \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{-y^2 - x^2}{y^3} = \frac{-y^2 - (1 - y^2)}{y^3} = \frac{-1}{y^3}$$

$$20. x^{2/3} + y^{2/3} = 1 \Rightarrow \frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} \left[\frac{2}{3} y^{-1/3} \right] = -\frac{2}{3} x^{-1/3} \Rightarrow y' = \frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\left(\frac{y}{x} \right)^{1/3};$$

$$\text{Differentiating again, } y'' = \frac{x^{1/3} \cdot \left(-\frac{1}{3} y^{-2/3} \right) y' + y^{1/3} \left(\frac{1}{3} x^{-2/3} \right)}{x^{2/3}} = \frac{x^{1/3} \cdot \left(-\frac{1}{3} y^{-2/3} \right) \left(-\frac{y^{1/3}}{x^{1/3}} \right) + y^{1/3} \left(\frac{1}{3} x^{-2/3} \right)}{x^{2/3}} \\ \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{3} x^{-2/3} y^{-1/3} + \frac{1}{3} y^{1/3} x^{-4/3} = \frac{y^{1/3}}{3x^{4/3}} + \frac{1}{3y^{1/3}x^{2/3}}$$

$$21. y^2 = e^{x^2} + 2x \Rightarrow 2yy' = 2x e^{x^2} + 2 \Rightarrow \frac{dy}{dx} = \frac{x e^{x^2} + 1}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{y(2x^2 e^{x^2} + e^{x^2}) - (x e^{x^2} + 1)y'}{y^2} \\ = \frac{y(2x^2 e^{x^2} + e^{x^2}) - (x e^{x^2} + 1) \cdot \frac{x e^{x^2} + 1}{y}}{y^2} = \frac{y^2(2x^2 e^{x^2} + e^{x^2}) - (x^2 e^{2x^2} + 2x e^{x^2} + 1)}{y^3} \\ = \frac{(2x^2 y^2 + y^2 - 2x) e^{x^2} - x^2 e^{2x^2} - 1}{y^3}$$

$$22. y^2 - 2x = 1 - 2y \Rightarrow 2y \cdot y' - 2 = -2y' \Rightarrow y'(2y + 2) = 2 \Rightarrow y' = \frac{1}{y+1} = (y+1)^{-1}; \text{ then } y'' = -(y+1)^{-2} \cdot y' \\ = -(y+1)^{-2} (y+1)^{-1} \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{-1}{(y+1)^3}$$

$$23. 2\sqrt{y} = x - y \Rightarrow y^{-1/2} y' = 1 - y' \Rightarrow y' (y^{-1/2} + 1) = 1 \Rightarrow \frac{dy}{dx} = y' = \frac{1}{y^{-1/2} + 1} = \frac{\sqrt{y}}{\sqrt{y} + 1}; \text{ we can} \\ \text{differentiate the equation } y' (y^{-1/2} + 1) = 1 \text{ again to find } y'': y' \left(-\frac{1}{2} y^{-3/2} y' \right) + (y^{-1/2} + 1) y'' = 0 \\ \Rightarrow (y^{-1/2} + 1) y'' = \frac{1}{2} [y']^2 y^{-3/2} \Rightarrow \frac{d^2y}{dx^2} = y'' = \frac{\frac{1}{2} \left(\frac{\sqrt{y}}{\sqrt{y} + 1} \right)^2 y^{-3/2}}{(y^{-1/2} + 1)} = \frac{1}{2y^{3/2} (y^{-1/2} + 1)^3} = \frac{1}{2(1 + \sqrt{y})^3}$$

$$24. xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow xy' + 2yy' = -y \Rightarrow y'(x + 2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)}; \frac{d^2y}{dx^2} = y'' \\ = \frac{-(x+2y)y' + y(1+2y')}{(x+2y)^2} = \frac{-(x+2y) \left[\frac{-y}{(x+2y)} \right] + y \left[1 + 2 \left(\frac{-y}{(x+2y)} \right) \right]}{(x+2y)^2} = \frac{\frac{1}{(x+2y)} [y(x+2y) + y(x+2y) - 2y^2]}{(x+2y)^2} \\ = \frac{2y(x+2y) - 2y^2}{(x+2y)^3} = \frac{2y^2 + 2xy}{(x+2y)^3} = \frac{2y(x+y)}{(x+2y)^3}$$

$$25. x^3 + y^3 = 16 \Rightarrow 3x^2 + 3y^2 y' = 0 \Rightarrow 3y^2 y' = -3x^2 \Rightarrow y' = -\frac{x^2}{y^2}; \text{ we differentiate } y^2 y' = -x^2 \text{ to find } y'':$$

$$y^2 y'' + y' [2y \cdot y'] = -2x \Rightarrow y^2 y'' = -2x - 2y [y']^2 \Rightarrow y'' = \frac{-2x - 2y \left(-\frac{x^2}{y^2} \right)^2}{y^2} = \frac{-2x - \frac{2x^4}{y^3}}{y^2} \\ = \frac{-2xy^3 - 2x^4}{y^5} \Rightarrow \left. \frac{d^2y}{dx^2} \right|_{(2,2)} = \frac{-32 - 32}{32} = -2$$

$$26. xy + y^2 = 1 \Rightarrow xy' + y + 2yy' = 0 \Rightarrow y'(x + 2y) = -y \Rightarrow y' = \frac{-y}{(x+2y)} \Rightarrow y'' = \frac{(x+2y)(-y') - (-y)(1+2y')}{(x+2y)^2}; \\ \text{since } y'|_{(0,-1)} = -\frac{1}{2} \text{ we obtain } y''|_{(0,-1)} = \frac{(-2) \left(\frac{1}{2} \right) - (-1)(0)}{4} = -\frac{1}{4}$$

$$27. y^2 + x^2 = y^4 - 2x \text{ at } (-2, 1) \text{ and } (-2, -1) \Rightarrow 2y \frac{dy}{dx} + 2x = 4y^3 \frac{dy}{dx} - 2 \Rightarrow 2y \frac{dy}{dx} - 4y^3 \frac{dy}{dx} = -2 - 2x \\ \Rightarrow \frac{dy}{dx} (2y - 4y^3) = -2 - 2x \Rightarrow \frac{dy}{dx} = \frac{x+1}{2y^3-y} \Rightarrow \left. \frac{dy}{dx} \right|_{(-2,1)} = -1 \text{ and } \left. \frac{dy}{dx} \right|_{(-2,-1)} = 1$$

28. $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1) \Rightarrow 2(x^2 + y^2) \left(2x + 2y \frac{dy}{dx} \right) = 2(x - y) \left(1 - \frac{dy}{dx} \right)$
 $\Rightarrow \frac{dy}{dx} [2y(x^2 + y^2) + (x - y)] = -2x(x^2 + y^2) + (x - y) \Rightarrow \frac{dy}{dx} = \frac{-2x(x^2 + y^2) + (x - y)}{2y(x^2 + y^2) + (x - y)} \Rightarrow \frac{dy}{dx} \Big|_{(1,0)} = -1$
 and $\frac{dy}{dx} \Big|_{(1,-1)} = 1$
29. $x^2 + xy - y^2 = 1 \Rightarrow 2x + y + xy' - 2yy' = 0 \Rightarrow (x - 2y)y' = -2x - y \Rightarrow y' = \frac{2x + y}{2y - x}$;
 (a) the slope of the tangent line $m = y' \Big|_{(2,3)} = \frac{7}{4} \Rightarrow$ the tangent line is $y - 3 = \frac{7}{4}(x - 2) \Rightarrow y = \frac{7}{4}x - \frac{1}{2}$
 (b) the normal line is $y - 3 = -\frac{4}{7}(x - 2) \Rightarrow y = -\frac{4}{7}x + \frac{29}{7}$
30. $x^2 + y^2 = 25 \Rightarrow 2x + 2yy' = 0 \Rightarrow y' = -\frac{x}{y}$;
 (a) the slope of the tangent line $m = y' \Big|_{(3,-4)} = -\frac{x}{y} \Big|_{(3,-4)} = \frac{3}{4} \Rightarrow$ the tangent line is $y + 4 = \frac{3}{4}(x - 3) \Rightarrow y = \frac{3}{4}x - \frac{25}{4}$
 (b) the normal line is $y + 4 = -\frac{4}{3}(x - 3) \Rightarrow y = -\frac{4}{3}x$
31. $x^2y^2 = 9 \Rightarrow 2xy^2 + 2x^2yy' = 0 \Rightarrow x^2yy' = -xy^2 \Rightarrow y' = -\frac{y}{x}$;
 (a) the slope of the tangent line $m = y' \Big|_{(-1,3)} = -\frac{y}{x} \Big|_{(-1,3)} = 3 \Rightarrow$ the tangent line is $y - 3 = 3(x + 1) \Rightarrow y = 3x + 6$
 (b) the normal line is $y - 3 = -\frac{1}{3}(x + 1) \Rightarrow y = -\frac{1}{3}x + \frac{8}{3}$
32. $y^2 - 2x - 4y - 1 = 0 \Rightarrow 2yy' - 2 - 4y' = 0 \Rightarrow 2(y - 2)y' = 2 \Rightarrow y' = \frac{1}{y-2}$;
 (a) the slope of the tangent line $m = y' \Big|_{(-2,1)} = -1 \Rightarrow$ the tangent line is $y - 1 = -1(x + 2) \Rightarrow y = -x - 1$
 (b) the normal line is $y - 1 = 1(x + 2) \Rightarrow y = x + 3$
33. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0 \Rightarrow 12x + 3y + 3xy' + 4yy' + 17y' = 0 \Rightarrow y'(3x + 4y + 17) = -12x - 3y$
 $\Rightarrow y' = \frac{-12x - 3y}{3x + 4y + 17}$;
 (a) the slope of the tangent line $m = y' \Big|_{(-1,0)} = \frac{-12x - 3y}{3x + 4y + 17} \Big|_{(-1,0)} = \frac{6}{7} \Rightarrow$ the tangent line is $y - 0 = \frac{6}{7}(x + 1)$
 $\Rightarrow y = \frac{6}{7}x + \frac{6}{7}$
 (b) the normal line is $y - 0 = -\frac{7}{6}(x + 1) \Rightarrow y = -\frac{7}{6}x - \frac{7}{6}$
34. $x^2 - \sqrt{3}xy + 2y^2 = 5 \Rightarrow 2x - \sqrt{3}xy' - \sqrt{3}y + 4yy' = 0 \Rightarrow y'(4y - \sqrt{3}x) = \sqrt{3}y - 2x \Rightarrow y' = \frac{\sqrt{3}y - 2x}{4y - \sqrt{3}x}$;
 (a) the slope of the tangent line $m = y' \Big|_{(\sqrt{3},2)} = \frac{\sqrt{3}y - 2x}{4y - \sqrt{3}x} \Big|_{(\sqrt{3},2)} = 0 \Rightarrow$ the tangent line is $y = 2$
 (b) the normal line is $x = \sqrt{3}$
35. $2xy + \pi \sin y = 2\pi \Rightarrow 2xy' + 2y + \pi(\cos y)y' = 0 \Rightarrow y'(2x + \pi \cos y) = -2y \Rightarrow y' = \frac{-2y}{2x + \pi \cos y}$;
 (a) the slope of the tangent line $m = y' \Big|_{(1, \frac{\pi}{2})} = \frac{-2y}{2x + \pi \cos y} \Big|_{(1, \frac{\pi}{2})} = -\frac{\pi}{2} \Rightarrow$ the tangent line is
 $y - \frac{\pi}{2} = -\frac{\pi}{2}(x - 1) \Rightarrow y = -\frac{\pi}{2}x + \pi$
 (b) the normal line is $y - \frac{\pi}{2} = \frac{2}{\pi}(x - 1) \Rightarrow y = \frac{2}{\pi}x - \frac{2}{\pi} + \frac{\pi}{2}$
36. $x \sin 2y = y \cos 2x \Rightarrow x(\cos 2y)2y' + \sin 2y = -2y \sin 2x + y' \cos 2x \Rightarrow y'(2x \cos 2y - \cos 2x)$
 $= -\sin 2y - 2y \sin 2x \Rightarrow y' = \frac{\sin 2y + 2y \sin 2x}{\cos 2x - 2x \cos 2y}$;
 (a) the slope of the tangent line $m = y' \Big|_{(\frac{\pi}{4}, \frac{\pi}{2})} = \frac{\sin 2y + 2y \sin 2x}{\cos 2x - 2x \cos 2y} \Big|_{(\frac{\pi}{4}, \frac{\pi}{2})} = \frac{\pi}{2} = 2 \Rightarrow$ the tangent line is
 $y - \frac{\pi}{2} = 2(x - \frac{\pi}{4}) \Rightarrow y = 2x$
 (b) the normal line is $y - \frac{\pi}{2} = -\frac{1}{2}(x - \frac{\pi}{4}) \Rightarrow y = -\frac{1}{2}x + \frac{5\pi}{8}$

$$37. y = 2 \sin(\pi x - y) \Rightarrow y' = 2[\cos(\pi x - y)] \cdot (\pi - y') \Rightarrow y'[1 + 2 \cos(\pi x - y)] = 2\pi \cos(\pi x - y) \\ \Rightarrow y' = \frac{2\pi \cos(\pi x - y)}{1 + 2 \cos(\pi x - y)};$$

$$(a) \text{ the slope of the tangent line } m = y'|_{(1,0)} = \frac{2\pi \cos(\pi x - y)}{1 + 2 \cos(\pi x - y)} \Big|_{(1,0)} = 2\pi \Rightarrow \text{the tangent line is}$$

$$y - 0 = 2\pi(x - 1) \Rightarrow y = 2\pi x - 2\pi$$

$$(b) \text{ the normal line is } y - 0 = -\frac{1}{2\pi}(x - 1) \Rightarrow y = -\frac{x}{2\pi} + \frac{1}{2\pi}$$

$$38. x^2 \cos^2 y - \sin y = 0 \Rightarrow x^2(2 \cos y)(-\sin y)y' + 2x \cos^2 y - y' \cos y = 0 \Rightarrow y'[-2x^2 \cos y \sin y - \cos y] \\ = -2x \cos^2 y \Rightarrow y' = \frac{2x \cos^2 y}{2x^2 \cos y \sin y + \cos y};$$

$$(a) \text{ the slope of the tangent line } m = y'|_{(0,\pi)} = \frac{2x \cos^2 y}{2x^2 \cos y \sin y + \cos y} \Big|_{(0,\pi)} = 0 \Rightarrow \text{the tangent line is } y = \pi$$

$$(b) \text{ the normal line is } x = 0$$

$$39. \text{ Solving } x^2 + xy + y^2 = 7 \text{ and } y = 0 \Rightarrow x^2 = 7 \Rightarrow x = \pm \sqrt{7} \Rightarrow (-\sqrt{7}, 0) \text{ and } (\sqrt{7}, 0) \text{ are the points where the} \\ \text{curve crosses the } x\text{-axis. Now } x^2 + xy + y^2 = 7 \Rightarrow 2x + y + xy' + 2yy' = 0 \Rightarrow (x + 2y)y' = -2x - y \\ \Rightarrow y' = -\frac{2x+y}{x+2y} \Rightarrow m = -\frac{2x+y}{x+2y} \Rightarrow \text{the slope at } (-\sqrt{7}, 0) \text{ is } m = -\frac{-2\sqrt{7}}{-\sqrt{7}} = -2 \text{ and the slope at } (\sqrt{7}, 0) \text{ is} \\ m = -\frac{2\sqrt{7}}{\sqrt{7}} = -2. \text{ Since the slope is } -2 \text{ in each case, the corresponding tangents must be parallel.}$$

$$40. \text{ Let } p \text{ and } q \text{ be integers with } q > 0 \text{ and suppose that } y = \sqrt[q]{x^p} = x^{p/q}. \text{ Then } y^q = x^p. \text{ Since } p \text{ and } q \text{ are integers and} \\ \text{assuming } y \text{ is a differentiable function of } x, \frac{d}{dx}(y^q) = \frac{d}{dx}(x^p) \Rightarrow qy^{q-1} \frac{dy}{dx} = px^{p-1} \Rightarrow \frac{dy}{dx} = \frac{px^{p-1}}{qy^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{y^{q-1}} \\ = \frac{p}{q} \cdot \frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q} \cdot \frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q} \cdot x^{p-1-(p-p/q)} = \frac{p}{q} \cdot x^{(p/q)-1}$$

$$41. y^4 = y^2 - x^2 \Rightarrow 4y^3 y' = 2yy' - 2x \Rightarrow 2(2y^3 - y)y' = -2x \Rightarrow y' = \frac{x}{y-2y^3}; \text{ the slope of the tangent line at} \\ \left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right) \text{ is } \frac{x}{y-2y^3} \Big|_{\left(\frac{\sqrt{3}}{4}, \frac{\sqrt{3}}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{\sqrt{3}}{2} - 6\frac{\sqrt{3}}{8}} = \frac{\frac{1}{4}}{\frac{1}{2} - \frac{3}{4}} = \frac{1}{2-3} = -1; \text{ the slope of the tangent line at } \left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right) \\ \text{is } \frac{x}{y-2y^3} \Big|_{\left(\frac{\sqrt{3}}{4}, \frac{1}{2}\right)} = \frac{\frac{\sqrt{3}}{4}}{\frac{1}{2} - \frac{1}{8}} = \frac{2\sqrt{3}}{4-2} = \sqrt{3}$$

$$42. x^3 + y^3 - 9xy = 0 \Rightarrow 3x^2 + 3y^2 y' - 9xy' - 9y = 0 \Rightarrow y'(3y^2 - 9x) = 9y - 3x^2 \Rightarrow y' = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}$$

$$(a) y'|_{(4,2)} = \frac{5}{4} \text{ and } y'|_{(2,4)} = \frac{4}{5};$$

$$(b) y' = 0 \Rightarrow \frac{3y - x^2}{y^2 - 3x} = 0 \Rightarrow 3y - x^2 = 0 \Rightarrow y = \frac{x^2}{3} \Rightarrow x^3 + \left(\frac{x^2}{3}\right)^3 - 9x\left(\frac{x^2}{3}\right) = 0 \Rightarrow x^6 - 54x^3 = 0 \\ \Rightarrow x^3(x^3 - 54) = 0 \Rightarrow x = 0 \text{ or } x = \sqrt[3]{54} = 3\sqrt[3]{2} \Rightarrow \text{there is a horizontal tangent at } x = 3\sqrt[3]{2}. \text{ To find the} \\ \text{corresponding } y\text{-value, we will use part (c).}$$

$$(c) \frac{dx}{dy} = 0 \Rightarrow \frac{y^2 - 3x}{3y - x^2} = 0 \Rightarrow y^2 - 3x = 0 \Rightarrow y = \pm \sqrt{3x}; y = \sqrt{3x} \Rightarrow x^3 + (\sqrt{3x})^3 - 9x\sqrt{3x} = 0 \\ \Rightarrow x^3 - 6\sqrt{3}x^{3/2} = 0 \Rightarrow x^{3/2}(x^{3/2} - 6\sqrt{3}) = 0 \Rightarrow x^{3/2} = 0 \text{ or } x^{3/2} = 6\sqrt{3} \Rightarrow x = 0 \text{ or } x = \sqrt[3]{108} = 3\sqrt[3]{4}.$$

Since the equation $x^3 + y^3 - 9xy = 0$ is symmetric in x and y , the graph is symmetric about the line $y = x$. That is, if (a, b) is a point on the folium, then so is (b, a) . Moreover, if $y'|_{(a,b)} = m$, then $y'|_{(b,a)} = \frac{1}{m}$. Thus, if the folium has a horizontal tangent at (a, b) , it has a vertical tangent at (b, a) so one might expect that with a horizontal tangent at $x = \sqrt[3]{54}$ and a vertical tangent at $x = 3\sqrt[3]{4}$, the points of tangency are $(\sqrt[3]{54}, 3\sqrt[3]{4})$ and $(3\sqrt[3]{4}, \sqrt[3]{54})$, respectively. One can check that these points do satisfy the equation $x^3 + y^3 - 9xy = 0$.

43. $x^2 + 2xy - 3y^2 = 0 \Rightarrow 2x + 2xy' + 2y - 6yy' = 0 \Rightarrow y'(2x - 6y) = -2x - 2y \Rightarrow y' = \frac{x+y}{3y-x} \Rightarrow$ the slope of the tangent line $m = y'|_{(1,1)} = \frac{x+y}{3y-x} \Big|_{(1,1)} = 1 \Rightarrow$ the equation of the normal line at $(1, 1)$ is $y - 1 = -1(x - 1) \Rightarrow y = -x + 2$. To find

where the normal line intersects the curve we substitute into its equation: $x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$

$$\Rightarrow x^2 + 4x - 2x^2 - 3(4 - 4x + x^2) = 0 \Rightarrow -4x^2 + 16x - 12 = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x - 3)(x - 1) = 0$$

$\Rightarrow x = 3$ and $y = -x + 2 = -1$. Therefore, the normal to the curve at $(1, 1)$ intersects the curve at the point $(3, -1)$.

Note that it also intersects the curve at $(1, 1)$.

44. $xy + 2x - y = 0 \Rightarrow x \frac{dy}{dx} + y + 2 - \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{1-x}$; the slope of the line $2x + y = 0$ is -2 . In order to be parallel, the normal lines must also have slope of -2 . Since a normal is perpendicular to a tangent, the slope of the tangent is $\frac{1}{2}$. Therefore, $\frac{y+2}{1-x} = \frac{1}{2} \Rightarrow 2y + 4 = 1 - x \Rightarrow x = -3 - 2y$. Substituting in the original equation, $y(-3 - 2y) + 2(-3 - 2y) - y = 0 \Rightarrow y^2 + 4y + 3 = 0 \Rightarrow y = -3$ or $y = -1$. If $y = -3$, then $x = 3$ and $y + 3 = -2(x - 3) \Rightarrow y = -2x + 3$. If $y = -1$, then $x = -1$ and $y + 1 = -2(x + 1) \Rightarrow y = -2x - 3$.

45. $xy^3 + x^2y = 6 \Rightarrow x \left(3y^2 \frac{dy}{dx} \right) + y^3 + x^2 \frac{dy}{dx} + 2xy = 0 \Rightarrow \frac{dy}{dx} (3xy^2 + x^2) = -y^3 - 2xy \Rightarrow \frac{dy}{dx} = \frac{-y^3 - 2xy}{3xy^2 + x^2}$
 $= -\frac{y^3 + 2xy}{3xy^2 + x^2}$; also, $xy^3 + x^2y = 6 \Rightarrow x(3y^2) + y^3 \frac{dx}{dy} + x^2 + y \left(2x \frac{dx}{dy} \right) = 0 \Rightarrow \frac{dx}{dy} (y^3 + 2xy) = -3xy^2 - x^2$
 $\Rightarrow \frac{dx}{dy} = -\frac{3xy^2 + x^2}{y^3 + 2xy}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$, so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a) .

46. $x^3 + y^2 = \sin^2 y \Rightarrow 3x^2 + 2y \frac{dy}{dx} = (2 \sin y)(\cos y) \frac{dy}{dx} \Rightarrow \frac{dy}{dx} (2y - 2 \sin y \cos y) = -3x^2 \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{2y - 2 \sin y \cos y}$
 $= \frac{3x^2}{2 \sin y \cos y - 2y}$; also, $x^3 + y^2 = \sin^2 y \Rightarrow 3x^2 \frac{dx}{dy} + 2y = 2 \sin y \cos y \Rightarrow \frac{dx}{dy} = \frac{2 \sin y \cos y - 2y}{3x^2}$; thus $\frac{dx}{dy}$ appears to equal $\frac{1}{\frac{dy}{dx}}$. The two different treatments view the graphs as functions symmetric across the line $y = x$ so their slopes are reciprocals of one another at the corresponding points (a, b) and (b, a) .

- 47-54. Example CAS commands:

Maple:

```
q1 := x^3-x*y+y^3 = 7;
pt := [x=2,y=1];
p1 := implicitplot( q1, x=-3..3, y=-3..3 );
p1;
eval( q1, pt );
q2 := implicitdiff( q1, y, x );
m := eval( q2, pt );
tan_line := y = 1 + m*(x-2);
p2 := implicitplot( tan_line, x=-5..5, y=-5..5, color=green );
p3 := pointplot( eval([x,y],pt), color=blue );
display( [p1,p2,p3], ="Section 2.8 #47(c)" );
```

Mathematica: (functions and x0 may vary):

Note use of double equal sign (logic statement) in definition of eqn and tanline.

```
<<Graphics`ImplicitPlot`
Clear[x, y]
{x0, y0}={1, Pi/4};
eqn=x + Tan[y/x]==2;
ImplicitPlot[eqn,{ x, x0 - 3, x0 + 3},{y, y0 - 3, y0 + 3}]
eqn/.{x -> x0, y -> y0}
eqn/.{ y -> y[x]}
```

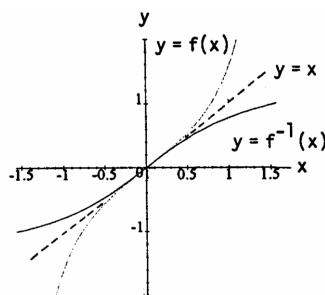
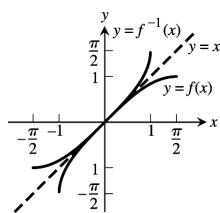
```

D[%, x]
Solve[%, y'[x]]
slope=y'[x]/.First[%]
m=slope/.{x → x0, y[x] → y0}
tanline=y==y0 + m (x - x0)
ImplicitPlot[{eqn, tanline}, {x, x0 - 3, x0 + 3},{y, y0 - 3, y0 + 3}]

```

2.9 INVERSE FUNCTIONS AND THEIR DERIVATIVES

1. Yes one-to-one, the graph passes the horizontal test.
2. Not one-to-one, the graph fails the horizontal test.
3. Not one-to-one since (for example) the horizontal line $y = 2$ intersects the graph twice.
4. Not one-to-one, the graph fails the horizontal test.
5. Domain: $-1 \leq x \leq 1$, Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
6. Domain: $-\infty < x < \infty$, Range: $-\frac{\pi}{2} < y \leq \frac{\pi}{2}$



7. Step 1: $y = x^2 + 1 \Rightarrow x^2 = y - 1 \Rightarrow x = \sqrt{y - 1}$
Step 2: $y = \sqrt{x - 1} = f^{-1}(x)$
8. Step 1: $y = x^2 \Rightarrow x = -\sqrt{y}$, since $x \leq 0$.
Step 2: $y = -\sqrt{x} = f^{-1}(x)$
9. Step 1: $y = x^3 - 1 \Rightarrow x^3 = y + 1 \Rightarrow x = (y + 1)^{1/3}$
Step 2: $y = \sqrt[3]{x + 1} = f^{-1}(x)$
10. Step 1: $y = x^2 - 2x + 1 \Rightarrow y = (x - 1)^2 \Rightarrow \sqrt{y} = x - 1$, since $x \geq 1 \Rightarrow x = 1 + \sqrt{y}$
Step 2: $y = 1 + \sqrt{x} = f^{-1}(x)$
11. Step 1: $y = (x + 1)^2 \Rightarrow \sqrt{y} = x + 1$, since $x \geq -1 \Rightarrow x = \sqrt{y} - 1$
Step 2: $y = \sqrt{x} - 1 = f^{-1}(x)$
12. Step 1: $y = x^{2/3} \Rightarrow x = y^{3/2}$
Step 2: $y = x^{3/2} = f^{-1}(x)$
13. Step 1: $y = x^5 \Rightarrow x = y^{1/5}$
Step 2: $y = \sqrt[5]{x} = f^{-1}(x)$

Domain and Range of f^{-1} : all reals;

$$f(f^{-1}(x)) = (x^{1/5})^5 = x \text{ and } f^{-1}(f(x)) = (x^5)^{1/5} = x$$

14. Step 1: $y = x^3 + 1 \Rightarrow x^3 = y - 1 \Rightarrow x = (y - 1)^{1/3}$

Step 2: $y = \sqrt[3]{x - 1} = f^{-1}(x)$;

Domain and Range of f^{-1} : all reals;

$$f(f^{-1}(x)) = ((x - 1)^{1/3})^3 + 1 = (x - 1) + 1 = x \text{ and } f^{-1}(f(x)) = ((x^3 + 1) - 1)^{1/3} = (x^3)^{1/3} = x$$

15. Step 1: $y = \frac{1}{x^2} \Rightarrow x^2 = \frac{1}{y} \Rightarrow x = \frac{1}{\sqrt{y}}$

Step 2: $y = \frac{1}{\sqrt{x}} = f^{-1}(x)$

Domain of f^{-1} : $x > 0$, Range of f^{-1} : $y > 0$;

$$f(f^{-1}(x)) = \frac{1}{(\frac{1}{\sqrt{x}})^2} = \frac{1}{(\frac{1}{x})} = x \text{ and } f^{-1}(f(x)) = \frac{1}{\sqrt{\frac{1}{x^2}}} = \frac{1}{(\frac{1}{x})} = x \text{ since } x > 0$$

16. Step 1: $y = \frac{1}{x^3} \Rightarrow x^3 = \frac{1}{y} \Rightarrow x = \frac{1}{y^{1/3}}$

Step 2: $y = \frac{1}{x^{1/3}} = \sqrt[3]{\frac{1}{x}} = f^{-1}(x)$;

Domain of f^{-1} : $x \neq 0$, Range of f^{-1} : $y \neq 0$;

