CHAPTER 2

MATHEMATICS FOR MICROECONOMICS

The problems in this chapter are primarily mathematical. They are intended to give students some practice with the concepts introduced in Chapter 2, but the problems in themselves offer few economic insights. Consequently, no commentary is provided. Results from some of the analytical problems are used in later chapters, however, and in those cases the student will be directed to here.

Solutions

2.1 U (x, y) = $4x^2 + 3y^2$

a. $\frac{\partial U}{\partial x} = 8x, \quad \frac{\partial U}{\partial y} = 6y$

b. 8, 12

c. dU = $\frac{\partial U}{\partial x}\, dx + \frac{\partial U}{\partial y}\, dy = 8x\ dx + 6y\ dy$

d. $\frac{dy}{dx}$ for dU = 0 $\quad 8x\ dx + 6y\ dy = 0$

$\frac{dy}{dx} = \frac{-8x}{6y} = \frac{-4x}{3y}$

e. $x = 1, \quad y = 2 \quad U = 4 \cdot 1 + 3 \cdot 4 = 16$

f. $\frac{dy}{dx} = \frac{-4(1)}{3(2)} = -2/3$

g. U = 16 contour line is an ellipse centered at the origin. With equation $4x^2 + 3y^2 = 16$, slope of the line at (x, y) is $\frac{dy}{dx} = -\frac{4x}{3y}$.

2.2 a. Profits are given by $\pi = R - C = -2q^2 + 40q - 100$

$\frac{d\pi}{dq} = -4q + 40 \quad q^* = 10$

$\pi^* = -2(10)^2 + 40(10) - 100 = 100$

b. $\frac{d^2\pi}{dq^2} = -4$ so profits are maximized

c. $MR = \frac{dR}{dq} = 70 - 2q$

$MC = \frac{dC}{dq} = 2q + 30$

so q^* = 10 obeys MR = MC = 50.

2.3 Substitution: $y = 1 - x$ so $f = xy = x - x^2$

$$\frac{\partial f}{\partial x} = 1 - 2x = 0$$

$x = 0.5, y = 0.5, f = 0.25$

Note: $f'' = -2 < 0$. This is a local and global maximum.

Lagrangian Method: $£ = xy + \lambda(1 - x - y)$

$$\frac{\partial £}{\partial x} = y - \lambda = 0$$

$$\frac{\partial £}{\partial y} = x - \lambda = 0$$

so, $x = y$.

using the constraint gives $x = y = 0.5, \quad xy = 0.25$

2.4 Setting up the Lagrangian: $£ = x + y + \lambda(0.25 - xy)$.

$$\frac{\partial £}{\partial x} = 1 - \lambda y$$

$$\frac{\partial £}{\partial y} = 1 - \lambda x$$

So, $x = y$. Using the constraint gives $xy = x^2 = 0.25, \quad x = y = 0.5$.

2.5 a. $f(t) = -0.5gt^2 + 40t$

$$\frac{df}{dt} = -g\,t + 40 = 0, \quad t^* = \frac{40}{g} \quad .$$

b. Substituting for t^*, $f(t^*) = -0.5g(40/g)^2 + 40(40/g) = 800/g$.

$$\frac{\partial f(t^*)}{\partial g} = -800/g^2 \, .$$

c. $\frac{\partial f}{\partial g} = -\frac{1}{2}(t^*)^2$ depends on g because t^* depends on g.

so $\frac{\partial f}{\partial g} = -0.5(t^*)^2 = -0.5(\frac{40}{g})^2 = \frac{-800}{g^2}$.

d. $800/32 = 25, \quad 800/32.1 = 24.92$ a reduction of .08. Notice that $-800/g^2 = 800/32^2 \approx -0.8$ so a 0.1 increase in g could be predicted to reduce height by 0.08 from the envelope theorem.

