

# Revised Answers Manual to an Introduction to Measure-Theoretic Probability

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## Chapter 1

### Certain Classes of Sets, Measurability, Pointwise Approximation

1. (i)  $x \in \lim_{n \rightarrow \infty} A_n$  if and only if  $x \in \cup_{n \geq 1} \cap_{j \geq n} A_j$ , so that  $x \in \cap_{j \geq n_0} A_j$  for some  $n_0 \geq 1$ , and then  $x \in A_j$  for all  $j \geq n_0$ , or  $x \in \cup_{j \geq n} A_j$  for all  $n \geq 1$ , so that  $x \in \cap_{n \geq 1} \cup_{j \geq n} A_j = \lim_{n \rightarrow \infty} A_n$ .
- (ii)  $(\lim_{n \rightarrow \infty} A_n)^c = (\cup_{n \geq 1} \cap_{j \geq n} A_j)^c = \cap_{n \geq 1} \cup_{j \geq n} A_j^c = \overline{\lim_{n \rightarrow \infty} A_n^c}$ ,  
 $(\overline{\lim_{n \rightarrow \infty} A_n})^c = (\cap_{n \geq 1} \cup_{j \geq n} A_j)^c = \cup_{n \geq 1} \cap_{j \geq n} A_j^c = \lim_{n \rightarrow \infty} A_n^c$ .  
 Let  $\lim_{n \rightarrow \infty} A_n = A$ . Then  $\lim_{n \rightarrow \infty} A_n^c = (\lim_{n \rightarrow \infty} A_n)^c = (\lim_{n \rightarrow \infty} A_n)^c = A^c$ , and  $\overline{\lim_{n \rightarrow \infty} A_n} = (\lim_{n \rightarrow \infty} A_n)^c = (\lim_{n \rightarrow \infty} A_n)^c = A^c$ , so that  $\lim_{n \rightarrow \infty} A_n^c$  exists and is  $A^c$ .
- (iii) To show that  $\lim_{n \rightarrow \infty} (A_n \cap B_n) = (\lim_{n \rightarrow \infty} A_n) \cap (\lim_{n \rightarrow \infty} B_n)$ .  
 Equivalently,

$$\bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} (A_j \cap B_j) = \left( \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j \right) \cap \left( \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} B_j \right).$$

Indeed, let  $x$  belong to the left-hand side. Then  $x \in \cap_{j=n_0}^{\infty} (A_j \cap B_j)$  for some  $n_0 \geq 1$ , hence  $x \in (A_j \cap B_j)$  for all  $j \geq n_0$ , and then  $x \in A_j$  and  $x \in B_j$  for all  $j \geq n_0$ . Hence  $x \in \cap_{j=n_0}^{\infty} A_j$  and  $x \in \cap_{j=n_0}^{\infty} B_j$ , so that  $x \in \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} A_j$  and  $x \in \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} B_j$ ; i.e.,  $x$  belongs to the right-hand side. Next, let  $x$  belong to the right-hand side. Then  $x \in \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} A_j$  and  $x \in \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} B_j$ , so that  $x \in \cap_{j=n_1}^{\infty} A_j$  and  $x \in \cap_{j=n_2}^{\infty} B_j$  for some  $n_1, n_2 \geq 1$ . Then  $x \in \cap_{j=n_0}^{\infty} A_j$  and  $x \in \cap_{j=n_0}^{\infty} B_j$  where  $n_0 = \max(n_1, n_2)$ , and hence  $x \in A_j$  and  $x \in B_j$  for all  $j \geq n_0$ . Thus,  $x \in (A_j \cap B_j)$  for all  $j \geq n_0$ , so that  $x \in \cap_{j=n_0}^{\infty} (A_j \cap B_j)$  and hence  $x \in \cup_{n=1}^{\infty} \cap_{j=n}^{\infty} (A_j \cap B_j)$ ; i.e.,  $x$  belongs to the left-hand side.

Next,  $\overline{\lim_{n \rightarrow \infty} (A_n \cup B_n)} = \overline{\lim_{n \rightarrow \infty} (A_n^c \cap B_n^c)^c} = [\lim_{n \rightarrow \infty} (A_n^c \cap B_n^c)]^c$  (by part (ii)), and this equals to  $[(\lim_{n \rightarrow \infty} A_n^c) \cap (\lim_{n \rightarrow \infty} B_n^c)]^c$  (by what we just proved), and this equals to  $[(\lim_{n \rightarrow \infty} A_n)^c \cap (\lim_{n \rightarrow \infty} B_n)^c]^c = (\lim_{n \rightarrow \infty} A_n) \cup (\lim_{n \rightarrow \infty} B_n)$ , as was to be seen.

- (iv) To show that:  $\overline{\lim}_{n \rightarrow \infty} (A_n \cap B_n) \subseteq (\overline{\lim}_{n \rightarrow \infty} A_n) \cap (\overline{\lim}_{n \rightarrow \infty} B_n)$  and  $\underline{\lim}_{n \rightarrow \infty} (A_n \cup B_n) \supseteq (\underline{\lim}_{n \rightarrow \infty} A_n) \cup (\underline{\lim}_{n \rightarrow \infty} B_n)$ .

Suffices to show:  $\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} (A_j \cap B_j) \subseteq \left( \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j \right) \cap \left( \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} B_j \right)$ .

Indeed, let  $x$  belong to the left-hand side. Then  $x \in \bigcup_{j=n}^{\infty} (A_j \cap B_j)$  for all  $n \geq 1$ , so that  $x \in (A_j \cap B_j)$  for some  $j \geq n$  and all  $n \geq 1$ . Then  $x \in A_j$  and  $x \in B_j$  for some  $j \geq n$  and all  $n \geq 1$ , hence  $x \in \bigcup_{j=n}^{\infty} A_j$  and  $x \in \bigcup_{j=n}^{\infty} B_j$  for all  $n \geq 1$ , so that  $x \in \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j$  and  $x \in \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} B_j$ , and hence  $x \in \left( \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j \right) \cap \left( \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} B_j \right)$ ; i.e.,  $x$  belongs to the right-hand side. So, the above inclusion is correct.

Also, to show that:  $\left( \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j \right) \cup \left( \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} B_j \right) \subseteq \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} (A_j \cup B_j)$ .

Indeed, let  $x$  belong to the left-hand side. Then  $x \in \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j$  or  $x \in \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} B_j$  or to both. Let  $x \in \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j$ . Then  $x \in \bigcap_{j=n_0}^{\infty} A_j$  for some  $n_0 \geq 1$ , hence  $x \in A_j$  for all  $j \geq n_0$ , and then  $x \in (A_j \cup B_j)$  for all  $j \geq n_0$ , so that  $x \in \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} (A_j \cup B_j)$ ; i.e.,  $x$  belongs to the right-hand side. Similarly if  $x \in \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} B_j$ .

An alternative proof of the second part is as follows:

$$\begin{aligned} \underline{\lim} (A_n \cup B_n) &= \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} (A_k \cup B_k) = \left[ \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (A_k^c \cap B_k^c) \right]^c \\ &= \left[ \overline{\lim} (A_k^c \cap B_k^c) \right]^c \supseteq \left[ (\overline{\lim} A_k^c) \cap (\overline{\lim} B_k^c) \right]^c \\ &\quad \text{(by the previous part)} \\ &= \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k^c \right)^c \cup \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k^c \right)^c \\ &= \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \right) \cup \left( \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} B_k \right) = (\underline{\lim} A_n) \cup (\underline{\lim} B_n). \end{aligned}$$

- (v) That the inverse inclusions in part (iv) need not hold is demonstrated by the following

*Counterexample:*

Let  $A_{2j-1} = A$ ,  $A_{2j} = A_0$  and  $B_{2j-1} = B$ ,  $B_{2j} = B_0$ ,  $j \geq 1$ , for some events  $A, A_0, B$  and  $B_0$ . Then:  $\underline{\lim}_{n \rightarrow \infty} A_n = A \cap A_0$ ,  $\overline{\lim}_{n \rightarrow \infty} A_n = A \cup A_0$ ,  $\underline{\lim}_{n \rightarrow \infty} B_n = B \cap B_0$ ,  $\overline{\lim}_{n \rightarrow \infty} B_n = B \cup B_0$ ,  $\underline{\lim}_{n \rightarrow \infty} (A_n \cap B_n) = (A \cap B) \cup (A_0 \cap B_0)$ ,  $\underline{\lim}_{n \rightarrow \infty} (A_n \cup B_n) = (A \cup B) \cap (A_0 \cup B_0)$ . Therefore  $(A \cup B) \cap (A_0 \cup B_0)$  need not contain  $(A \cup A_0) \cap (B \cup B_0)$ , and  $(A \cap A_0) \cup (B \cap B_0)$  need not contain  $(A \cup B) \cap (A_0 \cup B_0)$ .

As a concrete example, take  $\Omega = \mathfrak{R}$ ,  $A = (0, 1]$ ,  $A_0 = [2, 3]$ ,  $B = [1, 2]$ ,  $B_0 = [3, 4]$ . Then:  $(A \cup B) \cap (A_0 \cup B_0) = (0, 2]$ ,  $(A \cup A_0) \cap$

$(B \cup B_0) = ((0, 1] \cup [2, 3]) \cap ([1, 2] \cup [3, 4]) = \{1\} \cup \{3\} = \{1, 3\} \not\subseteq (0, 2]$ , and  $(A \cap A_0) \cup (B \cap B_0) = \emptyset \cup \emptyset = \emptyset$ ,  $(A \cup B) \cap (A_0 \cup B_0) = (0, 2] \cap [2, 4] = \{2\}$  not contained in  $\emptyset$ .

- (vi) If  $\lim_{n \rightarrow \infty} A_n = A$  and  $\lim_{n \rightarrow \infty} B_n = B$ , then by parts (iii) and (iv):  $\overline{\lim_{n \rightarrow \infty} (A_n \cap B_n)} \subseteq A \cap B$  and  $\underline{\lim_{n \rightarrow \infty} (A_n \cap B_n)} = A \cap B$ . Thus,  $A \cap B = \underline{\lim_{n \rightarrow \infty} (A_n \cap B_n)} \subseteq \overline{\lim_{n \rightarrow \infty} (A_n \cap B_n)} \subseteq A \cap B$ , so that  $\underline{\lim_{n \rightarrow \infty} (A_n \cap B_n)} = A \cap B$ . Likewise:  $A \cup B \subseteq \underline{\lim_{n \rightarrow \infty} (A_n \cup B_n)} \subseteq \overline{\lim_{n \rightarrow \infty} (A_n \cup B_n)} = A \cup B$ , so that  $\lim_{n \rightarrow \infty} (A_n \cup B_n) = A \cup B$ .
- (vii) Since  $A_n \Delta B = (A_n - B) + (B - A_n) = (A_n \cap B^c) + (B \cap A_n^c)$ , we have  $\lim_{n \rightarrow \infty} (A_n \cap B^c) = (\lim_{n \rightarrow \infty} A_n) \cap B^c = A \cap B^c$  by part (vi), and  $\lim_{n \rightarrow \infty} (B \cap A_n^c) = B \cap (\lim_{n \rightarrow \infty} A_n^c) = B \cap A^c$  by parts (vi) and (ii). Therefore, by part (vi) again,  $\lim_{n \rightarrow \infty} (A_n \Delta B) = \lim_{n \rightarrow \infty} [(A_n \cap B^c) + (B \cap A_n^c)] = \lim_{n \rightarrow \infty} (A_n \cap B^c) + \lim_{n \rightarrow \infty} (B \cap A_n^c) = (A \cap B^c) + (B \cap A^c) = A \Delta B$ .
- (viii)  $A_{2j-1} = B$ ,  $A_{2j} = C$ ,  $j \geq 1$ . Then, as in part (v),  $\underline{\lim_{n \rightarrow \infty} A_n} = B \cap C$  and  $\overline{\lim_{n \rightarrow \infty} A_n} = B \cup C$ . The  $\lim_{n \rightarrow \infty} A_n$  exists if and only if  $B \cap C = B \cup C$ , or  $B \cup C = (B \cap C^c) + (B^c \cap C) + (B \cap C) = B \cap C$ . Then, by the pairwise disjointness of  $B \cap C^c$ ,  $B^c \cap C$  and  $B \cap C$ , we have  $B \cap C^c = B^c \cap C = \emptyset$ . From  $B \cap C^c = \emptyset$ , it follows that  $B \subseteq C$ , and from  $B^c \cap C = \emptyset$ , it follows that  $C \subseteq B$ . Therefore  $B = C$ . Thus,  $\lim_{n \rightarrow \infty} A_n$  exists if and only if  $B = C$ . #

