Solution Manual for SIGNALS AND SYSTEMS USING MATLAB

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Chapter 0

From the Ground Up

0.1 Basic Problems

0.1 (a) i. $\mathcal{R}e(z) + \mathcal{I}m(v) = 8 - 2 = 6$

ii. $|z + v| = |17 + j1| = \sqrt{17^2 + 1}$

iii. $|zv| = |72 - j16 + j27 + 6| = |78 + j11| = \sqrt{78^2 + 11^2}$

iv. $\angle z + \angle v = \tan^{-1}(3/8) - \tan^{-1}(2/9)$

v. $|v/z| = |v|/|z| = \sqrt{85}/\sqrt{73}$

vi. $\angle(v/z) = -\tan^{-1}(2/9) - \tan^{-1}(3/8)$

(b) i. $z + v = 17 + j = \sqrt{17^2 + 1}e^{j\tan^{-1}(1/17)}$
ii. $zv = 78 + j11 = \sqrt{78^2 + 11^2}e^{j\tan^{-1}(11/78)}$
iii. $z^* = 8 - j3 = \sqrt{64 + 9}(e^{-j\tan^{-1}(3/8)})^* = \sqrt{73}e^{j\tan^{-1}(3/8)}$
iv. $zz^* = |z|^2 = 73$
v. $z - v = -1 + j5 = \sqrt{1 + 25}e^{-j\tan^{-1}(5)}$

0.2 (a) $z = 6e^{j\pi/4} = 6\cos(\pi/4) + j6\sin(\pi/4)$

i. $\mathcal{R}e(z) = 6\cos(\pi/4) = 3\sqrt{2}$

ii. $\mathcal{I}m(z) = 6\sin(\pi/4) = 3\sqrt{2}$

(b) i. Yes, $\mathcal{R}e(z) = 0.5(z + z^*) = 0.5(2\mathcal{R}e(z)) = \mathcal{R}e(z) = 8$

ii. Yes, $\mathcal{I}m(v) = -0.5j(v - v^*) = -0.5j(2j\mathcal{I}m(v)) = \mathcal{I}m(v) = -2$

iii. Yes, $\mathcal{R}e(z + v^*) = \mathcal{R}e(\mathcal{R}e(z) + \mathcal{R}e(v^*) + \mathcal{I}m(z) - \mathcal{I}m(v)) = \mathcal{R}e(z + v) = 17$

iv. Yes, $\mathcal{I}m(z + v^*) = \mathcal{I}m(17 + j5) = \mathcal{I}m(z - v) = \mathcal{I}m(-9 + j5) = 5$

0.3 (a) Representing the complex number $z = x + jy = |z|e^{j\theta}$ then $|x| = |z||\cos(\theta)|$ and since $|\cos(\theta)| \leq 1$ then $|x| \leq |z|$, the equality holds when $\theta = 0$ or when $z = x$, i.e., it is real.

(b) Adding two complex numbers is equivalent to adding two vectors to create a triangle with two sides the two vectors being added and the other side the vector resulting from the addition. Unless the two vector being added have the same angle, in which case $|z| + |v| = |z + v|$, it holds that $|z| + |v| > |z + v|$.

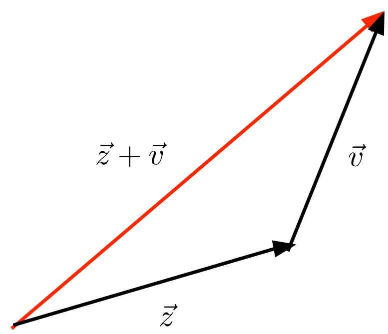


Figure 1: Addition of two vectors illustrating the triangular inequality.

(c) The answer to both is yes. Indeed,

(a) $|z + v| = \sqrt{13} \leq |z| + |v| = \sqrt{2} + \sqrt{5}$

(b) $|z - v| = |-1| = 1 \leq |z| + |v| = \sqrt{2} + \sqrt{5}$

0.4 (a) We have

i. $\cos(\theta - \pi/2) = 0.5(e^{j(\theta-\pi/2)} + e^{-j(\theta-\pi/2)}) = -j0.5(e^{j\theta} - e^{-j\theta}) = \sin(\theta)$

ii. $-\sin(\theta - \pi/2) = 0.5j(e^{j(\theta-\pi/2)} - e^{-j(\theta-\pi/2)}) = 0.5j(-j)(e^{j\theta} + e^{-j\theta}) = \cos(\theta)$

iii. $\sin(\theta + \pi/2) = (je^{j\theta} + je^{-j\theta})/(2j) = \cos(\theta)$

(b) i. $\cos(2\pi t)\sin(2\pi t) = (1/4j)(e^{j4\pi t} - e^{-j4\pi t})$ so that

$$\int_0^1 \cos(2\pi t)\sin(2\pi t)dt = \frac{1}{4j}\frac{e^{j4\pi t}}{4\pi j}|_0^1 + \frac{1}{4j}\frac{e^{-j4\pi t}}{4\pi j}|_0^1 = 0 + 0 = 0$$

ii. We have

$$\cos^2(2\pi t) = \frac{1}{4}(e^{j4\pi t} + 2 + e^{-j4\pi t}) = \frac{1}{2}(1 + \cos(4\pi t))$$

so that its integral is $1/2$ since the integral of $\cos(4\pi t)$ is over two of its periods and it is zero.

0.5 (a) i. $2\cos(\alpha+\beta) = e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)} = (e^{j\alpha}e^{j\beta}) + (e^{j\alpha}e^{j\beta})^* = 2\mathcal{R}e(e^{j\alpha}e^{j\beta})$ and $\mathcal{R}e[e^{j\alpha}e^{j\beta}] = \mathcal{R}e[(\cos(\alpha) + j\sin(\alpha))(\cos(\beta) + j\sin(\beta))] = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ so that

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

ii. $2j\sin(\alpha+\beta) = e^{j\alpha}e^{j\beta} - (e^{j\alpha}e^{j\beta})^* = 2j\mathcal{I}m[e^{j\alpha}e^{j\beta}]$, and the imaginary is $\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) = \sin(\alpha+\beta)$

(b)

$$\int_0^1 e^{j2\pi t}dt = \frac{e^{j2\pi t}}{j2\pi}|_0^1 = \frac{e^{j2\pi} - 1}{j2\pi} = 0$$

also

$$\int_0^1 e^{j2\pi t}dt = \int_0^1 \cos(2\pi t)dt + j\int_0^1 \sin(2\pi t)dt = 0 + j0$$

since the integrals of the sinusoids are over a period.

(c) i. Yes, $(-1)^n = (e^{j\pi})^n = e^{jn\pi} = \cos(n\pi) + j\sin(n\pi) = \cos(n\pi)$ since $\sin(n\pi) = 0$ for any integer n.

ii. Yes, $e^{j0} = -e^{j\pi}$ and $e^{j\pi/2} = -e^{j3\pi/2}$ so they add to zero.

0.6 (a) Using Euler's identity the product

$$\begin{aligned} e^{j\alpha}e^{j\beta} &= (\cos(\alpha) + j\sin(\alpha))(\cos(\beta) + j\sin(\beta)) \\ &= [\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + j[\sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)] \end{aligned}$$

while

$$e^{j(\alpha+\beta)} = \cos(\alpha+\beta) + j\sin(\alpha+\beta)$$

so that equating the real and imaginary parts of the above two equations we get the desired trigonometric identities.

(b) We have

$$\begin{aligned} \cos(\alpha)\cos(\beta) &= 0.5(e^{j\alpha} + e^{-j\alpha})\,0.5(e^{j\beta} + e^{-j\beta}) \\ &= 0.25(e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)}) + 0.25(e^{j(\alpha-\beta)} + e^{-j(\alpha-\beta)}) \\ &= 0.5\cos(\alpha+\beta) + 0.5\cos(\alpha-\beta) \end{aligned}$$

Now,

$$\begin{aligned} \sin(\alpha)\sin(\beta) &= \cos(\alpha - \pi/2)\cos(\beta - \pi/2) \\ &= 0.5\cos(\alpha - \pi/2 + \beta - \pi/2) + 0.5\cos(\alpha - \pi/2 - \beta + \pi/2) \\ &= 0.5\cos(\alpha + \beta - \pi) + 0.5\cos(\alpha - \beta) \\ &= -0.5\cos(\alpha+\beta) + 0.5\cos(\alpha-\beta) \end{aligned}$$

0.7 (a) Replacing $z_k = |\alpha|^{1/N} e^{j(\phi+2\pi k)/N}$ in z^N we get $z_k^N = |\alpha| e^{j(\phi+2\pi k)} = |\alpha| e^{j(\phi)} = \alpha$ for any value of $k = 0, \cdots, N-1$.

(b) Applying the above result we have:

- For $z^2 = 1 = 1e^{j2\pi}$ the roots are $z_k = 1e^{j(2\pi+2\pi k)/2}$, $k = 0, 1$. When $k = 0$, $z_0 = e^{j\pi} = -1$ and $z_1 = e^{j2\pi} = 1$.
- When $z^2 = -1 = 1e^{j\pi}$ the roots are $z_k = 1e^{j(\pi+2\pi k)/2}$, $k = 0, 1$. When $k = 0$, $z_0 = e^{j\pi/2} = j$, and $z_1 = e^{j3\pi/2} = -j$.
- For $z^3 = 1 = 1e^{j2\pi}$ the roots are $z_k = 1e^{j(2\pi+2\pi k)/3}$, $k = 0, 1, 2$. When $k = 0$, $z_0 = e^{j2\pi/3}$; for $k = 1$, $z_1 = e^{j4\pi/3} = e^{-j2\pi/3} = z_0^*$; and for $k = 2$, $z_2 = 1e^{j(2\pi)} = 1$.
- When $z^3 = -1 = 1e^{j\pi}$ the roots are $z_k = 1e^{j(\pi+2\pi k)/3}$, $k = 0, 1, 2$. When $k = 0$, $z_0 = e^{j\pi/3}$; for $k = 1$, $z_1 = e^{j\pi} = -1$; and for $k = 2$, $z_2 = 1e^{j(5\pi)/3} = 1e^{j(-\pi)/3} = z_0^*$

(c) Notice that the roots are equally spaced around a circle of radius r and that the complex roots appear as pairs of complex conjugate roots.

0.8 (a) We have

i. $z^3 = -1 = e^{j\pi(2k+1)}$ for $k = 0, 1, 2$, so roots are $z_k = e^{j\pi(2k+1)/3}$, $k = 0, 1, 2$, i.e., on a circle of unit radius and separated $\pi/3$ with one at -1.

ii. $z^2 = 1 = e^{j2\pi k}$ for $k = 0, \pm 1, \pm 2, \cdots$, so roots are $z_k = e^{j\pi k} =$, i.e., on a circle of unit radius and separated π , one at zero and the other at -1

iii. $z^2 + 3z + 1 = (z + 1.5)^2 + (1 - 1.5^2) = 0$ so the roots are $z_{1,2} = \pm\sqrt{(1.5^2 - 1)} - 1.5$

(b) The log (using the Naperian base) of a product is the sum of the logs of the terms in the product, and the log and the exponential are the opposite of each other so the last term in the equation.
Using the given expression for the log of a complex number (log of real numbers is a special case):

$$\log(-2) = \log(2e^{\pm j\pi}) = \log(2) \pm j\pi$$
$$\log(1 + j1) = \log(\sqrt{2}e^{j\pi/4}) = \log(\sqrt{2}) + j\pi/4 = 0.5\log(2) + j\pi/4$$
$$\log(2e^{j\pi/4}) = \log(2) + j\pi/4$$

(c) $z = -1 = 1e^{j\pi}$ thus

i. $\log(z) = \log(1e^{j\pi}) = \log(1) + j\pi = j\pi$

ii. From above result, $e^{\log(z)} = e^{j\pi} = -1$

(d) i. $z^2/4 = (2e^{j\pi/4})^2/4 = 4e^{j\pi/2}/4 = j$

ii. Yes, $(\cos(\pi/4)+j\sin(\pi/4))^2 = (z/2)^2$, and $(z/2)^2 = (\cos(\pi/4)+j\sin(\pi/4))^2 = \cos(\pi/2)+ j\sin(\pi/2) = j$

0.9 (a) If $w = e^z$ then

$$\log(w) = z = 1 + j1$$

given that the $\log$ and e functions are the inverse of each other.
The real and imaginary of w are

$$w = e^z = e^1 e^{j1} = \underbrace{e\cos(1)}_{\text{real part}} + j \underbrace{e\sin(1)}_{\text{imaginary part}}$$

(b) The imaginary parts are cancelled and the real parts added twice in

$$w + w^* = 2\mathcal{R}e[w] = 2e\cos(1)$$

(c) Replacing z

$$w = e^z = e^1 e^{j1}$$

so that $|w| = e$ and $\angle w = 1$.
Using the result in (a)

$$|\log(w)|^2 = |z|^2 = 2$$

(d) According to Euler's equation

$$\cos(1) = 0.5(e^j + e^{-j}) = 0.5\left(\frac{w}{e} + \frac{w^*}{e}\right)$$

which can be verified using $w + w^*$ obtained above.

0.10 (a) Shifting to the right a cosine by a fourth of its period we get a sinusoid, thus

$$\sin(\Omega_0 t) = \cos(\Omega_0(t - T_0/4)) = \cos(\Omega_0 t - \Omega_0 T_0/4) = \cos(\Omega_0 t - \pi/2)$$

since $\Omega_0 = 2\pi/T_0$ or $\Omega_0 T_0 = 2\pi$.

(b) The phasor that generates a sine is $Ae^{-j\pi/2}$ since

$$y(t) = \mathcal{R}e[Ae^{-j\pi/2}e^{j\Omega_0 t}] = \mathcal{R}e[Ae^{j(\Omega_0 t - \pi/2)}] = A\cos(\Omega_0 t - \pi/2)$$

which equals $A\sin(\Omega_0 t)$.

(c) The phasors corresponding to $-x(t) = -A\cos(\Omega_0 t) = A\cos(\Omega_0 t + \pi)$ is $Ae^{j\pi}$. For

$$-y(t) = -A\sin(\Omega_0 t) = -A\cos(\Omega_0 t - \pi/2) = A\cos(\Omega_0 t - \pi/2 + \pi) = A\cos(\Omega_0 t + \pi/2)$$

the phasor is $Ae^{j\pi/2}$. Thus, relating any sinusoid to the corresponding cosine, the magnitude and angle of this cosine gives the magnitude and phase of the phasor that generates the given sinusoid.

(d) If $z(t) = x(t) + y(t) = A\cos(\Omega_0 t) + A\sin(\Omega_0 t)$, the phasor corresponding to $z(t)$ is the sum of the phasors Ae^{j0}, corresponding to $A\cos(\Omega_0 t)$, with the phasor $Ae^{-j\pi/2}$, corresponding to $A\sin(\Omega_0 t)$, which gives $\sqrt{2}Ae^{-j\pi/4}$ (equivalently the sum of a vector with length A and angle 0 with another vector of length A and angle $-\pi/2$). We have that

$$z(t) = \mathcal{R}e\left[\sqrt{2}Ae^{-j\pi/4}e^{j\Omega_0 t}\right] = \sqrt{2}A\cos(\Omega_0 t - \pi/4)$$

(e) i. Phasor $4e^{j\pi/3}$

ii. $-4\sin(2t + \pi/3) = 4\cos(2t + \pi/3 + \pi/2)$ with phasor $4e^{j5\pi/6}$

iii. We have

$$\begin{aligned} 4\cos(2t+\pi/3) - 4\sin(2t+\pi/3) &= \mathcal{R}e[(4e^{j\pi/3} + 4e^{j(\pi/2+\pi/3)})e^{j2t}] \\ &= \mathcal{R}e[4e^{j\pi/3}\underbrace{(1+e^{j\pi/2})}_{\sqrt{2}e^{j\pi/4}}e^{j2t}] \\ &= \mathcal{R}e[4\sqrt{2}e^{j7\pi/12}e^{j2t}] \end{aligned}$$

so that the phasor is $4\sqrt{2}e^{j7\pi/12}$

0.11 (a) Assuming a maximum frequency of 22.05 kHz for the acoustic signal, the numbers of bytes (8 bits per byte) for two channels (stereo) and a 75 minutes recording is greater or equal to: $2 \times 22,050$ samples/channel/second $\times$ 2 bytes/sample $\times$ 2 channels $\times$ 75 minutes $\times$ 60 seconds/minute $= 7.938 \times 10^8$ bytes. Multiplying by 8 we get the number of bits. CD quality means that the signal is sampled at 44.1 kHz and each sample is represented by 16 bits or 2 bytes.

(b) The raw data would consist of 8 (bits/sample) $\times 10,000$ (samples/sec)$=80,000$ bits/sec. The vocoder is part of a larger unit called a digital signal processor chip set. It uses various procedures to reduce the number of bits that are transmitted while still keeping your voice recognizable. When there is silence it does not transmit, letting another signal use the channel during pauses.

(c) Texting between cell phones is possible by sending short messages (160 characters) using the short message services (SMS). Whenever your cell-phone communicates with the cell phone tower there is an exchange of messages over the control channel for localization, and call setup. This channel provides a pathway for SMS messages by sending packets of data. Except for the cost of storing messages, the procedure is rather inexpensive and convenient to users.

(d) For CD audio the sampling rate is 44.1 kHz with 16 bits/sample. For DVD audio the sampling rate is 192 kHz with 24 bits/sample. The sampling process requires getting rid of high frequencies in the signal, also each sample is only approximated by the binary representation, so analog recording could sound better in some cases.

(e) The number of pixels processed every second is: 352×240 pixels/frame $\times 60$ frames/sec.
The number of bits available for transmission every second is obtained by multiplying the above answer by 8 bits/pixel. There many compression methods JPEG, MPEG, etc.

0.12 (a) If $\alpha = 1$ then

$$S = \sum_{n=0}^{N-1} 1 = \underbrace{1 + 1 + \cdots + 1}_{N \text{ times}} = N$$

(b) The expression

$$\begin{aligned} S(1-\alpha) &= S - \alpha S \\ &= (1 + \alpha + \cdots + \alpha^{N-1}) - (\alpha + \alpha^2 + \cdots + \alpha^{N-1} + \alpha^N) \\ &= 1 - \alpha^N \end{aligned}$$

as the intermediate terms cancel. So that

$$S = \frac{1-\alpha^N}{1-\alpha}, \quad \alpha \neq 1$$

Since we do not want the denominator $1-\alpha$ to be zero, the above requires that $\alpha \neq 1$. If $\alpha = 1$ the sum was found in (a). As a finite sum, it exists for any finite values of α.
Putting (a) and (b) together we have

$$S = \begin{cases} (1-\alpha^N)/(1-\alpha) & \alpha \neq 1 \\ N & \alpha = 1 \end{cases}$$

(c) If N is infinite, the sum is of infinite length and we need to impose the condition that $|\alpha| < 1$ so that α^n decays as $n \to \infty$. In that case, the term $\alpha^N \to 0$ as $N \to \infty$, and the sum is

$$S = \frac{1}{1-\alpha} \qquad |\alpha| < 1$$

If $|\alpha| \geq 1$ this sum does not exist, i.e., it becomes infinite.
(d) The derivative becomes

$$S_1 = \frac{dS}{d\alpha} = \sum_{n=0}^{\infty} n\alpha^{n-1} = \frac{1}{(1-\alpha)^2}.$$

0.2 Problems using MATLAB

0.13 As we will see later, the sampling period of $x(t)$ with a frequency of $\Omega_{max} = 2\pi f_{max} = 2\pi$ should satisfy the Nyquist sampling condition

$$f_s = \frac{1}{T_s} \geq 2f_{max} = 2 \text{ samples/sec}$$

so $T_s \leq 1/2$ (sec/sample). Thus when $T_s = 0.1$ the continuous-time and the discrete-time signals look very much like each other, indicating the signals have the same information — such a statement will be justified in the chapter on sampling where we will show that the continuous-time signal can be recovered from the sampled signal. It is clear that when $T_s = 1$ the information is lost. Although it is not clear from the figure that when we let $T_s = 0.5$ the discrete-time signal keeps the information, this sampling period satisfies the Nyquist sampling condition and as such the original signal can be recovered from the sampled signal. The following MATLAB script is used.

```
% Pr. 0._13
clear all; clf
T=3; Tss= 0.0001; t=[0:Tss:T];
 xa=4*cos(2*pi*t);  % continuous-time signal
 xamin=min(xa);xamax=max(xa);
 figure(1)
 subplot(221)
 plot(t,xa); grid
 title('Continuous-time Signal');  ylabel('x(t)');  xlabel('t sec')
 axis([0  T 1.5*xamin 1.5*xamax])
 N=length(t);

 for k=1:3,
      if k==1,Ts= 0.1;  subplot(222)
        t1=[0:Ts:T];  n=1:Ts/Tss: N; xd=zeros(1,N);  xd(n)=4*cos(2*pi*t1);
        plot(t,xa); hold on; stem(t,xd);grid;hold off
        axis([0  T 1.5*xamin 1.5*xamax]); ylabel('x(0.1 n)'); xlabel('t')
        elseif  k==2, Ts=0.5;  subplot(223)
        t2=[0:Ts:T]; n=1:Ts/Tss: N;  xd=zeros(1,N);  xd(n)=4*cos(2*pi*t2);
        plot(t,xa); hold on; stem(t,xd); grid; hold off
        axis([0  T 1.5*xamin 1.5*xamax]); ylabel('x(0.5 n)'); xlabel('t')
        else,Ts=1; subplot(224)
        t3=[0:Ts:T];  n=1:Ts/Tss: N;  xd=zeros(1,N);  xd(n)=4*cos(2*pi*t3);
        plot(t,xa); hold on;  stem(t,xd); grid; hold off
         axis([0  T 1.5*xamin 1.5*xamax]); ylabel('x(n)'); xlabel('t')
      end
  end
```

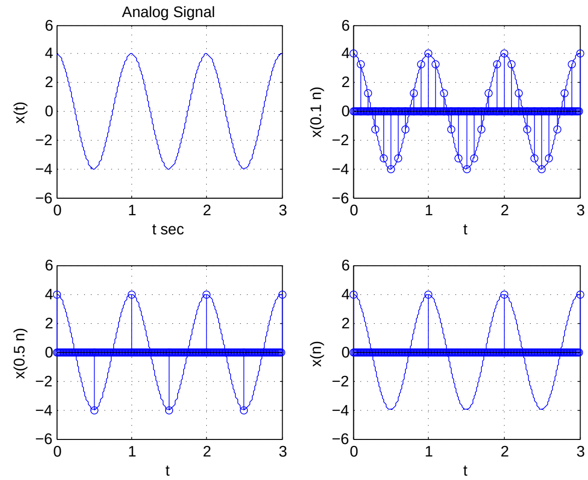


Figure 2: Problem 13: Analog continuous-time signal (top left); continuous-time and discrete-time signals superposed for $T_s = 0.1$ sec (top right) and $T_s = 0.5$ sec and $T_s = 1$ sec (bottom left to right).

