Chapter 3

Matrices

3.1 Practice Problems

1. (a) $T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3(2) + 2(-1) \\ -(2) + (-1) \\ -4(2) - 3(-1) \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -5 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 3 & 2 \\ -1 & 1 \\ -4 & -3 \end{bmatrix}$

   (c) Because $n = 3 > m = 2$, by Theorem 3.7 T is not onto. To determine if T is one-to-one, we row-reduce the corresponding augmented matrix:

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 1 & 0 \\ -4 & -3 & 0 \end{bmatrix} \underset{\sim}{\overset{(1/3)R_1+R_2 \to R_2}{(4/3)R_1+R_3 \to R_3}} \begin{bmatrix} 3 & 2 & 0 \\ 0 & \frac{5}{3} & 0 \\ 0 & -\frac{1}{3} & 0 \end{bmatrix}$$
$$\underset{\sim}{(1/5)R_2+R_3 \to R_3} \begin{bmatrix} 3 & 2 & 0 \\ 0 & \frac{5}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

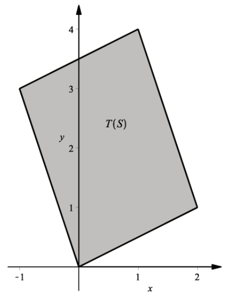
   Because $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$ has only the trivial solution, by Theorem 3.5 T is one-to-one.

2. (a) $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2) = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$

   (b) $T(3\mathbf{u}_1) = 3T(\mathbf{u}_1) = 3\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$

   (c) $T(2\mathbf{u}_1 - \mathbf{u}_2) = 2T(\mathbf{u}_1) - T(\mathbf{u}_2) = 2\begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$

3. Because $T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 + 2(0) \\ 0 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, T is not a linear transformation.

4. 

5. (a) False. $T\left(4\begin{bmatrix}0\\1\end{bmatrix}\right)=T\left(\begin{bmatrix}0\\4\end{bmatrix}\right)=\begin{bmatrix}0-2\,(4)\\3\,(0)+\sqrt{4}\end{bmatrix}=\begin{bmatrix}-8\\2\end{bmatrix}$, but $4T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=4\begin{bmatrix}0-2\,(1)\\3\,(0)+\sqrt{1}\end{bmatrix}=\begin{bmatrix}-8\\4\end{bmatrix}\neq T\left(4\begin{bmatrix}0\\1\end{bmatrix}\right)$.

(b) False. For example, $A=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$ is onto.

(c) True. By definition, a linear transformation is a function.

(d) False. For example, $T\left([x]\right)=\begin{bmatrix}x\\2x\end{bmatrix}$ satisfies $T\left(\mathbf{x}\right)=\mathbf{0}$ if and only $\mathbf{x}=\mathbf{0}$, but T is not onto.

3.1 Linear Transformations

1. $T(\mathbf{u}_1)=A\mathbf{u}_1=\begin{bmatrix}2&1\\-3&5\end{bmatrix}\begin{bmatrix}-4\\-2\end{bmatrix}=\begin{bmatrix}-10\\2\end{bmatrix}$, $T(\mathbf{u}_2)=A\mathbf{u}_2=\begin{bmatrix}2&1\\-3&5\end{bmatrix}\begin{bmatrix}1\\-6\end{bmatrix}=\begin{bmatrix}-4\\-33\end{bmatrix}$

2. $T(\mathbf{u}_1)=A\mathbf{u}_1=\begin{bmatrix}1&0\\2&-4\\3&3\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\-6\\9\end{bmatrix}$, $T(\mathbf{u}_2)=A\mathbf{u}_2=\begin{bmatrix}1&0\\2&-4\\3&3\end{bmatrix}\begin{bmatrix}-5\\0\end{bmatrix}=\begin{bmatrix}-5\\-10\\-15\end{bmatrix}$

3. $T(\mathbf{u}_1)=A\mathbf{u}_1=\begin{bmatrix}0&-4&2\\3&1&-2\end{bmatrix}\begin{bmatrix}3\\2\\1\end{bmatrix}=\begin{bmatrix}-6\\9\end{bmatrix}$, $T(\mathbf{u}_2)=A\mathbf{u}_2=\begin{bmatrix}0&-4&2\\3&1&-2\end{bmatrix}\begin{bmatrix}4\\-5\\-2\end{bmatrix}$ $=\begin{bmatrix}16\\11\end{bmatrix}$

4. $T(\mathbf{u}_1)=A\mathbf{u}_1=\begin{bmatrix}-2&5&-2\\0&-1&-2\\0&-1&-1\end{bmatrix}\begin{bmatrix}0\\7\\-2\end{bmatrix}=\begin{bmatrix}39\\-3\\-5\end{bmatrix}$, $T(\mathbf{u}_2)=A\mathbf{u}_2=\begin{bmatrix}-2&5&-2\\0&-1&-2\\0&-1&-1\end{bmatrix}\begin{bmatrix}3\\5\\-1\end{bmatrix}=\begin{bmatrix}21\\-3\\-4\end{bmatrix}$

5. We consider $T(\mathbf{x})=A\mathbf{x}=\mathbf{y}$, and row-reduce the corresponding augmented matrix:

$$\begin{bmatrix}1&-2&0&-3\\3&2&1&6\end{bmatrix}\underset{\sim}{\overset{-3R_1+R_2\to R_2}{}}\begin{bmatrix}1&-2&0&-3\\0&8&1&15\end{bmatrix}$$

Since there exists a solution $\mathbf{x}$ to $A\mathbf{x}=\mathbf{y}$, $\mathbf{y}$ is in the range of T.

6. We consider $T(\mathbf{x})=A\mathbf{x}=\mathbf{y}$, and row-reduce the corresponding augmented matrix:

$$\begin{bmatrix}1&-2&0&1\\3&2&1&-4\end{bmatrix}\underset{\sim}{\overset{-3R_1+R_2\to R_2}{}}\begin{bmatrix}1&-2&0&1\\0&8&1&-7\end{bmatrix}$$

Since there exists a solution $\mathbf{x}$ to $A\mathbf{x}=\mathbf{y}$, $\mathbf{y}$ is in the range of T.

7. We consider $T(\mathbf{x}) = A\mathbf{x} = \mathbf{y}$, and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{rrrr} 1 & -2 & 0 & 2 \\ 3 & 2 & 1 & 7 \end{array}\right] \overset{-3R_1+R_2\to R_2}{\sim} \left[\begin{array}{rrrr} 1 & -2 & 0 & 2 \\ 0 & 8 & 1 & 1 \end{array}\right]$$

Since there exists a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{y}$, $\mathbf{y}$ is in the range of T.

8. We consider $T(\mathbf{x}) = A\mathbf{x} = \mathbf{y}$, and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{rrrr} 1 & -2 & 0 & 4 \\ 3 & 2 & 1 & 5 \end{array}\right] \overset{-3R_1+R_2\to R_2}{\sim} \left[\begin{array}{rrrr} 1 & -2 & 0 & 4 \\ 0 & 8 & 1 & -7 \end{array}\right]$$

Since there exists a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{y}$, $\mathbf{y}$ is in the range of T.

9. $T(-2\mathbf{u}_1 + 3\mathbf{u}_2) = -2T(\mathbf{u}_1) + 3T(\mathbf{u}_2) = -2\left[\begin{array}{r} 2 \\ 1 \end{array}\right] + 3\left[\begin{array}{r} -3 \\ 2 \end{array}\right] = \left[\begin{array}{r} -13 \\ 4 \end{array}\right]$

10. $T(3\mathbf{u}_1 - 2\mathbf{u}_2) = 3T(\mathbf{u}_1) - 2T(\mathbf{u}_2) = 3\left[\begin{array}{r} 3 \\ -1 \\ -2 \end{array}\right] - 2\left[\begin{array}{r} 1 \\ 1 \\ 4 \end{array}\right] = \left[\begin{array}{r} 7 \\ -5 \\ -14 \end{array}\right]$

11. $T(-\mathbf{u}_1 + 4\mathbf{u}_2 - 3\mathbf{u}_3) = -T(\mathbf{u}_1) + 4T(\mathbf{u}_2) - 3T(\mathbf{u}_3)$
$= -\left[\begin{array}{r} -3 \\ 0 \end{array}\right] + 4\left[\begin{array}{r} 2 \\ -1 \end{array}\right] - 3\left[\begin{array}{r} 0 \\ 5 \end{array}\right] = \left[\begin{array}{r} 11 \\ -19 \end{array}\right]$

12. $T(\mathbf{u}_1 + 4\mathbf{u}_2 - 2\mathbf{u}_3) = T(\mathbf{u}_1) + 4T(\mathbf{u}_2) - 2T(\mathbf{u}_3) = \left[\begin{array}{r} 3 \\ -1 \\ -2 \end{array}\right] + 4\left[\begin{array}{r} 1 \\ 1 \\ 4 \end{array}\right] - 2\left[\begin{array}{r} 6 \\ 0 \\ 0 \end{array}\right] = \left[\begin{array}{r} -5 \\ 3 \\ 14 \end{array}\right]$

13. Linear transformation, with $A = \left[\begin{array}{rr} 3 & 1 \\ -2 & 4 \end{array}\right]$.

14. Not a linear transformation, since $T(2(1,1)) = T(2,2) = (2-2, 2(2)) = (0,4)$, but $2T(1,1) = 2(1-1, 1(1)) = 2(0,1) = (0,2)$.

15. Not a linear transformation, since $T(0(0,0,0)) = T(0,0,0) = (2\cos 0, 3\sin 0, 0) = (2,0,0)$, but $0\,(T(0,0,0)) = (0,0,0)$.

16. Linear transformation, with $A = \left[\begin{array}{rrr} 0 & -5 & 0 \\ 0 & 0 & 7 \end{array}\right]$.

17. Linear transformation, with $A = \left[\begin{array}{rrr} -4 & 0 & 1 \\ 6 & 5 & 0 \end{array}\right]$.

18. Not a linear transformation, since $T(0(0,0,0)) = T(0,0,0) = (0,4,0)$, but $0\,(T(0,0,0)) = (0,0,0)$.

19. Linear transformation, with $A = \left[\begin{array}{rr} 0 & \sin\frac{\pi}{4} \\ \ln 2 & 0 \end{array}\right]$.

20. Not a linear transformation, since $T(0,0) = (0,0,0)$ but $T(0,1) + T(0,-1) = (3,5,0) + (-3,5,0) = (0,10,0) \neq T(0,0)$.

21. We consider $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$, and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{rrr} 1 & -3 & b_1 \\ -2 & 5 & b_2 \end{array}\right] \overset{2R_1+R_2\to R_2}{\sim} \left[\begin{array}{rrc} 1 & -3 & b_1 \\ 0 & -1 & 2b_1 + b_2 \end{array}\right]$$

Since there exists a unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$, by The Unifying Theorem - Version 2, T is both one-to-one and onto.

22. We consider $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$, and row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{rrr} 3 & 2 & b_1 \\ 9 & 6 & b_2 \end{array}\right] \overset{-3R_1+R_2 \to R_2}{\sim} \left[\begin{array}{rrc} 3 & 2 & b_1 \\ 0 & 0 & -3b_1+b_2 \end{array}\right]$$

If $-3b_1 + b_2 \neq 0$, there does not exist a unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$. By The Unifying Theorem - Version 2, T is neither one-to-one nor onto.

23. Since $n = 2 < m = 3$, by Theorem 3.6 T is not one-to-one. To determine if T is onto, we row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{rrrr} 5 & 4 & -2 & b_1 \\ 3 & -1 & 0 & b_2 \end{array}\right] \overset{(-3/5)R_1+R_2 \to R_2}{\sim} \left[\begin{array}{rrrc} 5 & 4 & -2 & b_1 \\ 0 & -\frac{17}{5} & \frac{6}{5} & -\frac{3}{5}b_1+b_2 \end{array}\right]$$

Since there exists a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$, the columns of A span $\mathbb{R}^n$, and by Theorem 3.7 T is onto.

24. Since $n = 2 < m = 3$, by Theorem 3.6 T is not one-to-one. To determine if T is onto, we row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{rrrr} -1 & 3 & 2 & b_1 \\ 4 & -12 & -8 & b_2 \end{array}\right] \overset{4R_1+R_2 \to R_2}{\sim} \left[\begin{array}{rrrc} -1 & 3 & 2 & b_1 \\ 0 & 0 & 0 & 4b_1+b_2 \end{array}\right]$$

If $4b_1 + b_2 \neq 0$, there does not exist a unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$. By The Unifying Theorem - Version 2, T is neither one-to-one nor onto.

25. Since $n = 3 > m = 2$, by Theorem 3.7 T is not onto. To determine if T is one-to-one, we row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{rrr} 1 & -2 & 0 \\ -3 & 5 & 0 \\ 2 & -7 & 0 \end{array}\right] \overset{\substack{3R_1+R_2 \to R_2 \\ -2R_1+R_3 \to R_3}}{\sim} \left[\begin{array}{rrr} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & 0 \end{array}\right] \overset{-3R_2+R_3 \to R_3}{\sim} \left[\begin{array}{rrr} 1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Since $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$ has only the trivial solution, by Theorem 3.5 T is one-to-one.

26. Since $n = 3 > m = 2$, by Theorem 3.7 T is not onto. To determine if T is one-to-one, we row-reduce the corresponding augmented matrix:

$$\left[\begin{array}{rrr} 2 & -4 & 0 \\ 5 & -10 & 0 \\ -4 & 8 & 0 \end{array}\right] \overset{\substack{(-5/2)R_1+R_2 \to R_2 \\ 2R_1+R_3 \to R_3}}{\sim} \left[\begin{array}{rrr} 2 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Since $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$ has non-trivial solution, by Theorem 3.5 T is not one-to-one.

27. We consider $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$, and row-reduce the corresponding augmented matrix:

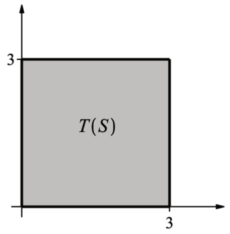
$$\left[\begin{array}{rrrr} 2 & 8 & 4 & b_1 \\ 3 & 2 & 3 & b_2 \\ 1 & 14 & 5 & b_3 \end{array}\right] \overset{\substack{(-3/2)R_1+R_2 \to R_2 \\ (-1/2)R_1+R_3 \to R_3}}{\sim} \left[\begin{array}{rrrr} 2 & 8 & 4 & b_1 \\ 0 & -10 & -3 & (-3/2)\,b_1+b_2 \\ 0 & 10 & 3 & (-1/2)\,b_1+b_3 \end{array}\right] \overset{R_2+R_3 \to R_3}{\sim} \left[\begin{array}{rrrr} 2 & 8 & 4 & b_1 \\ 0 & -10 & -3 & (-3/2)\,b_1+b_2 \\ 0 & 0 & 0 & -2b_1+b_2+b_3 \end{array}\right]$$

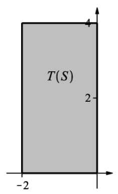
If $-2b_1 + b_2 + b_3 \neq 0$, there does not exist a unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$. By The Unifying Theorem - Version 2, T is neither one-to-one nor onto.

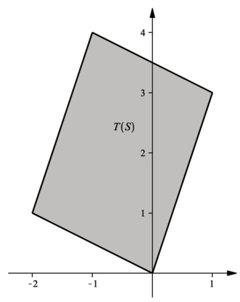
28. We consider $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$, and row-reduce the corresponding augmented matrix:

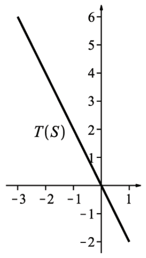
$$\left[\begin{array}{rrrr} 1 & 2 & -5 & b_1 \\ 3 & 7 & -8 & b_2 \\ -2 & -4 & 6 & b_3 \end{array}\right] \overset{\substack{-3R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrrr} 1 & 2 & -5 & b_1 \\ 0 & 1 & 7 & -3b_1+b_2 \\ 0 & 0 & -4 & 2b_1+b_3 \end{array}\right]$$

Since there exists a unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$, by The Unifying Theorem - Version 2, T is both one-to-one and onto.

29. 

30. 

31. 

32. 

33. $T(\mathbf{x}) = \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix} \mathbf{x}$

34. $T(\mathbf{x}) = \begin{bmatrix} 0 & 1 \\ 0 & 4 \\ 0 & 5 \end{bmatrix} \mathbf{x}$

35. $T(\mathbf{x}) = \begin{bmatrix} 7/3 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{x}$

36. $T(\mathbf{x}) = \begin{bmatrix} 7/3 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \mathbf{x}$

37. $T(\mathbf{x}) = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \mathbf{x}$

38. $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ -2 & 3 \end{bmatrix} \mathbf{x}$

39. (a) False. For instance $T : \mathbf{R}^2 \to \mathbf{R}^2$ defined by $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$ has $\text{range}(T) = \{\mathbf{0}\}$ but codomain equal to $\mathbf{R}^2$.
    (b) True, by Theorem 3.8.

40. (a) False. For instance $T : \mathbf{R}^2 \to \mathbf{R}$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 + x_2$ has $\text{range}(T) = \mathbf{R}$, which is not a subset of $\mathbf{R}^2$, the domain of T.
    (b) True, by definition of an onto transformation.

41. (a) True, by definition of the range of T.
    (b) True. Suppose $\mathbf{z}$ is in the codomain of W. Then $\mathbf{z}$ is in the codomain of S, and because S is onto, there exists $\mathbf{y}$ in the domain of S such that $S(\mathbf{y}) = \mathbf{z}$. Because $\mathbf{y}$ is in the domain of S, $\mathbf{y}$ is in the codomain of T, so because T is onto, there exists $\mathbf{x}$ in the domain of T such that $T(\mathbf{x}) = \mathbf{y}$. We now have $\mathbf{z} = S(\mathbf{y}) = S(T(\mathbf{x})) = W(\mathbf{x})$, and therefore W is onto.

42. (a) False. Suppose $T(\mathbf{x}) = 1$ for all $\mathbf{x} \in \mathbf{R}^1$.Then $T(1\,[1]) = 1\,(T([1]))$, but T is not linear.
    3.1.42b
    (b) False. For example, let $T([x_1]) = A[x_1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[x_1]$. Then the codomain of T is $\mathbf{R}^2$, but $\text{col}(\text{A}) = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \neq \mathbf{R}^2$.

43. (a) True. If T is linear, then $T(\mathbf{0}) = \mathbf{0}$ and so $\mathbf{b} = \mathbf{0}$. If $\mathbf{b} = \mathbf{0}$, then T is linear by Theorem 3.2.
    (b) True, by Theorem 3.9, (d) implies (c).

44. (a) False. If T is not one-to-one, the image is a segment, or just the origin. For instance, if $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then the image of the unit square is the origin.
    (b) True, by Theorem 3.9, (e) implies (b).

45. (a) False. W will be linear, but not necessarily one-to-one. Consider $T_2(\mathbf{x}) = -T_1(\mathbf{x})$ where T_1 is one-to-one.
    (b) True. Let A be $n \times m$. Since T is one-to-one, by Theorem 3.6, $n \geq m$. Since T is onto, by Theorem 3.7, $n \leq m$. Hence $n = m$, and A is a square matrix.

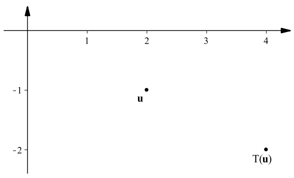
46. (a) False. W will be linear, but not necessarily onto. Consider $T_2(\mathbf{x}) = -T_1(\mathbf{x})$, where T_1 is onto.
    (b) False. For example, if $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $T(\mathbf{x}) = A\mathbf{x}$ is not one-to-one, and $A\mathbf{x} = \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ does not have a solution.

47. (a) False, by Theorem 3.5.
    (b) True, by Theorem 3.7(b).

48. (a) True, by The Unifying Theorem - Version 2.

(b) False. For example, if $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $T(\mathbf{x}) = A\mathbf{x}$ is not onto $\mathbf{R}^2$.

49. (a) $A = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}$

(b) 

50. (a)

$$
\begin{aligned}
T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) &= T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}\right) \\
&= \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\
&= T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
T\left(r\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) &= T\left(\begin{bmatrix} rx_1 \\ rx_2 \\ rx_3 \end{bmatrix}\right) \\
&= \begin{bmatrix} rx_1 \\ rx_2 \end{bmatrix} \\
&= r\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= rT\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right),
\end{aligned}
$$

hence T is a linear transformation.

(b) $T(\mathbf{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x}$

(c) $\mathbf{x} = (0, 0, x_3)$ where x_3 is any real number.

51. Let $\mathbf{u} = (u_1, \ldots, u_n)$.Then

$$\begin{aligned}
T(\mathbf{x}+\mathbf{y}) &= \mathbf{u} \cdot (\mathbf{x}+\mathbf{y}) \\
&= (u_1, \ldots, u_n) \cdot ((x_1, \ldots, x_n) + (y_1, \ldots, y_n)) \\
&= (u_1, \ldots, u_n) \cdot (x_1 + y_1, \ldots, x_n + y_n) \\
&= u_1(x_1 + y_1) + \cdots + u_n(x_n + y_n) \\
&= (u_1x_1 + u_1y_1) + \cdots + (u_nx_n + u_ny_n) \\
&= (u_1x_1 + \cdots + u_nx_n) + (u_1y_1 + \cdots + u_ny_n) \\
&= \mathbf{u} \cdot \mathbf{x} + \mathbf{u} \cdot \mathbf{y} \\
&= T(\mathbf{x}) + T(\mathbf{y}),
\end{aligned}$$

and

$$\begin{aligned}
T(r\mathbf{x}) &= \mathbf{u} \cdot (r\mathbf{x}) \\
&= (u_1, \ldots, u_n) \cdot (r(x_1, \ldots, x_n)) \\
&= (u_1, \ldots, u_n) \cdot (rx_1, \ldots, rx_n) \\
&= u_1(rx_1) + \cdots + u_n(rx_n) \\
&= r\,(u_1x_1 + \cdots + u_nx_n) \\
&= r\mathbf{u} \cdot \mathbf{x} \\
&= rT(\mathbf{x}).
\end{aligned}$$

Thus T is a linear transformation.

52. Let $\mathbf{u} = (u_1, u_2, u_3)$.Then

$$\begin{aligned}
T(\mathbf{x}+\mathbf{y}) &= \mathbf{u} \times (\mathbf{x}+\mathbf{y}) \\
&= (u_1, u_2, u_3) \times ((x_1, x_2, x_3) + (y_1, y_2, y_3)) \\
&= (u_1, u_2, u_3) \times (x_1 + y_1, x_2 + y_2, x_3 + y_3) \\
&= (u_2(x_3 + y_3) - u_3(x_2 + y_2), u_3(x_1 + y_1) - u_1(x_3 + y_3), u_1(x_2 + y_2) - u_2(x_1 + y_1)) \\
&= (u_2x_3 - u_3x_2, u_3x_1 - u_1x_3, u_1x_2 - u_2x_1) + (u_2y_3 - u_3y_2, u_3y_1 - u_1y_3, u_1y_2 - u_2y_1) \\
&= \mathbf{u} \times \mathbf{x} + \mathbf{u} \times \mathbf{y} \\
&= T(\mathbf{x}) + T(\mathbf{y}),
\end{aligned}$$

and

$$\begin{aligned}
T(r\mathbf{x}) &= \mathbf{u} \times (r\mathbf{x}) \\
&= (u_1, u_2, u_3) \times (r(x_1, x_2, x_3)) \\
&= (u_1, u_2, u_3) \times (rx_1, rx_2, rx_3) \\
&= (u_2\,(rx_3) - u_3\,(rx_2)\,, u_3\,(rx_1) - u_1\,(rx_3)\,, u_1\,(rx_2) - u_2\,(rx_1)) \\
&= (r\,(u_2x_3 - u_3x_2)\,, r\,(u_3x_1 - u_1x_3)\,, r\,(u_1x_2 - u_2x_1)) \\
&= r(u_2x_3 - u_3x_2, u_3x_1 - u_1x_3, u_1x_2 - u_2x_1) \\
&= r\,(\mathbf{u} \times \mathbf{x}) \\
&= rT(\mathbf{x}).
\end{aligned}$$

Thus T is a linear transformation.

53. Let $\mathbf{u}_1 = (1, 0, 0)$, $\mathbf{u}_2 = (0, 1, 0)$, and $\mathbf{u}_3 = (0, 0, 1)$. Since $T\,(\mathbf{u}_1)$, $T\,(\mathbf{u}_2)$, $T\,(\mathbf{u}_3)$ are three vectors in $\mathbf{R}^2$, they must be linearly dependent, and therefore there exist scalars c_1, c_2, and c_3 with at least one $c_i \neq 0$ and $c_1T\,(\mathbf{u}_1) + c_2T\,(\mathbf{u}_2) + c_3T\,(\mathbf{u}_3) = \mathbf{0}$. Since T is a linear transformation, $T\,(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3) = \mathbf{0}$. Also, since $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent and one of the $c_i \neq 0$, it follows that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \neq \mathbf{0}$. Noting that $T(\mathbf{0}) = \mathbf{0}$, $T\,(c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3) = \mathbf{0}$, and $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 \neq \mathbf{0}$, we conclude that T is not one-to-one.

54. Let $\mathbf{u}_1 = (1, 0)$ and $\mathbf{u}_2 = (0, 1)$. Since $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ are two vectors in $\mathbf{R}^4$, they do not span $\mathbf{R}^4$, and there exists a vector $\mathbf{v} \in \mathbf{R}^4$ which is not in the span of $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$. Let $\mathbf{u}$ be any vector in $\mathbf{R}^2$, then since $\mathbf{u}_1$ and $\mathbf{u}_2$ span $\mathbf{R}^2$ there are scalars c_1 and c_2 such that $\mathbf{u} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$. Since T is a linear transformation, $T(\mathbf{u}) = T(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) = c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) \neq \mathbf{v}$, since $\mathbf{v}$ is not in the span of $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$. Thus $\mathbf{v}$ is not in the range of T, and therefore T is not onto.

55. $T(\mathbf{0}) = T(\mathbf{0} + \mathbf{0}) = T(\mathbf{0}) + T(\mathbf{0}) \Rightarrow T(\mathbf{0}) = \mathbf{0}$, upon subtracting $T(\mathbf{0})$ from both sides.
(*Another proof, using the scalar property:* $T(\mathbf{0}) = T(2(\mathbf{0})) = 2T(\mathbf{0}) \Rightarrow T(\mathbf{0}) = \mathbf{0}$, *upon subtracting* $T(\mathbf{0})$ *from both sides.*)

56. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $A = [\mathbf{a}_1\ \mathbf{a}_2 \cdots \mathbf{a}_n]$. Then $T(\mathbf{u}) = A\mathbf{u} = u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_n\mathbf{a}_n = \mathbf{0}$, and since $\mathbf{u} \neq \mathbf{0}$ at least one of the $u_i \neq 0$. As a result, we may conclude that the columns of A are linearly dependent.

57. $T(r\mathbf{u}) = A(r\mathbf{u}) = r(A\mathbf{u}) = rT(\mathbf{u})$ for all scalars r and all vectors $\mathbf{u}$.

58. (a) $T(r\mathbf{x} + s\mathbf{y}) = T(r\mathbf{x}) + T(s\mathbf{y}) = rT(\mathbf{x}) + sT(\mathbf{y})$ for all scalars r and s and all vectors $\mathbf{x}$ and $\mathbf{y}$.
    (b) We have $T(\mathbf{x}+\mathbf{y}) = T(1(\mathbf{x})+1(\mathbf{y})) = 1T(\mathbf{x})+1T(\mathbf{y}) = T(\mathbf{x})+T(\mathbf{y})$, and $T(r\mathbf{x}) = T(r(\mathbf{x}+\mathbf{0})) = T(r\mathbf{x} + r\mathbf{0}) = rT(\mathbf{x}) + rT(\mathbf{0}) = rT(\mathbf{x}) + r\mathbf{0} = rT(\mathbf{x})$. Thus T is a linear transformation.

