Instructor's Manual

with complete solutions

for

Mathematics: A Discrete Introduction

Third Edition

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To Kelly, Jeff, and Andrew

Contents

Preface

This is the Instructor's Manual for *Mathematics: A Discrete Introduction*, third edition. This manual has two purposes. First, we give solutions for the problems in the text. The problems are on a variety of difficulty levels, and reading through the solutions will give you a sense of how difficult the problem is. Second, comments and helpful hints on teaching are given at the beginning of each section.

Sections are meant to be covered (roughly) at a rate of one per lecture. Of course, some sections can be covered more rapidly, while others may require two full lectures.

A semester course based on this text would be in two halves: core material and topics. The core consists of Sections 1 through 24 (optionally omitting Sections 18 and 19).

I hope you find this supplement helpful. Please send me your feedback by email to `ers@jhu.edu`. Thank you.

–ES

Chapter 1

Fundamentals

1 Joy

This section may be assigned as reading. The purpose of this section is to instill in students a sense of the pleasure mathematical work can bring. I recommend that you assign the one problem contained herein but not for course credit. Admonish your students thoroughly not to discuss the problem with each other or else they will spoil the experience.

1.1 Do not, under any circumstances, tell students the answer to this problem or you will rob them of the joy of discovering the answer themselves. Do not give them hints. Do not ask leading questions such as "What is the 24^{th} factor in the expression?"

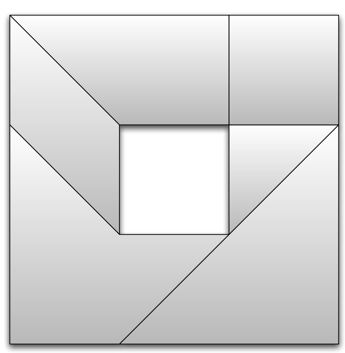
Let students work this out for themselves. Be encouraging and reassure them that when they have found the answer, they will know they are correct.

2 Speaking (and Writing) of Mathematics

This section may also be assigned as reading. The goal here is to emphasize that clarity of language is vital in mathematics. The extent to which students can articulate their thoughts is a good indicator of how well they understand them. Furthermore, the very act of putting their thoughts into clear sentences helps with the learning process.

Students often have the unhealthy notion that to do mathematics is to "grunt out" a series of equations with no words of explanation. They also believe that mathematics instructors will accept lousy writing, horrible penmanship, and innumerable crossouts on crumpled paper that has been torn out of a spiral notebook. We need to expect a lot better!

2.1 This diagram gives a solution to the puzzle.



Here are written instructions.

You will find before you six pieces: a large isosceles right triangle, a small isosceles right triangle, a square, a parallelogram, a trapezoid (with one side perpendicular to the parallel sides), and an oddly shaped (nonconvex) pentagon.

1. Place the square in the upper right with its sides vertical and horizontal.
2. Place the trapezoid to the left of the square so that its long parallel side aligns with the top of the square and its short parallel side aligns with the bottom of the square. Note that the side perpendicular to the parallel sides exactly abuts the left side of the square.
3. Place the parallelogram so that one of the long sides of the parallelogram matches the long non-parallel side of the trapezoid. Thus, the parallelogram is just below the left portion of the trapezoid. On the left, the short side of the parallelogram should complete a right angle with the long parallel side of the trapezoid.
4. Place the small right triangle below the square. One of the legs of the small right triangle exactly aligns with the bottom of the square and the other leg of the right triangle faces the inside of the puzzle. The hypotenuse of the small right triangle should slope from the lower left to the upper right.
5. Hold the large right triangle so that its legs are vertical and horizontal, and its right angle is to the lower right. Now place the large right triangle so that its upper $45°$ angle just touches the lower right corner of the square. Half of the hypotenuse of the large right triangle should align with all of the hypotenuse of the small right triangle.
6. Finally, hold the pentagon so that its right angle is in the lower left and the two sides that form that right angle are vertical and horizontal. It will now slide so that its right edge matches the lower half of the hypotenuse of the big right triangle and its upper edge meets up with the lower, long side of the parallelogram.

3 Definition

If we think of mathematics as a dramatic production, there are three main characters: Definition, Theorem, and Proof. (We can think of Counterexample as being the “evil” side of Proof. Important, supporting roles in this show are Conjecture and Example.) Thus the first few sections of this book are dedicated to introducing these main characters.

Perhaps conspicuous by its absence is Axiom. (The term is introduced later in the book.) This omission is intentional. In my philosophy of mathematics, axioms are a form of definition. For example, the axioms of Euclidean geometry form the *definition* of the Euclidean plane. Any collection of objects we call *points* and *lines* that satisfy the Euclidean axioms is “honored” with the title *Euclidean plane*. Likewise the so-called axioms of group theory are actually the definition of a group. I have found that introducing axioms as “unproved assumptions” undermines the truth and beauty of mathematics. It unnecessarily introduces some doubt and confusion in students’ minds.