2. (i) All three sets  $\underline{A}$ ,  $\overline{A}$ , and  $A$  (if it exists) are in  $\mathcal{A}$ , because they are expressed in terms of  $A_n$ ,  $n \geq 1$ , by means of countable operations.
- (ii) Let  $A_n \uparrow$ . Then  $\underline{\lim_{n \rightarrow \infty} A_n} = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j = \bigcup_{n=1}^{\infty} A_n$ , and  $\overline{\lim_{n \rightarrow \infty} A_n} = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j = \bigcup_{j=n}^{\infty} A_j = \bigcup_{j=1}^{\infty} A_j = \bigcup_{n=1}^{\infty} A_n$ , so that  $\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} A_n$ .  
If  $A_n \downarrow$ , then  $A_n^c \uparrow$  and hence  $\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j^c = \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j^c = \bigcup_{n=1}^{\infty} A_n^c$ , so that, by taking the complements,  $\bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j = \bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} A_j = \bigcap_{n=1}^{\infty} A_n$ , so that  $\lim_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} A_n$ . #
3. (i)  $\bigcap_{j \in I} \mathcal{F}_j \neq \emptyset$  since, e.g.,  $\Omega \in \mathcal{F}_j$ ,  $j \in I$ . Next, if  $A \in \bigcap_{j \in I} \mathcal{F}_j$  for all  $j \in I$ , and hence  $A^c \in \mathcal{F}_j$  for all  $j \in I$ , so that  $A^c \in \bigcap_{j \in I} \mathcal{F}_j$ . Finally, if  $A, B \in \bigcap_{j \in I} \mathcal{F}_j$ , then  $A, B \in \mathcal{F}_j$  for all  $j \in I$ , and hence  $A \cup B \in \mathcal{F}_j$  for all  $j \in I$ , so that  $A \cup B \in \bigcap_{j \in I} \mathcal{F}_j$ .
- (ii) If  $A_i \in \bigcap_{j \in I} \mathcal{A}_j$ ,  $i = 1, 2, \dots$ , then  $A_i \in \mathcal{A}_j$ ,  $i = 1, 2, \dots$ , for all  $j \in I$ , and hence  $\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}_j$  for all  $j \in I$ , so that  $\bigcup_{i=1}^{\infty} A_i \in \bigcap_{j \in I} \mathcal{A}_j$ . #
4. Let  $\Omega = \mathfrak{N}$ ,  $\mathcal{F} = \{A \subseteq \mathfrak{N}; \text{ either } A \text{ or } A^c \text{ is finite}\}$ , and let  $A_j = \{1, 2, \dots, j\}$ ,  $j \geq 1$ . Then  $\mathcal{F}$  is a field and  $A_j \in \mathcal{F}$ ,  $j \geq 1$ , but  $\bigcup_{j=1}^{\infty} A_j = \{1, 2, \dots\} \notin \mathcal{F}$ , because neither this set nor its complement is finite.  
Also, if  $B_j = \{j+1, j+2, \dots\}$ , then  $B_j \in \mathcal{F}_j$  since  $B_j^c$  is finite, whereas  $\bigcap_{j=1}^{\infty} B_j = \bigcap_{j=1}^{\infty} A_j^c = \left(\bigcup_{j=1}^{\infty} A_j\right)^c \notin \mathcal{F}$ , as it has been seen already. #

5. Clearly,  $\mathcal{C}$  is  $\neq \emptyset$ , every member of  $\mathcal{C}$  is a countable union of members of  $\mathcal{P}$ , and  $\mathcal{C}$  is the smallest  $\sigma$ -field containing  $\mathcal{P}$ , if indeed, is a  $\sigma$ -field. If  $B \in \mathcal{C}$ , then  $B = \cup_{i \in I} A_i$  for some  $I \subseteq \mathbb{N} = \{1, 2, \dots\}$ , and then  $B^c = \cup_{j \in J} A_j$ , where  $J = \mathbb{N} - I$ , so that  $B^c \in \mathcal{C}$ . Finally, if  $B_j \in \mathcal{C}$ ,  $j = 1, 2, \dots$ , then  $B_j = \cup_{i \in I_j} A_{ji}$ , where  $I_j \subseteq \mathbb{N}$  and  $I_i \cap I_j = \emptyset$ . Then  $\cup_{j=1}^{\infty} B_j = \cup_{j=1}^{\infty} \cup_{i \in I_j} A_{ji}$ , the union of members of  $\mathcal{P}$ , so that  $\cup_{j=1}^{\infty} B_j$  belongs in  $\mathcal{C}$ . #
6. Since  $\mathcal{C}_j$  and  $\mathcal{C}'_j \subseteq \mathcal{C}_0$ ,  $j = 1, \dots, 8$ , it follows that  $\sigma(\mathcal{C}_j)$  and  $\sigma(\mathcal{C}'_j) \subseteq \sigma(\mathcal{C}_0) = \mathcal{B}$ , so that it suffices to show that  $\mathcal{B} \subseteq \sigma(\mathcal{C}_j)$  and  $\mathcal{B} \subseteq \sigma(\mathcal{C}'_j)$ , which are implied, respectively, by  $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_j)$  and  $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}'_j)$ ,  $j = 1, \dots, 8$ . As an example, consider the classes mentioned in the hint.

So, to show that  $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_1)$ . In all that follows, all limits are taken as  $n \rightarrow \infty$ . Indeed, for  $y_n \downarrow y$ , we have  $(x, y_n) \in \mathcal{C}_1$  and  $\cap_{n=1}^{\infty} (x, y_n) = (x, y) \in \sigma(\mathcal{C}_1)$ . Likewise, for  $x_n \uparrow x$ , we have  $(x_n, y) \in \mathcal{C}_1$  and  $\cap_{n=1}^{\infty} (x_n, y) = [x, y) \in \sigma(\mathcal{C}_1)$ . Next, with  $x_n$  and  $y_n$  as above,  $(x_n, y_n) \in \mathcal{C}_1$  and  $\cap_{n=1}^{\infty} (x_n, y_n) = [x, y) \in \sigma(\mathcal{C}_1)$ . Also, for  $x_n \downarrow -\infty$ , we have  $(x_n, a) \in \mathcal{C}_1$  and  $\cap_{n=1}^{\infty} (x_n, a) = (-\infty, a) \in \sigma(\mathcal{C}_1)$ , and likewise  $(x_n, a] \in \mathcal{C}_1$  and  $\cup_{n=1}^{\infty} (x_n, a] = (-\infty, a] \in \sigma(\mathcal{C}_1)$ . Finally,  $(b, \infty) = (-\infty, b]^c \in \sigma(\mathcal{C}_1)$ , and  $[b, \infty) = (-\infty, b)^c \in \sigma(\mathcal{C}_1)$ . It follows that  $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_1)$ .

That  $\mathcal{C}_0 \in \sigma(\mathcal{C}'_1)$  is seen as follows. For  $(x, y)$ , there exist  $x_n$  and  $y_n$  rationals with  $x_n \downarrow x$  and  $y_n \uparrow y$ , so that  $(x, y) = \cup_{n=1}^{\infty} (x_n, y_n) \in \sigma(\mathcal{C}'_1)$ . Also, for  $y_n \downarrow y$ , we have  $(x, y_n) \in \sigma(\mathcal{C}'_1)$ , as was just proved, and then  $\cap_{n=1}^{\infty} (x, y_n) = (x, y] \in \sigma(\mathcal{C}'_1)$ . Likewise, with  $x_n \uparrow x$ , we have  $(x_n, y) \in \sigma(\mathcal{C}'_1)$  and then  $\cap_{n=1}^{\infty} (x_n, y) = [x, y) \in \sigma(\mathcal{C}'_1)$ . Also, with  $x_n \uparrow x$  and  $y_n \downarrow y$ , we have  $(x_n, y_n) \in \sigma(\mathcal{C}'_1)$ , and  $\cap_{n=1}^{\infty} (x_n, y_n) = [x, y) \in \sigma(\mathcal{C}'_1)$ . Likewise, with  $x_n \downarrow -\infty$ , we have  $(x_n, a) \in \sigma(\mathcal{C}'_1)$  and  $\cup_{n=1}^{\infty} (x_n, a) = (-\infty, a) \in \sigma(\mathcal{C}'_1)$ , whereas  $(x_n, a] \in \sigma(\mathcal{C}'_1)$ , so that  $\cup_{n=1}^{\infty} (x_n, a] = (-\infty, a] \in \sigma(\mathcal{C}'_1)$ . Finally,  $(b, \infty) = (-\infty, b]^c \in \sigma(\mathcal{C}'_1)$  since  $(-\infty, b] \in \sigma(\mathcal{C}'_1)$ , and  $[b, \infty) = (-\infty, b)^c \in \sigma(\mathcal{C}'_1)$  since  $(-\infty, b) \in \sigma(\mathcal{C}'_1)$ . It follows that  $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}'_1)$ .

A slightly alternative version of the proof follows. We will show (a)  $\sigma(\mathcal{C}_1) = \mathcal{B}$  and (b)  $\sigma(\mathcal{C}'_1) = \mathcal{B}$ .

- (a)  $\sigma(\mathcal{C}_1) = \mathcal{B}$ .

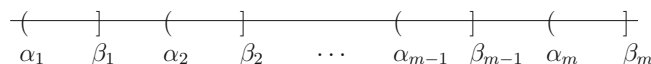
That  $\sigma(\mathcal{C}_1) \subseteq \mathcal{B}$  is clear; to show  $\mathcal{B} \subseteq \sigma(\mathcal{C}_1)$  it suffices to show that  $\mathcal{C}_0 \subseteq \sigma(\mathcal{C}_1)$ . To this end, we show that  $(x, y] \in \sigma(\mathcal{C}_1)$ . Indeed,  $(x, y + \frac{1}{n}) \in \mathcal{C}_1$ , so that  $\cap_{n=1}^{\infty} (x, y + \frac{1}{n}) = (x, y] \in \sigma(\mathcal{C}_1)$ . Next,  $(x - \frac{1}{n}, y) \in \mathcal{C}_1$ , so that  $\cap_{n=1}^{\infty} (x - \frac{1}{n}, y) = [x, y) \in \sigma(\mathcal{C}_1)$ . Also,  $(x - \frac{1}{n}, y + \frac{1}{n}) \in \mathcal{C}_1$ , so that  $\cap_{n=1}^{\infty} (x - \frac{1}{n}, y + \frac{1}{n}) = [x, y) \in \sigma(\mathcal{C}_1)$ . Next,  $(-n, x) \in \mathcal{C}_1$ , so that  $\cup_{n=1}^{\infty} (-n, x) = (-\infty, x) \in \sigma(\mathcal{C}_1)$ . Also,  $(-\infty, x + \frac{1}{n}] \in \mathcal{C}_1$ , so that  $\cap_{n=1}^{\infty} (-\infty, x + \frac{1}{n}] = (-\infty, x] \in \sigma(\mathcal{C}_1)$ . Likewise,  $(x, n) \in \mathcal{C}_1$ , so that  $\cup_{n=1}^{\infty} (x, n) = (x, \infty) \in \sigma(\mathcal{C}_1)$ ; and  $(x - \frac{1}{n}, \infty) \in \sigma(\mathcal{C}_1)$ , so that  $\cap_{n=1}^{\infty} (x - \frac{1}{n}, \infty) = [x, \infty) \in \sigma(\mathcal{C}_1)$ . The proof is complete.

- (b)  $\sigma(\mathcal{C}'_1) = \mathcal{B}$ .

Since, clearly,  $\sigma(\mathcal{C}'_1) \subseteq \sigma(\mathcal{C}_1)$ , it suffices to show that  $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}'_1)$ . For  $x, y \in \mathfrak{R}$  with  $x < y$ , there exist  $x_n \downarrow x$  and  $y_n \uparrow y$  with  $x_n, y_n$  rational numbers and  $x_n < y_n$  for each  $n$ . Since  $(x_n, y_n) \in \mathcal{C}'_1$ , it follows that  $\bigcup_{n=1}^{\infty} (x_n, y_n) = (x, y) \in \sigma(\mathcal{C}'_1)$ . So  $\mathcal{C}_1 \subseteq \sigma(\mathcal{C}'_1)$ , and hence  $\sigma(\mathcal{C}_1) \subseteq \sigma(\mathcal{C}'_1)$ . The proof is complete. #

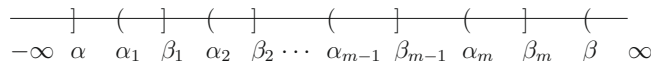
7. (i) Let  $A \in \mathcal{C}$ . Then there are the following possible cases:

(a)  $A = \sum_{i=1}^m I_i$ ,  $I_i = (\alpha_i, \beta_i]$ ,  $i = 1, \dots, m$ .