59. Suppose $T : \mathbf{R}^m \to \mathbf{R}^n$ is one-to-one, and let $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{w}$. Since there exists at most one vector whose image under T is $\mathbf{w}$, it follows that $\mathbf{u} = \mathbf{v}$. Now suppose $T(\mathbf{u}) = T(\mathbf{v})$ implies $\mathbf{u} = \mathbf{v}$. Let $\mathbf{w} \in \mathbf{R}^n$, and suppose $T(\mathbf{u}) = T(\mathbf{v}) = \mathbf{w}$. Then we must have $\mathbf{u} = \mathbf{v}$, and therefore there is at most one vector whose image under T is $\mathbf{w}$. Hence T is one-to-one.

60. Since $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly dependent, there exist scalars c_1 and c_2 such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{0}$ and with at least one $c_i \neq 0$. Apply the linear transformation T to the equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{0}$ to obtain $c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) = T(\mathbf{0}) = \mathbf{0}$. As one of the $c_i \neq 0$, this shows that $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ are linearly dependent.

61. Consider $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{0}$, and apply the linear transformation T to obtain $c_1T(\mathbf{u}_1) + c_2T(\mathbf{u}_2) = T(\mathbf{0}) = \mathbf{0}$. Since $T(\mathbf{u}_1)$ and $T(\mathbf{u}_2)$ are linearly independent, $c_1 = c_2 = 0$. This shows that $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent.

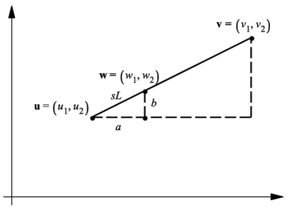
62. For example, if $T(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x}$ in $\mathbf{R}^2$, then T is a linear transformation. If $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, then $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, but $T(\mathbf{u}_1) = \mathbf{0}$ and $T(\mathbf{u}_2) = \mathbf{0}$ are linearly dependent.

63. Since $\mathbf{y}$ is in the range of T, there exists a vector $\mathbf{w}$ such that $T(\mathbf{w}) = \mathbf{y}$. For each $r \in \mathbf{R}$ define the vector $\mathbf{x}_r = \mathbf{w} + r\mathbf{u}$. Then $T(\mathbf{x}_r) = T(\mathbf{w} + r\mathbf{u}) = T(\mathbf{w}) + rT(\mathbf{u}) = \mathbf{y} + r\mathbf{0} = \mathbf{y}$ for every r. Moreover, each $\mathbf{x}_r$ is distinct, for if $\mathbf{x}_r = \mathbf{x}_s$, then $\mathbf{w} + r\mathbf{u} = \mathbf{w} + s\mathbf{u} \Rightarrow (r - s)\mathbf{u} = \mathbf{0} \Rightarrow r = s$, since $\mathbf{u} \neq \mathbf{0}$. Therefore there are infinitely many vectors $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$.

64. Suppose $u_1 < v_1$ and $u_2 < v_2$, and let $\mathbf{w} = (w_1, w_2)$ be a point on the segment joining $\mathbf{u}$ and $\mathbf{v}$ at a distance sL from $\mathbf{u}$, where L is the distance between $\mathbf{u}$ and $\mathbf{v}$, and $0 \leq s \leq 1$. Write $\mathbf{w} = \mathbf{u} + (a, b)$. By considering similar triangles, we have $a : sL = (v_1 - u_1) : L$, hence $a = s(v_1 - u_1)$. Likewise, we also have that $b : sL = (v_2 - u_2) : L$, hence $b = s(v_2 - u_2)$. We thus determine the vector

$$\begin{aligned}
\mathbf{w} &= (w_1, w_2) \\
&= (u_1 + a, u_2 + b) \\
&= (u_1 + s(v_1 - u_1), u_2 + s(v_2 - u_2)) \\
&= ((1 - s)u_1 + sv_1, (1 - s)u_2 + sv_2) \\
&= (1 - s)(u_1, u_2) + s(v_1, v_2) \\
&= (1 - s)\mathbf{u} + s\mathbf{v}
\end{aligned}$$

We conclude that the set of points joining $\mathbf{u}$ and $\mathbf{v}$ is the same as the set of points $(1 - s)\mathbf{u} + s\mathbf{v}$, $0 \leq s \leq 1$. The proof in the cases where $u_1 \geq v_1$ or $u_1 \geq v_1$ is handled similarly.



65. Let $\mathbf{u} = (1, 0)$ and $\mathbf{v} = (0, 1)$, and $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$. The unit square consists of all vectors $\mathbf{x} = s\mathbf{u} + t\mathbf{v}$ where $0 \le s \le 1$ and $0 \le t \le 1$. The image consists of all vectors $T(\mathbf{x}) = T(s\mathbf{u}+t\mathbf{v}) = s\,(A\mathbf{u})+t\,(A\mathbf{v}) = s\mathbf{a}_1 + t\mathbf{a}_2$. If the columns of A are linearly independent, then $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly independent, and neither vector is a multiple of the other. Thus $\{s\mathbf{a}_1 + t\mathbf{a}_2 : 0 \le s \le 1, 0 \le t \le 1\}$ is a parallelogram. If the columns of A are linearly dependent, then $\{\mathbf{a}_1, \mathbf{a}_2\}$ is linearly dependent, so one vector is a multiple of the other, and $\{s\mathbf{a}_1 + t\mathbf{a}_2 : 0 \le s \le 1, 0 \le t \le 1\}$ is a segment. In the linearly dependent case, if $\mathbf{a}_1 = \mathbf{a}_2 = \mathbf{0}$, then $\{s\mathbf{a}_1 + t\mathbf{a}_2 : 0 \le s \le 1, 0 \le t \le 1\} = \{\mathbf{0}\}$, a point.

66. (a) The number of 1's in the j^{th} row (or column) is the number of edges connected to node j. One can determine this by evaluating $T(\mathbf{x}) = A\mathbf{x}$, using $\mathbf{x} = (1, 1, 1, 1, 1)$, which gives each row sum:

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 2 \\ 2 \end{bmatrix}.$$

(b) The total number of graph edges will be the total number of ones in the adjacency matrix divided by two, since each edge corresponds to two vertices. In this case, we have $(3 + 3 + 2 + 2 + 2)/2 = 6$ edges in our graph.

67. (a) Let $p_1(x) = a_1x^2 + b_1x + c_1$ and $p_2(x) = a_2x^2 + b_2x + c_2$ be two polynomials of degree 2 or less, and r a constant. Then

$$\begin{aligned} (p_1 + p_2)(x) &= \left(a_1x^2 + b_1x + c_1\right) + \left(a_2x^2 + b_2x + c_2\right) \\ &= (a_1 + a_2)x^2 + (b_1 + b_2)\,x + (c_1 + c_2) \\ &\leftrightarrow \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \\ &\leftrightarrow p_1(x) + p_2(x) \end{aligned}$$

and

$$\begin{aligned} (rp_1)(x) &= r\left(a_1x^2 + b_1x + c_1\right) \\ &= (ra_1)\,x^2 + (rb_1)\,x + (rc_1) \\ &\leftrightarrow \begin{bmatrix} ra_1 \\ rb_1 \\ rc_1 \end{bmatrix} = r\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \\ &\leftrightarrow R\,(p_1(x)) \end{aligned}$$

Therefore addition of polynomials corresponds to addition of vectors, and scalar multiplication of polynomials corresponds to scalar multiplication of vectors.

(b) With p_1 and p_2 as above, we have

$$\begin{aligned}
T(p_1(x)+p_2(x)) &= \left(\left(a_1x^2+b_1x+c_1\right)+\left(a_2x^2+b_2x+c_2\right)\right)' \\
&= \left((a_1+a_2)x^2+(b_1+b_2)\,x+(c_1+c_2)\right)' \\
&= (2(a_1+a_2))x+(b_1+b_2) \\
&\leftrightarrow \begin{bmatrix} 0 \\ 2(a_1+a_2) \\ b_1+b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2a_2 \\ b_2 \end{bmatrix} \\
&\leftrightarrow (2a_1x+b_1)+(2a_2x+b_2) \\
&= \left(a_1x^2+b_1x+c_1\right)' + \left(a_2x^2+b_2x+c_2\right)' \\
&= T(p_1(x))+T(p_2(x))
\end{aligned}$$

and

$$\begin{aligned}
T(rp_1(x)) &= (rp_1(x))' \\
&= \left(ra_1x^2+rb_1x+rc_1\right)' \\
&= (2ra_1x+rb_1) \\
&\leftrightarrow \begin{bmatrix} 0 \\ 2ra_1 \\ rb_1 \end{bmatrix} = r\begin{bmatrix} 0 \\ 2a_1 \\ b_1 \end{bmatrix} \\
&\leftrightarrow R\,(2a_1x+b_1) \\
&= r\left(a_1x^2+b_1x+c_1\right)' \\
&= rT(p_1(x))
\end{aligned}$$

Thus T is a linear transformation.

(c) Since $p(x)=ax^2+bx+c$ has derivative $p'(x)=(2a)x+b$, we can represent T by

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\mathbf{x}$$

(d) T is not onto, as there is no polynomial $p(x)$ of degree 2 or less with $p'(x) = x^2$. T is not one-to-one, as $T(x^2) = T(x^2+1)$. One can also use the Unifying Theorem - Version 2, and the observation that the columns of the matrix A are linearly dependent to conclude that T is neither onto nor one-to-one.

68. (a) Let $p_1(x) = a_1x^3+b_1x^2+c_1x+d_1$ and $p_2(x) = a_2x^3+b_2x^2+c_2x+d_2$ be two polynomials of degree 3 or less, and r a constant. Then

$$\begin{aligned}
(p_1+p_2)\,(x) &= \left(a_1x^3+b_1x^2+c_1x+d_1\right)+\left(a_2x^3+b_2x^2+c_2x+d_2\right) \\
&= (a_1+a_2)x^3+(b_1+b_2)\,x^2+(c_1+c_2)\,x+(d_1+d_2) \\
&\leftrightarrow \begin{bmatrix} a_1+a_2 \\ b_1+b_2 \\ c_1+c_2 \\ d_1+d_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} \\
&\leftrightarrow p_1(x)+p_2(x)
\end{aligned}$$

and

$$\begin{aligned}(rp_1)(x) &= r\left(a_1x^3 + b_1x^2 + c_1x + d_1\right)\\ &= (ra_1)\,x^3 + (rb_1)\,x^2 + (rc_1)\,x + (rd_1)\\ &\leftrightarrow \begin{bmatrix} ra_1 \\ rb_1 \\ rc_1 \\ rd_1 \end{bmatrix} = r\begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix}\\ &\leftrightarrow R\,(p_1(x))\end{aligned}$$

Therefore addition of polynomials corresponds to addition of vectors, and scalar multiplication of polynomials corresponds to scalar multiplication of vectors.

(b) With p_1 and p_2 as above, we have

$$\begin{aligned}T(p_1(x)+p_2(x)) &= \left(\left(a_1x^3 + b_1x^2 + c_1x + d_1\right) + \left(a_2x^3 + b_2x^2 + c_2x + d_2\right)\right)'\\ &= \left((a_1+a_2)x^3 + (b_1+b_2)\,x^2 + (c_1+c_2)\,x + (d_1+d_2)\right)'\\ &= (3(a_1+a_2))x^2 + (2\,(b_1+b_2))x + (c_1+c_2)\\ &\leftrightarrow \begin{bmatrix} 0 \\ 3(a_1+a_2) \\ 2\,(b_1+b_2) \\ c_1+c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3a_1 \\ 2b_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3a_2 \\ 2b_2 \\ c_2 \end{bmatrix}\\ &\leftrightarrow \left(3a_1x^2 + 2b_1x + c_1\right) + \left(3a_2x^2 + 2b_2x + c_2\right)\\ &= \left(a_1x^3 + b_1x^2 + c_1x + d_1\right)' + \left(a_2x^3 + b_2x^2 + c_2x + d_2\right)'\\ &= T(p_1(x)) + T(p_2(x))\end{aligned}$$

and

$$\begin{aligned}T(rp_1(x)) &= (rp_1(x))'\\ &= \left(ra_1x^3 + rb_1x^2 + rc_1x + rd_1\right)'\\ &= \left(3ra_1x^2 + 2rb_1x + rc_1\right)\\ &\leftrightarrow \begin{bmatrix} 0 \\ 3ra_1 \\ 2rb_1 \\ rc_1 \end{bmatrix} = r\begin{bmatrix} 0 \\ 3a_1 \\ 2b_1 \\ c_1 \end{bmatrix}\\ &\leftrightarrow R\left(3a_1x^2 + 2b_1x + c_1\right)\\ &= r\left(a_1x^3 + b_1x^2 + c_1x + d_1\right)'\\ &= rT(p_1(x))\end{aligned}$$

Thus T is a linear transformation.

(c) Since $p(x) = ax^3 + bx^2 + cx + d$ has derivative $p'(x) = (3a)x^2 + (2b)x + c$, we can represent T by

$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x}$$

(d) T is not onto, as there is no polynomial $p(x)$ of degree 3 or less with $p'(x) = x^3$. T is not one-to-one, as $T(x^3) = T(x^3+1)$. One can also use the Unifying Theorem - Version 2, and the observation that the columns of the matrix A are linearly dependent to conclude that T is neither onto nor one-to-one.

69. (a) $T(x^2+\sin x)=\left(x^2+\sin x\right)'=2x+\cos x$

(b) i. $T\left(f(x)+g(x)\right)=\left(f(x)+g(x)\right)'=f'(x)+g'(x)=T\left(f(x)\right)+T\left(g(x)\right)$

ii. $T\left(rf(x)\right)=\left(rf(x)\right)'=rf'(x)=rT\left(f(x)\right)$

70. (a) $T\left(4x^3-6x^2+1\right)=\int_0^1\left(4x^3-6x^2+1\right)\,dx=\left(x^4-2x^3+x\right)|_0^1$
$=(1-2+1)-(0-0+0)=0$

(b) i. $T\left(f(x)+g(x)\right)=\int_0^1\left(f(x)+g(x)\right)\,dx=\int_0^1 f(x)\,dx+\int_0^1 g(x)\,dx$
$=T\left(f(x)\right)+T\left(g(x)\right)$

ii. $T\left(rf(x)\right)=\int_0^1\left(rf(x)\right)\,dx=r\int_0^1 f(x)\,dx=rT\left(f(x)\right)$

71. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}57&73\\93&101\\29&34\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, so

$$T\left(\begin{bmatrix}5\\6\end{bmatrix}\right)=\begin{bmatrix}57&73\\93&101\\29&34\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix}=\begin{bmatrix}723\\1071\\349\end{bmatrix}.$$

72. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}57&73\\93&101\\29&34\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, so

$$T\left(\begin{bmatrix}6\\10\end{bmatrix}\right)=\begin{bmatrix}57&73\\93&101\\29&34\end{bmatrix}\begin{bmatrix}6\\10\end{bmatrix}=\begin{bmatrix}1072\\1568\\514\end{bmatrix}.$$

73. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}57&73\\93&101\\29&34\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, so

$$T\left(\begin{bmatrix}8\\16\end{bmatrix}\right)=\begin{bmatrix}57&73\\93&101\\29&34\end{bmatrix}\begin{bmatrix}8\\16\end{bmatrix}=\begin{bmatrix}1624\\2360\\776\end{bmatrix}.$$

74. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}57&73\\93&101\\29&34\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$, so

$$T\left(\begin{bmatrix}12\\20\end{bmatrix}\right)=\begin{bmatrix}57&73\\93&101\\29&34\end{bmatrix}\begin{bmatrix}12\\20\end{bmatrix}=\begin{bmatrix}2144\\3136\\1028\end{bmatrix}.$$

75. Using a computer algebra system, the matrix has row-reduced echelon form

$$\begin{bmatrix}4&2&-5&2&6\\7&-2&0&-4&1\\0&3&-5&7&-1\end{bmatrix}\sim\begin{bmatrix}1&0&0&-6&13\\0&1&0&-19&45\\0&0&1&-\frac{64}{5}&\frac{136}{5}\end{bmatrix}$$

Hence T is onto, since $A\mathbf{x}=\mathbf{b}$ has solutions for all $\mathbf{b}$. T is not one-to-one, since $A\mathbf{x}=\mathbf{0}$ has nontrivial solutions.

76. Using a computer algebra system, the matrix has row-reduced echelon form

$$\begin{bmatrix}4&-2&5&2&1\\5&14&4&-5&8\\-1&6&-2&-3&2\end{bmatrix}\sim\begin{bmatrix}1&0&\frac{13}{11}&\frac{3}{11}&\frac{5}{11}\\0&1&-\frac{3}{22}&-\frac{5}{11}&\frac{9}{22}\\0&0&0&0&0\end{bmatrix}$$

Hence T is not onto, since $A\mathbf{x}=\mathbf{b}$ does not have solutions for all $\mathbf{b}$. T is not one-to-one, since $A\mathbf{x}=\mathbf{0}$ has nontrivial solutions.

77. Using a computer algebra system, the matrix has row-reduced echelon form

$$\begin{bmatrix} 2 & -1 & 4 & 0 \\ 3 & -3 & 1 & 1 \\ 1 & -1 & 8 & 3 \\ 0 & -2 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{37}{23} \\ 0 & 1 & 0 & -\frac{42}{23} \\ 0 & 0 & 1 & \frac{8}{23} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence T is not onto, since $A\mathbf{x} = \mathbf{b}$ does not have solutions for all $\mathbf{b}$. T is not one-to-one, since $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

78. Using a computer algebra system, the matrix has row-reduced echelon form

$$\begin{bmatrix} 3 & 2 & 0 & 5 \\ 0 & 1 & 2 & -3 \\ -2 & -1 & 3 & 1 \\ 4 & -2 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hence T is onto, since $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$. T is one-to-one, since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

79. Using a computer algebra system, the matrix has row-reduced echelon form

$$\begin{bmatrix} 2 & -3 & 5 & 1 \\ 6 & 0 & 3 & -2 \\ -4 & 2 & 1 & 1 \\ 8 & 2 & 3 & -4 \\ -1 & 2 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence T is not onto, since $A\mathbf{x} = \mathbf{b}$ does not have solutions for all $\mathbf{b}$. T is one-to-one, since $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

80. Using a computer algebra system, the matrix has row-reduced echelon form

$$\begin{bmatrix} 4 & 3 & -2 & 9 \\ -1 & 0 & 1 & -1 \\ 3 & 0 & -2 & 4 \\ 2 & -4 & 3 & 3 \\ 5 & -7 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence T is not onto, since $A\mathbf{x} = \mathbf{b}$ does not have solutions for all $\mathbf{b}$. T is not one-to-one, since $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

3.2 Practice Problems

1. (a) $A + B = \begin{bmatrix} 2 & 5 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & -9 \end{bmatrix}$. AC is not defined.

   (b) $B - 3I_2 = \begin{bmatrix} 3 & 1 \\ 4 & -5 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & -8 \end{bmatrix}$. DB is not defined.

   (c) $CB = \begin{bmatrix} 2 & 1 \\ 5 & 4 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & -5 \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ 31 & -15 \\ -4 & 5 \end{bmatrix}$; $A^2 = \begin{bmatrix} 2 & 5 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 19 & -10 \\ -6 & 31 \end{bmatrix}$

   (d) $C^T - D = \begin{bmatrix} 2 & 1 \\ 5 & 4 \\ 0 & -1 \end{bmatrix}^T - \begin{bmatrix} 2 & -2 & -3 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 0 \\ 1 & 4 & -1 \end{bmatrix} - \begin{bmatrix} 2 & -2 & -3 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 7 & 3 \\ 1 & 1 & -2 \end{bmatrix}$;

   $DC = \begin{bmatrix} 2 & -2 & -3 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 5 & 4 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 15 & 11 \end{bmatrix}$

2. Set $A^2 = A$,

$$\begin{bmatrix} a & 2 \\ -1 & 1 \end{bmatrix}^2 = \begin{bmatrix} a & 2 \\ -1 & 1 \end{bmatrix}$$
$$\begin{bmatrix} a^2-2 & 2a+2 \\ -a-1 & -1 \end{bmatrix} = \begin{bmatrix} a & 2 \\ -1 & 1 \end{bmatrix}$$

Because row 2, column 2 requires $-1 = 1$, we conclude that there are no solutions.

3. We have $T_1(\mathbf{x}) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\mathbf{x}$, and $T_2(\mathbf{x}) = \begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}\mathbf{x}$.

(a) $T_1(T_2(\mathbf{x})) = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\left(\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) =$
$\left(\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so $A = \begin{bmatrix} -3 & 5 \\ 4 & -6 \end{bmatrix}$.

(b) $T_2(T_2(\mathbf{x})) = \begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}\left(\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) =$
$\left(\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}\right)\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 & -6 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so $A = \begin{bmatrix} 4 & -6 \\ -2 & 4 \end{bmatrix}$.

4. (a) $-2R_1 + R_2 \to R_2$
(b) $4R_2 \to R_2$
(c) $3R_1 + R_2 \to R_2$
(d) $R_1 \leftrightarrow R_3$

5. (a) True, because $A^2 = AA$ is defined only if $n = m$.
(b) True, as the transpose operation distributes over addition and scalar multiplication.
(c) False. For example, $AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is not diagonal.
(d) False. For example, $E_1E_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is not an elementary matrix.

3.2 Matrix Algebra

1. (a) $A + B = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -3 & 5 \\ 0 & 4 \end{bmatrix}$
(b) $AB + I_2 = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -7 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -7 \\ 2 & 4 \end{bmatrix}$
(c) $A + C$ is not possible, since A and C are different sizes.

2. (a) AC is not possible, since A has 2 columns, and C has 3 rows.
(b) $C + D^T = \begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -2 \\ 0 & 5 \\ -3 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -1 & 9 \\ 0 & 2 \end{bmatrix}$
(c) $CB + I_2$ is not possible, since CB is a 3×2 matrix, and I_2 is a 2×2 matrix.

3. (a) $(AB)^T = \left(\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}\right)^T = \begin{bmatrix} -2 & -7 \\ 2 & 3 \end{bmatrix}^T = \begin{bmatrix} -2 & 2 \\ -7 & 3 \end{bmatrix}$

(b) CE is not defined, since C has 2 columns, and E has 3 rows.

(c) $(A-B)\,D = \left(\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix}$

$= \begin{bmatrix} -3 & -3 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -15 & 12 \\ 16 & -30 & -6 \end{bmatrix}$

4. (a) $A^3 = \left(\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\right)\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} =$

$\begin{bmatrix} 11 & -4 \\ -8 & 3 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -41 & 15 \\ 30 & -11 \end{bmatrix}$

(b) $BC^T = \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix}^T = \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 5 & -1 & 3 \\ 0 & 4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 16 & 12 \\ -10 & 22 & 9 \end{bmatrix}$

(c) $EC + I_3$ is not possible, since EC is a 3×2 matrix, and I_3 is a 3×3 matrix.

5. (a) $(C+E)\,B$ is not possible, since C and E are different sizes.

(b) $B\left(C^T + D\right) = \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}\left(\begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix}^T + \begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix}\right)$

$= \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}\left(\begin{bmatrix} 5 & -1 & 3 \\ 0 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix}\right)$

$= \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 6 & -1 & 0 \\ -2 & 9 & 2 \end{bmatrix} = \begin{bmatrix} -8 & 36 & 8 \\ -22 & 47 & 10 \end{bmatrix}$

(c) $E + CD = \begin{bmatrix} 1 & 4 & -5 \\ -2 & 1 & -3 \\ 0 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 4 & -5 \\ -2 & 1 & -3 \\ 0 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 0 & -15 \\ -9 & 20 & -1 \\ -3 & 15 & -12 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -20 \\ -11 & 21 & -4 \\ -3 & 17 & -6 \end{bmatrix}$

6. (a) $AD - C^T = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix}^T$

$= \begin{bmatrix} -5 & 5 & 8 \\ 4 & -5 & -5 \end{bmatrix} - \begin{bmatrix} 5 & -1 & 3 \\ 0 & 4 & 3 \end{bmatrix} = \begin{bmatrix} -10 & 6 & 5 \\ 4 & -9 & -8 \end{bmatrix}$

(b) $AB - DC = \begin{bmatrix} -3 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix}$

$= \begin{bmatrix} -2 & -7 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} -4 & -9 \\ -18 & 17 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 20 & -14 \end{bmatrix}$

(c) $DE + CB$ is not possible, since DE is a 2×3 matrix and CB is a 3×2 matrix.

7. $\begin{bmatrix} 2 & a \\ 3 & -2 \end{bmatrix}\begin{bmatrix} b & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2b-a & 2a-6 \\ 3b+2 & -13 \end{bmatrix} = \begin{bmatrix} 3 & -8 \\ 5 & c \end{bmatrix} \quad \Rightarrow$
$c = -13$, $3b+2 = 5 \ \Rightarrow\ b = 1$, $2a - 6 = -8 \ \Rightarrow\ a = -1$.

8. $\begin{bmatrix} 1 & 4 \\ a & 7 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ b & 3 \end{bmatrix} = \begin{bmatrix} 4b+2 & 11 \\ 2a+7b & 21-a \end{bmatrix} = \begin{bmatrix} 6 & d \\ 11 & c \end{bmatrix} \ \Rightarrow\ 4b+2 = 6 \ \Rightarrow\ b = 1,\ d = 11,$
$2a + 7b = 11 \ \Rightarrow\ 2a + 7(1) = 11 \ \Rightarrow\ a = 2,\ 21 - a = c \ \Rightarrow\ 21 - (2) = c \ \Rightarrow\ c = 19.$

9. $\begin{bmatrix} a & 3 & -2 \\ 3 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & b \\ c & 1 \end{bmatrix} = \begin{bmatrix} 2a-2c & 3b-a-2 \\ 4c+6 & 1-2b \end{bmatrix} = \begin{bmatrix} 4 & d \\ -6 & -5 \end{bmatrix} \Rightarrow$
$4c+6=-6 \Rightarrow c=-3$, $1-2b=-5 \Rightarrow b=3$, $2a-2c=4 \Rightarrow 2a-2(-3)=4 \Rightarrow a=-1$, $3b-a-2=d \Rightarrow 3(3)-(-1)-2=d \Rightarrow d=8$

10. $\begin{bmatrix} 1 & a \\ 0 & -2 \\ 5 & b \end{bmatrix} \begin{bmatrix} 3 & c & d \\ -2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 3-2a & a+c & 2a+d \\ 4 & -2 & -4 \\ 15-2b & b+5c & 2b+5d \end{bmatrix} = \begin{bmatrix} -3 & 3 & 7 \\ e & -2 & -4 \\ f & -2 & 1 \end{bmatrix} \Rightarrow$
$3-2a=-3 \Rightarrow a=3$, $a+c=3 \Rightarrow 3+c=3 \Rightarrow c=0$, $2a+d=7 \Rightarrow 2(3)+d=7 \Rightarrow d=1$, $4=e \Rightarrow e=4$, $b+2c=-2 \Rightarrow b+2(0)=-2 \Rightarrow b=-2$, $15-2b=f \Rightarrow 15-2(-2)=f \Rightarrow f=19$

11. $A^2 = \begin{bmatrix} 5 & -10 \\ a & -4 \end{bmatrix} \begin{bmatrix} 5 & -10 \\ a & -4 \end{bmatrix} = \begin{bmatrix} 25-10a & -10 \\ a & 16-10a \end{bmatrix}$. Setting this equal to A, we obtain $25-10a=5 \Rightarrow a=2$. We check that all entries of A^2 and A are equal when $a=2$.

