Solutions to Chapter 1

1. The intent of this rather vague problem is to get you to compare the two notions, probability as intuition and relative frequency theory. There are many possible answers to how to make the statement "Ralph is probably guilty of theft" have a numerical value in the relative frequency theory. First step is to define a repeatable experiment along with its outcomes. The favorable outcome in this case would be 'guilty.' Repeating this experiment a large number of times would then give the desired probability in a relative frequency sense. We thus see that it may entail a lot of work to attach an objective numerical value to such a subjective statement, if in fact it can be done at all.

   One possible approach would be to look through courthouse statistics for cases similar to Ralph's, similar both in terms of the case itself and the defendant. If we found a sufficiently large number of these cases, ten at least, we could then form the probability $p = n_F/n$, where n_F is the number of favorable (guilty) verdicts, and n is the total number of found cases. Here we effectively assume that the judge and jury are omniscient.

   Another possibility is to find a large number of people with personalities and backgrounds similar to Ralph's, and to expose them to a very similar situation in which theft is possible. The fraction of these people that then steal in relation to the total number of people, would then give an objective meaning to the phrase "Ralph is probably guilty of theft."

2. Note that $D \to 3$, but $3 \not\to D$, i.e., D implies 3 but not the other way around. Thus if we turn over card 2 and find a 3. So what? It was <u>never stated that</u> a $3 \to F$. Likewise, with card 3. On the other hand, if we turn over card 4 and find a D, then the rule is violated. Hence, we must turn over card 4 and card 1, of course.

3. First step here is to decide which kind of probability to use. Since no probabilities are explicitly given, it is reasonable to assume that all numbers are equally likely. Effectively we assume that the wheel is "fair." This then allows us to use the classical theory along with the axiomatic theory to solve this problem. Now we must find the corresponding probability model. We are told in the problem statement that the experiment is "spinning the wheel." We identify the pointed-to numbers as the outcomes ς. The sample space is thus $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.The total number of outcomes is then 9. The probability of each elemental event $\{i\}$ is then taken as $P[\{i\}] \triangleq p = 1/9$, as in the classical theory. We are also told in the problem statement that the contestant wins if an even number shows. The set of even numbers in Ω is $\{2, 4, 6, 8\}$. We can write this event as a disjoint union of four singleton (atomic) events

   $$\{2, 4, 6, 8\} = \{2\} \cup \{4\} \cup \{6\} \cup \{8\}.$$

   Now we can apply axiom 3 of probability to write

   $$\begin{aligned} P[\{2,4,6,8\}] &= P[\{2\}] + P[\{4\}] + P[\{6\}] + P[\{8\}] \\ &= \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} \\ &= \frac{4}{9}. \end{aligned}$$

   We have seen that some 'reasonable' assumptions are necessary to transform the given word problem into something that exactly corresponds to a probability model. It turns out that this is a general problem for such word problems, i.e. problems given in natural English.

4. The experiment involves flipping a fair coin 3 times. The outcome of each coin toss is either a head or a tail. Therefore, the sample space of the combined experiment that contains all the possible outcomes of the 3 tosses, is given by

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Since all the coins are fair, all the outcomes of the experiment are equally likely. The probability of each singleton event, i.e. an event with a single outcome, is then $\frac{1}{8}$. We are interested in finding the probability of the event A, which is the event of obtaining 2 heads and 1 tail. There are 3 favorable outcomes for this event given by $A = \{HHT, HTH, THH\}$. Therefore, $P[A] = P[\{HHT\} \cup \{HTH\} \cup \{THH\}] = P[HHT] + P[HTH] + P[THH] = \frac{3}{8}$. Note that we are able to write the probability of the event A as the sum of probability of the singleton events (from Axiom 3) because the singlteon events of any experiment are mutually exclusive. Why?

5. The experiment contains drawing two balls (with replacement) from an urn containing balls numbered 1, 2, and 3. The sample space of the experiment is given by

$$\Omega = \{11, 12, 13, 21, 22, 23, 31, 32, 33\}.$$

The event of drawing a ball twice is said to occur when one of the outcomes 11, 22, or 33 occurs. Therefore, the event of drawing 2 equal balls E, is given by $E = \{11, 22, 33\}$ and $P[E] = P[\{11\}] + P[\{22\}] + P[\{33\}]$. Since the balls are drawn at random, it can assumed that drawing each ball is equally likely. Therefore, the singleton events, or equivalently outcomes of the experiment, are equally likely. Hence, $P[E] = 3(\frac{1}{9}) = \frac{1}{3}$.

6. Let $b_1, b_2, \ldots, b_6$ represent the six balls. Each outcome will be represented by the two balls that were drawn. In the first experiment, the balls are drawn without replacement; hence, the two balls drawn cannot have the same index. Then the sample space containing all the outcomes is given by

$$\begin{aligned}
\Omega_1 &= \{b_1b_2, b_1b_3, b_1b_4, b_1b_5, b_1b_6, \\
&\quad b_2b_1, b_2b_3, b_2b_4, b_2b_5, b_2b_6, \\
&\quad b_3b_1, b_3b_2, b_3b_4, b_3b_5, b_3b_6, \\
&\quad b_4b_1, b_4b_2, b_4b_3, b_4b_5, b_4b_6, \\
&\quad b_5b_1, b_5b_2, b_5b_3, b_5b_4, b_5b_6, \\
&\quad b_6b_1, b_6b_2, b_6b_3, b_6b_4, b_6b_5\}.
\end{aligned}$$

This can be written compactly as

$$\Omega_1 = \{b_{(i,j)} | 1 \leq i \leq 6, 1 \leq j \leq 6, i \neq j\}.$$

If the first ball is replaced before the second draw, then in addition to the outcomes in the earlier part, there are outcomes where both the two balls drawn are the same. The sample space for the new experiment is given by

$$\Omega_2 = \Omega_1 \cup \{b_1b_1, b_2b_2, b_3b_3, b_4b_4, b_5b_5, b_6b_6\}.$$

This can also be written as $\Omega_2 = \{b_{(i,j)} | 1 \leq i \leq 6, 1 \leq j \leq 6\}$.

7. Let h_M be the height of the man and h_W be the height of the woman. Each outcome of the experiment can be expressed as a two-tuple (h_M, h_W). Thus

   (a) The sample space Ω is the set of all possible pairs of heights for the man and woman. This is given as
   $$\Omega = \{(h_M, h_W) : h_M > 0, h_W > 0\}.$$

   (b) The event E, which is a subset of Ω is given by
   $$E = \{(h_M, h_W) : h_M > 0, h_W > 0, h_M < h_W\}.$$

8. The word problem describes the physical experiment of drawing numbered balls from an urn. We need to find a corresponding mathematical model. First we form an appropriate event space with meaningful outcomes. Here the physical experiment is 'draw ball from urn,' so the outcome in words is 'particular labeled ball drawn,' which we can identify with its label. So we select as *outcome* in our mathematical model, the number on the drawn ball's face, i.e. the particular label. The outcomes are thus the integers 1,2,3,4,5,6,7,8, 9, and 10. The *sample space* is then $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and is the set of all ten outcomes. We are told that E is 'the event of drawing a ball numbered no greater than 5.' Thus we define in our event field $E = \{1, 2, 3, 4, 5\}$. The other event specified in the word problem is F 'the event of drawing a ball greater than 3 but less than 9.' In our mathematical event field this corresponds to $F = \{4, 5, 6, 7, 8\}$. Having constructed our sample space with indicated events, we can use elementary set theory to determine the following answers:
$$\begin{array}{rr} E^c = \{6,7,8,9,10\}, & F^c = \{1,2,3,9,10\}, \\ EF = \{4,5\}, & E \cup F = \{1,2,3,4,5,6,7,8\}, \\ EF^c = \{1,2,3\}, & E^cF = \{6,7,8\}, \\ E^c \cup F^c = \{1,2,3,6,7,8,9,10\}, & \\ (EF^c) \cup (E^cF) = \{1,2,3,6,7,8\}, & (EF) \cup (E^cF^c) = \{4,5,9,10\} \\ (E \cup F)^c = \{9,10\}, & (EF)^c = \{1,2,3,6,7,8,9,10\}. \end{array}$$
   The last part of the problem asks us to 'express these events in words.' Since we have a mathematical model, we should really more precisely ask what each of these events *corresponds to in words*. We know of course that E corresponds to 'drawing a ball numbered no greater than 5.' We can thus loosely write $E = \{$'drawing a ball numbered no greater than 5'$\}$, although in our mathematical model E is just the set of integers $\{1, 2, 3, 4, 5\}$. So when we write $E =\{$ 'drawing a ball numbered no greater than 5'$\}$, what we really mean is that the event E in our mathematical model corresponds to the physical event 'drawing a ball numbered no greater than 5' mentioned in the word problem. With this caveat in mind, we can then write:
$$\begin{aligned} E^c &= \{\text{'drawing a ball greater than 5'}\}, \\ F^c &= \{\text{'drawing a ball not in the range 4-8 inclusive'}\}, \\ EF &= \{\text{'drawing a ball greater than 3 and no greater than 5'}\}, \\ & \quad \text{etc.} \end{aligned}$$

9. The sample space containing four equally likely outcomes is given by $\Omega = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$. Two events $A = \{\zeta_1, \zeta_2\}$ and $B = \{\zeta_2, \zeta_3\}$ are given. The required events can be easily obtained by observation.