Then  $A^c = (-\infty, \alpha_1] + (\beta_1, \alpha_2] + \dots + (\beta_{m-1}, \alpha_m] + (\beta_m, \infty)$  and this is in  $\mathcal{C}$ .

- (b)  $A$  consists only of intervals of the form  $(-\infty, \alpha]$ . Then there can be only one such interval; i.e.,  $A = (-\infty, \alpha]$  and hence  $A^c = (\alpha, \infty)$  which is in  $\mathcal{C}$ .
- (c)  $A$  consists only of intervals of the form  $(\beta, \infty)$ . Then there can only be one such interval; i.e.,  $A = (\beta, \infty)$  so that  $A^c = (-\infty, \beta]$  which is in  $\mathcal{C}$ .
- (d)  $A$  consists only of intervals of the form  $(-\infty, \alpha]$  and  $(\beta, \infty)$ . Then  $A$  will be as follows:  $A = (-\infty, \alpha] + (\beta, \infty)$  ( $\alpha < \beta$ ), so that  $A^c = (\alpha, \infty) \cap (-\infty, \beta] = (\alpha, \beta]$  which is in  $\mathcal{C}$ .
- (e) Finally, let  $A$  consist of intervals of all forms. Then  $A$  is as below:



Then, clearly,

$$A^c = (\alpha, \alpha_1] + (\beta_1, \alpha_2] + \dots + (\beta_{m-1}, \alpha_m] + (\beta_m, \beta]$$

which is in  $\mathcal{C}$ . So,  $\mathcal{C}$  is closed under complementation. It is also closed under the union of two sets  $A$  and  $B$  in  $\mathcal{C}$ , because, clearly, the union of two such sets is also a member of  $\mathcal{C}$ . Thus,  $\mathcal{C}$  is a field. Next, let  $\mathcal{C}_2 = \{(\alpha, \beta]; \alpha, \beta \in \mathfrak{R}, \alpha < \beta\}$ . Then, by [Exercise 6](#),  $\sigma(\mathcal{C}_2) = \mathcal{B}$ . Also,  $\mathcal{C}_2 \subset \mathcal{C}$ , so that  $\mathcal{B} = \sigma(\mathcal{C}_2) \subseteq \sigma(\mathcal{C})$ . Furthermore,  $\mathcal{C} \subseteq \sigma(\mathcal{C}_0) = \mathcal{B}$  and hence  $\sigma(\mathcal{C}) \subseteq \mathcal{B}$ . It follows that  $\sigma(\mathcal{C}) = \mathcal{B}$ .

- (ii) If  $A \in \mathcal{C}$ , then  $A = \sum_{i=1}^m I_i$ , where  $I_i$ s are of the forms:  $(\alpha, \beta)$ ,  $(\alpha, \beta]$ ,  $[\alpha, \beta)$ ,  $[\alpha, \beta]$ ,  $(-\infty, \alpha)$ ,  $(-\infty, \alpha]$ ,  $(\beta, \infty)$ ,  $[\beta, \infty)$ . But  $(\alpha, \beta)^c = (-\infty, \alpha] + [\beta, \infty)$ ,  $(\alpha, \beta]^c = (-\infty, \alpha] + (\beta, \infty)$ ,  $[\alpha, \beta)^c = (-\infty, \alpha) + (\beta, \infty)$ ,  $[\alpha, \beta]^c = (-\infty, \alpha) + [\beta, \infty)$ ,  $(-\infty, \alpha)^c = [\alpha, \infty)$ ,  $(-\infty, \alpha]^c = (\alpha, \infty)$ ,  $(\beta, \infty)^c = (-\infty, \beta]$ , and  $[\beta, \infty)^c = (-\infty, \beta]$ . Then, considering all possibilities as in part (i), we conclude that  $A^c \in \mathcal{C}$  in all cases. Next, for  $A$  as above and  $B = \sum_{j=1}^n J_j$  with  $J_j$  being from among the above intervals, it follows that  $A \cup B$  is a finite sum of intervals as above,

and hence  $A \cup B \in \mathcal{C}$ . Thus,  $\mathcal{C}$  is a field. Finally, from  $\mathcal{C}_0 \subset \mathcal{C} \subset \mathcal{B}$ , it follows that  $\mathcal{B} = \sigma(\mathcal{C}_0) \subseteq \sigma(\mathcal{C}) \subseteq \mathcal{B}$ , so that  $\sigma(\mathcal{C}) = \mathcal{B}$ . #

8. Clearly,  $\mathcal{F}_A$  is  $\neq \emptyset$  since, for example,  $A = A \cap \Omega$  and hence  $A \in \mathcal{F}_A$ . Next, for  $B \in \mathcal{F}_A$ , it follows that  $B = A \cap C$ ,  $C \in \mathcal{F}$ , and  $B_A^c (= \text{complement of } B \text{ with respect to } A) = A \cap C^c \in \mathcal{F}_A$  since  $C^c \in \mathcal{F}$ . Finally, for  $B_1, B_2 \in \mathcal{F}_A$ , it follows that  $B_i = A \cap C_i$ ,  $C_i \in \mathcal{F}$ ,  $i = 1, 2$ , and then  $B_1 \cup B_2 = A \cap (C_1 \cup C_2) \in \mathcal{F}_A$ , since  $C_1 \cup C_2 \in \mathcal{F}$ . #
9. That  $\mathcal{A}_A \neq \emptyset$  and that it is closed under complementation is as in [Exercise 8](#). For  $B_i \in \mathcal{A}_A$ ,  $i = 1, 2, \dots$ , it follows that  $B_i = A \cap C_i$  for some  $C_i \in \mathcal{A}$ ,  $i \geq 1$ , and  $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} (A \cap C_i) = A \cap (\bigcup_{i=1}^{\infty} C_i) \in \mathcal{A}_A$  since  $\bigcup_{i=1}^{\infty} C_i \in \mathcal{A}$ . Thus,  $\mathcal{A}_A$  is a  $\sigma$ -field. Since  $\mathcal{F} \subseteq \mathcal{A}$ , it follows that  $\mathcal{F}_A \subseteq \mathcal{A}_A$  and hence  $\sigma(\mathcal{F}_A) \subseteq \mathcal{A}_A$ . Since for every  $\mathcal{F} \subseteq \mathcal{A}_i$ ,  $i \in I$ , it follows  $\mathcal{F}_A \subseteq \mathcal{A}_{i,A}$ ,  $i \in I$ , then  $\sigma(\mathcal{F}_A) \subseteq \bigcap_{i \in I} \mathcal{A}_{i,A}$ . Also,  $\sigma(\mathcal{F}_A) = \bigcap_{j \in J} \mathcal{A}_j^*$  for all  $\sigma$ -fields of subsets of  $A$  with  $\mathcal{A}_j^* \supseteq \mathcal{F}_A$ . In order to show that  $\sigma(\mathcal{F}_A) = \mathcal{A}_A$ , it must be shown that for every  $\sigma$ -field  $\mathcal{A}^*$  of subsets of  $A$  with  $\mathcal{A}^* \supseteq \mathcal{F}_A$ , we have  $\mathcal{A}^* \supseteq \mathcal{A}_A$ . That this is, indeed, the case is seen as follows. Define the class  $\mathcal{M}$  by:  $\mathcal{M} = \{C \in \mathcal{A}; A \cap C \in \mathcal{A}^*\}$ . Then, clearly,  $\mathcal{F} \subseteq \mathcal{M} \subseteq \mathcal{A}$  and  $\mathcal{M}_A (= \mathcal{M} \cap A) \subseteq \mathcal{A}^*$ . This is so because, for  $C \in \mathcal{F}$ , it follows that  $C \cap A \in \mathcal{F}_A$  and hence  $C \cap A \in \mathcal{A}^* (\supseteq \mathcal{F}_A)$ . Also, with  $\mathcal{M}_A = \{C \subseteq A; C = M \cap A, M \in \mathcal{M}\}$ , it follows that  $\mathcal{M}_A \subseteq \mathcal{A}^*$  from the definition of  $\mathcal{M}$ . We assert that  $\mathcal{M}$  is a monotone class. Indeed, let  $C_n \in \mathcal{M}$  with  $C_n \uparrow$  or  $C_n \downarrow$ . Then, for the case that  $C_n \uparrow$ ,  $A \cap (\lim_{n \rightarrow \infty} C_n) = A \cap (\bigcup_{n=1}^{\infty} C_n) = \bigcup_{n=1}^{\infty} (A \cap C_n) \in \mathcal{A}^*$  since  $A \cap C_n \in \mathcal{A}^*$ ,  $n \geq 1$ , so that  $\lim_{n \rightarrow \infty} C_n \in \mathcal{M}$ . Likewise, for  $C_n \downarrow$ ,  $A \cap (\lim_{n \rightarrow \infty} C_n) = A \cap (\bigcap_{n=1}^{\infty} C_n) = \bigcap_{n=1}^{\infty} (A \cap C_n) \in \mathcal{A}^*$  since  $A \cap C_n \in \mathcal{A}^*$ ,  $n \geq 1$ , so that  $\lim_{n \rightarrow \infty} C_n \in \mathcal{M}$ . So  $\mathcal{M}$  is a monotone class  $\supseteq \mathcal{F}$ , and hence  $\mathcal{M} \supseteq$  minimal monotone class  $\mathcal{M}_0$ , say,  $\supseteq \mathcal{F}$ . Since  $\mathcal{F}$  is a field, it follows that  $\mathcal{M}_0$  is a  $\sigma$ -field and indeed  $\mathcal{M}_0 = \mathcal{A}$  (by [Theorem 6](#)). Finally,  $\mathcal{A} = \mathcal{M}_0 \subseteq \mathcal{M}$  implies  $\mathcal{A}_A = \mathcal{M}_{0,A} \subseteq \mathcal{M}_A \subseteq \mathcal{A}^*$ , as was to be seen. #
10. Set  $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ , and let  $A \in \mathcal{F}$ . Then  $A \in \mathcal{A}_n$  for some  $n$ , so that  $A^c \in \mathcal{A}_n$  and hence  $A \in \mathcal{F}$ . Next, let  $A, B \in \mathcal{F}$ . Then  $A \in \mathcal{A}_{n_1}$ ,  $B \in \mathcal{A}_{n_2}$  for some  $n_1$  and  $n_2$ , and let  $n_0 = \max(n_1, n_2)$ . Then  $A, B \in \mathcal{A}_{n_0}$ , so that  $A \cup B \in \mathcal{A}_{n_0}$  and  $A \cup B \in \mathcal{F}$ . Then,  $A^c \in \mathcal{F}$  and  $A \cup B \in \mathcal{F}$ , so that  $\mathcal{F}$  is a field. It need not be a  $\sigma$ -field.  
*Counterexample:* Let  $\Omega = \mathfrak{R}$  and let  $\mathcal{A}_n = \{A \subseteq [-n, n]; \text{ either } A \text{ or } A^c \text{ is countable}\}$ ,  $n \geq 1$ . Then  $\mathcal{A}_n$  is a  $\sigma$ -field (by Example 8) and  $\mathcal{A}_n \uparrow$ . However,  $\mathcal{F}$  is not a  $\sigma$ -field because, if  $A_n = \{\text{rationals in } [-n, n]\}$ ,  $n \geq 1$ , and if we set  $A = \bigcup_{n=1}^{\infty} A_n$ , then  $A \notin \mathcal{F}$ , because otherwise  $A \in \mathcal{A}_n$  for some  $n$ , which cannot happen. #
11. Set  $\mathcal{M} \cap_{j \in I} \mathcal{M}_j$  and let  $A_n \in \mathcal{M}$ ,  $n \geq 1$ , where the  $A_n$ s form a monotone sequence. Then  $A_n \in \mathcal{M}_j$  for each  $j \in I$  and all  $n \geq 1$ , so that  $\lim_{n \rightarrow \infty} A_n$  is also in  $\mathcal{M}_j$ . Since this is true for all  $j \in I$ , it follows that  $\lim_{n \rightarrow \infty} A_n$  is in  $\mathcal{M}$ , and  $\mathcal{M}$  is a monotone class. #
12. Let  $\Omega = \{1, 2, \dots\}$ ,  $\mathcal{M} = \{\emptyset, \{1, \dots, n\}, \{n, n+1, \dots\}, n \geq 1, \Omega\}$ . Then  $\mathcal{M}$  is a monotone class, but not a field, because, e.g., if  $A = \{1, \dots, n\}$  and  $B = \{n-2, n-1, \dots\}$  ( $n \geq 3$ ), then  $A, B \in \mathcal{M}$ , but  $A \cap B = \{n-2, n-1, n\} \notin \mathcal{M}$ .

