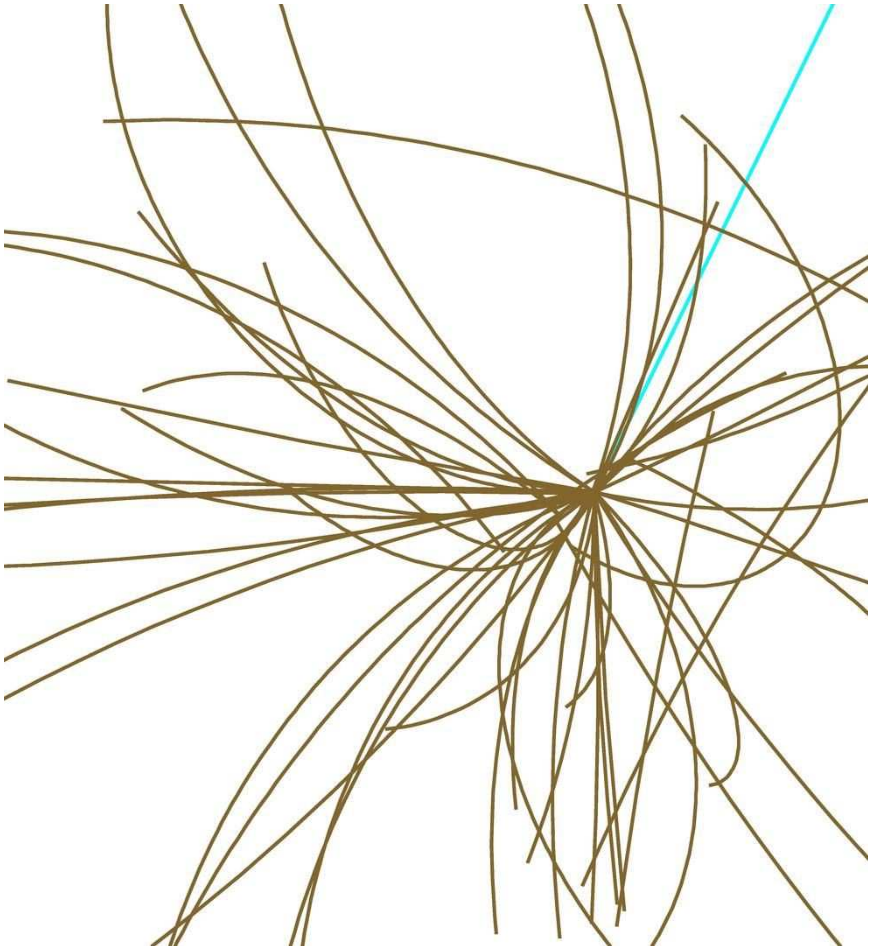

Particle Physics
Solutions to problems

Duncan Carlsmith

0.1 Preface

This supplement to Duncan Carlsmith, "Particle Physics," Pearson (2012) contains solutions to all of the end of chapter problems and is provided to instructors by the publisher upon request. The problem statements will be found in the book. Please send comments and corrections directly to the author at duncan@hep.wisc.edu.

Chapter 1

Introduction

Problem 1.1 Lepton numbers

The electron is (fortunately) stable. The μ^- and τ^- decay only by weak W^- emission since decay through Z, or γ emission leave the original lepton intact and would violate energy and momentum conservation. The W^- can decay to $\mu^-\bar{\nu}_\mu, e^-\bar{\nu}_e$ or quark pairs $q^{-1/3}\bar{q}^{-2/3}$ conserving lepton number. The only decay mode available to the muon is $\mu^- \to \nu_\mu(e^-\bar{\nu}_e)$ where lepton number is conserved. Decays including quarks can not conserve four momentum since there is no lighter hadron. $(m_\mu < m_\pi)$.

Diagrams for μ^- decays appear in Figure 1.1. There are three leading order contributions to $\mu^- \to e^-\bar{\nu}_e\nu_\mu\gamma$ corresponding to radiation of the photon by the μ^-, W^-, or e^-. In each of the three cases, the photon may materialize as an electron positron pair and contribute to $\mu^- \to e^-\bar{\nu}_e\nu_\mu e^+e^-$ as exemplified by Figure 1.1(e). Additional leading order contributions to $\mu^- \to e^-\bar{\nu}_e\nu_\mu e^+e^-$ result from replacing the photon by a Z or H but these are suppressed by the high value of m_Z or m_H.

For each of the allowed decay modes, the initial muon number of the μ^- appears in the final state as a ν_μ. The initial electron number is zero and, in the final state, the e^- and $\bar{\nu}_e$ have opposite electron number. A photon or e^+e^- pair carries no net lepton number. Hence, electron and muon number are both conserved in the allowed decays. In the decays $\mu^- \to e^-\gamma$, $\mu^- \to e^-e^+e^-$, $\mu^- \to e^-\gamma\gamma$, one unit of initial muon number disappears and one unit of net electron number appears in the final state. Hence both muon and electron number are violated. In $\tau^- \to e^-\mu^+\mu^-$ and

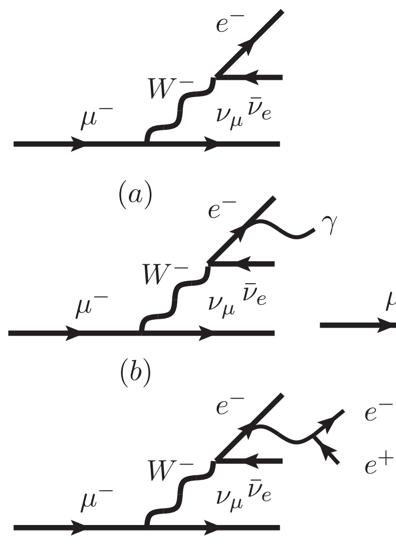

Figure 1.1: Fundamental diagrams for μ^- decays. (a) $\mu^- \to e^-\bar{\nu}_e\nu_\mu$ $(\Gamma_i/\Gamma \simeq 100\%)$, (b)(c)(d) $\mu^- \to e^-\bar{\nu}_e\nu_\mu\gamma$ $(\Gamma_i/\Gamma \simeq 1.4 \pm 0.04\%)$, and (e) $\mu^- \to e^-\bar{\nu}_e\nu_\mu e^+e^-$.

Mode	Fraction
$\mu^-\bar{\nu}_\mu\nu_\tau$	17.36%
$e^-\bar{\nu}_e\nu_\tau$	17.85%
$\pi^-\nu_\tau$	10.9%
$\pi^-\pi^0\nu_\tau$	25.94%
$\pi^-2\pi^0\nu_\tau$	9.3%
$3\pi^0\nu_\tau$	1.18%
$\pi^-3pi^0\nu_\tau$	1.18%
$\pi^-\pi^-\pi^+\nu_\tau$	9.32%
$2\pi^-\pi^0\pi^+\nu_\tau$	4.61%

Table 1.1: Principal decays modes of the τ^-.

$\tau^- \to e^-\pi^+K^-$, one unit of τ^- number disappears and one unit of electron number appears. In $\pi^0 \to \mu^+e^-$, one unit of electron number and negative one unit of muon number appear. The branching fractions from the PDG are $\Gamma_{\mu^-\to e^-\gamma}/\Gamma_\mu < 1.2 \times 10^{-11}$, $\Gamma_{\mu^-\to e^-e^+e^-}/\Gamma_\mu < 1.0 \times 10^{-12}$, $\Gamma_{\tau^-\to e^-\mu^+\mu^-}/\Gamma_\tau < 3.7 \times 10^{-8}$,$\Gamma_{\tau^-\to e^-\pi^+K^-}/\Gamma_\tau < 5.8 \times 10^{-8}$,$\Gamma_{\pi^0\to\mu^\pm e^\mp}/\Gamma_\pi < 3.6 \times 10^{-10}$.

Problem 1.2 The τ^- lepton

The principal decay modes are listed in Table **??**. The τ^- decays derive from the transition $\tau^- \to W^-\nu_\tau$ with the virtual W^- decaying to $\mu^-\bar{\nu}_\mu, e^-\bar{\nu}_e$ or $q^{-1/3}\bar{q^{-2/3}}$ where the quark pairs must be color neutral. The leptonic decays of the W^- account for 35% of the τ^- decay width. The quark pair combinations consistent with kinematics are $\bar{u}d$ and $\bar{u}s$ and $\bar{u}d$ is favored by the CKM matrix. If $\bar{u}d$ bind into the ground state, a single π^- is produced and accompanies the ν_τ. If $\bar{u}d$ bind into an excited state such as the ρ^- which decays strongly to $\pi^-\pi^0$, an additional π^0 is produced. Production of a resonance such as the ω which decays to three pions and non-resonant color processes must be responsible for decays with more than two pions. The single prong fraction is 85%. The three prong fraction is 15%. The five prong fraction is 0.1 %.

Problem 1.3 Quark and baryon number

When a γ, Z, H, or gluon is radiated or absorbed by a fermion, the fermion flavor is unchanged. When a fermion leg is crossed from initial to final state or from final state to initial state, it is interpreted as an antiparticle and its fermion flavor number is reversed so fermion flavor is conserved. The sum of quark fermion numbers is the quark number so this is conserved. Weak interactions of the $W^{\pm}$ bosons induce transitions between lepton pairs and between u-like and d-like quarks so conserve total lepton number (as well as individual lepton numbers) and total quark number. So from a fixed number of quarks and antiquarks in the form of mesons and baryons, standard model interactions can only induce transitions to states with the same quark number. Allowed initial states of mesons and baryons have quark number N_q equal to three times the baryon number of the initial state. Since quark number is conserved, the final state following any standard model interaction must contain N_q quarks which must and can combine to form baryons plus an arbitrary number of $q\bar{q}$ pairs which can combine into mesons or a combination of mesons and baryons. If three of the final additional quarks bind into a baryon, three antiquarks must be left unpaired and must bind into an antibaryon. In any event, net zero additional baryon number appears.