AB^c = set of outcomes in A and not in $B = \{\zeta_1\}$.
BA^c = set of outcomes in B and not in $A = \{\zeta_3\}$.
AB = set of outcomes in A and $B = \{\zeta_2\}$.
$A \cup B$ = set of outcomes in A or in $B = \{\zeta_1, \zeta_2, \zeta_3\}$.

10. $A = AB \cup AB^c$. This can be proved using the distributive law on

$$A = A\Omega = A(B \cup B^c) = AB \cup AB^c.$$

$A \cup B = (AB^c) \cup (BA^c) \cup (AB)$. Here we first write $A = A(B \cup B^c)$ and $B = B(A \cup A^c)$. Then we can write

$$\begin{aligned} A \cup B &= (A(B \cup B^c)) \cup (B(A \cup A^c)) \\ &= (AB \cup AB^c) \cup (BA \cup BA^c) \\ &= AB \cup AB^c \cup BA \cup BA^c \\ &= AB \cup AB^c \cup BA^c, \end{aligned}$$

using the above laws and formulas. Notice that the above two decompositions are into disjoint sets. From the third axiom of probability, we know that the probability of union of disjoint sets is the sum of the probabilities of the disjoint sets. Therefore, we can add the probabilities over the unions.

11. In a given random experiment there are four *equally likely* outcomes $\zeta_1, \zeta_2, \zeta_3$, and ζ_4. Let the event $A \triangleq \{\zeta_1, \zeta_2\}$.
$P[A] = P[\{\zeta_1, \zeta_2\}] = P[\{\zeta_1\}] + P[\{\zeta_2\}] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. $A^c = \{\zeta_3, \zeta_4\}$,
$P[A^c] = P[\{\zeta_3, \zeta_4\}] = P[\{\zeta_3\}] + P[\{\zeta_4\}] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.
Note that we are told that the four outcomes are equally likely. This means that the four singleton (atomic) events have equal probability. $P[A] = \frac{1}{2} = 1 - P[A^c] = 1 - \frac{1}{2}$.

12. (a) The three axioms of probability are given below

(a) [label=()]

(b) For any event A, the probability of the even occuring is always non-negative.

$$P[A] \geq 0.$$

This ensures that probability is never negative.

(c) The probability of occurence of the sample space event Ω is one.

$$P[\Omega] = 1.$$

This ensures that probability of no event exceeds one. The first two axioms ensures that the probability is a quantity between 0 and 1, inclusive.

(d) For any two events A, B that are disjoint, the probability of the union of the events is the sum of the probabilities of the two events.

$$P[A \cup B] = P[A] + P[B], \text{ when } AB = \phi.$$

This axiom tells us that the probability of any event can be obtained by the sum disjoint events that constitute the event.

(b) The event $A \cup B$ can be obtained as the disjoint union of the three sets AB, AB^c, A^cB. Hence by applying the third axiom of probability, we obtain

$$\begin{aligned} P[A \cup B] &= P[AB \cup (AB^c \cup A^cB)] \\ &= P[AB] + P[AB^c \cup A^cB] \\ &= P[AB] + P[AB^c] + P[A^cB]. \end{aligned}$$

Now the event A can be written as the disjoint union of AB and AB^c (Axiom 3). Therefore

$$P[A] = P[AB] + P[AB^c] \implies P[AB^c] = P[A] - P[AB]$$

Similarly

$$P[B] = P[AB] + P[A^cB] \implies P[A^cB] = P[B] - P[AB].$$

Therefore $P[A \cup B] = P[AB] + (P[A] - P[AB]) + (P[B] - P[AB]) = P[A] + P[B] - P[AB]$.

13. We first form our mathematical model by setting outcomes $\boldsymbol{\varsigma} = (\varsigma_1, \varsigma_2)$, where ς_1 corresponds to the label on the first ball drawn, and ς_2 corresponds to the label on the second ball drawn. We can also write the outcomes as strings $\boldsymbol{\varsigma} = \varsigma_1\varsigma_2$. The sample space Ω can then be identified with the 2-D array

$$\begin{array}{ccccc} 11 & 12 & 13 & 14 & 15 \\ 21 & 22 & 23 & 24 & 25 \\ 31 & 32 & 33 & 34 & 35 \\ 41 & 42 & 43 & 44 & 45 \\ 51 & 52 & 53 & 54 & 55 \end{array}.$$

There are thus 25 outcomes in the sample space. Now the word problem statement uses the phrase 'at random' to describe the drawing. This is a technical term that can be read 'equally likely.' Thus all the elementary events $\{\varsigma_1\varsigma_2\}$ in our mathematical model must have equal probability, i.e. $P[\{\varsigma_1\varsigma_2\}] = 1/25$. Armed thusly we can attack the given problem as follows. Define the event $E =$\{'sum of labels equals five'\}, or precisely $E = \{41, 32, 23, 14\}$. Then we decompose this event into four singleton events as

$$E = \{41\} \cup \{32\} \cup \{23\} \cup \{14\}.$$

Since different singleton events are disjoint, probability adds, and we have

$$\begin{aligned} P[E] &= \frac{1}{25} + \frac{1}{25} + \frac{1}{25} + \frac{1}{25} \\ &= \frac{4}{25}. \end{aligned}$$

"Dim" ignored that outcome ij is different (distinguishable) from outcome ji. "Dense" talked about the sums and correctly noted that there were nine of them. However, he incorrectly assumed that each sum was equally likely. Looking at our sample space above, we can see that the sum 2 has only one favorable outcome 11, while the sum 6 has five favorable outcomes, just looking at the anti-diagonals of this matrix.

14. First we show $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.
Let $x \in A \cap (B \cup C)$.
Then $x \in A$ and $x \in (B \cup C)$.
$x \in A$ and $x \in B$ or $x \in C$.

Say if $x \in B$. Then $x \in A$ and $x \in B$ (Step k)
Thus $x \in (A \cap B)$.
And therefore $x \in (A \cap B) \cup (A \cap C)$. Similar arguments can be made if we consider $x \in C$ in step k, in which case we will show that $x \in (A \cap C)$ and hence $x \in (A \cap B) \cup (A \cap C)$.
Thus we have shown that $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$.

Now we show that $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.
Suppose $x \in (A \cap B) \cup (A \cap C)$. Then $x \in (A \cap B)$ or $x \in (A \cap C)$.
Say $x \in (A \cap B)$
Then $x \in A$ and $x \in B$.
Or $x \in A$ and $x \in (B \cup C)$. Or in other words, $x \in A \cap (B \cup C)$.

Similar arguments can be used to show that if $x \in (A \cap C)$, then $x \in A \cap (B \cup C)$.
Thus $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$.

Thus we have shown that both sets are contained in each other. Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

15. We use the set identity $\Omega = A \cup A^c$. Since this union is disjoint, by the additivity of probability (i.e. axiom 3), we get $1 = P[\Omega] = P[A] + P[A^c]$, which with rearranging becomes the desired result.

16. (a) $A \cap C = \{1,2\} \cap \{4,5,6\} = \phi$. Therefore,

$$\begin{aligned}
P[A \cap C] &= P[\phi] \\
&= 1 - P[\Omega] \\
&\quad (\because \Omega \cap \Phi = \phi, 1 = P[\Omega \cup \phi] = P[\Omega] + P[\phi]) \\
&= 1 - 1 \text{ (because } P[\Omega] = 1) \\
&= 0.
\end{aligned}$$

(b) $P[A \cup B \cup C] = P[\{1,2\} \cup \{2,3\} \cup \{4,5,6\} = P[\{1,2,3,4,5,6\}] = P[\Omega] = 1$.

(c) We see that $B \cap C = \phi$ and so $P[BC] = 0$. For B and C to be independent, $P[BC] = P[B]P[C]$. Therefore, if either $P[B] = 0$ or $P[C] = 0$ or both are zeros, B and C will be independent.

17. This problem uses only set theory and just two axioms of probability to get these general results.

(a) We need to show $P[\phi] = 0$. We write the disjoint decomposition $\Omega = \Omega \cup \phi$ and then use the additivity of probability (axiom 3) to get

$$\begin{aligned}
P[\Omega] &= P[\Omega \cup \phi] \\
&= P[\Omega] + P[\phi].
\end{aligned}$$

So we must have $P[\phi] = 0$.

(b) Using set theory, we can write the disjoint decomposition

$$E = EF^c \cup EF.$$

Then by axiom 3, the additivity of probability, we have

$$\begin{aligned}
P[E] &= P[EF^c \cup EF] \\
&= P[EF^c] + P[EF],
\end{aligned}$$

or what is the same $P[EF^c] = P[E] - P[EF]$.

