An Introduction to Stochastic Modeling

Fourth Edition

Instructor Solutions Manual

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Chapter 1

2.1 $E[\mathbf{1}\{A_1\}] = Pr\{A_1\} = \frac{1}{13}$. Similarly, $E[\mathbf{1}\{A_1\}] = Pr\{A_k\} = \frac{1}{13}$ for $k = 1, \ldots, 13$. Then, because the expected value of a sum is always the sum of the expected values, $E[N] = E[\mathbf{1}\{A_1\}] + \cdots + E[\mathbf{1}\{A_{13}\}] = \frac{1}{13} + \cdots + \frac{1}{13} = 1$.

2.2 Let X be the first number observed and let Y be the second. We use the identity $(\Sigma x_i)^2 = \Sigma x_i^2 + \Sigma_{i \neq j} x_i x_j$ several times.

$$
\begin{aligned}
E[X] = E[Y] &= \frac{1}{N}\sum x_i;\\
Var[X] = Var[Y] &= \frac{1}{N}\sum x_i^2 - \left(\frac{1}{N}\sum x_i\right)^2 = \frac{(N-1)\sum x_i^2 - \sum_{i \neq j} x_i x_j}{N^2};\\
E[XY] &= \frac{\sum_{i \neq j} x_i x_j}{N(N-1)};\\
Cov[X, Y] = E[XY] - E[X]E[Y] &= \frac{\sum_{i \neq j} x_i x_j - (N-1)\sum x_i^2}{N^2(N-1)}\\
\text{ñ}_{X,Y} &= \frac{Cov[X, Y]}{\sigma_X \sigma_Y} = -\frac{1}{N-1}.
\end{aligned}
$$

2.3 Write $S_r = \xi_1 + \cdots + \xi_r$ where ξ_k is the number of additional samples needed to observe k distinct elements, assuming that $k-1$ district elements have already been observed. Then, defining $p_k = Pr[\xi_k = 1\} = 1 - \frac{k-1}{N}$ we have $Pr[\xi_k = n] = p_k (1 - p_k)^{n-1}$ for $n = 1, 2, \ldots$ and $E[\xi_k] = \frac{1}{p_k}$. Finally, $E[S_r] = E[\xi_1] + \cdots + E[\xi_r] = \frac{1}{p_1} + \cdots + \frac{1}{p_r}$ will verify the given formula.

2.4 Using an obvious notation, the event $\{N = n\}$ is equivalent to either $\overbrace{HTH\ldots HTT}^{n-1}$ or $\overbrace{THT\ldots THH}^{n-1}$ so $P_r\{N = n\} = 2 \times \left(\frac{1}{2}\right)^{n-1} \times \frac{1}{2} = \left(\frac{1}{2}\right)^{n-1}$ for $n = 2, 3, \ldots$; $Pr\{N \text{ is even}\} = \sum_{n=2,4,\ldots}\left(\frac{1}{2}\right)^{n-1} = \frac{2}{3}$ and $P_{\text{r}}\{N \leq 6\} = \sum_{n=2}^{6}\left(\frac{1}{2}\right)^{n-1} = \frac{31}{32}$. $Pr\{N \text{ is even and } N \leq 6\}\, 5 \sum_{m=2,4,6}\left(\frac{1}{2}\right)^{n-1} = \frac{21}{32}$.

2.5 Using an obvious notation, the probability that A wins on the $2n+1$ trial is $Pr\left\{\overbrace{A^cB^c\ldots A^cB^c}^{\textit{In losses}}A\right\} = [(1-p)(1-q)]^n p$, $n = 0, 1, \ldots Pr\{\text{A wins}\} = \sum_{n=0}^{\infty}[(1-p)(1-q)]^n p = \frac{p}{1-(1-p)(1-q)}$. $Pr\{A \text{ wins on } 2n+1 \text{ play}|\text{A wins}\} = (1-\pi)\pi^n$ where $\pi = (1-p)(1-q)$. $E\left[{}^{\#}\text{trials}|\text{A wins}\right] = \sum_{n=0}^{\infty}(2n+1)(1-\pi)\pi^n = 1 + \frac{2\pi}{1-\pi} = \frac{1+(1-p)(1-q)}{1-(1-p)(1-q)} = \frac{2}{1-(1-p)(1-q)} - 1$.