12. $A^3 = \left(\begin{bmatrix} -2 & 2 \\ -1 & a \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -1 & a \end{bmatrix} \right) \begin{bmatrix} -2 & 2 \\ -1 & a \end{bmatrix} = \begin{bmatrix} 2 & 2a-4 \\ 2-a & a^2-2 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ -1 & a \end{bmatrix}$
$= \begin{bmatrix} -2a & a(2a-4)+4 \\ -a^2+2a-2 & a\left(a^2-2\right)-2a+4 \end{bmatrix}$. Setting this equal to $2A = 2\begin{bmatrix} -2 & 2 \\ -1 & a \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ -2 & 2a \end{bmatrix}$ we obtain $-2a=-4 \Rightarrow a=2$. We check that all entries of A^3 and $2A$ are equal when $a=2$.

13. We first determine that $T_1(\mathbf{x}) = A_1\mathbf{x} = \begin{bmatrix} 3 & 5 \\ -2 & 7 \end{bmatrix}$ and $T_2(\mathbf{x}) = A_2\mathbf{x} = \begin{bmatrix} -2 & 9 \\ 0 & 5 \end{bmatrix}$.

(a) $T_1(T_2(\mathbf{x})) = T_1(A_2\mathbf{x}) = A_1(A_2\mathbf{x}) = (A_1A_2)\mathbf{x}$.
So $A = A_1A_2 = \begin{bmatrix} 3 & 5 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} -2 & 9 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -6 & 52 \\ 4 & 17 \end{bmatrix}$

(b) $T_2(T_1(\mathbf{x})) = T_2(A_1\mathbf{x}) = A_2(A_1\mathbf{x}) = (A_2A_1)\mathbf{x}$.
So $A = A_2A_1 = \begin{bmatrix} -2 & 9 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -24 & 53 \\ -10 & 35 \end{bmatrix}$

(c) $T_1(T_1(\mathbf{x})) = T_1(A_1\mathbf{x}) = A_1(A_1\mathbf{x}) = (A_1A_1)\mathbf{x}$.
So $A = A_1A_1 = \begin{bmatrix} 3 & 5 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -2 & 7 \end{bmatrix} = \begin{bmatrix} -1 & 50 \\ -20 & 39 \end{bmatrix}$

(d) $T_2(T_2(\mathbf{x})) = T_2(A_2\mathbf{x}) = A_2(A_2\mathbf{x}) = (A_2A_2)\mathbf{x}$.
So $A = A_2A_2 = \begin{bmatrix} -2 & 9 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -2 & 9 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 27 \\ 0 & 25 \end{bmatrix}$

14. We first determine that $T_1(\mathbf{x}) = A_1\mathbf{x} = \begin{bmatrix} -2 & 3 \\ 1 & 6 \end{bmatrix}$ and $T_2(\mathbf{x}) = A_2\mathbf{x} = \begin{bmatrix} 4 & -5 \\ 1 & 5 \end{bmatrix}$.

(a) $T_1(T_2(\mathbf{x})) = T_1(A_2\mathbf{x}) = A_1(A_2\mathbf{x}) = (A_1A_2)\mathbf{x}$.
So $A = A_1A_2 = \begin{bmatrix} -2 & 3 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 25 \\ 10 & 25 \end{bmatrix}$

(b) $T_2(T_1(\mathbf{x})) = T_2(A_1\mathbf{x}) = A_2(A_1\mathbf{x}) = (A_2A_1)\mathbf{x}$.
So $A = A_2A_1 = \begin{bmatrix} 4 & -5 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} -13 & -18 \\ 3 & 33 \end{bmatrix}$

(c) $T_1(T_1(\mathbf{x})) = T_1(A_1\mathbf{x}) = A_1(A_1\mathbf{x}) = (A_1A_1)\mathbf{x}$.
So $A = A_1A_1 = \begin{bmatrix} -2 & 3 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 39 \end{bmatrix}$

(d) $T_2(T_2(\mathbf{x})) = T_2(A_2\mathbf{x}) = A_2(A_2\mathbf{x}) = (A_2A_2)\mathbf{x}$.
So $A = A_2A_2 = \begin{bmatrix} 4 & -5 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & -5 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 11 & -45 \\ 9 & 20 \end{bmatrix}$

15. $(A+I)(A-I) = A(A-I) + I(A-I) = A(A) - A(I) + (A-I) = A^2 - A + A - I = A^2 - I$

16. $(A+I)\left(A^2+A\right) = A\left(A^2+A\right) + I\left(A^2+A\right) = A\left(A^2\right) + A(A) + \left(A^2+A\right)$
$= A^3 + A^2 + A^2 + A = A^3 + 2A^2 + A$

17. $\left(A+B^2\right)(BA-A) = A(BA-A) + B^2(BA-A) = A(BA) - A(A) + B^2(BA) - B^2A$
$= ABA - A^2 + B^3A - B^2A$

18. $A(A+B) + B(B-A) = A(A) + A(B) + B(B) - B(A) = A^2 + AB + B^2 - BA$

19. $(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = A^2 + AB + BA + B^2$. This only is equal to $A^2 + 2AB + B^2$ when $AB + BA = 2AB \quad \Leftrightarrow \quad AB = BA$, which in general is not true. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $(A+B)^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, but $A^2 + 2AB + B^2 =$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 + 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$.

20. $(A-B)^2 = (A-B)(A-B) = A(A-B) - B(A-B) = A^2 - AB - BA + B^2$. This only is equal to $A^2 - 2AB + B^2$ when $-AB - BA = -2AB \quad \Leftrightarrow \quad AB = BA$, which in general is not true. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $(A-B)^2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, but $A^2 - 2AB + B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 - 2\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$.

21. $(A-B)(A+B) = A(A+B) - B(A+B) = A^2 + AB - BA - B^2 = \left(A^2 - B^2\right) + AB - BA$, so $A^2 - B^2 = (A-B)(A+B) - AB + BA$. This only is equal to $(A-B)(A+B)$ when $-AB + BA = 0_{n\times n} \quad \Leftrightarrow$ $AB = BA$, which in general is not true. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A^2 - B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, but $(A-B)(A+B) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} =$ $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

22. $(A+B)\left(A^2 - AB + B^2\right) = A\left(A^2 - AB + B^2\right) + B\left(A^2 - AB + B^2\right) = A^3 - A^2B + AB^2 + BA^2 - BAB + B^3 = \left(A^3 + B^3\right) - A\left(AB - B^2\right) + B\left(A^2 - AB\right) = \left(A^3 + B^3\right) - A(A-B)B + BA(A-B)$, so $A^3 + B^3 = (A+B)\left(A^2 - AB + B^2\right) + [A(A-B)]B - B[A(A-B)]$. This only is equal to $(A+B)\left(A^2 - AB + B^2\right)$ when $[A(A-B)]B - B[A(A-B)] = 0_{n\times n} \quad \Leftrightarrow \quad [A(A-B)]B = B[A(A-B)]$, which in general is not true. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A^3 + B^3 =$ $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^3 + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, but $(A+B)\left(A^2 - AB + B^2\right) =$
$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2\right) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$.

23. AB is 4×5, 4 rows and 5 columns.

24. BA is not defined.

25. $E = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

26. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

27. $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

28. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

29. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

30. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$

31. $\{-2R_1 + R_2 \to R_2\} \leftrightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\{5R_3 \to R_3\} \leftrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Thus $B = E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

32. $\{-6R_2 + R_3 \to R_3\} \leftrightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{bmatrix}$ and $\{R_1 \leftrightarrow R_3\} \leftrightarrow E_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Thus $B = E_2E_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -6 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -6 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

33. $\{R_2 \leftrightarrow R_1\} \leftrightarrow E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\{3R_1 + R_2 \to R_2\} \leftrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus $B = E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

34. $\{-2R_1 \to R_1\} \leftrightarrow E_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $\{7R_2 + R_3 \to R_3\} \leftrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$. Thus $B = E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$.

35. $\{-3R_1 \to R_1\} \leftrightarrow E_1 = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $\{R_1 \leftrightarrow R_2\} \leftrightarrow E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

And $\{4R_1 + R_2 \to R_2\} \leftrightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Thus $B = E_3E_2E_1$

$= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

36. $\{-3R_1 + R_2 \to R_2\} \leftrightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $\{2R_1 + R_3 \to R_3\} \leftrightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

And $\{-R_2 + R_3 \to R_3\} \leftrightarrow E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. Thus $B = E_3E_2E_1$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -1 & 1 \end{bmatrix}.$$

37. $$A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 0 & 1 & 4 \\ -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & -2 & -1 & 3 \\ -2 & 0 & 1 & 4 \\ \hline -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$B = \begin{bmatrix} 2 & 0 & -1 & 1 \\ -3 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \\ 2 & 2 & -1 & -2 \end{bmatrix} = \left[\begin{array}{cc|cc} 2 & 0 & -1 & 1 \\ -3 & 1 & 2 & 1 \\ \hline 0 & -1 & -2 & 3 \\ 2 & 2 & -1 & -2 \end{array}\right] = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

(a) $$A - B = \left[\begin{array}{c|c} A_{11} - B_{11} & A_{12} - B_{12} \\ \hline A_{21} - B_{21} & A_{22} - B_{22} \end{array}\right] = \left[\begin{array}{cc|cc} -1 & -2 & 0 & 2 \\ 1 & -1 & -1 & 3 \\ \hline -1 & 3 & 0 & -3 \\ -2 & -1 & 3 & 3 \end{array}\right]$$

(b) $$AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 14 & 5 & -6 & -10 \\ 4 & 7 & -4 & -7 \\ \hline -8 & 4 & 9 & -5 \\ -1 & 1 & -3 & 5 \end{array}\right]$$

(c) $$BA = \left[\begin{array}{c|c} B_{11}A_{11} + B_{12}A_{21} & B_{11}A_{12} + B_{12}A_{22} \\ \hline B_{21}A_{11} + B_{22}A_{21} & B_{21}A_{12} + B_{22}A_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 3 & -5 & 2 & 7 \\ -7 & 11 & 2 & -4 \\ \hline 4 & -1 & 9 & -1 \\ -1 & -8 & -2 & 12 \end{array}\right]$$

38. $$A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 0 & 1 & 4 \\ -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & -2 & -1 & 3 \\ -2 & 0 & 1 & 4 \\ -1 & 2 & -2 & 0 \\ \hline 0 & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$$

$$B = \begin{bmatrix} 2 & 0 & -1 & 1 \\ -3 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \\ 2 & 2 & -1 & -2 \end{bmatrix} = \left[\begin{array}{ccc|c} 2 & 0 & -1 & 1 \\ -3 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \\ \hline 2 & 2 & -1 & -2 \end{array}\right] = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

(a) $$A + B = \left[\begin{array}{c|c} A_{11} + B_{11} & A_{12} + B_{12} \\ \hline A_{21} + B_{21} & A_{22} + B_{22} \end{array}\right] = \left[\begin{array}{ccc|c} 3 & -2 & -2 & 4 \\ -5 & 1 & 3 & 5 \\ -1 & 1 & -4 & 3 \\ \hline 2 & 3 & 1 & -1 \end{array}\right]$$

(b) $$AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right] = \left[\begin{array}{ccc|c} 14 & 5 & -6 & -10 \\ 4 & 7 & -4 & -7 \\ -8 & 4 & 9 & -5 \\ \hline -1 & 1 & -3 & 5 \end{array}\right]$$

(c) $BA = \left[\begin{array}{c|c} B_{11}A_{11} + B_{12}A_{21} & B_{11}A_{12} + B_{12}A_{22} \\ \hline B_{21}A_{11} + B_{22}A_{21} & B_{21}A_{12} + B_{22}A_{22} \end{array}\right] = \left[\begin{array}{rrr|r} 3 & -5 & 2 & 7 \\ -7 & 11 & 2 & -4 \\ 4 & -1 & 9 & -1 \\ \hline -1 & -8 & -2 & 12 \end{array}\right]$

39. $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 0 & 1 & 4 \\ -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \left[\begin{array}{rrr|r} 1 & -2 & -1 & 3 \\ \hline -2 & 0 & 1 & 4 \\ -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$

$B = \begin{bmatrix} 2 & 0 & -1 & 1 \\ -3 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \\ 2 & 2 & -1 & -2 \end{bmatrix} = \left[\begin{array}{rrr|r} 2 & 0 & -1 & 1 \\ \hline -3 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \\ 2 & 2 & -1 & -2 \end{array}\right] = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$

(a) $B - A = \left[\begin{array}{c|c} A_{11} - B_{11} & A_{12} - B_{12} \\ \hline A_{21} - B_{21} & A_{22} - B_{22} \end{array}\right] = \left[\begin{array}{rrr|r} 1 & 2 & 0 & -2 \\ \hline -1 & 1 & 1 & -3 \\ 1 & -3 & 0 & 3 \\ 2 & 1 & -3 & -3 \end{array}\right]$

(b) $AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right] = \left[\begin{array}{rrr|r} 14 & 5 & -6 & -10 \\ \hline 4 & 7 & -4 & -7 \\ -8 & 4 & 9 & -5 \\ -1 & 1 & -3 & 5 \end{array}\right]$

(c) $BA + A = \left[\begin{array}{c|c} B_{11}A_{11} + B_{12}A_{21} & B_{11}A_{12} + B_{12}A_{22} \\ \hline B_{21}A_{11} + B_{22}A_{21} & B_{21}A_{12} + B_{22}A_{22} \end{array}\right] + \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] =$

$\left[\begin{array}{rrr|r} 3 & -5 & 2 & 7 \\ \hline -7 & 11 & 2 & -4 \\ 4 & -1 & 9 & -1 \\ -1 & -8 & -2 & 12 \end{array}\right] + \left[\begin{array}{rrr|r} 1 & -2 & -1 & 3 \\ \hline -2 & 0 & 1 & 4 \\ -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{rrr|r} 4 & -7 & 1 & 10 \\ \hline -9 & 11 & 3 & 0 \\ 3 & 1 & 7 & -1 \\ -1 & -7 & 0 & 13 \end{array}\right]$

40. $A = \begin{bmatrix} 1 & -2 & -1 & 3 \\ -2 & 0 & 1 & 4 \\ -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix} = \left[\begin{array}{r|rrr} 1 & -2 & -1 & 3 \\ \hline -2 & 0 & 1 & 4 \\ -1 & 2 & -2 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]$

$B = \begin{bmatrix} 2 & 0 & -1 & 1 \\ -3 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \\ 2 & 2 & -1 & -2 \end{bmatrix} = \left[\begin{array}{r|rrr} 2 & 0 & -1 & 1 \\ \hline -3 & 1 & 2 & 1 \\ 0 & -1 & -2 & 3 \\ 2 & 2 & -1 & -2 \end{array}\right] = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$

(a) $A + B = \left[\begin{array}{c|c} A_{11} + B_{11} & A_{12} + B_{12} \\ \hline A_{21} + B_{21} & A_{22} + B_{22} \end{array}\right] = \left[\begin{array}{r|rrr} 3 & -2 & -2 & 4 \\ \hline -5 & 1 & 3 & 5 \\ -1 & 1 & -4 & 3 \\ 2 & 3 & 1 & -1 \end{array}\right]$

(b) $AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right] = \left[\begin{array}{r|rrr} 14 & 5 & -6 & -10 \\ \hline 4 & 7 & -4 & -7 \\ -8 & 4 & 9 & -5 \\ -1 & 1 & -3 & 5 \end{array}\right]$

(c) $BA = \left[\begin{array}{c|c} B_{11}A_{11} + B_{12}A_{21} & B_{11}A_{12} + B_{12}A_{22} \\ \hline B_{21}A_{11} + B_{22}A_{21} & B_{21}A_{12} + B_{22}A_{22} \end{array}\right] = \left[\begin{array}{r|rrr} 3 & -5 & 2 & 7 \\ \hline -7 & 11 & 2 & -4 \\ 4 & -1 & 9 & -1 \\ -1 & -8 & -2 & 12 \end{array}\right]$

41. (a) $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

42. (a) $E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

43. For example, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $BA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

44. For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, and $BA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

45. For example, $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

46. For example, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

47. For example, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

48. For example, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

49. For example, $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $AC = BC = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

50. For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Then $AC = BC = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix}$.

51. (a) False. Consider $A = [1]$, $B = [-1]$.
(b) True. $\left(A + B^T\right)^T = A^T + \left(B^T\right)^T = A^T + B$.

52. (a) True. If $i \neq j$, then $A_{ij} = B_{ij} = 0$, so $(A - B)_{ij} = 0$, and $A - B$ is diagonal.
(b) False. For example, $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ is not upper triangular.

53. (a) True. If $i < j$, then $(A^T)_{ij} = A_{ji} = 0$, since A is upper triangular.
(b) False. For example, $AB = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

54. (a) False. Consider $B = I_n$, then $AB = AI_n = A = I_nA = BA$.
(b) False. For example, $AA^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq I_2$.

55. (a) False. $C = [0]$, $I_1 = [1]$, but $C + I_1 = [1] \neq [0] = C$.
(b) True. The composition of linear transformations is a linear transformation.

56. (a) True. Since $(A + I_n)_{ij} = A_{ij} + (I_n)_{ij} = A_{ji} + (I_n)_{ji} = (A + I_n)_{ji}$, so $A + I_n$ is symmetric.
(b) True. If $i \neq j$, then $\left(B^T\right)_{ij} = B_{ji} = 0 = B_{ij}$, and $\left(B^T\right)_{ii} = B_{ii}$.

57. (a) True. Using Theorem 3.15(c), we have $(ABC)^T = ((AB)\,C)^T = C^T\,(AB)^T = C^T\left(B^TA^T\right) = C^TB^TA^T$.
(b) False. For example, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

58. (a) False. For example, $A = B = 0_{nn}$.
(b) False. For example, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $ABAB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, but $A^2B^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

59. (a) True. Using Theorem 3.15(a,c) and Theorem 3.11(a), we have $(AB + C)^T = (C + AB)^T = C^T + (AB)^T = C^T + B^TA^T$.
(b) False. For example, let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, but $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

60. (a) False. For example, the 2×2 elementary matrix corresponding to interchanging rows 1 and 2 is $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.
(b) True. If one multiplies row i by c, then the corresponding elementary matrix E will be the identity matrix, except for c in the i^{th} diagonal.

61. (a) False. If $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then $E_1E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, but $E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \neq E_1E_2$.

(b) True. If E is the matrix corresponding to the operation $R_i \leftrightarrow R_j$, then E^2 represents $R_i \leftrightarrow R_j$ executed twice. The result will restore both rows R_i and R_j to their original row, and thus E^2 will represent the identity operation. Hence $E^2 = I_n$.

62. (a) False. For example, if $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $(AB)^2 = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right)^2$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ but } A^2B^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) False. $(I_n + I_n)^3 = (2I_n)^3 = 2^3 I_n^3 = 8I_n \neq 3I_n$.

63. Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}$, $C = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nm} \end{bmatrix}$ and s and t scalars.

(a)
$$\begin{aligned} A + B &= \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \\ &= \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1m} + b_{1m} \\ \vdots & & \vdots \\ a_{n1} + b_{n1} & \cdots & a_{nm} + b_{nm} \end{bmatrix} = \begin{bmatrix} b_{11} + a_{11} & \cdots & b_{1m} + a_{1m} \\ \vdots & & \vdots \\ b_{n1} + a_{n1} & \cdots & b_{nm} + a_{nm} \end{bmatrix} \\ &= \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} + \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = B + A \end{aligned}$$

(b)
$$\begin{aligned} s(A + B) &= s \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} \right) \\ &= \begin{bmatrix} sa_{11} + sb_{11} & sa_{12} + sb_{12} & \cdots & sa_{1m} + sb_{1m} \\ sa_{21} + sb_{21} & sa_{22} + sb_{22} & \cdots & sa_{2m} + sb_{2m} \\ \vdots & \vdots & & \vdots \\ sa_{n1} + sb_{n1} & sa_{n2} + sb_{n2} & \cdots & sa_{nm} + sb_{nm} \end{bmatrix} \\ &= \begin{bmatrix} sa_{11} & sa_{12} & \cdots & sa_{1m} \\ sa_{21} & sa_{22} & \cdots & sa_{2m} \\ \vdots & \vdots & & \vdots \\ sa_{n1} & sa_{n2} & \cdots & sa_{nm} \end{bmatrix} + \begin{bmatrix} sb_{11} & sb_{12} & \cdots & sb_{1m} \\ sb_{21} & sb_{22} & \cdots & sb_{2m} \\ \vdots & \vdots & & \vdots \\ sb_{n1} & sb_{n2} & \cdots & sb_{nm} \end{bmatrix} \\ &= s \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} + s \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix} = sA + sB \end{aligned}$$

(c)
$$\begin{aligned}(s+t)A &= (s+t)\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm}\end{bmatrix} = \begin{bmatrix} (s+t)a_{11} & \cdots & (s+t)a_{1m} \\ \vdots & & \vdots \\ (s+t)a_{n1} & \cdots & (s+t)a_{nm}\end{bmatrix} \\ &= \begin{bmatrix} sa_{11}+ta_{11} & \cdots & sa_{1m}+ta_{1m} \\ \vdots & & \vdots \\ sa_{n1}+ta_{n1} & \cdots & sa_{nm}+ta_{nm}\end{bmatrix} = \begin{bmatrix} sa_{11} & \cdots & sa_{1m} \\ \vdots & & \vdots \\ sa_{n1} & \cdots & sa_{nm}\end{bmatrix} + \begin{bmatrix} ta_{11} & \cdots & ta_{1m} \\ \vdots & & \vdots \\ ta_{n1} & \cdots & ta_{nm}\end{bmatrix} \\ &= s\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm}\end{bmatrix} + t\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm}\end{bmatrix} = sA + tA\end{aligned}$$

(d)
$$\begin{aligned}(A+B)+C &= \left(\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm}\end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm}\end{bmatrix}\right) + \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nm}\end{bmatrix} \\ &= \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm}\end{bmatrix} + \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nm}\end{bmatrix} \\ &= \begin{bmatrix} (a_{11}+b_{11})+c_{11} & \cdots & (a_{1m}+b_{1m})+c_{1m} \\ \vdots & & \vdots \\ (a_{n1}+b_{n1})+c_{n1} & \cdots & (a_{nm}+b_{nm})+c_{nm}\end{bmatrix} \\ &= \begin{bmatrix} a_{11}+(b_{11}+c_{11}) & \cdots & a_{1m}+(b_{1m}+c_{1m}) \\ \vdots & & \vdots \\ a_{n1}+(b_{n1}+c_{n1}) & \cdots & a_{nm}+(b_{nm}+c_{nm})\end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm}\end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & \cdots & b_{1m}+c_{1m} \\ \vdots & & \vdots \\ b_{n1}+c_{n1} & \cdots & b_{nm}+c_{nm}\end{bmatrix} \\ &= \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm}\end{bmatrix} + \left(\begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm}\end{bmatrix} + \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nm}\end{bmatrix}\right) \\ &= A + (B+C)\end{aligned}$$

(f)
$$\begin{aligned}A + 0_{nm} &= \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm}\end{bmatrix} + \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0\end{bmatrix} \\ &= \begin{bmatrix} a_{11}+0 & \cdots & a_{1m}+0 \\ \vdots & & \vdots \\ a_{n1}+0 & \cdots & a_{nm}+0\end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm}\end{bmatrix} = A\end{aligned}$$

64. (a) Let $A = [a_{ij}]$ be $n \times m$, $B = [b_{ij}]$ be $m \times p$, and $C = [c_{ij}]$ be $p \times q$. Let $D = [d_{ij}] = BC$, $E = [e_{ij}] = A(BC) = AD$, $F = [f_{ij}] = AB$, and $G = [g_{ij}] = (AB)C = FC$. Then

$$\begin{aligned}e_{ij} &= a_{i1}d_{1j} + a_{i2}d_{2j} + \cdots + a_{im}d_{mj} \\ &= a_{i1}(b_{11}c_{1j} + b_{12}c_{2j} + \cdots + b_{1p}c_{pj}) + a_{i2}(b_{21}c_{1j} + b_{22}c_{2j} + \cdots + b_{2p}c_{pj}) \\ &\qquad + \cdots + a_{im}(b_{m1}c_{1j} + b_{m2}c_{2j} + \cdots + b_{mp}c_{pj}) \\ &= (a_{i1}b_{11} + a_{i2}b_{21} + \cdots + a_{im}b_{m1})c_{1j} + (a_{i1}b_{12} + a_{i2}b_{22} + \cdots + a_{im}b_{m2})c_{2j} \\ &\qquad + \cdots + (a_{i1}b_{1p} + a_{i2}b_{2p} + \cdots + a_{im}b_{mp})c_{pj} \\ &= f_{i1}c_{1j} + f_{i2}c_{2j} + \cdots + f_{ip}c_{pj} \\ &= g_{ij}.\end{aligned}$$

Therefore, $E = G$, and hence $A(BC) = (AB)C$.

(b) Let $A = [a_{ij}]$ be $n \times m$, $B = [b_{ij}]$ be $m \times p$, and $C = [c_{ij}]$ be $m \times p$. Let $D = [d_{ij}] = B + C$, $E = [e_{ij}] = A(B + C) = AD$, $F = [f_{ij}] = AB$, $G = [g_{ij}] = AC$, and $H = [h_{ij}] = AB + AC = F + G$. Then

$$\begin{aligned} e_{ij} &= a_{i1}d_{1j} + a_{i2}d_{2j} + \cdots + a_{im}d_{mj} \\ &= a_{i1}(b_{1j} + c_{1j}) + a_{i2}(b_{2j} + c_{2j}) + \cdots + a_{im}(b_{mj} + c_{mj}) \\ &= (a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}) + (a_{i1}c_{1j} + a_{i2}c_{2j} + \cdots + a_{im}c_{mj}) \\ &= f_{ij} + g_{ij} \\ &= h_{ij}. \end{aligned}$$

Therefore, $E = H$, and hence $A(B + C) = AB + AC$.

(d) Let $A = [a_{ij}]$ be $n \times m$, $B = [b_{ij}]$ be $m \times p$, and s a scalar. Let $D = [d_{ij}] = AB$, $E = [e_{ij}] = sD = s(AB)$, $F = [f_{ij}] = sA$, $G = [g_{ij}] = FB = (sA)B$, $H = [h_{ij}] = sB$, and $L = [l_{ij}] = AH = A(sB)$. Then

$$\begin{aligned} e_{ij} &= sd_{ij} \\ &= s(a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}) \\ &= s(a_{i1}b_{1j}) + s(a_{i2}b_{2j}) + \cdots + s(a_{im}b_{mj}) \\ &= (sa_{i1})b_{1j} + (sa_{i2})b_{2j} + \cdots + (sa_{im})b_{mj} \\ &= f_{i1}b_{1j} + f_{i2}b_{2j} + \cdots + f_{im}b_{mj} \\ &= g_{ij}. \end{aligned}$$

Therefore, $E = G$,and hence $s(AB) = (sA)B$. Likewise,

$$\begin{aligned} e_{ij} &= sd_{ij} \\ &= s(a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}) \\ &= s(a_{i1}b_{1j}) + s(a_{i2}b_{2j}) + \cdots + s(a_{im}b_{mj}) \\ &= a_{i1}(sb_{1j}) + a_{i2}(sb_{2j}) + \cdots + a_{im}(sb_{mj}) \\ &= a_{i1}h_{1j} + a_{i2}h_{2j} + \cdots + a_{im}h_{mj} \\ &= l_{ij}. \end{aligned}$$

Therefore, $E = L$,and hence $s(AB) = A(sB)$.