(c) Here we simply note $E \cup E^c = \Omega$ is a disjoint decomposition, so that again by axiom 3,

$$\begin{aligned} P[\Omega] &= P[E] + P[E^c] \\ &= 1, \quad \text{by axiom 2,} \end{aligned}$$

which is the same as $P[E] = 1 - P[E^c]$.

18. The outcome is the result of a probabilistic experiment. An event is a collection (set) of outcomes. The field of events is the complete collection of events that are relevant for the given probability problem.

19. We start with the mutually exclusive decomposition

$$A \cup B = AB^c \cup AB \cup A^cB,$$

yielding $P[A \cup B] = P[AB^c] + P[AB] + P[A^cB]$. Then consider the two simple disjoint decompositions

$$A = AB^c \cup AB \quad \text{and} \quad B = A^cB \cup AB,$$

which yield $P[A] = P[AB^c] + P[AB]$ and $P[B] = P[A^cB] + P[AB]$. Putting them all together, we have

$$\begin{aligned} P[A \cup B] &= P[AB^c] + P[AB] + P[A^cB] \\ &= (P[A] - P[AB]) + P[AB] + (P[B] - P[AB]) \\ &= P[A] + P[B] - P[AB]. \end{aligned}$$

20. From Eq. 1.4-3, we see that $E \oplus F = (E - F) \cup (F - E) = EF^c \cup E^cF$. We see that EF^c and E^cF are disjoint, i.e., $(EF^c) \cap (E^cF) = \phi$. Therefore, the probability of the union of EF^c and E^cF are the sum of the probabilities of the two events. In other words,

$$P(E \oplus F) = P(EF^c \cup E^cF) = P(EF^c) + P(E^cF).$$

21. We have already (Problem 17) seen that we can write $P[EF^c] = P[E] - P[EF]$ and $P[E^cF] = P[F] - P[EF]$. Therefore, $P(E \oplus F) = P(EF^c) + P(E^cF) = P[E] + P[F] - 2P[EF]$.

22. (a) For simplicity associate as follows: cat=1, dog=2, goat=3, and pig=4. The outcomes ξ then become the integers 1,2,3, and 4. The sample space $\Omega = \{1, 2, 3, 4\}$. For probability information we are given:

$$P[\{1, 2\}] = 0.9, P[\{3, 4\}] = 0.1, P[\{4\}] = 0.05, \text{ and } P[\{2\}] = 0.5.$$

Now for every event in our field of events, we must be able to specify the probability. This is equivalent to being able to supply the probability for all the singleton events. To see if we can do this, we note that singleton events $\{1\}$ and $\{3\}$ are missing probabilities, so we first write

$$\begin{aligned} \{1\} &= \{1, 2\} - \{2\}, \text{ so that} \\ P[\{1\}] &= P[\{1, 2\}] - P[\{2\}] = 0.9 - 0.5 = 0.4. \end{aligned}$$

Doing the same for the other missing singleton probability $P[\{3\}]$, we write

$$\begin{aligned}\{3\} &= \{3,4\} - \{4\}, \text{ so that} \\ P[\{3\}] &= P[\{3,4\}] - P[\{4\}] = 0.1 - 0.05 = 0.05.\end{aligned}$$

Thus we have enough probability information for all the singleton events, and hence all $16 = 2^4$ subsets of $\Omega = \{1,2,3,4\}$. The appropriate field $\mathcal{F}$ of events then consists of the following events along with their probabilities:

$$\begin{array}{cc}
\{1\}, & P[\{1\}] = 0.4, \\
\{2\}, & P[\{2\}] = 0.5, \\
\{3\}, & P[\{3\}] = 0.05, \\
\{4\}, & P[\{4\}] = 0.05, \\
\{1,2\}, & P[\{1,2\}] = 0.9, \\
\{1,3\}, & P[\{1,3\}] = 0.45, \\
\{1,4\}, & P[\{1,4\}] = 0.45, \\
\{2,3\}, & P[\{2,3\}] = 0.55, \\
\{2,4\}, & P[\{2,4\}] = 0.55, \\
\{3,4\}, & P[\{3,4\}] = 0.1, \\
\{1,2,3\}, & P[\{1,2,3\}] = 0.95, \\
\{1,2,4\}, & P\{1,2,4\}] = 0.95, \\
\{1,3,4\}, & P[\{1,3,4\}] = 0.5, \\
\{2,3,4\}, & P[\{2,3,4\}] = 0.6, \\
\{1,2,3,4\}(=\Omega), & P[\{1,2,3,4\}] = 1 = P[\Omega], \\
\phi, & P[\phi] = 0.
\end{array}$$

(b) Now the above is not an appropriate field of events if some of the events do not have known probabilities. So if $P[\text{'pig'} = \{4\}] = 0.05$ is removed, then we cannot determine the probabilities of some of the above events. In particular we cannot find $P[\{3\}]$. The alternative then is to treat $\{3,4\}$, whose probability is still given, as a singleton and form a smaller field with just the 8 events formed by unions of $\{1\}$, $\{2\}$, and $\{3,4\}$. The resulting field, along with its probabilities is as follows:

$$\begin{array}{cc}
\{1\}, & P[\{1\}] = 0.4, \\
\{2\}, & P[\{2\}] = 0.5, \\
\{1,2\}, & P[\{1,2\}] = 0.9, \\
\{3,4\}, & P[\{3,4\}] = 0.1, \\
\{1,3,4\}, & P[\{1,3,4\}] = 0.5, \\
\{2,3,4\}, & P[\{2,3,4\}] = 0.6, \\
\{1,2,3,4\}(=\Omega), & P[\{1,2,3,4\}] = 1 = P[\Omega], \\
\phi, & P[\phi] = 0.
\end{array}$$

23. First we show that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$.
Suppose $x \in A \cap (B \cup C)$
Then $x \in A$
Therefore $x \in (A \cup B)$, and $x \in (A \cup C)$
Hence, $x \in (A \cup B) \cap (A \cup C)$.

Now we show that $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$.
Suppose $x \in (A \cup B) \cap (A \cup C)$

Then $x \in (A \cup B)$ and $x \in (A \cup C)$
$x \in AorB$ and $x \in AorC$
If $x \in A$, then $x \in A \cup (B \cap C)$ (because $A \subset (A \cup (B \cap C))$)
If $x \in A$, then $x \in B$ and $x \in C$.
Or in other words, $x \in (B \cap C)$
$x \in A \cup (B \cap C)$.

Thus we have shown that both the sets are contained in each other. Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

24. The probability of A is $P[A] = P[\{\zeta_1, \zeta_2\}] = P[\{\zeta_1\}] + P[\{\zeta_2\}] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. The event (set) A^c in terms of the outcomes is $A^c = \{\zeta_3, \zeta_4\}$. The probabilty of A^c is $P[A^c] = P[\{\zeta_3, \zeta_4\}] = P[\{\zeta_3\}] + P[\{\zeta_4\}] = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$. Note that we are told that the four outcomes are equally likely. This means that the four singleton (atomic) events have equal probability. We verify $P[A] = \frac{1}{2} = 1 - P[A^c] = 1 - \frac{1}{2}$.

25. The composition of the urn is: (a), (a), (b), (b), (ab), (ab), (ab), (ab). $P[A] = 6/8$, $P[B] = 6/8$, $P[AB] = n_{ab}/n_T = 4/8$ is not equal to $P[A]P[B] = 9/16$. Therefore A and B are not independent.

26. Let $n_i, i = 1, 2$ represent the outcome of the ith toss. Since the tosses are independent:

$$P[n_1, n_2] = P[n_1]P[n_2] = \frac{1}{6} \cdot \frac{1}{6}$$

$$\begin{aligned} P[n_1 + n_2 = 7 | n_1 = 3] &= P[n_2 = 4 | n_1 = 3] \\ &= \frac{P[n_1 = 3, n_2 = 4]}{P[n_1 = 3]} \\ &= \frac{P[n_1 = 3]P[n_2 = 4]}{P[n_1 = 3]} \\ &\quad \text{(because tosses are independent)} \\ &= \frac{1}{6} \end{aligned}$$

27. Clearly

$$P[A] = \frac{4}{52} \quad \text{and} \quad P[B] = \frac{26}{52} = \frac{1}{2}.$$

Then $P[AB] = P[\{\text{pick one of two red aces in 52 cards}\}] = \frac{2}{52}$. Is $P[AB] = P[A]P[B]$? Now

$$\begin{aligned} P[AB] = \frac{2}{52} &= \frac{4}{52}\frac{1}{2} \\ &= P[A]P[B], \end{aligned}$$

so, yes A and B are independent events.

28. Since it is a fair die, the successive tosses are independent with probability $p = 1/6$ for each face. From the provided information, we equivalently want the probability of getting a total of 5 on the two remaining tosses. This can happen in just 4 equally likely outcomes, i.e. (4,1), (3,2), (2,3), and (1,4). The desired probability this then $4/36 = 1/9$.