2.6 a. This is the volume of a rectangular solid made from a piece of metal which is x by $3x$ with the defined corner squares removed.

b. $\frac{\partial V}{\partial t} = 3x^2 - 16xt + 12t^2 = 0$. Applying the quadratic formula to this expression yields $t = \frac{16x \pm \sqrt{256x^2 - 144x^2}}{24} = \frac{16x \pm 10.6x}{24} = 0.225x, \quad 1.11x$. To determine true maximum must look at second derivative -- $\frac{\partial^2 V}{\partial t^2} = -16x + 24t$ which is negative only for the first solution.

c. If $t = 0.225x, \quad V \approx 0.67x^3 - .04x^3 + .05x^3 \approx 0.68x^3$ so V increases without limit.

d. This would require a solution using the Lagrangian method. The optimal solution requires solving three non-linear simultaneous equations – a task not undertaken here. But it seems clear that the solution would involve a different relationship between t and x than in parts a-c.

2.7 a. Set up Lagrangian $£ = x_1 + 5\ln x_2 + \lambda(k - x_1 - x_2)$ yields the first order conditions:

$$\frac{\partial £}{\partial x_1} = 1 - \lambda = 0$$

$$\frac{\partial £}{\partial x_2} = \frac{5}{x_2} - \lambda = 0$$

$$\frac{\partial £}{\partial \lambda} = k - x_1 - x_2 = 0$$

Hence, $\lambda = 1 = 5/x_2$ or $x_2 = 5$. With k = 10, optimal solution is $x_1 = x_2 = 5$.

b. With $k = 4$, solving the first order conditions yields $x_2 = 5, x_1 = -1$.

c. Optimal solution is $x_1 = 0, x_2 = 4, y = 5\ln 4$. Any positive value for x_1 reduces y.

d. If $k = 20$, optimal solution is $x_1 = 15, x_2 = 5$. Because x_2 provides a diminishing marginal increment to y whereas x_1 does not, all optimal solutions require that, once x_2 reaches 5, any extra amounts be devoted entirely to x_1.

2.8 a. TC = $\int_0^q MCdq = \int_0^q (q+1)dq = \frac{q^2}{2} + q\Big|_o^q + C = \frac{q^2}{2} + q + C$

The constant of integration here is fixed costs.

b. By profit maximization, p = MC(q).
p = q +1 ; q = p -1 = 14
If the firm is just breaking even , profits = total revenue – total costs = 0

$$pq - TC(q) = 15*14 - (\frac{14^2}{2} + 14 + C) = 0$$
$$98 - C = 0$$
$$C = 98 = \text{Fixed Costs}$$

c. If p=20, q = 19. Following the same steps as in b., and using C=98, we get

$$\pi = pq - TC(q) = 20*19 - (\frac{19^2}{2} + 19 + 98) = 82.5$$

So, profits increase by 82.5

d. Assuming profit maximization, we have p = MC(q)

$$p = q + 1 \Rightarrow q = p - 1$$
$$\pi = pq - (\frac{q^2}{2} + q + 98) = p(p-1) - \left(\frac{(p-1)^2}{2} + (p-1) + 98\right) = \frac{(p-1)^2}{2} - 98$$

e. i. Using the above equation, π(20) - π(15) = 82.5

ii. $\pi(20) - \pi(15) = \int_{15}^{20} (p-1)dp = \frac{p^2}{2} - p \Big|_{15}^{20} = 82.5$

The 2 approaches above demonstrate the envelope theorem. In the first case, we optimize *q* first and then substitute it into the profit function. In the second case, we directly vary the parameter (i.e., *p*) and essentially move along the firm's supply curve.

Analytical Problems

2.9 **Concave and Quasiconcave Functions**

The proof is most easily accomplished through the use of the matrix algebra of quadratic forms. See, for example, Mas Colell et al., pp. 937-939. Intuitively, because concave functions lie below any tangent plane, their level curves must also be convex. But the converse is not true. Quasi-concave functions may exhibit "increasing returns to scale"; even though their level curves are convex, they may rise above the tangent plane when all variables are increased together. A counter example would be the Cobb-Douglas function which is always quasi-concave, but convex when α+β > 1.