And while we are speaking of philosophy, I try to be rather clear about mine in the text. To me, mathematics is a purely mental construction. Definitions, theorems, proofs, etc., are the invention of and exist only in the human mind. If you agree with me, great. If you disagree with me (e.g., you think proofs are discovered as opposed to created) so much the better! You can use the text in a point-counterpoint discussion. Ultimately, I do not think these philosophical questions have much bearing on the day-to-day work of mathematicians.

In any case, the essential point to convey from this section is that mathematical definitions must be much clearer than ordinary definitions. (Try having students define *chair* or *love*; as mathematicians we are lucky to be able to define our terms precisely.)

Furthermore, we cannot in this book define everything down to first principles. It is much too difficult for students on this level to go reduce everything to axiomatic set theory. It is reasonable for students to subsume sets and integers and to proceed from there.

3.1 (a) False. (b) True. (c) True. (d) True. (e) False. (f) False. (g) True. (h) True.

3.2 In the "official definition" (Definition 3.2) we have $0|0$, but in the alternative definition $0|0$ is false because $\frac{0}{0}$ is not an integer.

3.3 Only the first number, 21, is composite because $21 = 3 \times 7$ satisfies Definition 3.6. None of the other numbers are composite because they are not positive integers.

3.4 It is easiest if we define $\leq$ first. For integers x, y we say x is *less than or equal to* y, and we write $x \leq y$, provided $y - x$ is a natural number. We write $x < y$ provided $x \leq y$ and $x \neq y$. We write $x \geq y$ provided $y \leq x$. We write $x > y$ provided $y < x$.

3.5 Let x be an integer. Then x is also a rational number because we can write $x = \frac{x}{1}$, satisfying the definition of rational. On the other hand, $\frac{1}{2}$ is rational, but not an integer.

3.6 An integer x is called a *perfect square* provided there is an integer y such that $x = y^2$.

3.7 A number x is a *square root* of a number y provided $x^2 = y$.

3.8 The *perimeter* of a polygon is the sum of the lengths of its sides.

3.9 Suppose A, B, C are points in the plane. We say that C is *between* A and B provided the $d(A,C) + d(C,B) = d(A,B)$ where $d(\cdot,\cdot)$ denotes the distance between the two given points.

Suppose A, B, C are points in the plane. We say that they are collinear provided one of them is between the other two.

3.10 The *midpoint* of a line segment $\overline{AB}$ is a point C on the segment such that the distance from A to C equals the distance from C to B.

3.11 (a) A person X is called a *teenager* provided X is at least 13 years old and less than 20 years old.

(b) Person A is the *grandmother* of person B provided A is female and A is the parent of one of B's parents.

Alternatively: A person X is a *grandmother* if X is female and X is the parent of someone who is also a parent.

(c) A year is a *leap year* if it is 366 days long.

(d) A *dime* is a U.S. coin worth 10 cents, i.e., one-tenth of a dollar.

(e) A *palindrome* is a word that when written backwards spells the same word.

(f) A word X is called a *homophone* of a word Y if X and Y are spelled differently but are pronounced the same.

3.12 (a) 4. (b) 6. (c) $n+1$. (d) 4. (e) 9. (f) 49. (g) $(n+1)^2$. (h) 8. (i) 8. (j) $2^5 = 32$. (k) $8 \times 3 \times 2 \times 2 = 96$ because $8! = 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1$. (l) ∞.

The reason 30 and 42 have the same number of divisors is they are both the product of three distinct primes.

Prime factorization is developed formally later (see Section 39).

3.13 The first perfect number is 6. The next perfect number after 28 is 496. The divisors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, and 248, and they sum to 496.

The following is a *Mathematica* program to find the next perfect number after 28.

```
IsPerfect[n_] :=
   Apply[Plus, Divisors[n]]==2*n;
n = 29;
While[Not[IsPerfect[n]], n++];
n
```

3.14 Presumably, the umpire's word is law. Regardless of what really happened or what the umpire saw, if the umpire says "Out!" by *definition*, the runner is out. This is akin to mathematical definitions. A number is *prime* because we crafted the definition a certain way.

4 Theorem

If Definition, Theorem, and Proof are our main characters, the most glamorous role is played by Theorem.

It is important for students to understand what a mathematical Statement is, and that Theorems are *true* mathematical statements that can be proved (with Proof to be introduced in the next section). Philosophical discussions are unavoidable here! We must distinguish *theorem* from the scientists' *theory* and be utterly rigid about what *truth* is. Compare and contrast statements such as