In $\pi^- p$ collisions, the initial charge is zero and the initial baryon number is one. If the final state contains one $\bar{n}$ which has baryon number -1, it must contain two baryons such as pp, pn or nn. To conserve charge, the final states must therefore be $\pi^-\pi^- pp\bar{n}$, $\pi^- pn\bar{n}$, or $nn\bar{n}$. The last case could appear via annihilation of a $\bar{u}$ in the π^- with a u in the p producing n plus a gluon, the gluon fragmenting into $u\bar{u}$, two further gluons producing two $d\bar{d}$ pairs, these three quarks and antiquarks combining to form $n\bar{n}$.

Problem 1.4 Strangeness conservation

Diagrams for these processes appear in Figure 1.2.

Problem 1.5 Photon mass

The Earth's field limit corresponds to

$$m_\gamma = \frac{p}{L} = \frac{0.05 \times 0.2 \text{ GeV fm}}{10^{22} \text{ fm}} = 10^{-15} \text{ eV}.$$

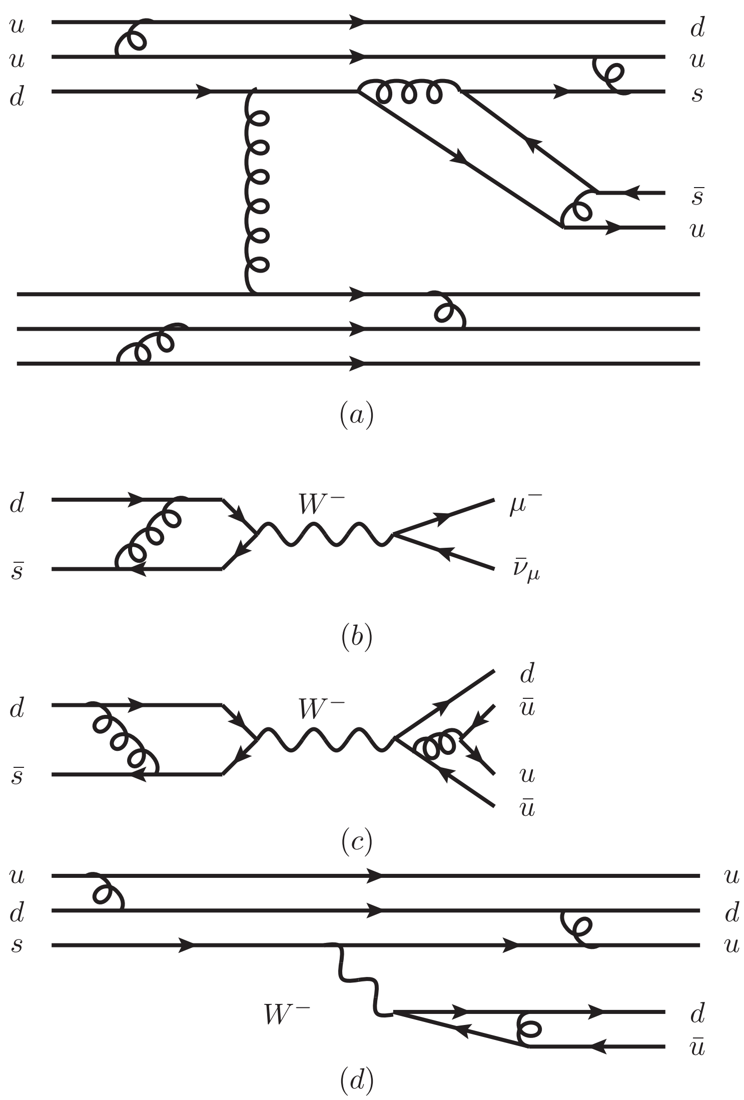

Figure 1.2: Strange quark processes. (a) $\mathrm{p} + \mathrm{Nucleus} \rightarrow \Lambda + \mathrm{K}^+ + \mathrm{Nucleus}$, (b) $K^- \rightarrow \mu^- \bar{\nu}_\mu$, (c) $K^- \rightarrow \pi^- \pi^0$, (d) $\Lambda \rightarrow p\pi^-$.

The Compton wavelength for photon mass $m_\gamma = 10^{-17}$ eV is

$$\lambda_C = \frac{0.2 \text{ GeV fm}}{10^{-26} \text{ GeV}} = 0.2 \times 10^{11} \text{ m} \simeq 0.1 \text{ a.u.}$$

Problem 1.6 Static weak interaction energy

a.) A particle of mass μ confined to a range r has momentum $p \sim r^{-1} = m_Z$ and, if nonrelativistic, has kinetic energy $K = p^2/(2\mu) = (2\mu r^2)^{-1} = m_Z^2/(2\mu)$. The potential energy is of order $U = -g^2/r = -g^2 m_Z$ and $K + U < 0$ gives the result. b.) $\mu = \alpha^{-1} m_Z \simeq$12 TeV. $m_Z = \alpha m_e = 3.7$ keV. c.) Since the weak interaction is short range,

$$\Delta E \simeq |\psi(0)|^2 \int d\mathbf{r}(-g^2 \frac{e^{-m_Z r}}{r}) = \frac{\alpha m_e^3}{\pi}(4\pi) \int_0^\infty dr \quad re^{-m_Z r}.$$

The integral is of the form

$$\int_0^\infty dr \quad re^{ar} = \frac{d}{da} \int_0^\infty e^a = a^{-2}.$$

Since the Rydberg is $E_0 = m_e \alpha^2/2$, we have

$$\Delta E \simeq 4\alpha m_e^3 m_Z^{-2} = 8\alpha(m_e/m_Z)^2 E_0.$$

Problem 1.7 Light hadron decays

Feynman diagrams for pion decay appear in Figure 1.3. The decay of the π^0 results from a second order electromagnetic process. The decay of the π^- results from a second order weak process. The decay of the ρ^+ results from a second order color process. In each case, additional color interactions bind the quarks into hadrons. The lifetime of the π^0 is $\tau_{\pi^0} = 8.4 \times 10^{-17}$ s. The similar process $\eta \to \gamma\gamma$ has a branching fraction of 0.71 and total width of 1.3 keV so partial width of 0.92 keV. The lifetime corresponding to the $\gamma\gamma$ decay is

$$\tau_\eta = \frac{197 \text{ MeV fm}}{(3 \times 10^{23} \text{ fm s}^{-1})(.92 \times 10^{-3} \text{ MeV})} = 0.71 \times 10^{-18} \text{ s}.$$

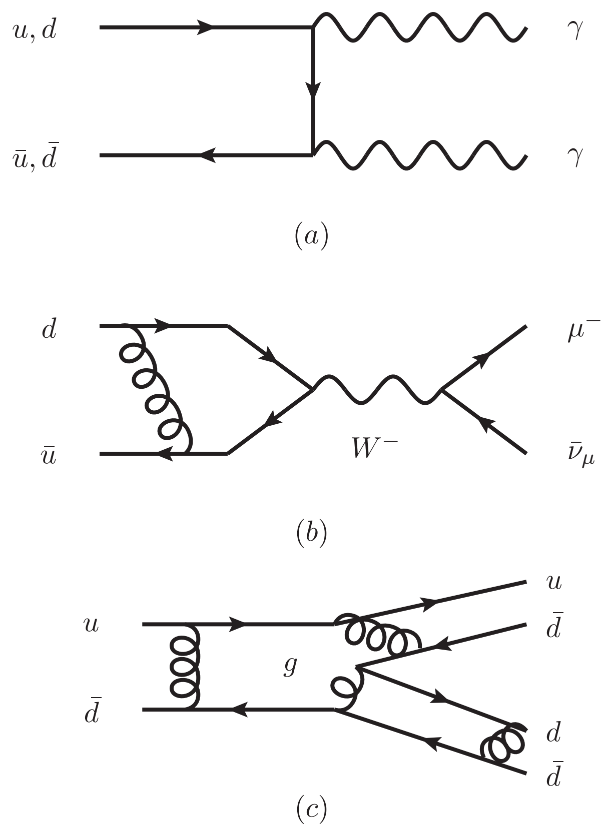

Figure 1.3: Feynman diagrams for (a) $\pi^0 \to \gamma\gamma$, (b) $\pi^- \to \mu^-\bar{\nu}_\mu$, and (c) $\rho^+ \to \pi^+\pi^0$.