2.6 Let N be the number of losses and let S be the sum. Then $Pr\{N = n, S = k\} = \left(\frac{1}{6}\right)^{n-1}\left(\frac{5}{6}P_k\right)$ where $p_3 = p_{11} = p_4 = p_{10} = \frac{1}{15}$; $p_5 = p_9 = p_6 = p_8 = \frac{2}{15}$ and $p_7 = \frac{3}{15}$. Finally $Pr\{S = k\} = \sum_{n=1}^{\infty} Pr\{N = n, S = k\} = p_k$. (It is *not* a correct *argument* to simply say $Pr\{S = k\} = Pr\{\text{Sum of 2 dice} = k|\text{Dice differ}\}$. Compare with Exercise II, 2.1.)

2.7 We are given that (*) $Pr\{U > \text{u}, \text{W} > \text{w}\} = [1 - F_u(u)][1 - F_w(w)]$ for all u, w. According to the definition for independence we wish to show that $Pr\{U \leq u, W \leq w\} = F_u(u)F_w(w)$ for all u, w. Taking complements and using the addition law

$$
\begin{aligned}
Pr\{U \leq u, W \leq w\} &= 1 - Pr\{U > u \text{ or } W > w\}\\
&= 1 - [Pr\{U > u\} + Pr\{W > w\} - Pr\{U > u, W > w\}]\\
&= 1 - [(1 - F_U(u)) + (1 - F_W(w)) - (1 - F_u(u))(1 - F_w(w))]\\
&= F_U(u)F_W(w) \text{ after simplification.}
\end{aligned}
$$

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2.8 **(a)** $E[Y] = E[a + bX] = \int (a + bx)dF_X(x) = a\int dF_X(x) + b\int x dF_X(x) = a + bE[X] = a + b\mu$. In words, (a) implies that the expected value of a constant times a random variable is the constant times the expected value of the random variable. So $E\left[b^2(X-\mu)^2\right] = b^2 E\left[(X-\mu)^2\right]$.

(b) $Var[Y] = E\left[(Y - E\{Y\})^2\right] = E\left[(a + bX - a - b\mu)^2\right] = E\left[b^2(X-\mu)^2\right] = b^2 E\left[(X-\mu)^2\right] = b^2\sigma^2$

2.9 Use the usual sums of numbers formula (See I, 6 if necessary) to establish

$$\sum_{k=1}^{n} k(n-k) = \frac{1}{6}n(n+1)(n-1); \text{ and}$$

$$\sum_{k=1}^{n} k^2(n-k) = n\sum k^2 - \sum k^3 = \frac{1}{12}n^2(n+1)(n-1), \text{ so}$$

$$E[X] = \frac{2}{n(n-1)}\sum k(n-k) = \frac{1}{3}(n+1)$$

$$E\left[X^2\right] = \frac{3}{n(n-1)}\sum k^2(n-k) = \frac{1}{6}n(n+1), \text{ and}$$

$$Var[X] = E\left[X^2\right] - (E[X])^2 = \frac{1}{18}(n+1)(n-2).$$

2.10 Observe, for example, $Pr\{Z = 4\} = Pr\{X = 3, Y = 1\} = \left(\frac{1}{2}\right)\left(\frac{1}{6}\right)$, using independence. Continuing in this manner,

z	1	2	3	4	5	6
$Pr\{Z = z\}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$

2.11 Observe, for example, $Pr\{W = z\} = Pr\{U = 0, V = 2\} + Pr\{U = 1, V = 1\} = \frac{1}{6} + \frac{1}{6} + \frac{1}{3}$. Continuing in this manner, arrive at

w	1	2	3	4
$Pr\{W = w\}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$

2.12 Changing any of the random variables by adding or subtracting a constant will not affect the covariance. Therefore, by replacing U with $U - E[U]$, if necessary, etc, we may assume, without loss of generality that all of the means are zero. Because the means are zero,
$Cov[X, Y] = E[XY] - E[X]E[Y] = E[XY] = E\left[UV - UW + VW - W^2\right] = -E\left[W^2\right] = -\sigma^2$. ($E[UV] = E[U]E[V] = 0$, etc.)

2.13
$$\begin{aligned} Pr\{v < V, U \le u\} &= Pr\{v < X \le u, v < Y \le u\} \\ &= Pr\{v < X \le u\}\, Pr\{v < Y \le u\} \text{ (by independence)} \\ &= (u-v)^2 \\ &= \iint\limits_{(u',v')v<v'\le u'\le u} f_{u,v}\left(u', v'\right) du'dv' \\ &= \int_v^u \left\{ \int_{v'}^u f_{u,v}\left(u', v'\right) du' \right\} dv'. \end{aligned}$$

The integrals are removed from the last expression by successive differentiation, first w.r.t. v (changing sign because v is a lower limit) than w.r.t. u. This tells us

$$f_{u,v}(u, v) = -\frac{d}{du}\frac{d}{dv}(u-v)^2 = 2 \text{ for } 0 < v \le u \le 1.$$