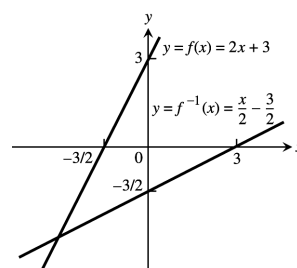
$$f(f^{-1}(x)) = \frac{1}{(\frac{1}{x^{1/3}})^3} = \frac{1}{x^{-1}} = x \text{ and } f^{-1}(f(x)) = (\frac{1}{x^3})^{-1/3} = (\frac{1}{x})^{-1} = x$$

17. (a) $y = 2x + 3 \Rightarrow 2x = y - 3$

$$\Rightarrow x = \frac{y}{2} - \frac{3}{2} \Rightarrow f^{-1}(x) = \frac{x}{2} - \frac{3}{2}$$

(c) $\left. \frac{df}{dx} \right|_{x=-1} = 2, \left. \frac{df^{-1}}{dx} \right|_{x=1} = \frac{1}{2}$

(b)

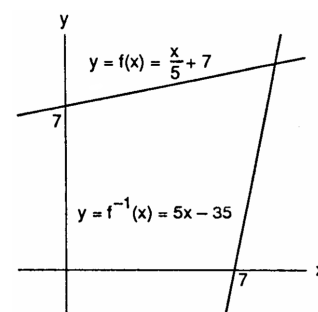


18. (a) $y = \frac{1}{5}x + 7 \Rightarrow \frac{1}{5}x = y - 7$

$$\Rightarrow x = 5y - 35 \Rightarrow f^{-1}(x) = 5x - 35$$

(c) $\left. \frac{df}{dx} \right|_{x=-1} = \frac{1}{5}, \left. \frac{df^{-1}}{dx} \right|_{x=34/5} = 5$

(b)

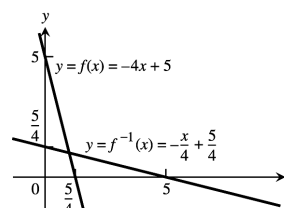


19. (a) $y = 5 - 4x \Rightarrow 4x = 5 - y$

$$\Rightarrow x = \frac{5}{4} - \frac{y}{4} \Rightarrow f^{-1}(x) = \frac{5}{4} - \frac{x}{4}$$

(c) $\left. \frac{df}{dx} \right|_{x=1/2} = -4, \left. \frac{df^{-1}}{dx} \right|_{x=3} = -\frac{1}{4}$

(b)



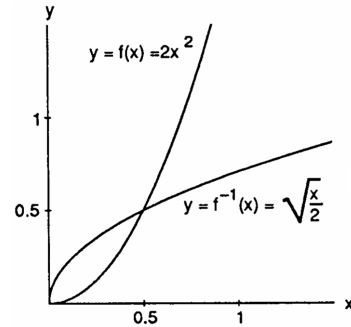
20. (a) $y = 2x^2 \Rightarrow x^2 = \frac{1}{2}y$

$$\Rightarrow x = \frac{1}{\sqrt{2}}\sqrt{y} \Rightarrow f^{-1}(x) = \sqrt{\frac{x}{2}}$$

(c) $\left.\frac{df}{dx}\right|_{x=5} = 4x|_{x=5} = 20,$

$$\left.\frac{df^{-1}}{dx}\right|_{x=50} = \frac{1}{2\sqrt{2}}x^{-1/2}\bigg|_{x=50} = \frac{1}{20}$$

(b)



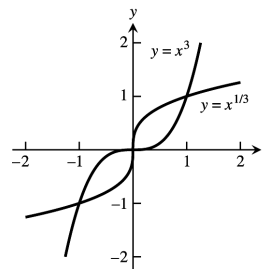
21. (a) $f(g(x)) = (\sqrt[3]{x})^3 = x, g(f(x)) = \sqrt[3]{x^3} = x$

(c) $f'(x) = 3x^2 \Rightarrow f'(1) = 3, f'(-1) = 3;$

$$g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(1) = \frac{1}{3}, g'(-1) = \frac{1}{3}$$

(d) The line $y = 0$ is tangent to $f(x) = x^3$ at $(0, 0)$;
the line $x = 0$ is tangent to $g(x) = \sqrt[3]{x}$ at $(0, 0)$

(b)



22. (a) $h(k(x)) = \frac{1}{4}((4x)^{1/3})^3 = x,$

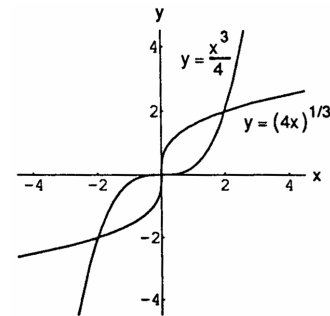
$$k(h(x)) = \left(4 \cdot \frac{x^3}{4}\right)^{1/3} = x$$

(c) $h'(x) = \frac{3x^2}{4} \Rightarrow h'(2) = 3, h'(-2) = 3;$

$$k'(x) = \frac{4}{3}(4x)^{-2/3} \Rightarrow k'(2) = \frac{1}{3}, k'(-2) = \frac{1}{3}$$

(d) The line $y = 0$ is tangent to $h(x) = \frac{x^3}{4}$ at $(0, 0)$;
the line $x = 0$ is tangent to $k(x) = (4x)^{1/3}$ at $(0, 0)$

(b)



23. $\frac{df}{dx} = 3x^2 - 6x \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(3)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=3}} = \frac{1}{9}$

24. $\frac{df}{dx} = 2x - 4 \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{x=f(5)} = \frac{1}{\left.\frac{df}{dx}\right|_{x=5}} = \frac{1}{6}$

25. (a) $y = mx \Rightarrow x = \frac{1}{m}y \Rightarrow f^{-1}(x) = \frac{1}{m}x$

(b) The graph of $y = f^{-1}(x)$ is a line through the origin with slope $\frac{1}{m}$.

26. $y = mx + b \Rightarrow x = \frac{y}{m} - \frac{b}{m} \Rightarrow f^{-1}(x) = \frac{1}{m}x - \frac{b}{m}$; the graph of $f^{-1}(x)$ is a line with slope $\frac{1}{m}$ and y-intercept $-\frac{b}{m}$.

27. Let $x_1 \neq x_2$ be two numbers in the domain of an increasing function f . Then, either $x_1 < x_2$ or $x_1 > x_2$ which implies $f(x_1) < f(x_2)$ or $f(x_1) > f(x_2)$, since $f(x)$ is increasing. In either case, $f(x_1) \neq f(x_2)$ and f is one-to-one. Similar arguments hold if f is decreasing.

28. $f(x)$ is increasing since $x_2 > x_1 \Rightarrow \frac{1}{3}x_2 + \frac{5}{6} > \frac{1}{3}x_1 + \frac{5}{6}; \frac{df}{dx} = \frac{1}{3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{(1/3)} = 3$

29. $f(x)$ is increasing since $x_2 > x_1 \Rightarrow 27x_2^3 > 27x_1^3; y = 27x^3 \Rightarrow x = \frac{1}{3}y^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{3}x^{1/3};$

$$\frac{df}{dx} = 81x^2 \Rightarrow \left.\frac{df^{-1}}{dx}\right|_{\frac{1}{3}x^{1/3}} = \frac{1}{81x^2}\bigg|_{\frac{1}{3}x^{1/3}} = \frac{1}{9x^{2/3}} = \frac{1}{9}x^{-2/3}$$

30. $f(x)$ is decreasing since $x_2 > x_1 \Rightarrow 1 - 8x_2^3 < 1 - 8x_1^3$; $y = 1 - 8x^3 \Rightarrow x = \frac{1}{2}(1 - y)^{1/3} \Rightarrow f^{-1}(x) = \frac{1}{2}(1 - x)^{1/3}$;
 $\frac{df}{dx} = -24x^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{-24x^2} \Big|_{\frac{1}{2}(1-x)^{1/3}} = \frac{-1}{6(1-x)^{2/3}} = -\frac{1}{6}(1-x)^{-2/3}$

31. $f(x)$ is decreasing since $x_2 > x_1 \Rightarrow (1 - x_2)^3 < (1 - x_1)^3$; $y = (1 - x)^3 \Rightarrow x = 1 - y^{1/3} \Rightarrow f^{-1}(x) = 1 - x^{1/3}$;
 $\frac{df}{dx} = -3(1 - x)^2 \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{-3(1-x)^2} \Big|_{1-x^{1/3}} = \frac{-1}{3x^{2/3}} = -\frac{1}{3}x^{-2/3}$

32. $f(x)$ is increasing since $x_2 > x_1 \Rightarrow x_2^{5/3} > x_1^{5/3}$; $y = x^{5/3} \Rightarrow x = y^{3/5} \Rightarrow f^{-1}(x) = x^{3/5}$;
 $\frac{df}{dx} = \frac{5}{3}x^{2/3} \Rightarrow \frac{df^{-1}}{dx} = \frac{1}{\frac{5}{3}x^{2/3}} \Big|_{x^{3/5}} = \frac{3}{5x^{2/5}} = \frac{3}{5}x^{-2/5}$

33. The function $g(x)$ is also one-to-one. The reasoning: $f(x)$ is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $-f(x_1) \neq -f(x_2)$ and therefore $g(x_1) \neq g(x_2)$. Therefore $g(x)$ is one-to-one as well.

34. The function $h(x)$ is also one-to-one. The reasoning: $f(x)$ is one-to-one means that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$, so $\frac{1}{f(x_1)} \neq \frac{1}{f(x_2)}$, and therefore $h(x_1) \neq h(x_2)$.

35. The composite is one-to-one also. The reasoning: If $x_1 \neq x_2$ then $g(x_1) \neq g(x_2)$ because g is one-to-one. Since $g(x_1) \neq g(x_2)$, we also have $f(g(x_1)) \neq f(g(x_2))$ because f is one-to-one. Thus, $f \circ g$ is one-to-one because $x_1 \neq x_2 \Rightarrow f(g(x_1)) \neq f(g(x_2))$.

36. Yes, g must be one-to-one. If g were not one-to-one, there would exist numbers $x_1 \neq x_2$ in the domain of g with $g(x_1) = g(x_2)$. For these numbers we would also have $f(g(x_1)) = f(g(x_2))$, contradicting the assumption that $f \circ g$ is one-to-one.

37. $(g \circ f)(x) = x \Rightarrow g(f(x)) = x \Rightarrow g'(f(x))f'(x) = 1$

38. The function $f(x) = x^n$ is differentiable everywhere, and $g(x) = x^{1/n}$ is the inverse of f , so by Theorem 4
 $g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n(g(x))^{n-1}} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{nx^{(n-1)/n}} = \frac{1}{nx^{1-1/n}} = \frac{1}{n}x^{(1/n)-1}$ provided $x \neq 0$.

39-46. Example CAS commands:

Maple:

```
with( plots );#41
f := x -> sqrt(3*x-2);
domain := 2/3 .. 4;
x0 := 3;
Df := D(f); # (a)
plot( [f(x),Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)","y=f'(x)"],
      title="#41(a) (Section 2.9)");
q1 := solve( y=f(x), x ); # (b)
g := unapply( q1, y );
m1 := Df(x0); # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0); # (d)
t2 := g(f(x0)) + m2*(x-f(x0));
y=t2;
domaing := map(f,domain); # (e)
```

```

p1 := plot( [f(x),x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] );
p2 := plot( g(x), x=domain, color=cyan, linestyle=3, thickness=4 );
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 );
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 );
p5 := plot( [ [x0,f(x0)], [f(x0),x0] ], color=green );
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#41(e) (Section 2.9)" );

```

Mathematica: (assigned function and values for a, b, and x0 may vary)

If a function requires the odd root of a negative number, begin by loading the RealOnly package that allows Mathematica to do this.

```

<<Miscellaneous`RealOnly`
Clear[x, y]
{a,b} = {-2, 1}; x0 = 1/2 ;
f[x_] = (3x + 2) / (2x - 11)
Plot[{f[x], f'[x]}, {x, a, b}]
solx = Solve[y == f[x], x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x-x0)
gtan[y_] = x0 + 1/ f'[x0] (y - y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]}, {x, a, b},
Epilog -> Line[{ {x0, y0}, {y0, x0} }], PlotRange -> { {a,b},{a,b} }, AspectRatio -> Automatic]

```

47-48. Example CAS commands:

Maple:

```

with( plots );
eq := cos(y) = x^(1/5);
domain := 0 .. 1;
x0 := 1/2;
f := unapply( solve( eq, y ), x ); # (a)
Df := D(f);
plot( [f(x),Df(x)], x=domain, color=[red,blue], linestyle=[1,3], legend=["y=f(x)", "y=f'(x)"],
title="#48(a) (Section 2.9)" );
q1 := solve( eq, x ); # (b)
g := unapply( q1, y );
m1 := Df(x0); # (c)
t1 := f(x0)+m1*(x-x0);
y=t1;
m2 := 1/Df(x0); # (d)
t2 := g(f(x0)) + m2*(x-f(x0));
y=t2;
domaing := map(f,domain); # (e)
p1 := plot( [f(x),x], x=domain, color=[pink,green], linestyle=[1,9], thickness=[3,0] );
p2 := plot( g(x), x=domain, color=cyan, linestyle=3, thickness=4 );
p3 := plot( t1, x=x0-1..x0+1, color=red, linestyle=4, thickness=0 );
p4 := plot( t2, x=f(x0)-1..f(x0)+1, color=blue, linestyle=7, thickness=1 );
p5 := plot( [ [x0,f(x0)], [f(x0),x0] ], color=green );
display( [p1,p2,p3,p4,p5], scaling=constrained, title="#48(e) (Section 2.9)" );

```

Mathematica: (assigned function and values for a, b, and x0 may vary)

For problems 47 and 48, the code is just slightly altered. At times, different "parts" of solutions need to be used, as in the definitions of f[x] and g[y]

```

Clear[x, y]
{a,b} = {0, 1}; x0 = 1/2 ;
eqn = Cos[y] == x1/5
soly = Solve[eqn, y]
f[x_] = y /. soly[[2]]
Plot[{f[x], f[x]}, {x, a, b}]
solx = Solve[eqn, x]
g[y_] = x /. solx[[1]]
y0 = f[x0]
ftan[x_] = y0 + f'[x0] (x - x0)
gtan[y_] = x0 + 1/f'[x0] (y - y0)
Plot[{f[x], ftan[x], g[x], gtan[x], Identity[x]}, {x, a, b},
Epilog -> Line[{x0, y0}, {y0, x0}], PlotRange -> {{a, b}, {a, b}}, AspectRatio -> Automatic]

```

2.10 LOGARITHMIC FUNCTIONS

1. (a) $\ln 0.75 = \ln \frac{3}{4} = \ln 3 - \ln 4 = \ln 3 - \ln 2^2 = \ln 3 - 2 \ln 2$
 (b) $\ln \frac{4}{9} = \ln 4 - \ln 9 = \ln 2^2 - \ln 3^2 = 2 \ln 2 - 2 \ln 3$
 (c) $\ln \frac{1}{2} = \ln 1 - \ln 2 = -\ln 2$ (d) $\ln \sqrt[3]{9} = \frac{1}{3} \ln 9 = \frac{1}{3} \ln 3^2 = \frac{2}{3} \ln 3$
 (e) $\ln 3\sqrt{2} = \ln 3 + \ln 2^{1/2} = \ln 3 + \frac{1}{2} \ln 2$
 (f) $\ln \sqrt{13.5} = \frac{1}{2} \ln 13.5 = \frac{1}{2} \ln \frac{27}{2} = \frac{1}{2} (\ln 3^3 - \ln 2) = \frac{1}{2} (3 \ln 3 - \ln 2)$
2. (a) $\ln \frac{1}{125} = \ln 1 - 3 \ln 5 = -3 \ln 5$ (b) $\ln 9.8 = \ln \frac{49}{5} = \ln 7^2 - \ln 5 = 2 \ln 7 - \ln 5$
 (c) $\ln 7\sqrt{7} = \ln 7^{3/2} = \frac{3}{2} \ln 7$ (d) $\ln 1225 = \ln 35^2 = 2 \ln 35 = 2 \ln 5 + 2 \ln 7$
 (e) $\ln 0.056 = \ln \frac{7}{125} = \ln 7 - \ln 5^3 = \ln 7 - 3 \ln 5$
 (f) $\frac{\ln 35 + \ln \frac{1}{5}}{\ln 25} = \frac{\ln 5 + \ln 7 - \ln 5}{2 \ln 5} = \frac{1}{2}$
3. (a) $\ln \sin \theta - \ln \left(\frac{\sin \theta}{5} \right) = \ln \left(\frac{\sin \theta}{\left(\frac{\sin \theta}{5} \right)} \right) = \ln 5$ (b) $\ln (3x^2 - 9x) + \ln \left(\frac{1}{3x} \right) = \ln \left(\frac{3x^2 - 9x}{3x} \right) = \ln (x - 3)$
 (c) $\frac{1}{2} \ln (4t^4) - \ln 2 = \ln \sqrt{4t^4} - \ln 2 = \ln 2t^2 - \ln 2 = \ln \left(\frac{2t^2}{2} \right) = \ln (t^2)$
4. (a) $\ln \sec \theta + \ln \cos \theta = \ln [(\sec \theta)(\cos \theta)] = \ln 1 = 0$
 (b) $\ln (8x + 4) - \ln 2^2 = \ln (8x + 4) - \ln 4 = \ln \left(\frac{8x+4}{4} \right) = \ln (2x + 1)$
 (c) $3 \ln \sqrt[3]{t^2 - 1} - \ln (t + 1) = 3 \ln (t^2 - 1)^{1/3} - \ln (t + 1) = 3 \left(\frac{1}{3} \right) \ln (t^2 - 1) - \ln (t + 1) = \ln \left(\frac{(t+1)(t-1)}{(t+1)} \right) = \ln (t - 1)$
5. (a) $e^{\ln 7.2} = 7.2$ (b) $e^{-\ln x^2} = \frac{1}{e^{\ln x^2}} = \frac{1}{x^2}$ (c) $e^{\ln x - \ln y} = e^{\ln(x/y)} = \frac{x}{y}$
6. (a) $e^{\ln(x^2+y^2)} = x^2 + y^2$ (b) $e^{-\ln 0.3} = \frac{1}{e^{\ln 0.3}} = \frac{1}{0.3}$ (c) $e^{\ln \pi x - \ln 2} = e^{\ln(\pi x/2)} = \frac{\pi x}{2}$
7. (a) $2 \ln \sqrt{e} = 2 \ln e^{1/2} = (2) \left(\frac{1}{2} \right) \ln e = 1$ (b) $\ln (\ln e^e) = \ln (e \ln e) = \ln e = 1$
 (c) $\ln e^{(-x^2-y^2)} = (-x^2 - y^2) \ln e = -x^2 - y^2$

8. (a) $\ln(e^{\sec \theta}) = (\sec \theta)(\ln e) = \sec \theta$ (b) $\ln e^{(e^x)} = (e^x)(\ln e) = e^x$
 (c) $\ln(e^{2 \ln x}) = \ln(e^{\ln x^2}) = \ln x^2 = 2 \ln x$
9. $\ln y = 2t + 4 \Rightarrow e^{\ln y} = e^{2t+4} \Rightarrow y = e^{2t+4}$ 10. $\ln y = -t + 5 \Rightarrow e^{\ln y} = e^{-t+5} \Rightarrow y = e^{-t+5}$
11. $\ln(y - 40) = 5t \Rightarrow e^{\ln(y-40)} = e^{5t} \Rightarrow y - 40 = e^{5t} \Rightarrow y = e^{5t} + 40$
12. $\ln(1 - 2y) = t \Rightarrow e^{\ln(1-2y)} = e^t \Rightarrow 1 - 2y = e^t \Rightarrow -2y = e^t - 1 \Rightarrow y = -\left(\frac{e^t - 1}{2}\right)$
13. $\ln(y - 1) - \ln 2 = x + \ln x \Rightarrow \ln(y - 1) - \ln 2 - \ln x = x \Rightarrow \ln\left(\frac{y-1}{2x}\right) = x \Rightarrow e^{\ln\left(\frac{y-1}{2x}\right)} = e^x \Rightarrow \frac{y-1}{2x} = e^x \Rightarrow y - 1 = 2xe^x \Rightarrow y = 2xe^x + 1$
14. $\ln(y^2 - 1) - \ln(y + 1) = \ln(\sin x) \Rightarrow \ln\left(\frac{y^2-1}{y+1}\right) = \ln(\sin x) \Rightarrow \ln(y - 1) = \ln(\sin x) \Rightarrow e^{\ln(y-1)} = e^{\ln(\sin x)} \Rightarrow y - 1 = \sin x \Rightarrow y = \sin x + 1$
15. (a) $e^{2k} = 4 \Rightarrow \ln e^{2k} = \ln 4 \Rightarrow 2k \ln e = \ln 4 \Rightarrow 2k = \ln 4 \Rightarrow k = \ln 2$
 (b) $100e^{10k} = 200 \Rightarrow e^{10k} = 2 \Rightarrow \ln e^{10k} = \ln 2 \Rightarrow 10k \ln e = \ln 2 \Rightarrow 10k = \ln 2 \Rightarrow k = \frac{\ln 2}{10}$
 (c) $e^{k/1000} = a \Rightarrow \ln e^{k/1000} = \ln a \Rightarrow \frac{k}{1000} \ln e = \ln a \Rightarrow \frac{k}{1000} = \ln a \Rightarrow k = 1000 \ln a$
16. (a) $e^{5k} = \frac{1}{4} \Rightarrow \ln e^{5k} = \ln 4^{-1} \Rightarrow 5k \ln e = -\ln 4 \Rightarrow 5k = -\ln 4 \Rightarrow k = -\frac{\ln 4}{5}$
 (b) $80e^k = 1 \Rightarrow e^k = 80^{-1} \Rightarrow \ln e^k = \ln 80^{-1} \Rightarrow k \ln e = -\ln 80 \Rightarrow k = -\ln 80$
 (c) $e^{(\ln 0.8)k} = 0.8 \Rightarrow (e^{\ln 0.8})^k = 0.8 \Rightarrow (0.8)^k = 0.8 \Rightarrow k = 1$
17. (a) $e^{-0.3t} = 27 \Rightarrow \ln e^{-0.3t} = \ln 3^3 \Rightarrow (-0.3t) \ln e = 3 \ln 3 \Rightarrow -0.3t = 3 \ln 3 \Rightarrow t = -10 \ln 3$
 (b) $e^{kt} = \frac{1}{2} \Rightarrow \ln e^{kt} = \ln 2^{-1} = kt \ln e = -\ln 2 \Rightarrow t = -\frac{\ln 2}{k}$
 (c) $e^{(\ln 0.2)t} = 0.4 \Rightarrow (e^{\ln 0.2})^t = 0.4 \Rightarrow 0.2^t = 0.4 \Rightarrow \ln 0.2^t = \ln 0.4 \Rightarrow t \ln 0.2 = \ln 0.4 \Rightarrow t = \frac{\ln 0.4}{\ln 0.2}$
18. (a) $e^{-0.01t} = 1000 \Rightarrow \ln e^{-0.01t} = \ln 1000 \Rightarrow (-0.01t) \ln e = \ln 1000 \Rightarrow -0.01t = \ln 1000 \Rightarrow t = -100 \ln 1000$
 (b) $e^{kt} = \frac{1}{10} \Rightarrow \ln e^{kt} = \ln 10^{-1} = kt \ln e = -\ln 10 \Rightarrow kt = -\ln 10 \Rightarrow t = -\frac{\ln 10}{k}$
 (c) $e^{(\ln 2)t} = \frac{1}{2} \Rightarrow (e^{\ln 2})^t = 2^{-1} \Rightarrow 2^t = 2^{-1} \Rightarrow t = -1$
19. $e^{\sqrt{t}} = x^2 \Rightarrow \ln e^{\sqrt{t}} = \ln x^2 \Rightarrow \sqrt{t} = 2 \ln x \Rightarrow t = 4(\ln x)^2$
20. $e^{x^2} e^{2x+1} = e^t \Rightarrow e^{x^2+2x+1} = e^t \Rightarrow \ln e^{x^2+2x+1} = \ln e^t \Rightarrow t = x^2 + 2x + 1$
21. (a) $5^{\log_5 7} = 7$ (b) $8^{\log_8 \sqrt{2}} = \sqrt{2}$ (c) $1.3^{\log_{1.3} 75} = 75$
 (d) $\log_4 16 = \log_4 4^2 = 2 \log_4 4 = 2 \cdot 1 = 2$ (e) $\log_3 \sqrt{3} = \log_3 3^{1/2} = \frac{1}{2} \log_3 3 = \frac{1}{2} \cdot 1 = \frac{1}{2} = 0.5$
 (f) $\log_4 \left(\frac{1}{4}\right) = \log_4 4^{-1} = -1 \log_4 4 = -1 \cdot 1 = -1$
22. (a) $2^{\log_2 3} = 3$ (b) $10^{\log_{10} (1/2)} = \frac{1}{2}$ (c) $\pi^{\log_\pi 7} = 7$
 (d) $\log_{11} 121 = \log_{11} 11^2 = 2 \log_{11} 11 = 2 \cdot 1 = 2$
 (e) $\log_{121} 11 = \log_{121} 121^{1/2} = \left(\frac{1}{2}\right) \log_{121} 121 = \left(\frac{1}{2}\right) \cdot 1 = \frac{1}{2}$
 (f) $\log_3 \left(\frac{1}{9}\right) = \log_3 3^{-2} = -2 \log_3 3 = -2 \cdot 1 = -2$

23. (a) Let $z = \log_4 x \Rightarrow 4^z = x \Rightarrow 2^{2z} = x \Rightarrow (2^z)^2 = x \Rightarrow 2^z = \sqrt{x}$
 (b) Let $z = \log_3 x \Rightarrow 3^z = x \Rightarrow (3^z)^2 = x^2 \Rightarrow 3^{2z} = x^2 \Rightarrow 9^z = x^2$
 (c) $\log_2 (e^{(\ln 2) \sin x}) = \log_2 2^{\sin x} = \sin x$

24. (a) Let $z = \log_5 (3x^2) \Rightarrow 5^z = 3x^2 \Rightarrow 25^z = 9x^4$
 (b) $\log_e (e^x) = x$
 (c) $\log_4 (2^{e^x \sin x}) = \log_4 4^{(e^x \sin x)/2} = \frac{e^x \sin x}{2}$

25. (a) $\frac{\log_2 x}{\log_3 x} = \frac{\ln x}{\ln 2} \div \frac{\ln x}{\ln 3} = \frac{\ln x}{\ln 2} \cdot \frac{\ln 3}{\ln x} = \frac{\ln 3}{\ln 2}$ (b) $\frac{\log_2 x}{\log_8 x} = \frac{\ln x}{\ln 2} \div \frac{\ln x}{\ln 8} = \frac{\ln x}{\ln 2} \cdot \frac{\ln 8}{\ln x} = \frac{3 \ln 2}{\ln 2} = 3$
 (c) $\frac{\log_x a}{\log_{x^2} a} = \frac{\ln a}{\ln x} \div \frac{\ln a}{\ln x^2} = \frac{\ln a}{\ln x} \cdot \frac{\ln x^2}{\ln a} = \frac{2 \ln x}{\ln x} = 2$

26. (a) $\frac{\log_9 x}{\log_3 x} = \frac{\ln x}{\ln 9} \div \frac{\ln x}{\ln 3} = \frac{\ln x}{2 \ln 3} \cdot \frac{\ln 3}{\ln x} = \frac{1}{2}$
 (b) $\frac{\log_{\sqrt{10}} x}{\log_{\sqrt{2}} x} = \frac{\ln x}{\ln \sqrt{10}} \div \frac{\ln x}{\ln \sqrt{2}} = \frac{\ln x}{(\frac{1}{2}) \ln 10} \cdot \frac{(\frac{1}{2}) \ln 2}{\ln x} = \frac{\ln 2}{\ln 10}$
 (c) $\frac{\log_a b}{\log_b a} = \frac{\ln b}{\ln a} \div \frac{\ln a}{\ln b} = \frac{\ln b}{\ln a} \cdot \frac{\ln b}{\ln a} = \left(\frac{\ln b}{\ln a}\right)^2$

27. $y = \ln 3x \Rightarrow y' = \left(\frac{1}{3x}\right)(3) = \frac{1}{x}$ 28. $y = \ln kx \Rightarrow y' = \left(\frac{1}{kx}\right)(k) = \frac{1}{x}$

29. $y = \ln(t^2) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^2}\right)(2t) = \frac{2}{t}$ 30. $y = \ln(t^{3/2}) \Rightarrow \frac{dy}{dt} = \left(\frac{1}{t^{3/2}}\right)\left(\frac{3}{2}t^{1/2}\right) = \frac{3}{2t}$

31. $y = \ln \frac{3}{x} = \ln 3x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{3x^{-1}}\right)(-3x^{-2}) = -\frac{1}{x}$

32. $y = \ln \frac{10}{x} = \ln 10x^{-1} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{10x^{-1}}\right)(-10x^{-2}) = -\frac{1}{x}$

33. $y = \ln(\theta + 1) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\theta + 1}\right)(1) = \frac{1}{\theta + 1}$ 34. $y = \ln(2\theta + 2) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{2\theta + 2}\right)(2) = \frac{1}{\theta + 1}$

35. $y = \ln x^3 \Rightarrow \frac{dy}{dx} = \left(\frac{1}{x^3}\right)(3x^2) = \frac{3}{x}$ 36. $y = (\ln x)^3 \Rightarrow \frac{dy}{dx} = 3(\ln x)^2 \cdot \frac{d}{dx}(\ln x) = \frac{3(\ln x)^2}{x}$