As another example, let  $\Omega = (0, 1)$  and  $\mathcal{M} = \{(0, 1 - \frac{1}{n}], n \geq 1, \Omega\}$ . Then  $\mathcal{M}$  is a monotone class and  $(0, \frac{1}{2}] \in \mathcal{M}$ , but  $(0, \frac{1}{2}]^c = (\frac{1}{2}, 1) \notin \mathcal{M}$ .

Still as a third example, let  $\Omega = \mathbb{R}$  and let  $\mathcal{M} = \{\emptyset, (0, n), (-n, 0), n \geq 1, (0, \infty), (-\infty, 0)\}$ . Then  $\mathcal{M}$  is a monotone class, but not a field since, for  $A = (-1, 0)$  and  $B = (0, 1)$ , we have  $A, B \in \mathcal{M}$ , but  $A \cup B = (-1, 1) \notin \mathcal{M}$ . #

13. (i) For  $\omega = (\omega_1, \omega_2) \in E^c$ , we have  $\omega \notin E = A \times B$ , so that either  $\omega_1 \notin A$  or  $\omega_2 \notin B$  or both. Let  $\omega_1 \notin A$ . Then  $\omega_1 \in A^c$  and  $(\omega_1, \omega_2) \in A^c \times \Omega_2$ , whether or not  $\omega_2 \in B$ . Hence  $E^c \subseteq (A \times B^c) + (A^c \times \Omega_2)$ . If  $\omega_1 \in A$ , then  $\omega_2 \notin B$ , so that  $(\omega_1, \omega_2) \in A \times B^c$  and  $E^c \subseteq (A \times B^c) + (A^c \times \Omega_2)$ . Next, if  $(\omega_1, \omega_2) \in A \times B^c$ , then  $\omega_1 \in A$  and  $\omega_2 \notin B$ , so that  $(\omega_1, \omega_2) \notin E$  and hence  $(\omega_1, \omega_2) \in E^c$ . If  $(\omega_1, \omega_2) \in A^c \times \Omega_2$ , then  $\omega_1 \notin A$  and hence  $(\omega_1, \omega_2) \notin A \times B = E$  whether or not  $\omega_2 \in B$ . Thus  $(\omega_1, \omega_2) \in E^c$ . In both cases,  $(A \times B^c) + (A^c \times \Omega_2) \supseteq E^c$  and equality follows. The second equality is entirely symmetric.
- (ii) Let  $(\omega_1, \omega_2) \in E_1 \cap E_2$ , so that  $(\omega_1, \omega_2) \in E_1$  and  $(\omega_1, \omega_2) \in E_2$  and hence  $\omega_1 \in A_1, \omega_2 \in B_1$ , and  $\omega_1 \in A_2, \omega_2 \in B_2$ . It follows that  $\omega_1 \in A_1 \cap A_2, \omega_2 \in B_1 \cap B_2$  and hence  $(\omega_1, \omega_2) \in (A_1 \cap A_2) \times (B_1 \cap B_2)$ . Next,  $(\omega_1, \omega_2) \in (A_1 \cap A_2) \times (B_1 \cap B_2)$ , so that  $\omega_1 \in A_1 \cap A_2$  and  $\omega_2 \in B_1 \cap B_2$ . Thus,  $\omega_1 \in A_1, \omega_1 \in A_2$  and  $\omega_2 \in B_1, \omega_2 \in B_2$ , so that  $(\omega_1, \omega_2) \in A_1 \cap B_1$  and  $(\omega_1, \omega_2) \in A_2 \cap B_2$ , or  $(\omega_1, \omega_2) \in E_1 \cap E_2$ , so that equality occurs. The second conclusion is immediate.
- (iii) Indeed,  $E_1 \cap F_1 = (A_1 \cap A'_1) \times (B_1 \cap B'_1)$  and  $E_2 \cap F_2 = (A_2 \cap A'_2) \times (B_2 \cap B'_2)$ , by part (ii), and the first equality follows. Next, again by part (ii), and replacing  $E_1$  by  $(A_1 \cap A'_1) \times (B_1 \cap B'_1)$  and  $E_2$  by  $(A_2 \cap A'_2) \times (B_2 \cap B'_2)$ , we obtain the second equality. The third equality is immediate. Finally, the last conclusion is immediate. #

14. (i) Either by the inclusion process or as follows:

$$\begin{aligned}
 & (A_1 \times B_1) - (A_2 \times B_2) \\
 &= (A_1 \times B_1) \cap (A_2 \times B_2)^c \\
 &= (A_1 \times B_1) \cap [(A_2 \times B_2^c) + (A_2^c \times \Omega_2)] \text{ (by Lemma 2)} \\
 &= (A_1 \times B_1) \cap (A_2 \times B_2^c) + (A_1 \times B_1) \cap (A_2^c \times \Omega_2) \\
 &= (A_1 \cap A_2) \times (B_1 \cap B_2^c) + (A_1 \cap A_2^c) \times (B_1 \cap \Omega_2) \text{ (clearly)} \\
 &= (A_1 \cap A_2) \times (B_1 - B_2) + (A_1 - A_2) \times B_1.
 \end{aligned}$$

- (ii) Let  $A \times B = \emptyset$ . Then  $(x, y) \in A \times B$ , so that  $x \in A$  and  $y \in B$ . Also,  $(x, y) \in \emptyset$  and this can happen only if at least one of  $A$  or  $B$  is  $= \emptyset$ . On the other hand, if at least one of  $A$  or  $B$  is  $= \emptyset$ , then, clearly,  $A \times B = \emptyset$ .
- (iii) Let  $A_1 \times B_1 \subseteq A_2 \times B_2$ . Then  $(x, y) \in A_1 \times B_1$ , so that  $x \in A_1$  and  $y \in B_1$ . Also,  $(x, y) \in A_2 \times B_2$  implies  $x \in A_2$  and  $y \in B_2$ . Thus,  $A_1 \subseteq A_2$  and  $B_1 \subseteq B_2$ . Next, let  $A_1 \subseteq A_2$  and  $B_1 \subseteq B_2$ . Then  $A_1 \times B_1 \subseteq A_2 \times B_2$  since  $(x, y) \in A_1 \times B_1$  if and only if  $x \in A_1$  and  $y \in B_1$ . Hence,  $x \in A_2$  and  $y \in B_2$  or  $(x, y) \in A_2 \times B_2$ .

- (iv)  $A_1 \times B_1 \neq \emptyset$  and  $A_2 \times B_2 \neq \emptyset$ . Then  $A_1 \times B_1 = A_2 \times B_2$  or  $A_1 \times B_1 \subseteq A_2 \times B_2$  and then (by (iii)),  $A_1 \subseteq A_2$  and  $B_1 \subseteq B_2$ . Also,  $A_2 \times B_2 = A_1 \times B_1$  or  $A_2 \times B_2 \subseteq A_1 \times B_1$ , and then (by (iii) again),  $A_2 \subseteq A_1$  and  $B_2 \subseteq B_1$ .

So, both  $A_1 \subseteq A_2$  and  $A_2 \subseteq A_1$ , and therefore  $A_1 = A_2$ . Likewise,  $B_1 \subseteq B_2$  and  $B_2 \subseteq B_1$  so that  $B_1 = B_2$ .

(v)

$$A \times B = (A_1 \times B_1) + (A_2 \times B_2) \quad (*)$$

From  $\emptyset = (A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2)$  and part (ii), we have that at least one of  $A_1 \cap A_2$ ,  $B_1 \cap B_2$  is  $\emptyset$ . Let  $A_1 \cap A_2 = \emptyset$ . Then the claim is that  $A = A_1 + A_2$ . In fact,  $(x, y) \in A \times B$  implies  $x \in A$  (and  $y \in B$ ). Also,  $(x, y)$  belonging to the right-hand side of (\*) implies  $(x, y) \in A_1 \times B_1$  or  $(x, y) \in A_2 \times B_2$ . Let  $(x, y) \in A_1 \times B_1$ . Then  $x \in A_1$  (and  $y \in B_1$ ), so that  $A \subseteq A_2$ . On the other hand,  $(x, y) \in A_2 \times B_2$  implies  $x \in A_2$  (and  $y \in B_2$ ), so that  $A \subseteq A_2$ . Thus,  $A \subseteq A_1 + A_2$ . Next, let again  $(x, y)$  belong to the right-hand side of (\*). Then  $(x, y) \in A_1 \times B_1$  or  $(x, y) \in A_2 \times B_2$ . Now  $(x, y) \in A_1 \times B_1$  implies that  $x \in A_1$  (and  $y \in B_1$ ). Also,  $(x, y)$  belonging to the left-hand side of (\*) implies  $(x, y) \in A \times B$ , so that  $x \in A$  (and  $y \in B$ ). Hence  $A_1 \subseteq A$ . Likewise,  $(x, y) \in A_2 \times B_2$  implies  $A_2 \subseteq A$ , so that  $A_1 + A_2 \subseteq A$ , and hence  $A = A_1 + A_2$ . Next, let  $A = A_1 + A_2$ . Then  $A \times B = (A_1 + A_2) \times B = (A_1 \times B) + (A_2 \times B)$ . Also,  $A \times B = (A_1 \times B_1) + (A_2 \times B_2)$ . Thus,  $(A_1 \times B) + (A_2 \times B) = (A_1 \times B_1) + (A_2 \times B_2)$ .  $(x, y)$  belonging to the left-hand side of (\*) implies  $(x, y) \in A_1 \times B$  or  $(x, y) \in A_2 \times B$ .  $(x, y) \in A_1 \times B$  yields  $y \in B$  (and  $x \in A_1$ ). Same if  $(x, y) \in A_2 \times B$ . Also,  $(x, y)$  belonging to the right-hand side of (\*) implies  $(x, y) \in A_1 \times B_1$  or  $(x, y) \in A_2 \times B_2$ . For  $(x, y) \in A_1 \times B_1$ , we have  $y \in B_1$  (and  $x \in A_1$ ), so that  $B \subseteq B_1$ . For  $(x, y) \in A_2 \times B_2$ , we have  $B \subseteq B_2$  likewise. Next, let again  $(x, y)$  belong to the right-hand side of (\*). Then  $(x, y) \in A_1 \times B_1$  or  $(x, y) \in A_2 \times B_2$ . For  $(x, y) \in A_1 \times B_1$ , we have  $y \in B_1$  (and  $x \in A_1$ ). Thus  $B_1 \subseteq B$ . For  $(x, y) \in A_2 \times B_2$ , we have  $B_2 \subseteq B$ . It follows that  $B = B_1 = B_2$ .