The rest masses $m_\pi = 135$ MeV and $m_\eta = 547$ MeV are converted to photon energy and a shorter lifetime is associated with higher energy. The lifetime of the π^- is 2.6×10^{-8} s. The lifetime of the K^- is 1.2×10^{-8} s and the lifetime associated with the decay to $\mu^-\bar{\nu}_\mu$ is 1.9×10^{-8} s. The masses are $m_\pi = 139$ MeV and $m_K = 493$ MeV. The CKM factor in the amplitude for $\pi^- \to \mu^-\bar{\nu}_\mu$ is $V_{ud} \simeq 1$. The CKM factor in the amplitude for $K^- \to \mu^-\bar{\nu}_\mu$ is $V_{su} \simeq 0.22$. These factors appear squared in the decay rate so the K^- lifetime is longer by a factor of 20 than might otherwise be expected. The decay width of the $\rho(770)$ is $\Gamma_\rho = 139$ MeV. The energy release is $m_\rho - 2m_\pi$=496 MeV. The decay width of the $K^+(890)$ is 46 MeV and the energy release is $m_{K^*} - m_K - m_\pi$ = 262 MeV. The smaller energy release implies a lifetime somewhat longer than that associated with the $\rho^- \to \pi^-\pi^0$ decay.

Problem 1.8 CKM matrix and heavy quark decay

The CKM matrix is

$$|V_{\mathrm{CKM}}| \equiv \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} = \begin{pmatrix} 0.97 & 0.22 & \sim 0.003 \\ 0.22 & 0.97 & \sim 0.04 \\ \sim 0.01 & 0.04 & 0.999 \end{pmatrix}$$

We have $m_t > m_b > m_c > m_s > m_u \simeq m_d$. If the CKM matrix were diagonal, t-quarks would decay to b-quarks which would then be stable and c-quarks would decay to s-quarks which would also be stable. Since m_u and m_d are both much smaller than Λ_{QCD}, their masses do not govern their dynamics and both would occur as presently observed. Starting with the t, all other things being equal, the ratio of decay rates

$$\frac{\Gamma_{t\to s}}{\Gamma_{t\to b}} = \frac{|V_{ts}|^2}{|V_{tb}|^2} \simeq 1.6 \times 10^{-3}$$

and $\Gamma_{t\to d}/\Gamma_{t\to b} \simeq 10^{-4}$. Hence the t-quark decays predominantly to a b-quark. In fact, since $m_t >> m_b$ and $m_t >> m_s$, the energy release and phase space factors cancel. The off-diagonal element $V_{cb} \simeq 0.04$ permits the the decay $b \to c$. The ratio

$$\frac{\Gamma_{b\to u}}{\Gamma_{b\to c}} = \frac{|V_{bu}|^2}{|V_{bc}|^2} \simeq 5 \times 10^{-3}$$

characterizes the fraction of b-quark decays proceeding directly to the first generation. Similarly, the c-quark decays predominantly to the s-quark. The s-quark can only decay to a u-quark.

Problem 1.9 Rare decays of t and Z

The diagrams for $t \to bW^+Z$ appear in Figure 1.4a. The process can be thought of as the leading order $t \to W^+b$ decay with an additional Z radiated from any one of the three charged particles. A diagram for $Z \to W^+\pi^-$ appears in Figure 1.4b. The masses are $m_t = 172.0 \pm 0.9 \pm 1.3$ GeV, $m_Z = 91.1876 \pm 0.0021$ GeV, $m_W = 80.4 \pm 0.02$ GeV, $m_b = 4.2 \pm 0.2$ GeV, $m_\pi = 0.139$ GeV. We have $m_Z + m_W$=171.58 GeV which is within one standard deviation of the measured value of m_t. We have $m_Z + m_W + m_b$=175 GeV so if the final state particles are real, the t-quark must be ever so slightly virtual. We have $m_W + m_\pi = 80.53$, some 10 GeV below m_Z so $Z \to W^+\pi^-$ can proceed with all particles real.

Problem 1.10 Fourth generation fermions

Assuming the fourth generation lepton number is conserved, the ω^- would decay exclusively to $\nu_\omega W^-$ with the W^- real and ν_ω would be stable. The decay $t \to bW^+ \to b\nu_\omega\omega^+$ would be allowed with the W^+ virtual. If the 4-dimensional quark CKM matrix were diagonal, then the x would decay exclusively by real W^+ emission to y which would be stable. The energy release would be $m_x - m_y - m_W = 170$ GeV. The decay rate scaled from t-decay would be naively

$$\Gamma_{x \to yW^+} \simeq \frac{\rho((m_x - m_y - m_W)}{\rho(m_t - m_W - m_b)} \Gamma_{t \to bW^+}$$

where ρ is the phase space factor. Assuming small off-diagonal matrix elements to the nearest neighbor generation, the x could decay to b by real W^+ emission with energy release $m_x - m_b - m_W \simeq 215$ GeV. The decay rate would be naively

$$\Gamma_{x \to bW^-} \simeq |V_{xb}|^2 \frac{\rho(m_x - m_b - m_W)}{\rho(m_t - m_W - m_b)} \Gamma_{t \to bW^+}.$$

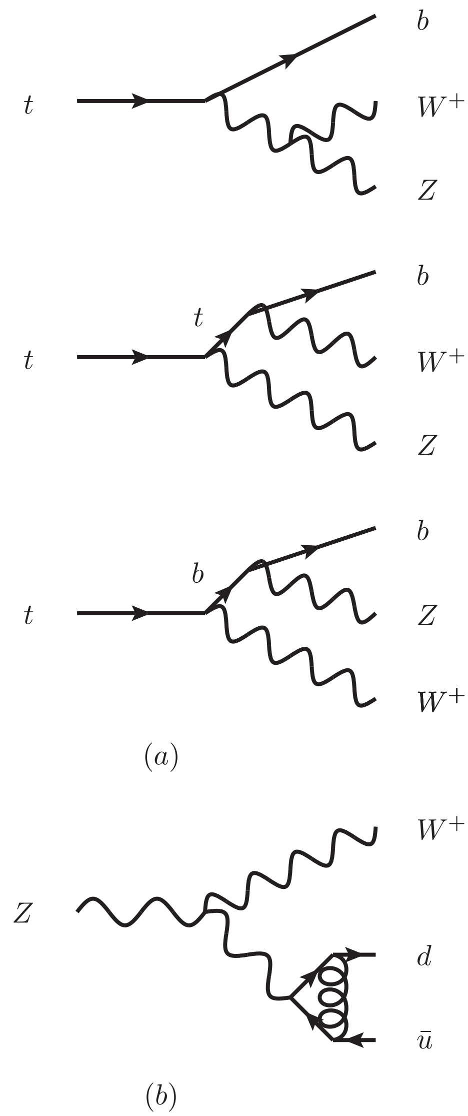

Figure 1.4: Feynman diagrams for a) $t \to W^+bZ$ and b) $Z \to W^+\pi^-$.

Neglecting the phase space difference, if $V_{xb} \simeq 0.1$, the branching fraction for this decay would be of order 1%. The nearest generation decay $y \to tW^-$ by real W^- emission is disallowed. The y can decay to virtual W and virtual t with rate suppressed by a factor $|V_{yt}|^2$. The decay $y \to cW^-$ to a real c and virtual W^- spans two generations and the decay rate is proportional to a much smaller factor $|V_{yc}|^2$. In both cases, the virtuality of the intermediate states as well as the CKM factors implies the y is relatively stable compared to the t-quark.

Problem 1.11 Exotic atoms

a) $(p\bar{p})_{\rm atom}$: $\mu = m_p/2 = 469$ MeV, $a_0 = 137(197$ MeV fm$)$ $/(\mu$ / 1 MeV$)$ = 57 fm, $E_0 = (1/2)\mu\alpha^2 = 12.5$ keV. $(\mu^+e^-)_{\rm atom}$: $\mu \simeq m_e = 0.511$ MeV, a_0 = 137(197 MeV fm) $/(\mu$ / 1 MeV$)$ = 54,000 fm, $E_0 = (1/2)\mu\alpha^2 = 13.6\times10^{-3}$ keV.
$(\pi^+e^-)_{\rm atom}$: $\mu \simeq m_e = 0.511$ MeV, $a_0 = 137(197$ MeV fm$)$ $/(\mu$ / 1 MeV$)$ = 54,000 fm, $E_0 = (1/2)\mu\alpha^2 = 13.6\times10^{-3}$ keV.
$(\pi^-p)_{\rm atom}$: $\mu \simeq m_\pi m_p/(m_\pi + m_p = 121$ MeV, $a_0 = 137(197$ MeV fm$)$ $/(\mu$ / 1 MeV$)$ = 222 fm, $E_0 = (1/2)\mu\alpha^2 = 3.22$ keV.
$(K^-p)_{\rm atom}$: $\mu \simeq m_K m_p/(m_K + m_p = 323$ MeV, $a_0 = 137(197$ MeV fm$)$ $/(\mu$ / 1 MeV$)$ = 84 fm, $E_0 = (1/2)\mu\alpha^2 = 8.60$ keV.
$(\pi^-\mu^+)_{\rm atom}$: $\mu \simeq m_\pi m_\mu/(m_\pi + m_\mu = 59.9$ MeV, $a_0 = 137(197$ MeV fm$)$ $/(\mu$ / 1 MeV$)$ = 450 fm, $E_0 = (1/2)\mu\alpha^2 = 1.60$ keV.
b) $E_\gamma =\simeq m_e$ and $E_0 = (1/4)m_e\alpha^2$ so $E_\gamma/E_0 = 4\alpha^{-2}$. $\lambda = m_e^{-1}$ and $a_0 = (\alpha m_e/2)^{-1}$ so $\lambda/a_0 = \alpha/2$.
c) We have from a), $a_0((\pi^-p)_{\rm atom}) = 222$ fm $\simeq (\alpha m_\pi)^{-1}$. The Bohr radius is $r_n = a_0 n^2$ so $a_0((\pi^-p)_{\rm atom})n^2 = a_0(H)$ implies $n = \sqrt{m_\pi/m_e} \simeq 16$. The x-ray energies are $E_n = E_0(1 - n^{-2})$ with $E_0 = 3..22$ keV.
d) $\tau = (197 \times 10^6)(\text{eV fm})(1\text{ eV})^{-1}(3 \times 10^{23}\text{ fm s}^{-1})^{-1} = 6 \times 10^{-16}$ s, much shorter than the weak decay lifetime.