(f) Let $A = [\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{array}]$, and $I_n = [\begin{array}{cccc} \mathbf{i}_1 & \mathbf{i}_2 & \cdots & \mathbf{i}_n \end{array}]$. Then if $B = [\begin{array}{cccc} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m \end{array}] = I_nA$, we have

$$\begin{aligned} \mathbf{b}_j = I_n\mathbf{a}_j &= a_{1j}\mathbf{i}_1 + a_{2j}\mathbf{i}_2 + \cdots + a_{nj}\mathbf{i}_n \\ &= a_{1j}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_{2j}\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_{nj}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = \mathbf{a}_j, \end{aligned}$$

hence $B = A$, so $A = IA$.

65. (a) Let $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}] = A + B$, $D = [d_{ij}] = A^T$, $E = [e_{ij}] = B^T$, $F = [f_{ij}] = D + E =$

$A^T + B^T$, and $G = [g_{ij}] = C^T = (A+B)^T$. Then

$$\begin{aligned} g_{ij} &= c_{ji} \\ &= a_{ji} + b_{ji} \\ &= d_{ij} + e_{ij} \\ &= f_{ij}, \end{aligned}$$

hence $G = F$, and so $(A+B)^T = A^T + B^T$.

(b) Let $A = [a_{ij}]$, $B = [b_{ij}] = sA$, $C = [c_{ij}] = A^T$, $D = [d_{ij}] = sC = sA^T$, and $E = [e_{ij}] = B^T = (sA)^T$. Then

$$\begin{aligned} e_{ij} &= b_{ji} \\ &= sa_{ji} \\ &= sc_{ij} \\ &= d_{ij}, \end{aligned}$$

hence $E = D$, and so $(sA)^T = sA^T$.

66. Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_m \end{bmatrix}$, and $I_n = \begin{bmatrix} \mathbf{i}_1 & \mathbf{i}_2 & \cdots & \mathbf{i}_n \end{bmatrix}$. If $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_m \end{bmatrix} = I_n A$, then

$$\begin{aligned} \mathbf{b}_j = I_n \mathbf{a}_j &= a_{1j}\mathbf{i}_1 + a_{2j}\mathbf{i}_2 + \cdots + a_{nj}\mathbf{i}_n \\ &= a_{1j}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + a_{2j}\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_{nj}\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} = \mathbf{a}_j, \end{aligned}$$

hence $B = A$, so $A = IA$

67. $(AB)^T = B^T A^T$ (by Theorem 3.15c) $= BA$ (since A and B are symmetric) $= AB$. Hence AB is symmetric.

68. Not necessarily.. Consider for example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Then $AD = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, but $DA = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 1 \end{bmatrix}$.

69. (a) If A is $n \times m$, then A^T is $m \times n$, and so $A^T A$ is $m \times m$.

(b) $\left(A^T A\right)^T = \left(A^T\right)\left(A^T\right)^T = A^T A$, hence $A^T A$ is symmetric.

70. Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}] = AB$. If $i \neq j$, then

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\ &= a_{ii}b_{ij} \text{ (since } a_{ik} = 0 \text{ if } i \neq k) \\ &= 0 \text{ (since } b_{ij} = 0 \text{ when } i \neq j) \end{aligned}$$

Thus AB is a diagonal matrix.

71. Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$. Then if $i > j$,

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\ &= a_{ii}b_{ij} + \cdots + a_{in}b_{nj} \text{ (since } a_{ik} = 0 \text{ if } i > k) \\ &= 0 \text{ (since } b_{kj} = 0 \text{ when } k \geq i > j). \end{aligned}$$

Therefore $C = AB$ is upper triangular.

72. Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}]$. Then if $i < j$,

$$\begin{aligned} c_{ij} &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} \\ &= a_{i1}b_{1j} + \cdots + a_{ii}b_{ij} \text{ (since } a_{ik} = 0 \text{ if } i < k) \\ &= 0 \text{ (since } b_{kj} = 0 \text{ when } k \leq i < j). \end{aligned}$$

Therefore $C = AB$ is lower triangular.

73. Proof by induction. Assume A^n is upper(lower) triangular, and note that when $n = 1$, A is upper(lower) triangular. Since $A^{n+1} = A^n A$, by exercise 57(58), since both A^n and A are upper(lower) triangular, A^{n+1} is upper(lower) triangular.

74. $\left(A + A^T\right)^T = A^T + \left(A^T\right)^T = A^T + A = A + A^T$.

75. (a) For example, $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$.

(b) Since $A^T = -A \Rightarrow A + A^T = 0_n$, and since the diagonal entry a_{ii} of A and A^T are the same, we have $a_{ii} + a_{ii} = 0$, and hence $a_{ii} = 0$.

76. (a) For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$, so A is idempotent.

(b) $(I - A)^2 = (I - A)(I - A) = I(I - A) - A(I - A) = I - A - A + A^2 = I - A - A + A$ (since $A^2 = A$) $= I - A$. Thus $I - A$ is idempotent.

77. Let $A = [a_{ij}], B = [b_{ij}] = A^T$, and $C = [c_{ij}] = B^T = \left(A^T\right)^T$. Then $c_{ij} = b_{ji} = a_{ij}$, hence $C = A$. Therefore $A = \left(A^T\right)^T$.

78. (a) For example, $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

(b) Let $A = [a_{ij}]$, $B = [b_{ij}]$, and $C = [c_{ij}] = A + B$. Then

$$\begin{aligned} \operatorname{tr}(A + B) &= \operatorname{tr}(C) \\ &= c_{11} + c_{22} + \cdots + c_{nn} \\ &= (a_{11} + b_{11}) + (a_{22} + b_{22}) + \cdots + (a_{nn} + b_{nn}) \\ &= (a_{11} + a_{22} + \cdots + a_{nn}) + (b_{11} + b_{22} + \cdots + b_{nn}) \\ &= \operatorname{tr}(A) + \operatorname{tr}(B) \end{aligned}$$

(c) Let $A = [a_{ij}]$ and $B = [b_{ij}] = A^T$. Then

$$\begin{aligned} \operatorname{tr}(A) &= a_{11} + a_{22} + \cdots + a_{nn} \\ &= b_{11} + b_{22} + \cdots + b_{nn} \\ &= \operatorname{tr}(B) \end{aligned}$$

(d) For example, $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Then $\operatorname{tr}(AB) =$
$\operatorname{tr}\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) = \operatorname{tr}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -2$. Also, $\operatorname{tr}(A) = 0$,
$\operatorname{tr}(B) = 0$, and $\operatorname{tr}(A)\operatorname{tr}(B) = 0(0) = 0$, so $\operatorname{tr}(AB) \neq \operatorname{tr}(A)\operatorname{tr}(B)$.

79. Let $A = \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 8000 \\ 1500 \\ 500 \end{bmatrix}$. Using a computer algebra system, the distribution after one year is $A\mathbf{x} = \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}\begin{bmatrix} 8000 \\ 1500 \\ 500 \end{bmatrix} = \begin{bmatrix} 6500 \\ 2200 \\ 1300 \end{bmatrix}$;
after two years $A(A\mathbf{x}) = \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}\begin{bmatrix} 6500 \\ 2200 \\ 1300 \end{bmatrix} = \begin{bmatrix} 5375 \\ 2760 \\ 1865 \end{bmatrix}$;
after three years $A\left(A^2\mathbf{x}\right) = \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}\begin{bmatrix} 5375 \\ 2760 \\ 1865 \end{bmatrix} \approx \begin{bmatrix} 4531 \\ 3208 \\ 2261 \end{bmatrix}$;
and after four years $A\left(A^3\mathbf{x}\right) \approx \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}\begin{bmatrix} 4531 \\ 3208 \\ 2261 \end{bmatrix} \approx \begin{bmatrix} 3898 \\ 3566 \\ 2535 \end{bmatrix}$.

80. Let $A = \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 5000 \\ 3000 \\ 2000 \end{bmatrix}$. Using a computer algebra system, the distribution after one year is $A\mathbf{x} = \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}\begin{bmatrix} 5000 \\ 3000 \\ 2000 \end{bmatrix} = \begin{bmatrix} 4250 \\ 3400 \\ 2350 \end{bmatrix}$;
after two years $A(A\mathbf{x}) = \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}\begin{bmatrix} 4250 \\ 3400 \\ 2350 \end{bmatrix} \approx \begin{bmatrix} 3688 \\ 3720 \\ 2592 \end{bmatrix}$;
after three years $A\left(A^2\mathbf{x}\right) \approx \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}\begin{bmatrix} 3688 \\ 3720 \\ 2592 \end{bmatrix} \approx \begin{bmatrix} 3266 \\ 3976 \\ 2758 \end{bmatrix}$;
and after four years $A\left(A^3\mathbf{x}\right) \approx \begin{bmatrix} .80 & .05 & .05 \\ .10 & .90 & .10 \\ .10 & .05 & .85 \end{bmatrix}\begin{bmatrix} 3266 \\ 3976 \\ 2758 \end{bmatrix} \approx \begin{bmatrix} 2949 \\ 4181 \\ 2870 \end{bmatrix}$.

81. The transition matrix is $A = \begin{bmatrix} .85 & .40 \\ .15 & .60 \end{bmatrix}$ and the initial distribution is $\mathbf{x} = \begin{bmatrix} 760 \\ 240 \end{bmatrix}$.
Using a computer algebra system, the distribution tomorrow is $A\mathbf{x} = \begin{bmatrix} .85 & .40 \\ .15 & .60 \end{bmatrix}\begin{bmatrix} 760 \\ 240 \end{bmatrix} = \begin{bmatrix} 742 \\ 258 \end{bmatrix}$,
the next day $A(A\mathbf{x}) = \begin{bmatrix} .85 & .40 \\ .15 & .60 \end{bmatrix}\begin{bmatrix} 742 \\ 258 \end{bmatrix} \approx \begin{bmatrix} 734 \\ 266 \end{bmatrix}$,
and the day after that $A\left(A^2\mathbf{x}\right) \approx \begin{bmatrix} .85 & .40 \\ .15 & .60 \end{bmatrix}\begin{bmatrix} 734 \\ 266 \end{bmatrix} \approx \begin{bmatrix} 730 \\ 270 \end{bmatrix}$.

82. The transition matrix is $A = \begin{bmatrix} .9 & .15 \\ .1 & .85 \end{bmatrix}$ and the initial distribution is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The distribution for the fourth person in the chain is $A^4\mathbf{x} = \begin{bmatrix} .9 & .15 \\ .1 & .85 \end{bmatrix}^4 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.727 \\ 0.273 \end{bmatrix}$ The probability that the fourth person in the chain hears the correct news is $0.727 = 72.7\%$.

83. (a) $A+B=\begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 3 & 3 & -1 \\ 6 & 8 & 1 & 1 \\ 5 & -3 & 1 & -2 \end{bmatrix}+\begin{bmatrix} -6 & 2 & -3 & 1 \\ -5 & 2 & 0 & 3 \\ 0 & 3 & -1 & 4 \\ 8 & 5 & -2 & 0 \end{bmatrix}=\begin{bmatrix} -4 & 1 & -3 & 5 \\ -5 & 5 & 3 & 2 \\ 6 & 11 & 0 & 5 \\ 13 & 2 & -1 & -2 \end{bmatrix}$

(b) $BA-I_4=\begin{bmatrix} -6 & 2 & -3 & 1 \\ -5 & 2 & 0 & 3 \\ 0 & 3 & -1 & 4 \\ 8 & 5 & -2 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 3 & 3 & -1 \\ 6 & 8 & 1 & 1 \\ 5 & -3 & 1 & -2 \end{bmatrix}-\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
$=\begin{bmatrix} -26 & -15 & 4 & -31 \\ 5 & 1 & 9 & -28 \\ 14 & -11 & 11 & -12 \\ 4 & -9 & 13 & 24 \end{bmatrix}$

(c) $D+C$ is not possible, since they are not the same size.

84. (a) $AC=\begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 3 & 3 & -1 \\ 6 & 8 & 1 & 1 \\ 5 & -3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 6 & 2 & 4 & 0 & 8 \\ 7 & 3 & 3 & 3 & 2 \end{bmatrix}=\begin{bmatrix} 27 & 11 & 12 & 10 & 7 \\ 26 & 6 & 15 & 9 & 31 \\ 65 & 13 & 29 & 41 & 40 \\ -13 & -7 & -3 & -13 & 0 \end{bmatrix}$

(b) C^T-D^T is not possible, since C^T is 5×4 and D^T is 4×5.

(c) $CB+I_2$ is not possible, since C has 5 columns, and B has 4 rows.

85. (a) $AB=\begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 3 & 3 & -1 \\ 6 & 8 & 1 & 1 \\ 5 & -3 & 1 & -2 \end{bmatrix}\begin{bmatrix} -6 & 2 & -3 & 1 \\ -5 & 2 & 0 & 3 \\ 0 & 3 & -1 & 4 \\ 8 & 5 & -2 & 0 \end{bmatrix}=\begin{bmatrix} 25 & 22 & -14 & -1 \\ -23 & 10 & -1 & 21 \\ -68 & 36 & -21 & 34 \\ -31 & -3 & -12 & 0 \end{bmatrix}$

(b) $CD=\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 6 & 2 & 4 & 0 & 8 \\ 7 & 3 & 3 & 3 & 2 \end{bmatrix}\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 1 & 3 \\ 0 & 7 & 1 & 4 \\ 3 & 6 & 9 & 2 \\ 1 & 4 & 7 & 1 \end{bmatrix}=\begin{bmatrix} 14 & 21 & 17 & 7 \\ 42 & 65 & 60 & 22 \\ 42 & 82 & 62 & 30 \\ 52 & 76 & 47 & 29 \end{bmatrix}$

(c) $(A-B)\,C^T$ is not possible, as $A-B$ has 4 columns and C^T has 5 rows.

86. (a) $B^4=\begin{bmatrix} -6 & 2 & -3 & 1 \\ -5 & 2 & 0 & 3 \\ 0 & 3 & -1 & 4 \\ 8 & 5 & -2 & 0 \end{bmatrix}^4=\begin{bmatrix} 1827 & -319 & 669 & -505 \\ 1972 & -220 & 832 & -459 \\ 1106 & -58 & 469 & -141 \\ -3071 & 850 & -1383 & 1011 \end{bmatrix}$

(b) BC^T is not possible, since B has 4 columns, and C^T has 5 rows.

(c) $D+I_4$ is not possible, since D is 5×4 and I_4 is 4×4.

87. (a) $(C+A)B$ is not possible since C and A are different sizes.

(b) $C\left(C^T+D\right)=\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 6 & 2 & 4 & 0 & 8 \\ 7 & 3 & 3 & 3 & 2 \end{bmatrix}\left(\begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 6 & 2 & 4 & 0 & 8 \\ 7 & 3 & 3 & 3 & 2 \end{bmatrix}^T+\begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 1 & 3 \\ 0 & 7 & 1 & 4 \\ 3 & 6 & 9 & 2 \\ 1 & 4 & 7 & 1 \end{bmatrix}\right)$
$=\begin{bmatrix} 21 & 40 & 41 & 29 \\ 61 & 120 & 124 & 84 \\ 66 & 146 & 182 & 106 \\ 74 & 138 & 123 & 109 \end{bmatrix}$

(c) $A+CD = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 0 & 3 & 3 & -1 \\ 6 & 8 & 1 & 1 \\ 5 & -3 & 1 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 1 & 1 & 1 \\ 5 & 1 & 2 & 4 & 3 \\ 6 & 2 & 4 & 0 & 8 \\ 7 & 3 & 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 5 & 2 & 0 & 0 \\ 2 & 5 & 1 & 3 \\ 0 & 7 & 1 & 4 \\ 3 & 6 & 9 & 2 \\ 1 & 4 & 7 & 1 \end{bmatrix}$

$= \begin{bmatrix} 16 & 20 & 17 & 11 \\ 42 & 68 & 63 & 21 \\ 48 & 90 & 63 & 31 \\ 57 & 73 & 48 & 27 \end{bmatrix}$

88. (a) $AB - D^T$ is not possible, since AB is 4×4 and D^T is 4×5.

(b) $AB - DC$ is not possible, since AB is 4×4 and DC is 5×5.

(c) $D + CB$ is not possible, since C has 5 columns and B has 4 rows.

3.3 Practice Problems

1. $\begin{bmatrix} 2 & 11 \\ 1 & 5 \end{bmatrix}^{-1} = \dfrac{1}{2(5) - 11(1)} \begin{bmatrix} 5 & -11 \\ -1 & 2 \end{bmatrix} = -1 \begin{bmatrix} 5 & -11 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 11 \\ 1 & -2 \end{bmatrix}$

2. The linear system is equivalent to $A\mathbf{x} = \mathbf{b}$, with $A = \begin{bmatrix} 2 & 11 \\ 1 & 5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Thus, $\mathbf{x} = A^{-1}\mathbf{b} =$ $\begin{bmatrix} 2 & 11 \\ 1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 & 11 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Therefore, $x_1 = -3$ and $x_2 = 1$.

3. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$. We determine A^{-1}:

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{bmatrix} \underset{(-5/3)R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & -\frac{1}{3} & -\frac{5}{3} & 1 \end{bmatrix}$$
$$\underset{6R_2+R_1 \to R_1}{\sim} \begin{bmatrix} 3 & 0 & -9 & 6 \\ 0 & -\frac{1}{3} & -\frac{5}{3} & 1 \end{bmatrix}$$
$$\underset{\substack{(1/3)R_1 \to R_1 \\ -3R_2 \to R_2}}{\sim} \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 5 & -3 \end{bmatrix},$$

so $A^{-1} = \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}$.
Consequently, $T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} = \left(\begin{bmatrix} -3x_1 + 2x_2 \\ 5x_1 - 3x_2 \end{bmatrix}\right)$.

4. $$\left[\begin{array}{rrrrrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -2 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \overset{\substack{-2R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrrrrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -4 & 1 & 2 & 0 & 1 \end{array}\right]$$

$$\overset{(4/5)R_2+R_3\to R_3}{\sim} \left[\begin{array}{rrrrrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & 0 & -\frac{3}{5} & \frac{2}{5} & \frac{4}{5} & 1 \end{array}\right]$$

$$\overset{5R_3\to R_3}{\sim} \left[\begin{array}{rrrrrr} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & 0 & -3 & 2 & 4 & 5 \end{array}\right]$$

$$\overset{\substack{(-2/3)R_3+R_2\to R_2 \\ (1/3)R_3+R_1\to R_1}}{\sim} \left[\begin{array}{rrrrrr} 1 & -2 & 0 & \frac{5}{3} & \frac{4}{3} & \frac{5}{3} \\ 0 & 5 & 0 & -\frac{10}{3} & -\frac{5}{3} & -\frac{10}{3} \\ 0 & 0 & -3 & 2 & 4 & 5 \end{array}\right]$$

$$\overset{(2/5)R_2+R_1\to R_1}{\sim} \left[\begin{array}{rrrrrr} 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 5 & 0 & -\frac{10}{3} & -\frac{5}{3} & -\frac{10}{3} \\ 0 & 0 & -3 & 2 & 4 & 5 \end{array}\right]$$

$$\overset{\substack{(1/5)R_2\to R_2 \\ (-1/3)R_3\to R_3}}{\sim} \left[\begin{array}{rrrrrr} 1 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & -\frac{4}{3} & -\frac{5}{3} \end{array}\right],$$

so $\left[\begin{array}{rrr} 1 & -2 & 1 \\ 2 & 1 & 0 \\ -2 & 0 & -1 \end{array}\right]^{-1} = \left[\begin{array}{rrr} \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{4}{3} & -\frac{5}{3} \end{array}\right]$.

5. (a) True. For if $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$.
   (b) True, because $(AB)^{-1} = B^{-1}A^{-1}$.
   (c) True, because A can be row-reduced to the identity matrix.
   (d) False. For example, if $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is singular, but $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has no solutions.

3.3 Inverses

1. $\begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}^{-1} = \dfrac{1}{7(1) - 3(2)} \left[\begin{array}{rr} 1 & -3 \\ -2 & 7 \end{array}\right] = \dfrac{1}{1} \left[\begin{array}{rr} 1 & -3 \\ -2 & 7 \end{array}\right] = \left[\begin{array}{rr} 1 & -3 \\ -2 & 7 \end{array}\right]$

2. $\left[\begin{array}{rr} 5 & -2 \\ -4 & 3 \end{array}\right]^{-1} = \dfrac{1}{5(3) - (-2)(-4)} \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} = \dfrac{1}{7} \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} \\ \frac{4}{7} & \frac{5}{7} \end{bmatrix}$

3. $\left[\begin{array}{rr} 2 & -5 \\ -4 & 10 \end{array}\right]^{-1}$ does not exist, since $2(10) - (-5)(-4) = 0$.

4. $\left[\begin{array}{rr} -6 & 2 \\ 5 & -1 \end{array}\right]^{-1} = \dfrac{1}{(-6)(-1) - (2)(5)} \left[\begin{array}{rr} -1 & -2 \\ -5 & -6 \end{array}\right] = \dfrac{1}{-4} \left[\begin{array}{rr} -1 & -2 \\ -5 & -6 \end{array}\right] = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{5}{4} & \frac{3}{2} \end{bmatrix}$

5. $$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 2 & 9 & 0 & 1 \end{bmatrix} \overset{-2R_1+R_2\to R_2}{\sim} \left[\begin{array}{rrrr} 1 & 4 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\overset{-4R_2+R_1\to R_1}{\sim} \left[\begin{array}{rrrr} 1 & 0 & 9 & -4 \\ 0 & 1 & -1 & 1 \end{array}\right],$$

so $\begin{bmatrix} 1 & 4 \\ 2 & 9 \end{bmatrix}^{-1} = \left[\begin{array}{rr} 9 & -4 \\ -2 & 1 \end{array}\right]$

6. $$\left[\begin{array}{rrrr} 4 & 13 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right] \overset{(-1/4)R_1+R_2\to R_2}{\sim} \left[\begin{array}{rrrr} 4 & 13 & 1 & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 \end{array}\right] \overset{52R_2+R_1\to R_1}{\sim} \left[\begin{array}{rrrr} 4 & 0 & -12 & 52 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 \end{array}\right] \overset{\substack{(1/4)R_1\to R_1 \\ -4R_2\to R_2}}{\sim} \left[\begin{array}{rrrr} 1 & 0 & -3 & 13 \\ 0 & 1 & 1 & -4 \end{array}\right],$$

so $\left[\begin{array}{rr} 4 & 13 \\ 1 & 3 \end{array}\right]^{-1} = \left[\begin{array}{rr} -3 & 13 \\ 1 & -4 \end{array}\right]$.

7. $$\left[\begin{array}{rrrrrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \overset{-R_1+R_3\to R_3}{\sim} \left[\begin{array}{rrrrrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \overset{-R_2+R_3\to R_3}{\sim} \left[\begin{array}{rrrrrr} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right],$$

and we conclude that the inverse does not exist, since the left part of the augmented matrix cannot be reduced to the identity matrix.

8. $$\left[\begin{array}{rrrrrr} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_1\leftrightarrow R_3}{\sim} \left[\begin{array}{rrrrrr} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{array}\right] \overset{-R_2+R_3\to R_3}{\sim} \left[\begin{array}{rrrrrr} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right] \overset{-R_3+R_1\to R_1}{\sim} \left[\begin{array}{rrrrrr} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right],$$

so $\left[\begin{array}{rrr} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right]^{-1} = \left[\begin{array}{rrr} -1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$.

9. $$\left[\begin{array}{rrrrrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{\substack{-3R_3+R_2\to R_2 \\ R_3+R_1\to R_1}}{\sim} \left[\begin{array}{rrrrrr} 1 & 2 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{-2R_2+R_1\to R_1}{\sim} \left[\begin{array}{rrrrrr} 1 & 0 & 0 & 1 & -2 & 7 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right],$$

so $\left[\begin{array}{rrr} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{array}\right]^{-1} = \left[\begin{array}{rrr} 1 & -2 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{array}\right]$.

10. $$\left[\begin{array}{rrrrrr} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 7 & 0 & 1 & 0 \\ -1 & -1 & 5 & 0 & 0 & 1 \end{array}\right] \overset{\substack{4R_1+R_2\to R_2 \\ R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrrrrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 & 0 & 1 \end{array}\right] \overset{-R_2+R_3\to R_3}{\sim} \left[\begin{array}{rrrrrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 4 & 1 & 0 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right]$$
$$\overset{\substack{-3R_3+R_2\to R_2 \\ R_3+R_1\Rightarrow R_1}}{\sim} \left[\begin{array}{rrrrrr} 1 & 2 & 0 & -2 & -1 & 1 \\ 0 & 1 & 0 & 13 & 4 & -3 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right] \overset{-2R_2+R_1\to R_1}{\sim} \left[\begin{array}{rrrrrr} 1 & 0 & 0 & -28 & -9 & 7 \\ 0 & 1 & 0 & 13 & 4 & -3 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right],$$

so $\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 7 \\ -1 & -1 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -28 & -9 & 7 \\ 13 & 4 & -3 \\ -3 & -1 & 1 \end{bmatrix}$.

11. $$\left[\begin{array}{rrr|rrr} 1 & -3 & 1 & 1 & 0 & 0 \\ 2 & -5 & 4 & 0 & 1 & 0 \\ -2 & 3 & -8 & 0 & 0 & 1 \end{array}\right] \underset{\substack{-2R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrr|rrr} 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & -3 & -6 & 2 & 0 & 1 \end{array}\right]$$
$$\underset{3R_2+R_3\to R_3}{\sim} \left[\begin{array}{rrr|rrr} 1 & -3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 0 & -4 & 3 & 1 \end{array}\right],$$
and we conclude that the inverse does not exist, since the left part of the augmented matrix cannot be reduced to the identity matrix.

12. $$\left[\begin{array}{rrr|rrr} 3 & -1 & 9 & 1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 1 & 0 \\ 2 & -2 & 10 & 0 & 0 & 1 \end{array}\right] \underset{\substack{(-1/3)R_1+R_2\to R_2 \\ (-2/3)R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrr|rrr} 3 & -1 & 9 & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & -\frac{1}{3} & 1 & 0 \\ 0 & -\frac{4}{3} & 4 & -\frac{2}{3} & 0 & 1 \end{array}\right]$$
$$\underset{-2R_2+R_3\to R_3}{\sim} \left[\begin{array}{rrr|rrr} 3 & -1 & 9 & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & -\frac{1}{3} & 1 & 0 \\ 0 & 0 & 2 & 0 & -2 & 1 \end{array}\right]$$
$$\underset{\substack{(-1/2)R_3+R_2\to R_2 \\ (-9/2)R_3+R_1\to R_1}}{\sim} \left[\begin{array}{rrr|rrr} 3 & -1 & 0 & 1 & 9 & -\frac{9}{2} \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{3} & 2 & -\frac{1}{2} \\ 0 & 0 & 2 & 0 & -2 & 1 \end{array}\right]$$
$$\underset{(-3/2)R_2+R_1\to R_1}{\sim} \left[\begin{array}{rrr|rrr} 3 & 0 & 0 & \frac{3}{2} & 6 & -\frac{15}{4} \\ 0 & -\frac{2}{3} & 0 & -\frac{1}{3} & 2 & -\frac{1}{2} \\ 0 & 0 & 2 & 0 & -2 & 1 \end{array}\right]$$
$$\underset{\substack{(1/3)R_1\to R_1 \\ (-3/2)R_2\to R_2 \\ (1/2)R_3\to R_3}}{\sim} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & 2 & -\frac{5}{4} \\ 0 & 1 & 0 & \frac{1}{2} & -3 & \frac{3}{4} \\ 0 & 0 & 1 & 0 & -1 & \frac{1}{2} \end{array}\right],$$
so $\begin{bmatrix} 3 & -1 & 9 \\ 1 & -1 & 4 \\ 2 & -2 & 10 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 2 & -\frac{5}{4} \\ \frac{1}{2} & -3 & \frac{3}{4} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$.