3.1 Z has a discrete uniform distribution on $0, 1, \dots, 9$.

3.2 In maximizing a continuous function, we often set the derivative equal to zero. In maximizing a function of a discrete variable, we equate the ratio of successive terms to one. More precisely, k^* is the smallest k for which $\frac{p(k+1)}{p(k)} < 1$, or, the smallest k for which $\frac{n-k}{k+1}\left(\frac{p}{1-p}\right) < 1$. Equivently, (b) $k^* = [(n+1)p]$ where $[x]$ = greatest integer $\le x$. for (a) let $n \to \infty, p \to 0, \lambda = np$. Then $k^* = [\lambda]$.

3.3 Recall that $e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$ and $e^{-\lambda} = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \cdots$ so that $\sinh\lambda \equiv \frac{1}{2}\left(e^\lambda - e^{-\lambda}\right) = \lambda + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots$ Then $Pr\{X \text{ is odd}\} = \sum_{k=1,3,5,\cdots} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda}\sinh(\lambda) = \frac{1}{2}\left(1 - e^{-2\lambda}\right)$.

3.4
$$E[V] = \sum_{k=0}^{\infty} \frac{1}{k+1}\frac{\lambda^k e^{-\lambda}}{k!} = \frac{e^{-\lambda}}{\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{(k+1)!}$$
$$= \frac{1}{\lambda}e^{-\lambda}\left(e^\lambda - 1\right) = \frac{1}{\lambda}\left(1 - e^{-\lambda}\right).$$

3.5
$$E[XY] = E[X(N-X)] = NE[X] - E\left[X^2\right]$$
$$= N^2p - \left[Np(1-p) + N^2p^2\right] = N^2p(1-p) - Np(1-p)$$

$Cov[X,Y] = E[XY] - E[X]E[Y] = -Np(1-p)$.

3.6 Your intuition should suggest the correct answers: (a) X_1 is binomially distributed with parameters M and π_1; (b) N is binomial with parameters M and $\pi_1 + \pi_2$; and (c) X_1, given $N = n$, is conditionally binominal with parameters n and $p = \pi_1/(\pi_1 + \pi_2)$. To derive these correct answers formally, begin with

$$Pr\{X_1 = i, X_2 = j, X_3 = k\} = \frac{M!}{i!\,j!\,k!}\pi_1^i \pi_2^j \pi_3^k;\ i + j + k = M.$$

Since $k = M - (i + j)$

$$Pr\{X_1 = i, X_2 = j\} = \frac{M!}{i!\,j!\,(M-i-j)!}\pi_1^i \pi_2^j \pi_3^{M-i-j};\ 0 \le i + j \le M.$$

(a)
$$Pr\{X_1 = i\} = \sum_j Pr\{X_1 = i, X_2 = j\}$$
$$= \frac{M!}{i!\,(M-i)!}\pi_1^i \sum_{j=0}^{M-i}\frac{(M-i)!}{j!\,(M-i-j)!}\pi_2^j \pi_3^{M-i-j}$$
$$= \binom{M}{i}\pi_1^i(\pi_2 + \pi_3)^{M-i},\ i = 0, 1, \dots, M.$$

(b) Observe that $N = n$ if and only if $X_3 = M - n$. Apply the results of (a) to X_3:

$$Pr\{N = n\} = Pr\{X_3 = M - n\} = \frac{M!}{n!\,(M-n)!}(\pi_1 + \pi_2)^n \pi_3^{M-n}$$

(c)
$$Pr\{X_1 = k | N = n\} = \frac{Pr\{X_1 = k, X_2 = n - k\}}{Pr\{N = n\}}$$
$$= \frac{\frac{M!}{k!(M-n)!(n-k)!}\pi_1^k \pi_2^{n-k}\pi_3^{M-n}}{\frac{M!}{n!(M-n)!}(\pi_1 + \pi_2)^n \pi_3^{M-n}}$$
$$= \frac{n!}{k!\,(n-k)!}\left(\frac{\pi_1}{\pi_1 + \pi_2}\right)^k\left(\frac{\pi_2}{\pi_1 + \pi_2}\right)^{n-k},\ k = 0, 1, \dots, n.$$

3.7 $$Pr\{Z=n\}=\sum_{k=0}^{n}Pr\{X=k\}Pr\{Y=n-k\}$$
$$=\sum_{k=0}^{n}\frac{\mu^k e^{-\mu}\upsilon^{(n-k)}e^{-\upsilon}}{k!\,(n-k)!}=e^{-(\mu+\upsilon)}\frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}\mu^k\upsilon^{n-k}$$
$$=\frac{e^{-(\mu+\upsilon)}(\mu+\upsilon)^n}{n!}\qquad \text{(Using binomial formula.)}$$

Z is Poisson distributed, parameter $\mu+\upsilon$.