29. We can look at the compound outcomes $\boldsymbol{\varsigma} = (\varsigma_1, \varsigma_2)$ as corresponding to the locations in the 9×9 array

$$\begin{array}{cccccccc}
11 & 21 & 31 & 41 & 51 & 61 & \cdots & 91 \\
12 & 22 & 32 & 42 & 52 & \cdots & & \vdots \\
13 & 23 & 33 & 43 & \cdots & & & \\
14 & 24 & 34 & \cdots & & & & \\
15 & 25 & \cdots & & \ddots & & & \\
16 & \cdots & & & & & & \\
\vdots & & & & & & & \\
 & & & & & & \ddots & \vdots \\
19 & \cdots & & & & & \cdots & 99
\end{array},$$

with 81 equally likely outcomes. We agree to call the sample space for the first experiment Ω_1, the sample space for the second experiment Ω_2, and the compound sample space simply Ω. To get the sum $\Sigma \triangleq N_1 + N_2 = 7$, we need one of the following outcomes

$$16, 25, 34, 43, 52, 61, \quad \text{located on a } 45^\circ \text{ diagonal in the above table.}$$

So there are 6 favorable outcomes for the event $\{\Sigma = 7\}$. The event $\{\Sigma = \text{odd}\}$ contains 40 outcomes and the event $\{\Sigma = \text{even}\}$ contains the remaining $81 - 40 = 41$ even-sum outcomes. Now the joint event $\{\Sigma = 7\} \cap \{\Sigma = \text{odd}\} = \{\Sigma = 7\}$ since the sum 7 is an odd number. We can now calculate the needed probabilities

$$P[\{\Sigma = \text{odd}\}] = \frac{40}{81} \quad \text{and} \quad P[\{\Sigma = 7\}] = \frac{6}{81}.$$

The answer for the first question is then

$$\begin{aligned}
P[\{\Sigma &= 7\}|\{\Sigma = \text{odd}\}] = \frac{P[\{\Sigma = 7\} \cap \{\Sigma = \text{odd}\}]}{P[\{\Sigma = \text{odd}\}]}, \quad \text{(by definition)} \\
&= P[\{\Sigma = 7\}]/P[\{\Sigma = \text{odd}\}], \quad \text{(by above result)} \\
&= 6/40.
\end{aligned}$$

The next question is to find $P[(\{N_1 > 7\} \times \Omega_2) \cup (\Omega_1 \times \{N_2 > 7\})|\{\Sigma > 10\}]$. For simplicity of notation, let's agree to write the compound events $\{N_1 > 7\} \times \Omega_2$ and $\Omega_1 \times \{N_2 > 7\}$ as simply $\{N_1 > 7\}$ and $\{N_2 > 7\}$, respectively, for the rest of this calculation. So we must count the relevant number of outcomes from the above 9×9 array, where the various sums are found on 45° diagonals. For the event $\{\Sigma > 10\}$, we count 36 outcomes. For the joint event $(\{N_1 > 7\} \cup \{N_2 > 7\}) \cap \{\Sigma > 10\}$, we find it easier to consider the set of outcomes that make up the remainder of the event $\{\Sigma > 10\}$, i.e. the event $\{N_1 \leq 7\} \cap \{N_2 \leq 7\} \cap \{\Sigma > 10\}$ which is equal, in words, to the event '$N_1 \leq 7$ and $N_2 \leq 7$ and $\Sigma > 10$ '. We could call this the *complement with respect to* $\{\Sigma > 10\}$ of the event $(\{N_1 > 7\} \cup \{N_2 > 7\}) \cap \{\Sigma > 10\}$. Anyway, we find from the 9×9 array that the number of outcomes in $\{N_1 \leq 7\} \cap \{N_2 \leq 7\}) \cap \{\Sigma > 10\}$ is composed of the following 10 cases: $\sum = 11 = 6 + 5 = 5 + 6 = 7 + 4 = 4 + 7$ and $\sum = 12 = 5 + 7 = 7 + 5 = 6 + 6$ and $\sum = 13 = 6 + 7 = 7 + 6$ and $\sum = 14 = 7 + 7$. So we subtract these 10 outcomes from the 36 outcomes in the event $\{\Sigma > 10\}$ to obtain 26 outcomes in the compound event $(\{N_1 > 7\} \cup \{N_2 > 7\}) \cap \{\Sigma > 10\}$. The relevant probabilities are then

$$P[\{\Sigma > 10\}] = \frac{36}{81} \quad \text{and} \quad P[(\{N_1 > 7\} \cup \{N_2 > 7\}) \cap \{\Sigma > 10\}] = \frac{26}{81}.$$

The desired conditional probability is then

$$P[(\{N_1 > 7\} \cup \{N_2 > 7\})|\{\Sigma > 10\}] = \frac{26/81}{36/81} = \frac{26}{36} \approx 0.72.$$

Finally to compute $P[\{\Sigma = \text{odd}\}|\{N_1 > 8\}]$, we proceed as follows. For the combined experiment, we know there is only one possibility for $N_1 > 8$ and that is $N_1 = 9$, along with any value for N_2. Thus there are 9 outcomes in the compound event $\{N_1 > 8\}$[1], so that it's probability is 9/81. Now the joint event $\{\Sigma = \text{odd}\} \cap \{N_1 > 8\} = \{N_1 = 9\} \cap \{\Sigma = \text{odd}\} = \{(9,2),(9,4),(9,6),(9,8)\}$ with four outcomes. Thus since all outcomes are equally likely, we have

$$P[\{\Sigma = \text{odd}\} \cap \{N_1 > 8\}] = \frac{4}{81}.$$

The desired conditional probability is then

$$\begin{aligned} P[\{\Sigma &= \text{odd}\}|\{N_1 > 8\}] = \frac{P[\{\Sigma = \text{odd}\} \cap \{N_1 > 8\}]}{P[\{N_1 > 8\}]} \\ &= \frac{4/81}{9/81} = \frac{4}{9} \approx 0.44. \end{aligned}$$

30. We are given that $P[D] = 0.001$, where D is the event 'disease is present.'. Let T denote the event 'test is positive,' so that T^c is the event 'test is negative.' We are additionally given $P[T|D] = 1$ and $P[T|D^c] = 0.005$. We are asked to compute $P[D|T]$, i.e. the probability that 'disease is present given the test is positive.' We use Bayes' rule and Theorem as follows

$$\begin{aligned} P[D|T] &= \frac{P[DT]}{P[T]} \\ &= \frac{P[T|D]P[D]}{P[T|D]P[D] + P[T|D^c]P[D^c]} \\ &= \frac{1 \times 0.001}{1 \times 0.001 + 0.005 \times 0.999} \\ &= \frac{1}{1 + 4.995} \approx 0.167. \end{aligned}$$

Thus in only about 17% of the cases will a positive test result actually confirm that you suffer from the disease. The other 83% of the time you will be needlessly worried!

31. Let S_1 denote the set of occupations and let S_2 denote the set of interests and/or hobbies. Then

$$\begin{aligned} S_1 &= \{\text{'office manager', 'engineer', 'doctor', 'teacher', ...}\}, \\ S_2 &= \{\text{'nat. defense', 'books', 'music', 'cooking',...}\}. \end{aligned}$$

Let X denote Henrietta's occupation and Y her interests. Then

$$\begin{aligned} &P[X = \text{'office manager'}, Y = \text{'nat. defense'}] \\ &\qquad = P[X = \text{'office manager'}]P[Y = \text{'nat. defense'}|X = \text{'office m} \\ &\qquad \leq P[X = \text{'office manager'}], \end{aligned}$$

since $0 \leq P[Y = \text{'nat. defense'}|X = \text{'office manager'}] \leq 1$.

[1] Remember, we decided above to write simply $\{N_1 > k\}$ for the compound event $\{N_1 > k\} \times \Omega_2$. This since, in this problem, we only compute probabilities for events in the compound experiment.

32. Directly from the problem statement

$$\begin{aligned} P[X=3] &= 3 \cdot P[X=1], \\ P[X=2] &= 2 \cdot P[X=1]. \end{aligned}$$

But we also know $P[X=3]+P[X=2]+P[X=1]=1$ which is always true by axiom 2 $P[\Omega]=1$. Therefore $P[X=1]=1/6$, $P[X=2]=1/3$, and $P[X=3]=1/2$. Using Bayes' Theorem, we then compute

$$\begin{aligned} P[X=1|Y=1] &= \frac{P[Y=1|X=1]P[X=1]}{\sum_{i=1}^{3} P[Y=1|X=i]P[X=i]} \\ &= \frac{(1-\alpha)1/6}{(1-\alpha)\frac{1}{6}+\frac{\beta}{2}\frac{1}{3}+\frac{\gamma}{2}\frac{1}{2}} \\ &= \frac{1-\alpha}{1-\alpha+\beta+\frac{3}{2}\gamma}. \end{aligned}$$

33. Let

$$\begin{aligned} A &\triangleq \{\text{examinee knows}\}, \\ B &\triangleq \{\text{examinee guesses}\}, \text{ and} \\ C &\triangleq \{\text{getting right answer}\}\ . \end{aligned}$$