37. $y = t(\ln t)^2 \Rightarrow \frac{dy}{dt} = (\ln t)^2 + 2t(\ln t) \cdot \frac{d}{dt}(\ln t) = (\ln t)^2 + \frac{2t \ln t}{t} = (\ln t)^2 + 2 \ln t$

38. $y = t\sqrt{\ln t} = t(\ln t)^{1/2} \Rightarrow \frac{dy}{dt} = (\ln t)^{1/2} + \frac{1}{2}t(\ln t)^{-1/2} \cdot \frac{d}{dt}(\ln t) = (\ln t)^{1/2} + \frac{t(\ln t)^{-1/2}}{2t} = (\ln t)^{1/2} + \frac{1}{2(\ln t)^{1/2}}$

39. $y = \frac{x^4}{4} \ln x - \frac{x^4}{16} \Rightarrow \frac{dy}{dx} = x^3 \ln x + \frac{x^4}{4} \cdot \frac{1}{x} - \frac{4x^3}{16} = x^3 \ln x$

40. $y = \frac{x^3}{3} \ln x - \frac{x^3}{9} \Rightarrow \frac{dy}{dx} = x^2 \ln x + \frac{x^3}{3} \cdot \frac{1}{x} - \frac{3x^2}{9} = x^2 \ln x$

41. $y = \frac{\ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (\ln t)(1)}{t^2} = \frac{1 - \ln t}{t^2}$

42. $y = \frac{1 + \ln t}{t} \Rightarrow \frac{dy}{dt} = \frac{t\left(\frac{1}{t}\right) - (1 + \ln t)(1)}{t^2} = \frac{1 - 1 - \ln t}{t^2} = -\frac{\ln t}{t^2}$

43. $y = \frac{\ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\frac{1}{x}\right) - (\ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{\frac{1}{x} + \frac{\ln x}{x} - \frac{\ln x}{x}}{(1 + \ln x)^2} = \frac{1}{x(1 + \ln x)^2}$

44. $y = \frac{x \ln x}{1 + \ln x} \Rightarrow y' = \frac{(1 + \ln x)\left(\ln x + x \cdot \frac{1}{x}\right) - (x \ln x)\left(\frac{1}{x}\right)}{(1 + \ln x)^2} = \frac{(1 + \ln x)^2 - \ln x}{(1 + \ln x)^2} = 1 - \frac{\ln x}{(1 + \ln x)^2}$

$$45. y = \ln(\ln x) \Rightarrow y' = \left(\frac{1}{\ln x}\right) \left(\frac{1}{x}\right) = \frac{1}{x \ln x}$$

$$46. y = \ln(\ln(\ln x)) \Rightarrow y' = \frac{1}{\ln(\ln x)} \cdot \frac{d}{dx}(\ln(\ln x)) = \frac{1}{\ln(\ln x)} \cdot \frac{1}{\ln x} \cdot \frac{d}{dx}(\ln x) = \frac{1}{x(\ln x)\ln(\ln x)}$$

$$47. y = \theta[\sin(\ln \theta) + \cos(\ln \theta)] \Rightarrow \frac{dy}{d\theta} = [\sin(\ln \theta) + \cos(\ln \theta)] + \theta \left[\cos(\ln \theta) \cdot \frac{1}{\theta} - \sin(\ln \theta) \cdot \frac{1}{\theta} \right] \\ = \sin(\ln \theta) + \cos(\ln \theta) + \cos(\ln \theta) - \sin(\ln \theta) = 2 \cos(\ln \theta)$$

$$48. y = \ln(\sec \theta + \tan \theta) \Rightarrow \frac{dy}{d\theta} = \frac{\sec \theta \tan \theta + \sec^2 \theta}{\sec \theta + \tan \theta} = \frac{\sec \theta (\tan \theta + \sec \theta)}{\tan \theta + \sec \theta} = \sec \theta$$

$$49. y = \ln \frac{1}{x\sqrt{x+1}} = -\ln x - \frac{1}{2} \ln(x+1) \Rightarrow y' = -\frac{1}{x} - \frac{1}{2} \left(\frac{1}{x+1}\right) = -\frac{2(x+1)+x}{2x(x+1)} = -\frac{3x+2}{2x(x+1)}$$

$$50. y = \frac{1}{2} \ln \frac{1+x}{1-x} = \frac{1}{2} [\ln(1+x) - \ln(1-x)] \Rightarrow y' = \frac{1}{2} \left[\frac{1}{1+x} - \left(\frac{1}{1-x}\right)(-1) \right] = \frac{1}{2} \left[\frac{1-x+1+x}{(1+x)(1-x)} \right] = \frac{1}{1-x^2}$$

$$51. y = \frac{1+\ln t}{1-\ln t} \Rightarrow \frac{dy}{dt} = \frac{(1-\ln t)\left(\frac{1}{t}\right) - (1+\ln t)\left(\frac{-1}{t}\right)}{(1-\ln t)^2} = \frac{\frac{1}{t} - \frac{\ln t}{t} + \frac{1}{t} + \frac{\ln t}{t}}{(1-\ln t)^2} = \frac{2}{t(1-\ln t)^2}$$

$$52. y = \sqrt{\ln \sqrt{t}} = (\ln t^{1/2})^{1/2} \Rightarrow \frac{dy}{dt} = \frac{1}{2} (\ln t^{1/2})^{-1/2} \cdot \frac{d}{dt}(\ln t^{1/2}) = \frac{1}{2} (\ln t^{1/2})^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{d}{dt}(t^{1/2}) \\ = \frac{1}{2} (\ln t^{1/2})^{-1/2} \cdot \frac{1}{t^{1/2}} \cdot \frac{1}{2} t^{-1/2} = \frac{1}{4t\sqrt{\ln \sqrt{t}}}$$

$$53. y = \ln(\sec(\ln \theta)) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\sec(\ln \theta)} \cdot \frac{d}{d\theta}(\sec(\ln \theta)) = \frac{\sec(\ln \theta) \tan(\ln \theta)}{\sec(\ln \theta)} \cdot \frac{d}{d\theta}(\ln \theta) = \frac{\tan(\ln \theta)}{\theta}$$

$$54. y = \ln \frac{\sqrt{\sin \theta \cos \theta}}{1+2 \ln \theta} = \frac{1}{2} (\ln \sin \theta + \ln \cos \theta) - \ln(1+2 \ln \theta) \Rightarrow \frac{dy}{d\theta} = \frac{1}{2} \left(\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} \right) - \frac{\frac{2}{\theta}}{1+2 \ln \theta} \\ = \frac{1}{2} \left[\cot \theta - \tan \theta - \frac{4}{\theta(1+2 \ln \theta)} \right]$$

$$55. y = \ln \left(\frac{x^2+1}{\sqrt{1-x}} \right) = 5 \ln(x^2+1) - \frac{1}{2} \ln(1-x) \Rightarrow y' = \frac{5 \cdot 2x}{x^2+1} - \frac{1}{2} \left(\frac{1}{1-x} \right) (-1) = \frac{10x}{x^2+1} + \frac{1}{2(1-x)}$$

$$56. y = \ln \sqrt{\frac{(x+1)^5}{(x+2)^3}} = \frac{1}{2} [5 \ln(x+1) - 3 \ln(x+2)] \Rightarrow y' = \frac{1}{2} \left(\frac{5}{x+1} - \frac{3}{x+2} \right) = \frac{5}{2} \left[\frac{(x+2)-4(x+1)}{(x+1)(x+2)} \right] \\ = -\frac{5}{2} \left[\frac{3x+2}{(x+1)(x+2)} \right]$$

$$57. y = \sqrt{x(x+1)} = (x(x+1))^{1/2} \Rightarrow \ln y = \frac{1}{2} \ln(x(x+1)) \Rightarrow 2 \ln y = \ln(x) + \ln(x+1) \Rightarrow \frac{2y'}{y} = \frac{1}{x} + \frac{1}{x+1} \\ \Rightarrow y' = \left(\frac{1}{2}\right) \sqrt{x(x+1)} \left(\frac{1}{x} + \frac{1}{x+1}\right) = \frac{\sqrt{x(x+1)}(2x+1)}{2x(x+1)} = \frac{2x+1}{2\sqrt{x(x+1)}}$$

$$58. y = \sqrt{(x^2+1)(x-1)^2} \Rightarrow \ln y = \frac{1}{2} [\ln(x^2+1) + 2 \ln(x-1)] \Rightarrow \frac{y'}{y} = \frac{1}{2} \left(\frac{2x}{x^2+1} + \frac{2}{x-1} \right) \\ \Rightarrow y' = \sqrt{(x^2+1)(x-1)^2} \left(\frac{x}{x^2+1} + \frac{1}{x-1} \right) = \sqrt{(x^2+1)(x-1)^2} \left[\frac{x^2-x+x^2+1}{(x^2+1)(x-1)} \right] = \frac{(2x^2-x+1)|x-1|}{\sqrt{x^2+1}(x-1)}$$

$$59. y = \sqrt{\frac{t}{t+1}} = \left(\frac{t}{t+1}\right)^{1/2} \Rightarrow \ln y = \frac{1}{2} [\ln t - \ln(t+1)] \Rightarrow \frac{1}{y} \frac{dy}{dt} = \frac{1}{2} \left(\frac{1}{t} - \frac{1}{t+1} \right) \\ \Rightarrow \frac{dy}{dt} = \frac{1}{2} \sqrt{\frac{t}{t+1}} \left(\frac{1}{t} - \frac{1}{t+1} \right) = \frac{1}{2} \sqrt{\frac{t}{t+1}} \left[\frac{1}{t(t+1)} \right] = \frac{1}{2\sqrt{t(t+1)^{3/2}}}$$

$$60. y = \sqrt{\frac{1}{t(t+1)}} = [t(t+1)]^{-1/2} \Rightarrow \ln y = \frac{1}{2} [\ln t + \ln(t+1)] \Rightarrow \frac{1}{y} \frac{dy}{dt} = -\frac{1}{2} \left(\frac{1}{t} + \frac{1}{t+1} \right) \\ \Rightarrow \frac{dy}{dt} = -\frac{1}{2} \sqrt{\frac{1}{t(t+1)}} \left[\frac{2t+1}{t(t+1)} \right] = -\frac{2t+1}{2(t^2+t)^{3/2}}$$

$$61. y = \sqrt{\theta + 3} (\sin \theta) = (\theta + 3)^{1/2} \sin \theta \Rightarrow \ln y = \frac{1}{2} \ln(\theta + 3) + \ln(\sin \theta) \Rightarrow \frac{1}{y} \frac{dy}{d\theta} = \frac{1}{2(\theta + 3)} + \frac{\cos \theta}{\sin \theta}$$

$$\Rightarrow \frac{dy}{d\theta} = \sqrt{\theta + 3} (\sin \theta) \left[\frac{1}{2(\theta + 3)} + \cot \theta \right]$$

$$62. y = (\tan \theta) \sqrt{2\theta + 1} = (\tan \theta)(2\theta + 1)^{1/2} \Rightarrow \ln y = \ln(\tan \theta) + \frac{1}{2} \ln(2\theta + 1) \Rightarrow \frac{1}{y} \frac{dy}{d\theta} = \frac{\sec^2 \theta}{\tan \theta} + \left(\frac{1}{2}\right) \left(\frac{2}{2\theta + 1}\right)$$

$$\Rightarrow \frac{dy}{d\theta} = (\tan \theta) \sqrt{2\theta + 1} \left(\frac{\sec^2 \theta}{\tan \theta} + \frac{1}{2\theta + 1} \right) = (\sec^2 \theta) \sqrt{2\theta + 1} + \frac{\tan \theta}{\sqrt{2\theta + 1}}$$

$$63. y = t(t+1)(t+2) \Rightarrow \ln y = \ln t + \ln(t+1) + \ln(t+2) \Rightarrow \frac{1}{y} \frac{dy}{dt} = \frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2}$$

$$\Rightarrow \frac{dy}{dt} = t(t+1)(t+2) \left(\frac{1}{t} + \frac{1}{t+1} + \frac{1}{t+2} \right) = t(t+1)(t+2) \left[\frac{(t+1)(t+2) + t(t+2) + t(t+1)}{t(t+1)(t+2)} \right] = 3t^2 + 6t + 2$$

$$64. y = \frac{1}{t(t+1)(t+2)} \Rightarrow \ln y = \ln 1 - \ln t - \ln(t+1) - \ln(t+2) \Rightarrow \frac{1}{y} \frac{dy}{dt} = -\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2}$$

$$\Rightarrow \frac{dy}{dt} = \frac{1}{t(t+1)(t+2)} \left[-\frac{1}{t} - \frac{1}{t+1} - \frac{1}{t+2} \right] = \frac{-1}{t(t+1)(t+2)} \left[\frac{(t+1)(t+2) + t(t+2) + t(t+1)}{t(t+1)(t+2)} \right]$$

$$= -\frac{3t^2 + 6t + 2}{(t^3 + 3t^2 + 2t)^2}$$

$$65. y = \frac{\theta + 5}{\theta \cos \theta} \Rightarrow \ln y = \ln(\theta + 5) - \ln \theta - \ln(\cos \theta) \Rightarrow \frac{1}{y} \frac{dy}{d\theta} = \frac{1}{\theta + 5} - \frac{1}{\theta} + \frac{\sin \theta}{\cos \theta}$$

$$\Rightarrow \frac{dy}{d\theta} = \left(\frac{\theta + 5}{\theta \cos \theta} \right) \left(\frac{1}{\theta + 5} - \frac{1}{\theta} + \tan \theta \right)$$

$$66. y = \frac{\theta \sin \theta}{\sqrt{\sec \theta}} \Rightarrow \ln y = \ln \theta + \ln(\sin \theta) - \frac{1}{2} \ln(\sec \theta) \Rightarrow \frac{1}{y} \frac{dy}{d\theta} = \left[\frac{1}{\theta} + \frac{\cos \theta}{\sin \theta} - \frac{(\sec \theta)(\tan \theta)}{2 \sec \theta} \right]$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{\theta \sin \theta}{\sqrt{\sec \theta}} \left(\frac{1}{\theta} + \cot \theta - \frac{1}{2} \tan \theta \right)$$

$$67. y = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} \Rightarrow \ln y = \ln x + \frac{1}{2} \ln(x^2+1) - \frac{2}{3} \ln(x+1) \Rightarrow \frac{y'}{y} = \frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)}$$

$$\Rightarrow y' = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}} \left[\frac{1}{x} + \frac{x}{x^2+1} - \frac{2}{3(x+1)} \right]$$

$$68. y = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}} \Rightarrow \ln y = \frac{1}{2} [10 \ln(x+1) - 5 \ln(2x+1)] \Rightarrow \frac{y'}{y} = \frac{5}{x+1} - \frac{5}{2x+1}$$

$$\Rightarrow y' = \sqrt{\frac{(x+1)^{10}}{(2x+1)^5}} \left(\frac{5}{x+1} - \frac{5}{2x+1} \right)$$

$$69. y = \sqrt[3]{\frac{x(x-2)}{x^2+1}} \Rightarrow \ln y = \frac{1}{3} [\ln x + \ln(x-2) - \ln(x^2+1)] \Rightarrow \frac{y'}{y} = \frac{1}{3} \left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1} \right)$$

$$\Rightarrow y' = \frac{1}{3} \sqrt[3]{\frac{x(x-2)}{x^2+1}} \left(\frac{1}{x} + \frac{1}{x-2} - \frac{2x}{x^2+1} \right)$$

$$70. y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \Rightarrow \ln y = \frac{1}{3} [\ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3)]$$

$$\Rightarrow y' = \frac{1}{3} \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}} \left(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3} \right)$$

$$71. y = \ln(\cos^2 \theta) \Rightarrow \frac{dy}{d\theta} = \frac{1}{\cos^2 \theta} \cdot 2 \cos \theta \cdot (-\sin \theta) = -2 \tan \theta$$

$$72. y = \ln(3\theta e^{-\theta}) = \ln 3 + \ln \theta + \ln e^{-\theta} = \ln 3 + \ln \theta - \theta \Rightarrow \frac{dy}{d\theta} = \frac{1}{\theta} - 1$$

$$73. y = \ln(3te^{-t}) = \ln 3 + \ln t + \ln e^{-t} = \ln 3 + \ln t - t \Rightarrow \frac{dy}{dt} = \frac{1}{t} - 1 = \frac{1-t}{t}$$

$$74. y = \ln(2e^{-t} \sin t) = \ln 2 + \ln e^{-t} + \ln \sin t = \ln 2 - t + \ln \sin t \Rightarrow \frac{dy}{dt} = -1 + \left(\frac{1}{\sin t} \right) \frac{d}{dt}(\sin t) = -1 + \frac{\cos t}{\sin t}$$

$$= \frac{\cos t - \sin t}{\sin t}$$

$$75. y = \ln \frac{e^\theta}{1+e^\theta} = \ln e^\theta - \ln(1+e^\theta) = \theta - \ln(1+e^\theta) \Rightarrow \frac{dy}{d\theta} = 1 - \left(\frac{1}{1+e^\theta}\right) \frac{d}{d\theta}(1+e^\theta) = 1 - \frac{e^\theta}{1+e^\theta} = \frac{1}{1+e^\theta}$$

$$76. y = \ln \frac{\sqrt{\theta}}{1+\sqrt{\theta}} = \ln \sqrt{\theta} - \ln(1+\sqrt{\theta}) \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\sqrt{\theta}}\right) \frac{d}{d\theta}(\sqrt{\theta}) - \left(\frac{1}{1+\sqrt{\theta}}\right) \frac{d}{d\theta}(1+\sqrt{\theta})$$

$$= \left(\frac{1}{\sqrt{\theta}}\right) \left(\frac{1}{2\sqrt{\theta}}\right) - \left(\frac{1}{1+\sqrt{\theta}}\right) \left(\frac{1}{2\sqrt{\theta}}\right) = \frac{(1+\sqrt{\theta}) - \sqrt{\theta}}{2\theta(1+\sqrt{\theta})} = \frac{1}{2\theta(1+\sqrt{\theta})} = \frac{1}{2\theta(1+\theta^{1/2})}$$

$$77. y = e^{(\cos t + \ln t)} = e^{\cos t} e^{\ln t} = te^{\cos t} \Rightarrow \frac{dy}{dt} = e^{\cos t} + te^{\cos t} \frac{d}{dt}(\cos t) = (1 - t \sin t) e^{\cos t}$$

$$78. y = e^{\sin t} (\ln t^2 + 1) \Rightarrow \frac{dy}{dt} = e^{\sin t} (\cos t) (\ln t^2 + 1) + \frac{2}{t} e^{\sin t} = e^{\sin t} [(\ln t^2 + 1) (\cos t) + \frac{2}{t}]$$

$$79. \ln y = e^y \sin x \Rightarrow \left(\frac{1}{y}\right) y' = (y' e^y) (\sin x) + e^y \cos x \Rightarrow y' \left(\frac{1}{y} - e^y \sin x\right) = e^y \cos x$$

$$\Rightarrow y' \left(\frac{1 - ye^y \sin x}{y}\right) = e^y \cos x \Rightarrow y' = \frac{ye^y \cos x}{1 - ye^y \sin x}$$

$$80. \ln xy = e^{x+y} \Rightarrow \ln x + \ln y = e^{x+y} \Rightarrow \frac{1}{x} + \left(\frac{1}{y}\right) y' = (1 + y') e^{x+y} \Rightarrow y' \left(\frac{1}{y} - e^{x+y}\right) = e^{x+y} - \frac{1}{x}$$

$$\Rightarrow y' \left(\frac{1 - ye^{x+y}}{y}\right) = \frac{xe^{x+y} - 1}{x} \Rightarrow y' = \frac{y(xe^{x+y} - 1)}{x(1 - ye^{x+y})}$$

$$81. x^y = y^x \Rightarrow \ln x^y = \ln y^x \Rightarrow y \ln x = x \ln y \Rightarrow y \cdot \frac{1}{x} + y' \cdot \ln x = x \cdot \frac{1}{y} \cdot y' + (1) \cdot \ln y \Rightarrow \ln x \cdot y' - \frac{x}{y} \cdot y' = \ln y - \frac{y}{x}$$

$$\Rightarrow y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}} = \frac{xy \ln y - y^2}{xy \ln x - x^2} = \frac{y}{x} \left(\frac{x \ln y - y}{y \ln x - x}\right)$$

$$82. \tan y = e^x + \ln x \Rightarrow (\sec^2 y) y' = e^x + \frac{1}{x} \Rightarrow y' = \frac{(xe^x + 1) \cos^2 y}{x}$$

$$83. y = 2^x \Rightarrow y' = 2^x \ln 2 \qquad 84. y = 3^{-x} \Rightarrow y' = 3^{-x} (\ln 3)(-1) = -3^{-x} \ln 3$$

$$85. y = 5^{\sqrt{s}} \Rightarrow \frac{dy}{ds} = 5^{\sqrt{s}} (\ln 5) \left(\frac{1}{2} s^{-1/2}\right) = \left(\frac{\ln 5}{2\sqrt{s}}\right) 5^{\sqrt{s}}$$

$$86. y = 2^{s^2} \Rightarrow \frac{dy}{ds} = 2^{s^2} (\ln 2) 2s = (\ln 2^2) (s 2^{s^2}) = (\ln 4) s 2^{s^2}$$

$$87. y = x^\pi \Rightarrow y' = \pi x^{(\pi-1)} \qquad 88. y = t^{1-e} \Rightarrow \frac{dy}{dt} = (1-e) t^{-e}$$

$$89. y = \log_2 5\theta = \frac{\ln 5\theta}{\ln 2} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 2}\right) \left(\frac{1}{5\theta}\right) (5) = \frac{1}{\theta \ln 2}$$

$$90. y = \log_3 (1 + \theta \ln 3) = \frac{\ln(1 + \theta \ln 3)}{\ln 3} \Rightarrow \frac{dy}{d\theta} = \left(\frac{1}{\ln 3}\right) \left(\frac{1}{1 + \theta \ln 3}\right) (\ln 3) = \frac{1}{1 + \theta \ln 3}$$

$$91. y = \log_4 x + \log_4 x^2 = \frac{\ln x}{\ln 4} + \frac{\ln x^2}{\ln 4} = \frac{\ln x}{\ln 4} + 2 \frac{\ln x}{\ln 4} = 3 \frac{\ln x}{\ln 4} \Rightarrow y' = \frac{3}{x \ln 4}$$

$$92. y = \log_{25} e^x - \log_5 \sqrt{x} = \frac{x \ln e}{\ln 25} - \frac{\ln x}{2 \ln 5} = \frac{x}{2 \ln 5} - \frac{\ln x}{2 \ln 5} = \left(\frac{1}{2 \ln 5}\right) (x - \ln x) \Rightarrow y' = \left(\frac{1}{2 \ln 5}\right) \left(1 - \frac{1}{x}\right) = \frac{x-1}{2x \ln 5}$$

$$93. y = \log_2 r \cdot \log_4 r = \left(\frac{\ln r}{\ln 2}\right) \left(\frac{\ln r}{\ln 4}\right) = \frac{\ln^2 r}{(\ln 2)(\ln 4)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 2)(\ln 4)}\right] (2 \ln r) \left(\frac{1}{r}\right) = \frac{2 \ln r}{r(\ln 2)(\ln 4)}$$

$$94. y = \log_3 r \cdot \log_9 r = \left(\frac{\ln r}{\ln 3}\right) \left(\frac{\ln r}{\ln 9}\right) = \frac{\ln^2 r}{(\ln 3)(\ln 9)} \Rightarrow \frac{dy}{dr} = \left[\frac{1}{(\ln 3)(\ln 9)}\right] (2 \ln r) \left(\frac{1}{r}\right) = \frac{2 \ln r}{r(\ln 3)(\ln 9)}$$

$$95. y = \log_3 \left(\left(\frac{x+1}{x-1} \right)^{\ln 3} \right) = \frac{\ln \left(\frac{x+1}{x-1} \right)^{\ln 3}}{\ln 3} = \frac{(\ln 3) \ln \left(\frac{x+1}{x-1} \right)}{\ln 3} = \ln \left(\frac{x+1}{x-1} \right) = \ln(x+1) - \ln(x-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x+1} - \frac{1}{x-1} = \frac{-2}{(x+1)(x-1)}$$

$$96. y = \log_5 \sqrt{\left(\frac{7x}{3x+2} \right)^{\ln 5}} = \log_5 \left(\frac{7x}{3x+2} \right)^{(\ln 5)/2} = \frac{\ln \left(\frac{7x}{3x+2} \right)^{(\ln 5)/2}}{\ln 5} = \left(\frac{\ln 5}{2} \right) \left[\frac{\ln \left(\frac{7x}{3x+2} \right)}{\ln 5} \right] = \frac{1}{2} \ln \left(\frac{7x}{3x+2} \right)$$

$$= \frac{1}{2} \ln 7x - \frac{1}{2} \ln(3x+2) \Rightarrow \frac{dy}{dx} = \frac{7}{2 \cdot 7x} - \frac{3}{2 \cdot (3x+2)} = \frac{(3x+2) - 3x}{2x(3x+2)} = \frac{1}{x(3x+2)}$$

$$97. y = \theta \sin(\log_7 \theta) = \theta \sin \left(\frac{\ln \theta}{\ln 7} \right) \Rightarrow \frac{dy}{d\theta} = \sin \left(\frac{\ln \theta}{\ln 7} \right) + \theta \left[\cos \left(\frac{\ln \theta}{\ln 7} \right) \right] \left(\frac{1}{\theta \ln 7} \right) = \sin(\log_7 \theta) + \frac{1}{\ln 7} \cos(\log_7 \theta)$$

$$98. y = \log_7 \left(\frac{\sin \theta \cos \theta}{e^\theta 2^\theta} \right) = \frac{\ln(\sin \theta) + \ln(\cos \theta) - \ln e^\theta - \ln 2^\theta}{\ln 7} = \frac{\ln(\sin \theta) + \ln(\cos \theta) - \theta - \theta \ln 2}{\ln 7}$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{\cos \theta}{(\sin \theta)(\ln 7)} - \frac{\sin \theta}{(\cos \theta)(\ln 7)} - \frac{1}{\ln 7} - \frac{\ln 2}{\ln 7} = \left(\frac{1}{\ln 7} \right) (\cot \theta - \tan \theta - 1 - \ln 2)$$

$$99. y = \log_5 e^x = \frac{\ln e^x}{\ln 5} = \frac{x}{\ln 5} \Rightarrow y' = \frac{1}{\ln 5}$$

$$100. y = \log_2 \left(\frac{x^2 e^2}{2\sqrt{x+1}} \right) = \frac{\ln x^2 + \ln e^2 - \ln 2 - \ln \sqrt{x+1}}{\ln 2} = \frac{2 \ln x + 2 - \ln 2 - \frac{1}{2} \ln(x+1)}{\ln 2}$$

$$\Rightarrow y' = \frac{2}{x \ln 2} - \frac{1}{2(\ln 2)(x+1)} = \frac{4(x+1) - x}{2x(x+1)(\ln 2)} = \frac{3x+4}{2x(x+1) \ln 2}$$

$$101. y = 3^{\log_2 t} = 3^{(\ln t)/(\ln 2)} \Rightarrow \frac{dy}{dt} = [3^{(\ln t)/(\ln 2)} (\ln 3)] \left(\frac{1}{t \ln 2} \right) = \frac{1}{t} (\log_2 3) 3^{\log_2 t}$$

$$102. y = 3 \log_8 (\log_2 t) = \frac{3 \ln(\log_2 t)}{\ln 8} = \frac{3 \ln \left(\frac{\ln t}{\ln 2} \right)}{\ln 8} \Rightarrow \frac{dy}{dt} = \left(\frac{3}{\ln 8} \right) \left[\frac{1}{(\ln t)(\ln 2)} \right] \left(\frac{1}{t \ln 2} \right) = \frac{3}{t(\ln t)(\ln 8)} = \frac{1}{t(\ln t)(\ln 2)}$$

$$103. y = \log_2 (8t^{\ln 2}) = \frac{\ln 8 + \ln(t^{\ln 2})}{\ln 2} = \frac{3 \ln 2 + (\ln 2)(\ln t)}{\ln 2} = 3 + \ln t \Rightarrow \frac{dy}{dt} = \frac{1}{t}$$

$$104. y = t \log_3 (e^{(\sin t)(\ln 3)}) = \frac{t \ln \left((e^{\ln 3})^{\sin t} \right)}{\ln 3} = \frac{t \ln (3^{\sin t})}{\ln 3} = \frac{t(\sin t)(\ln 3)}{\ln 3} = t \sin t \Rightarrow \frac{dy}{dt} = \sin t + t \cos t$$

$$105. y = (x+1)^x \Rightarrow \ln y = \ln(x+1)^x = x \ln(x+1) \Rightarrow \frac{y'}{y} = \ln(x+1) + x \cdot \frac{1}{(x+1)} \Rightarrow y' = (x+1)^x \left[\frac{x}{x+1} + \ln(x+1) \right]$$

$$106. y = x^{(x+1)} \Rightarrow \ln y = \ln x^{(x+1)} = (x+1) \ln x \Rightarrow \frac{y'}{y} = \ln x + (x+1) \left(\frac{1}{x} \right) = \ln x + 1 + \frac{1}{x} \Rightarrow y' = x^{(x+1)} \left(1 + \frac{1}{x} + \ln x \right)$$

$$107. y = (\sqrt{t})^t = (t^{1/2})^t = t^{t/2} \Rightarrow \ln y = \ln t^{t/2} = \left(\frac{t}{2} \right) \ln t \Rightarrow \frac{1}{y} \frac{dy}{dt} = \left(\frac{1}{2} \right) (\ln t) + \left(\frac{t}{2} \right) \left(\frac{1}{t} \right) = \frac{\ln t}{2} + \frac{1}{2}$$

$$\Rightarrow \frac{dy}{dt} = (\sqrt{t})^t \left(\frac{\ln t}{2} + \frac{1}{2} \right)$$

$$108. y = t^{\sqrt{t}} = t^{(t^{1/2})} \Rightarrow \ln y = \ln t^{(t^{1/2})} = (t^{1/2}) (\ln t) \Rightarrow \frac{1}{y} \frac{dy}{dt} = \left(\frac{1}{2} t^{-1/2} \right) (\ln t) + t^{1/2} \left(\frac{1}{t} \right) = \frac{\ln t + 2}{2\sqrt{t}}$$

$$\Rightarrow \frac{dy}{dt} = \left(\frac{\ln t + 2}{2\sqrt{t}} \right) t^{\sqrt{t}}$$

$$109. y = (\sin x)^x \Rightarrow \ln y = \ln(\sin x)^x = x \ln(\sin x) \Rightarrow \frac{y'}{y} = \ln(\sin x) + x \left(\frac{\cos x}{\sin x} \right) \Rightarrow y' = (\sin x)^x [\ln(\sin x) + x \cot x]$$

$$110. y = x^{\sin x} \Rightarrow \ln y = \ln x^{\sin x} = (\sin x)(\ln x) \Rightarrow \frac{y'}{y} = (\cos x)(\ln x) + (\sin x) \left(\frac{1}{x} \right) = \frac{\sin x + x(\ln x)(\cos x)}{x}$$

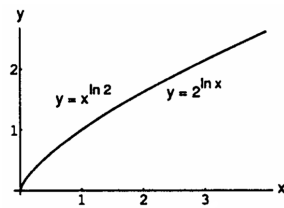
$$\Rightarrow y' = x^{\sin x} \left[\frac{\sin x + x(\ln x)(\cos x)}{x} \right]$$

$$111. y = x^{\ln x}, x > 0 \Rightarrow \ln y = (\ln x)^2 \Rightarrow \frac{y'}{y} = 2(\ln x) \left(\frac{1}{x}\right) \Rightarrow y' = (x^{\ln x}) \left(\frac{\ln x^2}{x}\right)$$

$$112. y = (\ln x)^{\ln x} \Rightarrow \ln y = (\ln x) \ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{x}\right) \ln(\ln x) + (\ln x) \left(\frac{1}{\ln x}\right) \frac{d}{dx}(\ln x) = \frac{\ln(\ln x)}{x} + \frac{1}{x} \\ \Rightarrow y' = \left(\frac{\ln(\ln x) + 1}{x}\right) (\ln x)^{\ln x}$$

113. (a) Begin with $y = \ln x$ and reduce the y -value by 3 $\Rightarrow y = \ln x - 3$.
 (b) Begin with $y = \ln x$ and replace x with $x - 1 \Rightarrow y = \ln(x - 1)$.
 (c) Begin with $y = \ln x$, replace x with $x + 1$, and increase the y -value by 3 $\Rightarrow y = \ln(x + 1) + 3$.
 (d) Begin with $y = \ln x$, reduce the y -value by 4, and replace x with $x - 2 \Rightarrow y = \ln(x - 2) - 4$.
 (e) Begin with $y = \ln x$ and replace x with $-x \Rightarrow y = \ln(-x)$.
 (f) Begin with $y = \ln x$ and switch x and $y \Rightarrow x = \ln y$ or $y = e^x$.