0.14 The derivative is

$$y(t) = \frac{dx(t)}{dt} = -8\pi \sin(2\pi t)$$

which has the same frequency as $x(t)$, thus the sampling period should be like in the previous problem, $T_s \leq 0.5$.

```
% Pr. 0_14
clear all
% actual derivative
Tss=0.0001;t1=0:Tss:3;
y=-8*pi*sin(2*pi*t1);
figure(2)
% forward difference
Ts=0.01;t=[0:Ts:3];N=length(t);
subplot(211)
xa=4*cos(2*pi*t);  % sampled signal
der1_x=forwardiff(xa,Ts,t,y,t1);

clear der1_x
% forward difference
Ts=0.1;t=[0:Ts:3];N=length(t);
subplot(212)
xa=4*cos(2*pi*t);  % sampled signal
der1_x=forwardiff(xa,Ts,t,y,t1);
```

The function *forwardiff* computes and plots the forward difference and the actual derivative.

```
function der=forwardiff(xa,Ts,t,y,t1)
% % forward difference
% % xa: sampled signal using Ts
% % y: actual derivative defined in t
 N=length(t);n=0:N-2;
 der=diff(xa)/Ts;
 stem(n*Ts,der,'filled');grid;xlabel('t, nT_s')
 hold on
 plot(t1,y,'r'); legend('forward difference','derivative')
 hold off
```

For $T_s = 0.1$ the finite difference looks like the actual derivative but shifted, while for $T_s = 0.01$ it does not.

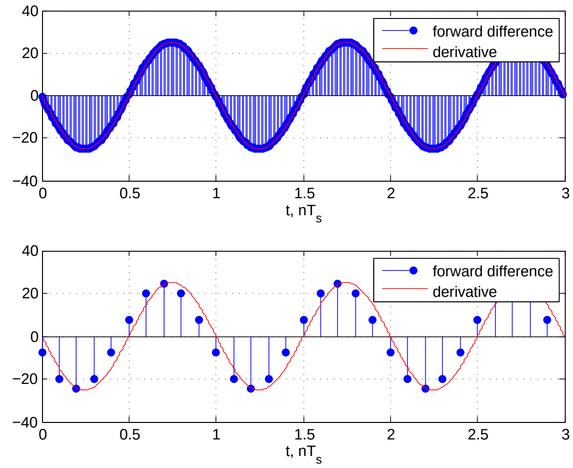


Figure 3: Problem 14: $T_s = 0.01$ sec (top) and $T_s = 0.1$ sec (bottom)

0.15 (a) The backward finite difference (let $T_s = 1$ for simplicity)

$$\Delta_1[x(n)] = x(n) - x(n-1)$$

is connected with the forward finite difference $\Delta[x(n)]$ given in the chapter as follows

$$\Delta_1[x(n+1)] = x(n+1) - x(n) = \Delta[x(n)]$$

That is, $\Delta[x(n)]$ is $\Delta_1[x(n)]$ shifted one sample to the left.
(b) (c) The average of the two finite differences gives

$$0.5\,\{\Delta_1[x(n)] + \Delta[x(n)]\} = 0.5[x(n+1) - x(n-1)]$$

which gives a better approximation to the derivative than either of the given finite differences. The following script is used to compute Δ_1 and the average.

```
%  Pr. 0_15
%  compares forward/backward differences
%  with new average difference
Ts=0.1;
for k=0:N-2,
   x1=4*cos(2*pi*(k-1)*Ts);
   x2=4*cos(2*pi*k*Ts);
   der_x(k+1)=x2-x1; % backward difference
end
der_x=der_x/Ts;
Tss=0.0001;t1=0:Tss:3;
y=-8*pi*sin(2*pi*t1); % actual derivative
n=0:N-2;
figure(3)
 subplot(211)
 stem(n*Ts,der_x,'k');grid
 hold on
 stem(n*Ts,der1_x,'b','filled') % derv1_x forward difference
                                 % from Pr.  0.2
 hold on
 plot(t1,y,'r'); xlabel('t, nT_s')
 legend('bck diff','forwd diff', 'derivative')
 hold off
 subplot(212)
 stem(n*Ts,0.5*(der_x+der1_x));grid;xlabel('t, nT_s') % average
 hold on
 plot(t1,y,'r')
 hold off
 legend('average diff','derivative')
```

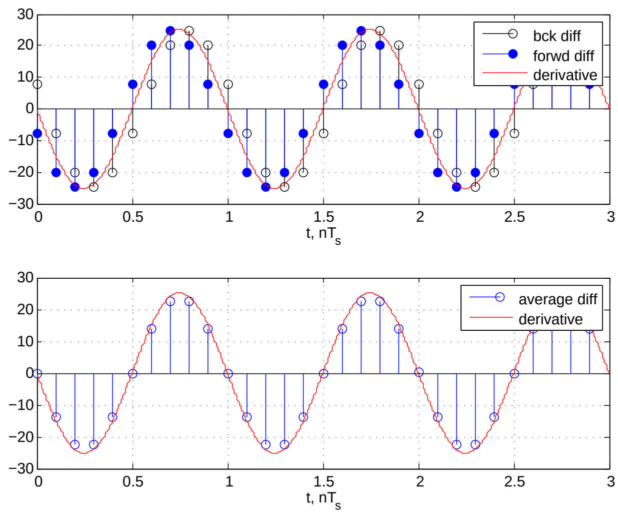


Figure 4: Problem 15: Comparison of different finite differences.

0.16 (a) According to Kirchoff's current law

$$i_s(t) = i_R(t) + i_L(t) = \frac{v_L(t)}{R} + i_L(t)$$

but $v_L(t) = L di_L(t)/dt$ so that the ordinary differential equation relating the input $i_s(t)$ to the output current in the inductor $i_L(t)$ is

$$\frac{di_L(t)}{dt} + i_L(t) = i_s(t)$$

after replacing $L = 1$ and $R = 1$. Notice that this d.e. is the dual of the one given in the Chapter, so that the difference equation is

$$i_L(nT_s) = \frac{T_s}{2+T_s}[i_s(nT_s) + i_s((n-1)T_s)] + \frac{2-T_s}{2+T_s} i_L((n-1)T_s) \qquad n \geq 1$$
$$i_L(0) = 0$$

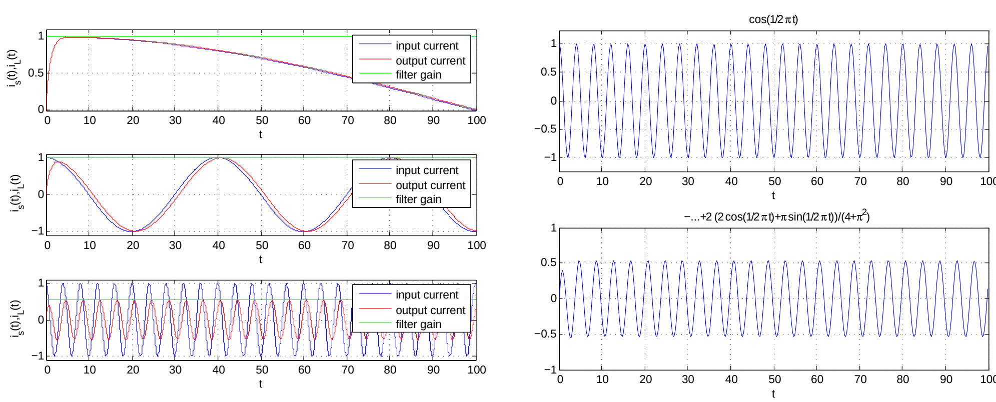


Figure 5: Problem 16: Left (top to bottom): solution of difference equation for $\Omega_0 = 0.005,\ 0.05,\ 0.5$ (rad/sec). Right: input (top), solution of ordinary differential equation (bottom).

(b)(c) The scripts to solve the difference and ordinary differential equations are the following.

```
% Pr.  0_16
clear all
% solution of difference equation
Ts=0.01;
t=[0:Ts:100];
figure(4)
for k=0:2;
    if k==0, subplot(311)
     elseif k==1, subplot(312)
     else, subplot(313)
    end
 W0= 0.005*10^k*pi; % frequency of source
 is=cos(W0*t); % source
```

```
 a=[1 (-2+Ts)/(2+Ts)]; % coefficients of i_L(n), i_L(n-1)
 b=[Ts/(2+Ts) Ts/(2+Ts)]; % coefficients of i_s(n), i_s(n-1)
 il=filter(b,a,is);  % current in inductor computed by
                     % MATLAB function 'filter'
 H=1/sqrt(1+W0^2)*ones(1,length(t)); % filter gain at W0
 plot(t,is,t,il,'r',t,H,'g'); xlabel('t'); ylabel('i_s(t),i_L(t)')
 axis([0 100 1.1*min(is) 1.1*max(is)])
 legend('input current','output current','filter gain'); grid
 pause(0.1)
end
%%%%
% solution of ordinary differential equation for cosine input of frequency  0.5pi
clear all
syms t x y
x=cos(0.5*pi*t);
y=dsolve('Dy+y=cos(0.5*pi*t)','y(0)=0','t')
figure(5)
 subplot(211)
 ezplot(x,[0 100]);grid
 subplot(212)
 ezplot(y,[0 100]);grid
 axis([0 100 -1 1])
```

0.17 (a) The distributive and the associative laws are equivalent to the ones for integrals, indeed

$$\sum_k ca_k = c(\cdots + a_{-1} + a_0 + a_1 + \cdots) = c\sum_k a_k$$

since c does not depend on k. Likewise

$$\sum_k [a_k + b_k] = (\cdots + a_{-1} + b_{-1} + a_0 + b_0 + a_1 + b_1 \cdots) = \sum_k a_k + \sum_k b_k$$

Finally, when adding a set of numbers the order in which they are added does not change the result. For instance,

$$a_0 + a_1 + a_2 + a_3 = a_0 + a_2 + a_1 + a_3$$

(b) Gauss' trick can be shown in general as follows. Let $S = \sum_{k=0}^{N} k$ then

$$2S = \sum_{k=0}^{N} k + \sum_{k=N}^{0} k$$

letting $\ell = -k + N$ in the second summation we have

$$2S = \sum_{k=0}^{N} k + \sum_{\ell=0}^{N} (N - \ell) = \sum_{k=0}^{N} (k + N - k) = N\sum_{k=0}^{N} 1 = N(N+1)$$

where we let the dummy variables of the two sums be equal. We thus have that for $N = 10^4$

$$S = \frac{N(N+1)}{2} = \frac{10^4(10^4+1)}{2} \approx 0.5 \times 10^8$$

(c) Using the above properties of the sum,

$$\begin{aligned} S_1 &= \sum_{k=0}^{N} (\alpha + \beta k) = \alpha \sum_{k=0}^{N} 1 + \beta \sum_{k=0}^{N} k \\ &= \alpha(N+1) + \beta \frac{N(N+1)}{2} \end{aligned}$$

(d) The following script computes numerically and symbolically the various sums.

```
% Pr.  0_17
clear all
% numeric
N=100;
S1=[0:1:N];
S2=[N:-1:0];
S=sum(S1+S2)/2
% symbolic
syms S1 N alpha beta k
simple(symsum(alpha+beta*k,0,N))
% computing sum for specific values of alpha, beta and N
subs(symsum(alpha+beta*k,0,N),{alpha,beta,N},{1,1,100})
```

```
 S = 5050

((2*alpha + N*beta)*(N + 1))/2

 5151
```

The answers shown at the bottom.

0.18 (a) The following figure shows the upper and lower bounds when approximating the integral of t:

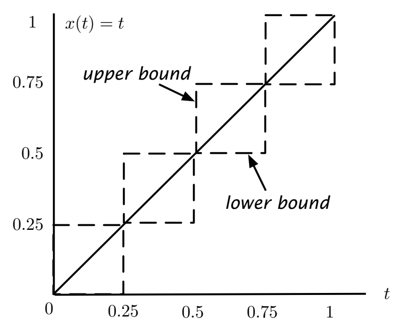


Figure 6: Problem 18: Upper and lower bounds of the integral of t when $N = 4$.

(b) (c) The lower bound for the integral is

$$\begin{aligned} S_\ell &= \sum_{n=1}^{N-1}(nT_s)T_s = T_s^2\sum_{n=1}^{N-1} n = T_s^2\sum_{\ell=0}^{N-2}(\ell+1) \\ &= T_s^2\left[\frac{(N-1)(N-2)}{2} + (N-1)\right] \end{aligned}$$

The definite integral is

$$\int_0^1 t dt = \frac{1}{2}$$

The upper bound is

$$S_u = \sum_{n=1}^{N}(nT_s)T_s = S_\ell + NT_s^2$$

Letting $NT_s = 1$, or $T_s = 1/N$ we have then that

$$\left[\frac{(N-1)(N-2)+2(N-1)}{2N^2}\right] \leq \frac{1}{2} \leq \left[\frac{(N-1)(N-2)+2(N-1)}{2N^2}\right] + \frac{1}{N}$$

for large N the upper and the lower bound tend to $1/2$.

The following script computes the lower and upper bound of the integral of t.

```
% Pr. 0_18
clear all
Ts=0.001;N=1/Ts;
% integral of t from 0 to 1 is 0.5
syms S1 n T k
% lower bound
 n=subs(N);T=subs(Ts);
 y=simple(symsum(k*T^2,1,n-1));
 yy=subs(y)
```

```
% upper bound
 z=simple(symsum(k*T^2,1,n));
 zz=subs(z)
% average
 int= 0.5*(yy+zz)
```

giving the following results (the actual integral is $1/2$).

```
yy = 0.4995
zz =  0.5005
int =  0.5000
```

(d) For $y(t) = t^2$, $0 \leq t \leq 1$, the following script computes the upper and the lower bounds and their average:

```
%% integral of t^2 from 0 to 1 is 0.333
% lower bound
 y1=simple(symsum(k^2*T^3,1,n-1));
 yy1=subs(y1)
%  upper bound
 z1=simple(symsum(k^2*T^3,1,n));
 zz1=subs(z1)
% average
 int= 0.5*(yy1+zz1)
```

giving the following results, in this case the value of the definite integral is $1/3$.

```
yy1 = 0.3328
zz1 = 0.3338
int =   0.3333
```

0.19 The indefinite integral equals $0.5t^2$. Computing it in $[0, 1]$ gives the same value as the sum of the integrals computed between $[0, 0.5]$ and $[0.5, 1]$.

As seen before, the sum

$$S = \sum_{n=0}^{100} n = \frac{100(101)}{2} = 5050$$

while

$$S_1 = S + 50 = 5100$$
$$S_2 = S$$

the first sum has an extra term when $n = 50$ while the other does not. To verify this use the following script:

```
% Pr. 0_19
clear all
N=100;
syms n,N
S=symsum(n,0,N)
S1=symsum(n,0,N/2)+symsum(n,N/2,N)
S2=symsum(n,0,N/2)+symsum(n,N/2+1,N)
```

giving

```
S = 5050
S1 = 5100
S2 = 5050
```

0.20 (a)(b) We have that

$$0 < e^{-\alpha t} < e^{-\beta t}$$

for $\alpha > \beta \geq 0$.

```
% Pr.  0_20
clear all
% compare two exponentials
 t=[0:0.001:10];
 x=exp(-0.5*t);
 x1=exp(-1*t);
 figure(6)
 plot(t,x,t,x1,'r');
 legend('Exponential Signal, a=-0.5','Exponential Signal, a=-1')
 grid
 axis([0 10 0 1.1 ]); xlabel('time')
```

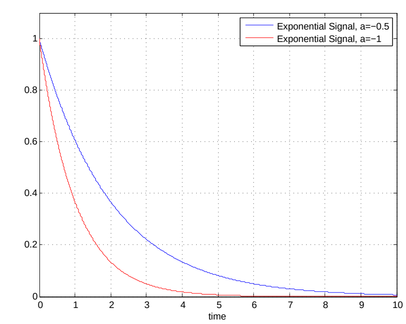


Figure 7: Problem 20: Comparison of exponentials $e^{-0.5t}$ and e^{-t} for $t \geq 0$ and 0 otherwise.

(c) Sampling $x(t) = e^{at}$ using $T_s = 1$, we get

$$x(t)|_{t=n} = e^{an} = \alpha^n$$

where $\alpha = e^a > 0$
(d) The voltage in the capacitor is given by

$$v_c(t) = \frac{1}{C}\int_0^t e^{-0.5\tau} d\tau + v_c(0)$$

with a initial voltage $v_c(0) = 0$. Letting $C = 1$, we have

$$v_c(t) = \frac{e^{-0.5\tau}}{-0.5}|_0^t = 2(1 - e^{-0.5t})$$

so that at $t = 1$ the voltage in the capacitor is $v_c(1) = 2 - 2e^{-0.5} = 0.79$.
(e) Letting $NT_s = 1$, the definite integral is approximated, from below, by

$$\sum_{n=0}^{N-1} T_s e^{-0.5(n+1)T_s}$$

if we let $\alpha = e^{-0.5T_s}$ the above sum becomes

$$T_s \sum_{n=0}^{N-1} \alpha^{n+1} = T_s \alpha \frac{1-\alpha^N}{1-\alpha}$$

which is computed using the following script:

```
% compute value of Int (the integral)
 N=1000;Ts=1/N;alpha=exp(- 0.5*Ts);
 Int=Ts*alpha*(1-alpha^N)/(1-alpha)

Int =  0.7867
```

approximating the analytic result found above.

0.21 (a) The point (1,1) in the two-dimensional plane corresponds to $z = 1 + j$. The magnitude and phase are

$$|z| = \sqrt{1+1} = \sqrt{2}$$
$$\angle z = \tan^{-1}(1) = \pi/4$$

(b) For the other complex numbers:

$$|w| = \sqrt{2}, \quad \angle w = \pi - \pi/4 = 3\pi/4$$
$$|v| = \sqrt{2}, \quad \angle v = \pi + \pi/4 = 5\pi/4$$
$$|u| = \sqrt{2}, \quad \angle u = -\pi/4$$

The sum of these complex numbers

$$z + w + v + u = 0$$

(c) The ratios

$$\frac{z}{w} = \frac{1+j}{-1+j} = \frac{\sqrt{2}e^{j\pi/4}}{\sqrt{2}e^{j3\pi/4}} = 1e^{-j\pi/2} = -j$$
$$\frac{w}{v} = \frac{-1+j}{-1-j} = \frac{\sqrt{2}e^{j3\pi/4}}{\sqrt{2}e^{j5\pi/4}} = 1e^{-j\pi/2} = -j$$
$$\frac{u}{z} = \frac{1-j}{1+j} = \frac{\sqrt{2}e^{-j\pi/4}}{\sqrt{2}e^{j\pi/4}} = 1e^{-j\pi/2} = -j$$

Also, multiplying numerator and denominator by the by the conjugate of the denominator we get the above results. For instance,

$$\frac{z}{w} = \frac{1+j}{-1+j} = \frac{(1+j)(-1-j)}{2} = \frac{-1-j-j-j^2}{2} = \frac{-2j}{2} = -j$$

and similarly for the others. Using these ratios we have

$$\frac{u}{w} = \frac{u}{z} \times \frac{z}{w} = (-j)(-j) = -1.$$

(d) $y = 10^{-6} = j10^{-6} = 10^{-6}z$ so that

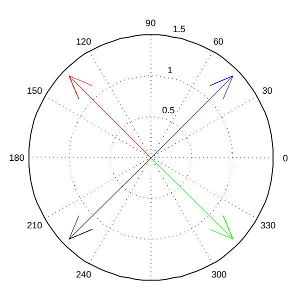
$$|y| = 10^{-6}|z| = 10^{-6}$$
$$\angle y = \pi/4$$

Although the magnitude of y is negligible, its phase is equal to that of z.