Problem 1.12 Muon catalyzed fusion

The μ^-dd ion is analogous to H_2^+. The μ^- bond length is $m_e/m_\mu = 0.0048$ times the Bohr radius of hydrogen forming a nucleus of charge $+e$ and mass $2m_d + m_\mu$ about which the electron orbits in the neutral molecule. The electronic excitation spectrum is similar to that of hydrogen. As an energy source, the number n of catalyzed fusion reactions in a muon lifetime must at least exceed the energy required to manufacture the muon. For $Q = 17.6$ MeV, $m_\mu = 105$ MeV, $n = m_\mu/Q = 6$.

Problem 1.13 Klein-Gordon equation for string

Newton's second law applied to an element of length dz reads

$$\lambda dz\ddot{y} = -\kappa dz y + T\frac{\partial^2 y}{\partial z^2}dz$$

where the net transverse component of the force of tension is $T[\theta(z+dz) - \theta(z)]$ with $\theta \simeq dy/dz$ in the small angle approximation. This can be written as a one dimensional Klein-Gordon equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{T}{\lambda}\frac{\partial^2 y}{\partial z^2} + \frac{\kappa}{\lambda}y = 0$$

and we can identify the wave speed as $c = \sqrt{T/\lambda}$. The static homogeneous equation

$$\frac{d^2 y}{dz^2} = L^{-2}y$$

with $L = \sqrt{T/\kappa}$ has the general solution

$$y = A\sinh(y/L) + B\cosh(y/L) = Ce^{y/L} + De^{-y/L}$$

where A, B, C, and D are constants. The bounded solution for a fixed displacement $y_0(z=0)$ is $y(z) = y_0 e^{-y/L}$ for $z > 0$ and $y(z) = y_0 e^{+z/L}$ for $z < 0$. A harmonic free solution $y(z,t) = y_0 e^{-i(\omega t - kz)}$ requires $-\omega^2 + c^2k^2 + \frac{\kappa}{\lambda}$ so, for fixed frequency,

$$k = \sqrt{\omega^2 - \frac{\kappa}{\lambda}}.$$

For $\omega > \sqrt{\kappa/\lambda}$, the wave vector is real.

Problem 1.14 Yukawa potential

Assuming the spherically symmetric form $\psi = f(r)/r$,

$$\nabla^2\phi = (1/r)\partial_r^2(r\phi) = f''/r \Rightarrow -f'' + m^2 f = 4\pi g\rho\ r$$

Solutions to the homogeneous equation are linear combinations of $e^{\pm mr}$. A particular solution for ρ constant is $4\pi g\rho r/m^2$. Outside a source, the solution subject to the boundary condition $f(\infty) = 0$ is $f(R < r) = ae^{-mr}$. For a point source with m=0, we expect $\phi = g/r$ so take $a = g$. To see that ge^{-mr}/r works at the origin, integrate the equation

$$-\nabla^2\phi + m^2\phi = 4\pi g\delta^3(\mathbf{x})$$

over a small sphere of radius r and use the divergence theorem

$$\int -\bigtriangledown^2 \phi dV = -\int \bigtriangledown\phi \cdot d\mathbf{s} = -\int (\frac{f'}{r} - \frac{f}{r^2})r^2 d\Omega = -4\pi r f' + 4\pi f(r).$$

Since f is smooth near the origin,

$$\int m^2\phi dV = m^2 4\pi \int drrf(r) \simeq 4\pi m^2 f(0)r^2/2$$

and

$$\int 4\pi g\delta(\mathbf{x})dV = 4\pi g.$$

In the limit of small r, we have the equality $4\pi g = 4\pi g$.

For a finite radius constant density source, the general interior solution is

$$f(r < R) = b\cosh(mr) + c\sinh(mr) + 4\pi g\rho r/m^2$$

and $b = 0$ for a finite ϕ at $r = 0$. At $r = R$, continuity of ϕ amounts to continuity of f and continuity of $\phi' = f'/r - f/r^2$ amounts to continuity f'. THe conditions of continuity are

$$c\sinh(mR) + 4\pi g\rho R/m^2 = ae^{-mR}$$

$$cm\cosh(mR) + 4\pi g\rho/m^2 = -mae^{-mR}$$

or with $x = mR$ and $k = 4\pi g\rho/m^3$,

$$c\sinh(x) + kx = ae^{-x}$$

$$c\cosh(x) + k = -ae^{-x}$$

from which

$$c = -\frac{k(1 + x)}{\sinh x + \cosh x}$$

and then

$$a = e^x(c \sinh x + kx).$$

The result may also be found by integration:

$$\psi(\mathbf{x}) = \int d^3\mathbf{y}\rho(\mathbf{y})g\frac{e^{-m|\mathbf{x}-\mathbf{y}|}}{|\ \mathbf{x}-\mathbf{y}\ |} = g\int y^2 dy d\phi d\cos\theta \frac{e^{-mr}}{r}$$

where $r^2 = x^2 + y^2 - 2xy\cos\theta$ and x is fixed in the integration. We can integrate first over rings within shells of fixed y with $2rdr = -2xyd\cos\theta$ and then over shell radii y to construct the field of a shell of finite thickness:

$$\begin{aligned}
\phi &= g\rho\int_{r_1}^{r_2} y^2 dy \int_0^{2\pi} d\phi \int [d\cos\theta = -\frac{rdr}{xy}]\frac{e^{-mr}}{r} \\
&= 4\pi\frac{g\rho}{2x}\int_{r_1}^{r_2} ydy\int_{r_{min}}^{r_{max}} dre^{-mr} \\
&= 4\pi\frac{g\rho}{2mx}\int_{r_1}^{r_2} ydy[e^{-mr_{max}} - e^{-mr_{min}}]
\end{aligned}$$

where $r_1 = 0$ and $r_2 = R$ are the minimum and maximum radius of a spherical shell of source and r_{min} and r_{max} are the minimum and maximum radii of infinitesimal shells for fixed x and y. Outside the source, we have $x > r_2$, $r_{min} = x - y$, $r_{max} = x + y$ so

$$\begin{aligned}
\phi &= 4\pi\frac{g\rho}{2mx}e^{-mx}\int_{r_1}^{r_2} dy\ y(e^{my} - e^{-my}) \\
&= 4\pi\frac{g\rho}{2mx}e^{-mx}\frac{d}{dm}\int_{r_1}^{r_2} dy 2\cosh(my) \\
&= 4\pi\frac{g\rho}{mx}e^{-mx}\ \frac{d}{dm}[\frac{\sinh(mr_2)}{m} - \frac{\sinh(mr_1)}{m}] \\
&= 4\pi\frac{g\rho}{m^3x}e^{-mx}[-\sinh(mr_2) + \sinh(mr_1) \\
&+ \ mr_2\cosh(mr_2) - mr_1 cosh(mr_1)].
\end{aligned}$$

Taking $r_1 = 0$ and $r_2 = R$, we obtain the potential outside a sphere. Using the Taylor series expansions for the hyperbolic functions, we can find that the exterior potential for $m = 0$ reduces to

$$\phi(x > R) \Rightarrow g\rho\frac{4\pi}{3}R^3\frac{1}{x}$$

as expected for a positive "charge" $g\rho V_{sphere}$ while for $m \Rightarrow \infty$ the potential vanishes outside since the force has infinitesimal range.

Interior to the shell, we would have $x < r_1$, $r_{min} = y - x$, $r_{max} = y + x$ and we find

$$\psi = 4\pi \frac{g\rho}{m^3 x} \sinh(mx)[(1 + mr_1)e^{-mr_1} - (1 + mr_2)e^{-mr_2}]$$

The potential at a point inside the source is the exterior potential at the outer surface of a shell of inner radius r_1 and outer radius x plus the interior potential at the inner surface of a shell of inner radius x and outer radius r_2. Taking a solid sphere, this prescription yields the interior result. By Taylor series expansion, for $m = 0$ we can find the expected harmonic interior potential while for $m \Rightarrow \infty$ the interior potential reduces to a constant value.

Problem 1.15 Pauli matrices

We have

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \; , \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \; , \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and these are easily seen to satisfy $\sigma_i^\dagger = (\sigma_i^*)^T = \sigma_i$ and $\mathrm{tr}\sigma_i = 0$. Also for example

$$\sigma_1\sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3 = i\epsilon_{123}\sigma_3.$$

Similar direct multiplications demonstrate the validity of $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$.

We have

$$\mathbf{a} \cdot \boldsymbol{\sigma} \mathbf{b} \cdot \boldsymbol{\sigma} = a_i b_j \sigma_i \sigma_j = a_i b_j (\delta_{ij} + i\epsilon_{ijk}\sigma_k) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot \mathbf{a} \times \mathbf{b}.$$

Hence $\mathbf{a} \cdot \boldsymbol{\sigma} \mathbf{a} \cdot \boldsymbol{\sigma} = \mathbf{a}^2$.