3.8 **(a)** X is the sum of N independent Bernoulli random variables, each with parameter p, and Y is the sum of M independent Bernoulli random variables each with the same parameter p. Z is the sum of $M+N$ independent Bernoulli random variables, each with parameter p.

(b) By considering the ways in which a committee of n people may be formed from a group comprised of M men and N women, establish the identity $\binom{M+N}{n}=\sum_{k=0}^{n}\binom{N}{k}\binom{M}{n-k}$.

Then

$$Pr\{Z=n\}=\sum_{k=0}^{n}Pr\{X=k\}Pr\{Y=n-K\}$$
$$=\sum_{k=0}^{n}\binom{N}{k}p^k(1-p)^{N-k}\binom{M}{n-k}p^{n-k}(1-p)^{M-n+k}$$
$$=\binom{M+N}{n}p^n(1-p)^{M+N-n}\text{ for }n=0,1,\ldots,M+N.$$

Note:

$$\binom{N}{k}=0\text{ for }k>N.$$

3.9 $$Pr\{X+Y=n\}=\sum_{k=0}^{n}Pr\{X=k,Y=n-k\}=\sum_{k=0}^{n}(1-\pi)\pi^k(1-\pi)\pi^{n-k}$$
$$=(1-\pi)^2\pi^n\sum_{k=0}^{n}1=(n+1)(1-\pi)^2\pi^n\text{ for }n\geqq 0.$$

3.10

k	Binomial $n=10\ p=.1$	Binomial $n=100\ p=.01$	Poisson $\lambda=1$
0	.349	.366	.368
1	.387	.370	.368
2	.194	.185	.184

3.11 $$Pr\{U=u,W=0\}=Pr\{X=u,Y=u\}=(1-\pi)^2\pi^{2u},u\geq 0.$$
$$Pr\{U=u,W=w>0\}=Pr\{X=u,Y=u+w\}+Pr\{Y=u,X=u+w\}=2(1-\pi)^2\pi^{2u+w}$$
$$Pr\{U=u\}=\sum_{w=0}^{\infty}Pr\{U=u,W=w\}=\pi^{2u}\left(1-\pi^2\right).$$
$$Pr\{W=0\}=\sum_{w=0}^{\infty}Pr\{U=u,W=0\}=(1-\pi)^2\Big/\left(1-\pi^2\right).$$
$$Pr\{W=w>0\}=2\left[(1-\pi)^2\Big/(1-\pi)^2\left(1-\pi^2\right)\right]\pi^w,\text{ and}$$
$$Pr\{U=u,W=w\}=Pr\{U=u\}Pr\{W=w\}\text{ for all }u,w.$$

3.12 Let X = number of calls to switch board in a minute. $Pr\{X \geq 7\} = 1 - \sum_{k=0}^{6} \frac{4^k e^{-4}}{k!} .111$.

3.13 Assume that inspected items are independently defective or good. Let X = # of defects in sample.

$$Pr\{X = 0\} = (.95)^{10} = .599$$
$$Pr\{X = 1\} = 10(.95)^9(.05) = .315$$
$$Pr\{X \geqq 2\} = 1 - (.599 + .315) = .086.$$

3.14 **(a)** $E[Z] = \frac{1-p}{p} = 9, Var[Z] = \frac{1-p}{p^2} = 90$

(b) $Pr\{Z > 10\} = (.9)^{10} = .349$.

3.15 $Pr\{X \leq 2\} = \left(1 + 2 + \frac{2^2}{2}\right) e^{-2} = 5e^{-2} = .677$.

3.16 **(a)** $p_0 = 1 - b \sum_{k=1}^{\infty} (1-p)^k = 1 - b\left(\frac{1-p}{p}\right)$.

(b) When $b = p$, then p_k is given by (3.4).
When $b = \frac{p}{1-p}$, then p_k is given by (3.5).

(c) $Pr\{N = n > 0\} = Pr\{X = 0, Z = n\} + Pr\{X = 1, Z = n-1\}$

$$= (1-\alpha)p(1-p)^n + \alpha p(1-p)^{n-1}$$
$$= \left[(1-\alpha)p + \alpha p \big/ (1-p)\right](1-p)^n$$

So $b = (1-\alpha)p + \alpha p \big/ (1-p)$.