The results are verified by the following script:

```
% Pr. 0_21
z=1+j; w=-1+j; v=-1-j;u=1-j;
figure(1)
compass(1,1)
hold on
compass(-1,1,'r')
hold on
compass(-1,-1,'k')
```

```
hold on
compass(1,-1,'g')
hold off
% part (a)
abs(z)
angle(z)
% part (b)
abs(w)
angle(w)
abs(v)
angle(v)
abs(u)
angle(u)
r=z+w+v+u
%part (c)
r1=z/w
r2=w/v
r3=u/z
r4=u/z
r5=u/w
figure(2)
compass(real(r1),imag(r1))
hold on
compass(real(r2),imag(r2),'r')
hold on
compass(real(r3),imag(r3),'k')
hold on
compass(real(r4),imag(r4),'g')
hold on
compass(real(r5),imag(r5),'b')
hold off
% part (c)
z
y=z*1e-16
abs(y)
angle(y)/pi
```



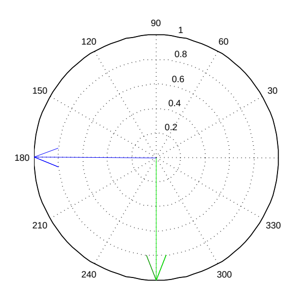


Figure 8: Problem 21: Results of complex calculations in parts (a) z, w, v, u and (b) $z/w, w/v, u/z, z/w$

0.22 (a)(b) Since

$$x(t) = (1 + jt)^2 = 1 + j2t + j^2t^2 = \underbrace{1 - t^2}_{\text{real}} + j\underbrace{2t}_{\text{imag.}}$$

its derivative with respect to t is

$$y(t) = \frac{dx(t)}{dt} = -2t + 2j = \frac{d\mathcal{R}e[x(t)]}{dt} + j\frac{d\mathcal{I}m[x(t)]}{dt}$$

```
% Pr.  0_22
clear all
 t=[-5: 0.001:5];
 x=(1+j*t).^2;
 xr=real(x);
 xi=imag(x);
 figure(7)
  subplot(211)
  plot(t,xr); title('Real part of x(t)'); grid
  subplot(212)
  plot(t,xi); title('Imaginary part of x(t)'); xlabel('Time'); grid
% Warning when plotting complex signals
figure(8)
 disp('Read warning. MATLAB is being nice with you, this time!')
 plot(t,x); title('COMPLEX Signal x(t)?'); xlabel('Time')
```

When plotting the complex function $x(t)$ as function of t, MATLAB ignores the imaginary part. One should not plot complex functions as functions of time as the results are not clear when using MATLAB. See Fig. 9 for plots.
(c) Using the rectangular expression of $x(t)$ we have

$$\int_0^1 x(t)dt = \int_0^1 (1 - t^2 + 2jt)dt = \int_0^1 (1 - t^2)dt + 2j\int_0^1 t\,dt = t - \frac{t^3}{3} + j\frac{2t^2}{2}\,|_0^1 = \frac{2}{3} + j1$$

(d) The integral

$$\int_0^1 x^*(t)dt = \int_0^1 (1 - t^2 - 2jt)dt = \int_0^1 (1 - t^2)dt - 2j\int_0^1 t\,dt = t - \frac{t^3}{3} - j\frac{2t^2}{2}\,|_0^1 = \frac{2}{3} - j1$$

which is the complex conjugate of the integral calculated in (c). So yes, the expression is true.

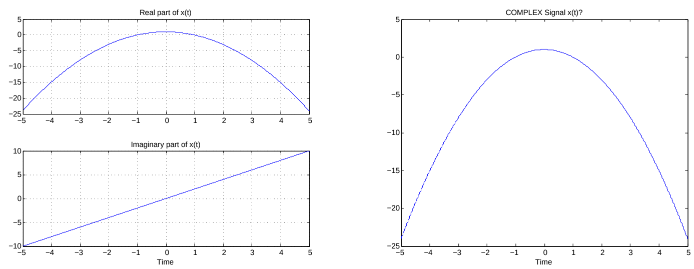


Figure 9: Problem 22: Real and imaginary parts of $x(t)$ (left); complex signal $x(t)$ ignoring imaginary part.

0.23 (a) Using Euler's identity

$$e^{j\pi n} = \cos(\pi n) + j\sin(\pi n) = \cos(\pi n) = (-1)^n$$

so it is a real signal.

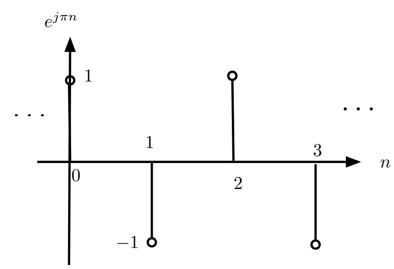


Figure 10: Problem 23: The complex exponential $e^{j\pi n} = \cos(\pi n)$ which is real.

(b) Replacing the sines by exponentials we have

$$\begin{aligned}
\sin(\alpha)\sin(\beta) &= \frac{1}{-4}(e^{j\alpha} - e^{-j\alpha})(e^{j\beta} - e^{-j\beta}) \\
&= \frac{1}{-4}\left[e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)} - e^{j(\alpha-\beta)} - e^{j(-\alpha+\beta)}\right] \\
&= \frac{1}{-4}\left[2\cos(\alpha+\beta) - 2\cos(\alpha-\beta)\right] \\
&= \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]
\end{aligned}$$

(c) Similarly

$$\begin{aligned}
\cos(\alpha)\sin(\beta) &= \frac{1}{4j}(e^{j\alpha} + e^{-j\alpha})(e^{j\beta} - e^{-j\beta}) \\
&= \frac{1}{4j}\left[e^{j(\alpha+\beta)} - e^{-j(\alpha+\beta)} - e^{j(\alpha-\beta)} + e^{j(-\alpha+\beta)}\right] \\
&= \frac{1}{4j}\left[2j\sin(\alpha+\beta) - 2j\sin(\alpha-\beta)\right] \\
&= \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]
\end{aligned}$$

If $\alpha = \beta$ then $\cos(\alpha)\sin(\alpha) = (1/2)\sin(2\alpha)$ since $\sin(0) = 0$. We have that $T_0 = 2$ is the period of $\sin(\pi t)$, $\cos(\pi t)$ as well as $\sin(2\pi t)$ (indeed, $\sin(2\pi(t+2)) = \sin(2\pi t + 4\pi) = \sin(2\pi t)$), therefore the integral

$$\int_0^{T_0} \sin(\pi t)\cos(\pi t)dt = 0.5 \underbrace{\int_0^2 \sin(2\pi t)dt}_{\text{area under two periods of } \sin(2\pi t)} = 0.$$

Thus $\sin(\pi t)$ and $\cos(\pi t)$ are orthogonal.

0.24

$$\cos(j\theta) = \frac{1}{2}(e^{-\theta} + e^{\theta}) = \cosh(\theta)$$

(b) The hyperbolic sine is defined as

$$\sinh(\theta) = \frac{1}{2}(e^{\theta} - e^{-\theta})$$

which is connected with the circular sine as follows

$$\sin(j\theta) = \frac{1}{2j}(e^{-\theta} - e^{\theta}) = j\sinh(\theta) \quad \Rightarrow \quad \sinh(\theta) = -j\sin(j\theta)$$

(c) Since $e^{\pm\theta} > 0$ then $\cosh(\theta) = \cosh(-\theta) > 0$, the smallest value is for $\theta = 0$ which gives $\cosh(0) = 1$

(d) Indeed,

$$\sinh(-\theta) = \frac{1}{2}(e^{-\theta} - e^{\theta}) = -\sinh(\theta)$$

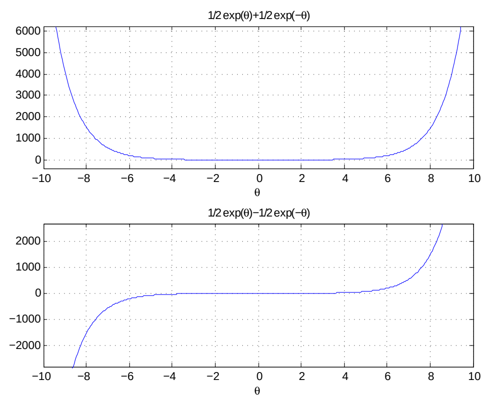


Figure 11: Problem 24: $\cosh(\theta)$ (top) and $\sinh(\theta)$ (bottom).

```
% Pr.  0_24
clear all
theta=sym('theta');
x= 0.5*(exp(-theta)+exp(theta));
y= 0.5*(exp(theta)-exp(-theta));
figure(9)
subplot(211)
ezplot(x,[-10,10])
grid
subplot(212)
ezplot(y,[-10,10])
grid
```

Chapter 1

Continuous–time Signals

1.1 Basic Problems

1.1 Notice that $0.5[x(t) + x(-t)]$, the even component of $x(t)$, is discontinuous at $t = 0$, it is 1 at $t = 0$ but 0.5 at $t \pm \epsilon$ for $\epsilon \to 0$. Likewise the odd component of $x(t)$, or $0.5[x(t) - x(-t)]$, must be zero at $t = 0$ so that when added to the even component one gets $x(t)$.
$z(t)$ equals $x(t)$. See Fig. 1.

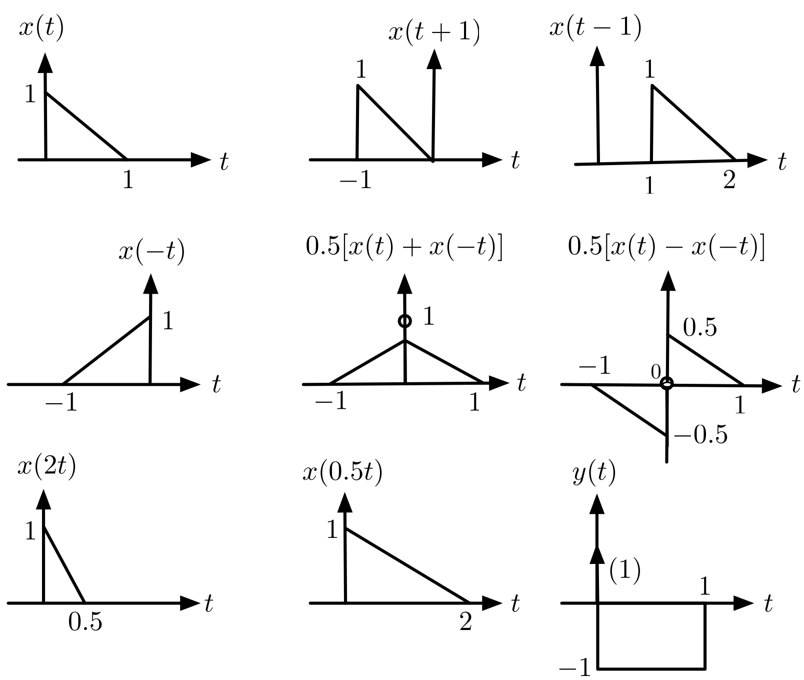


Figure 1.1: Problem 1

1.2 (a) If $x(t) = t$ for $0 \leq t \leq 1$, then $x(t+1)$ is $x(t)$ advanced by 1, i.e., shifted to the left by 1 so that $x(0) = 0$ occurs at $t = -1$ and $x(1) = 1$ occurs at $t = 0$.

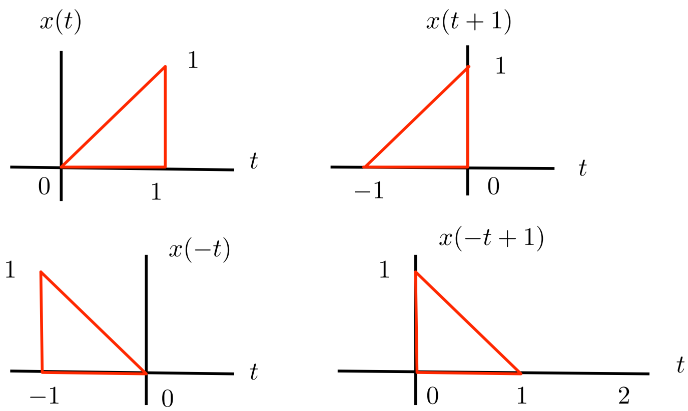


Figure 1.2: Problem 2: Original signal $x(t)$, shifted versions $x(t+1)$, $x(-t)$ and $x(-t+1)$.

The signal $x(-t)$ is the reversal of $x(t)$ and $x(-t+1)$ would be $x(-t)$ advanced to the right by 1. Indeed,

t	$x(-t+1)$
1	$x(0)$
0	$x(1)$
-1	$x(2)$

The sum $y(t) = x(t+1) + x(-t+1)$ is such that at $t = 0$ it is $y(0) = 2$; $y(t) = x(t+1)$ for $t < 0$; and $y(t) = x(-t+1)$ for $t > 0$. Thus,

$$\begin{aligned} &y(t) = x(t+1) = t+1 \qquad 0 \leq t+1 < 1 \quad \text{or} \quad -1 \quad \leq t < 0 \\ &y(0) = 2 \\ &y(t) = x(-t+1) = -t+1 \qquad 0 \leq -t+1 < 1 \quad \text{or} \quad 0 < t \leq 1 \end{aligned}$$

or

$$y(t) = \begin{cases} t+1 & -1 \leq t < 0 \\ 2 & t = 0 \\ -t+1 & 0 < t \leq 1 \end{cases}$$

(b) Except for the discontinuity at $t = 0$, $y(t)$ looks like the even triangle signal $\Lambda(t)$, their integrals are

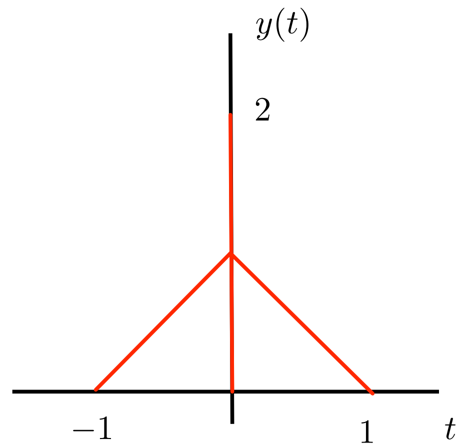


Figure 1.3: Problem 2: Triangular signal $y(t)$ with discontinuity at the origin.

identical as the discontinuity of $y(t)$ does not add any area.

1.3 (a) We have that

i. $x(t)$ is causal because it is zero for $t < 0$. It is neither even nor odd.

ii. Yes, the even component of $x(t)$ is

$$\begin{aligned} x_e(t) &= 0.5[x(-t) + x(t)] \\ &= 0.5[e^t u(-t) + e^{-t} u(t)] = 0.5e^{-|t|} \end{aligned}$$

(b) $x(t) = \cos(t) + j\sin(t)$ is a complex signal, $x_e(t) = 0.5[e^{jt} + e^{-jt}] = \cos(t)$ so $x_o(t) = j\sin(t)$.

(c) The product of the even signal $x(t)$ with the sine, which is odd, gives an odd signal and because of this symmetry the integral is zero.

(d) Yes, because $x(t) + x(-t) = 2x_e(t)$, i.e., twice the even component of $x(t)$, and multiplied by the sine it is an odd function.

1.4 The signal $x(t) = t[u(t) - u(t-1)]$ so that its reflection is

$$v(t) = x(-t) = -t[u(-t) - u(-t-1)]$$

and delaying $v(t)$ by 2 is

$$\begin{aligned} y(t) &= v(t-2) = -(t-2)[u(-(t-2)) - u(-(t-2)-1)] \\ &= (-t+2)[u(-t+2) - u(-t+1)] = (2-t)[u(t-1) - u(t-2)] \end{aligned}$$

On the other hand, the delaying of $x(t)$ by 2 gives

$$w(t) = x(t-2) = (t-2)[u(t-2) - u(t-3)]$$

which when reflected gives

$$z(t) = w(-t) = (-t-2)[u(-t-2) - u(-t-3)]$$

Comparing $y(t)$ and $z(t)$ we can see that these operations do not commute, that the order in which these operations are done cannot be changed, so that $y(t) \neq z(t)$ as shown in Fig. 1.4.

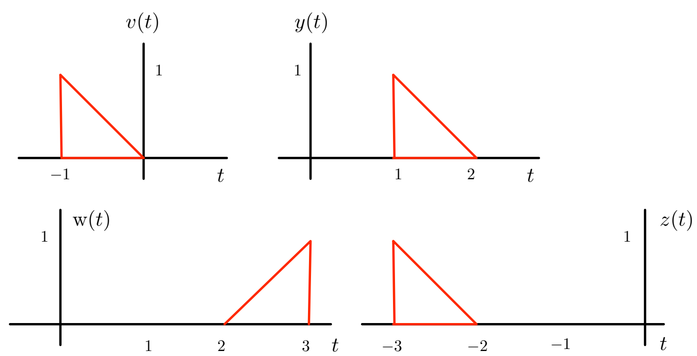


Figure 1.4: Problem 4: Reflection and delaying do not commute, $y(t) \neq z(t)$.

1.5 (a) $x(t)$ is called causal because it is zero for $t < 0$, it repeats every 0.5 sec. for $t \geq 0$.

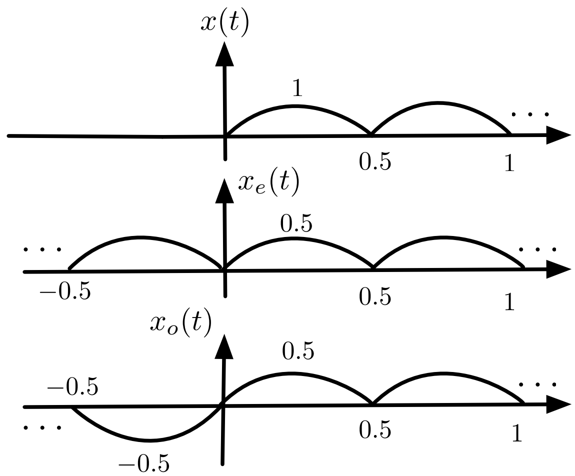


Figure 1.5: Problem 5.

(b) Even component $x_e(t) = 0.5[x(t) + x(-t)]$ is periodic of fundamental period $T_e = 0.5$. The odd component is $x_o(t) = 0.5[x(t) - x(-t)]$ is not periodic.

(c) $x_e(t)$ and $x_o(t)$ are non-causal signals as they are different from zero for negative times.

1.6 (a) Using $\Omega_0 = 2\pi f_0 = 2\pi/T_0$ for

i. $\cos(2\pi t)$: $\Omega_0 = 2\pi$ rad/sec, $f_0 = 1$ Hz and $T_0 = 1$ sec.

ii. $\sin(t - \pi/4)$: $\Omega_0 = 1$ rad/sec, $f_0 = 1/(2\pi)$ Hz and $T_0 = 2\pi$ sec.

iii. $\tan(\pi t) = \sin(\pi t)/\cos(\pi t)$: $\Omega_0 = \pi$ rad/sec, $f_0 = 1/2$ Hz and $T_0 = 2$ sec.

(b) The fundamental period of $\sin(t)$ is $T_0 = 2\pi$, and $T_1 = 2\pi/3$ is the fundamental period of $\sin(3t)$, $T_1/T_0 = 1/3$ so $3T_1 = T_0 = 2\pi$ is the fundamental period of $z(t)$.

(c) i. $y(t)$ is periodic of fundamental period $T_0 = 1$.

ii. $w(t) = x(2t)$ is $x(t)$ compressed by a factor of 2 so its fundamental period is $T_0/2 = 1/2$, the fundamental period of $z(t)$.

iii. $v(t)$ has same fundamental period as $x(t)$, $T_0 = 1$, indeed $v(t + kT_0) = 1/x(t + kT_0) = 1/x(t)$.

(d) i. $x(t) = 2\cos(t)$, $\Omega_0 = 2\pi f_0 = 1$ so $f_0 = 1/(2\pi)$

ii. $y(t) = 3\cos(2\pi t + \pi/4)$, $\Omega_0 = 2\pi f_0 = 2\pi$ so $f_0 = 1$

iii. $c(t) = 1/\cos(t)$, of fundamental period $T_0 = 2\pi$, so $f_0 = 1/(2\pi)$.