The exponential of a matrix can be defined by the series

$$e^{i\mathbf{a}\cdot\boldsymbol{\sigma}} = 1 + i\mathbf{a} \cdot \boldsymbol{\sigma} + \frac{1}{2!}(i\mathbf{a} \cdot \boldsymbol{\sigma})^2 + \frac{1}{3!}(i\mathbf{a} \cdot \boldsymbol{\sigma})^3 + \ldots$$

Since $(i\mathbf{a} \cdot \boldsymbol{\sigma})^2 = -\mathbf{a}^2$, collecting real and imaginary terms we have

$$\mathrm{Re}[e^{i\mathbf{a}\cdot\boldsymbol{\sigma}}] = 1 - \frac{1}{2!}(\mathbf{a})^2 + \ldots = cos|\mathbf{a}|$$

and

$$\mathrm{Im}[e^{i\mathbf{a}\cdot\boldsymbol{\sigma}}] = \mathbf{a}\cdot\boldsymbol{\sigma} - \frac{\mathbf{a}^2}{3!}(\mathbf{a}\cdot\boldsymbol{\sigma} + \ldots = i\mathbf{a}\cdot\boldsymbol{\sigma}\sin|\mathbf{a}|.$$

Problem 1.16 Spin precession

We have $\dot{\phi} = -i\omega\mathbf{n}\cdot\boldsymbol{\sigma}\phi$ so, if $\mathbf{m}$ is some constant vector, using $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, we find

$$\boldsymbol{\sigma}\cdot\mathbf{n}\boldsymbol{\sigma}\cdot\mathbf{m} - \boldsymbol{\sigma}\cdot\mathbf{m}\boldsymbol{\sigma}\cdot\mathbf{n} = +2i\mathbf{n}\times\mathbf{m}\cdot\boldsymbol{\sigma}$$

and

$$\begin{aligned} \frac{d}{dt} < \mathbf{s} > \cdot\mathbf{m} &= \dot{\phi}^\dagger\mathbf{s}\cdot\mathbf{m}\phi + \phi^\dagger\mathbf{s}\cdot\mathbf{m}\dot{\phi} \\ &= (-i\omega\boldsymbol{\sigma}\cdot\mathbf{n}\phi)^\dagger\frac{\boldsymbol{\sigma}}{2}\cdot\mathbf{m}\phi + \phi^\dagger\frac{\boldsymbol{\sigma}}{2}\cdot\mathbf{m}(-i\omega\boldsymbol{\sigma}\cdot\mathbf{n})\phi \\ &= i\frac{\omega}{2}\phi^\dagger[\boldsymbol{\sigma}\cdot\mathbf{n}\boldsymbol{\sigma}\cdot\mathbf{m} - \boldsymbol{\sigma}\cdot\mathbf{m}\boldsymbol{\sigma}\cdot\mathbf{n}]\phi \\ &= -\omega\phi^\dagger[\mathbf{n}\times\mathbf{m}\cdot\boldsymbol{\sigma}]\phi \\ &= \omega\phi^\dagger[\mathbf{n}\times\boldsymbol{\sigma}\cdot\mathbf{m}]\phi = 2\omega\mathbf{n}\times < \mathbf{s} > \cdot\mathbf{m}. \end{aligned}$$

For B= 1 T, with $g \simeq 2$ and $\alpha = 1$, the frequency is

$$\omega = \frac{2\mu_B B}{\hbar} = 2\cdot\frac{(5.79\times10^{-5}\ \mathrm{eV\ T^{-1}})(1\ \mathrm{T})}{6.58\times10^{-16}\ \mathrm{eV\ s}} = 1.76\times10^{11}\ \mathrm{s}^{-1}.$$

Problem 1.17 Planck scales

The numerical values follow with $G = 6.674\times10^{-11}$ m^3 kg^{-1} s^{-2}, $c = 2.997\times10^8$m s^{-1}, $\hbar = 1.054\times10^{-34}$ J s, and $k_B = 1.38\times10^{-23}$ J K^{-1}. The ratio of water mass density to the Planck mass density $\rho_{H_2O}/\rho_P = 5.1\times10^{93}$.

Problem 1.18 Magnetars

a) The magnetic mass density is

$$\frac{B^2}{2\mu_0 c^2} = \frac{10^{20}\ \mathrm{T}^2}{8\pi\times10^{-7}\ \mathrm{N\ A^{-2}}\times9\times10^{16}\ \mathrm{m\ s^{-2}}} = 4.4\times10^8\mathrm{kg\ m^{-3}}$$

while the density of lead is 1.1×10^4 kg m^{-3}. b) We have

$$2\mu_B B_c = \frac{e\hbar}{m_e} B = m_e c^2 \to B_c = \frac{m_e^2 c^2}{\hbar e} = 4.4 \times 10^9 \text{ T}.$$

c) Since $p = eBr = \hbar/r$, we have

$$r^2 = \frac{\hbar}{eB_c}\frac{B_c}{B} = \frac{\hbar^2}{m_e c}{}^2 \frac{B_c}{B}.$$

In terms of the Compton wavelength $\lambda_e = \hbar^2/(m_e c)$ and classical radius $r_e = \alpha/(m_e c^2)$,

$$r = \lambda_e \sqrt{B_c/B} = \alpha^{-1} r_e \sqrt{B_c/B}.$$

d) The field strength at which the gyration radius equals the Bohr radius is given by

$$a_0 = \alpha^{-2} r_e = \alpha^{-1} r_e \sqrt{B_c/B} \to B = \alpha^2 B_c.$$

Chapter 2

Relativity, accelerators, and collisions

Problem 2.1 Velocity addition

a) The ratio of the Lorentz transformation expressions is

$$u' = \frac{dx'}{dt'} = \frac{\frac{dx}{dt} - v}{1 - v\frac{dx}{dt}} = \frac{u - v}{1 - uv}.$$

b) If the velocity of one proton is $u = -u_p$ in the laboratory, its velocity as seen from a frame moving with velocity $v = +u_p$ is

$$u' = \frac{(-u_p) - (+u_p)}{1 - (-u_p)(+u_p)} = \frac{-2u_p}{1 + u_p^2}$$

which approaches -1 when $u_p \to 1$. The γ-factor of the second proton in the rest frame of the first is

$$\gamma' = (1 - u'^2)^{-1/2} = (1 + u_p^2)(1 - u_p^2)^{-1} = \gamma^2(1 + u_p^2)$$

where $\gamma = (1 - u_p^2)^{-1/2}$ so the energy is $E' = \gamma' m$. Applying the Lorentz transformation to the 4-momentum $p = (\gamma m, -\gamma m u_p)$, we have the same result

$$E' = \gamma(E + u_p p) = \gamma^2 m(1 + u_p^2) = \frac{1 + u_p^2}{1 - u_p^2}.$$

c) We have

$$d\mathbf{x}' = d\mathbf{x} - \gamma dt\mathbf{v} + \frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\mathbf{dx})\mathbf{v},\ dt' = \gamma dt - \gamma(\mathbf{v}\cdot\mathbf{dx})$$

so, dividing and writing $\mathbf{u} = d\mathbf{x}/dt$ and $\mathbf{u}' = d\mathbf{x}'/dt'$, we find

$$\mathbf{u}' = \frac{1}{\gamma(1+\mathbf{u}\cdot\mathbf{v})}[\mathbf{u} - \gamma\mathbf{v} + \frac{\gamma^2}{\gamma+1}(\mathbf{v}\cdot\mathbf{u})\mathbf{v}].$$

Problem 2.2 Relative velocity

In the rest frame of the first particle, $v_1^0 = 1$ and $\mathbf{v}_1 = 0$ while $v_2 = \gamma(v)(1,\mathbf{v})$ so $v_1 v_2 = \gamma(v)$. Hence

$$(1-v^2)^{-1/2} = v_1 v_2 \to v^2 = 1 - (v_1 v_2)^{-2}$$

and, since $v_1 = p_1/m_1$ and $v_2 = p_2/m_2$, the result follows.

Problem 2.3 Decay momentum

$$\begin{aligned} p^2 &= E_{2,cm}^2 - m_2^2 = \frac{1}{4s}[s + m_2^2 - m_1^2]^2 - 4sm_2^2] \\ &= \frac{1}{4s}[s^4 + m_2^4 + m_1^4 + 2sm_2^2 - 2sm_1^2 - 2m_1^2m_2^2 - 4sm_2^2] \\ &= \frac{1}{4s}[s^4 + m_2^4 + m_1^4 - 2sm_2^2 - 2sm_1^2 - 2m_1^2m_2^2]. \end{aligned}$$

The masses are m_{K^+}= 493.667 MeV, m_{π^+}=139.570, m_{π^0} = 134.976, m_μ = 105.658, and $p(K^+ \to \mu^+\nu_\mu) = 236$ MeV while $p(K^+ \to \pi^+\pi^0) = 205$ MeV.

