

SOLUTIONS TO PROBLEMS – Instructors' version (all problems)

PROBLEMS 1

1.1

$$(a) \bar{\nu}_e + e^+ \rightarrow \bar{\nu}_e + e^+; \quad (b) p + n \rightarrow p + n; \quad (c) p + \bar{p} \rightarrow e^+ + e^- + \gamma.$$

1.2 From (A),

$$\alpha_i \beta = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & -\sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix},$$

and similarly

$$\beta \alpha_i = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \sigma_i \\ -\sigma_i & \mathbf{0} \end{pmatrix}.$$

Hence $[\alpha_i, \beta] \equiv \alpha_i \beta + \beta \alpha_i = 0$, for $i = 1, 2, 3$. Using the same relations,

$$\alpha_i \alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} = \sigma_i \sigma_j \mathbf{1}$$

and likewise $\alpha_j \alpha_i = \sigma_j \sigma_i \mathbf{1}$. But it is straightforward to verify from (B) that $\sigma_i \sigma_j = -\sigma_j \sigma_i$ for $i \neq j$. So, $[\alpha_i, \alpha_j] = 0$, for $i \neq j = 1, 2, 3$.

1.3 Substituting the explicit forms of the Dirac matrices into (1.13) gives

$$\begin{aligned} (E - mc^2)u_1 - cp_z u_3 - c(p_x - ip_y)u_4 &= 0, \\ (E - mc^2)u_2 - c(p_x + ip_y)u_3 + cp_z u_4 &= 0, \\ (E + mc^2)u_3 - cp_z u_1 - c(p_x - ip_y)u_2 &= 0, \\ (E + mc^2)u_4 - c(p_x + ip_y)u_1 + cp_z u_2 &= 0. \end{aligned} \tag{A}$$

Case 1: $u_1 = 1, u_2 = 0, u_3 = a_1, u_4 = b_1$

In this case, the third and fourth of equations (A) give the values

$$u_3 = a_1 = \frac{cp_z}{E + mc^2}, \quad u_4 = b_1 = \frac{c(p_x + ip_y)}{E + mc^2}. \tag{B}$$

For $\mathbf{p} = \mathbf{0}$, the first equation in (A) with $u_1 = 1$ clearly requires $E = mc^2$, while substituting the relations (B) in the first equation of (A) leads to the requirement

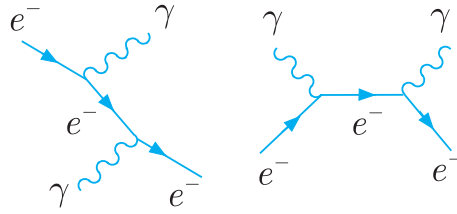
$$E - mc^2 = \frac{c^2 \mathbf{p}^2}{E + mc^2}.$$

Taken together, these are satisfied by $E = E_+ \equiv (c^2 p^2 + m^2 c^4)^{1/2}$, but not by $E = E_- \equiv -(c^2 p^2 + m^2 c^4)^{1/2}$.

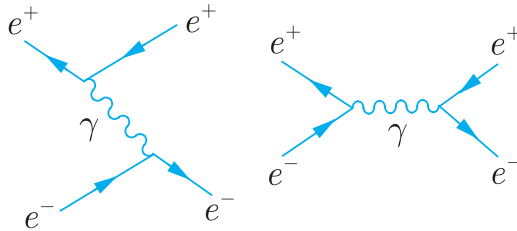
The other solutions are dealt with in a similar way, where the first two require $E = E_+ > 0$ and the second two require $E = E_- < 0$. All four solutions are tabulated below.

1	2	3	4
$\begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{(E_- - mc^2)} \\ \frac{c(p_x - ip_y)}{(E_- - mc^2)} \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x + ip_y)}{(E_- - mc^2)} \\ \frac{-cp_z}{(E_- - mc^2)} \end{pmatrix}$	$\begin{pmatrix} \frac{cp_z}{(E_+ + mc^2)} \\ \frac{c(p_x - ip_y)}{(E_+ + mc^2)} \\ 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \frac{c(p_x + ip_y)}{(E_+ + mc^2)} \\ \frac{-cp_z}{(E_+ + mc^2)} \\ 0 \\ 1 \end{pmatrix}$

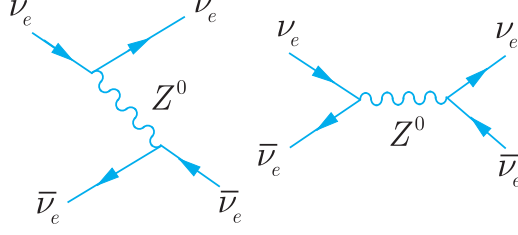
1.4 The topologically distinct diagrams for reactions (a) are given below.