To summarize:  $A_1 \cap A_2 = \emptyset$  implies  $A = A_1 + A_2$  and  $B = B_1 = B_2$ . Likewise,  $B_1 \cap B_2 = \emptyset$  implies  $B = B_1 + B_2$  and  $A = A_1 = A_2$ . Furthermore,  $A_1 \cap A_2 = \emptyset$  and  $B_1 \cap B_2 = \emptyset$  cannot happen simultaneously. Indeed,  $A_1 \cap A_2 = \emptyset$  implies  $A = A_1 + A_2$ , and  $B_1 \cap B_2 = \emptyset$  implies  $B = B_1 + B_2$ . Then  $A \times B = (A_1 + A_2) \times (B_1 + B_2) = (A_1 \times B_1) + (A_2 \times B_2) + (A_1 \times B_2) + (A_2 \times B_1)$ . Also,  $A \times B = (A_1 \times B_1) + (A_2 \times B_2)$ , so that:  $(A_1 \times B_1) + (A_2 \times B_2) + (A_1 \times B_2) + (A_2 \times B_1) = (A_1 \times B_1) + (A_2 \times B_2)$ . Then  $(A_1 \times B_2) + (A_2 \times B_1) = \emptyset$  implies  $(A_1 \times B_2) = (A_2 \times B_1) = \emptyset$ , so that at least one of  $A_1$ ,  $A_2$ ,  $B_1$ ,  $B_2 = \emptyset$  (by part (ii)). However, this is not possible by the fact that  $A_1 \times B_1 \neq \emptyset$ ,  $A_2 \times B_2 \neq \emptyset$ . #



15. (i) If either  $A$  or  $B = \emptyset$ , then, clearly,  $A \times B = \emptyset$ . Next, if  $A \times B = \emptyset$ , and  $A \neq \emptyset$  and  $B \neq \emptyset$ , then there exist  $\omega_1 \in A$  and  $\omega_2 \in B$ , so that  $(\omega_1, \omega_2) \in A \times B$ , a contradiction.
- (ii) Both directions of the first assertion are immediate. Without the assumption  $E_1$  and  $E_2 \neq \emptyset$ , the result need not be true. Indeed, let  $\Omega_1 = \Omega_2$ ,  $A_1 \neq \emptyset$ ,  $B_1 = A_2 = B_2 = \emptyset$ . Then  $E_1 = E_2 = \emptyset$ , but  $A_1 \not\subseteq A_2$ . #
16. (i) If at least one of  $A_1, \dots, A_n$  is  $= \emptyset$ , then, clearly,  $A_1 \times \dots \times A_n = \emptyset$ . Next, let  $E = \emptyset$  and suppose that  $A_i \neq \emptyset, i = 1, \dots, n$ . Then there exists  $\omega_i \in A_i, i = 1, \dots, n$ , so that  $(\omega_1, \dots, \omega_n) \in E$ , a contradiction.
- (ii) Let  $\omega = (\omega_1, \dots, \omega_n) \in E \cap F$ , or  $(\omega_1, \dots, \omega_n) \in (A_1 \times \dots \times A_n) \cap (B_1 \times \dots \times B_n)$ . Then  $(\omega_1, \dots, \omega_n) \in A_1 \times \dots \times A_n$  and  $(\omega_1, \dots, \omega_n) \in B_1 \times \dots \times B_n$ . It follows that  $\omega_i \in A_i$  and  $\omega_i \in B_i, i = 1, \dots, n$ , so that  $\omega_i \in A_i \cap B_i, i = 1, \dots, n$ , and hence  $(\omega_1, \dots, \omega_n) \in (A_1 \cap B_1) \times \dots \times (A_n \cap B_n)$ . Next, let  $(\omega_1, \dots, \omega_n) \in (A_1 \cap B_1) \times \dots \times (A_n \cap B_n)$ . Then  $\omega_i \in A_i \cap B_i, i = 1, \dots, n$ , so that  $\omega_i \in A_i$  and  $\omega_i \in B_i, i = 1, \dots, n$ . It follows that  $(\omega_1, \dots, \omega_n) \in A_1 \times \dots \times A_n$  and  $(\omega_1, \dots, \omega_n) \in B_1 \times \dots \times B_n$ , so that  $(\omega_1, \dots, \omega_n) \in (A_1 \times \dots \times A_n) \cap (B_1 \times \dots \times B_n)$ . #
17. We have  $E = F + G$  and  $E, F, G$  are all  $\neq \emptyset$ . This implies that  $A_i, B_i$ , and  $C_i, i = 1, \dots, n$  are all  $\neq \emptyset$ ; this is so by [Exercise 16\(i\)](#). Furthermore, by [Exercise 16\(ii\)](#):

$$F \cap G = (B_1 \times \dots \times B_n) \cap (C_1 \times \dots \times C_n) = (B_1 \cap C_1) \times \dots \times (B_n \cap C_n),$$

whereas  $F \cap G = \emptyset$ . It follows that  $B_j \cap C_j = \emptyset$  for at least one  $j, 1 \leq j \leq n$ . Without loss of generality, suppose that  $B_1 \cap C_1 = \emptyset$ . Then we shall show that  $A_1 = B_1 + C_1$  and  $A_i = B_i = C_i, i = 2, \dots, n$ . To this end, let  $\omega_j \in A_j, j = 1, \dots, n$ . Then  $(\omega_1, \dots, \omega_n) \in A_1 \times \dots \times A_n$  or  $(\omega_1, \dots, \omega_n) \in E$  or  $(\omega_1, \dots, \omega_n) \in (F + G)$ . Hence  $(\omega_1, \dots, \omega_n) \in F$  or  $(\omega_1, \dots, \omega_n) \in G$ . Let  $(\omega_1, \dots, \omega_n) \in F$ . Then  $(\omega_1, \dots, \omega_n) \in B_1 \times \dots \times B_n$  and hence  $\omega_1 \in B_1$  or  $\omega_1 \in (B_1 \cup C_1)$ , so that  $A_1 \subseteq B_1 \cup C_1$ . Likewise if  $(\omega_1, \dots, \omega_n) \in G$ . Next, let  $\omega_j \in B_j, j = 1, \dots, n$ . Then  $(\omega_1, \dots, \omega_n) \in B_1 \times \dots \times B_n$  or  $(\omega_1, \dots, \omega_n) \in F$  or  $(\omega_1, \dots, \omega_n) \in E$  or  $(\omega_1, \dots, \omega_n) \in (A_1 \times \dots \times A_n)$ , hence  $\omega_1 \in A_1$ , which implies that  $B_1 \subseteq A_1$ . By taking  $\omega_j \in C_j, j = 1, \dots, n$  and arguing as before, we conclude that  $C_1 \subseteq A_1$ . From  $B_1 \subseteq A_1$  and  $C_1 \subseteq A_1$ , we obtain  $B_1 \cup C_1 \subseteq A_1$ . Since also  $A_1 \subseteq B_1 \cup C_1$ , we get  $A_1 = B_1 \cup C_1$ . Since  $B_1 \cap C_1 = \emptyset$ , we have then  $A_1 = B_1 + C_1$ .

It remains for us to show that  $A_i = B_i = C_i, i = 2, \dots, n$ . Without loss of generality, it suffices to show that  $A_2 = B_2 = C_2$ , the remaining cases being treated symmetrically. As before, let  $\omega_j \in A_j, j = 1, \dots, n$ . Then  $(\omega_1, \dots, \omega_n) \in (A_1 \times \dots \times A_n)$  or  $(\omega_1, \dots, \omega_n) \in E$  or  $(\omega_1, \dots, \omega_n) \in (F + G)$ . Hence either  $(\omega_1, \dots, \omega_n) \in F$  or  $(\omega_1, \dots, \omega_n) \in G$ . Let  $(\omega_1, \dots, \omega_n) \in F$ . Then  $(\omega_1, \dots, \omega_n) \in B_1 \times \dots \times B_n$  and hence  $\omega_2 \in B_2$ , so that  $A_2 \subseteq B_2$ .

Likewise  $A_2 \subseteq C_2$  if  $(\omega_1, \dots, \omega_n) \in G$ . Next, let  $(\omega_1, \dots, \omega_n) \in B_1 \times \dots \times B_n$  or  $(\omega_1, \dots, \omega_n) \in F$  or  $(\omega_1, \dots, \omega_n) \in (F + G)$  or  $(\omega_1, \dots, \omega_n) \in E$  or  $(\omega_1, \dots, \omega_n) \in (A_1 \times \dots \times A_n)$  and hence  $\omega_2 \in A_2$ , so that  $B_2 \subseteq A_2$ . It follows that  $A_2 = B_2$ . We arrive at the same conclusion  $A_2 = B_2$  if we take  $(\omega_1, \dots, \omega_n) \in G$ . So, to sum it up,  $A_1 = B_1 + C_1$ , and  $A_2 = B_2 = C_2$ , and by symmetry,  $A_i = B_i = C_i, i = 3, \dots, n$ .

A variation to the above proof is as follows.

Let  $E = F + G$  or  $A_1 \times \dots \times A_n = (B_1 \times \dots \times B_n) + (C_1 \times \dots \times C_n)$ , and let  $(\omega_1, \dots, \omega_n) \in E$ . Then  $(\omega_1, \dots, \omega_n) \in A_1 \times \dots \times A_n$ , so that  $\omega_i \in A_i, i = 1, \dots, n$ . Then  $\omega_i \in B_i, i = 1, \dots, n$  or  $\omega_i \in C_i, i = 1, \dots, n$  (but not both). So,  $A_i = B_i \cup C_i, i = 1, \dots, n$  and  $A_j = B_j + C_j$  for at least one  $j$ . Consider the case  $n = 2$ , and without loss of generality suppose that  $A_1 = B_1 + C_1, A_2 = B_2 \cup C_2$ . Then, clearly:

$$\begin{aligned} A_1 \times A_2 &= (B_1 + C_1) \times (B_2 \cup C_2) \\ &= (B_1 \times B_2) \cup (C_1 \times C_2) \cup (B_1 \times C_2) \cup (C_1 \times B_2). \end{aligned}$$

However,  $A_1 \times A_2 = (B_1 \times B_2) + (C_1 \times C_2)$ , and this implies that  $B_1 \times C_2 \subseteq B_1 \times B_2$  and  $C_1 \times B_2 \subseteq C_1 \times C_2$ , hence  $C_2 \subseteq B_2$  and  $B_2 \subseteq C_2$ , so that  $B_2 = C_2 (= A_2)$ . Next, assume the assertion to be true for  $n$  and consider:

$$A_1 \times \dots \times A_n \times A_{n+1} = (B_1 \times \dots \times B_n \times B_{n+1}) + (C_1 \times \dots \times C_n \times C_{n+1}),$$

or  $A^n \times A_{n+1} = (B^n \times B_{n+1}) + (C^n \times C_{n+1})$ , where  $A^n = A_1 \times \dots \times A_n$ ,  $B^n = B_1 \times \dots \times B_n$  and  $C^n = C_1 \times \dots \times C_n$ . Apply the reasoning used in the case  $n = 2$  by replacing  $A_1$  by  $A^n$  and  $A_2$  by  $A_{n+1}$  (so that  $B_1, B_2$  and  $C_1, C_2$  are replaced, respectively, by  $B^n, B_{n+1}$  and  $C^n, C_{n+1}$ ) to get that:

$$A^n = B^n + C^n, A_{n+1} = B_{n+1} \cup C_{n+1}.$$

The first union is a “+” by the induction hypothesis. The second union may or may not be a “+” as of now. Then:

$$\begin{aligned} A^n \times A_{n+1} &= (B^n \cup C^n) \times (B_{n+1} \cup C_{n+1}) \\ &= (B^n \times B_{n+1}) \cup (C^n \times C_{n+1}) \cup (B^n \times C_{n+1}) \cup (C^n \times B_{n+1}). \end{aligned}$$

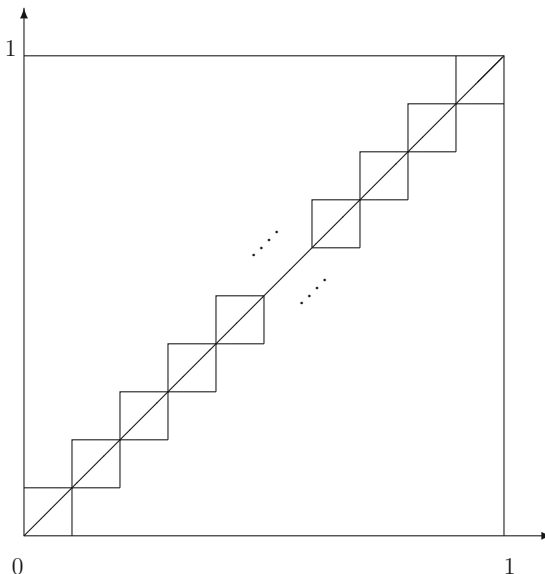
However,  $A^n \times A_{n+1} = (B^n \times B_{n+1}) + (C^n \times C_{n+1})$ . Therefore  $B^n \times C_{n+1} \subseteq B^n \times B_{n+1}$  and  $C^n \times B_{n+1} \subseteq C^n \times C_{n+1}$ , so that  $C_{n+1} \subseteq B_{n+1}$  and  $B_{n+1} \subseteq C_{n+1}$ , and hence  $B_{n+1} = C_{n+1}$ . The proof is completed. #

18. The only properties of the  $\sigma$ -fields  $\mathcal{A}_1$  and  $\mathcal{A}_2$  used in the proof of [Theorem 7](#) is that  $\mathcal{A}_i, i = 1, 2$  are closed under the intersection of two sets in them and also closed under complementations. Since these properties hold also for the case that  $\mathcal{A}_i, i = 1, 2$  are fields,  $\mathcal{F}_i, i = 1, 2$ , the proof is completed. #

19.  $\mathcal{C}$  as defined here need not be a  $\sigma$ -field. Here is a

*Counterexample:*  $\Omega_1 = \Omega_2 = [0, 1]$ . For  $n \geq 2$ , let  $I_{n1} = [0, \frac{1}{n}]$ ,  $I_{nj} = (\frac{j-1}{n}, \frac{j}{n}]$ ,  $j = 2, \dots, n$ , and set  $E_{nj} = I_{nj} \times I_{nj}, j = 1, \dots, n$ . Also, let

$Q_n = \sum_{j=1}^n E_{nj}$ ,  $n \geq 2$ . Then  $Q_n$  belongs to the field of all finite sums of rectangles. Furthermore, it is clear that  $\cap_{n=2}^{\infty} Q_n = D$ , where  $D$  is the main diagonal determined by the origin and the point  $(1,1)$ . (See picture below.) However,  $D$  is not in the class of all countable sums of rectangles, since it cannot be written as such.  $D$  is written as  $D = \cup_{x \in [0,1]} (x, x)$ , an uncountable union.