Problem 2.4 Three-body decay kinematics

The energy E_1 is maximal when m_2 is minimized. The pair mass is minimal when the total kinetic energy in their center of mass vanishes which

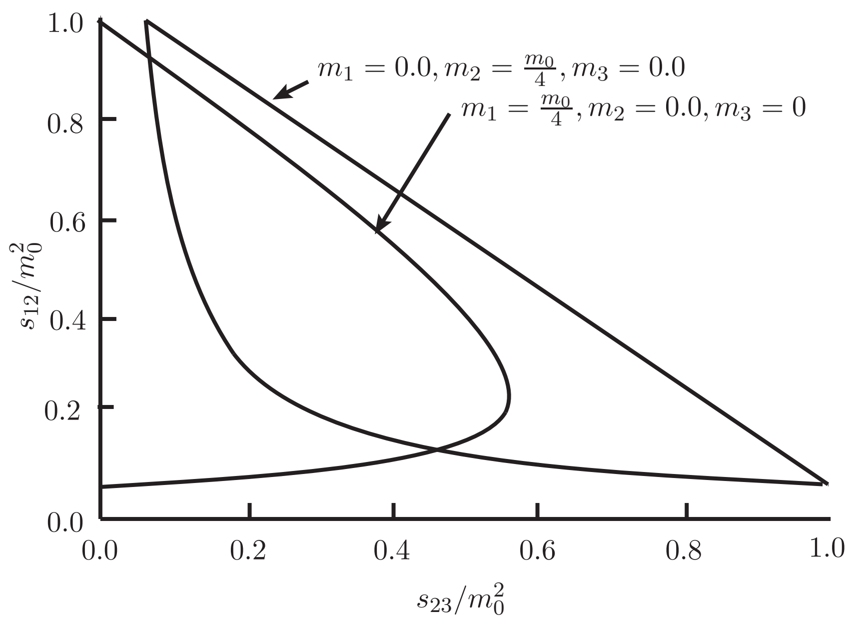

Figure 2.1: Boundaries in the Dalitz plot for the decay of a mass m_0 to masses m_1, m_2 and m_3 for the cases a) $m_1 = 0$, $m_2 = m_0/4$, $m_3 = 0$ and b) $m_1 = m_0/4$, $m_2 = 0$, $m_3 = 0$ in which two of the particles are massless.

implies they are moving with the same velocity. The pair mass in this case computed in their rest frame is $2m_\pi$ and the momentum corresponding to this pair mass is

$$p = \frac{1}{2m_K}[(m_K^2 - (3m_\pi)^2)(m_K^2 - m_\pi^2)]^{1/2}.$$

Using $m_K = 0.4976$ GeV, $m_\pi = 0.13498$ GeV, we have p=0.139 GeV.

Problem 2.5 Dalitz plot boundaries

See Figure 2.1.

Problem 2.6 Neutron Decay kinematics

a) The neutron total energy is $E_n = m_n + K$=1.43956 GeV so we can compute $\gamma = E_n/m_n = 1 + K/m_n$=1.532161 and then the velocity $v =$

$c\sqrt{1-1/\gamma^2}$=0.757640c = 2.2729196×10^8 m/s. b) Contracted by a factor γ so 0.653 fm. c) Energy conservation in the neutron rest frame gives $m_n = E_e + E_{\bar{\nu}} + m_p$ from which $E_e + E_{\bar{\nu}} = m_n - m_p$=1.29 MeV. d) The invariant mass of the electron plus neutrino is the energy in the frame in which the pair has zero momentum so $m_{e\bar{\nu}} \equiv m_n - m_p$. The energy in the laboratory frame is $E_{e\bar{\nu}} = \gamma m_{e\bar{\nu}} = \gamma(m_n - m_p)$=1.98 MeV. The proton kinetic energy is

$$K_p = E_p - m_p = (\gamma - 1)m_p = 0.4993.$$

Problem 2.7 Neutral pion mass

Neglecting the binding energy $E_{\rm binding} \simeq m_\pi/m_e \times$ 13.6 eV = 3.7 keV, the invariant mass of the pionic atom is $m_0 = m_\pi + m_p$ = 1077.8 MeV. For a two body decay $0 \Rightarrow 1 + 2$, the energy of particle 1 in the center of mass is

$$E_1 = \frac{{m_0}^2 + {m_1}^2 - {m_2}^2}{2m_0}$$

which gives E_{π^0} = 137.8 MeV and E_γ = 129.3 MeV. The laboratory is the center of mass so these particles are monoenergetic.

The π^0 is moving relative to the lab and the energies of the photons from $\pi^0 \to \gamma\gamma$ depend on the decay angle relative to the pion momentum vector. The lab energy is related to the rest frame energy, momentum, and decay angle relative to the π^0 direction by a Lorentz transformation:

$$E_L = \gamma(E \ + \ v \mid \mathbf{p} \mid \cos(\theta))$$

The minimum and maximum values are

$$E_L = \gamma(E \ \pm \ v \mid \mathbf{p} \mid).$$

The energy distribution is

$$\frac{dN}{dE_L} = \frac{dN}{d\cos(\theta)}\frac{d\cos(\theta)}{dE_L}$$

Since the lab energy is linear in $\cos(\theta)$, the energy distribution is flat for any two body decay which is isotropic in the rest frame regardless of the speed of the parent. For $\pi^0 \Rightarrow \gamma\gamma$, $E_\gamma =\mid \mathbf{p} \mid$, and the limiting energies are

$$E_\gamma^{lab} = \gamma(1 \pm v)\frac{M_\pi}{2}$$

where $\gamma = E_\pi/m_\pi = 1.021$. The corresponding speed is v=0.20 from which the photon energies are found to range between 55 and 83 MeV.

Problem 2.8 Energy transfer to an electron

The center of mass velocity is $v = p^{tot}/E^{tot} = p/(E + m_e)$ and $\gamma = (E + m_e)/\sqrt{(s)}$ with $s = m^2 + m_e^2 + 2m_eE$. The electron has energy $E' = \gamma m_e$ and momentum along the beam direction $-\gamma m_e v$ and can recoil in the center of mass frame with the same energy and reversed momentum. The momentum observed in the lab frame is then

$$E_{lab} = \gamma(E' + vp') = \gamma(\gamma m_e + v^2\gamma m_e) = m_e\frac{1 + v^2}{1 - v^2}.$$

The energy transfer is

$$\begin{aligned}\Delta E &= E_{lab} - m_e = m_e\frac{1 + v^2}{1 - v^2} - 1 \\ &= 2m_ev^2/(1 - v^2) = 2m_2v^2\gamma^2 \\ &= 2m_e(\frac{p}{E + m_e})^2(\frac{E + m_e}{\sqrt{s}})^2 = 2m_e\frac{p^2}{s}.\end{aligned}$$

For $m >> m_e$,we can neglect the electron mass squared in the denominator. Put $\gamma = E/m$. Then for $\gamma << m/m_e$, we find

$$\Delta E \approx 2m_e(p/m)^2.$$

For $\gamma \approx m/m_e$ we have

$$\Delta E \approx 2m_e(E/m)^2$$

while in the high energy limit $\gamma >> m/m_e$ we have

$$\Delta E \approx E$$

with $2m_e \approx 1$ MeV. The maximum energy transfer is about 0.113 GeV, 10.4 GeV, and 537 GeV for the cases listed. Expressed in terms of E, the fractional energy transfer is

$$\Delta E/E = \frac{1 - (\frac{m}{E})^2}{1 - \frac{1}{2}\frac{m}{E}\frac{m}{m_e} + \frac{1}{2}\frac{m_e}{2E}}$$

and approaches unity as $E \to \infty$.

Problem 2.9 Relativistic acceleration

For a constant force, we can integrate the relativistic form of Newton's Second Law of Motion and find (with $a = F/m$)

$$\begin{aligned} \frac{dp}{dt} &= F, \\ p &= \frac{mv}{\sqrt{1-v^2}} = Ft, \\ v(t) &= \frac{at}{\sqrt{1+(at)^2}} = \frac{dx}{dt}, \\ x(t) &= \frac{1}{a}(\sqrt{1+(at)^2} - 1). \end{aligned}$$

Solving for t in terms of displacement, we find $t = \sqrt{2(x/a) + (x/c)^2}$ with $a = F/m$. For small x and t, we have the nonrelativistic results $v = at$ and $x = (1/2)at^2$. For large x and t, we have $v = 1$.

Problem 2.10 Photon acceleration

The total 4-momentum is $p = p_\gamma + p_e$ so the invariant mass squared is

$$\begin{aligned} s = p^2 &= p_\gamma^2 + p_e^2 + 2p_\gamma \cdot p_e = m_\gamma^2 + m_e^2 + 2(E_\gamma E_e - \mathbf{p}_e \cdot \mathbf{p}_\gamma) \\ &= m_e^2 + 2E_\gamma(E_e + \mid \mathbf{p}_e \mid) \simeq m_e^2 + 4E_\gamma E_e = 0.461 \text{ MeV}^2. \end{aligned}$$

A system with energy E and mass m has $\gamma = E/m$ so the center of mass frame motion is characterized by

$$\gamma = \frac{E_\gamma + E_e}{\sqrt{s}} \simeq .74 \times 10^5 \; ; \; v = \sqrt{1 - \frac{1}{\gamma^2}} \simeq 1.$$

The photon energy in the center of mass is given by

$$E_\gamma^{cm} = \gamma E_\gamma(1+v) = \frac{s - m_e^2}{2\sqrt{s}} = 0.147 \text{ MeV}.$$

For elastic collisions, this expression applies both before and after the collision. Boost the reversed photon back to the lab frame to find

$$E_\gamma^{lab} = \gamma(E_\gamma^{cm} + vp_\gamma^{cm}) = \gamma E_\gamma^{cm}(1+v) = E_\gamma \gamma^2(1+v)$$

or $E_\gamma \simeq 0.74 \times 10^5 \times .147$ MeV $\times 2 = 22$ GeV. So a 50 GeV electron is (half) stopped by a visible photon.