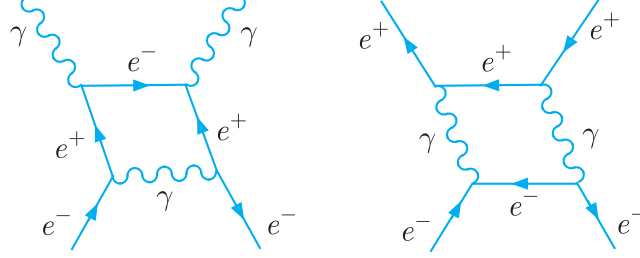
Those for (b) are



and those for (c) are shown below.



1.5 Two possible diagrams are shown in below. There are others.



1.6 For a spherically symmetric static solution we can set $\Psi(\mathbf{r}, t) = \phi(r)$, where $r = |\mathbf{r}|$, and use

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r},$$

giving

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} = \left(\frac{mc}{\hbar} \right)^2 \phi.$$

Substituting $\phi = u(r)/r$ gives

$$\frac{d^2 u(r)}{dr^2} = \left(\frac{mc}{\hbar} \right)^2 u(r)$$

and the result follows by solving for u , and imposing $\phi \rightarrow 0$ as $r \rightarrow \infty$.

1.7 Using spherical polar co-ordinates, we have

$$\mathbf{q} \cdot \mathbf{r} = |\mathbf{q}| r \cos \theta \quad \text{and} \quad d^3 \mathbf{r} = r^2 dr d\cos \theta d\phi.$$

Thus, from (1.32),

$$\mathcal{M}(q^2) = \frac{-g^2}{4\pi} \int_0^{2\pi} d\phi \int_0^\infty dr r^2 \frac{e^{-r/R} + 1}{r} \int_{-1}^1 d\cos \theta \exp(iqr \cos \theta / \hbar) = \frac{-g^2 \hbar^2}{|\mathbf{q}|^2 + M_X^2 c^2},$$

where we have used (1.26).

- 1.8** If we impose momentum conservation and neglect the momenta of the initial e^\pm , we have

$$e^+(m, \mathbf{0}) + e^-(m, \mathbf{0}) \rightarrow \gamma(E_k, \mathbf{k}) + \gamma(E_k, -\mathbf{k}),$$

where $E_k = |\mathbf{k}| = m$. Hence in Figure 1.9(a), the initial virtual process is

$$e^-(m, \mathbf{0}) \rightarrow e^-(E, \mathbf{k}) + \gamma(m, -\mathbf{k}),$$

where the energy of the virtual electron is

$$E = (k^2 + m^2)^{1/2} \approx \sqrt{2}m \approx m.$$

Hence $\Delta E = E \approx m$ and from the uncertainty principle, $r \approx 1/\Delta E \approx 1/m$. Restoring factors of \hbar and c gives $r \approx \hbar c/mc^2 = 368 \text{ fm}$.

- 1.9** The distance between the two vertices is given by

$$r \approx c\tau \approx \hbar c/\Delta E = 1/\Delta E,$$

where ΔE is the energy violation at the first vertex. On evaluating ΔE in each case, one obtains (a) $r \approx m^{-1}$ and (b) $r' \approx E^{-1}$, respectively, where E is the initial electron energy in the centre-of-mass frame. These are related by a Lorentz contraction $r' = r/\gamma$, where $\gamma = E/m$. (A resumé of special relativity is given in Appendix A.)

- 1.10** Set $\tau = (2/m\alpha^5)\hbar^a c^b$ and demand that τ has the dimensions of time. This gives $a = 1$ and $b = -2$ and hence $\tau = 1.245 \times 10^{-10} \text{ s}$.