13. $$\left[\begin{array}{rrrr|rrrr} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \underset{\substack{R_1\leftrightarrow R_2 \\ R_3\Leftrightarrow R_4}}{\sim} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right]$$
$$\underset{R_2\leftrightarrow R_3}{\sim} \left[\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{array}\right],$$
so $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

14. $$\begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & -3 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{-2R_2+R_4\to R_4}{\sim} \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{2R_4+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 0 & 0 & 2 \\ 0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{R_3+R_2\to R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 1 \end{bmatrix},$$

so $$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} -3 & 0 & 0 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}.$$

15. $$\begin{bmatrix} 1 & 3 & 1 & -4 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\substack{-R_4+R_3\to R_3 \\ -2R_4+R_2\to R_2 \\ 4R_4+R_1\to R_1}}{\sim} \begin{bmatrix} 1 & 3 & 1 & 0 & 1 & 0 & 0 & 4 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{\substack{2R_3+R_2\to R_2 \\ -R_3+R_1\Rightarrow R_1}}{\sim} \begin{bmatrix} 1 & 3 & 0 & 0 & 1 & 0 & -1 & 5 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{-3R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -3 & -7 & 17 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

so $$\begin{bmatrix} 1 & 3 & 1 & -4 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 & -7 & 17 \\ 0 & 1 & 2 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

16. $$\begin{bmatrix} 1 & -3 & 1 & -2 & 1 & 0 & 0 & 0 \\ 2 & -5 & 4 & -2 & 0 & 1 & 0 & 0 \\ -3 & 9 & -2 & 5 & 0 & 0 & 1 & 0 \\ 4 & -12 & 4 & -7 & 0 & 0 & 0 & 1 \end{bmatrix} \overset{\substack{-2R_1+R_2\to R_2 \\ 3R_1+R_3\to R_3 \\ -4R_4+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & -3 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 2 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -4 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{\substack{R_4+R_3\to R_3 \\ -2R_4+R_2\Rightarrow R_2 \\ 2R_4+R_1\to R_1}}{\sim} \begin{bmatrix} 1 & -3 & 1 & 0 & -7 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 & 6 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -4 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{\substack{-2R_3+R_2\to R_2 \\ -R_3+R_1\Rightarrow R_1}}{\sim} \begin{bmatrix} 1 & -3 & 0 & 0 & -6 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 & 8 & 1 & -2 & -4 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -4 & 0 & 0 & 1 \end{bmatrix}$$

$$\overset{3R_2+R_1\to R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 18 & 3 & -7 & -11 \\ 0 & 1 & 0 & 0 & 8 & 1 & -2 & -4 \\ 0 & 0 & 1 & 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -4 & 0 & 0 & 1 \end{bmatrix}$$

so $\begin{bmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & 4 & -2 \\ -3 & 9 & -2 & 5 \\ 4 & -12 & 4 & -7 \end{bmatrix}^{-1} = \begin{bmatrix} 18 & 3 & -7 & -11 \\ 8 & 1 & -2 & -4 \\ -1 & 0 & 1 & 1 \\ -4 & 0 & 0 & 1 \end{bmatrix}$.

17. The linear system is equivalent to $A\mathbf{x} = \mathbf{b}$, with $A = \begin{bmatrix} 4 & 13 \\ 1 & 3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.Thus $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 4 & 13 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 & 13 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 35 \\ -11 \end{bmatrix}$.Hence $x_1 = 35$ and $x_2 = -11$.

18. The linear system is equivalent to $A\mathbf{x} = \mathbf{b}$, with $A = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 7 \\ -1 & -1 & 5 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}$.Thus $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 7 \\ -1 & -1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -28 & -9 & 7 \\ 13 & 4 & -3 \\ -3 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 40 \\ -19 \\ 4 \end{bmatrix}$. Hence $x_1 = 40$, $x_2 = -19$ and $x_3 = 4$.

19. The linear system is equivalent to $A\mathbf{x} = \mathbf{b}$, with $A = \begin{bmatrix} 3 & -1 & 9 \\ 1 & -1 & 4 \\ 2 & -2 & 10 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$.Thus $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 3 & -1 & 9 \\ 1 & -1 & 4 \\ 2 & -2 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 2 & -\frac{5}{4} \\ \frac{1}{2} & -3 & \frac{3}{4} \\ 0 & -1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -\frac{15}{4} \\ \frac{29}{4} \\ \frac{5}{2} \end{bmatrix}$. Hence $x_1 = -\frac{15}{4}$, $x_2 = \frac{29}{4}$ and $x_3 = \frac{5}{2}$.

20. The linear system is equivalent to $A\mathbf{x} = \mathbf{b}$, with $A = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$.Thus $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 2 & 0 & 0 & -3 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 2 \\ -1 \end{bmatrix}$
$= \begin{bmatrix} 1 \\ -4 \\ -2 \\ 1 \end{bmatrix}$. Hence $x_1 = 1$, $x_2 = -4$, $x_3 = -2$, and $x_4 = 1$.

21. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$. We determine A^{-1}:

$$\begin{bmatrix} 4 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix} \overset{(-3/4)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 4 & 3 & 1 & 0 \\ 0 & -\frac{1}{4} & -\frac{3}{4} & 1 \end{bmatrix}$$
$$\overset{12R_2+R_1\to R_1}{\sim} \begin{bmatrix} 4 & 0 & -8 & 12 \\ 0 & -\frac{1}{4} & -\frac{3}{4} & 1 \end{bmatrix}$$
$$\overset{\substack{(1/4)R_1\to R_1 \\ -4R_2\to R_2}}{\sim} \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & -4 \end{bmatrix},$$

hence $A^{-1} = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$. Consequently, $T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} = \left(\begin{bmatrix} -2x_1 + 3x_2 \\ 3x_1 - 4x_2 \end{bmatrix}\right)$.

22. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & -5 \\ -1 & 4 \\ 1 & 1 \end{bmatrix}$. Since A is not square, A^{-1} does not exist, and hence T^{-1} does not exist.

23. $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & -5 \\ -2 & 10 \end{bmatrix}$. We seek to determine A^{-1}:

$$\begin{bmatrix} 1 & -5 & 1 & 0 \\ -2 & 10 & 0 & 1 \end{bmatrix} \underset{\sim}{2R_1+R_2 \to R_2} \begin{bmatrix} 1 & -5 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

Thus A^{-1} does not exist, and so T^{-1} does not exist.

24. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. We determine A^{-1}:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{-R_1+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{R_2+R_3 \to R_3} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\underset{\sim}{\substack{-R_3+R_2 \to R_2 \\ R_3+R_1 \to R_1}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{bmatrix}$$

$$\underset{\sim}{-R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$$

hence $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & -1 \end{bmatrix}$. Consequently, $T^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} = \left(\begin{bmatrix} x_2 + x_3 \\ x_1 - x_3 \\ x_1 - x_2 - x_3 \end{bmatrix}\right)$.

25. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -1 \end{bmatrix}$. Since A is not square, A^{-1} does not exist, and hence T^{-1} does not exist.

26. $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. We determine (if possible) A^{-1}:

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{-R_1+R_3 \to R_3} \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 2 & -1 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{2R_2+R_3 \to R_3} \begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 2 & 1 \end{bmatrix}$$

Since the left half of R_3 consists of zeroes, the left half of the matrix cannot be transformed to I_3 and hence A does not have an inverse.

27. $T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T_1(\mathbf{x}) = A_1\mathbf{x}$, where $A_1 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, and $T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T_2(\mathbf{x}) = A_2\mathbf{x}$, where $A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$.

(a) $T_1^{-1}(T_2(\mathbf{x})) = A\mathbf{x} = A_1^{-1}A_2\mathbf{x}$, so $A = A_1^{-1}A_2$. Now $A_1^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$,
so $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

(b) $T_1\left(T_2^{-1}(\mathbf{x})\right) = A\mathbf{x} = A_1A_2^{-1}\mathbf{x}$, so $A = A_1A_2^{-1}$. Now $A_2^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$,
so $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(c) $T_2^{-1}(T_1(\mathbf{x})) = A\mathbf{x} = A_2^{-1}A_1\mathbf{x}$, so $A = A_2^{-1}A_1$. Now $A_2^{-1} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$,
so $A = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

(d) $T_2\left(T_1^{-1}(\mathbf{x})\right) = A\mathbf{x} = A_2A_1^{-1}\mathbf{x}$, so $A = A_2A_1^{-1}$. Now $A_1^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$,
so $A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

28. $T_1\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T_1(\mathbf{x}) = A_1\mathbf{x}$, where $A_1 = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}$, and $T_2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T_2(\mathbf{x}) = A_2\mathbf{x}$, where $A_2 = \begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}$.

(a) $T_1^{-1}(T_2(\mathbf{x})) = A\mathbf{x} = A_1^{-1}A_2\mathbf{x}$, so $A = A_1^{-1}A_2$. Now $A_1^{-1} = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$, so
$A = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 38 \\ -5 & -21 \end{bmatrix}$.

(b) $T_1\left(T_2^{-1}(\mathbf{x})\right) = A\mathbf{x} = A_1A_2^{-1}\mathbf{x}$, so $A = A_1A_2^{-1}$. Now $A_2^{-1} = \begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -9 \\ -1 & 2 \end{bmatrix}$, so
$A = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}\begin{bmatrix} 5 & -9 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -17 \\ 13 & -22 \end{bmatrix}$.

(c) $T_2^{-1}(T_1(\mathbf{x})) = A\mathbf{x} = A_2^{-1}A_1\mathbf{x}$, so $A = A_2^{-1}A_1$. Now $A_2^{-1} = \begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -9 \\ -1 & 2 \end{bmatrix}$, so
$A = \begin{bmatrix} 5 & -9 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} -21 & -38 \\ 5 & 9 \end{bmatrix}$.

(d) $T_2\left(T_1^{-1}(\mathbf{x})\right) = A\mathbf{x} = A_2A_1^{-1}\mathbf{x}$, so $A = A_2A_1^{-1}$. Now $A_1^{-1} = \begin{bmatrix} 3 & 5 \\ 4 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix}$, so
$A = \begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} 7 & -5 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -22 & 17 \\ -13 & 10 \end{bmatrix}$.

29. $A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & 2 & 7 \\ 0 & 1 & 4 \end{array}\right]$, so

$$A^{-1}=\left[\begin{array}{c|c} A_{11}^{-1} & 0_{12} \\ \hline 0_{21} & A_{22}^{-1} \end{array}\right]=\left[\begin{array}{c|c} 1^{-1} & \begin{matrix}0 & 0\end{matrix} \\ \hline \begin{matrix}0\\0\end{matrix} & \begin{bmatrix}2 & 7\\1 & 4\end{bmatrix}^{-1} \end{array}\right]=\left[\begin{array}{c|rr} 1 & 0 & 0 \\ \hline 0 & 4 & -7 \\ 0 & -1 & 2 \end{array}\right].$$

30. $A=\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]=\left[\begin{array}{cc|c} 5 & 2 & 0 \\ 2 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$, so

$$A^{-1}=\left[\begin{array}{c|c} A_{11}^{-1} & 0_{12} \\ \hline 0_{21} & A_{22}^{-1} \end{array}\right]=\left[\begin{array}{c|c} \begin{bmatrix}5 & 2\\2 & 1\end{bmatrix}^{-1} & \begin{matrix}0\\0\end{matrix} \\ \hline \begin{matrix}0 & 0\end{matrix} & 1^{-1} \end{array}\right]=\left[\begin{array}{rr|c} 1 & -2 & 0 \\ -2 & 5 & 0 \\ \hline 0 & 0 & 1 \end{array}\right].$$

31. $A=\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]=\left[\begin{array}{cc|cc} 2 & 5 & 0 & 0 \\ 3 & 8 & 0 & 0 \\ \hline 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$, so

$$A^{-1}=\left[\begin{array}{c|c} A_{11}^{-1} & 0_{12} \\ \hline 0_{21} & A_{22}^{-1} \end{array}\right]=\left[\begin{array}{c|c} \begin{bmatrix}2 & 5\\3 & 8\end{bmatrix}^{-1} & \begin{matrix}0 & 0\\0 & 0\end{matrix} \\ \hline \begin{matrix}0 & 0\\0 & 0\end{matrix} & \begin{bmatrix}1 & 4\\1 & 3\end{bmatrix}^{-1} \end{array}\right]=\left[\begin{array}{rr|rr} 8 & -5 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ \hline 0 & 0 & -3 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right].$$

32. $A=\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]=\left[\begin{array}{rr|rr} 1 & 3 & 0 & 0 \\ 3 & 8 & 0 & 0 \\ \hline -1 & 2 & 2 & 5 \\ 4 & 3 & 1 & 3 \end{array}\right]$, so $A^{-1}=\left[\begin{array}{c|c} A_{11}^{-1} & 0_{12} \\ \hline -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{array}\right]$

$$=\left[\begin{array}{c|c} \begin{bmatrix}1 & 3\\3 & 8\end{bmatrix}^{-1} & \begin{matrix}0 & 0\\0 & 0\end{matrix} \\ \hline -\begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}^{-1}\begin{bmatrix}-1 & 2\\4 & 3\end{bmatrix}\begin{bmatrix}1 & 3\\3 & 8\end{bmatrix}^{-1} & \begin{bmatrix}2 & 5\\1 & 3\end{bmatrix}^{-1} \end{array}\right]=\left[\begin{array}{rr|rr} -8 & 3 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ \hline -157 & 60 & 3 & -5 \\ 60 & -23 & -1 & 2 \end{array}\right]$$

33. $A=\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]=\left[\begin{array}{rr|rrr} 1 & 3 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ \hline -1 & 2 & 1 & 2 & -2 \\ 4 & 3 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 1 \end{array}\right]$, so $A^{-1}=\left[\begin{array}{c|c} A_{11}^{-1} & 0_{12} \\ \hline -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{array}\right]$

$$=\left[\begin{array}{c|c} \begin{bmatrix}1 & 3\\3 & 8\end{bmatrix}^{-1} & \begin{matrix}0 & 0 & 0\\0 & 0 & 0\end{matrix} \\ \hline -\begin{bmatrix}1 & 2 & -2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}-1 & 2\\4 & 3\\1 & -2\end{bmatrix}\begin{bmatrix}1 & 3\\3 & 8\end{bmatrix}^{-1} & \begin{bmatrix}1 & 2 & -2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}^{-1} \end{array}\right]$$

$$=\left[\begin{array}{rr|rrr} -8 & 3 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ \hline -32 & 13 & 1 & -2 & 2 \\ 23 & -9 & 0 & 1 & 0 \\ 14 & -5 & 0 & 0 & 1 \end{array}\right].$$

34. $A=\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]=\left[\begin{array}{rr|rrr} 7 & 2 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ \hline 1 & 3 & 1 & 0 & 0 \\ -2 & 3 & 0 & 1 & -2 \\ 5 & -2 & 0 & 3 & -5 \end{array}\right]$, so $A^{-1}=\left[\begin{array}{c|c} A_{11}^{-1} & 0_{12} \\ \hline -A_{22}^{-1}A_{21}A_{11}^{-1} & A_{22}^{-1} \end{array}\right]$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^{-1} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \\ \hline -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 3 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ -2 & 3 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} 7 & 2 \\ 4 & 1 \end{bmatrix}^{-1} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 3 & -5 \end{bmatrix}^{-1} \end{array}\right]$$

$$= \left[\begin{array}{cc|ccc} -1 & 2 & 0 & 0 & 0 \\ 4 & -7 & 0 & 0 & 0 \\ \hline -11 & 19 & 1 & 0 & 0 \\ 96 & -173 & 0 & -5 & 2 \\ 55 & -99 & 0 & -3 & 1 \end{array}\right]$$

35. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

36. $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

37. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

38. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$, but $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$.

40. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then $AB = I_3$, but $BA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq I_4$.

41. For example, $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

42. For example, $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, then $BB^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is nonsingular.

43. (a) False. If A is invertible, then $A\mathbf{x} = \mathbf{b}$ will have one solution for all vectors $\mathbf{b}$.
    (b) True, by Definition 3.20 of invertible matrices.

44. (a) True. A matrix is equivalent to I_n if and only if it is invertible, and since A^{-1} is invertible, A^{-1} is equivalent to I_n.
    (b) False. By The Unifying Theorem (Version 3), if A is singular, and therefore not invertible, then the columns of A are not linearly independent.

45. (a) True. As shown in the example, the Caesar cipher corresponds to an invertible matrix.
    (b) True. $B^T \left(B^{-1}\right)^T = \left(B^{-1}B\right)^T = I^T = I$, so $\left(B^T\right)^{-1} = \left(B^{-1}\right)^T$.

46. (a) False. By The Unifying Theorem (Version 3), if the columns of A span $\mathbb{R}^n$ then A is invertible, so A is not singular.
    (b) False. The expression $\frac{A}{B}$ is not defined for matrices.

47. (a) True, since $\left(B^{-1}A^{-1}\right)(AB) = B^{-1}\left(A^{-1}A\right)B = B^{-1}I_nB = B^{-1}B = I_n$.
    (b) True. A is invertible if and only if $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$. This is only the case if every echelon form of A has a pivot in each row.

48. (a) False. For example, let $A = I_n$, and $B = -I_n$, then $A + B = 0_n$, which is not invertible.
    (b) True. The columns of B^T will span $\mathbf{R}^n$. By Theorem 3.27, B^T is invertible, so B is invertible.

49. (a) True, since $A(A^{-1}) = I_n$ we conclude that the inverse of A^{-1} is A, so $\left(A^{-1}\right)^{-1} = A$.
    (b) True. If S and T are invertible linear transformations represented by invertible matrices A and B respectively, then $S(T(\mathbf{x})) = S(B\mathbf{x}) = A(B\mathbf{x}) = (AB)\mathbf{x}$. So the composition of S and T is represented by the invertible matrix AB. Thus, the composition of S and T is an invertible linear transformation.

50. (a) False. Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Then $AB = 2I_3$, but $BA \neq 2I_3$.
    (b) False The columns of A^T will be linearly dependent, and therefore the matrix A^T is not invertible, by Theorem 3.27. Consequently A is not invertible.

51. $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow \left(A^{-1}A\right)X = A^{-1}B \Rightarrow I_nX = A^{-1}B \Rightarrow X = A^{-1}B$.

52. $BX = A + CX \Rightarrow BX - CX = A \Rightarrow (B - C)X = A \Rightarrow X = (B - C)^{-1}A$.

53. $B(X + A)^{-1} = C \Rightarrow (X + A)^{-1} = B^{-1}C \Rightarrow X + A = \left(B^{-1}C\right)^{-1} = C^{-1}B \Rightarrow X = C^{-1}B - A$.

54. $AX(D + BX)^{-1} = C \Rightarrow AX = C(D + BX) = CD + CBX \Rightarrow AX - CBX = CD \Rightarrow (A - CB)X = CD \Rightarrow X = (A - CB)^{-1}CD$.

55. If $A^{-1} = A$, then $A\left(A^{-1}\right) = A^2 = I_2$. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & d^2 + bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. From $ab + bd = 0$ we determine that either $b = 0$ or $d = -a$. From $ac + cd = 0$, we have that $c = 0$ or $d = -a$. *Case 1:* $d \neq -a$. Then $b = 0$ and $c = 0$, and so $a^2 + bc = 1 \Rightarrow a = \pm 1$. Since $d \neq -a$, we have from $d^2 + bc = 1$ that $d = \pm 1$, and hence either $a = d = 1$ or $a = d = -1$. We now have two matrices $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. *Case 2:* $d = -a$. Then we need to satisfy $a^2 + bc = 1$. If $b = 0$, then this implies $a = \pm 1$, so $d = \mp 1$, and we get the matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. If $b \neq 0$, then we may freely choose a and b, and obtain $c = \frac{1-a^2}{b}$, and the family of matrices $\begin{bmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{bmatrix}$.

56. A is not invertible. Consider the augmented matrix corresponding to $A\mathbf{x} = \mathbf{0}$. Apply the row operation $-R_1 + R_2 \to R_2$, which produces a new row 2 consisting entirely of zeroes. The row-reduced form will then have a zero row, and hence a free variable. Thus $A\mathbf{x} = \mathbf{0}$ will have non-trivial solutions, and hence A is not invertible.

57. With two equal columns, the columns of A are not linearly independent. By The Unifying Theorem A is not invertible.

58. $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ is invertible provided the columns of A are linearly independent, which will be the case as long as $a \neq 0$ and $d \neq 0$.

59. $A = \begin{bmatrix} 1 & 1 \\ c & c^2 \end{bmatrix}$ is invertible provided the columns of A are linearly independent, which will be the case if $c \neq c^2$. Thus we require that $c \neq 0$ and $c \neq 1$.

60. Since $(cA)^{-1} = c^{-1}A^{-1}$, we have $(2A)^{-1} = \frac{1}{2}\begin{bmatrix} 4 & -6 \\ 2 & 14 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 1 & 7 \end{bmatrix}$.

61. Suppose that $A\mathbf{x} = \mathbf{b}$ has solution $\mathbf{x}_b$. If A is $n \times n$ and not invertible, then it follows that the system $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x}_0$. Then $A(\mathbf{x}_b + \mathbf{x}_0) = A\mathbf{x}_b + A\mathbf{x}_0 = \mathbf{b} + \mathbf{0} = \mathbf{b}$. Therefore $A\mathbf{x} = \mathbf{b}$ does not have a unique solution, a contradiction.

62. $A^2 = \left(PDP^{-1}\right)\left(PDP^{-1}\right) = PD\left(P^{-1}P\right)DP^{-1} = P\left(D^2\right)P^{-1}$. $A^3 = A^2A$ $= \left(P\left(D^2\right)P^{-1}\right)\left(PDP^{-1}\right) = PD^2\left(P^{-1}P\right)DP^{-1} = PD^3P^{-1}$. By induction, we can establish the general result $A^n = PD^nP^{-1}$.

63. $AC = CB \Rightarrow C^{-1}(AC) = C^{-1}(CB) \Rightarrow C^{-1}AC = \left(C^{-1}C\right)B \Rightarrow C^{-1}AC = B$.

64. $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow \left(A^{-1}A\right)X = A^{-1}B \Rightarrow I_nX = A^{-1}B \Rightarrow X = A^{-1}B$.

65. $(B - C)A = 0_{nm} \Rightarrow ((B - C)A)A^{-1} = 0_{nm}A^{-1} \Rightarrow (B - C)\left(AA^{-1}\right) = 0_{nm} \Rightarrow (B - C)I_m = 0_{nm} \Rightarrow B - C = 0_{nm} \Rightarrow B = C$.

66. Since B is invertible, so is B^{-1}. Since AB is invertible, by Theorem 3.23 the product $(AB)B^{-1}$ is invertible. But $(AB)B^{-1} = A(BB^{-1}) = A$, so A is invertible.

67. Since B is singular, there exists $\mathbf{x} \neq \mathbf{0}$ such that $B\mathbf{x} = \mathbf{0}$. Thus, $(AB)\mathbf{x} = A(B\mathbf{x}) = A(\mathbf{0}) = \mathbf{0}$, and hence AB is singular.

68. Consider $(AB)\mathbf{x} = \mathbf{b}$. If this equation has a solution $\mathbf{x}$ for all $\mathbf{b}$, then $A(B\mathbf{x}) = \mathbf{b}$ shows that $A\mathbf{y} = \mathbf{b}$ has a solution $\mathbf{y}$ for all $\mathbf{b}$, specifically $\mathbf{y} = B\mathbf{x}$. But A is singular, so by the Unifying Theorem, there exists a $\mathbf{b}$ such that $A\mathbf{y} = \mathbf{b}$ has no solution $\mathbf{y}$, and hence the equation $(AB)\mathbf{x} = \mathbf{b}$ has no solution. Therefore AB is singular.

69. Let $\mathbf{x} = T(\mathbf{y})$, then $T^{-1}(r\mathbf{x}) = T^{-1}(rT(\mathbf{y})) = T^{-1}(T(r\mathbf{y})) = r\mathbf{y} = rT^{-1}(\mathbf{x})$.

70. (a) If $AC = AD$, then $A^{-1}(AC) = A^{-1}(AD) \Rightarrow \left(A^{-1}A\right)C = \left(A^{-1}A\right)D \Rightarrow A = D$

(b) If $AC = 0_{nm}$, then $A^{-1}(AC) = A^{-1}0_{nm} \Rightarrow \left(A^{-1}A\right)C = 0_{nm} \Rightarrow C = 0_{nm}$

71. $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 57 & 73 & 81 \\ 93 & 101 & 113 \\ 29 & 34 & 38 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} =$

$\begin{bmatrix} 57 & 73 & 81 \\ 93 & 101 & 113 \\ 29 & 34 & 38 \end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{116}\begin{bmatrix} 4 & 20 & -68 \\ 257 & 183 & -1092 \\ -233 & -179 & 1032 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Evaluate $A^{-1}\begin{bmatrix} 2150 \\ 3114 \\ 1027 \end{bmatrix} = \frac{1}{116}\begin{bmatrix} 4 & 20 & -68 \\ 257 & 183 & -1092 \\ -233 & -179 & 1032 \end{bmatrix}\begin{bmatrix} 2150 \\ 3114 \\ 1027 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 13 \end{bmatrix}$, so the desired production level is 9 j8's, 8 j8+'s, and 13 j9's.

72. $T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 57 & 73 & 81 \\ 93 & 101 & 113 \\ 29 & 34 & 38 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, so $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x} =$

$\begin{bmatrix} 57 & 73 & 81 \\ 93 & 101 & 113 \\ 29 & 34 & 38 \end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{116}\begin{bmatrix} 4 & 20 & -68 \\ 257 & 183 & -1092 \\ -233 & -179 & 1032 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Evaluate $A^{-1}\begin{bmatrix}2152\\3228\\1047\end{bmatrix}=\frac{1}{116}\begin{bmatrix}4&20&-68\\257&183&-1092\\-233&-179&1032\end{bmatrix}\begin{bmatrix}2152\\3228\\1047\end{bmatrix}=\begin{bmatrix}17\\4\\11\end{bmatrix}$, so the desired production level is 17 j8's, 4 j8+'s, and 11 j9's.

73. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}57&73&81\\93&101&113\\29&34&38\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, so $T^{-1}(\mathbf{x})=A^{-1}\mathbf{x}=$
$\begin{bmatrix}57&73&81\\93&101&113\\29&34&38\end{bmatrix}^{-1}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{1}{116}\begin{bmatrix}4&20&-68\\257&183&-1092\\-233&-179&1032\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$.
Evaluate $A^{-1}\begin{bmatrix}2946\\4254\\1404\end{bmatrix}=\frac{1}{116}\begin{bmatrix}4&20&-68\\257&183&-1092\\-233&-179&1032\end{bmatrix}\begin{bmatrix}2946\\4254\\1404\end{bmatrix}=\begin{bmatrix}12\\21\\9\end{bmatrix}$, so the desired production level is 12 j8's, 21 j8+'s, and 9 j9's.

74. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}57&73&81\\93&101&113\\29&34&38\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, so $T^{-1}(\mathbf{x})=A^{-1}\mathbf{x}=$
$\begin{bmatrix}57&73&81\\93&101&113\\29&34&38\end{bmatrix}^{-1}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\frac{1}{116}\begin{bmatrix}4&20&-68\\257&183&-1092\\-233&-179&1032\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$.
Evaluate $A^{-1}\begin{bmatrix}5062\\7302\\2413\end{bmatrix}=\frac{1}{116}\begin{bmatrix}4&20&-68\\257&183&-1092\\-233&-179&1032\end{bmatrix}\begin{bmatrix}5062\\7302\\2413\end{bmatrix}=\begin{bmatrix}19\\19\\32\end{bmatrix}$, so the desired production level is 19 j8's, 19 j8+'s, and 32 j9's.

75. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}29&18&50\\3&25&19\\4&6&9\end{bmatrix}\mathbf{x}$, $T^{-1}(\mathbf{y})=A^{-1}\mathbf{y}=\begin{bmatrix}29&18&50\\3&25&19\\4&6&9\end{bmatrix}^{-1}\mathbf{y}$
$=\begin{bmatrix}111&138&-908\\49&61&-401\\-82&-102&671\end{bmatrix}\mathbf{y}$, thus $T^{-1}\left(\begin{bmatrix}409\\204\\81\end{bmatrix}\right)=\begin{bmatrix}111&138&-908\\49&61&-401\\-82&-102&671\end{bmatrix}\begin{bmatrix}409\\204\\81\end{bmatrix}=\begin{bmatrix}3\\4\\5\end{bmatrix}$

76. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}29&18&50\\3&25&19\\4&6&9\end{bmatrix}\mathbf{x}$, $T^{-1}(\mathbf{y})=A^{-1}\mathbf{y}=\begin{bmatrix}29&18&50\\3&25&19\\4&6&9\end{bmatrix}^{-1}\mathbf{y}$
$=\begin{bmatrix}111&138&-908\\49&61&-401\\-82&-102&671\end{bmatrix}\mathbf{y}$, thus $T^{-1}\left(\begin{bmatrix}439\\147\\76\end{bmatrix}\right)=\begin{bmatrix}111&138&-908\\49&61&-401\\-82&-102&671\end{bmatrix}\begin{bmatrix}439\\147\\76\end{bmatrix}=\begin{bmatrix}7\\2\\4\end{bmatrix}$.

77. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}29&18&50\\3&25&19\\4&6&9\end{bmatrix}\mathbf{x}$, $T^{-1}(\mathbf{y})=A^{-1}\mathbf{y}=\begin{bmatrix}29&18&50\\3&25&19\\4&6&9\end{bmatrix}^{-1}\mathbf{y}$
$=\begin{bmatrix}111&138&-908\\49&61&-401\\-82&-102&671\end{bmatrix}\mathbf{y}$, thus $T^{-1}\left(\begin{bmatrix}1092\\589\\223\end{bmatrix}\right)=\begin{bmatrix}111&138&-908\\49&61&-401\\-82&-102&671\end{bmatrix}\begin{bmatrix}1092\\589\\223\end{bmatrix}=\begin{bmatrix}10\\14\\11\end{bmatrix}$.

78. $T(\mathbf{x})=A\mathbf{x}=\begin{bmatrix}29&18&50\\3&25&19\\4&6&9\end{bmatrix}\mathbf{x}$, $T^{-1}(\mathbf{y})=A^{-1}\mathbf{y}=\begin{bmatrix}29&18&50\\3&25&19\\4&6&9\end{bmatrix}^{-1}\mathbf{y}$
$=\begin{bmatrix}111&138&-908\\49&61&-401\\-82&-102&671\end{bmatrix}\mathbf{y}$, thus $T^{-1}\left(\begin{bmatrix}744\\428\\156\end{bmatrix}\right)=\begin{bmatrix}111&138&-908\\49&61&-401\\-82&-102&671\end{bmatrix}\begin{bmatrix}744\\428\\156\end{bmatrix}=\begin{bmatrix}0\\8\\12\end{bmatrix}$.

79. $A^{-1}\begin{bmatrix} 41 & 7 \\ 161 & 79 \\ -306 & -142 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -7 & 9 \\ -4 & 14 & -17 \end{bmatrix}^{-1} \begin{bmatrix} 41 & 7 \\ 161 & 79 \\ -306 & -142 \end{bmatrix} = \begin{bmatrix} 12 & 20 \\ 1 & 15 \\ 16 & 16 \end{bmatrix}$. Hence the decoded message is $\left\{\begin{bmatrix} 12 \\ 1 \\ 16 \end{bmatrix}, \begin{bmatrix} 20 \\ 15 \\ 16 \end{bmatrix}\right\} \to \left\{\begin{bmatrix} l \\ a \\ p \end{bmatrix}, \begin{bmatrix} t \\ o \\ p \end{bmatrix}\right\}$, *i.e.* "laptop".

80. $A^{-1}\begin{bmatrix} -30 & 1 \\ -45 & 22 \\ 96 & -39 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -7 & 9 \\ -4 & 14 & -17 \end{bmatrix}^{-1} \begin{bmatrix} -30 & 1 \\ -45 & 22 \\ 96 & -39 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 15 & 5 \\ 6 & 5 \end{bmatrix}$. Hence the decoded message is $\left\{\begin{bmatrix} 3 \\ 15 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 5 \end{bmatrix}\right\} \to \left\{\begin{bmatrix} c \\ o \\ f \end{bmatrix}, \begin{bmatrix} f \\ e \\ e \end{bmatrix}\right\}$, *i.e.* "coffee".

81. $A^{-1}\begin{bmatrix} 7 & -25 & 47 \\ 75 & -37 & 158 \\ -136 & 79 & -303 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -7 & 9 \\ -4 & 14 & -17 \end{bmatrix}^{-1} \begin{bmatrix} 7 & -25 & 47 \\ 75 & -37 & 158 \\ -136 & 79 & -303 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 24 \\ 9 & 12 & 1 \\ 14 & 5 & 13 \end{bmatrix}$. Hence the decoded message is $\left\{\begin{bmatrix} 6 \\ 9 \\ 14 \end{bmatrix}, \begin{bmatrix} 1 \\ 12 \\ 5 \end{bmatrix}, \begin{bmatrix} 24 \\ 1 \\ 13 \end{bmatrix}\right\} \to \left\{\begin{bmatrix} f \\ i \\ n \end{bmatrix}, \begin{bmatrix} a \\ l \\ e \end{bmatrix}, \begin{bmatrix} x \\ a \\ m \end{bmatrix}\right\}$, *i.e.* "final exam".

82. $A^{-1}\begin{bmatrix} 44 & -46 & 23 & 40 \\ 157 & -105 & 88 & 152 \\ -300 & 211 & -167 & -289 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ 2 & -7 & 9 \\ -4 & 14 & -17 \end{bmatrix}^{-1} \begin{bmatrix} 44 & -46 & 23 & 40 \\ 157 & -105 & 88 & 152 \\ -300 & 211 & -167 & -289 \end{bmatrix}$
$= \begin{bmatrix} 19 & 6 & 14 & 19 \\ 1 & 18 & 3 & 3 \\ 14 & 1 & 9 & 15 \end{bmatrix}$. Hence the decoded message is $\left\{\begin{bmatrix} 19 \\ 1 \\ 14 \end{bmatrix}, \begin{bmatrix} 6 \\ 18 \\ 1 \end{bmatrix}, \begin{bmatrix} 14 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 19 \\ 3 \\ 15 \end{bmatrix}\right\} \to$
$\left\{\begin{bmatrix} s \\ a \\ n \end{bmatrix}, \begin{bmatrix} f \\ r \\ a \end{bmatrix}, \begin{bmatrix} n \\ c \\ i \end{bmatrix}, \begin{bmatrix} s \\ c \\ o \end{bmatrix}\right\}$, *i.e.* "San Francisco".

83. $$A^{-1} = \begin{bmatrix} 3 & 1 & -2 & 0 \\ 2 & 2 & 5 & 1 \\ -3 & 0 & -2 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{8}{145} & -\frac{14}{145} & -\frac{23}{145} & \frac{4}{29} \\ \frac{67}{145} & \frac{64}{145} & \frac{43}{145} & -\frac{10}{29} \\ -\frac{27}{145} & \frac{11}{145} & -\frac{13}{145} & \frac{1}{29} \\ -\frac{3}{29} & -\frac{2}{29} & \frac{5}{29} & \frac{7}{29} \end{bmatrix}.$$

84. $$A^{-1} = \begin{bmatrix} 5 & 2 & -1 & 0 \\ 2 & -3 & 1 & 4 \\ 2 & 1 & -3 & 2 \\ 3 & 5 & -2 & -4 \end{bmatrix}^{-1} \text{ does not exist.}$$

85. $$A^{-1} = \begin{bmatrix} 5 & 1 & 2 & 1 & 2 \\ -3 & 2 & 2 & 1 & 0 \\ 2 & 3 & 1 & 0 & 1 \\ 5 & -1 & -1 & -1 & 3 \\ 0 & 0 & 3 & 2 & 1 \end{bmatrix}^{-1} \text{ does not exist.}$$

86. $$A^{-1} = \begin{bmatrix} 2 & 2 & 9 & 0 & 4 \\ 9 & 5 & 5 & 2 & 1 \\ 2 & 3 & 0 & 0 & 5 \\ 9 & 5 & 0 & 7 & 0 \\ 4 & 9 & 9 & 5 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{91}{12\,811} & \frac{1647}{12\,811} & -\frac{201}{12\,811} & \frac{248}{12\,811} & -\frac{1006}{12\,811} \\ -\frac{2394}{12\,811} & \frac{1017}{12\,811} & \frac{1346}{12\,811} & -\frac{1597}{12\,811} & \frac{1829}{12\,811} \\ \frac{1313}{12\,811} & -\frac{28}{12\,811} & -\frac{1070}{12\,811} & -\frac{82}{12\,811} & \frac{126}{12\,811} \\ \frac{1593}{12\,811} & -\frac{2844}{12\,811} & -\frac{703}{12\,811} & \frac{2652}{12\,811} & -\frac{13}{12\,811} \\ \frac{1400}{12\,811} & -\frac{1269}{12\,811} & \frac{1835}{12\,811} & \frac{859}{12\,811} & -\frac{695}{12\,811} \end{bmatrix}.$$

3.4 Practice Problems

1. (a) Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\mathbf{y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} -2 & 5 \\ 0 & 1 \end{bmatrix}\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, using back substitution, to obtain $\mathbf{x} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$.

   (b) Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}\mathbf{y} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} -1 \\ 5 \\ -6 \end{bmatrix}$. Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & -2 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} -1 \\ 5 \\ -6 \end{bmatrix}$, using back substitution, to obtain $\mathbf{x} = \begin{bmatrix} -\frac{7}{6} \\ -\frac{1}{2} \\ -3 \end{bmatrix}$.

2. (a) $\begin{bmatrix} 2 & 1 & -1 \\ 4 & 4 & 0 \\ -2 & 1 & 6 \end{bmatrix} \underset{\substack{-2R_1+R_2\to R_2 \\ R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & \bullet & 1 \end{bmatrix}$

   $\begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 2 & 5 \end{bmatrix} \underset{-R_2+R_3\to R_3}{\sim} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$

   Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

   (b) $\begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 0 & 2 \\ 2 & 3 & 2 & 4 \\ 2 & 1 & 2 & 2 \end{bmatrix} \underset{\substack{-R_1+R_3\to R_3 \\ -R_1+R_4\to R_4}}{\sim} \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & -1 & 3 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \bullet & 1 & 0 \\ 1 & \bullet & \bullet & 1 \end{bmatrix}$

   $\begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & -1 & 5 \\ 0 & 0 & -1 & 3 \end{bmatrix} \underset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & \bullet & 1 \end{bmatrix}$

   $\begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 3 \end{bmatrix} \underset{-R_3+R_4\to R_4}{\sim} \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

   Thus $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

   (c) $\begin{bmatrix} 2 & 1 & 0 & -2 \\ 2 & 2 & 3 & -1 \\ -4 & 0 & 7 & 8 \end{bmatrix} \underset{\substack{-R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 2 & 1 & 0 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 7 & 4 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & \bullet & 1 \end{bmatrix}$

   $\begin{bmatrix} 2 & 1 & 0 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 2 & 7 & 4 \end{bmatrix} \underset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} 2 & 1 & 0 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$

   Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 1 & 0 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

(d) $\begin{bmatrix} 2 & 1 & 2 \\ -2 & -2 & -1 \\ 4 & 2 & 7 \\ 2 & 0 & 9 \end{bmatrix} \underset{\sim}{\overset{\substack{R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3 \\ -R_1+R_4\to R_4}}{}} \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & -1 & 7 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & \bullet & 1 & 0 \\ 1 & \bullet & \bullet & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & -1 & 7 \end{bmatrix} \underset{\sim}{\overset{-R_2+R_4\to R_4}{}} \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & \bullet & 1 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \underset{\sim}{\overset{-2R_3+R_4\to R_4}{}} \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}$

Thus, $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$.

3. (a) $L=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

(c) $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} 2 & 1 & 0 & -2 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & \frac{1}{2} & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

(d) $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & \frac{1}{2} & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

3.4 LU Factorization

1. $\begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -3 \\ 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & -3 \\ -14 & -13 & 17 \end{bmatrix}$, so $a = 2$ and $b = -14$.

2. $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 3 \\ 8 & 1 & -11 \\ 4 & -1 & -5 \end{bmatrix}$, so $a = 0$, $b = -11$, and $c = 4$.

3. $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ a & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 0 & b \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 15 & b+6 \\ 5a & 2a+2b \end{bmatrix}$. Equating this to $\begin{bmatrix} 5 & c \\ 15 & 9 \\ 20 & 14 \end{bmatrix}$ we obtain that $c = 2$, $9 = b + 6 \Rightarrow b = 3$, and $20 = 5a \Rightarrow a = 4$. We check that $2a + 2b = 2(4) + 2(3) = 14$.

4. $\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix} \begin{bmatrix} 4 & 2 & 3 & b \\ 0 & 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 3 & b \\ 4a & 2a+2 & 3a+3 & ab+1 \end{bmatrix}$. Equating this to $\begin{bmatrix} 4 & c & 3 & 1 \\ 8 & 6 & d & 3 \end{bmatrix}$ we obtain $c = 2$, $b = 1$, $4a = 8 \Rightarrow a = 2$, and $d = 3a+3 = 3(2)+3 = 9$. We check that $6 = 2a+2 = 2(2)+2$ and $3 = ab + 1 = (2)(1) + 1$.

5. Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$. Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$, using back substitution, to obtain $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

6. Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} -7 \\ -17 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$. Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} 1 & 4 \\ 0 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$, using back substitution, to obtain $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

7. Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$. Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$, using back substitution, to obtain $\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$.

8. Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} -4 \\ 11 \\ 5 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$. Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}$, using back substitution, to obtain $\mathbf{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$.

9. Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & 4 & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, using back substitution, to obtain $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

10. Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 2 \\ 13 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$. Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, using Gaussian elimination, to obtain $\mathbf{x} = \begin{bmatrix} -\frac{3}{2} - \frac{5}{2}s_1 \\ -\frac{7}{2} - \frac{1}{2}s_1 \\ s_1 \end{bmatrix}$.

11. Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$.

Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, using back substitution, to obtain $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$.

12. Solve $L\mathbf{y} = \mathbf{b}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} \mathbf{y} = \begin{bmatrix} -1 \\ -3 \\ -2 \\ 3 \end{bmatrix}$, using back substitution, to obtain $\mathbf{y} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. Now solve $U\mathbf{x} = \mathbf{y}$, $\begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, using back substitution, to obtain $\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

13. $\begin{bmatrix} 1 & -4 \\ -2 & 9 \end{bmatrix} \overset{2R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet \\ -2 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$.

14. $\begin{bmatrix} 2 & 3 \\ 6 & 10 \end{bmatrix} \overset{-3R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = A_1 \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet \\ 3 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$.

15. $\begin{bmatrix} -2 & -1 & 1 \\ -6 & 0 & 4 \\ 2 & -2 & -1 \end{bmatrix} \overset{\substack{-3R_1+R_2\to R_2 \\ R_1+R_3\to R_3}}{\sim} \begin{bmatrix} -2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & -3 & 0 \end{bmatrix} \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ 3 & 1 & \bullet \\ -1 & -1 & \bullet \end{bmatrix}$

$\begin{bmatrix} -2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & -3 & 0 \end{bmatrix} \overset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} -2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ 3 & 1 & \bullet \\ -1 & -1 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

16. $\begin{bmatrix} -3 & 2 & 1 \\ -6 & 2 & 3 \\ 0 & -8 & 6 \end{bmatrix} \overset{-2R_1+R_2\to R_2}{\sim} \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & -8 & 6 \end{bmatrix} \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ 2 & 1 & \bullet \\ 0 & 4 & \bullet \end{bmatrix}$

$\begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & -8 & 6 \end{bmatrix} \overset{-4R_2+R_3\to R_3}{\sim} \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ 2 & 1 & \bullet \\ 0 & 4 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

17. $\begin{bmatrix} -1 & 0 & -1 & 2 \\ 1 & 3 & 2 & -2 \\ -2 & -9 & -3 & 3 \\ -1 & 9 & -2 & 5 \end{bmatrix} \overset{\substack{R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3 \\ -R_1+R_4\to R_4}}{\sim} \begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & -9 & -1 & -1 \\ 0 & 9 & -1 & 3 \end{bmatrix} \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet \\ 2 & -3 & \bullet & \bullet \\ 1 & 3 & \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & -9 & -1 & -1 \\ 0 & 9 & -1 & 3 \end{bmatrix} \overset{\substack{3R_2+R_3\to R_3 \\ -3R_2+R_4\to R_4}}{\sim} \begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -4 & 3 \end{bmatrix} \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet \\ 2 & -3 & 1 & \bullet \\ 1 & 3 & -2 & \bullet \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -4 & 3 \end{bmatrix} \overset{2R_3+R_4\to R_4}{\sim} \begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet \\ 2 & -3 & 1 & \bullet \\ 1 & 3 & -2 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 3 & -2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

18. $\begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 6 & -6 & -1 & -6 \\ -6 & 2 & -1 & -9 \end{bmatrix} \overset{\substack{2R_1+R_3\to R_3 \\ -2R_1+R_4\to R_4}}{\sim} \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & -2 & 1 & 2 \\ 0 & -2 & -3 & -17 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ -2 & -1 & \bullet & \bullet \\ 2 & -1 & \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & -2 & 1 & 2 \\ 0 & -2 & -3 & -17 \end{bmatrix} \overset{\substack{R_2+R_3\to R_3 \\ R_2+R_4\to R_4}}{\sim} \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -3 & -14 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ -2 & -1 & 1 & \bullet \\ 2 & -1 & -3 & \bullet \end{bmatrix}$

$\begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & -3 & -14 \end{bmatrix} \overset{3R_3+R_4\to R_4}{\sim} \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ -2 & -1 & 1 & \bullet \\ 2 & -1 & -3 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & -1 & 1 & 0 \\ 2 & -1 & -3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -3 & 2 & 1 & 4 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

19. $\begin{bmatrix} -1 & 2 & 1 & 3 \\ 4 & -7 & -7 & -17 \\ -2 & 6 & -3 & -2 \end{bmatrix} \overset{\substack{4R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3}}{\sim} \begin{bmatrix} -1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 2 & -5 & -8 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ -4 & 1 & \bullet \\ 2 & 2 & \bullet \end{bmatrix}$

$\begin{bmatrix} -1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 2 & -5 & -8 \end{bmatrix} \overset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} -1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ -4 & 1 & \bullet \\ 2 & 2 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

20. $\begin{bmatrix} -2 & 0 & -1 & 1 & 3 \\ 4 & 1 & -1 & 2 & -5 \\ -2 & 3 & -8 & 14 & 5 \end{bmatrix} \overset{\substack{2R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim} \begin{bmatrix} -2 & 0 & -1 & 1 & 3 \\ 0 & 1 & -3 & 4 & 1 \\ 0 & 3 & -7 & 13 & 2 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ -2 & 1 & \bullet \\ 1 & 3 & \bullet \end{bmatrix}$

$\begin{bmatrix} -2 & 0 & -1 & 1 & 3 \\ 0 & 1 & -3 & 4 & 1 \\ 0 & 3 & -7 & 13 & 2 \end{bmatrix} \overset{-3R_2+R_3\to R_3}{\sim} \begin{bmatrix} -2 & 0 & -1 & 1 & 3 \\ 0 & 1 & -3 & 4 & 1 \\ 0 & 0 & 2 & 1 & -1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ -2 & 1 & \bullet \\ 1 & 3 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -2 & 0 & -1 & 1 & 3 \\ 0 & 1 & -3 & 4 & 1 \\ 0 & 0 & 2 & 1 & -1 \end{bmatrix}$.

21. $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix} \overset{\substack{-R_1+R_2\to R_2 \\ -R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 1 & 1 & \bullet & \bullet \\ 1 & 2 & \bullet & \bullet \\ 0 & -1 & \bullet & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & -2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \overset{\substack{-2R_2+R_3\to R_3 \\ R_2+R_4\to R_4}}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 1 & 1 & \bullet & \bullet \\ 1 & 2 & 1 & \bullet \\ 0 & -1 & -1 & \bullet \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \overset{R_3+R_4\to R_4}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 1 & 1 & \bullet & \bullet \\ 1 & 2 & 1 & \bullet \\ 0 & -1 & -1 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

22. $$\begin{bmatrix} -1 & 2 & 4 \\ 4 & -6 & -17 \\ -3 & 2 & 15 \\ 2 & -4 & -9 \end{bmatrix} \overset{\substack{4R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3 \\ 2R_1+R_4\to R_4}}{\sim} \begin{bmatrix} -1 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ -4 & 1 & \bullet & \bullet \\ 3 & -2 & \bullet & \bullet \\ -2 & 0 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & -4 & 3 \\ 0 & 0 & -1 \end{bmatrix} \overset{2R_2+R_3\to R_3}{\sim} \begin{bmatrix} -1 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ -4 & 1 & \bullet & \bullet \\ 3 & -2 & 1 & \bullet \\ -2 & 0 & 8 & \bullet \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \overset{R_3+R_4\to R_4}{\sim} \begin{bmatrix} -1 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ -4 & 1 & \bullet & \bullet \\ 3 & -2 & 1 & \bullet \\ -2 & 0 & -1 & 1 \end{bmatrix}$$

Thus $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ -2 & 0 & -1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -1 & 2 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

23. $$\begin{bmatrix} -2 & 1 & 3 \\ 2 & 0 & 8 \\ -4 & 1 & 12 \\ 2 & 0 & -10 \\ -4 & 2 & 7 \end{bmatrix} \overset{\substack{R_1+R_2\to R_2 \\ -2R_1+R_3\to R_3 \\ R_1+R_4\to R_4 \\ -2R_1+R_5\to R_5}}{\sim} \begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 11 \\ 0 & -1 & 6 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet & \bullet \\ 2 & -1 & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet & \bullet \\ 2 & 0 & \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 11 \\ 0 & -1 & 6 \\ 0 & 1 & -7 \\ 0 & 0 & 1 \end{bmatrix} \overset{\substack{R_2+R_3\to R_3 \\ -R_2+R_4\to R_4}}{\sim} \begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 11 \\ 0 & 0 & 17 \\ 0 & 0 & -18 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet & \bullet \\ 2 & -1 & 1 & \bullet & \bullet \\ -1 & 1 & -\frac{18}{17} & \bullet & \bullet \\ 2 & 0 & \frac{1}{17} & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 11 \\ 0 & 0 & 17 \\ 0 & 0 & -18 \\ 0 & 0 & 1 \end{bmatrix} \overset{\substack{(18/17)R_3+R_4\to R_4 \\ (-1/17)R_3+R_5\to R_5}}{\sim} \begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 11 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet & \bullet \\ 2 & -1 & 1 & \bullet & \bullet \\ -1 & 1 & -\frac{18}{17} & 1 & 0 \\ 2 & 0 & \frac{1}{17} & 0 & 1 \end{bmatrix}$$

Thus $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ -1 & 1 & -\frac{18}{17} & 1 & 0 \\ 2 & 0 & \frac{1}{17} & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 11 \\ 0 & 0 & 17 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

24. $$\begin{bmatrix} 1 & 3 & 2 \\ -1 & -5 & -1 \\ 0 & -6 & -3 \\ 1 & 5 & 7 \\ -3 & 9 & 3 \end{bmatrix} \underset{\sim}{\overset{\substack{R_1+R_2\to R_2 \\ -R_1+R_4\to R_4 \\ 3R_1+R_5\to R_5}}{}} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & -6 & -3 \\ 0 & 2 & 5 \\ 0 & 18 & 9 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet & \bullet \\ 0 & 3 & \bullet & \bullet & \bullet \\ 1 & -1 & \bullet & \bullet & \bullet \\ -3 & -9 & \bullet & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & -6 & -3 \\ 0 & 2 & 5 \\ 0 & 18 & 9 \end{bmatrix} \underset{\sim}{\overset{\substack{-3R_2+R_3\to R_3 \\ R_2+R_4\to R_4 \\ 9R_2+R_5\to R_5}}{}} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -6 \\ 0 & 0 & 6 \\ 0 & 0 & 18 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet & \bullet \\ 0 & 3 & 1 & \bullet & \bullet \\ 1 & -1 & -1 & \bullet & \bullet \\ -3 & -9 & -3 & \bullet & \bullet \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -6 \\ 0 & 0 & 6 \\ 0 & 0 & 18 \end{bmatrix} \underset{\sim}{\overset{\substack{R_3+R_4\to R_4 \\ 3R_3+R_5\to R_5}}{}} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet & \bullet & \bullet \\ -1 & 1 & \bullet & \bullet & \bullet \\ 0 & 3 & 1 & \bullet & \bullet \\ 1 & -1 & -1 & 1 & 0 \\ -3 & -9 & -3 & 0 & 1 \end{bmatrix}$$

Thus $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 \\ -3 & -9 & -3 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

25. $L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

26. $L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$.

27. $L = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -2 & -1 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}$.

28. $L = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$.

29. $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} -2 & -1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix}$.

30. $L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$. We divide the rows of $U = \begin{bmatrix} -1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -2 & -1 & -3 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

31. $A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 2 & -2 \\ 0 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{7}{6} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}.$

32. $A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}^{-1}$
$= \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{1}{6} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -7 & -3 & 1 \end{bmatrix} = \begin{bmatrix} \frac{14}{3} & \frac{5}{3} & -\frac{1}{3} \\ \frac{11}{6} & \frac{5}{6} & -\frac{1}{6} \\ \frac{7}{2} & \frac{3}{2} & -\frac{1}{2} \end{bmatrix}.$

33. $A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}^{-1}$
$= \begin{bmatrix} 1 & 2 & -2 & -7 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -6 & -3 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 & -2 & -7 \\ 8 & 0 & -1 & -4 \\ -6 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$

34. $A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 4 \\ 2 & 1 \end{bmatrix}.$

35. $A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} -3 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}^{-1}$
$= \begin{bmatrix} -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & -\frac{1}{2} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 8 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -\frac{5}{3} & \frac{1}{3} \\ 3 & -\frac{3}{2} & \frac{1}{4} \\ 4 & -2 & \frac{1}{2} \end{bmatrix}.$

36. $A^{-1} = (LU)^{-1} = U^{-1}L^{-1} = \begin{bmatrix} -1 & 0 & -1 & 2 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 3 & -2 & 1 \end{bmatrix}^{-1}$
$= \begin{bmatrix} -1 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 1 & 0 \\ -2 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{9}{2} & 3 & \frac{5}{2} & \frac{3}{2} \\ \frac{1}{2} & -\frac{2}{3} & -\frac{1}{2} & -\frac{1}{6} \\ -\frac{1}{2} & 3 & \frac{3}{2} & \frac{1}{2} \\ -2 & 3 & 2 & 1 \end{bmatrix}$

37. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

38. $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$

39. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

40. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

41. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

42. $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = LDU = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

43. (a) False. If A is $m \times n$, then L is $m \times m$ and U is $m \times n$. (See answer to 19.)
    (b) True. If A is $n \times m$, and $A = LU$ is an LU factorization, then L is $n \times n$.