Problem 2.11 Synchrotron energy loss

Since **E**=0, and **v** is perpendicular to **B**, the acceleration in the instantaneous frame is $|\mathbf{a}| = |e\mathbf{E}'/m_e| = evB/m_e$. The Lamor formula gives

$$\frac{dE}{dt} = \frac{e^4\gamma^2 B^2 v^2}{6\pi\epsilon_0 c^3 m_e^2}$$

and, since $B = p/(eR) = \gamma v m_e/(eR)$, we have

$$\frac{dE}{dt} = \frac{e^2\gamma^4 R^{-2} v^4}{6\pi\epsilon_0 c^3} = \frac{2}{3}\frac{e^2}{4\pi\epsilon_0 \hbar c}(\frac{\gamma v}{c})^4 \hbar c^2.$$

Problem 2.12 Electron cooling

The γ factors must be equal so

$$E_e/m_e = E_{\bar{p}}/m_p \to E_e = E_{\bar{p}}/1837 = 4.3 \text{ MeV}.$$

[Ref: FERMILAB-CONF-06-317-AD.]

Problem 2.13 Van der Graaf accelerator

a) The electric field and potential difference are

$$E = kQ_{term}r^{-2} \Rightarrow \Delta V = kQ_{term}[r_{term}^{-1} - r_{vessel}^{-1}]$$

and the numerical values are

$$E = \frac{\Delta V}{r_{term}[1 - r_{term}/r_{vessel}]} = \frac{(25\ MV)}{(2.5\ m)[1/2]} = 20 \text{ MV m}^{-1}.$$

b) $E = pV_{min}/x_{min} = (760\text{ torr})(327\text{ V})/(0.567\ torr\text{ cm}) = 43.8\text{ MV m}^{-1}$.

Problem 2.14 Betatron

By symmetry, the electric field is azimuthal while the magnetic field has no azimuthal component. In cylindrical coordinates, we have

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\phi}\mathbf{e}_\phi + \dot{z}\mathbf{e}_z,$$

$$\dot{\mathbf{v}} = (\ddot{r} - r(\dot{\phi})^2)\dot{\mathbf{e}} + (r\ddot{\phi} + 2\dot{r}\dot{\phi})\mathbf{e}_\phi + \ddot{z}\mathbf{e}_z,$$

$$\mathbf{E} = E_\phi\mathbf{e}_\phi + E_z\mathbf{e}_z\ ;\ \mathbf{B} = B_r\mathbf{e}_r + B_z\mathbf{e}_z.$$

The equations of motion are

$$\frac{d\mathbf{p}}{dt} = \dot{E}\mathbf{v} + E\dot{\mathbf{v}} = e\mathbf{E} + e\mathbf{v}\times\mathbf{B}$$

$$\frac{dE}{dt} \equiv \dot{E} = e\mathbf{v}\cdot\mathbf{E}$$

The component equations are

$$\begin{aligned} \dot{E}\dot{r} + E(\ddot{r} - r(\dot{\phi})^2) &= er\dot{\phi}B_z, \\ \dot{E}r\dot{\phi} + E(r\ddot{\phi} + 2\dot{r}\dot{\phi}) &= eE_\phi + e(\dot{z}B_r - \dot{r}B_z) \\ \dot{E}\dot{z} + E\ddot{z} &= -er\dot{\phi}B_r. \end{aligned}$$

We can have a solution r = constant, z = constant in a symmetry plane where B_r=0. The equations of motion reduce to

$$-Er\dot{\phi} = erB_z\ ;\ \frac{d}{dt}Er\dot{\phi} = eE_\phi\ ;\ E = \frac{m}{\sqrt{1-(r\dot{\phi})^2}}.$$

The first equation expresses the confinement to a circle by the z component of magnetic field. The second is the azimuthal acceleration caused by the induced field. Faraday's Law gives

$$E_\phi = -\frac{1}{2\pi r}\frac{d\Phi(r)}{dt}.$$

Substituting the first equation and this expression into the second equation, the required result follows.

Problem 2.15 Strong focusing

The transverse components of the field are $B_x = B'y$ and $B_y = B'x$. First, we derive the field from Maxwell's equations using a magnetic potential function ϕ:

$$\nabla \times \mathbf{B} = 0 \ ; \ \ \Rightarrow \mathbf{B} = \nabla\phi$$

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \nabla^2\phi = 0$$

For a 2-dimensional magnetic field, in cylindrical coordinates, the potential has the form

$$\phi \sim \rho^{\pm n} e^{\pm in\theta}.$$

Look for solutions antisymmetric in θ and finite at $\rho = 0$. The dipole and quadrupole terms are

$$\begin{aligned}
\phi \ &= a_0 + a_1\rho\sin\theta + a_2\rho^2\sin 2\theta + \cdots \\
&= a_0 + a_1 y + 2a_2 xy + \cdots \\
\mathbf{B} \ &= a_1\hat{y} + 2a_2(y\mathbf{e}_x + x\mathbf{e}_y) + \cdots .
\end{aligned}$$

The quadrupole field is $\mathbf{B} = B'[y\mathbf{e}_x + x\mathbf{e}_y]$.

The relativistic vector equation of motion is

$$\frac{d\mathbf{p}}{dt} = q\mathbf{v} \times \mathbf{B} = qB'(-xv_z\mathbf{e}_x + yv_z\mathbf{e}_y + (xv_x - yv_y)\mathbf{e}_z).$$

For small displacements x, y and small transverse velocity components, neglect second order terms. Energy is conserved in a static magnetic field ($\frac{dE}{dt} = \mathbf{v} \cdot \mathbf{F} = 0$), so we can write

$$\frac{d\mathbf{p}}{dt} = \frac{dE\mathbf{v}}{dt} = E\frac{d\mathbf{v}}{dt} = E\frac{dz}{dt}\frac{d\mathbf{v}}{dz} = Ev_z\frac{d\mathbf{v}}{dz}$$

and express the equations of motion in terms of z as

$$\frac{dv_x}{dz} = \frac{qB'}{E}x; \frac{dv_y}{dz} = \frac{qB'}{E}y; \frac{dv_z}{dz} = 0.$$

We then use the expressions

$$\frac{dv_x}{dz} = \frac{d}{dz}\left(v_z\frac{dx}{dz}\right) = v_z\frac{d^2x}{dz^2}; \ \frac{dv_x}{dz} = v_z\frac{d^2y}{dz^2}$$

and define $k^2 = \frac{qB'}{Ev_z}$ to write

$$\frac{d^2x}{dz^2} = -k^2 x; \frac{d^2y}{dz^2} = k^2 y.$$

The solution for $k^2 > 0$ is

$$\begin{aligned} x(z) &= x_0(0)\cos(kz) + \tfrac{x'(0)}{k}\sin kz \ ; \ x' = \tfrac{dx}{dz} = \tan\theta_x \simeq \theta_x \\ y(z) &= y(0)\cosh(kz) + \tfrac{y'(0)}{k}\sinh kz \ ; \ y' = \tfrac{dy}{dz} = \tan\theta_y \simeq \theta_y. \end{aligned}$$

The $x(z)$ solution $x'(z) = -k\sin(kz)x(0) + \cos kz\, x'(0)$ in matrix form is

$$\begin{bmatrix} x(z) \\ x'(z) \end{bmatrix} = \begin{bmatrix} \cos kz & \frac{1}{k}\sin kz \\ -k\sin kz & \cos kz \end{bmatrix} \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = M_{\text{QUAD}} \begin{bmatrix} x(0) \\ x'(0) \end{bmatrix}.$$

The matrix transports a distance z down a quadrupole magnet. For $z = w$ with w the length of the magnet, we have the transport matrix for the magnet. For $k^2 > 0$, M is oscillatory corresponding to focusing. For $k^2 < 0$, write k $\rightarrow$ ik, cos(ikz)=cosh(kz), sin(ikz)=isinh(kz)

$$M_{\text{QUAD}} = \begin{pmatrix} \cosh kz & \frac{\sinh kz}{k} \\ +k\sinh kz & \cosh kz \end{pmatrix}.$$

In passing through a straight section, we have $x(z) = x(0) + zx'(0)$, $x'(z) = x'(0)$. The transfer matrix is

$$M = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}.$$

For a straight section of length followed by a focusing quadrupole, the transfer matrix is obtained by matrix multiplication and is

$$M = \begin{pmatrix} c_x & c_x z + \frac{s_x}{k} \\ -ks_x & c_x - zks_x \end{pmatrix}.$$

If $M_{22} = 0$, then x' after the magnet is independent of x' at the start of the straight section which locates the focal point a distance $f_x = k^{-1}\cot(kw) \simeq 1/k^2 w$ from the face of the magnet where w is the length of the quadrupole.

Problem 2.16 Hill's equation

We have

$$\begin{aligned} x &= Cx_0 + Sx_0' + S\int Cp - C\int Sp \\ x' &= C'x_0 + S'x_0' + S'\int Cp - C'\int Sp \\ x'' &= C''x_0 + S''x_0' + S''\int Cp - C''\int Sp + (S'C - C'S)p \end{aligned}$$

and, using $C'' + kC = S'' + kS = 0$, we have

$$x'' + kx = (S'C - C'S)p.$$

Again using $C'' + kC = S'' + kS = 0$, we find

$$\frac{d}{ds}(S'C - C'S) = 0$$

and the initial conditions on C and S imply $(S'C - C'S)|_{s=0} = 1$ so

$$x'' + kx = p.$$

Problem 2.17 FODO lattice magnet tolerance

See Figure 2.2.