2.10 **The Cobb-Douglas Function**

a. $f_1 = \alpha x_1^{\alpha-1} x_2^{\beta} > 0.$

$f_2 = \beta x_1^{\alpha} x_2^{\beta-1} > 0.$

$f_{11} = \alpha(\alpha - 1)\ x_1^{\alpha-2}\ x_1^{\beta} < 0.$

$f_{22} = \beta(\beta - 1)\ x_1^{\alpha}\ x_2^{\beta-2} < 0.$

$f_{12} = f_{21} = \alpha\beta x_1^{\alpha-1} x_2^{\beta-1} > 0.$

Clearly, all the terms in Equation 2.114 are negative.

b. If $y = c = x_1^{\alpha} x_2^{\beta}$

$x_2 = c^{1/\beta} x_1^{-\alpha/\beta}$ since α, β > 0, x_2 is a convex function of x_1 .

c. Using equation 2.98,

$$f_{11}f_{22} - f_{12}^2 = \alpha(\alpha-1)(\beta)(\beta-1)\, x_1^{2\alpha-2} x_2^{2\beta-2} - \alpha^2\beta^2 x_1^{2\alpha-2} x_2^{2\beta-2}$$

$= \alpha\beta(1-\beta-\alpha)\, x_1^{2\alpha-2} x_2^{2\beta-2}$ which is negative for α + β > 1.

2.11 The Power Function

a. Since $y' > 0, y'' < 0$, the function is concave.

b. Because $f_{11}, f_{22} < 0$, and $f_{12} = f_{21} = 0$, Equation 2.98 is satisfied and the function is concave. Because $f_1, f_2 > 0$ Equation 2.114 is also satisfied so the function is quasi-concave.

c. y is quasi-concave as is y^{γ} . But y^{γ} is not concave for $\gamma\delta > 1$. This can be shown most easily by $f(2x_1, 2x_2) = [(2x_1)^{\delta} + (2x_2)^{\delta}]^{\gamma} = 2^{\delta\gamma} f(x_1, x_2)$.

2.12 Taylor Approximations

a. From Equation 2.85, a function in one variable is concave if $f''(x) < 0$
Using the quadratic Taylor to approximate $f(x)$ near a point a:

$f(x) \approx f(a) + f'(a)(x-a) + 0.5 f''(a)(x-a)^2$

$\leq f(a) + f'(a)(x-a)$ (because $f''(a) < 0$ *and* $(x-a)^2 > 0$)

The RHS above is the equation of the line tangent to the point a and so, we have shown that any concave function must lie on or below the tangent to the function at that point.

b. From Equation 2.98, a function in 2 variables is concave if
$f_{11}f_{22} - f_{12}^2 > 0$ and we also know that due to the concavity of the function, $d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2 \leq 0$
So, 0.5($f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2) \leq 0$.

This is the third term of the quadratic Taylor expansion where $dx = x - a,\ dy - y - b$.

Thus, we have $f(x,y) \le f(a,b) + f_1(a,b)(x-a) + f_2(a,b)(y-b)$

Which shows that any concave function must lie on or below its tangent plane.

2.13 More on Expected Value

a. The tangent to $g(x)$ at the point $E(x)$ will have the form $c + dx \ge g(x)$ for all values of x and $c + dE(x) = g(E(x))$.

But, $E(g(x)) \le E(c + dx) = c + dE(x) = g(E(x))$.

b. Using the same procedure as before with $\ge$ instead of $\le$ and vice versa, we have the following proof:

The tangent to $g(x)$ at the point $E(x)$ will have the form $c + dx \le g(x)$ for all values of x and $c + dE(x) = g(E(x))$.