(e) $z_e(t)$ is periodic of fundamental period T_0, indeed

$$\begin{aligned} z_e(t + T_0) &= 0.5[z(t + T_0) + z(-t - T_0))] \\ &= 0.5[z(t) + z(-t)] \end{aligned}$$

Same for $z_o(t)$ since $z_o(t) = z(t) - z_e(t)$.

1.7 (a) i. $x(t) = \cos(t + \pi/4)$, $\Omega_0 = 1 = 2\pi/T_0$ so $T_0 = 2\pi$,
$x(t + kT_0) = \cos(t + k2\pi + \pi/4) = x(t)$

ii. $y(t) = 2 + \sin(2\pi t)$, $\Omega_0 = 2\pi$, $T_0 = 1$
$y(t + kT_0) = 2 + \sin(2\pi t + 2\pi k) = y(t)$

iii. $z(t) = 1 + (\cos(t)/\sin(3t))$, $T_0 = 2\pi$ fundamental period of cosine, $T_1 = 2\pi/3$ fundamental period of the sine, then $T_0/T_1 = 3$ or $T_0 = 3T_1 = 2\pi$ is the fundamental period of $z(t)$,

$$z(t + 2\pi k) = 1 + \frac{\cos(t + 2\pi k)}{\sin(3t + 6\pi k)} = z(t)$$

(b) i. $z_1(t)$ is periodic of period $10T_0$, indeed

$$\begin{aligned} z_1(t + 10T_0) &= x_1(t + 10T_0) + 2y_1(t + 10T_0) \\ &= x_1(t) + 2y_1(t) \end{aligned}$$

ii. $v_1(t)$ is periodic of fundamental period $10T_0$ as

$$v_1(t + 10T_0) = \frac{x_1(t + 10T_0)}{y_1(t + 10T_0)} = \frac{x_1(t)}{y_1(t)}$$

iii. $w_1(t)$ is periodic of fundamental period T_0, since $y_1(10T_0)$ is compressed by a factor of 10 so its fundamental period is T_0 the same as $x_1(t)$.

1.8 (a) $x(t)$ is a causal decaying exponential with energy

$$E_x = \int_0^\infty e^{-2t} dt = \frac{1}{2}$$

and zero power as

$$P_x = \lim_{T \to \infty} \frac{E_x}{2T} = 0$$

(b)

$$E_z = \int_{-\infty}^{\infty} e^{-2|t|} dt = 2 \underbrace{\int_0^\infty e^{-2t} dt}_{E_{z_1}}$$

(c) i. If $y(t) = \text{sign}[x_1(t)]$, it has the same fundamental period as $x_1(t)$, i.e., $T_0 = 1$ and $y(t)$ is a train of pulses so its energy is infinite, while

$$P_y = \int_0^1 1 \, dt = 1$$

ii. Since $x_2(t) = \cos(2\pi t - \pi/2) = \cos(2\pi(t - 1/4)) = x_1(t - 1/4)$, the energy and power of $x_2(t)$ coincide with those of $x_1(t)$.

(d) $v(t) = x_1(t) + x_2(t)$ is periodic of fundamental period $T_0 = 2\pi$, and its power is

$$P_v = \frac{1}{2\pi} \int_0^{2\pi} (\cos(t) + \cos(2t))^2 dt = \frac{1}{2\pi} \int_0^{2\pi} (\cos^2(t) + \cos^2(2t) + 2\cos(t)\cos(2t)) dt$$

Using

$$\begin{aligned} \cos^2(\theta) &= \frac{1}{2} + \frac{1}{2}\cos(2\theta) \\ \cos(\theta)\cos(\phi) &= \frac{1}{2}(\cos(\theta + \phi) + \cos(\theta - \phi)) \end{aligned}$$

we have

$$\begin{aligned} P_v &= \underbrace{\frac{1}{2\pi} \int_0^{2\pi} \cos^2(t) dt}_{P_{x_1}} + \underbrace{\frac{1}{2\pi} \int_0^{2\pi} \cos^2(2t) dt}_{P_{x_2}} + \underbrace{\frac{1}{2\pi} \int_0^{2\pi} 2\cos(t)\cos(2t)) dt}_{0} \\ &= \frac{1}{2} + \frac{1}{2} + 0 = 1 \end{aligned}$$

(e) Power of $x(t)$

$$\begin{aligned} P_x &= \frac{1}{T_0} \int_0^{T_0} x^2(t) dt \\ &= \int_0^1 \cos^2(2\pi t) dt \\ &= \int_0^1 (1/2 + \cos^2(4\pi t) dt = 0.5 + 0 = 0.5 \end{aligned}$$

Power of $f(t)$

$$\begin{aligned} P_f &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y^2(t)dt \\ &= \lim_{N\to\infty} \frac{1}{2(NT_0)}\int_{0}^{NT_0} y^2(t)dt \\ &= \frac{1}{2T_0}\int_0^{T_0} y^2(t)dt = 0.5P_s \end{aligned}$$

1.9 This problem can be done in the time domain or in the phasor domain. The series connection of the source $v_s(t) = \cos(t)$, the resistor R and the inductor L is equivalent to the connection of a phasor source $V_s = 1e^{j0}$, and impedances R and $j\Omega L = jL$ (the frequency of the source is $\Omega = 1$). The corresponding to the current across the resistor and the inductor, in steady state, is

$$I = \frac{V_s}{R + jL}$$

(a) $L = 1$, $R = 0$ —intuitively, the power used by the inductor is zero since only the resistor uses power.

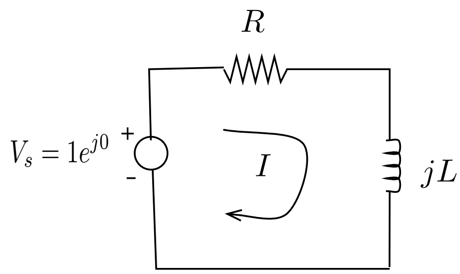


Figure 1.6: Problem 9: Phasor circuit.

In this case, the current $i(t)$ has a phasor

$$I = \frac{1}{j} = -j = 1e^{-j\pi/2}$$

so that the current across the inductor in steady state is given by

$$i(t) = \cos(t - \pi/2)$$

We can compute the average power P_a in time by finding the instantaneous power as

$$p(t) = i(t)v_s(t) = \cos(t - \pi/2)\cos(t) = \frac{1}{2}(\cos(\pi/2) + \cos(2t - \pi/2))$$

so that

$$\begin{aligned} P_a &= \frac{1}{T_0}\int_0^{T_0} p(t)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}[\cos(\pi/2) + \cos(2t - \pi/2)]dt = 0 \end{aligned}$$

since $\cos(\pi/2) = 0$ and the area under $\cos(2t - \pi/2)$ in a period is zero.

You probably remember from Circuits that the average power is computed using the equivalent expression

$$P_a = \frac{V_{sm}I_m}{2}\cos(\theta)$$

where V_{sm} and I_m are the peak-to-peak values of the phasors corresponding to V_s and I, and θ is the angle in the impedance of the inductor, i.e, $j1 = e^{j\pi/2}$ or $\theta = \pi/2$, and the average power is then

$$P_a = 0.5\cos(\pi/2) = 0$$

Confirming our intuition!
(b) For $L = 1$, $R = 1$, the phasor

$$I = \frac{V_s}{1+j} = \frac{\sqrt{2}}{2}e^{-j\pi/4}$$

and so in the phasor domain,

$$P_a = \frac{V_{sm}I_m}{2}\cos(\pi/4) = \frac{\sqrt{2}/2}{2}\sqrt{2}/2 = \frac{1}{4}$$

(c) $L = 0$, $R = 1$, in this case the power used by the resistor will be the power provided by the source. in this case the phasor for the current across the resistor is

$$I = V_s = 1e^{j0} \text{ so that } i(t) = \cos(t)$$

in the steady state. Thus,

$$\begin{aligned} P_a &= \frac{1}{T_0}\int_0^{T_0} p(t)dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}[\cos(0) + \cos(2t)]dt = 0.5 \end{aligned}$$

In the phasor domain, the average power is

$$P_a = \frac{V_{sm}^2}{2}\cos(0) = \frac{1}{2}$$

(d) The complex power supplied to the circuit is given by

$$P = \frac{1}{2}V_s I^* = \frac{1}{2}(IZ)I^* = \frac{|I|^2|Z|}{2}e^{j\theta}$$

where $Z = |Z|e^{j\theta} = R + j\Omega L$ is the input impedance.
Since $\Omega = 1$, then for

- $R = 0$, $L = 1$, $Z = j$, $I = -j$ so $P = \frac{1}{2}e^{j\pi/2} = 0 + j0.5$ and $P_a = \mathcal{R}e[P] = 0$.
- $R = 1$, $L = 1$, $Z = 1 + j$, $I = 1/(1 + j)$ so $|I|^2 = 1/2$, $Z = \sqrt{2}$, $\theta = \pi/4$ so that $P = 0.5(0.5)\sqrt{2}e^{j\pi/4} = 0.25\sqrt{2}(\cos(\pi/4) + j\sin(\pi/4))$ and $P_a = \mathcal{R}e[P] = 0.25$.
- $R = 1$, $L = 0$, $Z = 1$, $I = 1$ so $P = \frac{1}{2}e^{j0} = 0.5 + j0$ and $P_a = \mathcal{R}e[P] = 0.5$.

The real part of the complex power corresponds to the average power used by the resistors, while the imaginary part corresponds to the reactive power which is due to inductor and capacitors only.

1.10 (a) Let $x(t) = x_1(t) + x_2(t) = \cos(2\pi t) + 2\cos(\pi t)$, so that $x_1(t)$ is a cosine of frequency $\Omega_1 = 2\pi$ or period $T_1 = 1$, and $x_2(t)$ is a cosine of frequency $\Omega_2 = \pi$ or period $T_2 = 2$. The ratio of these periods $T_2/T_1 = 2/1$ is a rational number so $x(t)$ is periodic of fundamental period $T_0 = 2T_1 = T_2 = 2$.
The average power of $x(t)$ is given by

$$P_x = \frac{1}{T_0}\int_0^{T_0} x^2(t)dt = \frac{1}{2}\int_0^2 [x_1^2(t) + x_2^2(t) + 2x_1(t)x_2(t)]dt$$

Using the trigonometric identity $\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta)$ we have that the integral

$$\begin{aligned}\frac{1}{2}\int_0^2 2x_1(t)x_2(t)dt &= \frac{1}{2}\int_0^2 4\cos(2\pi t)\cos(\pi t)dt \\ &= \int_0^2 [\cos(\pi t) + \cos(3\pi t)]dt = 0\end{aligned}$$

since $\cos(\pi t) + \cos(3\pi t)$ is periodic of period 2 and so its area under a period is zero. Thus,

$$\begin{aligned}P_x &= \frac{1}{2}\int_0^2 [x_1^2(t) + x_2^2(t)]dt \\ &= \frac{1}{2}\int_0^2 x_1^2(t)dt + \frac{1}{2}2\int_0^1 x_2^2(t)]dt \\ &= P_{x_1} + P_{x_2}\end{aligned}$$

so that the power of $x(t)$ equals the sum of the powers of $x_1(t)$ and $x_2(t)$ which are sinusoids of different frequencies, and thus orthogonal as we will see later.
Finally,

$$\begin{aligned}P_x &= \frac{1}{2}\int_0^2 \cos^2(2\pi t)dt + \int_0^1 4\cos^2(\pi t)dt \\ &= \frac{1}{2}\int_0^2 [0.5 + 0.5\cos(4\pi t)]dt + \int_0^1 4[0.5 + 0.5\cos(2\pi t)]dt \\ &= 0.5 + 2 = 2.5\end{aligned}$$

remembering that the integrals of the cosines are zero (they are periodic of period 0.5 and 1 and the integrals compute their areas under one or more periods, so they are zero).
(b) The components of $y(t)$ have as periods $T_1 = 2\pi$ and $T_2 = 2$ so that $T_1/T_2 = \pi$ which is not rational so $y(t)$ is not periodic. In this case we need to find the power of $y(t)$ by finding the integral over an infinite support of $y^2(t)$ which will as before give

$$P_y = P_{y_1} + P_{y_2}$$

In the case of harmonically related signals we can use the periodicity and compute one integral. However, in either case the power superposition holds.

1.11 (a) Yes, expressing $e^{j2\pi t} = \cos(2\pi t) + j\sin(2\pi t)$, periodic of fundamental period $T_0 = 1$, then the integral is the area under the cosine and sine in one or more periods (which is zero) when $k \neq 0$ and integer. If $k = 0$, the integral is also zero.

(b) Yes, whether $t_0 = 0$ (first equation) or a value different from zero, the two integrals are equal as the area under a period is the same. In the case $x(t) = \cos(2\pi t)$, both integrals are zero.

(c) It is not true, $\cos(2\pi t)\delta(t-1) = \cos(2\pi)\delta(t-1) = \delta(t-1)$.

(d) It is true, considering $x(t)$ the product of $\cos(t)$ and $u(t)$ its derivative is

$$\begin{aligned}\frac{dx(t)}{dt} &= \frac{d\cos(t)}{dt}u(t) + \cos(t)\frac{du(t)}{dt}\\ &= -\sin(t)u(t) + \cos(0)\delta(t)\end{aligned}$$

(e) Yes,

$$\begin{aligned}\int_{-\infty}^{\infty}\left[e^{-t}u(t)\right]\delta(t-2)d\tau &= \int_{0}^{\infty}\left[e^{-2}\right]\delta(t-2)d\tau\\ &= e^{-2}\end{aligned}$$

(f) Yes,

$$\begin{aligned}\frac{dx(t)}{dt} &= 0.5[e^t u(t) + e^t\delta(t)] + 0.5[-e^{-t}u(t) + e^{-t}\delta(t)]\\ &= 0.5[e^t - e^{-t}]u(t) + \delta(t) = \sinh(t)u(t) + \delta(t)\end{aligned}$$

(g) The even component $x_e(t)$ is a periodic full-wave rectified signal of amplitude $1/2$ and fundamental period $T_1 = \pi$.
Power of $x(t)$

$$P_x = 0.5\left[\frac{1}{\pi}\int_0^{\pi} x^2(t)dt\right]$$

Power of $x_e(t)$

$$P_{x_e} = \frac{1}{\pi}\int_0^{\pi}(0.5x(t))^2 dt = 0.5P_x$$

1.12 (a) See Fig. 12a
$x(t) = |t| \underbrace{[u(t+2) - u(t-2)]}_{p(t)}$ Derivative

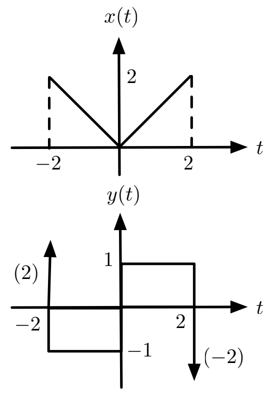


Figure 1.7: Problem 12

$$y(t) = \frac{dx(t)}{dt} = 2\delta(t+2) - u(t+2) + 2u(t) - u(t-2) - 2\delta(t-2)$$

(b) Integral

$$\int_{-\infty}^{t} y(t')dt' \;=\; \begin{cases} 0 & t < -2 \\ -t & -2 \leq t < 0 \\ t & 0 \leq t < 2 \\ 0 & t \geq 2 \end{cases}$$

which equals $x(t)$.

(c) Yes, because $x(t)$ is an even function of t.

1.13 (a) The signal $x(t)$ is

$$x(t) = \begin{cases} 0 & t < -1 \\ t+1 & -1 \leq t \leq 0 \\ -1 & 0 < t \leq 1 \\ 0 & t > 1 \end{cases}$$

there are discontinuities at $t = 0$ and at $t = 1$. The derivative

$$\begin{aligned} y(t) &= \frac{dx(t)}{dt} \\ &= u(t+1) - u(t) - 2\delta(t) + \delta(t-1) \end{aligned}$$

indicating the discontinuities at $t = 0$, a decrease from 1 to -1, and at $t = 1$ an increase from -1 to 0.

(b) The integral

$$\begin{aligned} \int_{-\infty}^{t} y(\tau)d\tau &= \int_{-\infty}^{t} [u(\tau+1) - u(\tau) \\ &\quad -2\delta(\tau) + \delta(\tau-1)]d\tau = x(t) \end{aligned}$$

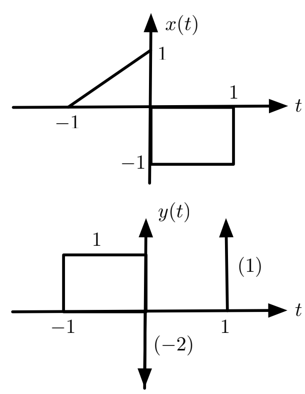


Figure 1.8: Problem 13

1.14 (a) $x(t)$, $-\infty < t < \infty$, is a continuous signal and its derivative exists and it is

$$y(t) = \frac{d\cos(\Omega_0 t)}{dt} = -\Omega_0 \sin(\Omega_0 t)$$

(b) $x_1(t)$ has a discontinuity at $t = 0$, and so its derivative will have a $\delta(t)$ function. Indeed, its derivative is

$$\begin{aligned} z(t) &= \frac{d\cos(\Omega_0 t)u(t)}{dt} \\ &= \frac{d\cos(\Omega_0 t)}{dt}u(t) + \cos(\Omega_0 t)\frac{du(t)}{dt} \\ &= -\Omega_0 \sin(\Omega_0 t)u(t) + \cos(\Omega_0 t)\delta(t) \\ &= -\Omega_0 \sin(\Omega_0 t)u(t) + \cos(0)\delta(t) \\ &= -\Omega_0 \sin(\Omega_0 t)u(t) + \delta(t) \end{aligned}$$

(c) The integral of $z(t)$ is zero for $t < 0$, and

$$\begin{aligned} \int_{-\infty}^{t} z(t')dt' &= \int_{0}^{t} -\Omega_0 \sin(\Omega_0 t')dt' + \int_{0-}^{t} \delta(t')dt' \\ &= [\cos(\Omega_0 t) - 1] + 1 = \cos(\Omega_0 t) \qquad t > 0 \end{aligned}$$

or $\cos(\Omega_0 t)u(t)$.

1.15 (a) The signal $x(t) = t$ for $0 \leq t \leq 1$, zero otherwise. Then

$$x(2t) = \begin{cases} 2t & 0 \leq 2t \leq 1 \text{ or } 0 \leq t \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

that is, the signal has been compressed — instead of being between 0 and 1, it is now between 0 and 0.5.

(b) Likewise, the signal

$$x(t/2) = \begin{cases} t/2 & 0 \leq t/2 \leq 1 \text{ or } 0 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

i.e., the signal has been expanded, its support has doubled.
The following figure illustrates the compressed and expanded signals $x(2t)$ and $x(t/2)$.

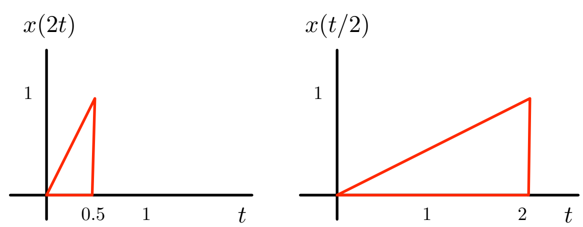


Figure 1.9: Problem 15: Compressed $x(2t)$, expanded $x(t/2)$ signals.

(c) If the acoustic signal is recorded in a tape, we can play it faster (contraction) or slower (expansion) than the speed at which it was recorded. Thus the signal can be made to last a desired amount of time, which might be helpful whenever an allocated time is reserved for broadcasting it.

1.16 (a) Because of the discontinuity of $x(t)$ at $t = 0$ the even component of $x(t)$ is a triangle with $x_e(0) = 1$, i.e.,

$$x_e(t) = \begin{cases} 0.5(1-t) & 0 < t \leq 1 \\ 0.5(1+t) & -1 \leq t < 0 \\ 1 & t = 0 \end{cases}$$

while the odd component is

$$x_o(t) = \begin{cases} 0.5(1-t) & 0 < t \leq 1 \\ -0.5(1+t) & -1 \leq t < 0 \\ 0 & t = 0 \end{cases}$$

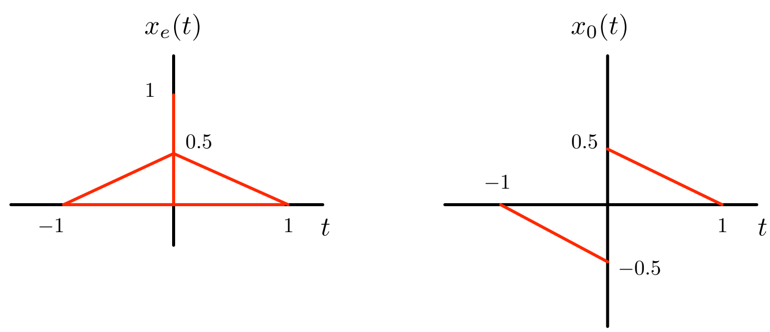


Figure 1.10: Problem 16: Even and odd decomposition of $x(t)$.