Problem 2.18 Quadrupole combinations

The forms for the matrices $\mathbf{d}_1\mathbf{f}\mathbf{d}_2$, $\mathbf{F}$, $\mathbf{M}_{FD}(k)$, and $\mathbf{M}_{DF}(k) = \mathbf{M}_{FD}(-k)$ follow by direct multiplication. The point to point imaging condition $d_1 = d_2$ for $\mathbf{M}_{FD}$ and $\mathbf{M}_{DF}$ gives a quadratic equation for $d = d_1 = d_2$. For that value of d, the magnification of $\mathbf{M}_{FD}$ is

$$\mu_{FD} = m_{11} + d_1 m_{21} = 1 + kL - \frac{1}{2}k^2L^2 + d(-k^2L).$$

and $\mu_{DF}(k) = \mu_{FD}(-k)$. Since μ_{FD} contains a term with an odd power of k, μ_{DF} can not equal μ_{FD}.

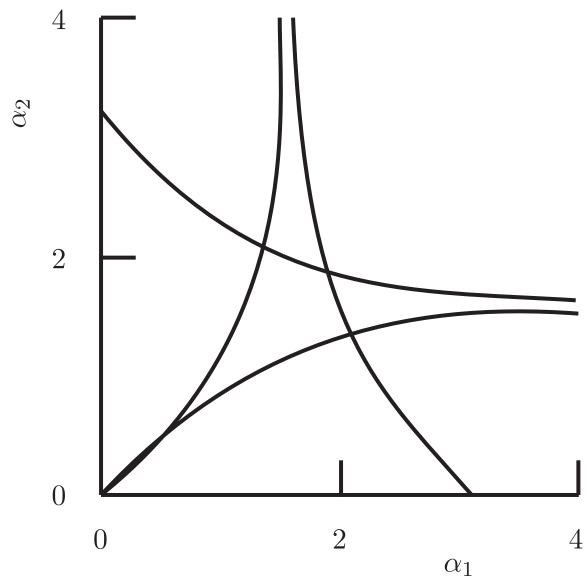

Figure 2.2: FODO lattice stability. The curves $\alpha_1 = \cos^{-1}(\pm 1/\cosh \alpha_2)$ and $\alpha_2 = \cos^{-1}(\pm 1/\cosh \alpha_1)$ define a bowtie shaped region of stability for a pair of disimilar quadrupole magnets.

Problem 2.19 Linear accelerator

The energy after crossing the nth gap is $E = E_0 + neV = \gamma m$ so the velocity is

$$v_n = \sqrt{1-(m/E)^2} = \sqrt{1-(\frac{m}{E_0+neV})^2}.$$

For tube length half a period, we have the sequence $l_n = v_n/2f$.

Problem 2.20 Radio frequency quadrupole accelerator

a) Gauss's Law requires

$$\partial_z E_z = -\partial_x E_x - \partial_y E_y = (E'_y - E'_x)\sin\omega t = 2E'_o\epsilon\sin(kz)\sin\omega t$$

and integration with respect to z gives $E_z = -2\epsilon\frac{E'_o}{k}\cos(kz)\sin(\omega t)$.
b) Suppose a particle arrives at z=0 with velocity v_0 at a time $t = -\phi/\omega$ so experiences an electric field

$$E_z = \epsilon\frac{E'_o}{k}\sin\phi.$$

To maintain the phase relationship for constant acceleration, we require $kz(t) - \omega t = 0$ so a synchronous velocity

$$v = dz/dt = \omega k^{-1}$$

if k is slowly varying. For a nonrelativistic ion, the acceleration is

$$\frac{dv}{dt} = v\frac{dv}{dz} = (\omega k^{-1})(\omega\frac{d(k^{-1})}{dz}) = (q/m)E_z = (q/m)\epsilon E'_0 k^{-1}\sin\phi$$

so $d(k^{-1})/dz = (q/m)\epsilon E'_o\sin\phi/\omega^2$ and the wavelength of the modulation should increase linearly with distance.

Problem 2.21 Synchrotron parameters

a) The radius of the orbit of a particle of charge e and momentum p in a uniform magnetic field of strength B is $R = p/eB$ in SI units. Write

$p(\text{kg } - \text{m s}^{-1}) = p(\text{GeV/c}) \times 10^9 \times e/3 \times 10^8$ so in convenient units $R(\text{m}) = p(\text{GeV/c})/0.3B(T)$. The circumference is $C = 2\pi\times$ 20,000 / (0.3 6.6)=63.5 km. The SSC design value 82.9 km allowed 19.4 km for other accelerator components and intersection regions.

b) The numerical values yield

$$10^{33}\ \text{cm}^{-2}\ \text{s}^{-1} \times 90 \times 10^{-27}\ \text{cm}^2 \times \pi \times 10^7\ \text{s yr}^{-1} \times 1/3 = 9.4 \times 10^{14}.$$

c) To reach 20 TeV requires 3.8×10^6 turns at 82.9 km/ 3 $\times 10^5$ km s^{-1}=0.28 ms per turn or 1050 seconds.

d) For a 60 km ring, the beam current is

$$I = 10^{14} \cdot 1.6 \times 10^{-19}\ \text{C} \cdot 3 \times 10^8\ \text{m s}^{-1} \times \frac{1}{6 \times 10^4\ \text{m}} = 80\ \text{mA}.$$

The stored energy is

$$E = 10^{14} \cdot 20 \times 10^{12}\ \text{eV} \cdot 1.6 \times 10^{-19}\ \text{J eV}^{-1} =\ 320\ \text{MJ}.$$

The synchrotron radiation loss per turn per particle is

$$dE = \frac{4\pi}{3}\frac{e^2}{R}\gamma^4 = \frac{4}{137}\frac{0.2\ \text{GeV fm}}{10^{19}\text{fm}} \cdot (20,000)^4 = 0.9 \times 10^{-4}\text{GeV}$$

so the total radiation power is

$$10^{14} \cdot 0.9 \times 10^5\ \text{eV}\frac{1}{2 \times 10^{-4}\ \text{s turn}^{-1}} \cdot 1.6 \times 10^{-19}\ \text{J eV}^{-1} =\ 7.2\ \text{kW}.$$

e) The cm energy squared is $s_1 = (E_p + E_{\bar{p}})^2\ =\ (1800\ \text{GeV})^2$. A collision of a an antiproton with a fixed proton has

$$s = (E + m_p)^2 - \mathbf{p}^2\ =\ 2m_p^2 + 2Em_p \approx 2Em_p$$

so the equivalent energy is $E\ =\ s_1/2m_p\ \approx\ 1800^2/2 = 1620$ TeV.

Problem 2.22 Antiproton source

The mass of an antiproton is $m_p = 1.67 \times 10^{-24}$ g so the total mass made in 50 weeks is

$$50 \times 4 \times 10^{13} 1.67 \times 10^{-24} = 334 \times 10^{-11}\ \text{g}$$

Since 1 TeV $= 10^{12}$ eV $\times 1.602 \times 10^{-19}$ JeV^{-1} $= 1.6 \times 10^{-7}$ J, the power is

$$4 \times 10^{13} \text{ collision wk}^{-1}\ 3.2 \times 10^{-7}\text{J collision}^{-1}\frac{1 \text{ wk}}{604,800 \text{ s}} = 21 \text{ W}.$$

Problem 2.23 Synchrotron radiation

The numerical results for synchrotron radiation at CESR follow from the formulae. The e^+ and e^- energy spread and correspondingly the spread in the center of mass energy is much larger than the natural width of the Υ resonance $\Gamma = 54$ keV.

Problem 2.24 Fission cross section and critical mass

a) The number density of nuclei is

$$\mathrm{n} = \mathrm{N_A}\rho/\mathrm{A} = (6.02 \times 10^{23} \text{ nuclei mole}^{-1})(19.6 \text{ g cm}^{-3})(1/238)\text{mole}^{-1} \text{ g}$$

or $\mathrm{n} = 4.95 \times 10^{22}$ cm^{-3}. The radius of U-235 is $\mathrm{R} = (1.25 \text{ fm})\mathrm{A}^{1/3} = 7.71$ fm so the cross section is

$$\sigma = \pi \mathrm{R}^2 = 189 \times 10^{-26} \text{ cm}^2.$$

The mean free path is $\lambda = 1/(\mathrm{n}\sigma) = 10.7$ cm. b) The mass of a sphere of this radius is

$$\mathrm{M} = \rho\frac{4\pi}{3}\lambda^3 = 100 \text{ kg}.$$

Problem 2.25 Hadron collision cross sections

For classical hard spheres, $\sigma = \pi(r_1 + r_2)^2$. From the pp and $p\bar{p}$ cross section, we infer r_p= .56 fm. Then r_π and $r_K \approx 0.3$ fm. The $\sigma_{hD}/\sigma_{hp} \simeq 2$ makes sense if $\sigma_{hp} \simeq \sigma_{hn}$ and the inelastic cross sections add. (Coherent shadowing effects in elastic scattering from deuterons is described in Fraunenfelder and Henley in the Glauber approximation.) The additivity of cross sections provided some early confirmation of the quark model of hadrons. For $A = 56$, R_{Fe}= 4.2 fm so $\pi R_{Fe}^2 \simeq .55$ b $>> r_p$. The number density n of nuclei for