- 1.11** Substituting the values for E and G_F , gives $\sigma = 3.60 \times 10^{-10} \text{ GeV}^{-2}$, and using the conversion factor $1 \text{ GeV}^{-2} = 0.389 \text{ m}$, given on the inside of the back cover, to convert from nu to SI units, we obtain $\sigma = 1.40 \times 10^{-10} \text{ mb} = 0.14 \text{ pb}$.

- 1.12** (a) In natural units $\alpha = e^2/4\pi\epsilon_0$, so we can write $r_c = \alpha/m_e$. Restoring factors of \hbar and c , we have

$$r_c = \alpha \frac{\hbar c}{m_e c^2} = \alpha \frac{1.973 \times 10^{-13} \text{ MeV m}}{0.51 \text{ MeV}} = 0.0282 \times 10^{-13} \text{ m} = 2.8 \text{ fm}.$$

(b) The electrostatic energy is

$$E = \frac{1}{2} \int \rho \phi \, dv,$$

where ϕ is the potential energy and the charge density for $r < R$ and zero for $r > R$. On a spherical shell of radius $r < R$, the potential energy is just the electrostatic potential due to the enclosed charge Q . Hence

$$\phi(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} = \frac{1}{4\pi\epsilon_0} \frac{er^2}{R^3}$$

and

$$E = \frac{1}{2} \frac{3e}{4\pi R^3} \frac{1}{4\pi\epsilon_0} \int_0^R \frac{er^2}{R^3} 4\pi r^2 \, dr = \frac{3}{10} \frac{e^2}{4\pi\epsilon_0 R} = \frac{3}{10} \frac{\alpha}{R} \text{ in nu.}$$

Setting $E = m$ then gives

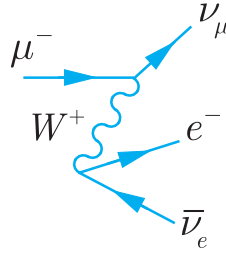
$$R = \frac{3}{10} \frac{\alpha}{m} = \frac{3r_c}{m},$$

where r_c is the ‘classical radius’ defined in (a) above.

PROBLEMS 2

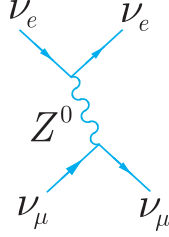
2.1 Weak interactions conserve baryon number B , charge Q and lepton numbers L_e , L_μ and L_τ . They need not conserve the quantum numbers S , C or \tilde{B} . Of the decays given, (a) violates L_μ conservation and (d) violates both L_μ and L_τ conservation. They are therefore both forbidden. Reactions (b) and (c) satisfy all the conservation laws and are allowed.

2.2 The basic processes are $\mu^- \rightarrow W^- \nu_\mu$ and $W^- \rightarrow e^- \bar{\nu}_e$. The other time-ordered diagram is shown below,

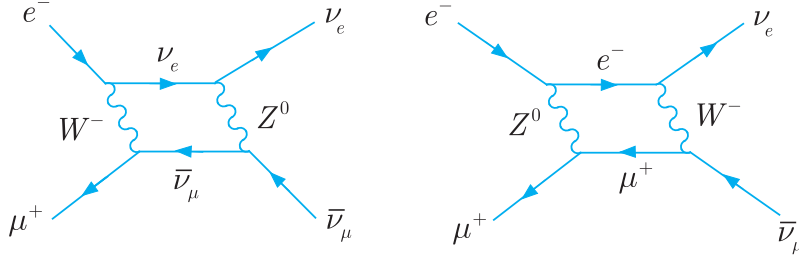


and the basic processes are $\text{vacuum} \rightarrow W^+ e^- \bar{\nu}_\mu$ and $\mu^- W^+ \rightarrow \bar{\nu}_\mu$.

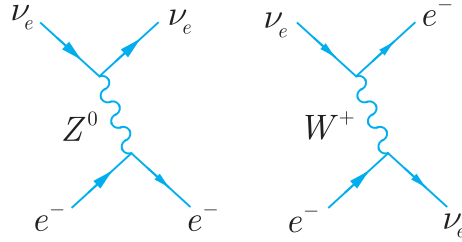
- 2.3** The lowest-order diagram involves the exchange of a single Z^0 boson, as shown below.