Then $P[A]=p, P[B]=1-p, P[C|A]=1$, and $P[C|B]=1/m$. So

$$\begin{aligned} P[A|C] &= \frac{P[C|A]P[A]}{P[C]} \\ &= \frac{1 \cdot p}{P[C|A]P[A]+P[C|B]P[B]} \\ &= \frac{p}{p+\frac{1}{m}(1-p)} \\ &= \frac{mp}{mp+(1-p)}. \end{aligned}$$

34. There are N contestants and only one most beautiful. Hence

$$P[\{\text{pick most beautiful}\}] = 1/N.$$

35. Let

$$\begin{aligned} \widetilde{A} &\triangleq \{\text{random drawn chip} \in A\}, \\ \widetilde{B} &\triangleq \{\text{random drawn chip} \in B\}, \text{ and} \\ \widetilde{C} &\triangleq \{\text{random drawn chip} \in C\}. \end{aligned}$$

Also, let $D \triangleq \{\text{random drawn chip is defective}\}$. Then

$$\begin{aligned} P[D] &= P[D|\widetilde{A}]P[\widetilde{A}]+P[D|\widetilde{B}]P[\widetilde{B}]+P[D|\widetilde{C}]P[\widetilde{C}] \\ &= 0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.40 \\ &= 0.0345. \end{aligned}$$

Hence

$$\begin{aligned} P[\widetilde{A}|D] &= \frac{P[D|\widetilde{A}]P[\widetilde{A}]}{P[D]} = \frac{0.05 \times 0.25}{0.0345} \doteq 0.363 \\ P[\widetilde{B}|D] &= \frac{P[D|\widetilde{B}]P[\widetilde{B}]}{P[D]} = \frac{0.04 \times 0.35}{0.0345} \doteq 0.406 \\ P[\widetilde{C}|D] &= \frac{P[D|\widetilde{C}]P[\widetilde{C}]}{P[D]} = \frac{0.02 \times 0.40}{0.0345} \doteq 0.232 \end{aligned}$$

36. From the example

$$P[C] \simeq \frac{k}{N} \log \frac{N}{k}$$

We set $x \triangleq k/N$ and construct the following table.

x	$P[C]$	x	$P[C]$
0.0	0.0	0.5	0.346
0.1	0.23	0.6	0.31
0.2	0.32	0.7	0.25
0.3	0.361	0.8	0.18
0.4	0.367	0.9	0.10

The peak is quite shallow, therefore the choice of k is not critical near the peak.

37. (a) If we associate the 103 villagers with $r = 103$ balls and the $n = 30$ tents with 30 cells, this becomes a classical occupancy problem.

(b) The result is given by Eq.1.8-6, which is repeated here as

$$\begin{aligned} \binom{n+r-1}{r} &= \binom{30+103-1}{103} \\ &= \frac{132!}{103!29!}. \end{aligned}$$

(c) The result is

$$\begin{aligned} \binom{r-1}{r-n} &= \binom{103-1}{103-30} \\ &= \frac{102!}{73!29!}. \end{aligned}$$

To obtain numerical evaluations of these factorial expressions, one might want to use Stirling's formulas:

$$n! \approx (2\pi)^{1/2} n^{n+1/2} e^{-n}.$$

38. The most natural set of outcomes here are the strings (or vectors) of length r, indicating where each ball has landed. There are n^r such strings. They are all equally likely. The number of favorable outcomes would be $r!$ since there are r choices for the first preselected location, $r-1$ choices for the second location, etc. The desired probability is then $P = r!/n^r$. Now, since the balls are indistinguishable, we could have considered the so-called distinguishable outcomes, $\binom{n+r-1}{r}$ in number, however from the description of the experiment in the problem statement, they would not all be equally likely. So we could not rely on classical theory then to give us the probabilities of these outcomes.

39. As in problem 1.38, the number of favorable ways is $r!$. However, the total number of ways is *not* n^r since cells can at most hold one ball. For the first ball, there are n cells; for the second ball, $n-1$ cells, etc. Thus

$$\begin{aligned} N_T &= n(n-1)\cdots(n-r+1) \\ &= \frac{n!}{(n-r)!}. \end{aligned}$$

Thus

$$\begin{aligned} P &= \frac{r!}{\left(\frac{n!}{(n-r)!}\right)} \\ &= \frac{r!(n-r)!}{n!} \\ &= \binom{n}{r}^{-1}. \end{aligned}$$

40. (a) Let the tribal leaders be the cells and the rifles be the balls. Then the three tribal leaders collecting the five rifles is the analog of putting five balls into three cells.

T1	0	0	0	0	0	0	1	**1**	**1**	**1**	1	2	**2**	**2**	2	3	**3**	3	4	4	5
T2	0	1	2	3	4	5	0	**1**	**2**	**3**	4	0	**1**	**2**	3	0	**1**	2	0	1	0
T3	5	4	3	2	1	0	4	**3**	**2**	**1**	0	3	**2**	**1**	0	2	**1**	0	1	0	0

(b) These are the distributions shown in non-bold. There are fifteen such distributions.

(c) Careful here! If we count only the outcomes in bold we shall get the wrong answer i.e., 6/21=0.286. The reason this answer is wrong is that the outcomes in the columns are not equally likely. The correct answer is computed using Eq.(1.8-9) i.e.,

41. (a) The probability that a specified number appears on the face of a dice is 1/6. Hence the probability of getting three specified numbers is or 1 in 216. Hence if you win you should get \$216 for every dollar bet. But the casino payout is only \$180:1.

(b) The face value of the first dice is irrelevant. The probability that the second dice matches the first is 1/6. The probability that the third dice matches the first is 1/6. Hence the probability of getting three unspecified matches is or 1 in 36.

(c) Let E_i denote that dice $i, i = 1, 2, 3$ shows a specified number. Then the probability that (at least) two specified numbers appear is

$$\begin{aligned} & P[E_1E_2E_3^c] + P[E_1E_3E_2^c] + P[E_3E_2E_1^c] + P[E_1E_2E_3] \\ = \;& 3 \times \frac{5}{6} \times \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} \\ = \;& 0.0741, \end{aligned}$$

or about 1 in 14. So per dollar bet you should get \$14 but the casino payout is only \$10.

d.-i. The next six parts can be solved by enumeration i.e., counting. However there is a systematic procedure based on the mathematical operation of convolution that can yield all of the answers from reading a graph. The details are given in Example 3.3.-5.

d. Refer to the table below:

Dice No.1	1	2	3	4	5	6
Dice No.2	1	2	3	4	5	6
Dice No.3	1	2	3	4	5	6

We note that there are only three ways of getting a 4: 1+1+2; 1+2+1,2+1+1. Hence the
probability that the sum equals 4 is $3/(6 \times 6 \times 6) = 1/72$. Thus the fair payout should be 1:72 instead of 1:60.

e. The number of ways of getting a 5 is 6: 3+1+1; 1+3+1; 1+1+3; 2+2+1; 2+1+2; 1+2+2. Hence the probability that the sum equals 5 is $6/(6 \times 6 \times 6) = 1/36$. A fair payout would be 1:36 instead of 1:30.

f.-i. follow the same enumeration.

j. Let's think of this a series of throws. The probability that the first throw matches one of the two specified numbers is 2/6. The probability that the next throw matches a specified number is 1/6. The last throw should not match either of the numbers. Its probability is 4/6. In a throw of three dice this can happen in three ways. Hence the probability is $3 \times \frac{2}{6} \times \frac{1}{6} \times \frac{4}{6} = 1/9$ or 1:9. But the payout is only 1:5.

42. For N packets there are $N!$ ways of arranging themselves, but only one way of doing it correctly. All the packet arrangements are equally likely. Hence

$$\begin{aligned} P[\{\text{correct reassembly}\}] &= 1/N! \\ &= (3628800)^{-1} \quad \text{for } N = 10, \\ &\approx 2.76 \times 10^{-7} \end{aligned}$$

43. For three packets, there are six different arrangements $(3 \cdot 2 \cdot 1 = 6)$, but only one correct one. Hence on any try

$$\begin{aligned} P_S &\triangleq P[\{\text{success}\}] \qquad \text{and} \qquad P_F \triangleq P[\{\text{failure}\}] \\ &= \frac{1}{6} \qquad\qquad\qquad\qquad = 1 - P_S = \frac{5}{6}. \end{aligned}$$

For a first correct reassembly on the n^{th} try, there must be $n-1$ failures followed by on success on the n^{th} try, thus

$$P(n) = \frac{1}{6}\left(\frac{5}{6}\right)^{n-1}, \qquad n \geq 1.$$

We note in passing that this is a valid PMF, i.e. it sums to one over its support $[1, \infty)$. To find the smallest n such that

$$\sum_{k=1}^{n} \frac{1}{6}\left(\frac{5}{6}\right)^{k-1} \geq 0.95,$$

we note that the complementary event is no successes in n trials, with probability $1 - \left(\frac{5}{6}\right)^n$. Thus we seek instead the smallest n such that $1 - \left(\frac{5}{6}\right)^n \leq 1 - 0.95 = 0.05$. Thus $n \simeq \frac{\ln(0.05)}{\ln(5/6)} \doteq 16.5$. So the answer is $n = 17$.