114. The functions $f(x) = x^{\ln 2}$ and $g(x) = 2^{\ln x}$ appear to have identical graphs for $x > 0$. This is no accident, because $x^{\ln 2} = e^{\ln 2 \cdot \ln x} = (e^{\ln 2})^{\ln x} = 2^{\ln x}$.



$$115. y = A \sin(\ln x) + B \cos(\ln x) \Rightarrow y' = A \cos(\ln x) \cdot \frac{1}{x} - B \sin(\ln x) \cdot \frac{1}{x} = (A \cos(\ln x) - B \sin(\ln x)) \cdot \frac{1}{x} \\ \Rightarrow y'' = (A \cos(\ln x) - B \sin(\ln x)) \cdot \frac{-1}{x^2} + (-A \sin(\ln x) \cdot \frac{1}{x} - B \cos(\ln x) \cdot \frac{1}{x}) \cdot \frac{1}{x} \\ = (-A(\cos(\ln x) + \sin(\ln x)) + B(\sin(\ln x) - \cos(\ln x))) \cdot \frac{1}{x^2} \\ \Rightarrow x^2 y'' + x y' + y = (-A(\cos(\ln x) + \sin(\ln x)) + B(\sin(\ln x) - \cos(\ln x))) + (A \cos(\ln x) - B \sin(\ln x)) \\ + (A \sin(\ln x) + B \cos(\ln x)) = 0$$

$$116. \lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{(n/x)}\right)^{(n/x)}\right]^x = e^x \text{ for any } x > 0.$$

$$117. (a) \text{ Amount} = 8\left(\frac{1}{2}\right)^{t/12}$$

$$(b) 8\left(\frac{1}{2}\right)^{t/12} = 1 \rightarrow \left(\frac{1}{2}\right)^{t/12} = \frac{1}{8} \rightarrow \left(\frac{1}{2}\right)^{t/12} = \left(\frac{1}{2}\right)^3 \rightarrow \frac{t}{12} = 3 \rightarrow t = 36$$

There will be 1 gram remaining after 36 hours.

$$118. y = y_0 e^{-0.18t} \text{ represents the decay equation; solving } (0.9)y_0 = y_0 e^{-0.18t} \Rightarrow t = \frac{\ln(0.9)}{-0.18} \approx 0.585 \text{ days}$$

2.11 INVERSE TRIGONOMETRIC FUNCTIONS

- | | | | | | |
|-------------------------|----------------------|----------------------|--------------------------|----------------------|----------------------|
| 1. (a) $\frac{\pi}{4}$ | (b) $-\frac{\pi}{3}$ | (c) $\frac{\pi}{6}$ | 2. (a) $-\frac{\pi}{4}$ | (b) $\frac{\pi}{3}$ | (c) $-\frac{\pi}{6}$ |
| 3. (a) $-\frac{\pi}{6}$ | (b) $\frac{\pi}{4}$ | (c) $-\frac{\pi}{3}$ | 4. (a) $\frac{\pi}{6}$ | (b) $-\frac{\pi}{4}$ | (c) $\frac{\pi}{3}$ |
| 5. (a) $\frac{\pi}{3}$ | (b) $\frac{3\pi}{4}$ | (c) $\frac{\pi}{6}$ | 6. (a) $\frac{2\pi}{3}$ | (b) $\frac{\pi}{4}$ | (c) $\frac{5\pi}{6}$ |
| 7. (a) $\frac{3\pi}{4}$ | (b) $\frac{\pi}{6}$ | (c) $\frac{2\pi}{3}$ | 8. (a) $\frac{\pi}{4}$ | (b) $\frac{5\pi}{6}$ | (c) $\frac{\pi}{3}$ |
| 9. (a) $\frac{\pi}{4}$ | (b) $-\frac{\pi}{3}$ | (c) $\frac{\pi}{6}$ | 10. (a) $-\frac{\pi}{4}$ | (b) $\frac{\pi}{3}$ | (c) $-\frac{\pi}{6}$ |

11. (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$

12. (a) $\frac{\pi}{4}$ (b) $\frac{5\pi}{6}$ (c) $\frac{\pi}{3}$

13. $\lim_{x \rightarrow 1^-} \sin^{-1} x = \frac{\pi}{2}$

14. $\lim_{x \rightarrow -1^+} \cos^{-1} x = \pi$

15. $\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$

16. $\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$

17. $\lim_{x \rightarrow \infty} \sec^{-1} x = \frac{\pi}{2}$

18. $\lim_{x \rightarrow -\infty} \sec^{-1} x = \lim_{x \rightarrow -\infty} \cos^{-1} \left(\frac{1}{x}\right) = \frac{\pi}{2}$

19. $\lim_{x \rightarrow \infty} \csc^{-1} x = \lim_{x \rightarrow \infty} \sin^{-1} \left(\frac{1}{x}\right) = 0$

20. $\lim_{x \rightarrow -\infty} \csc^{-1} x = \lim_{x \rightarrow -\infty} \sin^{-1} \left(\frac{1}{x}\right) = 0$

21. $y = \cos^{-1}(x^2) \Rightarrow \frac{dy}{dx} = -\frac{2x}{\sqrt{1-(x^2)^2}} = \frac{-2x}{\sqrt{1-x^4}}$

22. $y = \cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{|x| \sqrt{x^2-1}}$

23. $y = \sin^{-1} \sqrt{2t} \Rightarrow \frac{dy}{dt} = \frac{\frac{\sqrt{2}}{2}}{\sqrt{1-(\sqrt{2t})^2}} = \frac{\sqrt{2}}{\sqrt{1-2t^2}}$

24. $y = \sin^{-1}(1-t) \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-(1-t)^2}} = \frac{-1}{\sqrt{2t-t^2}}$

25. $y = \sec^{-1}(2s+1) \Rightarrow \frac{dy}{ds} = \frac{2}{|2s+1| \sqrt{(2s+1)^2-1}} = \frac{2}{|2s+1| \sqrt{4s^2+4s}} = \frac{1}{|2s+1| \sqrt{s^2+s}}$

26. $y = \sec^{-1} 5s \Rightarrow \frac{dy}{ds} = \frac{5}{|5s| \sqrt{(5s)^2-1}} = \frac{1}{|s| \sqrt{25s^2-1}}$

27. $y = \csc^{-1}(x^2+1) \Rightarrow \frac{dy}{dx} = -\frac{2x}{|x^2+1| \sqrt{(x^2+1)^2-1}} = \frac{-2x}{(x^2+1) \sqrt{x^4+2x^2}}$

28. $y = \csc^{-1}\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -\frac{\left(\frac{1}{2}\right)}{\left|\frac{x}{2}\right| \sqrt{\left(\frac{x}{2}\right)^2-1}} = \frac{-1}{|x| \sqrt{\frac{x^2-4}{4}}} = \frac{-2}{|x| \sqrt{x^2-4}}$

29. $y = \sec^{-1}\left(\frac{1}{t}\right) = \cos^{-1} t \Rightarrow \frac{dy}{dt} = \frac{-1}{\sqrt{1-t^2}}$

30. $y = \sin^{-1}\left(\frac{3}{t^2}\right) = \csc^{-1}\left(\frac{t^2}{3}\right) \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{2t}{3}\right)}{\left|\frac{t^2}{3}\right| \sqrt{\left(\frac{t^2}{3}\right)^2-1}} = \frac{-2t}{t^2 \sqrt{\frac{t^4-9}{9}}} = \frac{-6}{t \sqrt{t^4-9}}$

31. $y = \cot^{-1} \sqrt{t} = \cot^{-1} t^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)t^{-1/2}}{1+(t^{1/2})^2} = \frac{-1}{2\sqrt{t}(1+t)}$

32. $y = \cot^{-1} \sqrt{t-1} = \cot^{-1} (t-1)^{1/2} \Rightarrow \frac{dy}{dt} = -\frac{\left(\frac{1}{2}\right)(t-1)^{-1/2}}{1+[(t-1)^{1/2}]^2} = \frac{-1}{2\sqrt{t-1}(1+t-1)} = \frac{-1}{2t\sqrt{t-1}}$

33. $y = \ln(\tan^{-1} x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{1+x^2}\right)}{\tan^{-1} x} = \frac{1}{(\tan^{-1} x)(1+x^2)}$

34. $y = \tan^{-1}(\ln x) \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{x}\right)}{1+(\ln x)^2} = \frac{1}{x[1+(\ln x)^2]}$

35. $y = \csc^{-1}(e^t) \Rightarrow \frac{dy}{dt} = -\frac{e^t}{|e^t| \sqrt{(e^t)^2-1}} = \frac{-1}{\sqrt{e^{2t}-1}}$

36. $y = \cos^{-1}(e^{-t}) \Rightarrow \frac{dy}{dt} = -\frac{-e^{-t}}{\sqrt{1-(e^{-t})^2}} = \frac{e^{-t}}{\sqrt{1-e^{-2t}}}$

$$37. y = s\sqrt{1-s^2} + \cos^{-1} s = s(1-s^2)^{1/2} + \cos^{-1} s \Rightarrow \frac{dy}{ds} = (1-s^2)^{1/2} + s\left(\frac{1}{2}\right)(1-s^2)^{-1/2}(-2s) - \frac{1}{\sqrt{1-s^2}}$$

$$= \sqrt{1-s^2} - \frac{s^2}{\sqrt{1-s^2}} - \frac{1}{\sqrt{1-s^2}} = \sqrt{1-s^2} - \frac{s^2+1}{\sqrt{1-s^2}} = \frac{1-s^2-s^2-1}{\sqrt{1-s^2}} = \frac{-2s^2}{\sqrt{1-s^2}}$$

$$38. y = \sqrt{s^2-1} - \sec^{-1} s = (s^2-1)^{1/2} - \sec^{-1} s \Rightarrow \frac{dy}{dx} = \left(\frac{1}{2}\right)(s^2-1)^{-1/2}(2s) - \frac{1}{|s|\sqrt{s^2-1}} = \frac{s}{\sqrt{s^2-1}} - \frac{1}{|s|\sqrt{s^2-1}}$$

$$= \frac{s|s|-1}{|s|\sqrt{s^2-1}}$$

$$39. y = \tan^{-1} \sqrt{x^2-1} + \csc^{-1} x = \tan^{-1} (x^2-1)^{1/2} + \csc^{-1} x \Rightarrow \frac{dy}{dx} = \frac{\left(\frac{1}{2}\right)(x^2-1)^{-1/2}(2x)}{1+[(x^2-1)^{1/2}]^2} - \frac{1}{|x|\sqrt{x^2-1}}$$

$$= \frac{1}{x\sqrt{x^2-1}} - \frac{1}{|x|\sqrt{x^2-1}} = 0, \text{ for } x > 1$$

$$40. y = \cot^{-1} \left(\frac{1}{x}\right) - \tan^{-1} x = \frac{\pi}{2} - \tan^{-1} (x^{-1}) - \tan^{-1} x \Rightarrow \frac{dy}{dx} = 0 - \frac{-x^{-2}}{1+(x^{-1})^2} - \frac{1}{1+x^2} = \frac{1}{x^2+1} - \frac{1}{1+x^2} = 0$$

$$41. y = x \sin^{-1} x + \sqrt{1-x^2} = x \sin^{-1} x + (1-x^2)^{1/2} \Rightarrow \frac{dy}{dx} = \sin^{-1} x + x \left(\frac{1}{\sqrt{1-x^2}} \right) + \left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)$$

$$= \sin^{-1} x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}} = \sin^{-1} x$$

$$42. y = \ln(x^2+4) - x \tan^{-1} \left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = \frac{2x}{x^2+4} - \tan^{-1} \left(\frac{x}{2}\right) - x \left[\frac{\left(\frac{1}{2}\right)}{1+\left(\frac{x}{2}\right)^2} \right] = \frac{2x}{x^2+4} - \tan^{-1} \left(\frac{x}{2}\right) - \frac{2x}{4+x^2} = -\tan^{-1} \left(\frac{x}{2}\right)$$

43. The angle α is the large angle between the wall and the right end of the blackboard minus the small angle between the left end of the blackboard and the wall $\Rightarrow \alpha = \cot^{-1} \left(\frac{x}{15}\right) - \cot^{-1} \left(\frac{x}{3}\right)$.

44. (a) From the symmetry of the diagram, we see that $\pi - \sec^{-1} x$ is the vertical distance from the graph of $y = \sec^{-1} x$ to the line $y = \pi$ and this distance is the same as the height of $y = \sec^{-1} x$ above the x -axis at $-x$; i.e., $\pi - \sec^{-1} x = \sec^{-1} (-x)$.

(b) $\cos^{-1} (-x) = \pi - \cos^{-1} x$, where $-1 \leq x \leq 1 \Rightarrow \cos^{-1} \left(-\frac{1}{x}\right) = \pi - \cos^{-1} \left(\frac{1}{x}\right)$, where $x \geq 1$ or $x \leq -1$
 $\Rightarrow \sec^{-1} (-x) = \pi - \sec^{-1} x$

$$45. \text{ If } x = 1: \sin^{-1}(1) + \cos^{-1}(1) = \frac{\pi}{2} + 0 = \frac{\pi}{2}.$$

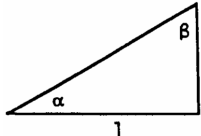
$$\text{ If } x = 0: \sin^{-1}(0) + \cos^{-1}(0) = 0 + \frac{\pi}{2} = \frac{\pi}{2}.$$

$$\text{ If } x = -1: \sin^{-1}(-1) + \cos^{-1}(-1) = -\frac{\pi}{2} + \pi = \frac{\pi}{2}.$$

The identity $\sin^{-1}(x) + \cos^{-1}(x) = \frac{\pi}{2}$ has been established for x in $(0, 1)$, by Figure 1.6.7. So now if x is in $(-1, 0)$, note that $-x$ is in $(0, 1)$, and we have that

$$\begin{aligned} \sin^{-1}(x) + \cos^{-1}(x) &= -\sin^{-1}(-x) + \cos^{-1}(x) && \text{since } \sin^{-1} \text{ is odd} \\ &= -\sin^{-1}(-x) + \pi - \cos^{-1}(-x) && \text{by Eq. 3, Section 1.6} \\ &= -(\sin^{-1}(-x) + \cos^{-1}(-x)) + \pi \\ &= -\frac{\pi}{2} + \pi \\ &= \frac{\pi}{2} \end{aligned}$$

This establishes the identity for all x in $[-1, 1]$.

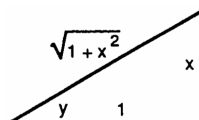
46.  $x \Rightarrow \tan \alpha = x \text{ and } \tan \beta = \frac{1}{x} \Rightarrow \frac{\pi}{2} = \alpha + \beta = \tan^{-1} x + \tan^{-1} \frac{1}{x}.$

47. (a) Defined; there is an angle whose tangent is 2.
 (b) Not defined; there is no angle whose cosine is 2.
 (c) Not defined; there is no angle whose sine is $\sqrt{2}$.

48. (a) Not defined; there is no angle whose secant is 0.
 (b) Defined; there is an angle whose cotangent is $-\frac{1}{2}$.
 (c) Not defined; there is no angle whose cosecant is $\frac{1}{2}$.

$$49. \csc^{-1} u = \frac{\pi}{2} - \sec^{-1} u \Rightarrow \frac{d}{dx} (\csc^{-1} u) = \frac{d}{dx} \left(\frac{\pi}{2} - \sec^{-1} u \right) = 0 - \frac{\frac{du}{dx}}{|u| \sqrt{u^2 - 1}} = - \frac{\frac{du}{dx}}{|u| \sqrt{u^2 - 1}}, |u| > 1$$

$$50. y = \tan^{-1} x \Rightarrow \tan y = x \Rightarrow \frac{d}{dx} (\tan y) = \frac{d}{dx} (x) \\ \Rightarrow (\sec^2 y) \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{(\sqrt{1+x^2})^2} \\ = \frac{1}{1+x^2}, \text{ as indicated by the triangle}$$

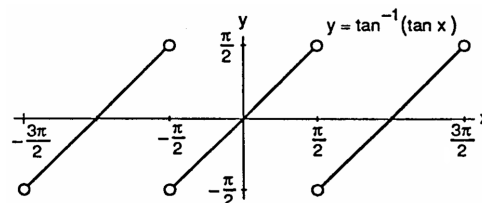


$$51. f(x) = \sec x \Rightarrow f'(x) = \sec x \tan x \Rightarrow \left. \frac{df^{-1}}{dx} \right|_{x=b} = \frac{1}{\left. \frac{df}{dx} \right|_{x=f^{-1}(b)}} = \frac{1}{\sec(\sec^{-1} b) \tan(\sec^{-1} b)} = \frac{1}{b(\pm \sqrt{b^2 - 1})}.$$

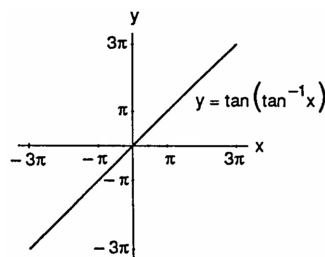
Since the slope of $\sec^{-1} x$ is always positive, we the right sign by writing $\frac{d}{dx} \sec^{-1} x = \frac{1}{|x| \sqrt{x^2 - 1}}$.

$$52. \cot^{-1} u = \frac{\pi}{2} - \tan^{-1} u \Rightarrow \frac{d}{dx} (\cot^{-1} u) = \frac{d}{dx} \left(\frac{\pi}{2} - \tan^{-1} u \right) = 0 - \frac{\frac{du}{dx}}{1+u^2} = - \frac{\frac{du}{dx}}{1+u^2}$$

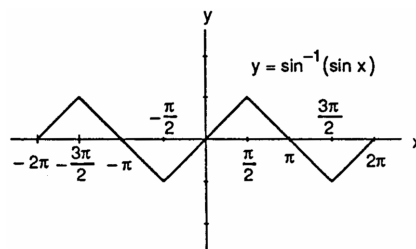
53. (a) Domain: all real numbers except those having the form $\frac{\pi}{2} + k\pi$ where k is an integer.
 Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



- (b) Domain: $-\infty < x < \infty$; Range: $-\infty < y < \infty$
 The graph of $y = \tan^{-1}(\tan x)$ is periodic, the graph of $y = \tan(\tan^{-1} x) = x$ for $-\infty \leq x < \infty$.

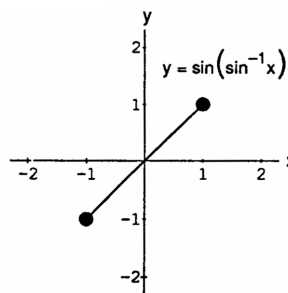


54. (a) Domain: $-\infty < x < \infty$; Range: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

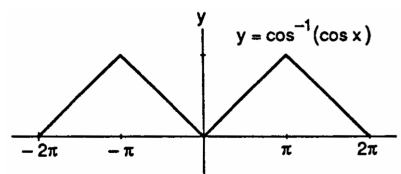


- (b) Domain:
- $-1 \leq x \leq 1$
- ; Range:
- $-1 \leq y \leq 1$

The graph of $y = \sin^{-1}(\sin x)$ is periodic; the graph of $y = \sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$.

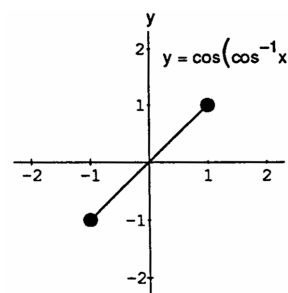


55. (a) Domain:
- $-\infty < x < \infty$
- ; Range:
- $0 \leq y \leq \pi$

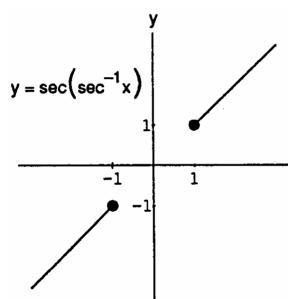


- (b) Domain:
- $-1 \leq x \leq 1$
- ; Range:
- $-1 \leq y \leq 1$

The graph of $y = \cos^{-1}(\cos x)$ is periodic; the graph of $y = \cos(\cos^{-1} x) = x$ for $-1 \leq x \leq 1$.



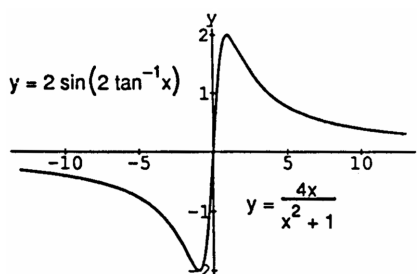
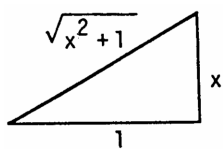
56. Since the domain of $\sec^{-1} x$ is $(-\infty, -1] \cup [1, \infty)$, we have $\sec(\sec^{-1} x) = x$ for $|x| \geq 1$. The graph of $y = \sec(\sec^{-1} x)$ is the line $y = x$ with the open line segment from $(-1, -1)$ to $(1, 1)$ removed.



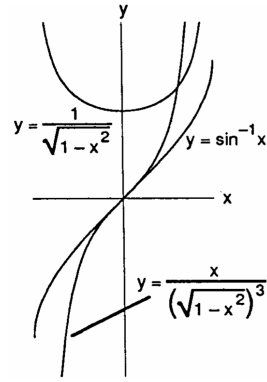
57. The graphs are identical for
- $y = 2 \sin(2 \tan^{-1} x)$

$$= 4 [\sin(\tan^{-1} x)] [\cos(\tan^{-1} x)] = 4 \left(\frac{x}{\sqrt{x^2 + 1}} \right) \left(\frac{1}{\sqrt{x^2 + 1}} \right)$$

$$= \frac{4x}{x^2 + 1} \text{ from the triangle}$$



58. The values of f increase over the interval $[-1, 1]$ because $f' > 0$, and the graph of f steepens as the values of f' increase towards the ends of the interval. The graph of f is concave down to the left of the origin where $f'' < 0$, and concave up to the right of the origin where $f'' > 0$. There is an inflection point at $x = 0$ where $f'' = 0$ and f' has a local minimum value.