(b) The energy of $x(t)$ is

$$\begin{aligned} \int_{-\infty}^{\infty} x^2(t)dt &= \int_{-\infty}^{\infty} [x_e(t) + x_o(t)]^2 dt \\ &= \int_{-\infty}^{\infty} x_e^2(t)dt + \int_{-\infty}^{\infty} x_o^2(t)dt + 2\int_{-\infty}^{\infty} x_e(t)x_o(t)dt \end{aligned}$$

where the last equation on the right is zero, given that the integrand is odd.
(c) The energy of $x(t) = 1 - t$, $0 \leq t \leq 1$ and zero otherwise, is given by

$$\int_{-\infty}^{\infty} x^2(t)dt = \int_0^1 (1-t)^2 dt = t - t^2 + \frac{t^3}{3} |_0^1 = \frac{1}{3}$$

The energy of the even component is

$$\int_{-\infty}^{\infty} x_e^2(t)dt = 0.25\int_{-1}^{0} (1+t)^2 dt + 0.25\int_0^1 (1-t)^2 dt = 0.5\int_0^1 (1-t)^2 dt$$

where the discontinuity at $t = 0$ does not change the above result. The energy of the odd component is

$$\int_{-\infty}^{\infty} x_o^2(t)dt = 0.25\int_{-1}^{0} (1+t)^2 dt + 0.25\int_0^1 (1-t)^2 dt = 0.5\int_0^1 (1-t)^2 dt$$

so that

$$E_x = E_{x_e} + E_{x_o}$$

1.17 (a) The function $g(t)$ corresponding to the first period of $x(t)$ is given by

$$g(t) = u(t) - 2u(t-1) + u(t-2)$$

(b) The periodic signal $x(t)$ is

$$\begin{aligned} x(t) = \quad & g(t) + g(t-2) + g(t-4) + \cdots \\ + \quad & g(t+2) + g(t+4) + \cdots = \sum_{k=-\infty}^{\infty} g(t+2k) \end{aligned}$$

(c) Yes, the signals $y(t)$, $z(t)$ and $v(t)$ are periodic of period $T_0 = 2$ as can be easily verified.
(d) The derivative of $x(t)$ is

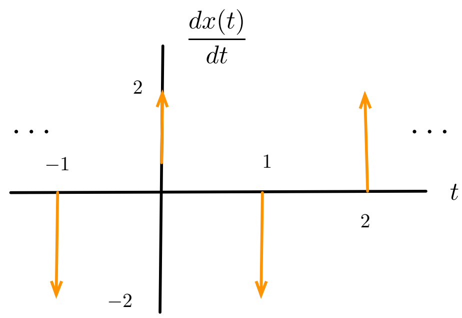


Figure 1.11: Problem 17: Derivative of $x(t)$.

$$\begin{aligned} w(t) = 2\delta(t) \quad & - \quad 2\delta(t-1) + 2\delta(t-2) + \cdots \\ & - \quad 2\delta(t+1) + 2\delta(t+2) + \cdots \end{aligned}$$

which can be seen to be periodic of period $T_0 = 2$.

1.18 (a) $\Omega_0 = 2\pi = 2\pi f_0$ (rad/sec), so $f_0 = 1/T_0 = 1$ (Hz) and $T_0 = 1$ sec.
The sum

$$\begin{aligned} z(t) &= x(t) + y(t) \\ &= (2\cos(2\pi t) + \cos(\pi t)) + j(2\sin(2\pi t) + \sin(\pi t) \end{aligned}$$

is also periodic of period $T_1 = 2$.
(b) $v(t) = x(t)y(t) = 2e^{j3\pi t}$ with frequency $\Omega_3 = 3\pi$ so that

$$T_3 = 2\pi/\Omega_3 = 2/3$$

1.19 (a) The derivative signal $y(t) = dx(t)/dt$ is a train of rectangular pulses. Indeed, if $x_1(t) = r(t) - 2r(t-0.5) + r(t-1)$ is the first period of $x(t)$ then

$$x(t) = \sum_{k=-\infty}^{\infty} x_1(t-k)$$

its derivative is

$$y(t) = \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} \frac{dx_1(t-k)}{dt}$$

where

$$\frac{dx_1(t-k)}{dt} = u(t-k) - 2u(t-0.5-k) + u(t-1-k)$$

(b) The signal $x(t) - 0.5$ has an average of zero, so its integral

$$z(t) = \lim_{N\to\infty} N \int_0^1 (x(t) - 0.5)dt = 0$$

(c) Neither is a finite energy signal.

1.2 Problems using MATLAB

1.20 The given signal $x(t) = e^{-|t|}$ is even, positive and decays to zero as $t \to \pm\infty$
(a) The signal is finite energy as

$$E_x = \int_{-\infty}^{\infty} x^2(t)dt \quad = \quad 2\int_0^{\infty} e^{-2t}dt = 2\frac{e^{-2t}}{-2}\,|_0^{\infty} = 1$$

(b) The signal $x(t)$ is absolutely integrable as

$$\int_{-\infty}^{\infty} |x(t)|dt \quad = \quad 2\int_0^{\infty} e^{-t}dt = 2\frac{e^{-2t}}{-1}\,|_0^{\infty} = 2$$

Notice that $0 < x^2(t) < x(t)$ and so the knowledge that $x(t)$ is absolutely integrable (i.e., that the above integral is finite) would imply that $x(t)$ has finite energy (i.e., the integral calculated in (b) is finite).
(c) The energy of $y(t)$ is

$$E_y = \int_0^{\infty} e^{-2t}\cos^2(2\pi t)dt < \int_0^{\infty} e^{-2t}dt = E_x/2 = 1/2$$

since $\cos^2(2\pi t) \leq 1$ (the decaying sinusoid is bounded by the envelope $e^{-2t}u(t)$).

```
% Pr. 1_20
clear all; clf
syms x y t z
x=exp(-abs(t));
% computation of integrals
% for increasing values of time
for k=1:100,
    zi=2*int(x,t,0,k/10);
    yi=2*int(x^2,t,0,k/10);
    vi=int((exp(-t)*cos(2*pi*t))^2,0,k/10);
    zz(k)=subs(zi);
    yy(k)=subs(yi);
    vv(k)=subs(vi);
end
t1=[1:100]/10;
figure(1)
 subplot(221)
 ezplot(x,[-10,10]);grid
 axis([-10 10 0 1]);title('x(t)=e^{-|t|}')
 subplot(222)
 plot(t1,zz);grid;title('integral of |x(t)|');xlabel('t')
 subplot(223)
 plot(t1,yy);grid;title('integral of |x(t)|^2');xlabel('t')
 subplot(224)
 plot(t1,vv);grid;title('integral of |e^{-t}cos(2\pit)|^2');xlabel('t')
figure(2)
 ezplot((exp(-t)*cos(2*pi*t))^2,[0,5]);grid
 axis([0 5 0 1])
 hold on
```

```
ezplot((exp(-t))^2,[0,5])
axis([0 5 0 1]);title('envelope of |y(t)|^2')
hold off
```

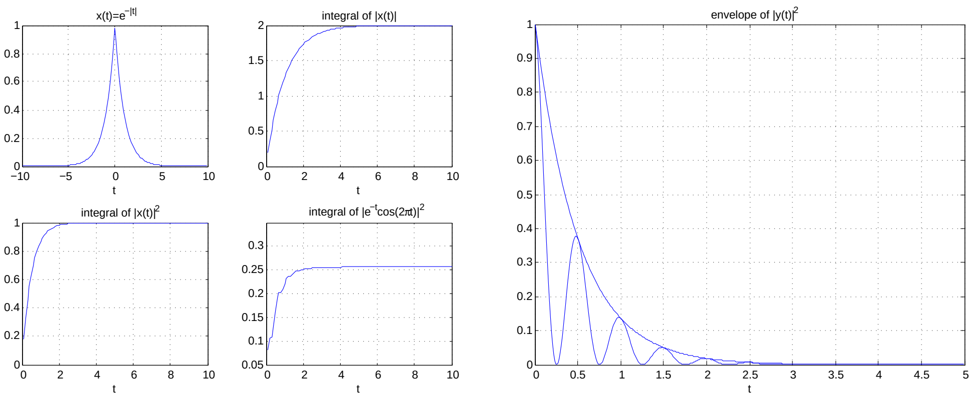


Figure 1.12: Problem 20: signal $x(t)$, and the integrals of $|x(t)|$, $|x(t)|^2$ and $|y(t)|^2$ (left). Right: envelope of $|y(t)|^2$.

(d) For a value C for the capacitor, considering the initial condition the source for the RC circuit the KVL equation for $t \geq 0$ is:

$$v_R(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = 1, \quad \text{or}$$
$$e^{-t} + \frac{1}{CR}\int_0^t e^{-\tau}d\tau = 1$$

after replacing the voltage and current in the resistor. Solving the integral we obtain

$$e^{-t} + \frac{1}{RC}(1 - e^{-t}) = 1$$

so that for $t = 0$ we get an identity indicating the initial condition is satisfied by the solution. For $t \to \infty$ we get $1/RC = 1$. So that $R = 1/C$ in general, for $C = 1$ mF then $R = 1$ KΩ and for $C = 1\mu = 10^{-6}$F, then $R = 10^6\Omega$ or 1 MΩ.

1.21 (a) The signal $x_1(t) = 4\cos(\pi t)$ has frequency $\Omega_1 = 2\pi/2$ so that the period of $x_1(t)$ is $T_1 = 2$. Likewise the signal $x_2(t) = -\sin(3\pi t + \pi/2)$ has frequency $\Omega_2 = 3\pi = 2\pi/(2/3)$ so that it is periodic of period $T_2 = 2/3$. The signal $x(t)$ is periodic of fundamental period $T_0 = 2$ as the ratio $T_1/T_2 = 2/(2/3) = 3$ so that $T_0 = 3T_2 = T_1 = 2$.

(b) The ratio of the two periods is

$$\frac{T_1}{T_2} = \frac{3}{3 \times 4} = \frac{1}{4}$$

so that

$$T_0 = 4T_1 = T_2$$

is the period of $x(t) = x_1(t) + x_2(t)$.

(c) In general, if the ratio of the periods of two periodic signals is

$$\frac{T_1}{T_2} = \frac{M}{K}$$

for integers M and K, not divisible by each other, then $T_0 = KT_1 = MT_2$ is the period of the sum of the periodic signals. If the ratio is not rational (i.e., M and/or K are not integers) then the sum of the two periodic signals is not periodic.

The following script is used to show that $x_1(t) + x_2(t)$ is periodic, while $x_3(t) + x_4(t)$ is not.

```
% Pr. 1_21
clear all; clf
syms x1 x2 x3 x4 t
x1=4*cos(2*pi*t); x2=-sin(3*pi*t+pi/2);
x3=4*cos(2*t);x4=x2;
figure(3)
  subplot(211)
  ezplot(x1+x2,[0 10]);grid
  subplot(212)
  ezplot(x3+x4,[0 10]);grid
```

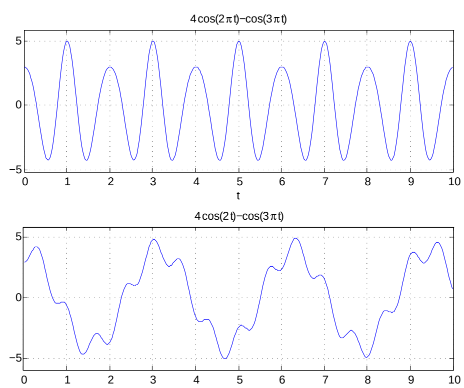


Figure 1.13: Problem 21: periodic $x_1(t) + x_2(t)$ (top), non–periodic $x_3(t) + x_4(t)$ (bottom).

1.22 (a) The triangular pulse has a width of 2Δ and a height of $1/\Delta$, its area is 1. The following MATLAB script can be used to see the limit as $\Delta \to 0$

```
% Pr. 1_22
clear all; clf
% part (a)
delta=0.1;
t=[-delta:0.05:delta];N=length(t);
lambda=zeros(1,N);
figure(5)
for k=1:6,
    lambda=(1-abs(t/delta))/delta;
    delta=delta/2;
    plot(t,lambda);xlabel('t')
    axis([-0.1 0.1 0 330]);grid
    hold on
    pause(0.5)
end
grid
hold off
```

(b) The signal $S_\Delta(t) = 1/\Delta s(t/\Delta)$ so that

$$S_\Delta(t) = \frac{1}{\Delta}\,\frac{\sin(\pi t/\Delta)}{\pi t/\Delta} = \frac{\sin(\pi t/\Delta)}{\pi t}$$

and so

$$S_\Delta(0) = \lim_{t\to\ 0}(\pi/\Delta)\,\frac{\cos(\pi t/\Delta)}{\pi} = 1/\Delta$$

and $S_\Delta(t)$ is zero at

$$\pi t/\Delta = \pm k\pi \qquad k \neq 0 \text{ integer}$$

or $t = \pm k\Delta$ and finally the integral

$$\int_{-\infty}^{\infty} S_\Delta(t)dt = \int_{-\infty}^{\infty} \frac{\sin(\tau\pi)}{\pi\Delta\tau}\Delta d\tau = 1$$

where we used $\tau = t/\Delta$. The following script illustrates the limit as $\Delta \to 0$.

```
% part (b)
 syms S t
 delta=1;
 figure(6)
 for k=1:4,
    delta=delta/k;
    S=(1/delta)*sinc(t/delta);
    ezplot(S,[-2 2])
    axis([-2 2  -8 30])
    hold on
    I=subs(int(S,t,-100*delta, 100*delta)) % area under sinc
    pause(0.5)
 end
 grid;xlabel('t')
 hold off
```

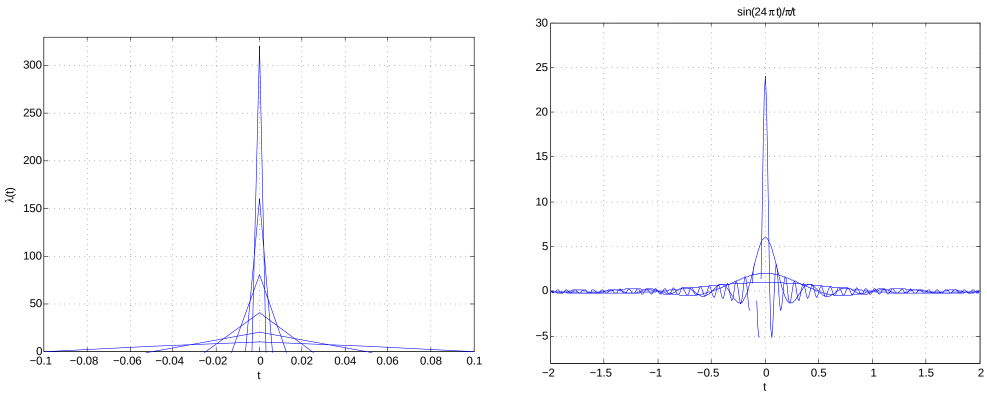


Figure 1.14: Problem 22: approximation of $\delta(t)$ using triangular (left) or sinc (right) functions

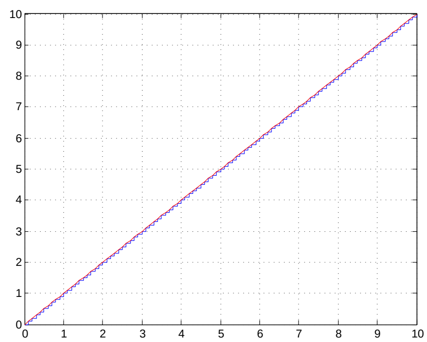
1.23 (a) The sampling signal is a sequence of unit impulses at uniform times kT, for $t > 0$. The integral

$$ss_T(t) = \int_{-\infty}^{t} \sum_{k=0}^{\infty} \delta(t - kT)dt = \sum_{k=0}^{\infty} \int_{-\infty}^{t} \delta(t - kT)dt = \sum_{k=0}^{\infty} u(t - kT)$$

This signal is called the "stairway to the stars" ($ss_T(t)$) for obvious reasons.
(b)(c) The following script will display the signal $ss(t)$ and the conversion to analog (just uncomment the desired signal and comment the not desired signal).

```
% Pr. 1_23 parts (b) and (c)
clear all; clf
T=0.1; t=0:T:10;
%f=t;  % part (b)
f=cos(2*pi*t); % part (c)
figure(6)
stairs(t,f);grid
%axis([0 10 0 10])  % part (b)
axis([0 10 -1.1 1.1]) % part (c)
hold on
plot(t,f,'r')
hold off
```

When $f(t) = t + 1$ the output looks like a digitized line with unit slope and cut at 1 (see figure below), similarly when $f(t) = \cos(2\pi t)$ the output looks like a digitized sinusoid.



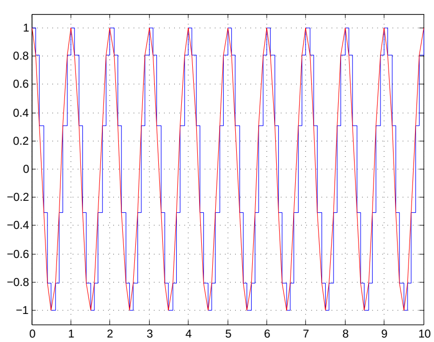


Figure 1.15: Problem 23: 'Digitized' ramp and cosine signals using 'stairway to stars'

1.24 (a) The expanded signal $x(t/2)$ is periodic. The first period of $x(t)$ is $x_1(t)$ for $0 \leq t \leq 2$, and so the period of $x(t/2)$ is $x_1(t/2)$ which is supported in $0 \leq t/2 \leq 2$ or $0 \leq t \leq 4$, so the period of $x(t/2)$ is 4.

(b) The compressed signal $x(2t)$ is periodic. The first period of $x(t)$, $x_1(t)$ for $0 \leq t \leq 2$, becomes $x_1(2t)$ for $0 \leq 2t \leq 2$ or $0 \leq t \leq 1$, its support is halved. So the period of $x(2t)$ is 1.

```
% Pr. 1_24, part(b)
clear all; clf
t=0:0.002:8;
t1=0:0.001:8; t2=0:0.004:8;
x=cos(pi*t);
x1=cos(pi*t1/2);
x2=cos(pi*2*t2);
figure(7)
 subplot(211)
 plot(t1,x1)
 hold on
 plot(t,x,'r')
 xlabel('t (sec)')
 ylabel('x(t/2), x(t)')
 legend('expanded signal', 'original signal')
 subplot(212)
 plot(t2,x2)
 hold on
 plot(t,x,'r')
 xlabel('t (sec) ')
 ylabel('x(2t), x(t)')
 hold off
 legend('compressed signal', 'original signal')
```

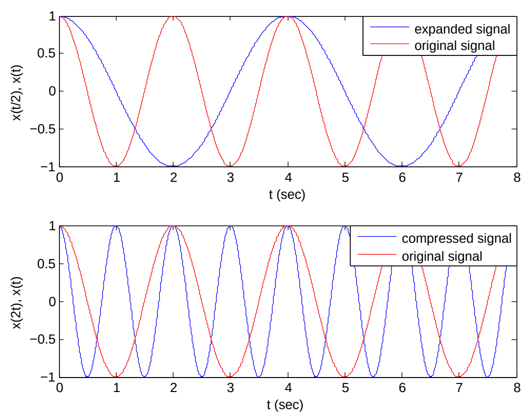


Figure 1.16: Problem 24: expanded and compressed sinusoids vs original sinusoid.

1.25 (a) The power of the full-wave rectified signal is

$$P_y = \int_0^1 |\sin(\pi t)|^2 dt$$

because the period of $y(t)$ is $T = 1$. A simpler expression for $\sin^2(\pi t)$ can be computed using Euler's equation

$$\begin{aligned} \sin^2(\pi t) &= \left[\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\right]^2 \\ &= \frac{-1}{4}\left[e^{j2\pi t} - 2 + e^{-j2\pi t}\right] \\ &= 0.5(1 - \cos(2\pi t)) \end{aligned}$$

Since $\cos(2\pi t)$ has a period 1 its integral over a period is zero, thus

$$P_y = 0.5$$

(b) A pulse $\rho(t) = u(t) - u(t-1)$ covers one of the periods of $y(t)$ and thus the area under the full-wave rectified signal is $P_y < 1$ the area of the pulse squared.
(c) The following script is used to calculate the power which is found to be 1/2

```
%  Pr. 1_25, part (c)
clear all;clf
syms x t
x=sin(pi*t); T=1;
figure(8)
ezplot(x^2,[0,5*T]);grid
P=int(x^2,t,0,T)/T
```

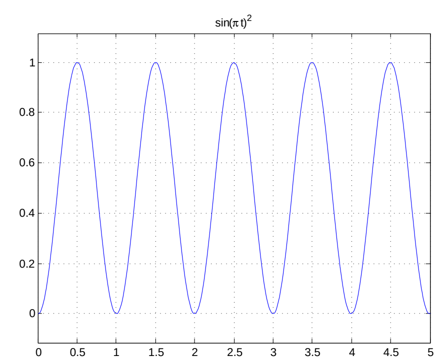


Figure 1.17: Problem 25: magnitude squared signal used to compute power.