44.

$$\begin{aligned}\sum_{k=0}^{n} b(k;n,p) &= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-1} \\ &= (p+q)^n \\ &= (p+(1-p))^n \\ &= 1^n = 1.\end{aligned}$$

45. (a) The probability that a BM gets destroyed is

$$\begin{aligned}1 - P[\{\text{both AMM miss}\}] &= 1-(0.2)(0.2) \\ &= 0.96.\end{aligned}$$

Hence for all BMs to get destroyed, we need six wins in six tries:

$$\begin{aligned}\binom{6}{6}(0.96)^6(0.04)^0 &= (0.96)^6 \\ &\simeq 0.783.\end{aligned}$$

(b) $P[\{\text{at least one BM gets through }\}] = 1 - P[\{\text{all are destroyed}\}] \simeq 1 - 0.783 = 0.217.$

(c)

$$\begin{aligned}P[\{\text{exactly one gets through}\}] &= \binom{6}{5}(0.96)^5(0.04)^1 \\ &= 6(0.96)^5(0.04) \\ &\simeq 0.196.\end{aligned}$$

46. We want to compute

$$\begin{aligned}& P[\{\text{only one BM gets through}\}|\{\text{target destroyed}\}] \\ =\ & P[\{\text{only one BM gets through}\}|\{\text{at least one BM gets through}\}] \\ =\ & \frac{P[\{\text{only one BM gets through}\},\{\text{at least one BM gets through}\}]}{P[\{\text{at least one BM gets through}\}]} \\ =\ & \frac{P[\{\text{only one BM gets through}\}]}{P[\{\text{at least one BM gets through}\}]} \\ =\ & \frac{0.2}{0.217} \simeq 0.922.\end{aligned}$$

47. Let

$$\begin{aligned}A &= \{\text{Event that a chip meets specs}\}, \\ B &= \{\text{Event that a chip needs rework}\}, \quad \text{and} \\ C &= \{\text{Event that a chip is discarded}\}.\end{aligned}$$

We have $P[A] = 0.85, P[B] = 0.10$, and $P[C] = 0.05$. The multinomial law applies here!

(a)

$$\begin{aligned} P[\{\text{all chips meet specs}\}] &= \frac{10!}{10!0!0!}(0.85)^{10}(0.10)^{0}(0.05)^{0} \\ &\simeq 0.197. \end{aligned}$$

(b)

$$\begin{aligned} P[\{\text{ two or more discards}\}] &= 1 - P[\{\text{no discards}\}] - P[\{\text{one discard}\}] \\ &\triangleq P_2. \end{aligned}$$

Now $P[\{\text{a chip is discarded}\}] = P[C] = 0.05$, so $P[C^c] = 0.95$, thus

$$\begin{aligned} P_2 &= 1 - \binom{10}{0}(0.95)^{10}(0.05)^{0} - \binom{10}{1}(0.95)^{9}(0.05)^{1} \\ &\simeq 1 - 0.599 - 0.315 = 0.086. \end{aligned}$$

(c)

$$\begin{aligned} P[\{\text{8 meet specs, 1 needs rework, 1 discard}\}] &= \frac{10!}{8!1!1!}(0.85)^{8}(0.10)^{1}(0.05)^{1} \\ &= 0.45(0.85)^{8} \\ &\simeq 0.123. \end{aligned}$$

48. Let

$$\begin{aligned} A &= \{\text{Event that call is to report fire emergency}\}, \\ B &= \{\text{Event that call is to police}\}, \quad \text{and} \\ C &= \{\text{Event that call is for ambulance}\}. \end{aligned}$$

We have $P[A] = 0.15, P[B] = 0.60, P[C] = 0.25$, and the sequence 02030202030102030202 contains 1 A, 6 Bs and 3 Cs.

(a) $P[BCBBCABCBB] = 0.15^1 \times 0.60^6 \times 0.25^3 = 1.1 \times 10^{-4}$

(b) The number of distinguishable sequences is just the multinomial coefficient

$$\frac{10!}{6!3!1!} = \frac{10 \times 9 \times 8 \times 7}{3 \times 2 \times 1} = 840$$

(c) The probability that the 10 calls involve six calls to the police, three for ambulance and one to the subdepartment:

$$\frac{10!}{6!3!1!} \times 0.15^1 \times 0.60^6 \times 0.25^3 = 0.092$$

49. We use the Poisson approximation to the binomial: Eq. 1.10-2, with $p = \frac{1}{1000} = 10^{-3}, n = 100$, and $np = 0.1$. Then

$$\begin{aligned} P[\{\text{at least one diamond is found}\}] &= 1 - P[\{\text{no diamonds are found}\}] \\ &\simeq 1 - \frac{(0.1)^0}{0!}e^{-0.1} \\ &\simeq 1 - 0.9 = 0.1. \end{aligned}$$

50. Use Eq. (1.10-7) from the text with $t = 0$, $t + \tau = 10$, and $\lambda(\tau)$ as given. This gives

$$\int_0^{10} [1 - e^{-u/10}]\, du = 10e^{-1} = 3.68.$$

Thus,

$$P[k \text{ clicks in 10 seconds }] = e^{-3.68} \frac{1}{k!} (3.68)^k.$$

51. If all the tickets are in one lottery, then $P[\text{'win'}] = \frac{50}{100} = \frac{1}{2}$. If one ticket in each of 50 lotteries, the the probability of a win in any one lottery is $p = \frac{1}{100}$, but we have 50 chances to win. Hence $P[\text{'at least one win'}] = 1 - P[\text{'no win'}]$, where

$$\begin{aligned} P[\text{'no win'}] &= \binom{50}{0} p^0 (1-p)^{50} \\ &= \binom{50}{0} \left(\frac{1}{100}\right)^0 (\frac{99}{100})^{50} \\ &\doteq 0.605. \end{aligned}$$

Hence, taking one ticket in each of 50 lotteries, $P[\text{'at least one win'}] \doteq 1 - 0.605 = 0.395 < \frac{1}{2}$.

52. If 50 tickets in one lottery, then $E[G_1] = G_1 p = 100 \cdot (\frac{1}{2}) = 50$. If one ticket in each of 50 lotteries, we would have

$$\begin{aligned} E[G_{50}] &= \sum_{i=1}^{50} 100i \binom{50}{i} \left(\frac{1}{100}\right)^i \left(\frac{99}{100}\right)^{50-i} \\ &= \frac{50}{100} 100 \left(\sum_{i'=0}^{49} \binom{49}{i'} \left(\frac{1}{100}\right)^{i'} \left(\frac{99}{100}\right)^{49-i'} \right), \quad \text{with } i' = i - 1, \\ &= \frac{50}{100} 100 \times 1, \quad \text{since the sum in parentheses is 1,} \\ &= 50. \end{aligned}$$

53. (a) A closed circuit can occur as

$$(A_2A_4 \cup A_3A_5)\, A_1A_6 = A_1A_2A_4A_6 \cup A_1A_3A_5A_6.$$

(b) Now in general $P[A \cup B] = P[A] + P[B] - PAB]$, thus

$$\begin{aligned} P[\{\text{at least one closed path}\}] &= P[A_1A_2A_4A_6] + P[A_1A_3A_5A_6] - P[A_1A_2A_3A_4A_5A_6] \\ &= 2p^4 - p^6 \\ &= 2p^4(1 - \frac{1}{2}p^2). \end{aligned}$$

54. (a) Events associated with disjoint time intervals, under Poisson law, are independent. The number of cars arriving at a tollbooth in the time interval $(0, T)$ at a rate of λ per minute is such that $P[k \text{ cars arrive in } (0, T)] = e^{-\lambda T} \frac{[\lambda T]^k}{k!}$. Let us define the events:

$$\begin{aligned} A &\triangleq \{\, n_1 \text{ cars arrive in } (0, t_1)\}, \\ B &\triangleq \{\, n_2 \text{ cars arrive in } (t_1, T)\}, \text{ and} \\ C &\triangleq \{\, n_1 + n_2 \text{ cars arrive in } (0, T)\}. \end{aligned}$$

We are asked to find $P[A|C]$. From the definition of conditional probability, we know that this equals $\frac{P[AC]}{P[C]}$. The event AC is the event that n_1 cars arrive in $(0, t_1)$ and $n_1 + n_2$ cars arrive in $(0, T)$. This is the same as saying that n_1 cars arrive in $(0, t_1)$ and n_2 cars arrive in (t_1, T), which is nothing but the event AB. Therefore, $AC = AB$. But from the Poisson law (given), we know that $P[AB] = P[A]P[B]$, because A and B are events on disjoint time intervals. Therefore,

$$\begin{aligned} P[A|C] &= \frac{P[AC]}{P[C]} = \frac{P[AB]}{P[C]} = \frac{P[A]P[B]}{P[C]} \\ &= \frac{\frac{(\lambda t_1)^{n_1} e^{-\lambda t_1}}{n_1!} \frac{(\lambda(T-t_1))^{n_2} e^{-\lambda(T-t_1)}}{n_2!}}{\frac{(\lambda T)^{n_1+n_2} e^{-\lambda T}}{(n_1+n_2)!}} \\ &= \frac{t_1^{n_1}(T-t_1)^{n_2}}{T^{n_1+n_2}} \frac{(n_1+n_2)!}{n_1! n_2!}, \end{aligned}$$

and that is independent of λ.

b. Substituting $T = 2$, $t_1 = 1$, and $n_1 = n_2 = 5$, we get

$$P[5 \text{ cars in } (0,1) | 10 \text{ cars in } (0,2)] = \frac{10!}{5!5!} \frac{1}{2^{10}} \approx 0.25$$

.