2.12 RELATED RATES

- $A = \pi r^2 \Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$
- $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$
- $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt}$
 - $V = \pi r^2 h \Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dh}{dt} + 2\pi r h \frac{dr}{dt}$
- $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt}$
 - $V = \frac{1}{3}\pi r^2 h \Rightarrow \frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt}$
 - $\frac{dV}{dt} = \frac{1}{3}\pi r^2 \frac{dh}{dt} + \frac{2}{3}\pi r h \frac{dr}{dt}$
- $\frac{dV}{dt} = 1 \text{ volt/sec}$
 - $\frac{dI}{dt} = -\frac{1}{3} \text{ amp/sec}$
 - $\frac{dV}{dt} = R \left(\frac{dI}{dt} \right) + I \left(\frac{dR}{dt} \right) \Rightarrow \frac{dR}{dt} = \frac{1}{I} \left(\frac{dV}{dt} - R \frac{dI}{dt} \right) \Rightarrow \frac{dR}{dt} = \frac{1}{I} \left(\frac{dV}{dt} - \frac{V}{I} \frac{dI}{dt} \right)$
 - $\frac{dR}{dt} = \frac{1}{2} \left[1 - \frac{12}{2} \left(-\frac{1}{3} \right) \right] = \left(\frac{1}{2} \right) (3) = \frac{3}{2} \text{ ohms/sec, } R \text{ is increasing}$
- $P = RI^2 \Rightarrow \frac{dP}{dt} = I^2 \frac{dR}{dt} + 2RI \frac{dI}{dt}$
 - $P = RI^2 \Rightarrow 0 = \frac{dP}{dt} = I^2 \frac{dR}{dt} + 2RI \frac{dI}{dt} \Rightarrow \frac{dR}{dt} = -\frac{2RI}{I^2} \frac{dI}{dt} = -\frac{2 \left(\frac{P}{I} \right)}{I^2} \frac{dI}{dt} = -\frac{2P}{I^3} \frac{dI}{dt}$
- $s = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2 + y^2}} \frac{dx}{dt}$
 - $s = \sqrt{x^2 + y^2} = (x^2 + y^2)^{1/2} \Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2 + y^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2 + y^2}} \frac{dy}{dt}$
 - $s = \sqrt{x^2 + y^2} \Rightarrow s^2 = x^2 + y^2 \Rightarrow 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow 2s \cdot 0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow \frac{dx}{dt} = -\frac{y}{x} \frac{dy}{dt}$
- $s = \sqrt{x^2 + y^2 + z^2} \Rightarrow s^2 = x^2 + y^2 + z^2 \Rightarrow 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt}$
 $\Rightarrow \frac{ds}{dt} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \frac{dy}{dt} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \frac{dz}{dt}$
 - From part (a) with $\frac{dx}{dt} = 0 \Rightarrow \frac{ds}{dt} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} \frac{dy}{dt} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \frac{dz}{dt}$
 - From part (a) with $\frac{ds}{dt} = 0 \Rightarrow 0 = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} \Rightarrow \frac{dx}{dt} + \frac{y}{x} \frac{dy}{dt} + \frac{z}{x} \frac{dz}{dt} = 0$
- $A = \frac{1}{2} ab \sin \theta \Rightarrow \frac{dA}{dt} = \frac{1}{2} ab \cos \theta \frac{d\theta}{dt}$
 - $A = \frac{1}{2} ab \sin \theta \Rightarrow \frac{dA}{dt} = \frac{1}{2} ab \cos \theta \frac{d\theta}{dt} + \frac{1}{2} b \sin \theta \frac{da}{dt}$
 - $A = \frac{1}{2} ab \sin \theta \Rightarrow \frac{dA}{dt} = \frac{1}{2} ab \cos \theta \frac{d\theta}{dt} + \frac{1}{2} b \sin \theta \frac{da}{dt} + \frac{1}{2} a \sin \theta \frac{db}{dt}$
- Given $A = \pi r^2$, $\frac{dr}{dt} = 0.01 \text{ cm/sec}$, and $r = 50 \text{ cm}$. Since $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$, then $\frac{dA}{dt} \Big|_{r=50} = 2\pi(50) \left(\frac{1}{100} \right) = \pi \text{ cm}^2/\text{min}$.
- Given $\frac{d\ell}{dt} = -2 \text{ cm/sec}$, $\frac{dw}{dt} = 2 \text{ cm/sec}$, $\ell = 12 \text{ cm}$ and $w = 5 \text{ cm}$.
 - $A = \ell w \Rightarrow \frac{dA}{dt} = \ell \frac{dw}{dt} + w \frac{d\ell}{dt} \Rightarrow \frac{dA}{dt} = 12(2) + 5(-2) = 14 \text{ cm}^2/\text{sec}$, increasing

$$(b) P = 2\ell + 2w \Rightarrow \frac{dP}{dt} = 2 \frac{d\ell}{dt} + 2 \frac{dw}{dt} = 2(-2) + 2(2) = 0 \text{ cm/sec, constant}$$

$$(c) D = \sqrt{w^2 + \ell^2} = (w^2 + \ell^2)^{1/2} \Rightarrow \frac{dD}{dt} = \frac{1}{2} (w^2 + \ell^2)^{-1/2} \left(2w \frac{dw}{dt} + 2\ell \frac{d\ell}{dt} \right) \Rightarrow \frac{dD}{dt} = \frac{w \frac{dw}{dt} + \ell \frac{d\ell}{dt}}{\sqrt{w^2 + \ell^2}} \\ = \frac{(5)(2) + (12)(-2)}{\sqrt{25 + 144}} = -\frac{14}{13} \text{ cm/sec, decreasing}$$

$$12. (a) V = xyz \Rightarrow \frac{dV}{dt} = yz \frac{dx}{dt} + xz \frac{dy}{dt} + xy \frac{dz}{dt} \Rightarrow \frac{dV}{dt} \Big|_{(4,3,2)} = (3)(2)(1) + (4)(2)(-2) + (4)(3)(1) = 2 \text{ m}^3/\text{sec}$$

$$(b) S = 2xy + 2xz + 2yz \Rightarrow \frac{dS}{dt} = (2y + 2z) \frac{dx}{dt} + (2x + 2z) \frac{dy}{dt} + (2x + 2y) \frac{dz}{dt} \\ \Rightarrow \frac{dS}{dt} \Big|_{(4,3,2)} = (10)(1) + (12)(-2) + (14)(1) = 0 \text{ m}^2/\text{sec}$$

$$(c) \ell = \sqrt{x^2 + y^2 + z^2} = (x^2 + y^2 + z^2)^{1/2} \Rightarrow \frac{d\ell}{dt} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2 + y^2 + z^2}} \frac{dy}{dt} + \frac{z}{\sqrt{x^2 + y^2 + z^2}} \frac{dz}{dt} \\ \Rightarrow \frac{d\ell}{dt} \Big|_{(4,3,2)} = \left(\frac{4}{\sqrt{29}} \right) (1) + \left(\frac{3}{\sqrt{29}} \right) (-2) + \left(\frac{2}{\sqrt{29}} \right) (1) = 0 \text{ m/sec}$$

$$13. \text{ Given: } \frac{dx}{dt} = 5 \text{ ft/sec, the ladder is 13 ft long, and } x = 12, y = 5 \text{ at the instant of time}$$

$$(a) \text{ Since } x^2 + y^2 = 169 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt} = -\left(\frac{12}{5} \right) (5) = -12 \text{ ft/sec, the ladder is sliding down the wall}$$

$$(b) \text{ The area of the triangle formed by the ladder and walls is } A = \frac{1}{2} xy \Rightarrow \frac{dA}{dt} = \left(\frac{1}{2} \right) \left(x \frac{dy}{dt} + y \frac{dx}{dt} \right). \text{ The area is changing at } \frac{1}{2} [12(-12) + 5(5)] = -\frac{119}{2} = -59.5 \text{ ft}^2/\text{sec}.$$

$$(c) \cos \theta = \frac{x}{13} \Rightarrow -\sin \theta \frac{d\theta}{dt} = \frac{1}{13} \cdot \frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = -\frac{1}{13 \sin \theta} \cdot \frac{dx}{dt} = -\left(\frac{1}{5} \right) (5) = -1 \text{ rad/sec}$$

$$14. s^2 = y^2 + x^2 \Rightarrow 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow \frac{ds}{dt} = \frac{1}{s} \left(x \frac{dx}{dt} + y \frac{dy}{dt} \right) \Rightarrow \frac{ds}{dt} = \frac{1}{\sqrt{169}} [5(-442) + 12(-481)] = -614 \text{ knots}$$

$$15. \text{ Let } s \text{ represent the distance between the girl and the kite and } x \text{ represents the horizontal distance between the girl and kite} \\ \Rightarrow s^2 = (300)^2 + x^2 \Rightarrow \frac{ds}{dt} = \frac{x}{s} \frac{dx}{dt} = \frac{400(25)}{500} = 20 \text{ ft/sec}.$$

$$16. \text{ When the diameter is 3.8 in., the radius is 1.9 in. and } \frac{dr}{dt} = \frac{1}{3000} \text{ in/min. Also } V = 6\pi r^2 \Rightarrow \frac{dV}{dt} = 12\pi r \frac{dr}{dt} \\ \Rightarrow \frac{dV}{dt} = 12\pi(1.9) \left(\frac{1}{3000} \right) = 0.0076\pi. \text{ The volume is changing at about } 0.0239 \text{ in}^3/\text{min}.$$

$$17. V = \frac{1}{3} \pi r^2 h, h = \frac{3}{8} (2r) = \frac{3r}{4} \Rightarrow r = \frac{4h}{3} \Rightarrow V = \frac{1}{3} \pi \left(\frac{4h}{3} \right)^2 h = \frac{16\pi h^3}{27} \Rightarrow \frac{dV}{dt} = \frac{16\pi h^2}{9} \frac{dh}{dt}$$

$$(a) \frac{dh}{dt} \Big|_{h=4} = \left(\frac{9}{16\pi^2} \right) (10) = \frac{90}{256\pi} \approx 0.1119 \text{ m/sec} = 11.19 \text{ cm/sec}$$

$$(b) r = \frac{4h}{3} \Rightarrow \frac{dr}{dt} = \frac{4}{3} \frac{dh}{dt} = \frac{4}{3} \left(\frac{90}{256\pi} \right) = \frac{15}{32\pi} \approx 0.1492 \text{ m/sec} = 14.92 \text{ cm/sec}$$

$$18. (a) V = \frac{1}{3} \pi r^2 h \text{ and } r = \frac{15h}{2} \Rightarrow V = \frac{1}{3} \pi \left(\frac{15h}{2} \right)^2 h = \frac{75\pi h^3}{4} \Rightarrow \frac{dV}{dt} = \frac{225\pi h^2}{4} \frac{dh}{dt} \Rightarrow \frac{dh}{dt} \Big|_{h=5} = \frac{4(-50)}{225\pi(5)^2} = \frac{-8}{225\pi} \\ \approx -0.0113 \text{ m/min} = -1.13 \text{ cm/min}$$

$$(b) r = \frac{15h}{2} \Rightarrow \frac{dr}{dt} = \frac{15}{2} \frac{dh}{dt} \Rightarrow \frac{dr}{dt} \Big|_{h=5} = \left(\frac{15}{2} \right) \left(\frac{-8}{225\pi} \right) = \frac{-4}{15\pi} \approx -0.0849 \text{ m/sec} = -8.49 \text{ cm/sec}$$

$$19. (a) V = \frac{\pi}{3} y^2 (3R - y) \Rightarrow \frac{dV}{dt} = \frac{\pi}{3} [2y(3R - y) + y^2(-1)] \frac{dy}{dt} \Rightarrow \frac{dy}{dt} = \left[\frac{\pi}{3} (6Ry - 3y^2) \right]^{-1} \frac{dV}{dt} \Rightarrow \text{at } R = 13 \text{ and } y = 8 \text{ we have } \frac{dy}{dt} = \frac{1}{144\pi} (-6) = \frac{-1}{24\pi} \text{ m/min}$$

$$(b) \text{ The hemisphere is on the circle } r^2 + (13 - y)^2 = 169 \Rightarrow r = \sqrt{26y - y^2} \text{ m}$$

$$(c) r = (26y - y^2)^{1/2} \Rightarrow \frac{dr}{dt} = \frac{1}{2} (26y - y^2)^{-1/2} (26 - 2y) \frac{dy}{dt} \Rightarrow \frac{dr}{dt} = \frac{13 - y}{\sqrt{26y - y^2}} \frac{dy}{dt} \Rightarrow \frac{dr}{dt} \Big|_{y=8} = \frac{13 - 8}{\sqrt{26 \cdot 8 - 64}} \left(\frac{-1}{24\pi} \right) \\ = \frac{-5}{288\pi} \text{ m/min}$$

$$20. \text{ If } V = \frac{4}{3} \pi r^3, S = 4\pi r^2, \text{ and } \frac{dV}{dt} = kS = 4k\pi r^2, \text{ then } \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow 4k\pi r^2 = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = k, \text{ a constant.} \\ \text{Therefore, the radius is increasing at a constant rate.}$$

21. If $V = \frac{4}{3}\pi r^3$, $r = 5$, and $\frac{dV}{dt} = 100\pi \text{ ft}^3/\text{min}$, then $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = 1 \text{ ft/min}$. Then $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi(5)(1) = 40\pi \text{ ft}^2/\text{min}$, the rate at which the surface area is increasing.

22. Let s represent the length of the rope and x the horizontal distance of the boat from the dock.

(a) We have $s^2 = x^2 + 36 \Rightarrow \frac{ds}{dt} = \frac{s}{x} \frac{dx}{dt} = \frac{s}{\sqrt{s^2 - 36}} \frac{ds}{dt}$. Therefore, the boat is approaching the dock at

$$\left. \frac{dx}{dt} \right|_{s=10} = \frac{10}{\sqrt{10^2 - 36}} (-2) = -2.5 \text{ ft/sec.}$$

(b) $\cos \theta = \frac{x}{r} \Rightarrow -\sin \theta \frac{d\theta}{dt} = -\frac{x}{r^2} \frac{dr}{dt} \Rightarrow \frac{d\theta}{dt} = \frac{6}{r^2 \sin \theta} \frac{dr}{dt}$. Thus, $r = 10$, $x = 8$, and $\sin \theta = \frac{6}{10}$
 $\Rightarrow \frac{d\theta}{dt} = \frac{6}{10^2 (\frac{6}{10})} \cdot (-2) = -\frac{3}{20} \text{ rad/sec}$

23. Let s represent the distance between the bicycle and balloon, h the height of the balloon and x the horizontal distance between the balloon and the bicycle. The relationship between the variables is $s^2 = h^2 + x^2$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{s} \left(h \frac{dh}{dt} + x \frac{dx}{dt} \right) \Rightarrow \frac{ds}{dt} = \frac{1}{85} [68(1) + 51(17)] = 11 \text{ ft/sec.}$$

24. (a) Let h be the height of the coffee in the pot. Since the radius of the pot is 3, the volume of the coffee is

$$V = 9\pi h \Rightarrow \frac{dV}{dt} = 9\pi \frac{dh}{dt} \Rightarrow \text{the rate the coffee is rising is } \frac{dh}{dt} = \frac{1}{9\pi} \frac{dV}{dt} = \frac{10}{9\pi} \text{ in/min.}$$

(b) Let h be the height of the coffee in the pot. From the figure, the radius of the filter $r = \frac{h}{2} \Rightarrow V = \frac{1}{3}\pi r^2 h$
 $= \frac{\pi h^3}{12}$, the volume of the filter. The rate the coffee is falling is $\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt} = \frac{4}{25\pi} (-10) = -\frac{8}{5\pi} \text{ in/min.}$

25. Let $P(x, y)$ represent a point on the curve $y = x^2$ and θ the angle of inclination of a line containing P and the origin. Consequently, $\tan \theta = \frac{y}{x} \Rightarrow \tan \theta = \frac{x^2}{x} = x \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{dx}{dt} \Rightarrow \frac{d\theta}{dt} = \cos^2 \theta \frac{dx}{dt}$. Since $\frac{dx}{dt} = 10 \text{ m/sec}$ and $\cos^2 \theta|_{x=3} = \frac{x^2}{y^2 + x^2} = \frac{3^2}{9^2 + 3^2} = \frac{1}{10}$, we have $\left. \frac{d\theta}{dt} \right|_{x=3} = 1 \text{ rad/sec.}$

26. $y = (-x)^{1/2}$ and $\tan \theta = \frac{y}{x} \Rightarrow \tan \theta = \frac{(-x)^{1/2}}{x} \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{(\frac{1}{2})(-x)^{-1/2}(-1)x - (-x)^{1/2}(1)}{x^2} \frac{dx}{dt}$
 $\Rightarrow \frac{d\theta}{dt} = \left(\frac{\frac{-x}{2\sqrt{-x}} - \sqrt{-x}}{x^2} \right) (\cos^2 \theta) \left(\frac{dx}{dt} \right)$. Now, $\tan \theta = \frac{2}{-4} = -\frac{1}{2} \Rightarrow \cos \theta = -\frac{2}{\sqrt{5}} \Rightarrow \cos^2 \theta = \frac{4}{5}$. Then
 $\frac{d\theta}{dt} = \left(\frac{\frac{4}{-2} - 2}{16} \right) \left(\frac{4}{5} \right) (-8) = \frac{2}{5} \text{ rad/sec.}$

27. The distance from the origin is $s = \sqrt{x^2 + y^2}$ and we wish to find $\left. \frac{ds}{dt} \right|_{(5,12)}$
 $= \frac{1}{2} (x^2 + y^2)^{-1/2} \left(2x \frac{dx}{dt} + 2y \frac{dy}{dt} \right) \Big|_{(5,12)} = \frac{(5)(-1) + (12)(-5)}{\sqrt{25 + 144}} = -5 \text{ m/sec}$

28. When s represents the length of the shadow and x the distance of the man from the streetlight, then $s = \frac{3}{5}x$.

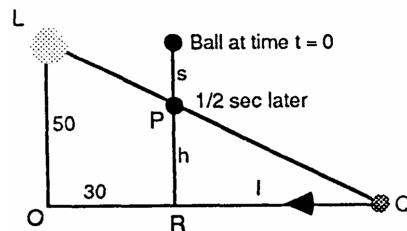
If I represents the distance of the tip of the shadow from the streetlight, then $I = s + x \Rightarrow \frac{dI}{dt} = \frac{ds}{dt} + \frac{dx}{dt}$

(which is velocity not speed) $\Rightarrow \left| \frac{dI}{dt} \right| = \left| \frac{3}{5} \frac{dx}{dt} + \frac{dx}{dt} \right| = \left| \frac{8}{5} \right| \left| \frac{dx}{dt} \right| = \frac{8}{5} |-5| = 8 \text{ ft/sec}$, the speed the tip of the shadow is moving along the ground. $\frac{ds}{dt} = \frac{3}{5} \frac{dx}{dt} = \frac{3}{5} (-5) = -3 \text{ ft/sec}$, so the length of the shadow is decreasing at a rate of 3 ft/sec.

29. Let $s = 16t^2$ represent the distance the ball has fallen, h the distance between the ball and the ground, and I the distance between the shadow and the point directly beneath the ball. Accordingly, $s + h = 50$ and since the triangle LOQ and triangle PRQ are similar we have

$$I = \frac{30h}{50-h} \Rightarrow h = 50 - 16t^2 \text{ and } I = \frac{30(50 - 16t^2)}{50 - (50 - 16t^2)}$$

$$= \frac{1500}{16t^2} - 30 \Rightarrow \frac{dI}{dt} = -\frac{1500}{8t^3} \Rightarrow \left. \frac{dI}{dt} \right|_{t=\frac{1}{2}} = -1500 \text{ ft/sec.}$$



30. Let s = distance of car from foot of perpendicular in the textbook diagram $\Rightarrow \tan \theta = \frac{s}{132} \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = \frac{1}{132} \frac{ds}{dt}$
 $\Rightarrow \frac{d\theta}{dt} = \frac{\cos^2 \theta}{132} \frac{ds}{dt}$; $\frac{ds}{dt} = -264$ and $\theta = 0 \Rightarrow \frac{d\theta}{dt} = -2$ rad/sec. A half second later the car has traveled 132 ft
 right of the perpendicular $\Rightarrow |\theta| = \frac{\pi}{4}$, $\cos^2 \theta = \frac{1}{2}$, and $\frac{ds}{dt} = 264$ (since s increases) $\Rightarrow \frac{d\theta}{dt} = \frac{(\frac{1}{2})}{132} (264) = 1$ rad/sec.
31. The volume of the ice is $V = \frac{4}{3} \pi r^3 - \frac{4}{3} \pi 4^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \left. \frac{dr}{dt} \right|_{r=6} = \frac{-5}{72\pi}$ in./min when $\frac{dV}{dt} = -10$ in³/min, the
 thickness of the ice is decreasing at $\frac{5}{72\pi}$ in./min. The surface area is $S = 4\pi r^2 \Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt} \Rightarrow \left. \frac{dS}{dt} \right|_{r=6} = 48\pi \left(\frac{-5}{72\pi} \right)$
 $= -\frac{10}{3}$ in²/min, the outer surface area of the ice is decreasing at $\frac{10}{3}$ in²/min.
32. Let s represent the horizontal distance between the car and plane while r is the line-of-sight distance between
 the car and plane $\Rightarrow 9 + s^2 = r^2 \Rightarrow \frac{ds}{dt} = \frac{r}{\sqrt{r^2 - 9}} \frac{dr}{dt} \Rightarrow \left. \frac{ds}{dt} \right|_{r=5} = \frac{5}{\sqrt{16}} (-160) = -200$ mph
 $\Rightarrow$ speed of plane + speed of car = 200 mph $\Rightarrow$ the speed of the car is 80 mph.
33. When x represents the length of the shadow, then $\tan \theta = \frac{80}{x} \Rightarrow \sec^2 \theta \frac{d\theta}{dt} = -\frac{80}{x^2} \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{-x^2 \sec^2 \theta}{80} \frac{d\theta}{dt}$.
 We are given that $\frac{d\theta}{dt} = 0.27^\circ = \frac{3\pi}{2000}$ rad/min. At $x = 60$, $\cos \theta = \frac{3}{5} \Rightarrow$
 $\left| \frac{dx}{dt} \right| = \left| \frac{-x^2 \sec^2 \theta}{80} \frac{d\theta}{dt} \right| \bigg|_{\left(\frac{d\theta}{dt} = \frac{3\pi}{2000} \text{ and } \sec \theta = \frac{5}{3} \right)} = \frac{3\pi}{16}$ ft/min ≈ 0.589 ft/min ≈ 7.1 in./min.
34. Let a represent the distance between point O and ship A, b the distance between point O and ship B, and
 D the distance between the ships. By the Law of Cosines, $D^2 = a^2 + b^2 - 2ab \cos 120^\circ$
 $\Rightarrow \frac{dD}{dt} = \frac{1}{2D} \left[2a \frac{da}{dt} + 2b \frac{db}{dt} + a \frac{db}{dt} + b \frac{da}{dt} \right]$. When $a = 5$, $\frac{da}{dt} = 14$, $b = 3$, and $\frac{db}{dt} = 21$, then $\frac{dD}{dt} = \frac{413}{2D}$
 where $D = 7$. The ships are moving $\frac{dD}{dt} = 29.5$ knots apart.

2.13 LINEARIZATION AND DIFFERENTIALS

- $f(x) = x^3 - 2x + 3 \Rightarrow f'(x) = 3x^2 - 2 \Rightarrow L(x) = f'(2)(x - 2) + f(2) = 10(x - 2) + 7 \Rightarrow L(x) = 10x - 13$ at $x = 2$
- $f(x) = \sqrt{x^2 + 9} = (x^2 + 9)^{1/2} \Rightarrow f'(x) = \left(\frac{1}{2} \right) (x^2 + 9)^{-1/2} (2x) = \frac{x}{\sqrt{x^2 + 9}} \Rightarrow L(x) = f'(-4)(x + 4) + f(-4)$
 $= -\frac{4}{5}(x + 4) + 5 \Rightarrow L(x) = -\frac{4}{5}x + \frac{9}{5}$ at $x = -4$
- $f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - x^{-2} \Rightarrow L(x) = f'(1)(x - 1) + f(1) = 2 + 0(x - 1) = 2$
- $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3x^{2/3}} \Rightarrow L(x) = f'(-8)(x - (-8)) + f(-8) = \frac{1}{12}(x + 8) - 2 \Rightarrow L(x) = \frac{1}{12}x - \frac{4}{3}$
- $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f(\pi) + f'(\pi)(x - \pi) = 0 + 1(x - \pi) = x - \pi$
- (a) $f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow L(x) = f(0) + f'(0)(x - 0) = x \Rightarrow L(x) = x$
 (b) $f(x) = \cos x \Rightarrow f'(x) = -\sin x \Rightarrow L(x) = f(0) + f'(0)(x - 0) = 1 \Rightarrow L(x) = 1$
 (c) $f(x) = \tan x \Rightarrow f'(x) = \sec^2 x \Rightarrow L(x) = f(0) + f'(0)(x - 0) = x \Rightarrow L(x) = x$
 (d) $f(x) = e^x \Rightarrow f'(x) = e^x \Rightarrow L(x) = f(0) + f'(0)(x - 0) = x + 1 \Rightarrow L(x) = x + 1$
 (e) $f(x) = \ln(1 + x) \Rightarrow f'(x) = \frac{1}{1+x} \Rightarrow L(x) = f(0) + f'(0)(x - 0) = x \Rightarrow L(x) = x$
- $f(x) = x^2 + 2x \Rightarrow f'(x) = 2x + 2 \Rightarrow L(x) = f'(0)(x - 0) + f(0) = 2(x - 0) + 0 \Rightarrow L(x) = 2x$ at $x = 0$
- $f(x) = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow L(x) = f'(1)(x - 1) + f(1) = (-1)(x - 1) + 1 \Rightarrow L(x) = -x + 2$ at $x = 1$
- $f(x) = 2x^2 + 4x - 3 \Rightarrow f'(x) = 4x + 4 \Rightarrow L(x) = f'(-1)(x + 1) + f(-1) = 0(x + 1) + (-5) \Rightarrow L(x) = -5$ at $x = -1$

$$10. f(x) = 1 + x \Rightarrow f'(x) = 1 \Rightarrow L(x) = f'(8)(x - 8) + f(8) = 1(x - 8) + 9 \Rightarrow L(x) = x + 1 \text{ at } x = 8$$

$$11. f(x) = \sqrt[3]{x} = x^{1/3} \Rightarrow f'(x) = \left(\frac{1}{3}\right) x^{-2/3} \Rightarrow L(x) = f'(8)(x - 8) + f(8) = \frac{1}{12}(x - 8) + 2 \Rightarrow L(x) = \frac{1}{12}x + \frac{4}{3} \text{ at } x = 8$$

$$12. f(x) = \frac{x}{x+1} \Rightarrow f'(x) = \frac{(1)(x+1) - (1)(x)}{(x+1)^2} = \frac{1}{(x+1)^2} \Rightarrow L(x) = f'(1)(x - 1) + f(1) = \frac{1}{4}(x - 1) + \frac{1}{2} \\ \Rightarrow L(x) = \frac{1}{4}x + \frac{1}{4} \text{ at } x = 1$$

$$13. f(x) = e^{-x} \Rightarrow f'(x) = -e^{-x} \Rightarrow L(x) = f(0) + f'(0)(x - 0) = -x + 1$$

$$14. f(x) = \sin^{-1}x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow L(x) = f(0) + f'(0)(x - 0) = x$$

$$15. f'(x) = k(1+x)^{k-1}. \text{ We have } f(0) = 1 \text{ and } f'(0) = k. L(x) = f(0) + f'(0)(x - 0) = 1 + k(x - 0) = 1 + kx$$

$$16. (a) f(x) = (1-x)^6 = [1 + (-x)]^6 \approx 1 + 6(-x) = 1 - 6x$$

$$(b) f(x) = \frac{2}{1-x} = 2[1 + (-x)]^{-1} \approx 2[1 + (-1)(-x)] = 2 + 2x$$

$$(c) f(x) = (1+x)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)x = 1 - \frac{x}{2}$$

$$(d) f(x) = \sqrt{2+x^2} = \sqrt{2}\left(1 + \frac{x^2}{2}\right)^{1/2} \approx \sqrt{2}\left(1 + \frac{1}{2}\frac{x^2}{2}\right) = \sqrt{2}\left(1 + \frac{x^2}{4}\right)$$

$$(e) f(x) = (4+3x)^{1/3} = 4^{1/3}\left(1 + \frac{3x}{4}\right)^{1/3} \approx 4^{1/3}\left(1 + \frac{1}{3}\frac{3x}{4}\right) = 4^{1/3}\left(1 + \frac{x}{4}\right)$$

$$(f) f(x) = \left(1 - \frac{1}{2+x}\right)^{2/3} = \left[1 + \left(-\frac{1}{2+x}\right)\right]^{2/3} \approx 1 + \frac{2}{3}\left(-\frac{1}{2+x}\right) = 1 - \frac{2}{6+3x}$$

$$17. (a) (1.0002)^{50} = (1 + 0.0002)^{50} \approx 1 + 50(0.0002) = 1 + .01 = 1.01$$

$$(b) \sqrt[3]{1.009} = (1 + 0.009)^{1/3} \approx 1 + \left(\frac{1}{3}\right)(0.009) = 1 + 0.003 = 1.003$$

$$18. f(x) = \sqrt{x+1} + \sin x = (x+1)^{1/2} + \sin x \Rightarrow f'(x) = \left(\frac{1}{2}\right)(x+1)^{-1/2} + \cos x \Rightarrow L_f(x) = f'(0)(x - 0) + f(0)$$

$$= \frac{3}{2}(x - 0) + 1 \Rightarrow L_f(x) = \frac{3}{2}x + 1, \text{ the linearization of } f(x); g(x) = \sqrt{x+1} = (x+1)^{1/2} \Rightarrow g'(x)$$

$$= \left(\frac{1}{2}\right)(x+1)^{-1/2} \Rightarrow L_g(x) = g'(0)(x - 0) + g(0) = \frac{1}{2}(x - 0) + 1 \Rightarrow L_g(x) = \frac{1}{2}x + 1, \text{ the linearization of } g(x);$$

$$h(x) = \sin x \Rightarrow h'(x) = \cos x \Rightarrow L_h(x) = h'(0)(x - 0) + h(0) = (1)(x - 0) + 0 \Rightarrow L_h(x) = x, \text{ the linearization of } h(x).$$

$L_f(x) = L_g(x) + L_h(x)$ implies that the linearization of a sum is equal to the sum of the linearizations.