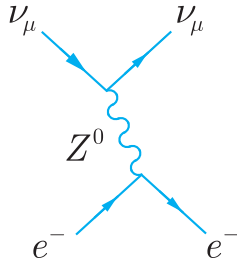
- 2.4** The two diagrams are shown below.



- 2.5** The electron neutrino may interact with electrons via both Z^0 and W^- exchange:



whereas, because of lepton number conservation, the muon neutrino can only interact via Z^0 exchange:



- 2.6** Firstly restore factors of \hbar and c by writing $L_0 = 4E\hbar^a c^b / \Delta m_{ij}^2$ and find a and b by demanding that the right-hand side has the dimensions of a length. This gives $a = 1$ and $b = -3$, so that $L_0 = 4E(\hbar c) / \Delta m_{ij}^2 c^4$. Then if L_0 is expressed in km, E in GeV and Δm_{ij}^2 in $(\text{eV}/c^2)^2$, we have

$$L_0 = \frac{4E \times (1.97 \times 10^{-13}) \times 10^{18}}{\Delta m_{ij}^2} = \frac{E}{1.27 \Delta m_{ij}^2} \text{ km}.$$

2.7 By energy conservation

$$E_+ = E_\nu + m_p - m_n - K_n = 1.71 \text{ MeV} - K_n,$$

where $K_n = E_n - m_n$ is the kinetic energy of the recoil neutron. Since $K_n > 0$, this implies $E_+ \leq 1.71 \text{ MeV}$ and hence $|\mathbf{p}_+| < 1.63 \text{ MeV}$, using $E^2 = p^2 + m^2$. By momentum conservation $\mathbf{p}_n = \mathbf{p}_\nu - \mathbf{p}_+$, so that

$$|\mathbf{p}_n| \leq |\mathbf{p}_\nu| + |\mathbf{p}_+| = 4.63 \text{ MeV},$$

neglecting the neutrino mass, and $K_n = \mathbf{p}_n^2 / 2m_n \leq 0.011 \text{ MeV}$. Hence $E_+ = E_\nu + m_p - m_n = 1.71 \text{ MeV}$ to within 0.01 MeV, which is less than 1%.

2.8 From (2.30) and (2.31), the necessary condition is

$$\cos(2\theta_{12}) = A = -\frac{2\sqrt{2}G_F N_e E}{\Delta m_{12}^2}.$$

Since $N_e \leq N_0$, this requires

$$E \geq -\frac{\Delta m_{12}^2 \cos(2\theta_{12})}{2\sqrt{2}G_F N_0}. \quad (1)$$

Using the conversion factor $1 \text{ GeV}^{-1} = 0.197 \text{ fm}$, gives $1 \text{ m}^{-1} = 0.197 \times 10^{-12} \text{ MeV}$ and hence $N_0 = 2.3 \times 10^{-7} \text{ MeV}^3$ in natural units. In these units, $\Delta m_{12}^2 = 7.5 \times 10^{-17} \text{ MeV}^2$, $G_F = 1.17 \times 10^{-11} \text{ MeV}^{-2}$, and $\tan^2 \theta_{12} = 0.444$ gives $\cos 2\theta_{12} = 0.385$. Substituting into (1) then gives $E \geq 3.8 \text{ MeV}$.

2.9 Using the results of Problem 2.6, together with the squared mass differences

$$|\Delta m_{13}|^2 \approx 2 \times 10^{-3} (\text{eV}/c^2)^2 \text{ and } |\Delta m_{21}|^2 \approx 7.5 \times 10^{-5} (\text{eV}/c^2)^2,$$

we have

$$L_{13} \approx 1.2 \text{ km}, \quad L_{12} \approx 31.5 \text{ km}.$$

Hence the magnitudes of amplitudes of the associated transitions (2.30a) are

$$P_{13} \approx 0.09 \sin^2(L/1.2) \quad \text{and} \quad P_{12} \approx 0.9 \sin^2(L/32),$$

where L is in km and we have used $\sin^2(2\theta_{12}) \approx 0.08$ and $\sin^2(2\theta_{13}) \approx 0.9$.