55. The probability of a patient dying without the monitoring system is:

$$P_B = 0.1/2 = 0.05$$

The probability of a patient dying with the monitoring system is:

$$P(B, M) = P(B)P(M) = 0.05 \times 0.1 = 0.005$$

B and M are independent events.
Thus, Prof. X's argument is wrong.

56. (a) At each attempt, the probability of successful transmission is p^N. The repeated experiments are Bernoulli trials. Now the event $S(m)$ ={at least one successful transmission occurs in m attempts}. Also define $F(m) \triangleq$ {no successful transmission occurs in m attempts}. Then these events are mutually exclusive, so

$$\begin{aligned} P(m) &\triangleq P[S(m)] = 1 - P[F(m)] \\ &= 1 - \binom{m}{0} (p^N)^0 (1-p^N)^m \\ &= 1 - (1-p^N)^m. \end{aligned}$$

(b) For an individual receiver, we need the probability of at least one successful transmission in m attempts (trials). This is just the answer to part a, with p substituted for p^N, i. e. $1 - (1-p)^m$. Next consider the event $S_D(m) \triangleq${For every receiver, at least one successful transmission occurs in m attempts}. We have

$$P_D(m) \triangleq P[S_D(m)] = [1 - (1-p)^m]^N,$$

since there are N independent receivers.
If $p = 0.9, N = 5$, and $m = 2$, then,

$$P(2) \approx 0.832 \quad \text{and} \quad P_D(2) \approx 0.951.$$

57. The sample space for the compound experiment is

$$\Omega = \{(x_1, x_2, ..., x_{100}) : 2 \leq x_i \leq 12, 1 \leq i \leq 100\}.$$

For the individual experiment with the two die, we can write the sample space as the locations (ξ_1, ξ_2) in the 6×6 table

(ξ_1, ξ_2)	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

,

where we have entered in each cell the sum of the die's upward faces. Now we set the event $A \triangleq \{\text{the sum is } 7\}$ and find $P[A] = 6/36 = 1/6 \triangleq p$. As for the compound experiment consisting of $N = 100$ tries, it is seen to be Bernoulli trials with $n = 100$ and $p = 1/6$. So the answer for '10 seven's in 100 tries' is $b(7; 100, 1/6) = \binom{100}{10} p^{10}(1-p)^{90}$. We can evaluate this simply using the Poisson approximation with $a = np = 100/6$. Then

$$\begin{aligned} P[10 \text{ seven's in } 100 \text{ tries}] &\approx \frac{a^{10}e^{-a}}{10!} \\ &= \frac{(100/6)^{10}\, e^{-100/6}}{10!} \\ &\approx 0.0264. \end{aligned}$$

58. b) We do part (b) first. From the landlord's viewpoint, the following applies. If lease includes *free repairs*, then the cost of the two "Cloggers" versus one "NeverFail" is the same, so it doesn't matter. If repairs are not free and are the same for the "Cloggers" as for the "NeverFail," then clearly the "NeverFail" is the cheaper to lease.

a) From the tenants' point of view:

$$\begin{aligned} P[\{\text{at least one "Clogger" on}\}] &= 1 - P[\{\text{both fail}\}] \\ &= 1 - (0.4)^2 \\ &= 0.84, \text{ while} \end{aligned}$$

$$\begin{aligned} P[\{\text{"NeverFail" on}\}] &= 0.80 \\ &< 0.84. \end{aligned}$$

Therefore, the two "Cloggers" are better since at least one of them will be working 84% of the time.

59. This problem typifies the faulty reasoning exhibited by many people not familiar with probability and, in particular, the idea of *independent events*. While it is true that $P[\{2 \text{ bombs on board}\}] = 10^{-4}$, the issue is $P[\{\text{terrorist bomb on board}\}|\{\text{"nervous" has a bomb on board}\}] = P[\{\text{terrorist bomb on board}\}] = 10^{-2}$, since the events $A = [\{\text{terrorist bomb on board}\}]$ and $B = [\{\text{"nervous" has a bomb on board}\}]$ are independent. Hence $P[A|B] = P[A]$; therefore, no protection!

60. Let $E = \{\text{event of successful transmission on short path}\}$;$F = \{\text{event of successful transmission on a long path}\}$. Then $P[\{\text{successful transmission}\}] = 1 - P[E^cF^c]$ and $P[E^c] = 1 - p^3$, while $P[F^c] = 1 - p^5$, where $p \triangleq 1 - q$. Therefore

$$\begin{aligned} P[\{\text{successful transmission}\}] &= 1 - P[E^cF^c] \\ &= 1 - (1 - p^3)(1 - p^5) \\ &= p^3 + p^5 - p^8. \end{aligned}$$

61. In this case the telephone company might find the union directive unreasonable. Here is why:

$$\begin{aligned} P[\text{overtime}] &= \sum_{n=5761}^{\infty} \frac{(720 \times 8)^n}{n!} e^{-720\times 8} \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{l_1}^{l_2} e^{-\frac{1}{2}x^2} dx \\ &\quad \text{(by the Gaussian approximation to Poisson),} \end{aligned}$$

where

$$l_1 = \frac{5761 - 5760 - 0.5}{\sqrt{5760}} \quad \text{and} \quad l_2 = \frac{\infty - 5760 + 0.5}{\sqrt{5760}}.$$

Hence

$$\begin{aligned} P[\text{overtime}] &\approx \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{1}{2}x^2} dx \\ &= \operatorname{erf}(\infty) = \frac{1}{2}. \end{aligned}$$

So approximately half the time, Curtis will collect overtime.

62. The sample space for the compound experiment is

$$\Omega = \{(x_1, x_2, ..., x_{80}) : 2 \le x_i \le 4, 1 \le i \le 80\}.$$

For the individual experiment with the two die, we can write the sample space as the locations (ξ_1, ξ_2) in the 2×2 table

(ξ_1, ξ_2)	1	2
1	2	3
2	3	4

,

where we have entered in each cell the sum of the coin's upward sides. Now we set the event $A \triangleq \{\text{the sum is 2}\}$ and find $P[A] = 1/4 \triangleq p$. As for the compound experiment consisting of $N = 80$ tries, it is seen to be Bernoulli trials with $n = 80$ and $p = 1/4$. So the answer for '10 two's in 80 tries' is $b(10;80,1/4) = \begin{pmatrix} 80 \\ 10 \end{pmatrix} p^{10}(1-p)^{70}$. We can evaluate an approximation simply using the Poisson approximation with $a = np = 80/4 = 20$. Then

$$\begin{aligned} P\acute{[}10 \text{ two's in 80 tries}] &\approx \frac{a^{10}e^{-a}}{10!} \\ &= \frac{(20)^{10}\, e^{-20}}{10!} \\ &\approx 0.0058. \end{aligned}$$

Incidently the exact answer for the binomial $b(10;80,1/4)$ is 0.0028. The Poisson approximation is only marginal here since $p = 1/4$ is not really $<< 1$.

63. Since arrivals in disjoint intervals are independent under the Poisson law, it follows that we equivalently want the probability of 5 cars arriving in the second 2 minutes. This is given as

$$P[5 \text{ cars in 2 minutes}] = \frac{(2\lambda)^5}{5!}e^{-2\lambda}.$$

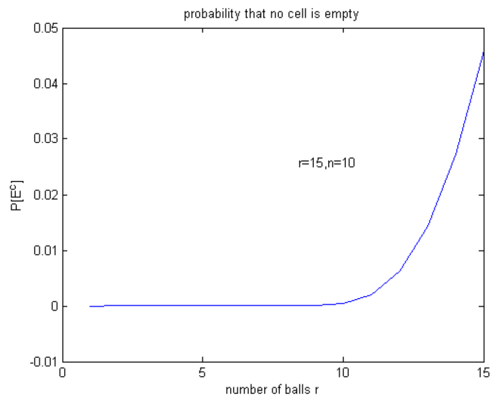
64. Unfortunately, very small. The reader should recognize that this is an occupancy problem with the candies being the "balls" and the students being the "cells." The appropriate formula is Eq. 1.8-9, which gives the probability that no cell is empty. Hence, with $r = 15, n = 10$,