1.26 The sampling rate F_s in sample/second is given with the discretized signal. To get one second of the signal we need to take $N = F_s$ samples from the given signal. The corresponding number of samples NN for $\tau = 0.5$ sec. is then calculated and the signal $y(t)$ computed and displayed as function of time as shown in the following script. For $F_s = 8,192$ samples/sec, $NN = 4,096$ samples

```
% Pr. 1_26
clear all; clf
load handel; Fs % test signal and sampling freq
N=Fs; y=y(1:N)'; % one second of handel
NN=fix(0.5*Fs)  % delay in samples
% delaying signals
t=0:1/Fs:(N-1)/Fs;
tt=0:1/Fs:(N-1)/Fs+2*NN/Fs;
 y1=[y zeros(1,2*NN)];
 y2=0.8*[zeros(1,NN) y zeros(1,NN)];
 y3=0.5*[zeros(1,2*NN) y];
yy=y1+y2+y3;
figure(9)
  subplot(211)
  plot(t,y); title('original signal');grid
  subplot(212)
  plot(tt,yy); title('multipath signal');grid
  xlabel('t (sec)')
sound(yy,Fs)
```

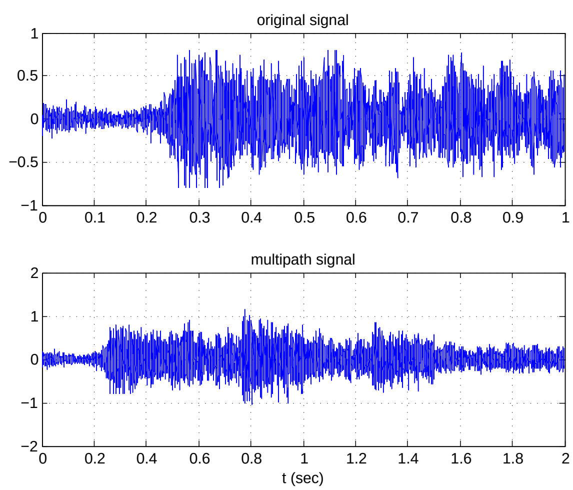


Figure 1.18: Problem 26: original 'handel' signal (top); two-path affected signal (bottom).

1.27 (a) (b) Adding 1 to the phasor $0.7e^{j\phi t}$ gives a phasor of continuously varying magnitude and phase. Part (a) of the script below shows it.
We have

$$1 + 0.7e^{j\phi t} = 1 + 0.7\cos(\phi t) + j0.7\sin(\phi t) = A(t)e^{j\theta(t)}$$

where

$$A(t) = \sqrt{(1 + 0.7\cos(\phi t))^2 + (0.7\sin(\phi t))^2} = \sqrt{1.49 + 1.4\cos(\phi t)}$$

and

$$\theta(t) = \tan^{-1}\left[\frac{0.7\sin(\phi t)}{1 + 0.7\cos(\phi t)}\right]$$

which are computed as indicated in the script below.
(c) In this case we consider the effects of having two paths, the attenuation and the delays in time and in frequency.

```
% Pr. 1_27
clear all; clf
 % part (a)
 t1=0;T=0.5;
 m=1;
 figure(10)
 for k=1:512,
     B=0.7*exp(j*pi*t1/100);
     A=1+B;
     A1(k)=abs(A);
     Theta(k)=angle(A)*180/pi;
     if k==20*m,
       compass(real(A),imag(A),'r')
       hold on
       compass(real(B),imag(B))
       hold on
       compass(1,0,'k')
       legend('A=B+1','B','1')
       m=m+1;
       pause(0.1)
     else
       t1=t1+T;
       hold off
     end
 end
 t=0:T:511*T;
 % part (b)
 figure(11)
   subplot(211)
   plot(t,A1);title('Magnitude of 1+e^{j\phi t}');grid
   axis([0 max(t) 0 1.1*max(A1)])
   subplot(212)
   plot(t,Theta);title('Phase (degrees) of 1+e^{j\phi t}');grid
   axis([0 max(t) 1.1*min(Theta) 1.1*max(Theta)]);xlabel('t')
 % part (c)
 y0=0.7*exp(j*(pi+pi/100)*t);
 y1=real(exp(j*pi*t)+[zeros(1,100) y0(1:length(y0)-100)]);
```

```
t1=0:T:(length(y1)-1)*T;
figure(12)
  plot(t1,y1);title('Multi-path effects');grid
  axis([0 max(t1) 1.1*min(y1) 1.1*max(y1)]); ylabel('y_1(t)');xlabel('t')
```

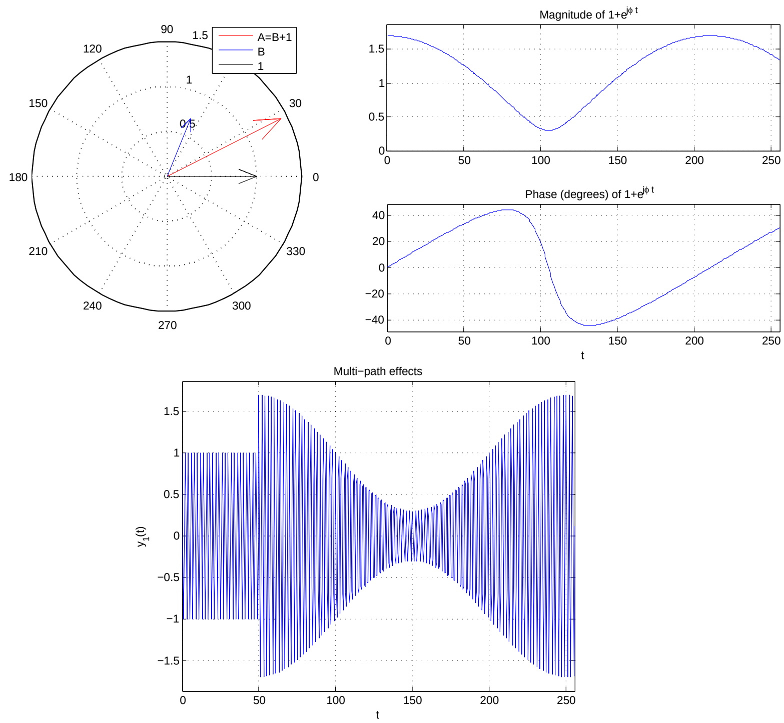


Figure 1.19: Problem 27: phasor plot (top–left); magnitude and phase of $1 + e^{j\phi t}$ (top–right); resulting signal due to multipath (bottom).

1.28 (a) The following script generates the signal $y(t)$ for $NP = 101$ players, and $\Delta = 0.02$ Hz (changing the NP to 51 we obtain the corresponding signal).

```
% Pr. 1_28
clear all; clf
NP=101 % number of players
% NP=51
A=10; delta=2/(NP-1);
F=160-(NP-1)/2*delta:delta:160+(NP-1)/2*delta;
t=0:0.1:200;
y=zeros(1,length(t));
figure(13)
for k=1:NP,
   y=y+A*cos(2*pi*F(k)*t);
   plot(t,y);grid
   pause(0.1)
end
ylabel('y(t)'); xlabel('t')
```

The final signal looks like a sequence of very narrow pulses.
(b) In this part, one can think of a multipath with NP paths, with no attenuation but a different Doppler shift, ranging from -1 Hz to 1 Hz, in increments of 0.02 Hz.

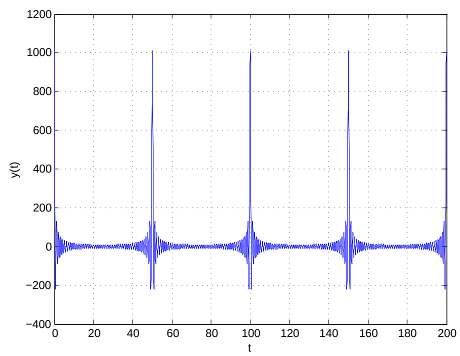


Figure 1.20: Problem 28: pulsation effect when $NP = 101$ and $\Delta = 0.02$ Hz.

1.29 (a)(b) The following script generates the chirps

```
% Pr. 1_29
clear all;clf
t=0:0.05:40;
% chirps
y=cos(2*t+t.^2/4);
y1=cos(2*t- 2*sin(t));
figure(14)
 subplot(211)
 plot(t,y); title('linear chirp')
 axis([0 20 1.1*min(y) 1.1*max(y)]);grid
 subplot(212)
 plot(t,y1);title('sinusoidal chirp');xlabel('t')
 axis([0 20 1.1*min(y1) 1.1*max(y1)]);grid
% instantaneous frequencies
IF=2+2*t/4;
IF1=2-2*cos(2*t);
figure(15)
 subplot(211)
 plot(t,IF);title('IF of linear chirp')
 ylabel('frequency'); xlabel('t');grid
 subplot(212
 plot(t,IF1);title('IF of sinusoidal chirp')
 ylabel('frequency');xlabel('t');grid
```

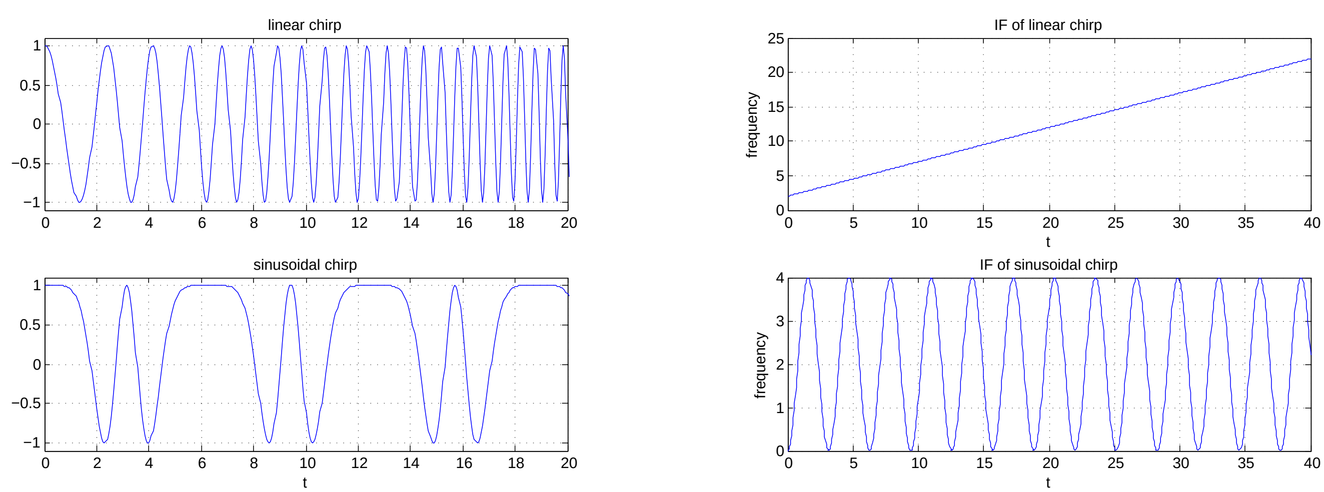


Figure 1.21: Problem 29: linear and sinusoidal chirps (left) and their corresponding instantaneous frequencies (right).

Chapter 2

Continuous–time Systems

2.1 Basic Problems

2.1 (a) The $y(t)$-$x(t)$ relation is a line through the origin between -10 to 10 and a constant before and after that. The system is non-linear, for instance if $x(t) = 7$ the output is $y(t) = 700$ but if we double the input, the output is not $2y(t) = 1400$ but 1000.
(b) If the inputs is always between -10 and 10 the system behaves like a linear system. In this case the output is chopped whenever $x(t)$ is above 10 or below -10. Se Fig. 2.1.
(c) Whenever the input goes below -10 or above 10 the output is -1000 and 1000, otherwise the output is $2000\cos(2\pi t)u(t)$.
(d) If the input is delayed by 2 the clipping will still occur, simply at a later time. So the system is time invariant.

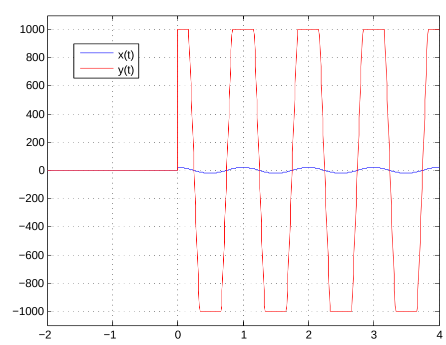


Figure 2.1: Problem 1: input and output of amplifier.

2.2 (a) Input $x_1(t) = \delta(t)$ gives

$$y_1(t) = \int_{t-1}^{t} \delta(\tau)d\tau + 2 = \begin{cases} 2 & t < 0 \\ 3 & 0 \leq t \leq 1 \\ 2 & t > 1 \end{cases}$$

$x_2(t) = 2x_1(t)$ gives

$$y_2(t) = 2\int_{t-1}^{t} \delta(\tau)d\tau + 2 = \begin{cases} 2 & t < 0 \\ 4 & 0 \leq t \leq 1 \\ 2 & t > 1 \end{cases}$$

Since $y_2(t) \neq 2y(t)$ system is non-linear.

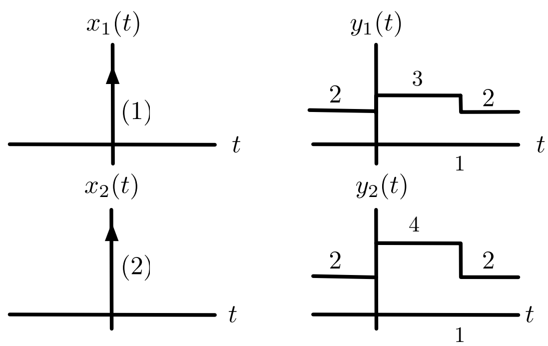


Figure 2.2: Problem 2

(b) If $x_3(t) = u(t) - u(t-1)$ then $y_3(t) = 2 + r(t) - 2r(t-1) + r(t-2)$. If $x_4(t) = x_3(t-1)$ then the corresponding output is $y_3(t-1)$, so the system is time-invariant.

(c) Non-causal, although $y(t)$ depends on present and past inputs, it is not zero when $x(t) = 0$, due to the bias of 2.

(d) If $|x(t)| < M$ we have

$$|y(t)| \leq \int_{t-1}^{t} |x(\tau)|d\tau + 2 < M + 2 < \infty$$

The system is BIBO stable.

2.3 (a) $x(t) = \sin(2\pi t)u(t)$ is zero for $t < 0$ and repeats periodically every $T_0 = 1$. Thus,

$$y(t) = \sum_{k=0}^{\infty} p(t-k), \quad p(t) = u(t) - 2u(t-0.5) + u(t-1)$$

If the input is $2x(t)$ the output is the same as before, so the system is non-linear.

(b) The system is time-invariant: if input $x(t-\tau)$ the output is $y(t-\tau)$ for any value of τ.

2.4 (a) Derivative

$$\frac{dz(t)}{dt} = w(t) - w(t-1)$$

which excludes the initial condition of 2. System is LTI if initial condition is zero.

(b) i. If input is $i(t-\mu)$ then the output is letting $\eta = \tau - \mu$

$$\int_0^t i(\tau - \mu) d\tau = \int_{-\mu}^0 i(\eta) d\eta + \int_0^{t-\mu} i(\eta) d\eta = v_c(t-\mu)$$

that is, provided that $i(t) = 0$ for $t < 0$, the system is time-invariant.

ii. If $i(t) = u(t)$ then $v_c(t) = \int_0^t u(\tau)d\tau = r(t)$. If we shift the inputs $i_1(t) = i(t-1) = u(t-1)$ the previous output is shifted, so system is time-invariant.

(c) If $x(t) = u(t)$ then $y(t) = \sin(2\pi t)u(t)$ while corresponding to $x(t-0.5) = u(t-0.5)$ is $y_1(t) = \sin(2\pi t)u(t-0.5)$ indicating the system is not time-invariant as $y_1(t)$ is not $y(t-0.5)$.

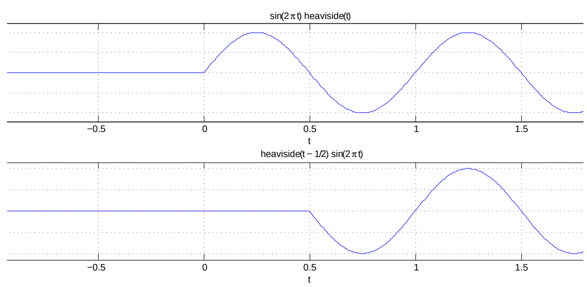


Figure 2.3: Problem 4(c)

2.5 (a) See Fig. 1. The circuit is a series connection of a voltage source $x(t)$ with a resistor $R = 1/2\ \Omega$, and capacitor $C = 1$F. Indeed, the mesh current is $i(t) = dy(t)/dt$ so

$$x(t) = Ri(t) + y(t) = Rdy(t)/dt + y(t)$$

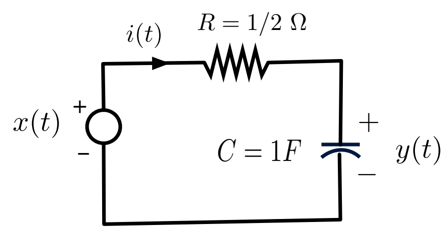


Figure 2.4: Problem 5

(b) The output is

$$y(t) = e^{-2t}\, 0.5\, e^{2\tau}|_0^t = 0.5(1 - e^{-2t})u(t)$$

and

$$\begin{aligned} \frac{dy(t)}{dt} &= e^{-2t}u(t) + 0.5(1 - e^{-2t})\delta(t) \\ &= e^{-2t}u(t) \\ \frac{dy(t)}{dt} + 2y(t) &= e^{-2t}u(t) + u(t) - e^{-2t}u(t) \\ &= u(t) \end{aligned}$$

2.6 (a) i. $z(t) = Av(t) + B$, system is linear if $B = 0$, non-linear otherwise.

ii. $z(t) = v(t)\cos(\Omega_0 t)$ is linear but time-varying.

iii. If $f(t) = v(t) = u(t) - u(t-1)$, $B = 0$ then $z(t) = u(t) - u(t-1)$, and if we shift $v(t)$ so the input is $v_1(t) = u(t-2) - u(t-3)$ the output is $z_1(t) = v_1(t)f(t) = 0$ which is different from $z(t-2)$, so the system is time-varying.

(b) i. The system is linear:

$$\frac{1}{T}\int_{t-T}^{t}[Ax_1(\tau) + Bx_2(\tau)]d\tau = \frac{A}{T}\int_{t-T}^{t} x_1(\tau)d\tau + \frac{B}{T}\int_{t-T}^{t} x_2(\tau)d\tau = Ay_1(t) + By_2(t)$$

where $y_i(t)$, $i = 1, 2$, are the corresponding outputs to $x_i(t)$,$i = 1, 2$.

ii. If $x(t) = u(t)$ then $y(t) = r(t) - r(t-1)$ and if $x_1(t) = u(t-2)$ the corresponding output is $y_1(t) = y(t-2)$. System is time-invariant.
In general, if $x_1(t) = x(t-\lambda)$ the output is

$$\frac{1}{T}\int_{t-T}^{t} \underbrace{x_1(\tau)}_{x(t-\lambda)} d\tau = \frac{1}{T}\int_{t-T-\lambda}^{t-\lambda} x(\nu)d\nu$$

(using $\nu = \tau - \lambda$) which is the same as $y(t-\lambda)$, so time-invariant.

iii. $y(t)$ depends on present and past inputs, and zero if input is zero, so the system is causal.

2.7 (a) The charge is

$$q(t) = C(t)v(t)$$

so that

$$i(t) = \frac{dq(t)}{dt} = C(t)\frac{dv(t)}{dt} + v(t)\frac{dC(t)}{dt}$$

(b) If $C(t) = 1 + \cos(2\pi t)$ and $v(t) = \cos(2\pi t)$, the current is

$$\begin{aligned} i_1(t) &= C(t)\frac{dv(t)}{dt} + v(t)\frac{dC(t)}{dt} \\ &= (1 + \cos(2\pi t))(-2\pi \sin(2\pi t)) - \cos(2\pi t)(2\pi \sin(2\pi t)) \\ &= -2\pi \sin(2\pi t)[1 + 2\cos(2\pi t)] \end{aligned}$$

(c) When the input is

$$v(t - 0.25) = \cos(2\pi(t - 1/4)) = \sin(2\pi t)$$

the output current is

$$\begin{aligned} i_2(t) &= C(t)\frac{dv(t-0.25)}{dt} + v(t-0.25)\frac{dC(t)}{dt} \\ &= (1 + \cos(2\pi t))(2\pi \cos(2\pi t)) - 2\pi \sin^2(2\pi t) \\ &= 2\pi \cos(2\pi t) + 2\pi[\cos^2(2\pi t) - \sin^2(2\pi t)] \end{aligned}$$

which is not

$$i_1(t - 0.25) = 2\pi \cos(2\pi t)[1 + \sin(2\pi t)]$$

so the system is time varying.