$$19. y = x^3 - 3\sqrt{x} = x^3 - 3x^{1/2} \Rightarrow dy = (3x^2 - \frac{3}{2}x^{-1/2}) dx \Rightarrow dy = \left(3x^2 - \frac{3}{2\sqrt{x}}\right) dx$$

$$20. y = \frac{2x}{1+x^2} \Rightarrow dy = \left(\frac{(2)(1+x^2) - (2x)(2x)}{(1+x^2)^2}\right) dx = \frac{2-2x^2}{(1+x^2)^2} dx$$

$$21. 2y^{3/2} + xy - x = 0 \Rightarrow 3y^{1/2} dy + y dx + x dy - dx = 0 \Rightarrow (3y^{1/2} + x) dy = (1 - y) dx \Rightarrow dy = \frac{1-y}{3\sqrt{y}+x} dx$$

$$22. xy^2 - 4x^{3/2} - y = 0 \Rightarrow y^2 dx + 2xy dy - 6x^{1/2} dx - dy = 0 \Rightarrow (2xy - 1) dy = (6x^{1/2} - y^2) dx \Rightarrow dy = \frac{6\sqrt{x}-y^2}{2xy-1} dx$$

$$23. y = \sin(5\sqrt{x}) = \sin(5x^{1/2}) \Rightarrow dy = (\cos(5x^{1/2}))\left(\frac{5}{2}x^{-1/2}\right) dx \Rightarrow dy = \frac{5\cos(5\sqrt{x})}{2\sqrt{x}} dx$$

$$24. y = 4 \tan\left(\frac{x^3}{3}\right) \Rightarrow dy = 4\left(\sec^2\left(\frac{x^3}{3}\right)\right)(x^2) dx \Rightarrow dy = 4x^2 \sec^2\left(\frac{x^3}{3}\right) dx$$

$$25. y = e^{\sqrt{x}} \Rightarrow dy = \frac{e^{\sqrt{x}}}{2\sqrt{x}} dx$$

$$26. y = x e^{-x} \Rightarrow dy = (-x e^{-x} + e^{-x}) dx = (1 - x) e^{-x} dx$$

$$27. y = \ln(1 + x^2) \Rightarrow dy = \frac{2x}{1+x^2} dx$$

$$28. y = \sec^{-1}(e^{-x}) \Rightarrow dy = \frac{1}{e^{-x}\sqrt{(e^{-x})^2 - 1}} \cdot (-e^{-x}) dx = \frac{-1}{\sqrt{(\frac{1}{e^x})^2 - 1}} dx = \frac{-e^x}{\sqrt{1 - e^{2x}}} dx$$

$$29. f(x) = x^2 + 2x, x_0 = 1, dx = 0.1 \Rightarrow f'(x) = 2x + 2$$

$$(a) \Delta f = f(x_0 + dx) - f(x_0) = f(1.1) - f(1) = 3.41 - 3 = 0.41$$

$$(b) df = f'(x_0) dx = [2(1) + 2](0.1) = 0.4$$

$$(c) |\Delta f - df| = |0.41 - 0.4| = 0.01$$

$$30. f(x) = x^3 - x, x_0 = 1, dx = 0.1 \Rightarrow f'(x) = 3x^2 - 1$$

$$(a) \Delta f = f(x_0 + dx) - f(x_0) = f(1.1) - f(1) = .231$$

$$(b) df = f'(x_0) dx = [3(1)^2 - 1](.1) = .2$$

$$(c) |\Delta f - df| = |.231 - .2| = .031$$

$$31. f(x) = x^{-1}, x_0 = 0.5, dx = 0.1 \Rightarrow f'(x) = -x^{-2}$$

$$(a) \Delta f = f(x_0 + dx) - f(x_0) = f(.6) - f(.5) = -\frac{1}{3}$$

$$(b) df = f'(x_0) dx = (-4) \left(\frac{1}{10}\right) = -\frac{2}{5}$$

$$(c) |\Delta f - df| = \left| -\frac{1}{3} + \frac{2}{5} \right| = \frac{1}{15}$$

$$32. f(x) = x^3 - 2x + 3, x_0 = 2, dx = 0.1 \Rightarrow f'(x) = 3x^2 - 2$$

$$(a) \Delta f = f(x_0 + dx) - f(x_0) = f(2.1) - f(2) = 1.061$$

$$(b) df = f'(x_0) dx = (10)(0.10) = 1$$

$$(c) |\Delta f - df| = |1.061 - 1| = .061$$

$$33. V = \frac{4}{3} \pi r^3 \Rightarrow dV = 4\pi r^2 dr$$

$$34. S = 6x^2 \Rightarrow dS = 12x_0 dx$$

$$35. V = \pi r^2 h, \text{ height constant} \Rightarrow dV = 2\pi r h dr$$

$$36. S = \pi r \sqrt{r^2 + h^2} = \pi r (r^2 + h^2)^{1/2}, h \text{ constant} \Rightarrow \frac{dS}{dr} = \pi (r^2 + h^2)^{1/2} + \pi r \cdot r (r^2 + h^2)^{-1/2} \\ \Rightarrow \frac{dS}{dr} = \frac{\pi (r^2 + h^2) + \pi r^2}{\sqrt{r^2 + h^2}} \Rightarrow dS = \frac{\pi (2r^2 + h^2)}{\sqrt{r_0^2 + h^2}} dr, h \text{ constant}$$

$$37. \text{ Given } r = 2 \text{ m, } dr = .02 \text{ m}$$

$$(a) A = \pi r^2 \Rightarrow dA = 2\pi r dr = 2\pi(2)(.02) = .08\pi \text{ m}^2$$

$$(b) \left(\frac{.08\pi}{4\pi}\right)(100\%) = 2\%$$

$$38. C = 2\pi r \text{ and } dC = 2 \text{ in.} \Rightarrow dC = 2\pi dr \Rightarrow dr = \frac{1}{\pi} \Rightarrow \text{the diameter grew about } \frac{2}{\pi} \text{ in.}; A = \pi r^2 \Rightarrow dA = 2\pi r dr \\ = 2\pi(5) \left(\frac{1}{\pi}\right) = 10 \text{ in.}^2$$

$$39. \text{ The volume of a cylinder is } V = \pi r^2 h. \text{ When } h \text{ is held fixed, we have } \frac{dV}{dr} = 2\pi r h, \text{ and so } dV = 2\pi r h dr. \text{ For } h = 30 \text{ in.,} \\ r = 6 \text{ in., and } dr = 0.5 \text{ in., the volume of the material in the shell is approximately } dV = 2\pi r h dr = 2\pi(6)(30)(0.5) \\ = 180\pi \approx 565.5 \text{ in.}^3.$$

40. Let θ = angle of elevation and h = height of building. Then $h = 30 \tan \theta$, so $dh = 30 \sec^2 \theta d\theta$. We want $|dh| < 0.04h$, which gives: $|30 \sec^2 \theta d\theta| < 0.04|30 \tan \theta| \Rightarrow \frac{1}{\cos^2 \theta} |d\theta| < \frac{0.04 \sin \theta}{\cos \theta} \Rightarrow |d\theta| < 0.04 \sin \theta \cos \theta \Rightarrow |d\theta| < 0.04 \sin \frac{5\pi}{12} \cos \frac{5\pi}{12} = 0.01$ radian. The angle should be measured with an error of less than 0.01 radian (or approximately 0.57 degrees), which is a percentage error of approximately 0.76%.
41. $V = \pi h^3 \Rightarrow dV = 3\pi h^2 dh$; recall that $\Delta V \approx dV$. Then $|\Delta V| \leq (1\%)(V) = \frac{(1)(\pi h^3)}{100} \Rightarrow |dV| \leq \frac{(1)(\pi h^3)}{100} \Rightarrow |3\pi h^2 dh| \leq \frac{(1)(\pi h^3)}{100} \Rightarrow |dh| \leq \frac{1}{300} h = \left(\frac{1}{3}\right)\% h$. Therefore the greatest tolerated error in the measurement of h is $\frac{1}{3}\%$.
42. (a) Let D_i represent the inside diameter. Then $V = \pi r^2 h = \pi \left(\frac{D_i}{2}\right)^2 h = \frac{\pi D_i^2 h}{4}$ and $h = 10 \Rightarrow V = \frac{5\pi D_i^2}{2} \Rightarrow dV = 5\pi D_i dD_i$. Recall that $\Delta V \approx dV$. We want $|\Delta V| \leq (1\%)(V) \Rightarrow |dV| \leq \left(\frac{1}{100}\right) \left(\frac{5\pi D_i^2}{2}\right) = \frac{\pi D_i^2}{40} \Rightarrow 5\pi D_i dD_i \leq \frac{\pi D_i^2}{40} \Rightarrow \frac{dD_i}{D_i} \leq \frac{1}{200}$. The inside diameter must be measured to within 0.5%.
- (b) Let D_e represent the exterior diameter, h the height and S the area of the painted surface. $S = \pi D_e h \Rightarrow dS = \pi h dD_e \Rightarrow \frac{dS}{S} = \frac{dD_e}{D_e}$. Thus for small changes in exterior diameter, the approximate percentage change in the exterior diameter is equal to the approximate percentage change in the area painted, and to estimate the amount of paint required to within 5%, the tanks's exterior diameter must be measured to within 5%.
43. $V = \pi r^2 h$, h is constant $\Rightarrow dV = 2\pi r h dr$; recall that $\Delta V \approx dV$. We want $|\Delta V| \leq \frac{1}{1000} V \Rightarrow |dV| \leq \frac{\pi r^2 h}{1000} \Rightarrow |2\pi r h dr| \leq \frac{\pi r^2 h}{1000} \Rightarrow |dr| \leq \frac{r}{2000} = (0.05\%)r \Rightarrow$ a .05% variation in the radius can be tolerated.
44. (a) $T = 2\pi \left(\frac{L}{g}\right)^{1/2} \Rightarrow dT = 2\pi \sqrt{L} \left(-\frac{1}{2} g^{-3/2}\right) dg = -\pi \sqrt{L} g^{-3/2} dg$
- (b) If g increases, then $dg > 0 \Rightarrow dT < 0$. The period T decreases and the clock ticks more frequently. Both the pendulum speed and clock speed increase.
- (c) $0.001 = -\pi \sqrt{100} (980^{-3/2}) dg \Rightarrow dg \approx -0.977 \text{ cm/sec}^2 \Rightarrow$ the new $g \approx 979 \text{ cm/sec}^2$
45. The error in measurement $dx = (1\%)(10) = 0.1 \text{ cm}$; $V = x^3 \Rightarrow dV = 3x^2 dx = 3(10)^2(0.1) = 30 \text{ cm}^3 \Rightarrow$ the percentage error in the volume calculation is $\left(\frac{30}{1000}\right)(100\%) = 3\%$
46. $A = s^2 \Rightarrow dA = 2s ds$; recall that $\Delta A \approx dA$. Then $|\Delta A| \leq (2\%)A = \frac{2s^2}{100} = \frac{s^2}{50} \Rightarrow |dA| \leq \frac{s^2}{50} \Rightarrow |2s ds| \leq \frac{s^2}{50} \Rightarrow |ds| \leq \frac{s^2}{(2s)(50)} = \frac{s}{100} = (1\%)s \Rightarrow$ the error must be no more than 1% of the true value.
47. Given $D = 100 \text{ cm}$, $dD = 1 \text{ cm}$, $V = \frac{4}{3} \pi \left(\frac{D}{2}\right)^3 = \frac{\pi D^3}{6} \Rightarrow dV = \frac{\pi}{2} D^2 dD = \frac{\pi}{2} (100)^2(1) = \frac{10^4 \pi}{2}$. Then $\frac{dV}{V} (100\%) = \left[\frac{\frac{10^4 \pi}{2}}{\frac{\pi D^3}{6}}\right] (10^2\%) = \left[\frac{\frac{10^6 \pi}{2}}{\frac{10^6 \pi}{6}}\right] \% = 3\%$
48. $V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi \left(\frac{D}{2}\right)^3 = \frac{\pi D^3}{6} \Rightarrow dV = \frac{\pi D^2}{2} dD$; recall that $\Delta V \approx dV$. Then $|\Delta V| \leq (3\%)V = \left(\frac{3}{100}\right) \left(\frac{\pi D^3}{6}\right) = \frac{\pi D^3}{200} \Rightarrow |dV| \leq \frac{\pi D^3}{200} \Rightarrow \left|\frac{\pi D^2}{2} dD\right| \leq \frac{\pi D^3}{200} \Rightarrow |dD| \leq \frac{D}{100} = (1\%)D \Rightarrow$ the allowable percentage error in measuring the diameter is 1%.
49. $E(x) = f(x) - g(x) \Rightarrow E(x) = f(x) - m(x - a) - c$. Then $E(a) = 0 \Rightarrow f(a) - m(a - a) - c = 0 \Rightarrow c = f(a)$. Next we calculate m : $\lim_{x \rightarrow a} \frac{E(x)}{x - a} = 0 \Rightarrow \lim_{x \rightarrow a} \frac{f(x) - m(x - a) - c}{x - a} = 0 \Rightarrow \lim_{x \rightarrow a} \left[\frac{f(x) - f(a)}{x - a} - m\right] = 0$ (since $c = f(a)$) $\Rightarrow f'(a) - m = 0 \Rightarrow m = f'(a)$. Therefore, $g(x) = m(x - a) + c = f'(a)(x - a) + f(a)$ is the linear approximation, as claimed.

50. (a) i. $Q(a) = f(a)$ implies that $b_0 = f(a)$.
 ii. Since $Q'(x) = b_1 + 2b_2(x - a)$, $Q'(a) = f'(a)$ implies that $b_1 = f'(a)$.
 iii. Since $Q''(x) = 2b_2$, $Q''(a) = f''(a)$ implies that $b_2 = \frac{f''(a)}{2}$.

In summary, $b_0 = f(a)$, $b_1 = f'(a)$, and $b_2 = \frac{f''(a)}{2}$.

(b) $f(x) = (1 - x)^{-1}$

$$f'(x) = -1(1 - x)^{-2}(-1) = (1 - x)^{-2}$$

$$f''(x) = -2(1 - x)^{-3}(-1) = 2(1 - x)^{-3}$$

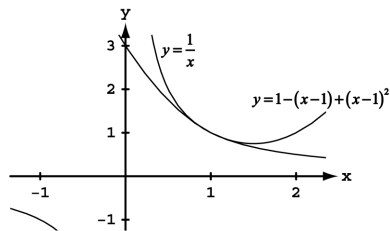
Since $f(0) = 1$, $f'(0) = 1$, and $f''(0) = 2$, the coefficients are $b_0 = 1$, $b_1 = 1$, $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 + x + x^2$.

(c) $g(x) = x^{-1}$

$$g'(x) = -1x^{-2}$$

$$g''(x) = 2x^{-3}$$

Since $g(1) = 1$, $g'(1) = -1$, and $g''(1) = 2$, the coefficients are $b_0 = 1$, $b_1 = -1$, $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 - (x - 1) + (x - 1)^2$.



[-1.35, 3.35] by [-1.25, 3.25]

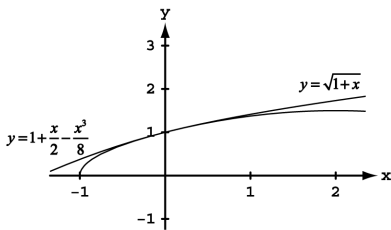
As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(d) $h(x) = (1 + x)^{1/2}$

$$h'(x) = \frac{1}{2}(1 + x)^{-1/2}$$

$$h''(x) = -\frac{1}{4}(1 + x)^{-3/2}$$

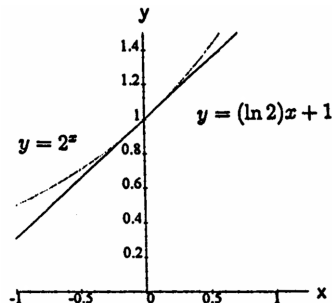
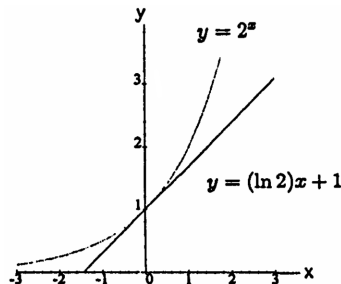
Since $h(0) = 1$, $h'(0) = \frac{1}{2}$, and $h''(0) = -\frac{1}{4}$, the coefficients are $b_0 = 1$, $b_1 = \frac{1}{2}$, $b_2 = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$. The quadratic approximation is $Q(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$.



As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

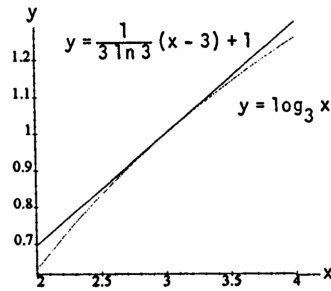
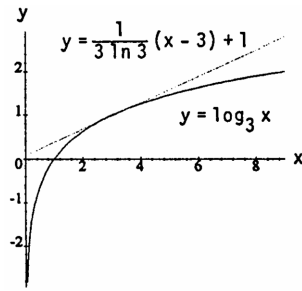
51. (a) $f(x) = 2^x \Rightarrow f'(x) = 2^x \ln 2$; $L(x) = (2^0 \ln 2)x + 2^0 = x \ln 2 + 1 \approx 0.69x + 1$

(b)



52. (a) $f(x) = \log_3 x \Rightarrow f'(x) = \frac{1}{x \ln 3}$, and $f(3) = \frac{\ln 3}{\ln 3} \Rightarrow L(x) = \frac{1}{3 \ln 3} (x - 3) + \frac{\ln 3}{\ln 3} = \frac{x}{3 \ln 3} - \frac{1}{\ln 3} + 1$
 $\approx 0.30x + 0.09$

(b)



53-58. Example CAS commands:

Maple:

```
with(plots):
a:= 1: f:=x -> x^3 + x^2 - 2*x;
plot(f(x), x=-1..2);
diff(f(x),x);
fp := unapply ("",x);
L:=x -> f(a) + fp(a)*(x - a);
plot({f(x), L(x)}, x=-1..2);
err:=x -> abs(f(x) - L(x));
plot(err(x), x=-1..2, title = #absolute error function#);
err(-1);
```

Mathematica: (function, x1, x2, and a may vary):

```
Clear[f, x]
{x1, x2} = {-1, 2}; a = 1;
f[x_]:=x^3 + x^2 - 2x
Plot[f[x], {x, x1, x2}]
lin[x_]:=f[a] + f'[a](x - a)
Plot[{f[x], lin[x]}, {x, x1, x2}]
err[x_]:=Abs[f[x] - lin[x]]
Plot[err[x], {x, x1, x2}]
err/N
```

After reviewing the error function, plot the error function and epsilon for differing values of epsilon (eps) and delta (del)

```
eps = 0.5; del = 0.4
Plot[{err[x], eps}, {x, a - del, a + del}]
```

CHAPTER 2 PRACTICE AND ADDITIONAL EXERCISES

- $y = x^5 - 0.125x^2 + 0.25x \Rightarrow \frac{dy}{dx} = 5x^4 - 0.25x + 0.25$
- $y = 3 - 0.7x^3 + 0.3x^7 \Rightarrow \frac{dy}{dx} = -2.1x^2 + 2.1x^6$
- $y = x^3 - 3(x^2 + \pi^2) \Rightarrow \frac{dy}{dx} = 3x^2 - 3(2x + 0) = 3x^2 - 6x = 3x(x - 2)$
- $y = x^7 + \sqrt{7}x - \frac{1}{\pi+1} \Rightarrow \frac{dy}{dx} = 7x^6 + \sqrt{7}$

$$5. y = (x+1)^2(x^2+2x) \Rightarrow \frac{dy}{dx} = (x+1)^2(2x+2) + (x^2+2x)(2(x+1)) = 2(x+1)[(x+1)^2 + x(x+2)] \\ = 2(x+1)(2x^2+4x+1)$$

$$6. y = (2x-5)(4-x)^{-1} \Rightarrow \frac{dy}{dx} = (2x-5)(-1)(4-x)^{-2}(-1) + (4-x)^{-1}(2) = (4-x)^{-2}[(2x-5) + 2(4-x)] \\ = 3(4-x)^{-2}$$

$$7. y = (\theta^2 + \sec \theta + 1)^3 \Rightarrow \frac{dy}{d\theta} = 3(\theta^2 + \sec \theta + 1)^2(2\theta + \sec \theta \tan \theta)$$

$$8. y = \left(-1 - \frac{\csc \theta}{2} - \frac{\theta^2}{4}\right)^2 \Rightarrow \frac{dy}{d\theta} = 2\left(-1 - \frac{\csc \theta}{2} - \frac{\theta^2}{4}\right)\left(\frac{\csc \theta \cot \theta}{2} - \frac{\theta}{2}\right) = \left(-1 - \frac{\csc \theta}{2} - \frac{\theta^2}{4}\right)(\csc \theta \cot \theta - \theta)$$

$$9. s = \frac{\sqrt{t}}{1+\sqrt{t}} \Rightarrow \frac{ds}{dt} = \frac{(1+\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} - \sqrt{t} \left(\frac{1}{2\sqrt{t}}\right)}{(1+\sqrt{t})^2} = \frac{(1+\sqrt{t}) - \sqrt{t}}{2\sqrt{t}(1+\sqrt{t})^2} = \frac{1}{2\sqrt{t}(1+\sqrt{t})^2}$$

$$10. s = \frac{1}{\sqrt{t}-1} \Rightarrow \frac{ds}{dt} = \frac{(\sqrt{t}-1)(0) - 1\left(\frac{1}{2\sqrt{t}}\right)}{(\sqrt{t}-1)^2} = \frac{-1}{2\sqrt{t}(\sqrt{t}-1)^2}$$

$$11. y = 2 \tan^2 x - \sec^2 x \Rightarrow \frac{dy}{dx} = (4 \tan x)(\sec^2 x) - (2 \sec x)(\sec x \tan x) = 2 \sec^2 x \tan x$$

$$12. y = \frac{1}{\sin^2 x} - \frac{2}{\sin x} = \csc^2 x - 2 \csc x \Rightarrow \frac{dy}{dx} = (2 \csc x)(-\csc x \cot x) - 2(-\csc x \cot x) = (2 \csc x \cot x)(1 - \csc x)$$

$$13. s = \cos^4(1-2t) \Rightarrow \frac{ds}{dt} = 4 \cos^3(1-2t)(-\sin(1-2t))(-2) = 8 \cos^3(1-2t) \sin(1-2t)$$

$$14. s = \cot^3\left(\frac{2}{t}\right) \Rightarrow \frac{ds}{dt} = 3 \cot^2\left(\frac{2}{t}\right)\left(-\csc^2\left(\frac{2}{t}\right)\right)\left(\frac{-2}{t^2}\right) = \frac{6}{t^2} \cot^2\left(\frac{2}{t}\right) \csc^2\left(\frac{2}{t}\right)$$

$$15. s = (\sec t + \tan t)^5 \Rightarrow \frac{ds}{dt} = 5(\sec t + \tan t)^4(\sec t \tan t + \sec^2 t) = 5(\sec t)(\sec t + \tan t)^5$$

$$16. s = \csc^5(1-t+3t^2) \Rightarrow \frac{ds}{dt} = 5 \csc^4(1-t+3t^2)(-\csc(1-t+3t^2) \cot(1-t+3t^2))(-1+6t) \\ = -5(6t-1) \csc^5(1-t+3t^2) \cot(1-t+3t^2)$$

$$17. r = \sqrt{2\theta \sin \theta} = (2\theta \sin \theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = \frac{1}{2}(2\theta \sin \theta)^{-1/2}(2\theta \cos \theta + 2 \sin \theta) = \frac{\theta \cos \theta + \sin \theta}{\sqrt{2\theta \sin \theta}}$$

$$18. r = 2\theta\sqrt{\cos \theta} = 2\theta(\cos \theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = 2\theta\left(\frac{1}{2}\right)(\cos \theta)^{-1/2}(-\sin \theta) + 2(\cos \theta)^{1/2} = \frac{-\theta \sin \theta}{\sqrt{\cos \theta}} + 2\sqrt{\cos \theta} \\ = \frac{2 \cos \theta - \theta \sin \theta}{\sqrt{\cos \theta}}$$

$$19. r = \sin \sqrt{2\theta} = \sin(2\theta)^{1/2} \Rightarrow \frac{dr}{d\theta} = \cos(2\theta)^{1/2} \left(\frac{1}{2}(2\theta)^{-1/2}(2)\right) = \frac{\cos \sqrt{2\theta}}{\sqrt{2\theta}}$$

$$20. r = \sin(\theta + \sqrt{\theta+1}) \Rightarrow \frac{dr}{d\theta} = \cos(\theta + \sqrt{\theta+1}) \left(1 + \frac{1}{2\sqrt{\theta+1}}\right) = \frac{2\sqrt{\theta+1}+1}{2\sqrt{\theta+1}} \cos(\theta + \sqrt{\theta+1})$$

$$21. y = \frac{1}{2}x^2 \csc \frac{2}{x} \Rightarrow \frac{dy}{dx} = \frac{1}{2}x^2\left(-\csc \frac{2}{x} \cot \frac{2}{x}\right)\left(\frac{-2}{x^2}\right) + \left(\csc \frac{2}{x}\right)\left(\frac{1}{2} \cdot 2x\right) = \csc \frac{2}{x} \cot \frac{2}{x} + x \csc \frac{2}{x}$$

$$22. y = 2\sqrt{x} \sin \sqrt{x} \Rightarrow \frac{dy}{dx} = 2\sqrt{x}(\cos \sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) + (\sin \sqrt{x})\left(\frac{2}{2\sqrt{x}}\right) = \cos \sqrt{x} + \frac{\sin \sqrt{x}}{\sqrt{x}}$$

$$23. y = x^{-1/2} \sec(2x)^2 \Rightarrow \frac{dy}{dx} = x^{-1/2} \sec(2x)^2 \tan(2x)^2 (2(2x) \cdot 2) + \sec(2x)^2 \left(-\frac{1}{2} x^{-3/2}\right) \\ = 8x^{1/2} \sec(2x)^2 \tan(2x)^2 - \frac{1}{2} x^{-3/2} \sec(2x)^2 = \frac{1}{2} x^{1/2} \sec(2x)^2 [16 \tan(2x)^2 - x^{-2}] \text{ or } \frac{1}{2x^{3/2}} \sec(2x)^2 [16x^2 \tan(2x)^2 - 1]$$

$$24. y = \sqrt{x} \csc(x+1)^3 = x^{1/2} \csc(x+1)^3 \\ \Rightarrow \frac{dy}{dx} = x^{1/2} (-\csc(x+1)^3 \cot(x+1)^3) (3(x+1)^2) + \csc(x+1)^3 \left(\frac{1}{2} x^{-1/2}\right) \\ = -3\sqrt{x} (x+1)^2 \csc(x+1)^3 \cot(x+1)^3 + \frac{\csc(x+1)^3}{2\sqrt{x}} = \frac{1}{2} \sqrt{x} \csc(x+1)^3 \left[\frac{1}{x} - 6(x+1)^2 \cot(x+1)^3\right] \\ \text{or } \frac{1}{2\sqrt{x}} \csc(x+1)^3 [1 - 6x(x+1)^2 \cot(x+1)^3]$$

$$25. y = 5 \cot x^2 \Rightarrow \frac{dy}{dx} = 5 (-\csc^2 x^2) (2x) = -10x \csc^2(x^2)$$

$$26. y = x^2 \cot 5x \Rightarrow \frac{dy}{dx} = x^2 (-\csc^2 5x) (5) + (\cot 5x)(2x) = -5x^2 \csc^2 5x + 2x \cot 5x$$

$$27. y = x^2 \sin^2(2x^2) \Rightarrow \frac{dy}{dx} = x^2 (2 \sin(2x^2)) (\cos(2x^2)) (4x) + \sin^2(2x^2) (2x) = 8x^3 \sin(2x^2) \cos(2x^2) + 2x \sin^2(2x^2)$$

$$28. y = x^{-2} \sin^2(x^3) \Rightarrow \frac{dy}{dx} = x^{-2} (2 \sin(x^3)) (\cos(x^3)) (3x^2) + \sin^2(x^3) (-2x^{-3}) = 6 \sin(x^3) \cos(x^3) - 2x^{-3} \sin^2(x^3)$$

$$29. s = \left(\frac{4t}{t+1}\right)^{-2} \Rightarrow \frac{ds}{dt} = -2 \left(\frac{4t}{t+1}\right)^{-3} \left(\frac{(t+1)(4) - (4t)(1)}{(t+1)^2}\right) = -2 \left(\frac{4t}{t+1}\right)^{-3} \frac{4}{(t+1)^2} = -\frac{(t+1)}{8t^3}$$

$$30. s = \frac{-1}{15(15t-1)^3} = -\frac{1}{15} (15t-1)^{-3} \Rightarrow \frac{ds}{dt} = -\frac{1}{15} (-3)(15t-1)^{-4} (15) = \frac{3}{(15t-1)^4}$$

$$31. y = \left(\frac{\sqrt{x}}{x+1}\right)^2 \Rightarrow \frac{dy}{dx} = 2 \left(\frac{\sqrt{x}}{x+1}\right) \cdot \frac{(x+1)\left(\frac{1}{2\sqrt{x}}\right) - (\sqrt{x})(1)}{(x+1)^2} = \frac{(x+1) - 2x}{(x+1)^3} = \frac{1-x}{(x+1)^3}$$

$$32. y = \left(\frac{2\sqrt{x}}{2\sqrt{x}+1}\right)^2 \Rightarrow \frac{dy}{dx} = 2 \left(\frac{2\sqrt{x}}{2\sqrt{x}+1}\right) \left(\frac{(2\sqrt{x}+1)\left(\frac{1}{\sqrt{x}}\right) - (2\sqrt{x})\left(\frac{1}{\sqrt{x}}\right)}{(2\sqrt{x}+1)^2}\right) = \frac{4\sqrt{x}\left(\frac{1}{\sqrt{x}}\right)}{(2\sqrt{x}+1)^3} = \frac{4}{(2\sqrt{x}+1)^3}$$

$$33. y = \sqrt{\frac{x^2+x}{x^2}} = \left(1 + \frac{1}{x}\right)^{1/2} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \left(1 + \frac{1}{x}\right)^{-1/2} \left(-\frac{1}{x^2}\right) = -\frac{1}{2x^2 \sqrt{1 + \frac{1}{x}}}$$

$$34. y = 4x\sqrt{x + \sqrt{x}} = 4x(x + x^{1/2})^{1/2} \Rightarrow \frac{dy}{dx} = 4x \left(\frac{1}{2}\right) (x + x^{1/2})^{-1/2} \left(1 + \frac{1}{2} x^{-1/2}\right) + (x + x^{1/2})^{1/2} (4) \\ = (x + \sqrt{x})^{-1/2} \left[2x \left(1 + \frac{1}{2\sqrt{x}}\right) + 4(x + \sqrt{x})\right] = (x + \sqrt{x})^{-1/2} (2x + \sqrt{x} + 4x + 4\sqrt{x}) = \frac{6x + 5\sqrt{x}}{\sqrt{x + \sqrt{x}}}$$

$$35. r = \left(\frac{\sin \theta}{\cos \theta - 1}\right)^2 \Rightarrow \frac{dr}{d\theta} = 2 \left(\frac{\sin \theta}{\cos \theta - 1}\right) \left[\frac{(\cos \theta - 1)(\cos \theta) - (\sin \theta)(-\sin \theta)}{(\cos \theta - 1)^2}\right] \\ = 2 \left(\frac{\sin \theta}{\cos \theta - 1}\right) \left(\frac{\cos^2 \theta - \cos \theta + \sin^2 \theta}{(\cos \theta - 1)^2}\right) = \frac{2 \sin \theta (1 - \cos \theta)}{(\cos \theta - 1)^3} = \frac{-2 \sin \theta}{(\cos \theta - 1)^2}$$

$$36. r = \left(\frac{\sin \theta + 1}{1 - \cos \theta}\right)^2 \Rightarrow \frac{dr}{d\theta} = 2 \left(\frac{\sin \theta + 1}{1 - \cos \theta}\right) \left[\frac{(1 - \cos \theta)(\cos \theta) - (\sin \theta + 1)(\sin \theta)}{(1 - \cos \theta)^2}\right] \\ = \frac{2(\sin \theta + 1)}{(1 - \cos \theta)^3} (\cos \theta - \cos^2 \theta - \sin^2 \theta - \sin \theta) = \frac{2(\sin \theta + 1)(\cos \theta - \sin \theta - 1)}{(1 - \cos \theta)^3}$$