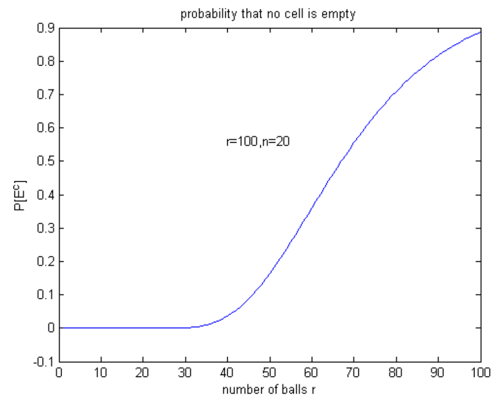
$$\begin{aligned} P[\{\text{no student is without a candy}\}] &= \sum_{i=0}^{10} \binom{10}{i} (-1)^i (1 - \frac{i}{10})^{15} \\ &\approx 0.05. \end{aligned}$$

A MATLAB function to do this problem is as follows:

```
function [tries,prob] = occupancy(balls,cells)
% Here #balls=r and #cells=n. This function then
% calculates the answer to the occupancy problem in
% Example 1.8-5, specifically Eq. 1.8-9. This function
% is used in the solution to Problem 1.64 .
%
tries=1:balls; prob=zeros(1,balls);
c=zeros(1,cells); d=zeros(1,cells);
term=zeros(1,cells);
for m=1:balls
for k=1:cells
c(k)=((-1)^k)*prod(1:cells)/(prod(1:k)*prod (1:cells-k));
d(k)=(1-(k/cells))^m;
term(k)=c(k)*d(k);
end
prob (m)=1+sum(term);
end
plot(tries,prob)
xlabel('number of balls r');
ylabel('P[E^c]');
title('probability that no cell is empty');
end
```

Two example runs follow. The first is for $r = 15, n = 10$, yielding the answer to this problem at $n = 10$. The second run is for a larger case with $r = 100$ and $n = 20$.





65. These are repeated Bernoulli trials resulting in the Binomial distribution with $n = 1000$ and $p = 0.001$. Let X_i be the individual random variables, taking on value 1 for an erroneous line and 0 for an error-free line. Then we can write the sum or total of the errors as

$$Z = \sum_{i=1}^{n} X_i.$$

Then Z is Binomial with $\mu_Z = np = 1$ and $\sigma_Z^2 = npq = 0.999$. We can use the Poisson approximation to the Binomial with $a = np = 1$ here. Then

$$\begin{aligned} P[2 &\leq Z \leq 1000] = 1 - P_Z(0) - P_Z(1) \\ &\approx 1 - e^{-a} - ae^{-a} \\ &= 1 - 2e^{-1} \\ &\doteq 0.264. \end{aligned}$$

The CLT approximation gives a Normal distribution with mean $\mu = 1$ and $\sigma = \sqrt{0.999} = 0.9995$. However, it is not as accurate here since the mean μ_Z is only approximately one standard deviation away from 0, the minimum value of a Bernoulli random variable. Calculating the CLT approximate answer, we find

$$\begin{aligned} P[2 \leq Z \leq 1000] &\approx \frac{1}{\sqrt{2\pi \times 0.999}} \int_2^{1000} e^{-\frac{1}{2}\left[\frac{z-1}{\sqrt{0.999}}\right]^2} dz \\ &\approx \frac{1}{\sqrt{2\pi}} \int_{1.0005}^{+\infty} e^{-\frac{1}{2}x^2} dx \\ &= 0.5 - \text{erf}\,(1.0005) \\ &\doteq 0.5 - 0.341 \\ &= 0.159, \quad \text{not very accurate here.} \end{aligned}$$

66. This is a classic problem and the solution is unexpected. Let $A = \{$event that *no* two people have their birthdays on the same date$\}$ Let $B = \{$event that at least two people have their birthdays on the same date$\}$. Then $B = A^c$ and

$$\begin{aligned} P[B] &= P[A^c] \\ &= 1 - P[A], \end{aligned}$$

where

$$
\begin{aligned}
P[A] &= \frac{\text{no. of ways } A \text{ can occur}}{\text{no. of all possible outcomes}} \\
&= \frac{n_A}{n_T},
\end{aligned}
$$

where $n_A = 365(365-1)(365-2)\cdots(365-(n-1))$ and $n_T = (365)^2$. Thus

$$
\begin{aligned}
P[A] &= \frac{n_A}{n_T} \\
&= 1\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)...\left(1-\frac{n-1}{365}\right) = \frac{1}{2}?.
\end{aligned}
$$

Taking logarithms, we have

$$
\ln 1 + \ln\left(1-\frac{1}{365}\right) + \ln\left(1-\frac{2}{365}\right) + \cdots + \ln\left(1-\frac{n-1}{365}\right) = -0.7?.
$$

Then, upon using $\ln(1-x) \simeq x$ for x small, when $n/365$ is small, we get

$$
\begin{aligned}
-\left(\frac{1+2+\cdots+(n-1)}{365}\right) &= -\frac{n(n-1)}{2(365)} \\
&\simeq -0.7.
\end{aligned}
$$

So we must set

$$
\frac{n(n-1)}{2(365)} = 0.7
$$

and solve for n, resulting in the quadratic equation

$$
n^2 - n - 511 = 0,
$$

whose positive root is $n = 22.6$. Rounding up to the next integer, we get the answer at just 23 people necessary for the probability to be one-half or greater that at least two people will have their birthday on the same date.

67. By the problem statement, we have a Binomial probability law $b(k;N,p)$ with $N = 10$ and $p = P[\text{defect}] = 0.02$. So the probability of more than one defect in the sample is given as

$$
\begin{aligned}
P[\text{more than 1 defect}] &= \sum_{k=2}^{10} b(k;10,0.02) \\
&= 1 - \sum_{k=0}^{1} b(k;10,0.02) \\
&= 1 - (0.98)^{10} - \binom{10}{1}(0.02)(0.98)^9 \\
&= 1 - (0.98)^{10} - 0.2(0.98)^9 \\
&= 1 - 1.18(0.98)^9 \\
&= 0.0162
\end{aligned}
$$

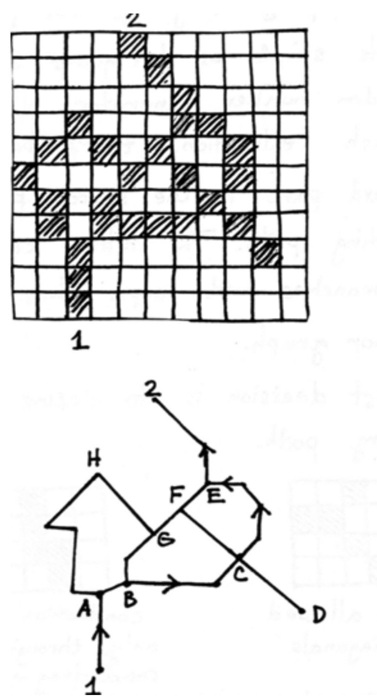
68. The programming of this problem is quite simple as it requires applying a random number generator N times for each realization ... that's basically it. The hard part is the search for a percolating path. The lattice contains both branches and loops. Thus it is netither tree, nor graph. The first decision is to define a conducting path, and there are two choices:

conduction allowed through diagonal elements

conduction allowed only through hor. and vert. elements

Two choices for conducting paths.

The time-consuming part is the search. Thus if you come to a node (junction) and the path you choose doesn't lead anywhere, you must be careful to return to the note and try the other path.



An Example.

A possible path is 1ABCE2. Note the dead-end at D and the possibility of endless looping if you are not careful. Since $N \leq 50$, relatively simple search techniques should work. A good MS thesis. Good luck!

69. Let:

$$\begin{aligned} A &= \{\text{door picked by you}\}, \\ B &= \{\text{door picked by MC}\}, \\ C &= \{\text{remaining door}\}, \text{ and} \\ D &= \{\text{door that leads to Rexis}\}. \end{aligned}$$

Then $AD = \{\text{door picked by you leads to Rexis}\}$, $BD = \{\text{door picked by MC leads to Rexis}\}$, and $CD = \{\text{remaining door leads to Rexis}\}$. Then

$$P[AD = \frac{1}{3} \quad \text{and} \quad P[BD \cup CD] = \frac{2}{3}.$$

But, since $BD \cap CD = \phi$,

$$\begin{aligned} P[BD] + P[CD] &= P[BD \cup CD] \\ &= \frac{2}{3}, \end{aligned}$$

and the MC always chooses the wrong door, so that $P[BD] = 0$, and hence $P[CD] = \frac{2}{3}$. Therefore, you should switch to door C, as it will double your probability of winning the Rexis!

70. (a) This is Bernoulli trials. Thus $P_1[E] = \binom{10}{4}\left(\frac{1}{2}\right)^{10}$ and $P_2[E] = \binom{10}{4}p^4(1-p)^6$.

(b) The likelihood ratio is given as

$$\begin{aligned} L &= P_1[E]/P_2[E] \\ &= \binom{10}{4}\left(\frac{1}{2}\right)^{10} / \binom{10}{4} p^4(1-p)^6 \\ &= \left(\frac{1}{2}\right)^{10} p^{-4}(1-p)^{-6} \\ &= \left(\frac{1}{2p}\right)^4 \left(\frac{1}{2(1-p)}\right)^6. \end{aligned}$$