2.8 (a) The system is LTI since the input $x(t)$ and the output $y(t)$ are related by a convolution integral with $h(t-\tau) = e^{-(t-\tau)}u(t-\tau)$ or $h(t) = e^{-t}u(t)$.
Another way: to show that the system is linear let the input be $x_1(t) + x_2(t)$, and $x_1(t)$ and $x_2(t)$ have as outputs

$$y_i(t) = \int_0^t e^{-(t-\tau)} x_i(\tau) d\tau \qquad i = 1, 2$$

The output for $x_1(t) + x_2(t)$ is

$$\int_0^t e^{-(t-\tau)}(x_1(\tau) + x_2(\tau))d\tau = y_1(t) + y_2(t)$$

To show the time invariance let the input be $x(t - t_0)$, its output will be

$$\begin{aligned}\int_0^t e^{-(t-\tau)} x(\tau - t_0) d\tau &= \int_{-t_0}^{0} e^{-((t-t_0)-\mu)} x(\mu) d\mu + \int_0^{t-t_0} e^{-((t-t_0)-\mu)} x(\mu) d\mu \\ &= \int_0^{t-t_0} e^{-((t-t_0)-\mu)} x(\mu) d\mu = y(t - t_0)\end{aligned}$$

by letting $\mu = \tau - t_0$ and using the causality of the input. The system is then TI.

Finally the impulse response is found by letting $x(t) = \delta(t)$ so that the output is

$$h(t) = \int_0^t e^{-(t-\tau)} \delta(\tau) d\tau = \int_0^t e^{-(t-0)} \delta(\tau) d\tau = \begin{cases} e^{-t} \times 1 = e^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Yes, this system is causal as the output $y(t)$ depends on present and past values of the input.
(c) Letting $x(t) = u(t)$, the unit-step response is

$$s(t) = \int_0^t e^{-t+\tau} u(\tau) d\tau = e^{-t} \int_0^t e^{\tau} d\tau = 1 - e^{-t}$$

for $t \geq 0$ and zero otherwise. The impulse response as indicated before is $h(t) = ds(t)/dt = e^{-t}u(t)$. The BIBO stability of the system is then determined by checking whether the impulse response is absolutely integrable or not,

$$\int_{-\infty}^{\infty} |h(t)| dt = \int_0^{\infty} e^{-t} dt = -e^{-t} \,|_0^{\infty} \, = 1$$

so yes it is BIBO stable.
(d) Using superposition, the response to the pulse $x_1(t) = u(t) - u(t-1)$ would be

$$y_1(t) = y(t) - y(t-1) = (1 - e^{-t})u(t) - (1 - e^{-(t-1)})u(t-1)$$

which starts at zero, grows to a maximum of $1 - e^{-1}$ at $t = 1$ and goes down to zero as $t \to \infty$.

2.9 (a) Letting $x(t) = \delta(t)$ the impulse response is

$$\begin{aligned} h(t) &= \frac{1}{T}\int_{t-T/2}^{t+T/2} \delta(\tau)d\tau \\ &= \frac{1}{T}\int_{t-T/2}^{t} \delta(\tau)d\tau + \frac{1}{T}\int_{t}^{t+T/2} \delta(\tau)d\tau \end{aligned}$$

If $t > 0$, and $t - T/2 < 0$ the first integral includes 0, while the second does not. Thus

$$h(t) = \frac{1}{T}\int_{t-T/2}^{t} \delta(\tau)d\tau + 0 = \frac{1}{T} \qquad t > 0 \text{ and } t - T/2 < 0, \text{ or } \; 0 < t < T/2$$

Likewise when $t < 0$ then $t - T/2 < -T/2$ and $t + T/2 < T/2$ the reverse of the previous case happens and so

$$h(t) = 0 + \frac{1}{T}\int_{t}^{t+T/2} \delta(\tau)d\tau = \frac{1}{T} \qquad t < 0 \text{ and } t + T/2 > 0, \text{ or } \; -T/2 < t < 0$$

so that

$$h(t) = \frac{1}{T}[u(t + T/2) - u(t - T/2)]$$

indicating that the system is non-causal as $h(t) \neq 0$ for $t < 0$.
(b) If $x(t) = u(t)$ then the output of the averager is

$$y(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} u(\tau)d\tau$$

If $t + T/2 < 0$ then $y(t) = 0$ since the argument of the unit step signal is negative. If $t + T/2 \geq 0$ and $t - T/2 < 0$ then

$$y(t) = \int_{0}^{t+T/2} u(\tau)d\tau = \frac{1}{T}(t + T/2)$$

and finally when $t - T/2 \geq 0$ then

$$y(t) = \frac{1}{T}\int_{t-T/2}^{t+T/2} u(\tau)d\tau = 1$$

The unit-step response of the noncausal averager is

$$y(t) = \begin{cases} 0 & t < -T/2 \\ \frac{1}{T}(t + T/2) & -T/2 \leq t < T/2 \\ 1 & t \geq T/2 \end{cases}$$

2.10 The input to all the systems is $x(t) = \cos(t), -\infty < t < \infty$
(a) The system is non-linear, as the output

$$y(t) = \cos^2(t) = 0.5(1 + \cos(2t))$$

has frequency components of frequencies 0 and 2 (rad/sec) which are not in the input.
(b) The output is

$$y(t) = 0.5\cos(t) + 0.5\cos(t - 1)$$

having the same frequencies as the input so it is LTI.
(c) The output

$$y(t) = \cos(t)u(t)$$

is not LTI. This is not a periodic signal, and it has frequencies different from the one at the input due to the multiplication by the $u(t)$.
(d) The output is

$$y(t) = 0.5\sin(\tau)|_{t-2}^{t} = 0.5\sin(t) - 0.5\sin(t - 2)$$

having the same frequency as the input so it is LTI.

2.11 (a) Since $f(t)$ is not a constant, the system is a modulator thus linear but time varying. Linearity is clearly satisfied. If $x(t) \neq 0$ is the input and we shift it to get as input $x(t-11)$ the corresponding output is zero different from $y(t-11)$. Thus the system is time varying. Since $y(t)$ depends on $x(t)$ the system is causal. For $x(t)$ bounded, i.e., $|x(t)| < M < \infty$, the output is also bounded, $|y(t)| < M|f(t)| < \infty$ so the system is BIBO stable.
(b) The modulated signal is

$$x(t)f(t) = 2[\cos((\pi/2 + 6\pi/7)t) + \cos((6\pi/7 - \pi/2)t) = 2(\cos(19\pi t/14) + \cos(5\pi t/14)$$

with periods of $T_0 = 28/19$ and $T_1 = 28/5$ for the two components. The ratio

$$\frac{T_0}{T_1} = \frac{5}{9}$$

i.e., it is rational so the modulated signal is periodic of period $5T_1 = 19T_0 = 28$, which is easily verified. The frequencies at the output are not present at the input so the system is linear but not time–invariant ($f(t)$ is a function of t).
(c) If $x(t) = u(t)$, the modulated signal is $y(t) = u(t) - u(t-2)$, and if we shift the input so that it is $x(t-3) = u(t-3)$ the corresponding output is $u(t-3)[u(t) - u(t-2)] = 0$ different from the previous output shifted by 3, therefore the system is time-varying.

2.12 (a) If $y(0) = 0$ the system is linear, indeed for an input $\alpha x_1(t) + \beta x_2(t)$ with $y_1(t)$ the response due to $x_1(t)$ and $y_2(t)$ the response due to $x_2(t)$ we have

$$\int_0^t e^{-(t-\tau)}[\alpha a x_1(\tau) + \beta x_2(\tau)]d\tau = \alpha y_1(t) + \beta y_2(t)$$

If $y(0) \neq 0$, the output for input $\alpha x_1(t)$ is

$$y(0)e^{-t} + \int_0^t e^{-(t-\tau)}\alpha x_1(\tau)d\tau = y(0)e^{-t} + \alpha y_1(t)$$

which is not $\alpha y_1(t)$ thus it is not linear.

(b) If the input is $x(t) = 0$, then $y(t) = y(0)e^{-t}u(t)$ is the zero-input response, due completely to the initial condition. If $y(0) = 0$ the response

$$y(t) = \int_0^t e^{-(t-\tau)}x(\tau)d\tau$$

(which is the convolution integral of the impulse response $h(t) = e^{-t}u(t)$ with $x(t)$) is the zero-state response.

(c) The impulse response, obtained when $y(0) = 0$, $x(t) = \delta(t)$, and $y(t) = h(t)$ is

$$h(t) = \int_{0-}^t e^{-(t-\tau)}\delta(\tau)d\tau = e^{-t}\int_{0-}^t e^0\delta(\tau)d\tau = \begin{cases} e^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(d) If $x(t) = u(t)$ and $y(0) = 0$, then $y(t) = s(t)$ given by

$$s(t) = \int_0^t e^{-(t-\tau)}d\tau = (1 - e^{-t})u(t)$$

Notice the relation between the unit-step and the impulse response:

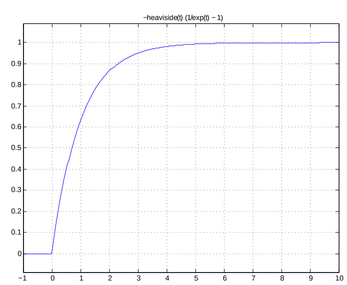


Figure 2.5: Problem 12

$$\begin{aligned} \frac{ds(t)}{dt} &= \delta(t) - e^{-t}\delta(t) + e^{-t}u(t) \\ &= e^{-t}u(t) = h(t) \end{aligned}$$

2.13 (a) To find the differential equation let $y(0) = 0$, so that

$$y(t) = 2e^{-2t} \int_0^t e^{2\tau} x(\tau) d\tau$$

and its derivative is

$$\frac{dy(t)}{dt} = -4e^{-2t} \int_0^t e^{2\tau} x(\tau) d\tau + 2e^{-2t} e^{2t} x(t) = -2y(t) + 2x(t)$$

giving the differential equation

$$\frac{dy(t)}{dt} + 2y(t) = 2x(t)$$

(b) Finding the integral after replacing $x(\tau) = u(\tau)$ we have that

$$y(t) = y(0)e^{-2t} + 2e^{-2t} \frac{e^{2\tau}}{2} \Big|_0^t = y(0)e^{-t} + (1 - e^{-2t})$$

so that in the steady state, i.e., when $t \to \infty$, the output is 1, independent of the value of the initial condition.

(c) From the given input-output equation, letting $x(t) = \delta(t)$ and $y(0) = 0$ the output is the impulse response of the system

$$h(t) = 2 \int_0^t e^{-2(t-\tau)} \delta(\tau) d\tau = 2e^{-2t} \int_0^t \delta(\tau) d\tau = 2e^{-2t} u(t)$$

Computing the convolution integral as

$$y(t) = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) d\tau$$

it can be obtained graphically by reflecting the input $x(t) = u(t)$ and shifting it linearly from $-\infty$ to ∞ to get

- for $t < 0$ then $y(t) = 0$
- for $t \geq 0$

$$y(t) = \int_0^t 2e^{-2\tau} d\tau = 1 - e^{-t}$$

and so in the steady state the output goes to 1.

(d) The zero-input response is

$$y_{zi}(t) = y(0)e^{-2t} u(t)$$

and it is bounded for any finite value of the initial condition $y(0)$, in particular for $y(0) = 1$, therefore the system that depends on the initial condition is BIBO.

2.14 (a) Yes. Using the convolution integral the output is

$$y(t) = \int_{-\infty}^{\infty} \underbrace{h(\tau)}_{u(\tau)-u(\tau-1)} x(t-\tau)d\tau = \int_0^1 x(t-\tau)d\tau = \int_{t-1}^{t} x(\eta)d\eta$$

where we changed the variable to $\eta = t - \tau$.

(b) If $x(t) = u(t)$ then the step-response is

$$y(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 1 & t \geq 1. \end{cases}$$

i.e., the unit-step response is $s(t) = r(t) - r(t-1)$ and the impulse response is

$$h(t) = \frac{ds(t)}{dt} = u(t) - u(t-1)$$

2.15 (a) $x_1(t) = x(t) - x(t-2)$ so $y_1(t) = y(t) - y(t-2)$, two triangular pulses, the second multiplied by -1.

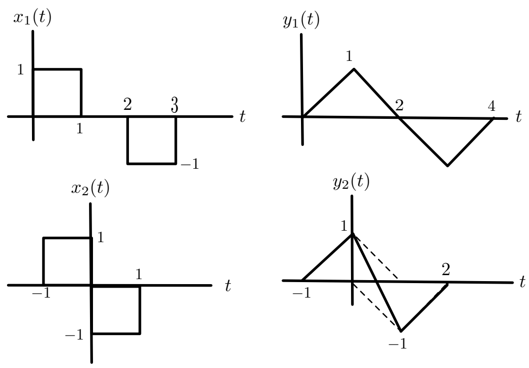


Figure 2.6: Problem 15

(b) $x_2(t) = x(t+1) - x(t)$ then $y_2(t) = y(t+1) - y(t)$ (they overlap between 0 and 1).

(c) $x_3(t) = \delta(t) - \delta(t-1)$ so $y_3(t) = dy(t)/dt = u(t) - 2u(t-1) + u(t-2)$. Considering that the output of $x(t)$ is $y(t)$, i.e., $y(t) = \mathcal{S}[x(t)]$, and that the integrator and the differentiator are LTI systems Fig. 2.7 shows how to visualize the result in this problem by considering that you can change the order of the cascading of LTI systems.

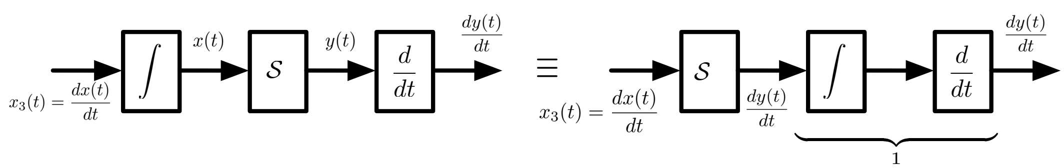


Figure 2.7: Problem 15

2.16 (a) If $x(t) = \sum_{k=0}^{9} \delta(t - kT)$ then by superposition and time-invariance

$$y(t) = \sum_{k=0}^{9} h(t - kT)$$

(b) If $T = 1$, $y(t) = u(t) - u(t - 10)$, while when $t = 0.5$

$$y(t) = \sum_{k=0}^{9} h(t - 0.5k) = \sum_{k=0}^{9} [u(t - 0.5k) - u(t - 0.5k - 1)]$$

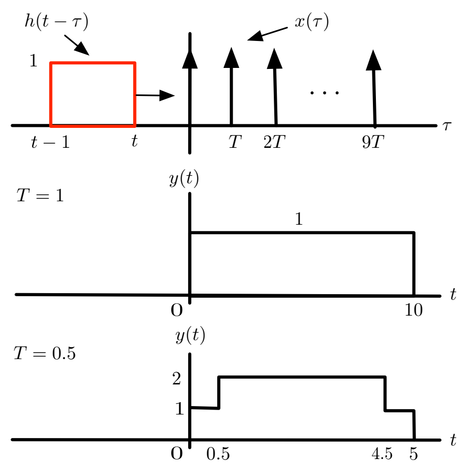


Figure 2.8: Problem 16

2.17 (a) By the definition of the derivative

$$\begin{aligned}\frac{d}{dt}\left[\int_0^t f(\tau)d\tau\right] &= \lim_{h\to 0}\frac{1}{h}\left[\int_0^{t+h} f(\tau)d\tau - \int_0^t f(\tau)d\tau\right]\\ &= \lim_{h\to 0}\frac{1}{h}\left[\int_t^{t+h} f(\tau)d\tau\right]\\ &= \lim_{h\to 0}\frac{f(t)h}{h} = f(t)\end{aligned}$$

(b) $y(t)$ satisfies the ordinary differential equation, indeed

$$\begin{aligned}\frac{dy(t)}{dt} &= -ay(0)e^{-at} + \frac{d}{dt}\left[e^{-at}\int_0^t e^{a\tau}x(\tau)d\tau\right]\\ &= -a\underbrace{[y(0)e^{-at} + e^{-at}\int_0^t e^{a\tau}x(\tau)d\tau]}_{y(t)} + x(t) = -ay(t) + x(t)\end{aligned}$$

(c) Multiplying the two terms of the ordinary differential equation by e^{at} we get

$$\underbrace{e^{at}dy(t)/dt + ae^{at}y(t)}_{d(e^{at}y(t))/dt} = e^{at}x(t)$$

$$e^{at}y(t) = \int_0^t e^{a\tau}x(\tau)d\tau + y(0)$$

solving for $y(t)$ we obtain the solution.

2.18 (a) For $x_1(t) = u(t) - u(t-2)$ and zero initial conditions, we can use the convolution integral

$$\begin{aligned} y_1(t) &= \int_0^t h(\tau)x_1(t-\tau)d\tau \\ &= \sum_{k=0}^{\infty} \int_0^t h_1(\tau - 2k)x(t-\tau)d\tau \end{aligned}$$

for $k = 0$, we find graphically the integral to be

$$z(t) = \int_0^t h_1(\tau)x_1(t-\tau)d\tau = r(t) - 2r(t-1) + 2r(t-3) - r(t-4)$$

graphically. So that

$$y_1(t) = \sum_{k=0}^{\infty} z(t-2k) = r(t) - 2r(t-1) + r(t-2)$$

(b) For $x_2(t) = \delta(t) - \delta(t-2)$, the output is

$$\begin{aligned} y_2(t) &= h(t) - h(t-2) \\ &= \sum_{k=0}^{\infty} h_1(t-2k) - \sum_{k=0}^{\infty} h_1(t-2(k+1)) \\ &= h_1(t) + \sum_{k=1}^{\infty} h_1(t-2k) - \sum_{k'=1}^{\infty} h_1(t-2k') \\ &= h_1(t) \end{aligned}$$

where we changed to the variable $k' = k+1$ in the second summation.
Also since $x_2(t) = dx_1(t)/dt$ the output

$$y_2(t) = \frac{dy_1(t)}{dt} = u(t) - 2u(t-1) + u(t-2) = h_1(t)$$

2.19 (a) The output is the sum of two modulated signals, and as such it is time-varying as each modulator is time-varying.
(b) If we let $m_1(t) = m_2(t) = m(t)$, the output is equal to the sum of two modulators so it is linear since the modulators are liner. Also

$$s(t) = m(t)(\cos(\Omega_c t) + \sin(\Omega_c t)) = \sqrt{2}m(t)\cos(\Omega_c t - \pi/4)$$

i.e., one modulator with a phase. This system is known to be linear.

2.20 (a) We can either find the impulse response or show that the output is bounded for any input signal that is bounded. To find the impulse response change the variable to $\sigma = t - \tau$ which gives

$$y(t) = \frac{-1}{T}\int_T^0 x(t-\sigma)d\sigma = \int_0^T \frac{1}{T}x(t-\sigma)d\sigma$$

which is a convolution integral with $h(t) = (1/T)[u(t) - u(t-T)]$. This impulse response is absolutely integrable as

$$\int_0^\infty |h(t)|dt = \int_0^T \frac{1}{T}dt = 1$$

Likewise, if we assume that $x(t)$ is bounded, i.e., there is a value $M < \infty$ such that $|x(t)| < M$, then

$$|y(t)| \leq \frac{1}{T}\int_{t-T}^t |x(\tau)|d\tau \leq \frac{M}{T}\int_{t-T}^t d\tau = M < \infty$$

so system is BIBO stable.

(b) The ramp is not bounded, there is no M such that $|r(t)| < M < \infty$. So when we compute the output of the stable system to the ramp we do not expect it to be bounded either. Indeed

$$y(t) = \frac{1}{T}\int_{t-T}^t \tau d\tau = \frac{t^2 - (t-T)^2}{2T} = t - \frac{T}{2}$$

increases as t increases, so it is unbound.

2.21 (a) The following block diagram represents the echo system:

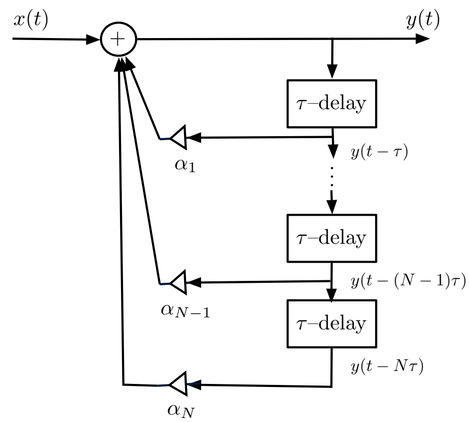


Figure 2.9: Problem 21: recursive model for echo system.