For L of order 150–200 km the amplitude of the oscillations of P_{13} is ten times smaller than those of P_{12} , so that the former may be neglected to a first approximation. For an L of 1 km, $P_{12} \approx 0.9 \times 10^{-3}$, which is small compared to the amplitude of the oscillation of P_{13} , so that P_{12} can, to a good approximation, be neglected. For an L of 10 km, $P_{12} \approx 0.09$, which is comparable to the amplitude of the oscillation of P_{13} , so that a two-component mixing model would fail badly.

- 2.10** From the solution to Problem 2.6, we have, for maximal mixing ($\theta = \pi/4$), $P(\bar{\nu}_e \rightarrow \bar{\nu}_x) = \sin^2[1.27 \Delta(m^2 c^4) L/E]$ where L is measured in metres, E in MeV, $\Delta(m^2 c^4)$ in $(\text{eV})^2$ and $\Delta m^2 \equiv m^2(\bar{\nu}_e) - m^2(\bar{\nu}_x)$. If $P(\bar{\nu}_e \rightarrow \bar{\nu}_e) = 0.9 \pm 0.1$, then at 95% confidence level, $0.3 \geq P(\bar{\nu}_e \rightarrow \bar{\nu}_x) \geq 0$ and hence $0 \leq \Delta(m^2 c^4) \leq 3.8 \times 10^{-3} (\text{eV})^2$.

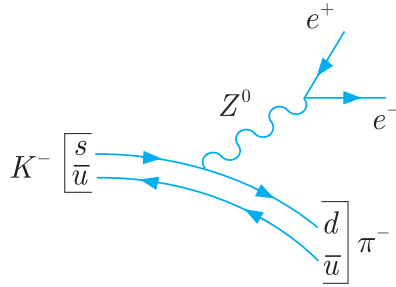
- 2.11** From the data given, the total number of nucleons is given by

$$N = \frac{2 \times 10^{30}}{(0.938 \times 10^9)(1.78 \times 10^{-36})} = 1.2 \times 10^{57}$$

and hence $n = 8.4 \times 10^{38} \text{ km}^{-3}$. Also the mean energy of the neutrinos from reaction (2.35) is 0.26 MeV, so the cross-section is $\sigma = 1.8 \times 10^{-46} \text{ m}^2$. Thus, finally, $\lambda \approx 6.6 \times 10^{12} \text{ km}$, i.e. about 10^7 times the solar radius.

PROBLEMS 3

- 3.1** (a) forbidden – violates baryon number conservation; (b) forbidden – violates charge conservation; (c) forbidden – violates muon lepton number conservation; (d) allowed by the diagram:



- 3.2** (a) Involves neutrinos and is therefore a weak interaction. (b) Involves photons and is therefore electromagnetic. (c) Conserves all quantum numbers and is therefore a

strong interaction. (d) Violates both strangeness and charm and is therefore a weak interaction. (e) Conserves all quantum numbers and is a strong interaction. (f) Involves electrons and positrons and is therefore an electromagnetic interaction.

- 3.3** The final-state pions have equal energies $E_\pi = E_K/2 = 247 \text{ MeV}$ and $\gamma = E_\pi/m_\pi c^2 = 1.76$. Hence the lifetime in flight is $\tau = \gamma\tau_0 = 4.6 \times 10^{-8} \text{ s}$ and the π^+ velocity is $v = c(1 - 1/\gamma^2)^{-1/2} = 1.21c$. The mean distance travelled is thus $\tau v = 17 \text{ m}$.