*Note:* In the picture, the first rectangle  $E_{n1} = [0, \frac{1}{n}] \times [0, \frac{1}{n}]$ , and the subsequent rectangles  $E_{nj}$  are:  $E_{nj} = (\frac{j-1}{n}, \frac{j}{n}] \times (\frac{j-1}{n}, \frac{j}{n}]$ ,  $j = 2, 3, \dots, n$ . #

20. That  $\mathcal{C} \neq \emptyset$  is obvious. For  $A \in \mathcal{C}$ , there exists  $A' \in \mathcal{A}'$  such that  $A = X^{-1}(A')$ . Then  $A^c = [X^{-1}(A')]^c = X^{-1}[(A')^c]$  with  $(A')^c \in \mathcal{A}'$ . Thus  $A^c \in \mathcal{C}$ . Finally, if  $A_j \in \mathcal{C}$ ,  $j = 1, 2, \dots$ , then  $A_j = X^{-1}(A'_j)$  with  $A'_j \in \mathcal{A}'$ , and hence  $\cup_{j=1}^{\infty} A_j = \cup_{j=1}^{\infty} X^{-1}(A'_j) = X^{-1}(\cup_{j=1}^{\infty} A'_j)$  with  $\cup_{j=1}^{\infty} A'_j \in \mathcal{A}'$ , so that  $\cup_{j=1}^{\infty} A_j \in \mathcal{C}$ , and  $\mathcal{C}$  is a  $\sigma$ -field. #
21. That  $\mathcal{C}' \neq \emptyset$  is obvious. For  $A' \in \mathcal{C}'$ , there exists  $A \in \mathcal{A}$  such that  $A = X^{-1}(A')$ . Then  $X^{-1}[(A')^c] = [X^{-1}(A')]^c = A^c \in \mathcal{A}$ , so that  $(A')^c \in \mathcal{C}'$ . Finally, for  $A'_j \in \mathcal{C}'$ ,  $j = 1, 2, \dots$ , there exists  $A_j \in \mathcal{A}$  such that  $A_j = X^{-1}(A'_j)$  and  $X^{-1}(\cup_{j=1}^{\infty} A'_j) = \cup_{j=1}^{\infty} X^{-1}(A'_j) = \cup_{j=1}^{\infty} A_j \in \mathcal{A}$ , so that  $\cup_{j=1}^{\infty} A'_j \in \mathcal{C}'$ . It follows that  $\mathcal{C}'$  is a  $\sigma$ -field. #
22. A simple example is the following. Let  $\Omega = \{a, b, c, d\}$ ,  $\mathcal{A} = \{\emptyset, \{a\}, \{b, c, d\}, \Omega\}$ ,  $X(a) = X(b) = 1$ ,  $X(c) = 2$ ,  $X(d) = 3$ . Then  $\Omega' = \{1, 2, 3\}$  and  $X(\{a\}) = \{1\}$ ,  $X(\{b, c, d\}) = \{1, 2, 3\}$ , so that  $\mathcal{C}' = \{\emptyset, \{1\}, \{1, 2, 3\}\}$  which is not a  $\sigma$ -field. #
23. Let  $X = \sum_{i=1}^n \alpha_i I_{A_i}$  and suppose that  $A_i \in \mathcal{A}$ ,  $i = 1, \dots, n$ . Then for any  $B \in \mathcal{B}$ ,  $X^{-1}(B) = \cup A_i$  where the union is taken over those  $i$ s for which  $\alpha_i \in B$ .

Since this union is in  $\mathcal{A}$ , it follows that  $X$  is a r.v. Next, let  $X$  be a r.v. Then, by assuming without loss of generality that  $\alpha_i \neq \alpha_j, i \neq j$ , we have  $X^{-1}(\{\alpha_i\}) = A_i \in \mathcal{A}$  since  $\{\alpha_i\} \in \mathcal{B}, i = 1, \dots, n$ . Clearly, the same reasoning applies when  $X = \sum_{i=1}^{\infty} \alpha_i I_{A_i}$ . #

24. Let  $\omega$  belong to the right-hand side. Then  $X(\omega) < r$  and  $Y(\omega) < x - r$  for some  $r \in Q$ , so that  $X(\omega) + Y(\omega) < x$  and hence  $\omega$  belongs to the left-hand side. Next, let  $\omega$  belong to the left-hand side, so that  $X(\omega) + Y(\omega) < x$  or  $X(\omega) < x - Y(\omega)$ . But then there exists  $r \in Q$  such that  $X(\omega) < r < x - Y(\omega)$  or  $X(\omega) < r$  and  $r < x - Y(\omega)$  or  $X(\omega) < r$  and  $Y(\omega) < x - r$ , so that  $\omega$  belongs to the right-hand side. #
25. If  $X$  is a r.v., then so is  $|X|$ , because for all  $x \geq 0$ , we have  $|X|^{-1}((-\infty, x)) = \{|X| < x\} = \{-x < X < x\} \in \mathcal{A}$ , since  $X$  is a r.v. That the converse is not necessarily true is seen by the following simple example. Take  $\Omega = \{a, b, c, d\}$ ,  $\mathcal{A} = \{\emptyset, \{a, b\}, \{c, d\}, \Omega\}$ , and define  $X$  by:  $X(a) = -1, X(b) = 1, X(c) = -2, X(d) = 2$ . Then  $\Omega' = \{-2, -1, 1, 2\}$ , and let  $\mathcal{A}' = \mathcal{P}(\Omega')$ . We have  $|X|^{-1}(\{1\}) = \{a, b\}, |X|^{-1}(\{2\}) = \{c, d\}, |X|^{-1}(\{-2\}) = |X|^{-1}(\{-1\}) = \emptyset$ , and all these sets are in  $\mathcal{A}$ , so that  $|X|$  is measurable. However,  $X^{-1}(\{-1\}) = \{a\}$  and  $X^{-1}(\{-2\}) = \{c\}$ , none of which belongs in  $\mathcal{A}$ , so that  $X$  is not measurable.

As another example, let  $B$  be a non-Borel set in  $\mathbb{R}$ , and define  $X$  by:  $X(\omega) = 1, \omega \in B$ , and  $X(\omega) = -1, \omega \in B^c$ . Then  $X$  is not  $\mathcal{B}$ -measurable as  $X^{-1}(\{1\}) = B \notin \mathcal{B}$ , but  $|X|^{-1}(\{1\}) = \mathbb{R} \in \mathcal{B}$ . #

26.  $X + Y$  is measurable by Exercise 24. Next,  $(-Y \leq y) = (Y \geq -y) \in \mathcal{A}$ , so that  $-Y$  is measurable. Then  $X + (-Y) = X - Y$  is measurable. Now, if  $Z$  is measurable, then so is  $Z^2$  because, for  $z \geq 0$ ,  $(Z^2 \leq z) = (-\sqrt{z} \leq Z \leq \sqrt{z}) \in \mathcal{A}$ . Thus, if  $X, Y$  are measurable, then so are  $(X + Y)^2$  and  $(X - Y)^2$ , and therefore so is:  $(X + Y)^2 - (X - Y)^2$ . But  $(X + Y)^2 - (X - Y)^2 = 4XY$ . Thus,  $4XY$  is measurable, and then so is, clearly,  $XY$ .
- Finally, if  $P(Y \neq 0) = 1$ , then, for  $y \neq 0$ ,  $(\frac{1}{y} \leq y) = (Y \geq \frac{1}{y}) \in \mathcal{A}$ , so that  $\frac{1}{y}$  is measurable. Thus,  $X$  and  $Y$  are measurable, and  $P(Y \neq 0) = 1$ , so that  $X$  and  $\frac{1}{Y}$  are measurable. Then  $X \times \frac{1}{Y} = \frac{X}{Y}$  is measurable. #
27. Since  $\sigma(\mathcal{T}_m) = \mathcal{B}^m$ , it suffices to show (by Theorem 2) that  $f^{-1}(\mathcal{T}_m) \subseteq \mathcal{B}^m$  for  $f$  to be measurable. By continuity of  $f$ ,  $f^{-1}(\mathcal{T}_m) \subseteq \mathcal{T}_n \subseteq \mathcal{B}^n$ , since  $\sigma(\mathcal{T}_n) = \mathcal{B}^n$ . Thus,  $f$  is measurable. Then, for  $B \in \mathcal{B}^m$ ,  $[f(X)]^{-1} = X^{-1}[f^{-1}(B)] \in \mathcal{A}$ , since  $f^{-1}(B) \in \mathcal{B}^n$  and  $X$  is measurable. #
28. For any r.v.  $Z$ , it holds:  $Z = Z^+ - Z^-$  and  $|Z| = Z^+ + Z^-$ . Hence  $Z^+ = \frac{1}{2}(|Z| + Z), Z^- = \frac{1}{2}(|Z| - Z)$ . Applying this to  $X, Y$  and  $X + Y$ , we get:

$$X^+ = \frac{1}{2}(|X| + X), Y^+ = \frac{1}{2}(|Y| + Y), (X + Y)^+ = \frac{1}{2}[|X + Y| + (X + Y)].$$

Hence

$$X^+ + Y^+ = \frac{1}{2}[|X| + |Y| + (X + Y)] \geq \frac{1}{2}[|X + Y| + (X + Y)] = (X + Y)^+.$$

Likewise,

$$X^- = \frac{1}{2}(|X| - X), \quad Y^- = \frac{1}{2}(|Y| - Y), \quad (X + Y)^- = \frac{1}{2}[|X + Y| - (X + Y)]$$

and hence

$$X^- + Y^- = \frac{1}{2}[(|X| + |Y|) - (X + Y)] \geq \frac{1}{2}[|X + Y| - (X + Y)] = (X + Y)^-.$$

*Alternative proof:*

Let  $X + Y \leq 0$ . Then  $(X + Y)^+ = 0 = 0 + 0 \leq X^+ + Y^+$ . Let  $X + Y > 0$ . Then  $(X + Y)^+ = X + Y \leq X^+ + Y^+$ , because  $X = X^+ - X^- \leq X^+$  and  $Y = Y^+ - Y^- \leq Y^+$ . Thus,  $(X + Y)^+ \leq X^+ + Y^+$ . Again, let  $X + Y < 0$ . Then  $(X + Y)^- = -(X + Y) = -X - Y \leq X^- + Y^-$ , because  $X = X^+ - X^-$  or  $-X = X^- - X^+ \leq X^-$  and  $Y = Y^+ - Y^-$  or  $-Y = Y^- - Y^+ \leq Y^-$ . Next, let  $X + Y \geq 0$ . Then  $(X + Y)^- = 0 = 0 + 0 \leq X^- + Y^-$ , so that  $(X + Y)^- \leq X^- + Y^-$ . So, again:  $(X + Y)^+ \leq X^+ + Y^+$  and  $(X + Y)^- \leq X^- + Y^-$ . #