(b) When $N = 1$, $\tau = 1$ and $\alpha = 0.1$ the input/output equation for the echo system is

$$y(t) = x(t) + 0.1y(t-1)$$

To check the LTI we need an explicit expression of the output in terms of the input. This can be obtained by recursively replacing $y(t-k)$, $k = 1, 2, \cdots$ in the right by the a delayed version of the input/output equation, i.e.,

$$\begin{aligned} y(t) &= x(t) + 0.1y(t-1) = x(t) + 0.1x(t-1) + 0.01x(t-2)\cdots \\ &= \sum_{k=0}^{\infty}(0.1)^k x(t-k) \end{aligned}$$

Using this expression we have that if the input is $x(t) = x_1(t) + x_2(t)$, with corresponding outputs $y_1(t)$ and $y_2(t)$, then

$$\sum_{k=0}^{\infty}(0.1)^k[x_1(t-k) + x_2(t-k)] = y_1(t) + y_2(t)$$

so the echo system is linear. If we shift the input to $x(t-n_0)$ then the output will be

$$\sum_{k=0}^{\infty}(0.1)^k x(t-n_0-k) = y(t-n_0)$$

i.e., time-invariant. The system is then overall LTI.
(c) The non-recursive model can be written as

$$z(t) = \sum_{k=0}^{M}\beta_k x(t-k\tau) \qquad \beta_0 = 1$$

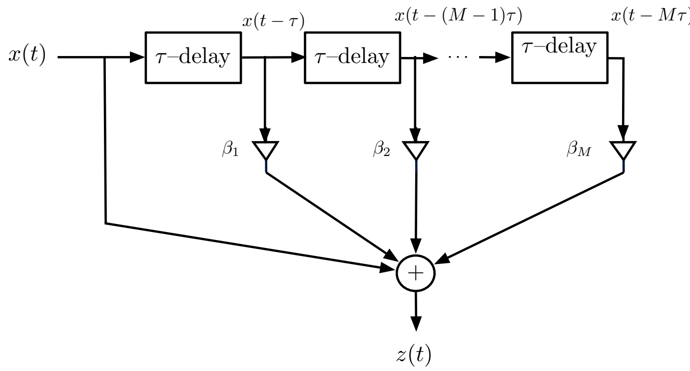


Figure 2.10: Problem 21: non-recursive model for echo system.

We have a finite representation of the output in terms of the inputs, attenuated and shifted in time. The system resembles that in (b), and as in there it can be shown to be LTI.

2.2 Problems using MATLAB

2.22 (a) The output voltage when the switch closes at $t = 0$ is

$$v_o(t) = -R(t)u(t) = -(1 + 0.5\cos(20\pi t))u(t)$$

The initial value of the voltage is $v(0) = -1.5$.
(b) If the switch closes at $t = 50$ msec, the output voltage is

$$\begin{aligned} v_{o1}(t) &= -R(t)u(t - 50 \times 10^{-3}) \\ &= \begin{cases} -(1 + 0.5\cos(20\pi t)) & t \geq 50 \times 10^{-3} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

with an initial value of $v_{o1}(50 \times 10^{-3}) = -0.5$.
(c) The initial values are different, and $v_{01}(t) \neq v_o(t - 50 \times 10^{-3})$ so the system is time varying.

The following script is used for the plotting in (a) and (b)

```
%% Pr 2_22
t=0:0.01:0.2;M=length(t);
v0=-(1+0.5*cos(20*pi*t));
t1=0:0.01:0.05;
N=length(t1);
v01=[zeros(1,N) v0(N+1:M)];
figure(1)
subplot(211)
plot(t,v0); grid; axis([0 max(t) -1.7 0.2]); ylabel('v_0(t)')
subplot(212)
plot(t,v01);  grid;axis([0 max(t) -1.7 0.2]); xlabel('t'); ylabel('v_{01}(t)')
```

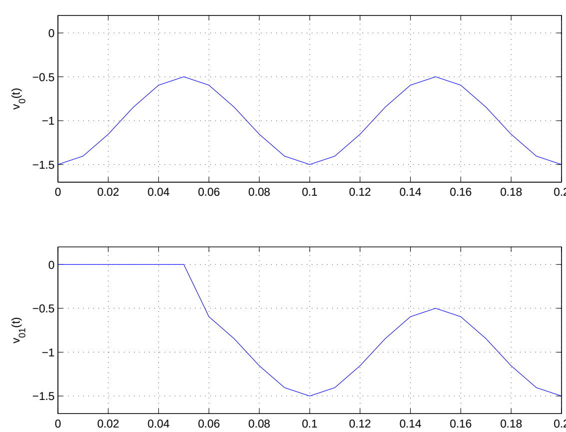


Figure 2.11: Problem 22

2.23 (a) The zener diode clips any signal that exceeds a set threshold, in this case 0.5, thus if we have a sinusoid with amplitude 0.3, less than the threshold, the output of the zerner diode would be the same as the input. When the amplitude of the input is 1 the output is a sinusoid clipped at 0.5, but when it is 0.3 is not a clipped sinuosoid. Thus, the system is non-linear as scaling does not hold.

(b) The zener diode would do the same thresholding to a signal $x(t)$ or to a shifted version of it $x(t - \tau)$, so the system is time invariant.

```
%% Pr 2_23
 t=0:0.001:4;
 vs=cos(pi*t); x=vs;
 % vs=0.3*cos(pi*t);x=vs;  % for a second input get rid of %
 N=length(t);
 for k=1:N,
     if abs(vs(k))>=0.5,
         if vs(k)<0,
             vs(k)=-0.5;
         else
             vs(k)=0.5;
         end
     end
 end
 y=vs;
 figure(2)
 plot(t,x); grid; hold on
 plot(t,y,'or'); axis([0 4 -1.1 1.1]); legend('x(t)','y(t)'); xlabel('t')
 hold off
```

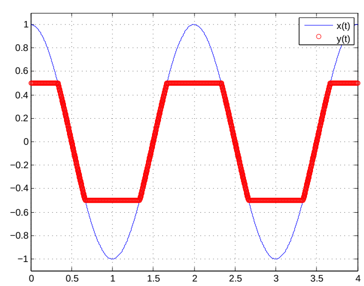


Figure 2.12: Problem 23: input $x(t)$ and output $y(t)$ of the zener diode.

2.24 (a) No, the diode is non-linear. For $v(t) > 0$ doubling the voltage does not double the current $i(t)$ because the voltage-current relation is not a 45^o line.

(b) To get the current to be close to zero when the voltage is negative we would need that $I_s \to 0$ because $e^{qv(t)/kT} - 1 \leq 0$. If the constant $q/kT \to \infty$ the current would be very large for small values of the voltage. The ideal diode is an approximation to the p-n diode with the voltage-current plot given by the script below. The ideal diode is non-linear, the voltage-current plot is not a line of slope 45^o. The following is the plot for the given values of I_s and the constant q/kT.

```
%% Pr. 2_24
     v=-0.1:0.01:1;
     Is=0.0001;kqT=0.026;
     i=Is*(exp(v/kqT)-1);
     figure(1)
     plot(v,i);grid
     axis([-0.1 1.2 -5 1.1*max(i)])
```

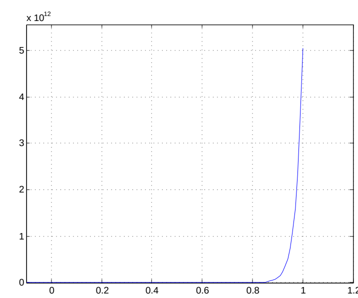


Figure 2.13: Problem 24: i-v response of the p-n diode.

(c) This is a half-wave rectifier used to obtain dc sources. The output is the positive part of the sinusoid and zero whenever the sinusoid is negative.

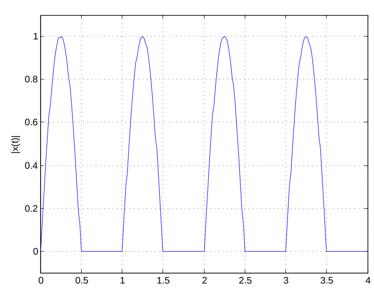


Figure 2.14: Problem 24: output of the half-wave rectifier.

2.25

```
% Pr 2_25
clear all; clf
Ts=0.01; delay=1; Tend=20;
t=0:Ts:Tend;
%x=cos(2*pi*t).*(ustep(t,0)-ustep(t,-20));
%x=sin(2*pi*t).*exp(-0.1*t).*(ustep(t,0)-ustep(t,-20));
x=ramp(t,1,0)+ramp(t,-2,-2)+ramp(t,1,-8);
h=exp(-t);
y=Ts*conv(x,h);
% plots
t1=0:Ts:length(y)*Ts-Ts;
figure(1)
subplot(311)
plot(t,x); axis([0 20 -5 3]);grid;ylabel('x(t)');
subplot(312)
plot(t,h); axis([0 20 -0.1 1]);grid;ylabel('h(t)');
subplot(313)
plot(t1,y);
grid
```

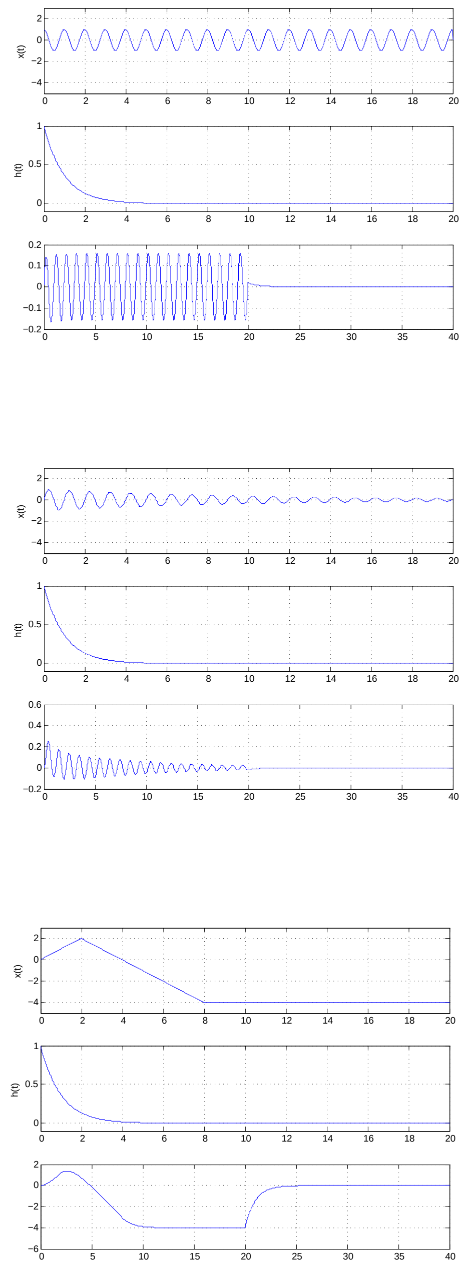


Figure 2.15: Problem 25: input, impulse response and output of convolution integral.

2.26 This averager is a LTI system

(a) The average signal for $T = 1$ and $x(t) = u(t)$ is

$$y(t) = \int_{t-1}^{t} u(\tau)d\tau = \begin{cases} 0 & t < 0 \\ \int_0^t d\tau = t & 0 \leq t < 1 \\ \int_{t-1}^t d\tau = 1 & t \geq 1 \end{cases}$$

Using the LTI of the system, the response to $x_1(t) = x(t) - x(t-1) = u(t) - u(t-1)$ is then

$$y_1(t) = y(t) - y(t-1) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 - (t-1) = 2 - t & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

which is a triangular signal.

(b) The implementation of the averager can be seen as the sliding, from left to right, of a rectangular window of width T and amplitude 1, and adding the values of the signal in that window to get the output. If we let $T = T_0$, the period of the sinusoid $x(t) = \cos(2\pi t/T_0)u(t)$ after zero, the output of the averager is zero for $t \leq 0$. There is then a transient for $0 \leq t \leq T_0$, and the output will be zero for $t \geq t_0$, as the average of the sinusoids is zero.

(c) The following script is used to implement this convolution in (a) and (b).

```
%% Pr. 2_26
     N=1000; Ts=1/N;
     tt=0:Ts:2;  x=cos(2*pi*tt);
     h=[ones(1,N) zeros(1,N+1)];
     y=conv(x,h)*Ts;t=0:Ts:(length(y)-1)*Ts;
     figure(5)
     subplot(211)
     plot(tt,x);hold on;plot(tt,h,'or'); legend('x(t)','h(t)');
     axis([0 2  -1.1 1.1]); grid
     subplot(212)
     plot(t(1:length(t)/2),y(1:length(t)/2));grid
     axis([0 2 1.1*min(y) 1.1*max(y)]); ylabel('x*h'); xlabel('t')
```

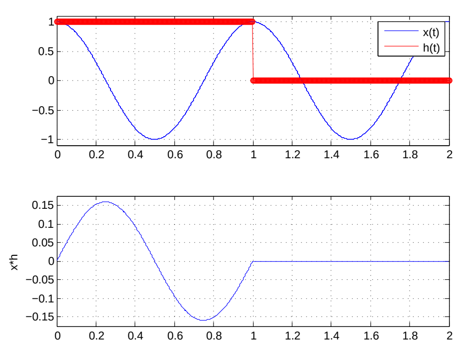


Figure 2.16: Problem 26:Sinusoid and impulse response (top), result of convolution.

2.27 (a) The system with the sinc impulse response is non-causal because $h(t) \neq 0$ for $t < 0$. As such an ideal low-pass filter cannot be used for real-time processing as it would require future inputs, not available in real-time. This can be seen by looking at the convolution with a causal signal $x(t)$:

$$
\begin{aligned}
y(t) &= \int_0^\infty x(\tau)h(t-\tau)d\tau \\
&= \int_0^t x(\tau)h(t-\tau)d\tau + \int_t^\infty x(\tau)h(t-\tau)d\tau
\end{aligned}
$$

where the upper limit of the top integral is due to $h(\tau)$ having an infinite support, and the lower limit because $x(t)$ is causal. The second bottom integral shows that the output depends on future values of the input.

(b) For the low-pass filter to be BIBO stable the impulse response $h(t)$ must be absolutely integrable, i.e.,

$$
\int_{-\infty}^{\infty} |\sin(t)/t| dt < \infty
$$

To approximtely compute this integral we use the following script. The integral is bounded, so the filter is BIBO stable.

```
%% Pr 2_27
clear all; clf
syms x t x1
x=abs(sinc(t/pi));
for k=1:30,
x1=int(x,t,0,k);
xx(k)=subs(2*x1);
end
n=1:30;
figure(1)
subplot(211)
ezplot(x,[-30,30]); axis([-30 30 -0.1 1.2]);grid
subplot(212)
stem(n,xx); axis([1 30 0 8]);grid
```

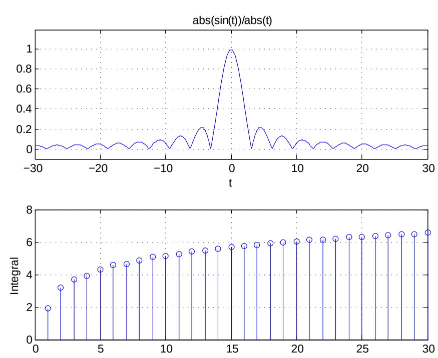


Figure 2.17: Problem 27: absolute value of sinc function and its integral values for $t = n$ sec.

2.28
```
%% Pr. 2_28
clear all; clf
% parts (a)--(c)
     Ts=0.01;
     t=0:0.01:100; N=length(t)
     p=[20*ones(1,4000) -10*ones(1,2000) zeros(1,N-6000)];
     P=1.1*abs(min(p))
     x=(p+P).*cos(2*pi*t);
     figure(6)
       subplot(221)
       plot(t,p); ylabel('p(t)')
       subplot(222)
       plot(t,x); ylabel('x(t)')
     y=abs(x);
     subplot(223)
     plot(t,y); ylabel('y(t)')
     h=exp(-0.8*t);
     z=conv(h,y)*Ts;  z=z*20/15;
     subplot(224)
     plot(t,z(1:length(t))-P); ylabel('z(t)'); xlabel('t')
% part (d)
     Ts=0.01;
     t=0:0.01:100;N=length(t)
     p=2*cos(0.2*pi*t);
     P=abs(min(p))
     x=(p+P).*cos(10*pi*t);
     figure(7)
       subplot(221)
       plot(t,p+P); ylabel('p(t)+P(t)')
       subplot(222)
       plot(t,x); ylabel('x(t)')
     y=abs(x);
       subplot(223)
       plot(t,y); ylabel('y(t)'); xlabel('t')
       h=exp(-0.8*t);
     z=conv(h,y)*Ts;z=z-P+0.4;z=z*2/max(z);
       subplot(224)
       plot(t,z(1:length(t)));grid; ylabel('z(t)'); xlabel('t')
```

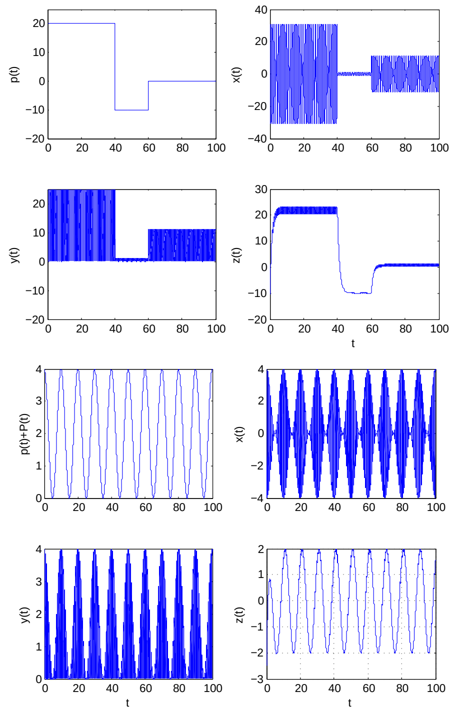


Figure 2.18: Problem 28: modulation and envelope detection for pulses and for sinusoid.

2.29

```
%% Pr. 2_29
clear all; clf
% part (a)
     Ts=0.01;
     t=0:Ts:10;
     omega_c=2*pi; nu=10;
     m=cos(t);      %% message
     im=sin(t);     %% integral of message
     y=cos(omega_c*t +2*pi*nu*im);       %% FM1 with nu=10
     y1=cos(omega_c*t +2*pi*nu*im/10);   %% FM2 with nu/10
     figure(9)
       subplot(311)
       plot(t,m); ylabel('m(t)')
       subplot(312)
       plot(t,y); ylabel('y(t)')
       subplot(313)
       plot(t,y1); ylabel('y_1(t)')
% part(b)
     N=length(t)
     for k=1:N,
         if m(k)>=0,
             m1(k)=1;                            %% message
             y(k)=cos(omega_c*k*Ts +2*pi*nu*k*Ts/10); %% FM
         else
             m1(k)=-1;
         y(k)=cos(omega_c*k*Ts -2*pi*nu*k*Ts/10);

         end
     end
     figure(10)
       subplot(211)
       plot(t,m1); ; ylabel('m_1(t)')
       axis([0 10 1.1*min(m1) 1.1*max(m1)])
       subplot(212)
       plot(t,y);  ylabel('y(t)')
       axis([0 10 1.1*min(y) 1.1*max(y)])
```

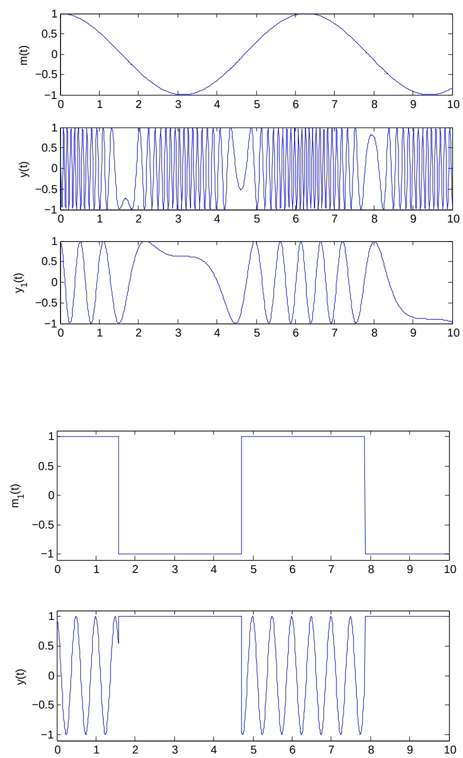


Figure 2.19: Problem 29: message signal $m(t)$ and FM signals for $\nu = 10$ and 1 (top); message $m_1(t)$ and FM signal for $\nu = 1$ (bottom)