- 3.4** The quantum numbers are:

$$X^0 : B = 1, S = -1, C = 0, \tilde{B} = 0 ; Y^- : B = 1, S = -2, C = 0, \tilde{B} = 0 .$$

From their charges, and the definitions of \tilde{B} , S , C and B , it follows that $X^0 = uds$ and $Y^- = dss$. The decay $Y^- \rightarrow \Lambda + \pi^-$ violates strangeness conservation and is a weak interaction, so we expect $\tau = 10^{-6} - 10^{-13} \text{ s}$. (The Y^- is in fact the so-called $\Xi^-(1321)$ state with a lifetime $1.6 \times 10^{-10} \text{ s}$)

- 3.5** The quantum number combination $(2, 1, 0, 1, 0)$ corresponds to a baryon qqq . It has $S = 0$, $C = 1$ and $B = 0$, so must be of the form cxy , where x and y are u or d quarks. The charge $Q = 2$ requires both x and y to be u quarks, i.e. cuu . The others are established by similar arguments and so the full set is:

$$\begin{aligned} (2, 1, 0, 1, 0) &= cuu, & (0, 1, -2, 1, 0) &= css, & (0, 0, 1, 0, -1) &= b\bar{s}, \\ (0, -1, 1, 0, 0) &= \bar{s}\bar{d}\bar{u}, & (0, 1, -1, 1, 0) &= csd, & (-1, 1, -3, 0, 0) &= sss. \end{aligned}$$

They are called Σ_c^{++} , Ω_c^0 , \bar{B}_s^0 , $\bar{\Lambda}$, Σ_c^0 and Ω^- , respectively.

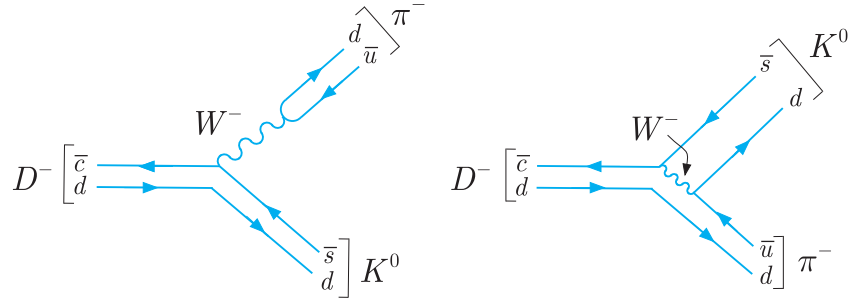
- 3.6** From their quark compositions, the Σ^+ , Σ^0 and Σ^- have $B = 1$, $S = -1$. We need to check the reactions for B , Q and S conservation. Reactions (b) and (d) conserve all three quantum numbers and so are allowed in strong interactions. Reactions (a) and (c) violate S and Q conservation respectively, and are forbidden in strong interactions.

- 3.7** The $\Sigma^0(1193)$ has $B = 1$, $Q = 0$, $S = -1$, which must all be conserved in a strong interaction. The lightest two-body hadron state with these quantum numbers is $\Lambda\pi^0$, but $m(\Sigma^0) < m(\Lambda) + m(\pi^0)$. Hence strong interaction decays are forbidden by energy conservation. The decay $\Sigma^0 \rightarrow \Lambda\gamma$ conserves all three quantum numbers and is first-order in the electromagnetic interaction. Other electromagnetic decays, such as $\Sigma^0 \rightarrow \Lambda e^+e^-$, are higher order, and suppressed by higher powers of α . For strong decays $\tau \sim 10^{-22} - 10^{-24} \text{ s}$ from Table 3.4. Hence for the decay $\Sigma^0 \rightarrow \Lambda\gamma$,

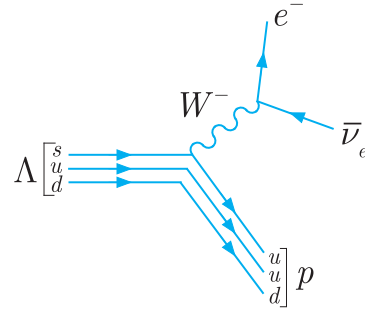
$\tau(\Lambda\gamma) \sim \tau/\alpha \sim 10^{-19} - 10^{-22} \text{ s}$, with $\alpha \sim 10^{-2} - 10^{-3}$. (The measured value is $7.4 \times 10^{-20} \text{ s}$.)

3.8 The $\Sigma^+(1189)$ has $B=1, Q=1$, which must be conserved in all interactions. The only possible hadron final states with these quantum numbers are $n\pi^+, p\pi^0$, i.e the decay modes are $\Sigma^+ \rightarrow n\pi^+, p\pi^0$. Since $\Sigma^+(1189)$ has $S=-1$ and the final states have $S=0$, strangeness is violated and the decays must be via the weak interaction. From Table 3.4, the lifetime is expected to be in the range $(10^{-7} - 10^{-13}) \text{ s}$. (The actual value is $8 \times 10^{-11} \text{ s}$.)