29. (i) From the definition of  $B_m$ , we have:  $B_1 = A_1$ , and for  $m \geq 2$ ,  $B_m = A_1^c \cap \dots \cap A_{m-1}^c \cap A_m$ .
- (ii) For  $i \neq j$  (e.g.,  $i < j$ ),  $B_i$  is either  $A_1$  (for  $i = 1$ ) or  $B_i = A_1^c \cap \dots \cap A_{i-1}^c \cap A_i$ , whereas  $B_j = A_1^c \cap \dots \cap A_{j-1}^c \cap A_j$ , and  $B_i \cap B_j = \emptyset$ , because  $B_i$  contains  $A_i$  and  $B_j$  contains  $A_i^c$  (since  $i \leq j - 1$ ).
- (iii) Let  $\omega = \sum_{m=1}^{\infty} B_m$ . Then either  $\omega \in B_1 = A_1$ , and hence  $\omega \in \bigcup_{n=1}^{\infty} A_n$ , or  $\omega \notin A_i$ ,  $i = 1, \dots, n - 1$  and  $\omega \in A_n$ , so that  $\omega \in \bigcup_{n=1}^{\infty} A_n$ . Thus,  $\sum_{m=1}^{\infty} B_m \subseteq \bigcup_{n=1}^{\infty} A_n$ . Next, let  $\omega \in \bigcup_{n=1}^{\infty} A_n$ . Then either  $\omega \in A_1 = B_1$ , so that  $\omega \in \sum_{m=1}^{\infty} B_m$ , or  $\omega \notin A_i$ ,  $i = 1, \dots, n - 1$  and  $\omega \in A_n$ . Then  $\omega \in B_n$ , so that  $\omega \in \sum_{m=1}^{\infty} B_m$ . #
30. (i) We have  $\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$ , so that  $\omega \in (\lim_{n \rightarrow \infty} A_n)$  or  $\omega \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$ , therefore  $\omega \in \bigcap_{k=n_0}^{\infty} A_k$  for some  $n_0$ , and hence  $\omega \in A_k$  for all  $k \geq n_0$ . Next, let  $\omega \in A_n$  for all but finitely many  $n$ s; i.e.,  $\omega \in A_n$  for all  $n \geq n_0$ . Then  $\omega \in \bigcap_{k=n_0}^{\infty} A_k$  and hence  $\omega \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$ , which completes the proof.
- (ii) Here  $\overline{\lim}_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ , and hence  $\omega \in (\overline{\lim}_{n \rightarrow \infty} A_n)$  or  $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$  implies that  $\omega \in \bigcup_{k=n}^{\infty} A_k$  for  $n \geq 1$ . From  $\omega \in \bigcup_{k=1}^{\infty} A_k$ , let  $k_1$  be the first  $k$  for which  $\omega \in A_{k_1}$ . Next, consider  $\bigcup_{k=k_1+1}^{\infty} A_k$ , and from  $\omega \in \bigcup_{k=k_1+1}^{\infty} A_k$ , let  $k_2$  be the first  $k$  ( $\geq k_1 + 1$ ) for which  $\omega \in A_{k_2}$ . Continuing like this, we get that  $\omega$  belongs to infinitely many  $A_n$ s. In the other way around, if  $\omega$  belongs to infinitely many  $A_n$ s, that means that there exist  $1 < k_1 < k_2 < \dots$  such that  $\omega \in A_{k_j}$ ,  $j = 1, 2, \dots$ . Then  $\omega \in \bigcup_{k=k_j}^{\infty} A_k$ ,  $j \geq 1$ , and hence  $\omega \in \bigcup_{k=n}^{\infty} A_k$  for  $1 \leq n \leq k_1$  and  $k_j < n < k_{j+1}$ ,  $j \geq 1$ . Thus,  $\omega \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$  and the result follows. #
31. From  $A_k \subseteq B_k$ ,  $k \geq 1$ , we have  $\bigcup_{k=n}^{\infty} A_k \subseteq \bigcup_{k=n}^{\infty} B_k$ ,  $n \geq 1$ , and hence  $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k \subseteq \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k$  or  $\overline{\lim}_{n \rightarrow \infty} A_n \subseteq \overline{\lim}_{n \rightarrow \infty} B_n$  or  $(A_n \text{ i.o.}) \subseteq (B_n \text{ i.o.})$  (by Exercise 2). #

32. We have  $\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$  and  $\bigcap_{k=n}^{\infty} A_k = \bigcap_{k=n}^{\infty} \{r \in (1 - \frac{1}{k+1}, 1 + \frac{1}{k}) ; r \in Q\} = \{1\}$  for all  $n$ , so that  $\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k = \{1\}$ ; i.e.,  $\lim_{n \rightarrow \infty} A_n = \{1\}$ . Next,  $\overline{\lim}_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$  and  $\bigcup_{k=n}^{\infty} A_k = \bigcup_{k=n}^{\infty} \{r \in (1 - \frac{1}{k+1}, 1 + \frac{1}{k}) ; r \in Q\} = \{r \in (1 - \frac{1}{n+1}, 1 + \frac{1}{n}) ; r \in Q\}$ , so that  $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} \{r \in (1 - \frac{1}{n+1}, 1 + \frac{1}{n}) ; r \in Q\} = \{1\}$ . Thus,  $\lim_{n \rightarrow \infty} A_n = \overline{\lim}_{n \rightarrow \infty} A_n = \{1\} = \lim_{n \rightarrow \infty} A_n$ . #
33. Here  $\lim_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$ , and consider the  $\bigcap_{k=n}^{\infty} A_k$  for  $n$  odd or even.

Then

$$\bigcap_{k=2n-1}^{\infty} A_k = \left( \bigcap_{\substack{k \text{ odd} \\ \geq 2n-1}} A_k \right) \cap \left( \bigcap_{\substack{k \text{ even} \\ \geq 2n}} A_k \right),$$

and

$$A_{2n-1} \cap A_{2n+1} \cap \dots = [-1, \frac{1}{2n-1}] \cap [-1, \frac{1}{2n+1}] \cap \dots = [-1, 0], A_{2n} \cap A_{2n+2} \cap \dots = [0, \frac{1}{2n}] \cap [0, \frac{1}{2n+2}] \cap \dots = \{0\}, \text{ so that } \bigcap_{k=2n-1}^{\infty} A_k = [-1, 0] \cap \{0\} = \{0\}.$$

Next,

$$\bigcap_{k=2n}^{\infty} A_k = \left( \bigcap_{\substack{k \text{ even} \\ \geq 2n}} A_k \right) \cap \left( \bigcap_{\substack{k \text{ odd} \\ \geq 2n+1}} A_k \right),$$

and

$$A_{2n} \cap A_{2n+2} \cap \dots = [0, \frac{1}{2n}] \cap [0, \frac{1}{2n+2}] \cap \dots = \{0\}, A_{2n+1} \cap A_{2n+3} \cap \dots = [-1, \frac{1}{2n+1}] \cap [-1, \frac{1}{2n+3}] \cap \dots = [-1, 0], \text{ so that } \bigcap_{k=2n}^{\infty} A_k = \{0\} \cap [-1, 0] = \{0\}.$$

It follows that  $\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_n = \{0\} = \lim_{n \rightarrow \infty} A_n$ .

Next,  $\overline{\lim}_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$ , and consider the  $\bigcup_{k=n}^{\infty} A_k$  for odd and even values of  $n$ . We have

$$\bigcup_{k=2n-1}^{\infty} A_k = \left( \bigcup_{\substack{k \text{ odd} \\ \geq 2n-1}} A_k \right) \cup \left( \bigcup_{\substack{k \text{ even} \\ \geq 2n}} A_k \right),$$

and

$$A_{2n-1} \cup A_{2n+1} \cup \dots = [-1, \frac{1}{2n-1}] \cup [-1, \frac{1}{2n+1}] \cup \dots = [-1, \frac{1}{2n-1}], A_{2n} \cup A_{2n+2} \cup \dots = [0, \frac{1}{2n}] \cup [0, \frac{1}{2n+2}] \cup \dots = [0, \frac{1}{2n}], \text{ so that } \bigcup_{k=2n-1}^{\infty} A_k = [-1, \frac{1}{2n-1}] \cup [0, \frac{1}{2n}] = [-1, \frac{1}{2n-1}]. \text{ Next,}$$

$$\bigcup_{k=2n}^{\infty} A_k = \left( \bigcup_{\substack{k \text{ even} \\ \geq 2n}} A_k \right) \cup \left( \bigcup_{\substack{k \text{ odd} \\ \geq 2n+1}} A_k \right),$$

and

$$A_{2n} \cup A_{2n+2} \cup \dots = [0, \frac{1}{2n}] \cup [0, \frac{1}{2n+2}] \cup \dots = [0, \frac{1}{2n}], A_{2n+1} \cup A_{2n+3} \cup \dots = [-1, \frac{1}{2n+1}] \cup [-1, \frac{1}{2n+3}] \cup \dots = [-1, \frac{1}{2n+1}], \text{ so that } \bigcup_{k=2n}^{\infty} A_k = [0, \frac{1}{2n}] \cup [-1, \frac{1}{2n+1}]$$

$[-1, \frac{1}{2n+1}] = [-1, \frac{1}{2n})$ . It follows that

$$\begin{aligned}\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k &= [-1, 1] \cap [-1, \frac{1}{2}) \cap [-1, \frac{1}{3}] \cap [-1, \frac{1}{4}) \cap \dots \\ &= [-1, 0] = \overline{\lim_{n \rightarrow \infty} A_n}.\end{aligned}$$

So,  $\lim_{n \rightarrow \infty} A_n = \{0\}$  and  $\overline{\lim_{n \rightarrow \infty} A_n} = [-1, 0]$ , so that the  $\lim_{n \rightarrow \infty} A_n$  does not exist. #

34. (i) We have:

$$\{[0, 1), [1, 2), \dots, [n-1, n)\} \subset \{[0, 1), [1, 2), \dots, [n-1, n), [n, n+1)\}$$

and hence  $\mathcal{A}_n \subseteq \mathcal{A}_{n+1}$ . That  $\mathcal{A}_n \subset \mathcal{A}_{n+1}$  follows by the fact that, e.g.,  $[n, n+1)$  cannot belong in  $\mathcal{A}_n$  since all members of  $\mathcal{A}_n$  are  $\subseteq [0, n)$ .

(ii) Let  $A_1 \in \mathcal{A}_1$ ,  $A_2 \in \mathcal{A}_2$  but not in  $\mathcal{A}_1, \dots, A_n \in \mathcal{A}_n$  but not in  $\mathcal{A}_{n-1}, \dots$ , and set  $A = \bigcup_{i=1}^{\infty} A_i$ . Then  $A \notin \bigcup_{n=1}^{\infty} \mathcal{A}_n$ , because otherwise,  $A \in \mathcal{A}_n$  for some  $n$ . However, this is not possible since  $\bigcup_{i=n+1}^{\infty} A_i \notin \mathcal{A}_n$ .

(iii)  $\mathcal{A}_1 = \{\emptyset, [0, 1), [0, 1)^c = (-\infty, 0) \cup [1, \infty), \Re\}$ ,  $\mathcal{A}_2 = \{\emptyset, [0, 1), [1, 2), (-\infty, 0) \cup [1, \infty), (-\infty, 1) \cup [2, \infty), [0, 2), (-\infty, 0) \cup [2, \infty), \Re\}$ . #

35. (i) First, observe that all intersections  $A'_1 \cap \dots \cap A'_n$  are pairwise disjoint, so that their unions are, actually, sums. Next, if  $A$  and  $B$  are in  $\mathcal{C}$ , it is clear that  $A \cup B$  is a sum of intersections  $A'_1 \cap \dots \cap A'_n$  (the sum of those intersections in  $A$  and those intersections in  $B$ ), so that  $A \cup B$  is in  $\mathcal{C}$ . Now, if  $A \in \mathcal{C}$ , then  $A^c$  is the sum of all those intersections  $A'_1 \cap \dots \cap A'_n$  which are not part of  $A$ . Hence  $A^c$  is also in  $\mathcal{C}$ , and  $\mathcal{C}$  is a field.

(ii) In forming  $A'_1 \cap \dots \cap A'_n$ , we have 2 choices at each one of the  $n$  steps. Thus, there are  $2^n$  sets of the form  $A'_1 \cap \dots \cap A'_n$ . Next, in forming their sums, we select  $k$  of those members at a time, where  $k = 0, 1, \dots, 2^n$ . Therefore the total number of sums is:  $\binom{2^n}{0} + \binom{2^n}{1} + \dots + \binom{2^n}{2^n} = 2^{2^n}$ . #

36. (i) If  $\omega \in A$ , then  $f(\omega) \in f(A)$  and  $\omega \in f^{-1}[f(A)]$ . For a concrete example, take  $f : \Re \rightarrow [0, 1)$  where  $f(x) = x^2$ , and let  $A = [0, 1)$ . Then  $f(A) = f([0, 1)) = [0, 1)$ , and  $f^{-1}([0, 1)) = (-1, 1)$ . It follows that  $f^{-1}[f(A)] = f^{-1}([0, 1)) = (-1, 1) \supset [0, 1) = A$ .

(ii) Let  $\omega' \in f[f^{-1}(B)]$  which implies that there exists  $\omega \in f^{-1}(B)$  such that  $f(\omega) = \omega'$ . Also,  $\omega \in f^{-1}(B)$  implies that  $f(\omega) \in B$ . Since also  $f(\omega) = \omega'$ , it follows that  $\omega' \in B$ . Thus  $f[f^{-1}(B)] \subseteq B$ .