4.1 $$E\left[e^{\lambda Z}\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{1}{2}z^2 + \lambda z} dz = e^{\frac{1}{2}\lambda^2} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\frac{1}{2}(z-\lambda)^2} dz \right\} = e^{\frac{1}{2}\lambda^2}.$$

4.2 **(a)** $PrW > \frac{1}{\theta} = e^{-\theta/\theta} = e^{-1} = .368 \ldots$.

(b) Mode = 0.

4.3 $X - \theta$ and $Y - \theta$ are both uniform over $\left[-\frac{1}{2}, \frac{1}{2}\right]$, independent of θ, and $W = X - Y = (X - \theta) - (Y - \theta)$. Therefore the distribution of W is independent of θ and we may determine it assuming $\theta = 0$. Also, the density of W is symmetric since that of both X and Y are.

$$Pr\{W > w\} = Pr\{X > Y + w\} = \frac{1}{2}(1-w)^2, \quad w > 0$$

So $f_w(w) = 1 - w$ for $0 \leq w \leq 1$ and $f_w(w) = 1 - |w|$ for $-1 \leq w \leq +1$

4.4 $\mu_c = .010; \sigma_c^2 = (.005)^2, Pr\{C < 0\} = Pr\left\{\frac{C - .010}{.005} < \frac{-.010}{.005}\right\} = Pr\{Z < -2\} = .0228$.

4.5 $Pr\{Z < Y\} = \int_0^{\infty} \left\{\int_x^{\infty} 3e^{-3y} dy\right\} 2e^{-2x} dx = \frac{2}{5}$.

5.1 $Pr\{N > k\} = Pr\{X_1 \leq \xi, \ldots, X_k \leq \xi\} = [F(\xi)]^k, k = 0, 1, \ldots$
$Pr\{N = k\} = Pr\{N > k-1\} - Pr\{N > k\} = [1 - F(\xi)]F(\xi)^{k-1}, k = 1, 2, \ldots$

5.2 $Pr\{Z > z\} = Pr\{X_1 > z, \ldots, X_n > z\} = Pr\{X_1 > z\} \cdot \cdots \cdot Pr\{X_n > z\}$
$= e^{-\lambda z} \cdot \cdots \cdot e^{-\lambda z} = e^{-n\lambda z}, z > 0$.

Z is exponentially distributed, parameter $n\lambda$.

5.3 $Pr\{X > k\} = \sum_{I=k+1}^{\infty} p(1-p)^l = p(1-p)^{k+1}, k = 0, 1, \ldots$

$$E[X] = \sum_{k=0}^{\infty} Pr\{X > k\} = \frac{1-p}{p}.$$

5.4 Write $V = V^+ - V^-$ when $V^+ = \max\{V, 0\}$ and $V^- = \max\{-V, 0\}$. Then $Pr\left\{V^+ > v\right\} = 1 - F_v(v)$ and $Pr\left\{V^- > v\right\} = F_v(-v)$ for $v > 0$. Use (5.3) on V^+ and V^- together with $E[V] = E\left[V^+\right] - E\left[V^-\right]$. Mean does not exit if $E\left[V^+\right] = E\left[V^-\right] = \infty$.

5.5 $E\left[W^2\right] = \int_0^\infty P\left\{W^2 > t\right\} dt = \int_0^\infty \left[1 - F_w\left(\sqrt{t}\right)\right] dt = \int_0^\infty 2y\left[1 - F_w(y)\right] dy$ by letting $y = \sqrt{t}$.

5.6 $Pr\{V > t\} = \int_t^\infty \lambda e^{-\lambda v} dv = e^{-\lambda t}$; $E[V] = \int_0^\infty Pr\{V > t\} dt = \frac{1}{\lambda} \int_0^\infty \lambda e^{-\lambda i} dt = \frac{1}{\lambda}$.

5.7 $Pr\{V > v\} = Pr\{X_1 > v, \ldots, X_n > v\} = Pr\{X_1 > v\} \cdot \cdots \cdot Pr\{X_n > v\}$

$$= e^{-\lambda_1 v} \cdot \cdots \cdot e^{-\lambda_n v} = e^{-(\lambda_1 + \cdots + \lambda n)v}, v > 0.$$

V is exponentially distributed with parameter $\sum_i^\lambda$.

5.8

Spares	3	2	1	0
A				
B				
Mean	$\frac{1}{2\lambda}$	$\frac{1}{2\lambda}$	$\frac{1}{2\lambda}$	$\frac{1}{2\lambda}$

Expected flash light operating duration $= \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} + \frac{1}{2\lambda} = \frac{2}{\lambda} = 2$ Expected battery operating durations!