3.9 (a) The quark compositions are: $D^- = d\bar{c}$; $K^0 = d\bar{s}$; $\pi^- = d\bar{u}$ and since the dominant decay of a c -quark is $c \rightarrow s$, we can have either of the diagrams shown below.

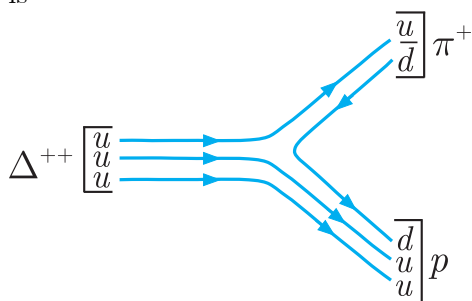


(b) The quark compositions are: $\Lambda = sud$; $p = uud$ and since the dominant decay of an s -quark is $s \rightarrow u$, we have the diagram:

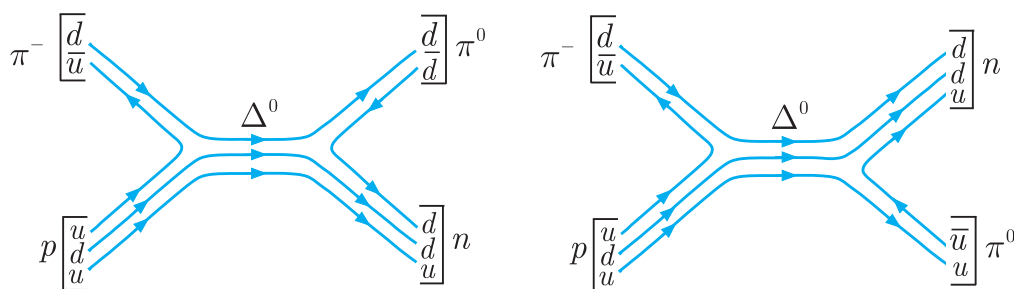


3.10 For $\gamma \approx 10$, $\tau \approx c$ and the average distance $d \approx c\gamma\tau \approx 3 \times 10^{-14} \text{ m} = 30 \text{ fm}$, if we assume a lifetime for the particle at rest of 10^{-23} s . This is much smaller than the best experimental resolution given, but larger than the range of the strong interaction. This is important if the decay of, for example, the X^- produced in reaction (3.26a) is to be treated as the decay of a free particle, since this requires that it should be sufficiently far from any other hadrons that are present (e.g. the proton in (3.26a)).

- 3.11** (a) The decay $\Delta^{++} \rightarrow \pi^+ p$ in terms of quarks is $uuu \rightarrow u\bar{d} + uud$, so a $d\bar{d}$ is created and the quark diagram is



- (b) The corresponding diagrams for $\pi^- p \rightarrow \Delta^0 \rightarrow \pi^0 n$ are



- 3.12** An argument similar to that given in Section 3.6 gives the following allowed combinations:

<i>Baryons</i>		<i>Mesons</i>	
<i>C</i>	<i>Q</i>	<i>C</i>	<i>Q</i>
3	2	1	1,0
2	2,1	0	1,0,-1
1	2,1,0	-1	0,-1
0	2,1,0,-1		

- 3.13** The first two quantum number combinations are compatible with the assignments

$$(1,0,0,1,1) = c\bar{b}, \quad (-1,1,-2,0,-1) = ssb$$

and can exist within the simple quark model. There are no combinations $q\bar{q}$ or qqq which are compatible with the second two combinations, so these cannot exist within the quark model. The combination $(0,0,1,0,1)$ must be a meson $q\bar{q}$ because $B=0$, but must contain both an \bar{s} antiquark and a \bar{b} antiquark since $S=\tilde{B}=1$. These are incompatible requirements. The combination $(-1,1,0,1,-1)$ must be a baryon qqq of the form xcb (where $x=u$ or d) since $B=1$, $S=0$, $C=1$ and $\tilde{B}=-1$. The