Now, $E(g(x)) \ge E(c + dx) = c + dE(x) = g(E(x))$.

c. Let $u = 1 - F(x),\quad du = -f(x),\ x = v,\ dv = dx$. Apply Equation 2.136.

$$\int_0^\infty (1 - F(x))dx = (1 - F(x))x\Big|_0^\infty - \int_0^\infty - f(x)xdx = 0 + E(x) = E(x)$$

d.

$$RHS = \frac{E(x)}{t} = \frac{1}{t}\left(\int_0^t xf(x)dx + \int_t^\infty xf(x)dx\right)$$

$$\ge \frac{1}{t}\left(\int_t^\infty xf(x)dx\right) \ge \frac{1}{t}\left(\int_t^\infty tf(x)dx\right) = \int_t^\infty f(x)dx = P(x \ge t) = LHS$$

e. 1. $\int_{-\infty}^\infty f(x)dx = \int_1^\infty 2x^{-3}dx = 2\left[\frac{x^{-2}}{-2}\right]_1^\infty = 1$

2. $F(x) = \int_{-\infty}^x f(x)dx = \int_1^x 2x^{-3}dx = 2\left[\frac{x^{-2}}{-2}\right]_1^x = 1 - x^{-2}$

3. E(x) = $\int_{-\infty}^\infty (1 - F(x))dx = \int_1^\infty x^{-2}dx = -x^{-1}\Big|_1^\infty = 1$

4.

$$P(x \geq t) = 1 - F(t) = t^{-2} = \frac{1}{t^2}$$

$$\frac{E(x)}{t} = \frac{1}{t}$$

$$Since\ x \geq 1,\ \frac{1}{t^2} \leq \frac{E(x)}{t}$$

Thus, Markov's inequality holds.

f. 1. $\int_{-\infty}^{\infty} f(x)dx = \int_{-1}^{2} \frac{x^2}{3}dx = \frac{1}{9}\left[x^3\right]_{-1}^{2} = 1$

2. $E(x) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-1}^{2} \frac{x^3}{3}dx = \frac{1}{12}\left[x^4\right]_{-1}^{2} = \frac{16-1}{12} = \frac{5}{4}$

3. $P(-1 \leq x \leq 0) = \int_{-1}^{0} \frac{x^2}{3}dx = \frac{1}{9}\left[x^3\right]_{-1}^{0} = \frac{1}{9}$

4. $f(x|A) = \frac{P(x\ and\ A)}{P(A)} = \frac{\frac{x^2}{3}}{\frac{8}{9}} = \frac{3x^2}{8}; 0 \leq x \leq 2$

5. $E(x|A) = \int_{-\infty}^{\infty} xf(x|A)dx = \int_{0}^{2} \frac{3x^3}{8}dx = \frac{3}{32}\left[x^4\right]_{0}^{2} = \frac{3}{2}$

6. Eliminating the lowest values for x should increase the expected value of the remaining values.

2.14 More on Variances and Covariances

a.

$$Var(x) = E[(x - E(x))^2] = E(x^2 - 2xE(x) + (E(x))^2)$$

$$= E(x^2) - 2E(xE(x)) + E((E(x))^2) = E(x^2) - 2(E(x)E(x)) + (E(x))^2$$

$$= E(x^2) - (E(x))^2$$

b.

$$Cov(x, y) = E[(x - E(x))(y - E(y))]$$

$$= E[xy - xE(y) - yE(x) + E(x)E(y)]$$

$$= E(xy) - E(x)E(y) - E(y)E(x) + E(x)E(y)$$

$$= E(xy) - E(x)E(y)$$

c. $Var(ax \pm by) = E[(ax \pm by)^2] - (E(ax \pm by))^2$ (From part a.)

$= E(a^2x^2 \pm 2axby + b^2y^2) - (aE(x) \pm bE(y))^2$

$= a^2E(x^2) \pm 2abE(xy) + b^2E(y^2) - a^2(E(x))^2 \mp 2abE(x)E(y) - b^2(E(y))^2$

$= a^2Var(x) + b^2Var(y) \pm 2abCov(x, y)$

(From results of parts a. and b.)

d. $E(.5x + .5y) = .5E(x) + .5E(y) = E(x)$

Remember that if 2 random variables x and y are independent, then Cov(x, y) = 0

$Var(.5x + .5y) = .25Var(x) + .25Var(y) + 0$

$= .5Var(x)$

If x and y characterize 2 different assets with the properties $E(x) = E(y)$, $\text{var}(x) = \text{var}(y)$ we have shown that the variance of a diversified portfolio is only half as large as for a portfolio invested in only one of the assets.