44. (a) True. If row exchanges are required then the matrix L will require row exchanges, and L will not be a lower triangular matrix. If row exchanges are not required, this follows by construction as shown.
    (b) False. For example, consider the LU factorization $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

45. (a) False. We would also need that the diagonal entries of A all be non-zero, so that row exchanges are not necessary.
    (b) False. If A is 4×3, then L is 4×4.

46. (a) False. For example, $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ does not have an LU factorization, but $A^{-1} = A$ exists, so A is nonsingular.
    (b) True. If the matrix E results from the interchange of 2 rows, then $EE = I$, and E is invertible. If E results from multiplying row i by $c \neq 0$, then the matrix E' obtained from multiplying row i by c^{-1} will satisfy $E'E = I$, and thus E is invertible. Finally, if E results from the row operation $cR_i + R_j \to R_j$, then E' obtained from the row operation $-cR_i + R_j \to R_j$ will satisfy $E'E = I$, and E is invertible.

47. (a) False. For example $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is one LU factorization, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is another.
    (b) False. For example, consider the LU factorization $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

48. (a) False. For example, consider the LU factorization $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
    (b) False. For example, consider the LU factorization $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

49. Let $D = [a_{ij}]$ be an $n \times n$ diagonal matrix and $U = [u_{ij}]$ an $m \times n$ upper triangular matrix. Let $B = [b_{ij}] = DU$. Then

$$\begin{aligned} b_{ij} &= d_{i1}u_{1j} + d_{i2}u_{2j} + \cdots + d_{in}u_{nj} \\ &= d_{ii}u_{ij} \end{aligned}$$

since $d_{kj} = 0$ if $k \neq j$. Thus the j^{th} column of row i of the product DU is given by the product of the diagonal d_{ii} in row i with the entry u_{ij} in row i and column j of U. Thus the entire i^{th} row of DU is given by multiplication with the diagonal entry d_{ii} in the i^{th} row of D.

50. Write U as a block matrix $\left[\begin{array}{c} U_{11} \\ \hline U_{21} \end{array}\right]$ where U_{11} is a $(n-k)\times m$ matrix and U_{21} is a $k\times m$ zero matrix. Let $L_1 = [\ L_{11} \mid L_{12}\ ]$ and $L_2 = [\ L_{11} \mid M_{12}\ ]$, where L_{11} is $n\times(n-k)$. Then $L_1U = [\ L_{11} \mid L_{12}\ ]\left[\begin{array}{c} U_{11} \\ \hline U_{21} \end{array}\right] = L_{11}U_{11} + L_{12}U_{21} = L_{11}U_{11}$ and $L_2U = [\ L_{11} \mid M_{12}\ ]\left[\begin{array}{c} U_{11} \\ \hline U_{21} \end{array}\right] = L_{11}U_{11} + L_{12}M_{21} = L_{11}U_{11}$. Thus $L_1U = L_2U$.

51. True. For example, consider the LU factorization $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. More generally, if A is an $n\times n$ upper triangular matrix with ones on the diagonal, then $A = LU = IA$ is an LU factorization of A with $L = I$.

52. (a) Proof by induction on the size n of the $n\times n$ matrices L_1 and L_2. If $n=1$, then $L_2 = L_1 = [1]$, so $L_2L_1 = [1]$, a unit lower triangular matrix. Now suppose L_2 and L_1 are both $n\times n$ unit lower triangular matrices, and that the product of $(n-1)\times(n-1)$ unit lower triangular matrices is a unit lower triangular matrix. Partition $L_2 = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{v} & M \end{array}\right]$ and $L_1 = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{w} & N \end{array}\right]$ where both M and N are $(n-1)\times(n-1)$ unit lower triangular matrices. Then $L_2L_1 = \left[\begin{array}{c|c} 1(1)+\mathbf{0w} & 1\,(\mathbf{0})+\mathbf{0}N \\ \hline \mathbf{v}(1)+M\mathbf{w} & \mathbf{v0}+MN \end{array}\right] = \left[\begin{array}{c|c} 1 & \mathbf{0} \\ \hline \mathbf{v}+M\mathbf{w} & MN \end{array}\right]$, which is a unit lower triangular matrix, since by the induction hypothesis the product MN is a unit lower triangular matrix.

(b) Proof by induction on k, the number of factors. If $k=1$, then the product is simply L_1 which is a unit lower triangular matrix. Suppose we have k factors, and that the product of $k-1$ unit lower triangular matrices is a unit lower triangular matrix. Then $L_kL_{k-1}\cdots L_1 = L_k\,(L_{k-1}\cdots L_1)$ is a unit lower triangular matrix since by the induction hypothesis $L_{k-1}\cdots L_1$ is a unit lower triangular matrix, and by part (a) the product of two unit lower triangular matrices is a unit lower triangular matrix.

(c) Since L_i is lower triangular, there is a sequence of row operations of the form $cR_j + R_k \to R_k$, with $j<k$, that transforms L_i into the identity matrix I_n. Each of these row operations corresponds to multiplication by an elementary matrix which is unit lower triangular. Hence there is a sequence of elementary matrices such that $E_kE_{k-1}\cdots E_1L_i = I_n$, and thus $L_i^{-1} = E_kE_{k-1}\cdots E_1$. Since each E_j is a unit lower triangular matrix, by part (b) we conclude that L_i^{-1} is a unit lower triangular matrix.

(d) Since $(L_k\cdots L_2L_1)^{-1} = L_1^{-1}L_2^{-1}\cdots L_k^{-1}$, the result follows from parts (b) and (c), since we have the product of unit lower triangular matrices.

53. Using a computer algebra system, it is determined that A does not have a LU factorization. Also note that A can not be reduced to the a lower triangular matrix without interchanging rows.

54. Using a computer algebra system, we determine $L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{5} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{5}{12} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{24}{19} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{19}{5} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{15}{77} & 1 \end{bmatrix}$, and $U =$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{12}{5} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{19}{12} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{5}{19} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{77}{5} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{139}{77} \end{bmatrix}.$$

55. Using a computer algebra system, we determine that $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{3}{2} & \frac{2}{7} & -\frac{1}{5} & 1 \end{bmatrix}$,

$$U = \begin{bmatrix} 10 & 2 & 0 & -4 & 2 \\ 0 & 0 & -14 & 7 & 21 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & -16 \end{bmatrix}.$$

56. Using a computer algebra system we determine that $L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ -\frac{1}{3} & -\frac{2}{5} & 1 & 0 \\ -\frac{1}{3} & \frac{3}{5} & -\frac{7}{4} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -15 & -3 & 21 \\ 0 & 5 & 15 \\ 0 & 0 & 16 \\ 0 & 0 & 0 \end{bmatrix}$.

3.5 Practice Problems

1. (a) Stochastic, because each entry is nonnegative and all column sums are one.
   (b) Not stochastic, because the third column does not sum to one.

2. (a) From column 1, we would need $a = 0.7$, but then row 1 has sum 1.4. Therefore, A cannot be a doubly stochastic matrix.
   (b) Setting column and row sums equal to one, we first obtain $a = 0.4$ (from row 1), $b = 0.3$ (from column 1), $d = 0.4$ (from column 2), and we obtain $c = 0.3$ from row 3. We check that the resulting matrix $\begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.3 & 0.4 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$ is doubly stochastic.

3. $\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix}$; $\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0.4 & 0.7 \\ 0.6 & 0.3 \end{bmatrix} \begin{bmatrix} 0.55 \\ 0.45 \end{bmatrix} = \begin{bmatrix} 0.535 \\ 0.465 \end{bmatrix}$.

4. (a) Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\begin{bmatrix} -0.75 & 0.5 & 0 \\ 0.75 & -0.5 & 0 \end{bmatrix} \overset{R_1+R_2 \to R_2}{\sim} \begin{bmatrix} -0.75 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

   and we obtain $\mathbf{x} = s \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$. Setting the column sum of $\mathbf{x}$ equal to one, we need $s = \frac{3}{5}$, and so $\mathbf{x} = \frac{3}{5} \begin{bmatrix} 2/3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$.

(b) Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\left[\begin{array}{rrrr} -0.8 & 0.5 & 0.5 & 0 \\ 0.4 & -0.5 & 0 & 0 \\ 0.4 & 0 & -0.5 & 0 \end{array}\right] \overset{\substack{(1/2)R_1+R_2\to R_2 \\ (1/2)R_1+R_3\to R_3}}{\sim} \left[\begin{array}{rrrr} -0.8 & 0.5 & 0.5 & 0 \\ 0 & -0.25 & 0.25 & 0 \\ 0 & 0.25 & -0.25 & 0 \end{array}\right]$$
$$\overset{R_2+R_3\to R_3}{\sim} \left[\begin{array}{rrrr} -0.8 & 0.5 & 0.5 & 0 \\ 0 & -0.25 & 0.25 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and we obtain $\mathbf{x} = s\left[\begin{array}{r} 5/4 \\ 1 \\ 1 \end{array}\right]$. Setting the column sum of $\mathbf{x}$ equal to one, we need $s = \frac{4}{13}$, and so $\mathbf{x} = \frac{4}{13}\left[\begin{array}{r} 5/4 \\ 1 \\ 1 \end{array}\right] = \left[\begin{array}{c} \frac{5}{13} \\ \frac{4}{13} \\ \frac{4}{13} \end{array}\right]$.

5. (a) False. Entries may be 0.
   (b) True, by Theorem 3.29(c).
   (c) True, by Theorem 3.31(a).
   (d) False. For example, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, then $AA^T = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ is not doubly stochastic.

3.5 Markov Chains

1. Stochastic, since each entry is nonnegative and all column sums are one.
2. Not stochastic, since there is a negative entry.
3. Stochastic, since each entry is nonnegative and all column sums are one.
4. Not stochastic, since the first and third columns sums are not one.
5. Setting column sums equal to one, we obtain $a = 0.35$ and $b = 0.55$.
6. Setting column sums equal to one, we obtain $a = 0.45$, $b = 0.05$, and $c = 0.4$.
7. Setting column sums equal to one, we obtain $a = \frac{8}{13}$, $b = \frac{1}{7}$, and $c = \frac{1}{10}$.
8. Setting column sums equal to one, we obtain $a = 0.45$, $b = 0.15$, $c = 0.1$, and $d = 0.55$.
9. Setting column and row sums equal to one, we obtain $a = 0.7$ and $b = 0.7$.
10. Setting column and row sums equal to one, we obtain $a = 0.6$ and $b = 0.4$.
11. Setting column and row sums equal to one, we first obtain $a = 0.5$ (from row 1 or column 2), $c = 0.5$ (from column 3), and $d = 0.4$ (from row 3). Then we obtain $b = 0.4$ from either row 2 or column 1.
12. From column 2, we would need $c = 0.4$, but then row 2 has sum 0.8. Hence A cannot be a doubly stochastic matrix.
13. $\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 0.52 \\ 0.48 \end{bmatrix}$, $\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0.52 \\ 0.48 \end{bmatrix} = \begin{bmatrix} 0.392 \\ 0.608 \end{bmatrix}$, $\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 0.2 & 0.6 \\ 0.8 & 0.4 \end{bmatrix}\begin{bmatrix} 0.392 \\ 0.608 \end{bmatrix} = \begin{bmatrix} 0.4432 \\ 0.5568 \end{bmatrix}$.

14. $\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.36 \\ 0.64 \end{bmatrix}$, $\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \begin{bmatrix} 0.36 \\ 0.64 \end{bmatrix} = \begin{bmatrix} 0.372 \\ 0.628 \end{bmatrix}$, $\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} 0.5 & 0.3 \\ 0.5 & 0.7 \end{bmatrix} \begin{bmatrix} 0.372 \\ 0.628 \end{bmatrix} = \begin{bmatrix} 0.3744 \\ 0.6256 \end{bmatrix}$.

15. $\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} \frac{1}{3} & \frac{2}{5} \\ \frac{2}{3} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{30} \\ \frac{19}{30} \end{bmatrix}$, $\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} \frac{1}{3} & \frac{2}{5} \\ \frac{2}{3} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{11}{30} \\ \frac{19}{30} \end{bmatrix} = \begin{bmatrix} \frac{169}{450} \\ \frac{281}{450} \end{bmatrix}$, $\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} \frac{1}{3} & \frac{2}{5} \\ \frac{2}{3} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} \frac{169}{450} \\ \frac{281}{450} \end{bmatrix} = \begin{bmatrix} \frac{2531}{6750} \\ \frac{4219}{6750} \end{bmatrix}$.

16. $\mathbf{x}_1 = A\mathbf{x}_0 = \begin{bmatrix} \frac{1}{4} & \frac{3}{7} \\ \frac{3}{4} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{31}{84} \\ \frac{53}{84} \end{bmatrix}$, $\mathbf{x}_2 = A\mathbf{x}_1 = \begin{bmatrix} \frac{1}{4} & \frac{3}{7} \\ \frac{3}{4} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} \frac{31}{84} \\ \frac{53}{84} \end{bmatrix} = \begin{bmatrix} \frac{853}{2352} \\ \frac{1499}{2352} \end{bmatrix}$,
$\mathbf{x}_3 = A\mathbf{x}_2 = \begin{bmatrix} \frac{1}{4} & \frac{3}{7} \\ \frac{3}{4} & \frac{4}{7} \end{bmatrix} \begin{bmatrix} \frac{853}{2352} \\ \frac{1499}{2352} \end{bmatrix} = \begin{bmatrix} \frac{23959}{65856} \\ \frac{41897}{65856} \end{bmatrix}$.

17. Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\begin{bmatrix} -0.2 & 0.5 & 0 \\ 0.2 & -0.5 & 0 \end{bmatrix} \overset{R_1+R_2\to R_2}{\sim} \begin{bmatrix} -0.2 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we obtain $\mathbf{x} = s\begin{bmatrix} 2.5 \\ 1 \end{bmatrix}$. Setting the column sum of $\mathbf{x}$ equal to one, we need $s = \frac{1}{3.5}$, and so $\mathbf{x} = \frac{1}{3.5}\begin{bmatrix} 2.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.71429 \\ 0.28571 \end{bmatrix}$.

18. Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\begin{bmatrix} -0.7 & 0.6 & 0 \\ 0.7 & -0.6 & 0 \end{bmatrix} \overset{R_1+R_2\to R_2}{\sim} \begin{bmatrix} -0.7 & 0.6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we obtain $\mathbf{x} = s\begin{bmatrix} \frac{0.6}{0.7} \\ 1 \end{bmatrix} = s\begin{bmatrix} 0.85714 \\ 1 \end{bmatrix}$. Setting the column sum of $\mathbf{x}$ equal to one, we need $s = \frac{1}{1.85714}$, and so $\mathbf{x} = \frac{1}{1.85714}\begin{bmatrix} 0.85714 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.46154 \\ 0.53846 \end{bmatrix}$.

19. Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\begin{bmatrix} -0.6 & 0.5 & 0.3 & 0 \\ 0.2 & -0.7 & 0.4 & 0 \\ 0.4 & 0.2 & -0.7 & 0 \end{bmatrix} \overset{\substack{(1/3)R_1+R_2\to R_2 \\ (2/3)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} -0.6 & 0.5 & 0.3 & 0 \\ 0 & -0.53333 & 0.5 & 0 \\ 0 & 0.53333 & -0.5 & 0 \end{bmatrix}$$

$$\overset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} -0.6 & 0.5 & 0.3 & 0 \\ 0 & -0.53333 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and we obtain $\mathbf{x} = s\begin{bmatrix} 1.2813 \\ 0.93751 \\ 1 \end{bmatrix}$. Setting the column sum of $\mathbf{x}$ equal to one, we need $s = \frac{1}{3.2188}$, and so $\mathbf{x} = \frac{1}{3.2188}\begin{bmatrix} 1.2813 \\ 0.93751 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.39806 \\ 0.29126 \\ 0.31068 \end{bmatrix}$.

20. Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\begin{bmatrix} -0.7 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 \end{bmatrix} \underset{\sim}{\substack{(2/7)R_1+R_2\to R_2 \\ (5/7)R_1+R_3\to R_3}} \begin{bmatrix} -0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and we obtain $\mathbf{x} = s_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Setting the column sum of $\mathbf{x}$ equal to one, we need $s_1 + s_2 = 1$, and so we can write $\mathbf{x} = \begin{bmatrix} 0 \\ 1-s \\ s \end{bmatrix}$ for any $0 \le s \le 1$.

21. A is not regular since every power A^k of an upper triangular matrix will be upper triangular, and hence will have a zero entry.

22. A is not regular since every power A^k of a lower triangular matrix will be lower triangular, and hence will have a zero entry.

23. A is not regular since every power A^k of a block lower triangular matrix will be block lower triangular, and hence will have a zero entry.

24. $A^2 = \begin{bmatrix} 0 & 0.2 & 0.5 \\ 0.9 & 0 & 0.5 \\ 0.1 & 0.8 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0.23 & 0.4 & 0.1 \\ 0.05 & 0.58 & 0.45 \\ 0.72 & 0.02 & 0.45 \end{bmatrix}$; thus A is regular since A is stochastic with A^2 containing all positive entries.

25. $A = \begin{bmatrix} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 & 0.3 \\ 0.4 & 0.4 & 0.4 & 0.4 \end{bmatrix}$.

26. $A = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$.

27. $A = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$.

28. $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$.

29. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then $\mathbf{x}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{x}_0$, and we obtain a Markov chain which cycles between $\mathbf{x}_1$ and $\mathbf{x}_0$, and therefore does not converge to a steady-state vector.

30. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ has only $\mathbf{x}_0 = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ that generates a Markov chain with steady-state vector. For all other vectors $\mathbf{x}_0 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ we obtain the cycle $\mathbf{x}_1 = \begin{bmatrix} a_2 \\ a_3 \\ a_1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} a_3 \\ a_1 \\ a_2 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \mathbf{x}_0$.

31. (a) False. For example, $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ is stochastic, but $A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is not.

(b) False. For example, let $A = \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix}$, then A *is* stochastic but $A^T = \begin{bmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{bmatrix}$ is not.

32. (a) False. If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ then $A\mathbf{x} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) True. If $A\mathbf{x} = \mathbf{x}$, then $\mathbf{x} = A^{-1}\mathbf{x}$, and so $\mathbf{x}$ is a steady-state vector of A^{-1}.

33. (a) False. If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ then A and B are stochastic, but $AB^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is not.

(b) False. For example, $A = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}$ is regular, because $A^2 = \begin{bmatrix} 0 & 1/2 \\ 1 & 1/2 \end{bmatrix}^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$.

34. (a) True. Since row i of A is column i of $A^T = A$, each row sum of A must equal one, and hence A is doubly stochastic.

(b) False. For example, let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is a steady-state vector for $B = A^2$. However, $A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is not a steady-state vector for A.

35. (a) False. If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then the Markov chain cycles: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ldots,$ and hence does not converge.

(b) True. If $A - I_3$ has nonnegative entries, then the diagonal of A must consist of ones. That implies that $A = I_3$, and so $A - I_3$ is the zero matrix, which is not stochastic.

36. (a) True. The general 2×2 stochastic matrix has the form $A = \begin{bmatrix} a & 1-b \\ 1-a & b \end{bmatrix}$, where $0 \leq a \leq 1$ and $0 \leq b \leq 1$. Then if $\mathbf{x} = \begin{bmatrix} \frac{1-b}{2-a-b} \\ \frac{1-a}{2-a-b} \end{bmatrix}$, we can check that $A\mathbf{x} = \mathbf{x}$ and that $\mathbf{x}$ is a probability vector. Hence every 2×2 stochastic matrix has a steady-state vector.

(b) True. $\frac{1}{2}(A + B)$ will consist of nonnegative entries, and the columns of $\frac{1}{2}(A + B)$ will add to $\frac{1}{2}(1 + 1) = 1$.

37. Let A be a stochastic matrix and $\mathbf{x}$ an initial state vector. Let $\mathbf{y} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$. Since each column of A has sum one, we have $\mathbf{y}A = \mathbf{y}$. Thus $\mathbf{y}(A\mathbf{x}) = (\mathbf{y}A)\mathbf{x} = \mathbf{y}\mathbf{x} = 1$, since the sum of the entries of $\mathbf{x}$ is one. This shows that the sum of the entries of $A\mathbf{x}$ is one, and we may conclude that each state vector is a probability vector.

38. Let A and B be stochastic matrices. Since the entries of A and B are non-negative, the entries of AB must be non-negative. Let $\mathbf{y} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$. Since the column sums in A and B are all one, we have $\mathbf{y}(AB) = (\mathbf{y}A)B = \mathbf{y}B = \mathbf{y}$, and therefore each column sum in AB is one. Hence AB is a stochastic matrix.

39. We show A^k is stochastic using induction on the power k. If $k = 1$, then $A^1 = A$ is stochastic. Assume A^{k-1} is stochastic. Then $A^k = A\left(A^{k-1}\right)$ is the product of stochastic matrices, and hence stochastic. Thus A^k is stochastic for every integer $k \geq 1$.

40. By Theorem 3.29, A^2 is a stochastic matrix. Since A is regular, A^k has strictly positive entries for some $k \geq 1$. Hence $(A^k)(A^k)$ also has strictly positive entries. Therefore $A^{2k} = (A^2)^k$ has strictly positive entries, so A^2 is regular.

41. Since all column and row sums are one, we have $a+b=a+c=1$, and thus $b=c=1-a$. Also, $b+d=1$, so $d=1-b=1-(1-a)=a$.

42. $A\mathbf{x}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}x_2\\x_1\end{bmatrix}$, and thus for any initial state vector we obtain the cycle $\begin{bmatrix}x_1\\x_2\end{bmatrix}$, $\begin{bmatrix}x_2\\x_1\end{bmatrix}$, $\begin{bmatrix}x_1\\x_2\end{bmatrix}$, $\begin{bmatrix}x_2\\x_1\end{bmatrix}$,... This can only converge if $x_1=x_2$ and requiring $x_1+x_2=1$ we obtain $\mathbf{x}=\begin{bmatrix}0.5\\0.5\end{bmatrix}$ as the only initial state vector which leads to a steady-state solution.

43. Consider $A^{k+1}=\left(A^k\right)A$, with $A^k=[b_{ij}]$. The entry in the i^{th} row and j^{th} column of A^{k+1} will have the form
$$b_{i1}a_{1j}+b_{i2}a_{2j}+\cdots+b_{in}a_{nj}$$
None of these terms is negative, and at least one of these must be positive, since every column of A has at least one positive entry, and every entry in A^k is positive. Thus we conclude that every entry of A^{k+1} is positive. By induction we establish that A^{k+l} is regular for all $l\geq 1$, i.e. A^{k+1}, A^{k+2}, ... all have strictly positive entries.

44. The product of lower triangular matrices is lower triangular, and the product of upper triangular matrices is upper triangular. Hence for all k, A^k will be lower (upper) triangular if A is lower (upper) triangular. If A is $n\times n$ with $n\geq 2$, this will mean that every A^k will contain at least one entry which is zero, and thus A will not be regular.

45. (a) Every entry of A is non-negative, and each column sum is one.

(b) We have $A=\begin{bmatrix}\alpha&0\\1-\alpha&1\end{bmatrix}$, $A^2=\begin{bmatrix}\alpha&0\\1-\alpha&1\end{bmatrix}^2=\begin{bmatrix}\alpha^2&0\\1-\alpha^2&1\end{bmatrix}$, A^3 $=\begin{bmatrix}\alpha&0\\1-\alpha&1\end{bmatrix}\begin{bmatrix}\alpha^2&0\\1-\alpha^2&1\end{bmatrix}=\begin{bmatrix}\alpha^3&0\\1-\alpha^3&1\end{bmatrix}$, and in general $A^k=\begin{bmatrix}\alpha^k&0\\1-\alpha^k&1\end{bmatrix}$. Since we have a zero entry for every A^k, A is not regular.

(c) $\lim\limits_{k\to\infty}A^k=\lim\limits_{k\to\infty}\begin{bmatrix}\alpha^k&0\\1-\alpha^k&1\end{bmatrix}=\begin{bmatrix}0&0\\1&1\end{bmatrix}$, since for $0<\alpha<1$, $\lim\limits_{k\to\infty}\alpha^k=0$.

(d) $A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\alpha&0\\1-\alpha&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$. And $A\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\alpha&0\\1-\alpha&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$ $=\begin{bmatrix}\alpha x_1\\x_2-x_1(\alpha-1)\end{bmatrix}\neq\begin{bmatrix}x_1\\x_2\end{bmatrix}$ unless $x_1=0$, since $\alpha\neq 0$. Thus $\begin{bmatrix}0\\1\end{bmatrix}$ is the unique steady-state vector of A.

46. (a) $A=\begin{bmatrix}0.85&0.40\\0.15&0.60\end{bmatrix}$

(b) $A^5\begin{bmatrix}760\\240\end{bmatrix}=\begin{bmatrix}0.85&0.40\\0.15&0.60\end{bmatrix}^5\begin{bmatrix}760\\240\end{bmatrix}=\begin{bmatrix}727.88\\272.12\end{bmatrix}$, so 728 employees will be at work five days from now.

(c) $\mathbf{x}=\begin{bmatrix}\frac{8000}{11}\\\frac{3000}{11}\end{bmatrix}\approx\begin{bmatrix}727.27\\272.73\end{bmatrix}$

47. (a) $A=\begin{bmatrix}0.9&0.15\\0.1&0.85\end{bmatrix}$

(b) $A^6\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0.9&0.15\\0.1&0.85\end{bmatrix}^6\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}0.67119\\0.32881\end{bmatrix}$, so the probability that the sixth person in the chain hears the wrong news is 0.32881.

(c) $\mathbf{x} = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$

48. (a) $A = \begin{bmatrix} 0.6 & 0.15 \\ 0.4 & 0.85 \end{bmatrix}$

(b) $A^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.15 \\ 0.4 & 0.85 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.42 \\ 0.58 \end{bmatrix}$, so the probability of rain Thursday is 0.42.

(c) $A^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.15 \\ 0.4 & 0.85 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.24788 \\ 0.75213 \end{bmatrix}$, so the probability of no rain on Monday is 0.75213.

(d) $A \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.6 & 0.15 \\ 0.4 & 0.85 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.7 \end{bmatrix} = \begin{bmatrix} 0.285 \\ 0.715 \end{bmatrix}$, so the probability of rain tomorrow is 0.285.

(e) $\mathbf{x} = \begin{bmatrix} 0.27273 \\ 0.72727 \end{bmatrix}$

49. (a) $A = \begin{bmatrix} 0.35 & 0.8 \\ 0.65 & 0.2 \end{bmatrix}$

(b) i. $A^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.35 & 0.8 \\ 0.65 & 0.2 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.6425 \\ 0.3575 \end{bmatrix}$, so the probability that she will go to McDonald's two Sundays from now is 0.3575.

ii. $A^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.35 & 0.8 \\ 0.65 & 0.2 \end{bmatrix}^3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.51088 \\ 0.48913 \end{bmatrix}$, so the probability that she will go to McDonald's two Sundays from now is 0.48913.

(c) $A^2 \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.35 & 0.8 \\ 0.65 & 0.2 \end{bmatrix}^2 \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} = \begin{bmatrix} 0.521 \\ 0.479 \end{bmatrix}$, so the probability that his third fast food experience will be at Krusty's will be 0.521.

(d) $\mathbf{x} = \begin{bmatrix} 0.55172 \\ 0.44828 \end{bmatrix}$

50. (a) The transition matrix is $A = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix}$.
The probability that a Wahoo is determined two pastries following a Zot is determined by
$A^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix}^2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.54 \\ 0.46 \end{bmatrix}$; thus the probability is 0.54.

(b) $A^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{bmatrix}^3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.589 \\ 0.411 \end{bmatrix}$, so the probability a Zot emerges three pastries later is 0.411.