For a concrete example, let  $f : \Re \rightarrow \Re$  with  $f(x) = x^2$ . Take  $B = (c-1, c+1)$ , so that  $f^{-1}[(c-1, c+1)] = \Re$  and  $f(\Re) = \{c\} \subset (c-1, c+1)$ . That is,  $f[f^{-1}(B)] = \{c\} \subset (c-1, c+1) = B$ . #

37. (i) Since  $X^{-1}(\{-1\}) = A_1$ ,  $X^{-1}(\{1\}) = A_1^c \cap A_2$ , and  $X^{-1}(\{0\}) = A_1^c \cap A_2^c$ , and  $A_1, A_1^c \cap A_2, A_1^c \cap A_2^c$  are in  $\mathcal{A}$ ,  $X$  is a r.v.

- (ii) We have  $X^{-1}(\{-1\}) = \{a, b\}$ ,  $X^{-1}(\{1\}) = \{c\}$ ,  $X^{-1}(\{2\}) = \{d\}$ , and neither  $\{c\}$  nor  $\{d\}$  are in  $\mathcal{A}$ . Then  $X$  is not  $\mathcal{A}$ -measurable.
- (iii) We have  $X^{-1}(\{-2\}) = \{-2\}$ ,  $X^{-1}(\{-1\}) = \{-1\}$ ,  $X^{-1}(\{0\}) = \{0\}$ ,  $X^{-1}(\{1\}) = \{1\}$ ,  $X^{-1}(\{2\}) = \{2\}$ , so that  $X^{-1}(\mathcal{B})$  is the field induced in  $\Omega$  by the partition:  $\{\{-2\}, \{-1\}, \{0\}, \{1\}, \{2\}\}$ .  
The values taken on by  $X^2$  are 0, 1, 4, and  $(X^2)^{-1}(\{0\}) = \{0\}$ ,  $(X^2)^{-1}(\{1\}) = \{-1, 1\}$ ,  $(X^2)^{-1}(\{4\}) = \{-2, 2\}$ , so that the field induced by  $X^2$  is the one generated by the sets  $\{0\}$ ,  $\{-1, 1\}$ ,  $\{-2, 2\}$ , and it is, clearly, strictly contained in the one induced by  $X$ . #
38. For a fixed  $k$ , let  $\mathcal{A}_{k,n} = (X_k, \dots, X_{k+n-1})^{-1}(\mathcal{B})$ . Then the  $\sigma$ -fields  $\mathcal{A}_{k,n}$ ,  $n \geq 1$ , form a nondecreasing sequence and therefore  $\mathcal{F}_k = \bigcup_{n=1}^{\infty} \mathcal{A}_{k,n}$  is a field (but it may fail to be a  $\sigma$ -field; see [Exercise 10](#) in this chapter) and  $\mathcal{B}_k = \sigma(\mathcal{F}_k)$ . Likewise,  $\mathcal{B}_l = \sigma(\mathcal{F}_l)$  where  $\mathcal{F}_l = \bigcup_{n=1}^{\infty} \mathcal{A}_{l,n}$ .  
However,  $\bigcup_{n=k}^{\infty} \mathcal{A}_n \supseteq \bigcup_{n=l}^{\infty} \mathcal{A}_n$ , so that  $\mathcal{B}_k = \sigma(\bigcup_{n=k}^{\infty} \mathcal{A}_n) \supseteq \sigma(\bigcup_{n=l}^{\infty} \mathcal{A}_n) = \mathcal{B}_l$ . This is so by the way the  $\sigma$ -fields  $\mathcal{B}_k$  and  $\mathcal{B}_l$  are generated (see [Theorem 2\(ii\)](#) in this chapter). #
39. Since  $S_k$  is a function of the  $X_j$ s,  $j = 1, \dots, k$ ,  $k = 1, \dots, n$  it follows that  $\sigma(S_k) \subseteq \sigma(X_1, \dots, X_n)$ ,  $k = 1, \dots, n$ . Hence  $\bigcup_{k=1}^n \sigma(S_k) \subseteq \sigma(X_1, \dots, X_n)$  and then  $\sigma(\bigcup_{k=1}^n \sigma(S_k)) \subseteq \sigma(X_1, \dots, X_n)$  or  $\sigma(S_1, \dots, S_n) \subseteq \sigma(X_1, \dots, X_n)$ . Next,  $X_k = S_k - S_{k-1}$ ,  $k = 1, \dots, n$  ( $S_0 = 0$ ), so that  $X_k$  is a function of the  $S_j$ s,  $k = 1, \dots, n$ . Then, as above,  $\sigma(X_1, \dots, X_n) \subseteq \sigma(S_1, \dots, S_n)$ , and equality follows. #
40. Consider the function  $f : \Re \rightarrow \Re$  defined by  $y = f(x) = x + c$ . Then, clearly,  $f(B) = B_c$ . The existing inverse of  $f$ ,  $f^{-1}$ , is given by:  $x = f^{-1}(y) = y - c$ , and it is clear that  $(f^{-1})(B_c) = B$ . By setting  $g = f^{-1}$ , so that  $g^{-1} = f$ , we have that  $g^{-1}(B) (= f(B)) = B_c$ . So,  $g^{-1}$  is continuous and hence measurable, and  $g^{-1}(B) = B_c$ . Since  $B$  is measurable then so is  $B_c$ . #
41. (i) Clearly,  $\mathcal{F} \neq \emptyset$ . Next, to show that  $\mathcal{F}$  is closed under complementation. Indeed, if  $A \in \mathcal{F}$ , then

$$A = \bigcup_{i=1}^n A_i = \bigcup_{i=1}^n \bigcap_{j=1}^{m_i} A_i^j \\ = (A_1^1 \cap \dots \cap A_1^{m_1}) \cup \dots \cup (A_n^1 \cap \dots \cap A_n^{m_n})$$

with all  $A_1^1, \dots, A_1^{m_1}, \dots, A_n^1, \dots, A_n^{m_n}$  in  $\mathcal{F}_1$ , so that

$$A^c = [A_1^1 \cap \dots \cap A_1^{m_1} \cup \dots \cup (A_n^1 \cap \dots \cap A_n^{m_n})]^c \\ = [(A_1^1)^c \cup \dots \cup (A_1^{m_1})^c] \cap \dots \cap [(A_n^1)^c \cup \dots \cup (A_n^{m_n})^c] \\ = \bigcup_{i_1=1}^{m_1} \dots \bigcup_{i_n=1}^{m_n} [(A_1^{i_1})^c \cap \dots \cap (A_n^{i_n})^c].$$

The fact that  $A_1^{i_1}, \dots, A_n^{i_n}$  are in  $\mathcal{F}_1$  implies that  $(A_1^{i_1})^c, \dots, (A_n^{i_n})^c$  are also in  $\mathcal{F}_1$ , as follows from the definition of  $\mathcal{F}_1$ . So,  $A^c$  is a finite union of a finite intersection of members of  $\mathcal{F}_1$ , and hence  $A^c \in \mathcal{F}_3 (= \mathcal{F})$ ,



by the definition of  $\mathcal{F}_3$ . Next, let  $A, B \in \mathcal{F}$ . To show that  $A \cup B \in \mathcal{F}$ . Indeed,  $A, B \in \mathcal{F}$  implies that  $A = A_1 \cup \dots \cup A_m = (A_1^1 \cap \dots \cap A_1^{k_1}) \cup \dots \cup (A_m^1 \cap \dots \cap A_m^{k_m})$  with  $A_i^1, \dots, A_i^{k_i}$  in  $\mathcal{F}_1, i = 1, \dots, m$ ,  $B = B_1 \cup \dots \cup B_n = (B_1^1 \cap \dots \cap B_1^{l_1}) \cup \dots \cup (B_n^1 \cap \dots \cap B_n^{l_n})$  with  $B_j^1, \dots, B_j^{l_j}$  in  $\mathcal{F}_1, j = 1, \dots, n$ , so that

$$\begin{aligned} A \cup B &= [(A_1^1 \cap \dots \cap A_1^{k_1}) \cup \dots \cup (A_m^1 \cap \dots \cap A_m^{k_m})] \cup \\ &\quad [(B_1^1 \cap \dots \cap B_1^{l_1}) \cup \dots \cup (B_n^1 \cap \dots \cap B_n^{l_n})] \\ &= (A_1^1 \cap \dots \cap A_1^{k_1}) \cup \dots \cup (A_m^1 \cap \dots \cap A_m^{k_m}) \cup \\ &\quad (B_1^1 \cap \dots \cap B_1^{l_1}) \cup \dots \cup (B_n^1 \cap \dots \cap B_n^{l_n}), \end{aligned}$$

which is a finite union of finite intersections of members of  $\mathcal{F}_1$ . It follows that  $A \cup B$  is in  $\mathcal{F}_3 (= \mathcal{F})$ , so that  $\mathcal{F}$  is a field.

- (ii) Trivially,  $\mathcal{C} \subseteq \mathcal{F}$ , so that  $\mathcal{F}(\mathcal{C}) \subseteq \mathcal{F}$ . To show that  $\mathcal{F} \subseteq \mathcal{F}(\mathcal{C})$ . Let  $A \in \mathcal{F}$ . Then, by part (i),  $A = (A_1^1 \cap \dots \cap A_1^{m_1}) \cup \dots \cup (A_n^1 \cap \dots \cap A_n^{m_n})$  with all  $A_1^1, \dots, A_1^{m_1}, \dots, A_n^1, \dots, A_n^{m_n}$  in  $\mathcal{F}_1$ . Clearly,  $\mathcal{F}_1 \subseteq \mathcal{F}(\mathcal{C})$  by the definition of  $\mathcal{F}_1$ . Thus,  $A_i^1, \dots, A_i^{m_i}$  are in  $\mathcal{F}(\mathcal{C})$ , for  $i = 1, \dots, n$ , and then the intersections  $A_i^1 \cap \dots \cap A_i^{m_i}, i = 1, \dots, n$  are in  $\mathcal{F}(\mathcal{C})$ , and therefore so is their union  $(A_1^1 \cap \dots \cap A_1^{m_1}) \cup \dots \cup (A_n^1 \cap \dots \cap A_n^{m_n})$ . Since this union is  $A$ , it follows that  $A \in \mathcal{F}(\mathcal{C})$ . Thus,  $\mathcal{F} \subseteq \mathcal{F}(\mathcal{C})$ , and the proof is completed. #

*Remark:* In Exercise 41, in the proof that  $A \in \mathcal{F}$  implies  $A^c \in \mathcal{F}$ , the following property was used (in a slightly different notation for simplification); namely,  $(C_1^1 \cup \dots \cup C_1^{m_1}) \cap \dots \cap (C_n^1 \cup \dots \cup C_n^{m_n}) = \bigcup_{i_1=1}^{m_1} \dots \bigcup_{i_n=1}^{m_n} (C_1^{i_1} \cap \dots \cap C_n^{i_n})$ .

This is justified as follows: Let  $\omega$  belong to the right-hand side. Then  $\omega$  belongs to at least one of the  $m_1 \times \dots \times m_n$  members of the union, for example,  $\omega \in (C_1^{i'_1} \cap \dots \cap C_n^{i'_n})$  for some  $1 \leq i'_1 \leq m_1, \dots, 1 \leq i'_n \leq m_n$ . But then  $\omega \in (C_1^1 \cup \dots \cup C_1^{m_1}), \dots, \omega \in (C_n^1 \cup \dots \cup C_n^{m_n})$ , and therefore  $\omega \in [(C_1^1 \cup \dots \cup C_1^{m_1}) \cap \dots \cap (C_n^1 \cup \dots \cup C_n^{m_n})]$ , or  $\omega$  belongs to the left-hand side. Next, let  $\omega$  belong to the left-hand side. Then  $\omega \in (C_1^1 \cup \dots \cup C_1^{m_1}), \dots, \omega \in (C_n^1 \cup \dots \cup C_n^{m_n})$ , so that  $\omega \in C_1^{i'_1}, \dots, \omega \in C_n^{i'_n}$  for some  $1 \leq i'_1 \leq m_1, \dots, 1 \leq i'_n \leq m_n$ . But then  $\omega \in (C_1^{i'_1} \cap \dots \cap C_n^{i'_n})$ , and  $C_1^{i'_1} \cap \dots \cap C_n^{i'_n}$  is one of the  $m_1 \times \dots \times m_n$  members of the union on the right-hand side. It follows that  $\omega$  belongs to the right-hand side, and the justification is completed. #

42. Let  $A \in \mathcal{A}$ . Then  $A = \bigcup_{i=1}^{\infty} A_i, A_i = A_i^1 \cap A_i^2 \cap \dots$  with  $A_i^1, A_i^2, \dots$  in  $\mathcal{A}_1, i \geq 1$ . Then

$$A^c = \left( \bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c = \bigcap_{i=1}^{\infty} (A_i^1 \cap A_i^2 \cap \dots)^c$$