$$37. y = (2x+1)\sqrt{2x+1} = (2x+1)^{3/2} \Rightarrow \frac{dy}{dx} = \frac{3}{2} (2x+1)^{1/2} (2) = 3\sqrt{2x+1}$$

$$38. y = 20(3x-4)^{1/4}(3x-4)^{-1/5} = 20(3x-4)^{1/20} \Rightarrow \frac{dy}{dx} = 20 \left(\frac{1}{20}\right) (3x-4)^{-19/20} (3) = \frac{3}{(3x-4)^{19/20}}$$

$$39. y = 3(5x^2 + \sin 2x)^{-3/2} \Rightarrow \frac{dy}{dx} = 3\left(-\frac{3}{2}\right)(5x^2 + \sin 2x)^{-5/2}[10x + (\cos 2x)(2)] = \frac{-9(5x + \cos 2x)}{(5x^2 + \sin 2x)^{5/2}}$$

$$40. y = (3 + \cos^3 3x)^{-1/3} \Rightarrow \frac{dy}{dx} = -\frac{1}{3}(3 + \cos^3 3x)^{-4/3}(3 \cos^2 3x)(-\sin 3x)(3) = \frac{3 \cos^2 3x \sin 3x}{(3 + \cos^3 3x)^{4/3}}$$

$$41. y = 10e^{-x/5} \Rightarrow \frac{dy}{dx} = (10)\left(-\frac{1}{5}\right)e^{-x/5} = -2e^{-x/5}$$

$$42. y = \sqrt{2}e^{\sqrt{2}x} \Rightarrow \frac{dy}{dx} = (\sqrt{2})(\sqrt{2})e^{\sqrt{2}x} = 2e^{\sqrt{2}x}$$

$$43. y = \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} \Rightarrow \frac{dy}{dx} = \frac{1}{4}[x(4e^{4x}) + e^{4x}(1)] - \frac{1}{16}(4e^{4x}) = xe^{4x} + \frac{1}{4}e^{4x} - \frac{1}{4}e^{4x} = xe^{4x}$$

$$44. y = x^2e^{-2/x} = x^2e^{-2x^{-1}} \Rightarrow \frac{dy}{dx} = x^2[(2x^{-2})e^{-2x^{-1}}] + e^{-2x^{-1}}(2x) = (2 + 2x)e^{-2x^{-1}} = 2e^{-2/x}(1 + x)$$

$$45. y = \ln(\sin^2 \theta) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sin \theta)(\cos \theta)}{\sin^2 \theta} = \frac{2 \cos \theta}{\sin \theta} = 2 \cot \theta$$

$$46. y = \ln(\sec^2 \theta) \Rightarrow \frac{dy}{d\theta} = \frac{2(\sec \theta)(\sec \theta \tan \theta)}{\sec^2 \theta} = 2 \tan \theta$$

$$47. y = \log_2\left(\frac{x^2}{2}\right) = \frac{\ln\left(\frac{x^2}{2}\right)}{\ln 2} \Rightarrow \frac{dy}{dx} = \frac{1}{\ln 2}\left(\frac{x}{\left(\frac{x^2}{2}\right)}\right) = \frac{2}{(\ln 2)x}$$

$$48. y = \log_5(3x - 7) = \frac{\ln(3x-7)}{\ln 5} \Rightarrow \frac{dy}{dx} = \left(\frac{1}{\ln 5}\right)\left(\frac{3}{3x-7}\right) = \frac{3}{(\ln 5)(3x-7)}$$

$$49. y = 8^{-t} \Rightarrow \frac{dy}{dt} = 8^{-t}(\ln 8)(-1) = -8^{-t}(\ln 8) \quad 50. y = 9^{2t} \Rightarrow \frac{dy}{dt} = 9^{2t}(\ln 9)(2) = 9^{2t}(2 \ln 9)$$

$$51. y = 5x^{3.6} \Rightarrow \frac{dy}{dx} = 5(3.6)x^{2.6} = 18x^{2.6}$$

$$52. y = \sqrt{2}x^{-\sqrt{2}} \Rightarrow \frac{dy}{dx} = (\sqrt{2})\left(-\sqrt{2}\right)x^{(-\sqrt{2}-1)} = -2x^{(-\sqrt{2}-1)}$$

$$53. y = (x+2)^{x+2} \Rightarrow \ln y = \ln(x+2)^{x+2} = (x+2) \ln(x+2) \Rightarrow \frac{y'}{y} = (x+2)\left(\frac{1}{x+2}\right) + (1) \ln(x+2) \\ \Rightarrow \frac{dy}{dx} = (x+2)^{x+2} [\ln(x+2) + 1]$$

$$54. y = 2(\ln x)^{x/2} \Rightarrow \ln y = \ln[2(\ln x)^{x/2}] = \ln(2) + \left(\frac{x}{2}\right) \ln(\ln x) \Rightarrow \frac{y'}{y} = 0 + \left(\frac{x}{2}\right) \left[\frac{\left(\frac{1}{\ln x}\right)}{\ln x}\right] + (\ln(\ln x))\left(\frac{1}{2}\right) \\ \Rightarrow y' = \left[\frac{1}{2 \ln x} + \left(\frac{1}{2}\right) \ln(\ln x)\right] 2(\ln x)^{x/2} = (\ln x)^{x/2} \left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

$$55. y = \sin^{-1} \sqrt{1-u^2} = \sin^{-1}(1-u^2)^{1/2} \Rightarrow \frac{dy}{du} = \frac{\frac{1}{2}(1-u^2)^{-1/2}(-2u)}{\sqrt{1-[(1-u^2)^{1/2}]^2}} = \frac{-u}{\sqrt{1-u^2}\sqrt{1-(1-u^2)}} = \frac{-u}{|u|\sqrt{1-u^2}} \\ = \frac{-u}{u\sqrt{1-u^2}} = \frac{-1}{\sqrt{1-u^2}}, 0 < u < 1$$

$$56. y = \sin^{-1}\left(\frac{1}{\sqrt{v}}\right) = \sin^{-1}v^{-1/2} \Rightarrow \frac{dy}{dv} = \frac{-\frac{1}{2}v^{-3/2}}{\sqrt{1-(v^{-1/2})^2}} = \frac{-1}{2v^{3/2}\sqrt{1-v^{-1}}} = \frac{-1}{2v^{3/2}\sqrt{\frac{v-1}{v}}} = \frac{-\sqrt{v}}{2v^{3/2}\sqrt{v-1}} \\ = \frac{-1}{2v\sqrt{v-1}}$$

$$57. y = \ln(\cos^{-1} x) \Rightarrow y' = \frac{\left(\frac{-1}{\sqrt{1-x^2}}\right)}{\cos^{-1} x} = \frac{-1}{\sqrt{1-x^2} \cos^{-1} x}$$

$$58. y = z \cos^{-1} z - \sqrt{1 - z^2} = z \cos^{-1} z - (1 - z^2)^{1/2} \Rightarrow \frac{dy}{dz} = \cos^{-1} z - \frac{z}{\sqrt{1 - z^2}} - \left(\frac{1}{2}\right) (1 - z^2)^{-1/2} (-2z) \\ = \cos^{-1} z - \frac{z}{\sqrt{1 - z^2}} + \frac{z}{\sqrt{1 - z^2}} = \cos^{-1} z$$

$$59. y = t \tan^{-1} t - \left(\frac{1}{2}\right) \ln t \Rightarrow \frac{dy}{dt} = \tan^{-1} t + t \left(\frac{1}{1+t^2}\right) - \left(\frac{1}{2}\right) \left(\frac{1}{t}\right) = \tan^{-1} t + \frac{t}{1+t^2} - \frac{1}{2t}$$

$$60. y = (1 + t^2) \cot^{-1} 2t \Rightarrow \frac{dy}{dt} = 2t \cot^{-1} 2t + (1 + t^2) \left(\frac{-2}{1+4t^2}\right)$$

$$61. y = z \sec^{-1} z - \sqrt{z^2 - 1} = z \sec^{-1} z - (z^2 - 1)^{1/2} \Rightarrow \frac{dy}{dz} = z \left(\frac{1}{|z| \sqrt{z^2 - 1}}\right) + (\sec^{-1} z) (1) - \frac{1}{2} (z^2 - 1)^{-1/2} (2z) \\ = \frac{z}{|z| \sqrt{z^2 - 1}} - \frac{z}{\sqrt{z^2 - 1}} + \sec^{-1} z = \frac{1 - z}{\sqrt{z^2 - 1}} + \sec^{-1} z, z > 1$$

$$62. y = 2\sqrt{x-1} \sec^{-1} \sqrt{x} = 2(x-1)^{1/2} \sec^{-1} (x^{1/2}) \\ \Rightarrow \frac{dy}{dx} = 2 \left[\left(\frac{1}{2}\right) (x-1)^{-1/2} \sec^{-1} (x^{1/2}) + (x-1)^{1/2} \left(\frac{\left(\frac{1}{2}\right) x^{-1/2}}{\sqrt{x} \sqrt{x-1}}\right) \right] = 2 \left(\frac{\sec^{-1} \sqrt{x}}{2\sqrt{x-1}} + \frac{1}{2x}\right) = \frac{\sec^{-1} \sqrt{x}}{\sqrt{x-1}} + \frac{1}{x}$$

$$63. y = \csc^{-1} (\sec \theta) \Rightarrow \frac{dy}{d\theta} = \frac{-\sec \theta \tan \theta}{|\sec \theta| \sqrt{\sec^2 \theta - 1}} = -\frac{\tan \theta}{|\tan \theta|} = -1, 0 < \theta < \frac{\pi}{2}$$

$$64. y = (1 + x^2) e^{\tan^{-1} x} \Rightarrow y' = 2xe^{\tan^{-1} x} + (1 + x^2) \left(\frac{e^{\tan^{-1} x}}{1+x^2}\right) = 2xe^{\tan^{-1} x} + e^{\tan^{-1} x}$$

$$65. xy + 2x + 3y = 1 \Rightarrow (xy' + y) + 2 + 3y' = 0 \Rightarrow xy' + 3y' = -2 - y \Rightarrow y'(x+3) = -2 - y \Rightarrow y' = -\frac{y+2}{x+3}$$

$$66. x^2 + xy + y^2 - 5x = 2 \Rightarrow 2x + \left(x \frac{dy}{dx} + y\right) + 2y \frac{dy}{dx} - 5 = 0 \Rightarrow x \frac{dy}{dx} + 2y \frac{dy}{dx} = 5 - 2x - y \Rightarrow \frac{dy}{dx} (x + 2y) \\ = 5 - 2x - y \Rightarrow \frac{dy}{dx} = \frac{5 - 2x - y}{x + 2y}$$

$$67. x^3 + 4xy - 3y^{4/3} = 2x \Rightarrow 3x^2 + \left(4x \frac{dy}{dx} + 4y\right) - 4y^{1/3} \frac{dy}{dx} = 2 \Rightarrow 4x \frac{dy}{dx} - 4y^{1/3} \frac{dy}{dx} = 2 - 3x^2 - 4y \\ \Rightarrow \frac{dy}{dx} (4x - 4y^{1/3}) = 2 - 3x^2 - 4y \Rightarrow \frac{dy}{dx} = \frac{2 - 3x^2 - 4y}{4x - 4y^{1/3}}$$

$$68. 5x^{4/5} + 10y^{6/5} = 15 \Rightarrow 4x^{-1/5} + 12y^{1/5} \frac{dy}{dx} = 0 \Rightarrow 12y^{1/5} \frac{dy}{dx} = -4x^{-1/5} \Rightarrow \frac{dy}{dx} = -\frac{1}{3} x^{-1/5} y^{-1/5} = -\frac{1}{3(xy)^{1/5}}$$

$$69. (xy)^{1/2} = 1 \Rightarrow \frac{1}{2} (xy)^{-1/2} \left(x \frac{dy}{dx} + y\right) = 0 \Rightarrow x^{1/2} y^{-1/2} \frac{dy}{dx} = -x^{-1/2} y^{1/2} \Rightarrow \frac{dy}{dx} = -x^{-1} y \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$70. x^2 y^2 = 1 \Rightarrow x^2 \left(2y \frac{dy}{dx}\right) + y^2 (2x) = 0 \Rightarrow 2x^2 y \frac{dy}{dx} = -2xy^2 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$71. y^2 = \frac{x}{x+1} \Rightarrow 2y \frac{dy}{dx} = \frac{(x+1)(1) - (x)(1)}{(x+1)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2y(x+1)^2}$$

$$72. y^2 = \left(\frac{1+x}{1-x}\right)^{1/2} \Rightarrow y^4 = \frac{1+x}{1-x} \Rightarrow 4y^3 \frac{dy}{dx} = \frac{(1-x)(1) - (1+x)(-1)}{(1-x)^2} \Rightarrow \frac{dy}{dx} = \frac{1}{2y^3(1-x)^2}$$

$$73. e^{x+2y} = 1 \Rightarrow e^{x+2y} \left(1 + 2\frac{dy}{dx}\right) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1}{2}$$

$$74. y^2 = 2e^{-1/x} \Rightarrow 2y \frac{dy}{dx} = 2e^{-1/x} \frac{d}{dx}(-x^{-1}) = \frac{2e^{-1/x}}{x^2} \Rightarrow \frac{dy}{dx} = \frac{e^{-1/x}}{yx^2}$$

$$75. \ln\left(\frac{x}{y}\right) = 1 \Rightarrow \frac{1}{x/y} \frac{d}{dx}\left(\frac{x}{y}\right) = 0 \Rightarrow \frac{y(1) - x \frac{dy}{dx}}{y^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

$$76. x \sin^{-1} y = 1 + x^2 \Rightarrow y = \sin(x^{-1} + x) \Rightarrow \frac{dy}{dx} = \cos(x^{-1} + x) \frac{d}{dx}(x^{-1} + x) = (1 - x^{-2})\cos(x^{-1} + x) \\ = \left(\frac{x^2-1}{x}\right)\cos\left(\frac{x^2+1}{x}\right)$$

$$77. y e^{\tan^{-1} x} = 2 \Rightarrow y = 2e^{-\tan^{-1} x} \Rightarrow \frac{dy}{dx} = 2e^{-\tan^{-1} x} \frac{d}{dx}(-\tan^{-1} x) = -2e^{-\tan^{-1} x} \left(\frac{1}{1+x^2}\right) = -\frac{2e^{-\tan^{-1} x}}{1+x^2}$$

$$78. x^y = \sqrt{2} \Rightarrow \ln(x^y) = \ln(2^{1/2}) \Rightarrow y \ln x = \frac{\ln 2}{2} \Rightarrow \frac{d}{dx}(y \ln x) = 0 \Rightarrow y\left(\frac{1}{x}\right) + \ln x \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x \ln x}$$

$$79. p^3 + 4pq - 3q^2 = 2 \Rightarrow 3p^2 \frac{dp}{dq} + 4\left(p + q \frac{dp}{dq}\right) - 6q = 0 \Rightarrow 3p^2 \frac{dp}{dq} + 4q \frac{dp}{dq} = 6q - 4p \Rightarrow \frac{dp}{dq}(3p^2 + 4q) = 6q - 4p \\ \Rightarrow \frac{dp}{dq} = \frac{6q - 4p}{3p^2 + 4q}$$

$$80. q = (5p^2 + 2p)^{-3/2} \Rightarrow 1 = -\frac{3}{2}(5p^2 + 2p)^{-5/2} \left(10p \frac{dp}{dq} + 2 \frac{dp}{dq}\right) \Rightarrow -\frac{2}{3}(5p^2 + 2p)^{5/2} = \frac{dp}{dq}(10p + 2) \\ \Rightarrow \frac{dp}{dq} = -\frac{(5p^2 + 2p)^{5/2}}{3(5p + 1)}$$

$$81. r \cos 2s + \sin^2 s = \pi \Rightarrow r(-\sin 2s)(2) + (\cos 2s) \left(\frac{dr}{ds}\right) + 2 \sin s \cos s = 0 \Rightarrow \frac{dr}{ds}(\cos 2s) = 2r \sin 2s - 2 \sin s \cos s \\ \Rightarrow \frac{dr}{ds} = \frac{2r \sin 2s - \sin 2s}{\cos 2s} = \frac{(2r - 1)(\sin 2s)}{\cos 2s} = (2r - 1)(\tan 2s)$$

$$82. 2rs - r - s + s^2 = -3 \Rightarrow 2\left(r + s \frac{dr}{ds}\right) - \frac{dr}{ds} - 1 + 2s = 0 \Rightarrow \frac{dr}{ds}(2s - 1) = 1 - 2s - 2r \Rightarrow \frac{dr}{ds} = \frac{1 - 2s - 2r}{2s - 1}$$

$$83. (a) x^3 + y^3 = 1 \Rightarrow 3x^2 + 3y^2 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x^2}{y^2} \Rightarrow \frac{d^2y}{dx^2} = \frac{y^2(-2x) - (-x^2)\left(2y \frac{dy}{dx}\right)}{y^4} \\ \Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy^2 + (2yx^2)\left(-\frac{x^2}{y^2}\right)}{y^4} = \frac{-2xy^2 - \frac{2x^4}{y}}{y^4} = \frac{-2xy^3 - 2x^4}{y^5}$$

$$(b) y^2 = 1 - \frac{2}{x} \Rightarrow 2y \frac{dy}{dx} = \frac{2}{x^2} \Rightarrow \frac{dy}{dx} = \frac{1}{yx^2} \Rightarrow \frac{dy}{dx} = (yx^2)^{-1} \Rightarrow \frac{d^2y}{dx^2} = -(yx^2)^{-2} \left[y(2x) + x^2 \frac{dy}{dx}\right] \\ \Rightarrow \frac{d^2y}{dx^2} = \frac{-2xy - x^2\left(\frac{1}{yx^2}\right)}{y^2x^4} = \frac{-2xy^2 - 1}{y^3x^4}$$

$$84. (a) x^2 - y^2 = 1 \Rightarrow 2x - 2y \frac{dy}{dx} = 0 \Rightarrow -2y \frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$(b) \frac{dy}{dx} = \frac{x}{y} \Rightarrow \frac{d^2y}{dx^2} = \frac{y(1) - x \frac{dy}{dx}}{y^2} = \frac{y - x\left(\frac{x}{y}\right)}{y^2} = \frac{y^2 - x^2}{y^3} = \frac{-1}{y^3} \text{ (since } y^2 - x^2 = -1)$$

$$85. (a) \text{ Let } h(x) = 6f(x) - g(x) \Rightarrow h'(x) = 6f'(x) - g'(x) \Rightarrow h'(1) = 6f'(1) - g'(1) = 6\left(\frac{1}{2}\right) - (-4) = 7$$

$$(b) \text{ Let } h(x) = f(x)g^2(x) \Rightarrow h'(x) = f(x)(2g(x))g'(x) + g^2(x)f'(x) \Rightarrow h'(0) = 2f(0)g(0)g'(0) + g^2(0)f'(0) \\ = 2(1)(1)\left(\frac{1}{2}\right) + (1)^2(-3) = -2$$

$$(c) \text{ Let } h(x) = \frac{f(x)}{g(x)+1} \Rightarrow h'(x) = \frac{(g(x)+1)f'(x) - f(x)g'(x)}{(g(x)+1)^2} \Rightarrow h'(1) = \frac{(g(1)+1)f'(1) - f(1)g'(1)}{(g(1)+1)^2} = \frac{(5+1)\left(\frac{1}{2}\right) - 3(-4)}{(5+1)^2} = \frac{5}{12}$$

$$(d) \text{ Let } h(x) = f(g(x)) \Rightarrow h'(x) = f'(g(x))g'(x) \Rightarrow h'(0) = f'(g(0))g'(0) = f'(1)\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{4}$$

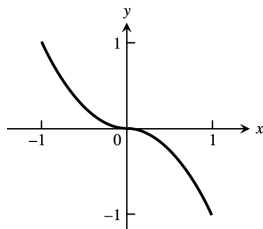
$$(e) \text{ Let } h(x) = g(f(x)) \Rightarrow h'(x) = g'(f(x))f'(x) \Rightarrow h'(0) = g'(f(0))f'(0) = g'(1)f'(0) = (-4)(-3) = 12$$

$$(f) \text{ Let } h(x) = (x + f(x))^{3/2} \Rightarrow h'(x) = \frac{3}{2}(x + f(x))^{1/2}(1 + f'(x)) \Rightarrow h'(1) = \frac{3}{2}(1 + f(1))^{1/2}(1 + f'(1)) \\ = \frac{3}{2}(1 + 3)^{1/2}\left(1 + \frac{1}{2}\right) = \frac{9}{2}$$

$$(g) \text{ Let } h(x) = f(x + g(x)) \Rightarrow h'(x) = f'(x + g(x))(1 + g'(x)) \Rightarrow h'(0) = f'(g(0))(1 + g'(0)) = f'(1)\left(1 + \frac{1}{2}\right) = \left(\frac{1}{2}\right)\left(\frac{3}{2}\right) \\ = \frac{3}{4}$$

86. (a) Let $h(x) = \sqrt{x}f(x) \Rightarrow h'(x) = \sqrt{x}f'(x) + f(x) \cdot \frac{1}{2\sqrt{x}} \Rightarrow h'(1) = \sqrt{1}f'(1) + f(1) \cdot \frac{1}{2\sqrt{1}} = \frac{1}{5} + (-3)\left(\frac{1}{2}\right) = -\frac{13}{10}$
 (b) Let $h(x) = (f(x))^{1/2} \Rightarrow h'(x) = \frac{1}{2}(f(x))^{-1/2}(f'(x)) \Rightarrow h'(0) = \frac{1}{2}(f(0))^{-1/2}f'(0) = \frac{1}{2}(9)^{-1/2}(-2) = -\frac{1}{3}$
 (c) Let $h(x) = f(\sqrt{x}) \Rightarrow h'(x) = f'(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} \Rightarrow h'(1) = f'(\sqrt{1}) \cdot \frac{1}{2\sqrt{1}} = \frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$
 (d) Let $h(x) = f(1 - 5 \tan x) \Rightarrow h'(x) = f'(1 - 5 \tan x)(-5 \sec^2 x) \Rightarrow h'(0) = f'(1 - 5 \tan 0)(-5 \sec^2 0) = f'(1)(-5) = \frac{1}{5}(-5) = -1$
 (e) Let $h(x) = \frac{f(x)}{2 + \cos x} \Rightarrow h'(x) = \frac{(2 + \cos x)f'(x) - f(x)(-\sin x)}{(2 + \cos x)^2} \Rightarrow h'(0) = \frac{(2 + 1)f'(0) - f(0)(0)}{(2 + 1)^2} = \frac{3(-2)}{9} = -\frac{2}{3}$
 (f) Let $h(x) = 10 \sin\left(\frac{\pi x}{2}\right)f^2(x) \Rightarrow h'(x) = 10 \sin\left(\frac{\pi x}{2}\right)(2f(x)f'(x)) + f^2(x)\left(10 \cos\left(\frac{\pi x}{2}\right)\right)\left(\frac{\pi}{2}\right) \Rightarrow h'(1) = 10 \sin\left(\frac{\pi}{2}\right)(2f(1)f'(1)) + f^2(1)\left(10 \cos\left(\frac{\pi}{2}\right)\right)\left(\frac{\pi}{2}\right) = 20(-3)\left(\frac{1}{5}\right) + 0 = -12$
87. $x = t^2 + \pi \Rightarrow \frac{dx}{dt} = 2t$; $y = 3 \sin 2x \Rightarrow \frac{dy}{dx} = 3(\cos 2x)(2) = 6 \cos 2x = 6 \cos(2t^2 + 2\pi) = 6 \cos(2t^2)$; thus,
 $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = 6 \cos(2t^2) \cdot 2t \Rightarrow \frac{dy}{dt} \Big|_{t=0} = 6 \cos(0) \cdot 0 = 0$
88. $t = (u^2 + 2u)^{1/3} \Rightarrow \frac{dt}{du} = \frac{1}{3}(u^2 + 2u)^{-2/3}(2u + 2) = \frac{2}{3}(u^2 + 2u)^{-2/3}(u + 1)$; $s = t^2 + 5t \Rightarrow \frac{ds}{dt} = 2t + 5$
 $= 2(u^2 + 2u)^{1/3} + 5$; thus $\frac{ds}{du} = \frac{ds}{dt} \cdot \frac{dt}{du} = \left[2(u^2 + 2u)^{1/3} + 5\right]\left(\frac{2}{3}\right)(u^2 + 2u)^{-2/3}(u + 1)$
 $\Rightarrow \frac{ds}{du} \Big|_{u=2} = \left[2(2^2 + 2(2))^{1/3} + 5\right]\left(\frac{2}{3}\right)(2^2 + 2(2))^{-2/3}(2 + 1) = 2(2 \cdot 8^{1/3} + 5)(8^{-2/3}) = 2(2 \cdot 2 + 5)\left(\frac{1}{4}\right) = \frac{9}{2}$
89. $\frac{dw}{ds} = \frac{dw}{dr} \cdot \frac{dr}{ds} = \left[\cos\left(e^{\sqrt{r}}\right)\left(e^{\sqrt{r}} \frac{1}{2\sqrt{r}}\right)\right]\left[3 \cos\left(s + \frac{\pi}{6}\right)\right]$ at $x = 0$, $r = 3 \sin \frac{\pi}{6} = \frac{3}{2}$
 $\Rightarrow \frac{dw}{ds} = \cos\left(e^{\sqrt{3/2}}\right)\left(e^{\sqrt{3/2}} \frac{1}{2\sqrt{3/2}}\right)(3 \cos(\frac{\pi}{6})) = \frac{3\sqrt{3}e^{\sqrt{3/2}}}{4\sqrt{3/2}} \cos\left(e^{\sqrt{3/2}}\right) = \frac{3\sqrt{2}e^{\sqrt{3/2}}}{4} \cos\left(e^{\sqrt{3/2}}\right)$
90. $\frac{dt}{dt} = \frac{d\theta}{d\theta} \frac{d\theta}{dt}$; $\frac{d\theta}{d\theta} = \frac{1}{3}(\theta^2 + 7)^{-2/3}(2\theta)$; $\theta^2 e^t + \theta = 1 \Rightarrow \frac{d}{dt}(\theta^2 e^t + \theta) = \frac{d}{dt}(1) \Rightarrow \theta^2 e^t + 2\theta \frac{d\theta}{dt} e^t + \frac{d\theta}{dt} = 0$
 $\Rightarrow (1 + 2\theta e^t) \frac{d\theta}{dt} = -\theta^2 e^t \Rightarrow \frac{d\theta}{dt} = -\frac{\theta^2 e^t}{1 + 2\theta e^t} \Rightarrow \frac{dr}{dt} = \left[\frac{2\theta}{3(\theta^2 + 7)^{2/3}}\right]\left[-\frac{\theta^2 e^t}{1 + 2\theta e^t}\right] = -\frac{2\theta^3 e^t}{3(1 + 2\theta e^t)(\theta^2 + 7)^{2/3}}$
 At $t = 0$, $\theta^2 + \theta - 1 = 0 \Rightarrow \theta = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow \frac{dr}{dt} = -\frac{2\left(\frac{-1 \pm \sqrt{5}}{2}\right)^3}{3\left(1 + (-1 \pm \sqrt{5})\right)\left(\left(\frac{-1 \pm \sqrt{5}}{2}\right)^2 + 7\right)^{2/3}}$
91. $y^3 + y = 2 \cos x \Rightarrow 3y^2 \frac{dy}{dx} + \frac{dy}{dx} = -2 \sin x \Rightarrow \frac{dy}{dx}(3y^2 + 1) = -2 \sin x \Rightarrow \frac{dy}{dx} = \frac{-2 \sin x}{3y^2 + 1} \Rightarrow \frac{dy}{dx} \Big|_{(0,1)} = \frac{-2 \sin(0)}{3 + 1} = 0$; $\frac{d^2y}{dx^2} = \frac{(3y^2 + 1)(-2 \cos x) - (-2 \sin x)(6y \frac{dy}{dx})}{(3y^2 + 1)^2} \Rightarrow \frac{d^2y}{dx^2} \Big|_{(0,1)} = \frac{(3 + 1)(-2 \cos 0) - (-2 \sin 0)(6 \cdot 0)}{(3 + 1)^2} = -\frac{1}{2}$
92. $x^{1/3} + y^{1/3} = 4 \Rightarrow \frac{1}{3}x^{-2/3} + \frac{1}{3}y^{-2/3} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y^{2/3}}{x^{2/3}} \Rightarrow \frac{dy}{dx} \Big|_{(8,8)} = -1$; $\frac{dy}{dx} = \frac{-y^{2/3}}{x^{2/3}}$
 $\Rightarrow \frac{d^2y}{dx^2} = \frac{(x^{2/3})\left(-\frac{2}{3}y^{-1/3} \frac{dy}{dx}\right) - (-y^{2/3})\left(\frac{2}{3}x^{-1/3}\right)}{(x^{2/3})^2} \Rightarrow \frac{d^2y}{dx^2} \Big|_{(8,8)} = \frac{(8^{2/3})\left[-\frac{2}{3} \cdot 8^{-1/3} \cdot (-1)\right] + (8^{2/3})\left(\frac{2}{3} \cdot 8^{-1/3}\right)}{8^{4/3}}$
 $= \frac{\frac{1}{3} + \frac{1}{3}}{8^{2/3}} = \frac{\frac{2}{3}}{\frac{4}{8}} = \frac{1}{6}$
93. $f(t) = \frac{1}{2t+1}$ and $f(t+h) = \frac{1}{2(t+h)+1} \Rightarrow \frac{f(t+h)-f(t)}{h} = \frac{\frac{1}{2(t+h)+1} - \frac{1}{2t+1}}{h} = \frac{2t+1 - (2t+2h+1)}{(2t+2h+1)(2t+1)h} = \frac{-2h}{(2t+2h+1)(2t+1)h} = \frac{-2}{(2t+2h+1)(2t+1)} \Rightarrow f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h)-f(t)}{h} = \lim_{h \rightarrow 0} \frac{-2}{(2t+2h+1)(2t+1)} = \frac{-2}{(2t+1)^2}$
94. $g(x) = 2x^2 + 1$ and $g(x+h) = 2(x+h)^2 + 1 = 2x^2 + 4xh + 2h^2 + 1 \Rightarrow \frac{g(x+h)-g(x)}{h} = \frac{(2x^2 + 4xh + 2h^2 + 1) - (2x^2 + 1)}{h} = \frac{4xh + 2h^2}{h} = 4x + 2h \Rightarrow g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h)-g(x)}{h} = \lim_{h \rightarrow 0} (4x + 2h) = 4x$

95. (a)

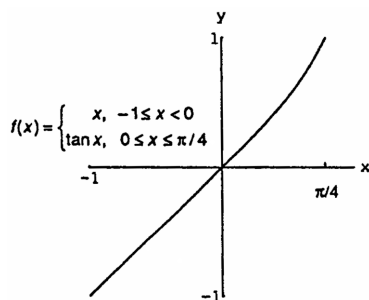


$$f(x) = \begin{cases} x^2, & -1 \leq x < 0 \\ -x^2, & 0 \leq x < 1 \end{cases}$$

(b) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^2 = 0$ and $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} -x^2 = 0 \Rightarrow \lim_{x \rightarrow 0} f(x) = 0$. Since $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$ it follows that f is continuous at $x = 0$.