(c) The steady-state vector is $\mathbf{x} = \begin{bmatrix} 0.61538 \\ 0.38462 \end{bmatrix}$, hence the long-term probability that a randomly emerging pastry will be a Wahoo is 0.61538.

51. (a) The transition matrix is $A = \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.3 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0.1 \end{bmatrix}$.
The probability that a book is at C after two more circulations is determined by
$A^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0.3 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0.1 \end{bmatrix}^2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.54 \\ 0.21 \end{bmatrix}$; thus the probability is 0.21.

(b) $A^3\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0.4&0.1&0.2\\0.3&0.7&0.7\\0.3&0.2&0.1\end{bmatrix}^3\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0.164\\0.64\\0.196\end{bmatrix}$, so the probability that the book is at B after three more circulations is 0.64.

(c) $\mathbf{x}=\begin{bmatrix}0.17105\\0.63158\\0.19737\end{bmatrix}$

52. (a) We compute $A^k\mathbf{x}_0$ for a sufficiently large value of k;

$$A^{10}\mathbf{x}_0=\begin{bmatrix}1&0&1/3\\0&1&1/3\\0&0&1/3\end{bmatrix}^{10}\begin{bmatrix}0.5\\0.25\\0.25\end{bmatrix}=\begin{bmatrix}0.62500\\0.37500\\4.2338\times10^{-6}\end{bmatrix}\approx\begin{bmatrix}0.625\\0.375\\0\end{bmatrix}$$

(b) We compute $A^k\mathbf{x}_0$ for a sufficiently large value of k;

$$A^{10}\mathbf{x}_0=\begin{bmatrix}1&0&1/3\\0&1&1/3\\0&0&1/3\end{bmatrix}^{10}\begin{bmatrix}0.2\\0.6\\0.2\end{bmatrix}=\begin{bmatrix}0.30000\\0.70000\\3.387\times10^{-6}\end{bmatrix}\approx\begin{bmatrix}0.3\\0.7\\0\end{bmatrix}$$

53. $A^9\mathbf{x}_0=\begin{bmatrix}0.2&0.3&0.1&0.4\\0.3&0.5&0.6&0.2\\0.1&0.1&0.2&0.2\\0.4&0.1&0.1&0.2\end{bmatrix}^9\begin{bmatrix}0.25\\0.25\\0.25\\0.25\end{bmatrix}=\begin{bmatrix}0.266666636\\0.399999867\\0.133333363\\0.200000134\end{bmatrix}$

and $A^{10}\mathbf{x}_0=\begin{bmatrix}0.2&0.3&0.1&0.4\\0.3&0.5&0.6&0.2\\0.1&0.1&0.2&0.2\\0.4&0.1&0.1&0.2\end{bmatrix}^{10}\begin{bmatrix}0.25\\0.25\\0.25\\0.25\end{bmatrix}=\begin{bmatrix}0.266666677\\0.399999968\\0.133333349\\0.200000004\end{bmatrix}$ agree to six decimal places, and $k=9$ is the smallest integer for which this is true. The steady-state vector is $\mathbf{x}=\begin{bmatrix}\frac{4}{15}\\\frac{2}{5}\\\frac{2}{15}\\\frac{1}{5}\end{bmatrix}$.

54. $A^{24}\mathbf{x}_0=\begin{bmatrix}0&1&0.2&0.5\\0.2&0&0.3&0\\0.5&0&0.4&0.5\\0.3&0&0.1&0\end{bmatrix}^{24}\begin{bmatrix}0.1\\0.2\\0.3\\0.4\end{bmatrix}=\begin{bmatrix}0.317002777471\\0.175792530929\\0.374639796701\\0.132564894900\end{bmatrix}$

and $A^{25}\mathbf{x}_0=\begin{bmatrix}0&1&0.2&0.5\\0.2&0&0.3&0\\0.5&0&0.4&0.5\\0.3&0&0.1&0\end{bmatrix}^{25}\begin{bmatrix}0.1\\0.2\\0.3\\0.4\end{bmatrix}=\begin{bmatrix}0.317002937719\\0.175792494504\\0.374639754866\\0.132564812911\end{bmatrix}$ agree to six decimal places, and $k=24$ is the smallest integer for which this is true. The steady state vector is $\mathbf{x}=\begin{bmatrix}\frac{110}{347}\\\frac{61}{347}\\\frac{130}{347}\\\frac{46}{347}\end{bmatrix}$.

55. $A\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ so $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ has itself as its steady-state vector. Also $A\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ so $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ has itself as its steady-state vector.

56. $A\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ so $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ has itself as its steady-state vector. Also $A\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ so $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ has

itself as its steady-state vector.

Chapter 3 Supplementary Exercises

1. $T(\mathbf{u}_1) = A\mathbf{u}_1 = \begin{bmatrix} -2 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -7 \\ 12 \end{bmatrix}$;
   $T(\mathbf{u}_2) = A\mathbf{u}_2 = \begin{bmatrix} -2 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \end{bmatrix}$

2. $T(\mathbf{u}_1) = A\mathbf{u}_1 = \begin{bmatrix} 3 & 0 \\ -2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -8 \end{bmatrix}$;
   $T(\mathbf{u}_2) = A\mathbf{u}_2 = \begin{bmatrix} 3 & 0 \\ -2 & -1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -8 \\ 18 \end{bmatrix}$

3. $T(\mathbf{u}_1) = A\mathbf{u}_1 = \begin{bmatrix} 5 & 1 & 0 \\ -1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \end{bmatrix}$;
   $T(\mathbf{u}_2) = A\mathbf{u}_2 = \begin{bmatrix} 5 & 1 & 0 \\ -1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -9 \end{bmatrix}$

4. $T(\mathbf{u}_1) = A\mathbf{u}_1 = \begin{bmatrix} -1 & 0 & 4 \\ 3 & 2 & 0 \\ 2 & 2 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \\ 1 \end{bmatrix}$;
   $T(\mathbf{u}_2) = A\mathbf{u}_2 = \begin{bmatrix} -1 & 0 & 4 \\ 3 & 2 & 0 \\ 2 & 2 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 2 \end{bmatrix}$

5. $T(\mathbf{u}_1 - \mathbf{u}_2) = T(\mathbf{u}_1) - T(\mathbf{u}_2) = \begin{bmatrix} 3 \\ -4 \end{bmatrix} - \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \end{bmatrix}$

6. $T(2\mathbf{u}_1 - 3\mathbf{u}_2) = 2T(\mathbf{u}_1) - 3T(\mathbf{u}_2) = 2\begin{bmatrix} 3 \\ -4 \end{bmatrix} - 3\begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} -15 \\ -11 \end{bmatrix}$

7. $T(\mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3) = T(\mathbf{u}_1) + T(\mathbf{u}_2) - T(\mathbf{u}_3) =$
   $\begin{bmatrix} 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -6 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$

8. $T(-\mathbf{u}_1 + 2\mathbf{u}_2 - \mathbf{u}_3) = -T(\mathbf{u}_1) + 2T(\mathbf{u}_2) - T(\mathbf{u}_3) =$
   $-\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -6 \\ 3 \end{bmatrix} - \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} -19 \\ 7 \end{bmatrix}$

9. We consider $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$, and row-reduce the corresponding augmented matrix:
$$\begin{bmatrix} -1 & 1 & b_1 \\ 3 & -3 & b_2 \end{bmatrix} \overset{3R_1+R_2\to R_2}{\sim} \begin{bmatrix} -1 & 1 & b_1 \\ 0 & 0 & 3b_1 + b_2 \end{bmatrix}$$
   If $3b_1 + b_2 \neq 0$, there does not exist a unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$. By The Unifying Theorem, T is neither one-to-one nor onto.

10. Since $n = 2 < m = 3$, by Theorem 3.6 T is not one-to-one. To determine if T is onto, we row-reduce the corresponding augmented matrix:
$$\begin{bmatrix} 1 & 3 & -1 & b_1 \\ 2 & 6 & 2 & b_2 \end{bmatrix} \overset{-2R_1+R_2\to R_2}{\sim} \begin{bmatrix} 1 & 3 & -1 & b_1 \\ 0 & 0 & 4 & b_2 - 2b_1 \end{bmatrix}$$

Since there exists a solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ for all $\mathbf{b}$, the columns of A span $\mathbf{R}^n$, and, by Theorem 3.7, T is onto.

11. Since $n = 3 > m = 2$, by Theorem 3.7, T is not onto. To determine if T is one-to-one, we row-reduce the corresponding augmented matrix:

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & -2 & 0 \\ 1 & 4 & 0 \end{bmatrix} \underset{\substack{(-3/2)R_1+R_2\to R_2 \\ (-1/2)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -\frac{7}{2} & 0 \\ 0 & \frac{7}{2} & 0 \end{bmatrix} \underset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} 2 & 1 & 0 \\ 0 & -\frac{7}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since $T(\mathbf{x}) = A\mathbf{x} = \mathbf{0}$ has only the trivial solution, by Theorem 3.5, T is one-to-one.

12. We consider $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$, and row-reduce the corresponding augmented matrix:

$$\begin{bmatrix} 3 & 0 & -2 & b_1 \\ 1 & -2 & 4 & b_2 \\ 0 & 2 & 5 & b_3 \end{bmatrix} \underset{(-1/3)R_1+R_2\to R_2}{\sim} \begin{bmatrix} 3 & 0 & -2 & b_1 \\ 0 & -2 & \frac{14}{3} & b_2 - \frac{1}{3}b_1 \\ 0 & 2 & 5 & b_3 \end{bmatrix} \underset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} 3 & 0 & -2 & b_1 \\ 0 & -2 & \frac{14}{3} & b_2 - \frac{1}{3}b_1 \\ 0 & 0 & \frac{29}{3} & b_2 - \frac{1}{3}b_1 + b_3 \end{bmatrix}$$

Since there exists a unique solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$, by The Unifying Theorem, T is both one-to-one and onto.

13. $B - A = \begin{bmatrix} 3 & -4 \\ 0 & 6 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}$; BC is not defined;
$DE = \begin{bmatrix} 3 & -2 & -5 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & -3 \\ 2 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 25 & -11 \\ 12 & -9 & 8 \end{bmatrix}$

14. $5I_2 + A = 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 9 & -2 \\ 1 & 10 \end{bmatrix}$; $CD = \begin{bmatrix} -2 & 3 \\ 1 & 2 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} 3 & -2 & -5 \\ 1 & 3 & 4 \end{bmatrix}$
$= \begin{bmatrix} -3 & 13 & 22 \\ 5 & 4 & 3 \\ 14 & -24 & -46 \end{bmatrix}$; $EC = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & -3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 2 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ -15 & 18 \\ 15 & -20 \end{bmatrix}$

15. DA is not defined; $E^2 = \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & -3 \\ 2 & -5 & 4 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & -3 \\ 2 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 18 & 9 & 2 \\ -6 & 24 & -21 \\ 16 & -31 & 33 \end{bmatrix}$;
$A^3 = \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 38 & -118 \\ 59 & 97 \end{bmatrix}$

16. $3D^T + C = 3\begin{bmatrix} 3 & -2 & -5 \\ 1 & 3 & 4 \end{bmatrix}^T + \begin{bmatrix} -2 & 3 \\ 1 & 2 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ -5 & 11 \\ -9 & 8 \end{bmatrix}$; $AB + DC = \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 0 & 6 \end{bmatrix} +$
$\begin{bmatrix} 3 & -2 & -5 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 2 \\ 6 & -4 \end{bmatrix} = \begin{bmatrix} -26 & -3 \\ 28 & 19 \end{bmatrix}$; $B^2 + A^2 = \begin{bmatrix} 3 & -4 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 0 & 6 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 1 & 5 \end{bmatrix} =$
$\begin{bmatrix} 23 & -54 \\ 9 & 59 \end{bmatrix}$

17. We first determine that $T_1(\mathbf{x}) = A_1\mathbf{x} = \begin{bmatrix} 4 & -3 \\ -1 & 5 \end{bmatrix}\mathbf{x}$ and $T_2(\mathbf{x}) = A_2\mathbf{x} = \begin{bmatrix} 6 & 2 \\ 1 & -5 \end{bmatrix}\mathbf{x}$. We have $T_1(T_2(\mathbf{x})) = T_1(A_2\mathbf{x}) = A_1(A_2\mathbf{x}) = (A_1A_2)\mathbf{x}$. So $A = A_1A_2 = \begin{bmatrix} 4 & -3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 1 & -5 \end{bmatrix} = \begin{bmatrix} 21 & 23 \\ -1 & -27 \end{bmatrix}$.

18. We first determine that $T_1(\mathbf{x}) = A_1\mathbf{x} = \begin{bmatrix} 4 & -3 \\ -1 & 5 \end{bmatrix}\mathbf{x}$. We have $T_1(T_1(\mathbf{x})) = T_1(A_1\mathbf{x}) = A_1(A_1\mathbf{x}) = (A_1A_1)\mathbf{x}$. So $A = A_1A_1 =$
$\begin{bmatrix} 4 & -3 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 19 & -27 \\ -9 & 28 \end{bmatrix}$.

19. We first determine that $T_1(\mathbf{x}) = A_1\mathbf{x} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -2 \\ 3 & 0 & 1 \end{bmatrix}\mathbf{x}$ and $T_2(\mathbf{x}) = A_2\mathbf{x} = \begin{bmatrix} -1 & 2 & -1 \\ 4 & -1 & 0 \\ 3 & 1 & -1 \end{bmatrix}\mathbf{x}$.
We have $T_2(T_1(\mathbf{x})) = T_2(A_1\mathbf{x}) = A_2(A_1\mathbf{x}) = (A_2A_1)\mathbf{x}$. So $A = A_2A_1 =$
$\begin{bmatrix} -1 & 2 & -1 \\ 4 & -1 & 0 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & -2 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 7 & -6 \\ 2 & -7 & 6 \\ 2 & 0 & 0 \end{bmatrix}$.

20. We first determine that $T_2(\mathbf{x}) = A_2\mathbf{x} = \begin{bmatrix} -1 & 2 & -1 \\ 4 & -1 & 0 \\ 3 & 1 & -1 \end{bmatrix}\mathbf{x}$. We have $T_2(T_2(\mathbf{x})) = T_2(A_2\mathbf{x}) = A_2(A_2\mathbf{x}) = (A_2A_2)\mathbf{x}$. So $A = A_2A_2 =$
$\begin{bmatrix} -1 & 2 & -1 \\ 4 & -1 & 0 \\ 3 & 1 & -1 \end{bmatrix}\begin{bmatrix} -1 & 2 & -1 \\ 4 & -1 & 0 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 2 \\ -8 & 9 & -4 \\ -2 & 4 & -2 \end{bmatrix}$.

21. $E = \begin{bmatrix} -5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

22. $E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

23. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1 \end{bmatrix}$

24. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$

25. $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$

26. $E = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

27. $E = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

28. $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -5 & -1 & 0 & 1 \end{bmatrix}$

29. $\begin{bmatrix} 2 & 3 \\ 1 & 5 \end{bmatrix}^{-1} = \dfrac{1}{2(5) - 3(1)} \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 5 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{5}{7} & -\frac{3}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}$

30. $\begin{bmatrix} -4 & 0 \\ 6 & 8 \end{bmatrix}^{-1} = \dfrac{1}{(-4)(8) - 0(6)} \begin{bmatrix} 8 & 0 \\ -6 & -4 \end{bmatrix} = -\frac{1}{32} \begin{bmatrix} 8 & 0 \\ -6 & -4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & 0 \\ \frac{3}{16} & \frac{1}{8} \end{bmatrix}$

31. $$\begin{bmatrix} 1 & -1 & -3 & 1 & 0 & 0 \\ 2 & 1 & -4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{(-2/3)R_2+R_3\to R_3} \begin{bmatrix} 1 & -1 & -3 & 1 & 0 & 0 \\ 0 & 3 & 2 & -2 & 1 & 0 \\ 0 & 0 & -\frac{4}{3} & \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

$$\underset{\sim}{\substack{(3/2)R_3+R_2\to R_2 \\ (-9/4)R_3+R_1\to R_1}} \begin{bmatrix} 1 & -1 & 0 & -2 & \frac{3}{2} & -\frac{9}{4} \\ 0 & 3 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & -\frac{4}{3} & \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

$$\underset{\sim}{(1/3)R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & 0 & -2 & \frac{3}{2} & -\frac{7}{4} \\ 0 & 3 & 0 & 0 & 0 & \frac{3}{2} \\ 0 & 0 & -\frac{4}{3} & \frac{4}{3} & -\frac{2}{3} & 1 \end{bmatrix}$$

$$\underset{\sim}{\substack{(1/3)R_2\to R_2 \\ (-3/4)R_3\to R_3}} \begin{bmatrix} 1 & 0 & 0 & -2 & \frac{3}{2} & -\frac{7}{4} \\ 0 & 1 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & -1 & \frac{1}{2} & -\frac{3}{4} \end{bmatrix}$$

so $\begin{bmatrix} 1 & -1 & -3 \\ 2 & 1 & -4 \\ 0 & 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & \frac{3}{2} & -\frac{7}{4} \\ 0 & 0 & \frac{1}{2} \\ -1 & \frac{1}{2} & -\frac{3}{4} \end{bmatrix}$.

32. $$\begin{bmatrix} 3 & 2 & -2 & 1 & 0 & 0 \\ -1 & -2 & 3 & 0 & 1 & 0 \\ 1 & -2 & 4 & 0 & 0 & 1 \end{bmatrix} \underset{\sim}{R_1\leftrightarrow R_3} \begin{bmatrix} 1 & -2 & 4 & 0 & 0 & 1 \\ -1 & -2 & 3 & 0 & 1 & 0 \\ 3 & 2 & -2 & 1 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{\substack{R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3}} \begin{bmatrix} 1 & -2 & 4 & 0 & 0 & 1 \\ 0 & -4 & 7 & 0 & 1 & 1 \\ 0 & 8 & -14 & 1 & 0 & -3 \end{bmatrix}$$

$$\underset{\sim}{2R_2+R_3\to R_3} \begin{bmatrix} 1 & -2 & 4 & 0 & 0 & 1 \\ 0 & -4 & 7 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -1 \end{bmatrix}$$

and we conclude that the inverse does not exist, since the left part of the augmented matrix cannot be reduced to the identity matrix.

33. A is not a square matrix, so the inverse of A does not exist.

34. A is not a square matrix, so the inverse of A does not exist.

35. $$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{2R_2+R_3 \to R_3}{\sim} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 5 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{R_3 \leftrightarrow R_4}{\sim} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 7 & 5 & 0 & 2 & 1 & 0 \end{bmatrix}$$

$$\underset{-7R_3+R_4 \to R_4}{\sim} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 2 & 1 & -7 \end{bmatrix}$$

$$\underset{\substack{(1/2)R_4+R_3 \to R_3 \\ R_4+R_2 \to R_2}}{\sim} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 3 & 1 & -7 \\ 0 & 0 & 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & -2 & 0 & 2 & 1 & -7 \end{bmatrix}$$

$$\underset{\substack{-3R_3+R_2 \to R_2 \\ 2R_3+R_1 \to R_1}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 2 & 1 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & -2 & 0 & 2 & 1 & -7 \end{bmatrix}$$

$$\underset{(-1/2)R_4 \to R_4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 2 & 1 & -5 \\ 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & 1 & 0 & -1 & -\frac{1}{2} & \frac{7}{2} \end{bmatrix}$$

so $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 2 & 1 & -5 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{5}{2} \\ 0 & -1 & -\frac{1}{2} & \frac{7}{2} \end{bmatrix}$.

36. $$\begin{bmatrix} -1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & -1 & 0 & 0 & 1 & 0 \\ -2 & 4 & 4 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \underset{\substack{-R_1+R_3 \to R_3 \\ -2R_1+R_4 \to R_4}}{\sim} \begin{bmatrix} -1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 & -2 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{-R_2+R_4 \to R_4}{\sim} \begin{bmatrix} -1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & -1 & 0 & 1 \end{bmatrix}$$

$$\underset{-R_3+R_4 \to R_4}{\sim} \begin{bmatrix} -1 & 1 & 2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -2 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & -1 & 1 \end{bmatrix}$$

and we conclude that the inverse does not exist, since the left part of the augmented matrix cannot be reduced to the identity matrix.

37. $\begin{bmatrix} 1 & 5 \\ -3 & 8 \end{bmatrix} \underset{3R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 1 & 5 \\ 0 & 23 \end{bmatrix} = A_1 \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet \\ -3 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 5 \\ 0 & 23 \end{bmatrix}$.

38. $\begin{bmatrix} 3 & -4 \\ 12 & 7 \end{bmatrix} \underset{-4R_1+R_2 \to R_2}{\sim} \begin{bmatrix} 3 & -4 \\ 0 & 23 \end{bmatrix} = A_1 \quad \Rightarrow L = \begin{bmatrix} 1 & \bullet \\ 4 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & -4 \\ 0 & 23 \end{bmatrix}$.

39. $\begin{bmatrix} -2 & 3 & 1 \\ 2 & -1 & 3 \\ 4 & -2 & 1 \end{bmatrix} \underset{\substack{R_1+R_2\to R_2 \\ 2R_1+R_3\to R_3}}{\sim} \begin{bmatrix} -2 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 4 & 3 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ -1 & 1 & \bullet \\ -2 & \bullet & 1 \end{bmatrix}$

$\underset{-2R_2+R_3\to R_3}{\sim} \begin{bmatrix} -2 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -5 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ -1 & 1 & \bullet \\ -2 & 2 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} -2 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & -5 \end{bmatrix}$.

40. $\begin{bmatrix} 3 & -1 & 1 \\ 9 & 2 & 0 \\ 0 & -5 & 6 \end{bmatrix} \underset{-3R_1+R_3\to R_3}{\sim} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & -5 & 6 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ 3 & 1 & \bullet \\ 0 & \bullet & 1 \end{bmatrix}$

$\underset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} 3 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 3 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ 0 & 1 & \bullet \\ 3 & -1 & 1 \end{bmatrix}$

Thus $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 3 \end{bmatrix}$.

41. $\begin{bmatrix} 2 & 3 \\ -8 & 7 \end{bmatrix} \underset{4R_1+R_2\to R_2}{\sim} \begin{bmatrix} 2 & 3 \\ 0 & 19 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$

We divide the rows of $\begin{bmatrix} 2 & 3 \\ 0 & 19 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 2 & 0 \\ 0 & 19 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$.

42. $\begin{bmatrix} -1 & 3 \\ 7 & 2 \end{bmatrix} \underset{7R_1+R_2\to R_2}{\sim} \begin{bmatrix} -1 & 3 \\ 0 & 23 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 \\ -7 & 1 \end{bmatrix}$

We divide the rows of $\begin{bmatrix} -1 & 3 \\ 0 & 23 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} -1 & 0 \\ 0 & 23 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$.

43. $\begin{bmatrix} -1 & 2 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & -2 \end{bmatrix} \underset{\substack{2R_1+R_2\to R_2 \\ 3R_1+R_3\to R_3}}{\sim} \begin{bmatrix} -1 & 2 & 0 \\ 0 & 5 & 1 \\ 0 & 8 & -2 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ -2 & 1 & \bullet \\ -3 & \bullet & 1 \end{bmatrix}$

$\underset{(-8/5)R_2+R_3\to R_3}{\sim} \begin{bmatrix} -1 & 2 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & -\frac{18}{5} \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 8/5 & 1 \end{bmatrix}$

We divide the rows of $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & -\frac{18}{5} \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -\frac{18}{5} \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 1 \end{bmatrix}$.

44. $\begin{bmatrix} 2 & 0 & 1 \\ -8 & 5 & 1 \\ 6 & -1 & 4 \end{bmatrix} \underset{\substack{4R_1+R_2\to R_2 \\ -3R_1+R_3\to R_3}}{\sim} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 5 \\ 0 & -1 & 1 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & \bullet & \bullet \\ -4 & 1 & \bullet \\ 3 & \bullet & 1 \end{bmatrix}$

$\underset{(1/5)R_2+R_3\to R_3}{\sim} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & -1/5 & 1 \end{bmatrix}$

We divide the rows of $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ by the diagonal entries to obtain $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

45. Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\begin{bmatrix} -0.4 & 0.5 & 0 \\ 0.4 & -0.5 & 0 \end{bmatrix} \overset{R_1+R_2\to R_2}{\sim} \begin{bmatrix} 0.4 & -0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we obtain $\mathbf{x} = s\begin{bmatrix} \frac{0.4}{0.5} \\ 1 \end{bmatrix} = s\begin{bmatrix} 4/5 \\ 1 \end{bmatrix}$. Setting the column sum of $\mathbf{x}$ equal to 1, we need $s = \frac{1}{9/5}$, and so $\mathbf{x} = \frac{1}{9/5}\begin{bmatrix} 4/5 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{9} \\ \frac{5}{9} \end{bmatrix}$.

46. Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\begin{bmatrix} -0.8 & 0.6 & 0 \\ 0.8 & -0.6 & 0 \end{bmatrix} \overset{R_1+R_2\to R_2}{\sim} \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we obtain $\mathbf{x} = s\begin{bmatrix} \frac{0.6}{0.8} \\ 1 \end{bmatrix} = s\begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$. Setting the column sum of $\mathbf{x}$ equal to 1, we need $s = \frac{1}{7/4}$, and so $\mathbf{x} = \frac{1}{7/4}\begin{bmatrix} 3/4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}$.

47. Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\begin{bmatrix} -0.7 & 0.5 & 0.3 & 0 \\ 0.3 & -0.8 & 0.4 & 0 \\ 0.4 & 0.3 & -0.7 & 0 \end{bmatrix} \overset{\substack{(3/7)R_1+R_2\to R_2 \\ (4/7)R_1+R_3\to R_3}}{\sim} \begin{bmatrix} -0.7 & 0.5 & 0.3 & 0 \\ 0 & -0.586 & 0.529 & 0 \\ 0 & 0.586 & -0.529 & 0 \end{bmatrix}$$

$$\overset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} -0.7 & 0.5 & 0.3 & 0 \\ 0 & -0.586 & 0.529 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and we obtain $\mathbf{x} = s\begin{bmatrix} 1.073 \\ 0.902 \\ 1.0 \end{bmatrix}$. Setting the column sum of $\mathbf{x}$ equal to 1, we need $s = \frac{1}{2.976}$, and so $\mathbf{x} = \frac{1}{2.976}\begin{bmatrix} 1.073 \\ 0.902 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0.361 \\ 0.303 \\ 0.336 \end{bmatrix}$.

48. Solve $(A - I)\,\mathbf{x} = \mathbf{0}$ by row-reducing the augmented matrix.

$$\begin{bmatrix} -0.7 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0.7 & 0 & 0 & 0 \end{bmatrix} \overset{R_1+R_3\to R_3}{\sim} \begin{bmatrix} -0.7 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\overset{R_2+R_3\to R_3}{\sim} \begin{bmatrix} -0.7 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and we obtain $\mathbf{x} = s\begin{bmatrix} 0 \\ 0 \\ 1.0 \end{bmatrix}$. Setting the column sum of $\mathbf{x}$ equal to 1, we need $s = 1$, and so $\mathbf{x} = \frac{1}{1}\begin{bmatrix} 0 \\ 0 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

49. A is lower triangular, so A^k will be lower triangular for all k. Therefore, A is not regular.

50. $A^k = A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for every k, so A is not regular.

51. A is block upper triangular, so A^k will be block upper triangular for all k. Therefore, A is not regular.

52. $A^4 = \begin{bmatrix} 0 & .3 & .5 \\ .8 & 0 & 0 \\ .2 & .7 & 0 \end{bmatrix}^4 = \begin{bmatrix} 0.1156 & 0.203 & 0.14 \\ 0.224 & 0.0816 & 0.136 \\ 0.2464 & 0.2164 & 0.034 \end{bmatrix}$,so A is regular.