(c) $\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} (2x) = 0$ and $\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} (-2x) = 0 \Rightarrow \lim_{x \rightarrow 0} f'(x) = 0$. Since this limit exists, it follows that f is differentiable at $x = 0$.

96. (a)



$$f(x) = \begin{cases} x, & -1 \leq x < 0 \\ \tan x, & 0 \leq x \leq \pi/4 \end{cases}$$

(b) $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x = 0$ and $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \tan x = 0 \Rightarrow \lim_{x \rightarrow 0} f(x) = 0$. Since $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$, it follows that f is continuous at $x = 0$.

(c) $\lim_{x \rightarrow 0^-} f'(x) = \lim_{x \rightarrow 0^-} 1 = 1$ and $\lim_{x \rightarrow 0^+} f'(x) = \lim_{x \rightarrow 0^+} \sec^2 x = 1 \Rightarrow \lim_{x \rightarrow 0} f'(x) = 1$. Since this limit exists it follows that f is differentiable at $x = 0$.

97. $y = \frac{x}{2} + \frac{1}{2x-4} = \frac{1}{2}x + (2x-4)^{-1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} - 2(2x-4)^{-2}$; the slope of the tangent is $-\frac{3}{2} \Rightarrow -\frac{3}{2}$
 $= \frac{1}{2} - 2(2x-4)^{-2} \Rightarrow -2 = -2(2x-4)^{-2} \Rightarrow 1 = \frac{1}{(2x-4)^2} \Rightarrow (2x-4)^2 = 1 \Rightarrow 4x^2 - 16x + 16 = 1$
 $\Rightarrow 4x^2 - 16x + 15 = 0 \Rightarrow (2x-5)(2x-3) = 0 \Rightarrow x = \frac{5}{2}$ or $x = \frac{3}{2} \Rightarrow (\frac{5}{2}, \frac{9}{4})$ and $(\frac{3}{2}, -\frac{1}{4})$ are points on the curve where the slope is $-\frac{3}{2}$.

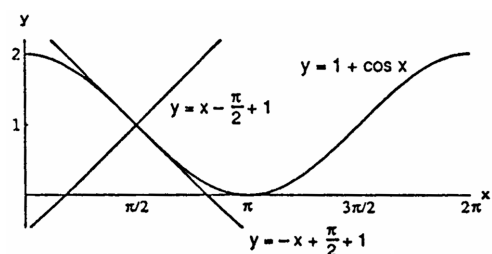
98. $y = x - e^{-x}$; $\frac{dy}{dx} = 1 + e^{-x} = 2 \Rightarrow e^{-x} = 1 \Rightarrow x = 0 \Rightarrow y = 0 - e^0 = -1$. Therefore, the curve has a tangent with a slope of 2 at the point $(0, -1)$.

99. $y = 2x^3 - 3x^2 - 12x + 20 \Rightarrow \frac{dy}{dx} = 6x^2 - 6x - 12$

(a) The tangent is perpendicular to the line $y = 1 - \frac{x}{24}$ when $\frac{dy}{dx} = -\left(-\frac{1}{24}\right) = 24$; $6x^2 - 6x - 12 = 24$
 $\Rightarrow x^2 - x - 2 = 4 \Rightarrow x^2 - x - 6 = 0 \Rightarrow (x-3)(x+2) = 0 \Rightarrow x = -2$ or $x = 3 \Rightarrow (-2, 16)$ and $(3, 11)$ are points where the tangent is perpendicular to $y = 1 - \frac{x}{24}$.

(b) The tangent is parallel to the line $y = \sqrt{2} - 12x$ when $\frac{dy}{dx} = -12 \Rightarrow 6x^2 - 6x - 12 = -12 \Rightarrow x^2 - x = 0$
 $\Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1 \Rightarrow (0, 20)$ and $(1, 7)$ are points where the tangent is parallel to $y = \sqrt{2} - 12x$.

100. $y = 1 + \cos x \Rightarrow \frac{dy}{dx} = -\sin x \Rightarrow \frac{dy}{dx}\bigg|_{(\frac{\pi}{2}, 1)} = -1$
 $\Rightarrow$ the tangent at $(\frac{\pi}{2}, 1)$ is the line $y - 1 = -(x - \frac{\pi}{2})$
 $\Rightarrow y = -x + \frac{\pi}{2} + 1$; the normal at $(\frac{\pi}{2}, 1)$ is
 $y - 1 = (1)(x - \frac{\pi}{2}) \Rightarrow y = x - \frac{\pi}{2} + 1$



101. $y = x^2 + C \Rightarrow \frac{dy}{dx} = 2x$ and $y = x \Rightarrow \frac{dy}{dx} = 1$; the parabola is tangent to $y = x$ when $2x = 1 \Rightarrow x = \frac{1}{2} \Rightarrow y = \frac{1}{2}$;
 thus, $\frac{1}{2} = (\frac{1}{2})^2 + C \Rightarrow C = \frac{1}{4}$
102. $y = x^3 \Rightarrow \frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx}\bigg|_{x=a} = 3a^2 \Rightarrow$ the tangent line at (a, a^3) is $y - a^3 = 3a^2(x - a)$. The tangent line intersects
 $y = x^3$ when $x^3 - a^3 = 3a^2(x - a) \Rightarrow (x - a)(x^2 + xa + a^2) = 3a^2(x - a) \Rightarrow (x - a)(x^2 + xa - 2a^2) = 0$
 $\Rightarrow (x - a)^2(x + 2a) = 0 \Rightarrow x = a$ or $x = -2a$. Now $\frac{dy}{dx}\bigg|_{x=-2a} = 3(-2a)^2 = 12a^2 = 4(3a^2)$, so the slope at $x = -2a$ is 4
 times as large as the slope at (a, a^3) where $x = a$.
103. $x^2 + 2y^2 = 9 \Rightarrow 2x + 4y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{2y} \Rightarrow \frac{dy}{dx}\bigg|_{(1,2)} = -\frac{1}{4} \Rightarrow$ the tangent line is $y - 2 = -\frac{1}{4}(x - 1) = -\frac{1}{4}x + \frac{9}{4}$
 and the normal line is $y - 2 = 4(x - 1) = 4x - 2$.
104. $e^x + y^2 = 2 \Rightarrow \frac{d}{dx}(e^x + y^2) = \frac{d}{dx}(2) \Rightarrow e^x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{e^x}{2y} \Rightarrow m_{\tan} = \frac{dy}{dx}\bigg|_{(0,1)} = -\frac{e^0}{2(1)} = -\frac{1}{2}$;
 $m_{\perp} = -\frac{1}{m_{\tan}} = 2$; tangent line: $y - 1 = -\frac{1}{2}(x - 0) \Rightarrow y = 1 - \frac{x}{2}$; normal line: $y - 1 = 2(x - 0) \Rightarrow y = 2x + 1$
105. $xy + 2x - 5y = 2 \Rightarrow (x \frac{dy}{dx} + y) + 2 - 5 \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(x - 5) = -y - 2 \Rightarrow \frac{dy}{dx} = \frac{-y-2}{x-5} \Rightarrow \frac{dy}{dx}\bigg|_{(3,2)} = 2$
 $\Rightarrow$ the tangent line is $y - 2 = 2(x - 3) = 2x - 4$ and the normal line is $y - 2 = -\frac{1}{2}(x - 3) = -\frac{1}{2}x + \frac{7}{2}$.
106. $x + \sqrt{xy} = 6 \Rightarrow 1 + \frac{1}{2\sqrt{xy}}(x \frac{dy}{dx} + y) = 0 \Rightarrow x \frac{dy}{dx} + y = -2\sqrt{xy} \Rightarrow \frac{dy}{dx} = \frac{-2\sqrt{xy}-y}{x} \Rightarrow \frac{dy}{dx}\bigg|_{(4,1)} = -\frac{5}{4}$
 $\Rightarrow$ the tangent line is $y - 1 = -\frac{5}{4}(x - 4) = -\frac{5}{4}x + 6$ and the normal line is $y - 1 = \frac{4}{5}(x - 4) = \frac{4}{5}x - \frac{11}{5}$.
107. $\lim_{x \rightarrow 0} \frac{\sin x}{2x^2 - x} = \lim_{x \rightarrow 0} \left[\left(\frac{\sin x}{x} \right) \cdot \frac{1}{(2x-1)} \right] = (1) \left(\frac{1}{-1} \right) = -1$
108. $\lim_{x \rightarrow 0} \frac{3x - \tan 7x}{2x} = \lim_{x \rightarrow 0} \left(\frac{3x}{2x} - \frac{\sin 7x}{2x \cos 7x} \right) = \frac{3}{2} - \lim_{x \rightarrow 0} \left(\frac{1}{\cos 7x} \cdot \frac{\sin 7x}{7x} \cdot \frac{1}{(\frac{7}{x})} \right) = \frac{3}{2} - \left(1 \cdot 1 \cdot \frac{7}{2} \right) = -2$
109. $\lim_{r \rightarrow 0} \frac{\sin r}{\tan 2r} = \lim_{r \rightarrow 0} \left(\frac{\sin r}{r} \cdot \frac{2r}{\tan 2r} \cdot \frac{1}{2} \right) = \left(\frac{1}{2} \right) (1) \lim_{r \rightarrow 0} \frac{\cos 2r}{(\frac{\sin 2r}{2r})} = \left(\frac{1}{2} \right) (1) \left(\frac{1}{1} \right) = \frac{1}{2}$
110. $\lim_{\theta \rightarrow 0} \frac{\sin(\sin \theta)}{\theta} = \lim_{\theta \rightarrow 0} \left(\frac{\sin(\sin \theta)}{\sin \theta} \right) \left(\frac{\sin \theta}{\theta} \right) = \lim_{\theta \rightarrow 0} \frac{\sin(\sin \theta)}{\sin \theta}$. Let $x = \sin \theta$. Then $x \rightarrow 0$ as $\theta \rightarrow 0$
 $\Rightarrow \lim_{\theta \rightarrow 0} \frac{\sin(\sin \theta)}{\sin \theta} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$
111. $\lim_{\theta \rightarrow (\frac{\pi}{2})^-} \frac{4 \tan^2 \theta + \tan \theta + 1}{\tan^2 \theta + 5} = \lim_{\theta \rightarrow (\frac{\pi}{2})^-} \frac{4 + \frac{1}{\tan \theta} + \frac{1}{\tan^2 \theta}}{1 + \frac{5}{\tan^2 \theta}} = \frac{(4+0+0)}{(1+0)} = 4$

112. $\lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x} = \lim_{x \rightarrow 0} \frac{x \sin x}{2(1 - \cos x)} = \lim_{x \rightarrow 0} \frac{x \sin x}{2(2 \sin^2(\frac{x}{2}))} = \lim_{x \rightarrow 0} \left[\frac{\frac{x}{2} \cdot \frac{x}{2}}{\sin^2(\frac{x}{2})} \cdot \frac{\sin x}{x} \right]$
 $= \lim_{x \rightarrow 0} \left[\frac{(\frac{x}{2})}{\sin(\frac{x}{2})} \cdot \frac{(\frac{x}{2})}{\sin(\frac{x}{2})} \cdot \frac{\sin x}{x} \right] = (1)(1)(1) = 1$
113. $y = \frac{2(x^2+1)}{\sqrt{\cos 2x}} \Rightarrow \ln y = \ln \left(\frac{2(x^2+1)}{\sqrt{\cos 2x}} \right) = \ln(2) + \ln(x^2+1) - \frac{1}{2} \ln(\cos 2x) \Rightarrow \frac{y'}{y} = 0 + \frac{2x}{x^2+1} - \left(\frac{1}{2}\right) \frac{(-2 \sin 2x)}{\cos 2x}$
 $\Rightarrow y' = \left(\frac{2x}{x^2+1} + \tan 2x \right) y = \frac{2(x^2+1)}{\sqrt{\cos 2x}} \left(\frac{2x}{x^2+1} + \tan 2x \right)$
114. $y = \sqrt[10]{\frac{3x+4}{2x-4}} \Rightarrow \ln y = \ln \sqrt[10]{\frac{3x+4}{2x-4}} = \frac{1}{10} [\ln(3x+4) - \ln(2x-4)] \Rightarrow \frac{y'}{y} = \frac{1}{10} \left(\frac{3}{3x+4} - \frac{2}{2x-4} \right)$
 $\Rightarrow y' = \frac{1}{10} \left(\frac{3}{3x+4} - \frac{1}{x-2} \right) y = \sqrt[10]{\frac{3x+4}{2x-4}} \left(\frac{1}{10} \right) \left(\frac{3}{3x+4} - \frac{1}{x-2} \right)$
115. $y = \left[\frac{(t+1)(t-1)}{(t-2)(t+3)} \right]^5 \Rightarrow \ln y = 5 [\ln(t+1) + \ln(t-1) - \ln(t-2) - \ln(t+3)] \Rightarrow \left(\frac{1}{y} \right) \left(\frac{dy}{dt} \right)$
 $= 5 \left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3} \right) \Rightarrow \frac{dy}{dt} = 5 \left[\frac{(t+1)(t-1)}{(t-2)(t+3)} \right]^5 \left(\frac{1}{t+1} + \frac{1}{t-1} - \frac{1}{t-2} - \frac{1}{t+3} \right)$
116. $y = \frac{2u^2}{\sqrt{u^2+1}} \Rightarrow \ln y = \ln 2 + \ln u + u \ln 2 - \frac{1}{2} \ln(u^2+1) \Rightarrow \left(\frac{1}{y} \right) \left(\frac{dy}{du} \right) = \frac{1}{u} + \ln 2 - \frac{1}{2} \left(\frac{2u}{u^2+1} \right)$
 $\Rightarrow \frac{dy}{du} = \frac{2u^2}{\sqrt{u^2+1}} \left(\frac{1}{u} + \ln 2 - \frac{u}{u^2+1} \right)$
117. $y = (\sin \theta)^{\sqrt{\theta}} \Rightarrow \ln y = \sqrt{\theta} \ln(\sin \theta) \Rightarrow \left(\frac{1}{y} \right) \left(\frac{dy}{d\theta} \right) = \sqrt{\theta} \left(\frac{\cos \theta}{\sin \theta} \right) + \frac{1}{2} \theta^{-1/2} \ln(\sin \theta)$
 $\Rightarrow \frac{dy}{d\theta} = (\sin \theta)^{\sqrt{\theta}} \left(\sqrt{\theta} \cot \theta + \frac{\ln(\sin \theta)}{2\sqrt{\theta}} \right)$
118. $y = (\ln x)^{1/\ln x} \Rightarrow \ln y = \left(\frac{1}{\ln x} \right) \ln(\ln x) \Rightarrow \frac{y'}{y} = \left(\frac{1}{\ln x} \right) \left(\frac{1}{\ln x} \right) \left(\frac{1}{x} \right) + \ln(\ln x) \left[\frac{-1}{(\ln x)^2} \right] \left(\frac{1}{x} \right)$
 $\Rightarrow y' = (\ln x)^{1/\ln x} \left[\frac{1 - \ln(\ln x)}{x(\ln x)^2} \right]$
119. (a) $S = 2\pi r^2 + 2\pi rh$ and h constant $\Rightarrow \frac{dS}{dt} = 4\pi r \frac{dr}{dt} + 2\pi h \frac{dr}{dt} = (4\pi r + 2\pi h) \frac{dr}{dt}$
(b) $S = 2\pi r^2 + 2\pi rh$ and r constant $\Rightarrow \frac{dS}{dt} = 2\pi r \frac{dh}{dt}$
(c) $S = 2\pi r^2 + 2\pi rh \Rightarrow \frac{dS}{dt} = 4\pi r \frac{dr}{dt} + 2\pi \left(r \frac{dh}{dt} + h \frac{dr}{dt} \right) = (4\pi r + 2\pi h) \frac{dr}{dt} + 2\pi r \frac{dh}{dt}$
(d) S constant $\Rightarrow \frac{dS}{dt} = 0 \Rightarrow 0 = (4\pi r + 2\pi h) \frac{dr}{dt} + 2\pi r \frac{dh}{dt} \Rightarrow (2r + h) \frac{dr}{dt} = -r \frac{dh}{dt} \Rightarrow \frac{dr}{dt} = \frac{-r}{2r+h} \frac{dh}{dt}$
120. $S = \pi r \sqrt{r^2 + h^2} \Rightarrow \frac{dS}{dt} = \pi r \cdot \frac{(r \frac{dr}{dt} + h \frac{dh}{dt})}{\sqrt{r^2 + h^2}} + \pi \sqrt{r^2 + h^2} \frac{dr}{dt};$
(a) h constant $\Rightarrow \frac{dh}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi r^2 \frac{dr}{dt}}{\sqrt{r^2 + h^2}} + \pi \sqrt{r^2 + h^2} \frac{dr}{dt} = \left[\pi \sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}} \right] \frac{dr}{dt}$
(b) r constant $\Rightarrow \frac{dr}{dt} = 0 \Rightarrow \frac{dS}{dt} = \frac{\pi rh}{\sqrt{r^2 + h^2}} \frac{dh}{dt}$
(c) In general, $\frac{dS}{dt} = \left[\pi \sqrt{r^2 + h^2} + \frac{\pi r^2}{\sqrt{r^2 + h^2}} \right] \frac{dr}{dt} + \frac{\pi rh}{\sqrt{r^2 + h^2}} \frac{dh}{dt}$
121. $\frac{dR_1}{dt} = -1$ ohm/sec, $\frac{dR_2}{dt} = 0.5$ ohm/sec; and $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \Rightarrow \frac{-1}{R^2} \frac{dR}{dt} = \frac{-1}{R_1^2} \frac{dR_1}{dt} - \frac{1}{R_2^2} \frac{dR_2}{dt}$. Also,
 $R_1 = 75$ ohms and $R_2 = 50$ ohms $\Rightarrow \frac{1}{R} = \frac{1}{75} + \frac{1}{50} \Rightarrow R = 30$ ohms. Therefore, from the derivative equation,
 $\frac{-1}{(30)^2} \frac{dR}{dt} = \frac{-1}{(75)^2} (-1) - \frac{1}{(50)^2} (0.5) = \left(\frac{1}{5625} - \frac{1}{5000} \right) \Rightarrow \frac{dR}{dt} = (-900) \left(\frac{5000-5625}{5625 \cdot 5000} \right) = \frac{9(625)}{50(5625)} = \frac{1}{50} = 0.02$ ohm/sec.
122. $\frac{dR}{dt} = 3$ ohms/sec and $\frac{dX}{dt} = -2$ ohms/sec; $Z = \sqrt{R^2 + X^2} \Rightarrow \frac{dZ}{dt} = \frac{R \frac{dR}{dt} + X \frac{dX}{dt}}{\sqrt{R^2 + X^2}}$ so that $R = 10$ ohms and
 $X = 20$ ohms $\Rightarrow \frac{dZ}{dt} = \frac{(10)(3) + (20)(-2)}{\sqrt{10^2 + 20^2}} = \frac{-1}{\sqrt{5}} \approx -0.45$ ohm/sec.

123. Given $\frac{dx}{dt} = 10$ m/sec and $\frac{dy}{dt} = 5$ m/sec, let D be the distance from the origin $\Rightarrow D^2 = x^2 + y^2 \Rightarrow 2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} \Rightarrow D \frac{dD}{dt} = x \frac{dx}{dt} + y \frac{dy}{dt}$. When $(x, y) = (3, -4)$, $D = \sqrt{3^2 + (-4)^2} = 5$ and $5 \frac{dD}{dt} = (5)(10) + (-4)(5) \Rightarrow \frac{dD}{dt} = \frac{10}{5} = 2$. Therefore, the particle is moving away from the origin at 22 m/sec (because the distance D is increasing).

124. Let D be the distance from the origin. We are given that $\frac{dD}{dt} = 11$ units/sec. Then $D^2 = x^2 + y^2$
 $= x^2 + (x^{3/2})^2 = x^2 + x^3 \Rightarrow 2D \frac{dD}{dt} = 2x \frac{dx}{dt} + 3x^2 \frac{dx}{dt} = x(2 + 3x) \frac{dx}{dt}$; $x = 3 \Rightarrow D = \sqrt{3^2 + 3^3} = 6$
 and substitution in the derivative equation gives $(2)(6)(11) = (3)(2 + 9) \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = 4$ units/sec.

125. (a) From the diagram we have $\frac{10}{h} = \frac{4}{r} \Rightarrow r = \frac{2}{5} h$.

(b) $V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left(\frac{2}{5} h\right)^2 h = \frac{4\pi h^3}{75} \Rightarrow \frac{dV}{dt} = \frac{4\pi h^2}{25} \frac{dh}{dt}$, so $\frac{dV}{dt} = -5$ and $h = 6 \Rightarrow \frac{dh}{dt} = -\frac{125}{144\pi}$ ft/min.

126. From the sketch in the text, $s = r\theta \Rightarrow \frac{ds}{dt} = r \frac{d\theta}{dt} + \theta \frac{dr}{dt}$. Also $r = 1.2$ is constant $\Rightarrow \frac{dr}{dt} = 0$
 $\Rightarrow \frac{ds}{dt} = r \frac{d\theta}{dt} = (1.2) \frac{d\theta}{dt}$. Therefore, $\frac{ds}{dt} = 6$ ft/sec and $r = 1.2$ ft $\Rightarrow \frac{d\theta}{dt} = 5$ rad/sec

127. (a) From the sketch in the text, $\frac{d\theta}{dt} = -0.6$ rad/sec and $x = \tan \theta$. Also $x = \tan \theta \Rightarrow \frac{dx}{dt} = \sec^2 \theta \frac{d\theta}{dt}$; at point A, $x = 0 \Rightarrow \theta = 0 \Rightarrow \frac{dx}{dt} = (\sec^2 0)(-0.6) = -0.6$. Therefore the speed of the light is $0.6 = \frac{3}{5}$ km/sec when it reaches point A.

(b) $\frac{(3/5) \text{ rad}}{\text{sec}} \cdot \frac{1 \text{ rev}}{2\pi \text{ rad}} \cdot \frac{60 \text{ sec}}{\text{min}} = \frac{18}{\pi}$ revs/min

128. From the figure, $\frac{a}{r} = \frac{b}{BC} \Rightarrow \frac{a}{r} = \frac{b}{\sqrt{b^2 - r^2}}$. We are given that r is constant. Differentiation gives,

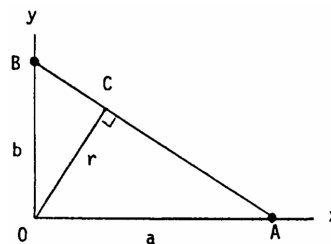
$$\frac{1}{r} \cdot \frac{da}{dt} = \frac{(\sqrt{b^2 - r^2}) \left(\frac{db}{dt}\right) - (b) \left(\frac{-r}{\sqrt{b^2 - r^2}}\right) \left(\frac{db}{dt}\right)}{b^2 - r^2}. \text{ Then,}$$

$$b = 2r \text{ and } \frac{db}{dt} = -0.3r$$

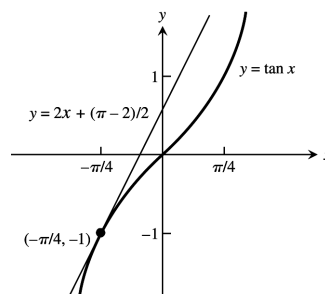
$$\Rightarrow \frac{da}{dt} = r \left[\frac{\sqrt{(2r)^2 - r^2} (-0.3r) - (2r) \left(\frac{-r(-0.3r)}{\sqrt{(2r)^2 - r^2}}\right)}{(2r)^2 - r^2} \right]$$

$$= \frac{\sqrt{3r^2} (-0.3r) + \frac{4r^2(0.3r)}{\sqrt{3r^2}}}{3r} = \frac{(3r^2)(-0.3r) + (4r^2)(0.3r)}{3\sqrt{3}r^2} = \frac{0.3r}{3\sqrt{3}} = \frac{r}{10\sqrt{3}} \text{ m/sec. Since } \frac{da}{dt} \text{ is positive,}$$

the distance OA is increasing when $OB = 2r$, and B is moving toward O at the rate of $0.3r$ m/sec.

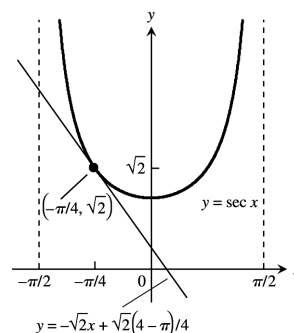


129. (a) If $f(x) = \tan x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec^2 x$,
 $f(-\frac{\pi}{4}) = -1$ and $f'(-\frac{\pi}{4}) = 2$. The linearization of $f(x)$ is $L(x) = 2(x + \frac{\pi}{4}) + (-1) = 2x + \frac{\pi-2}{2}$.



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- (b) If $f(x) = \sec x$ and $x = -\frac{\pi}{4}$, then $f'(x) = \sec x \tan x$,
 $f(-\frac{\pi}{4}) = \sqrt{2}$ and $f'(-\frac{\pi}{4}) = -\sqrt{2}$. The
 linearization of $f(x)$ is $L(x) = -\sqrt{2}(x + \frac{\pi}{4}) + \sqrt{2}$
 $= -\sqrt{2}x + \frac{\sqrt{2}(4-\pi)}{4}$.



130. $f(x) = \sqrt{x+1} + \sin x - 0.5 = (x+1)^{1/2} + \sin x - 0.5 \Rightarrow f'(x) = (\frac{1}{2})(x+1)^{-1/2} + \cos x$
 $\Rightarrow L(x) = f'(0)(x-0) + f(0) = 1.5(x-0) + 0.5 \Rightarrow L(x) = 1.5x + 0.5$, the linearization of $f(x)$.
131. $C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$, $S = 4\pi r^2 = \frac{C^2}{\pi}$, and $V = \frac{4}{3}\pi r^3 = \frac{C^3}{6\pi^2}$. It also follows that $dr = \frac{1}{2\pi} dC$, $dS = \frac{2C}{\pi} dC$ and $dV = \frac{C^2}{2\pi^2} dC$. Recall that $C = 10$ cm and $dC = 0.4$ cm.
- (a) $dr = \frac{0.4}{2\pi} = \frac{0.2}{\pi}$ cm $\Rightarrow (\frac{dr}{r})(100\%) = (\frac{0.2}{\pi})(\frac{2\pi}{10})(100\%) = (.04)(100\%) = 4\%$
- (b) $dS = \frac{20}{\pi}(0.4) = \frac{8}{\pi}$ cm $\Rightarrow (\frac{dS}{S})(100\%) = (\frac{8}{\pi})(\frac{\pi}{100})(100\%) = 8\%$
- (c) $dV = \frac{10^2}{2\pi^2}(0.4) = \frac{20}{\pi^2}$ cm $\Rightarrow (\frac{dV}{V})(100\%) = (\frac{20}{\pi^2})(\frac{6\pi^2}{1000})(100\%) = 12\%$
132. Similar triangles yield $\frac{35}{h} = \frac{15}{6} \Rightarrow h = 14$ ft. The same triangles imply that $\frac{20+a}{h} = \frac{a}{6} \Rightarrow h = 120a^{-1} + 6$
 $\Rightarrow dh = -120a^{-2} da = -\frac{120}{a^2} da = (-\frac{120}{a^2})(\pm \frac{1}{12}) = (-\frac{120}{15^2})(\pm \frac{1}{12}) = \pm \frac{2}{45} \approx \pm .0444$ ft $= \pm 0.53$ inches.