Chapter 1

Probability and Distributions

1.2.1 Part (c): $C_1 \cap C_2 = \{(x, y) : 1 < x < 2, 1 < y < 2\}$.

1.2.3 $C_1 \cap C_2 = \{\text{mary,mray}\}$.

1.2.6 $\lim_{k\to\infty} C_k = \{x : 0 < x < 3\}$. Note: neither the number 0 nor the number 3 is in any of the sets C_k, $k = 1, 2, 3, \ldots$

1.2.7 Part (b): $\lim_{k\to\infty} C_k = \phi$, because no point is in all the sets C_k, $k = 1, 2, 3, \ldots$

1.2.9 Because $f(x) = 0$ when $1 \le x < 10$,

$$Q(C_3) = \int_0^{10} f(x)\,dx = \int_0^1 6x(1-x)\,dx = 1.$$

1.2.11 Part (c): Draw the region C carefully, noting that $x < 2/3$ because $3x/2 < 1$. Thus

$$Q(C) = \int_0^{2/3} \left[\int_{x/2}^{3x/2} dy\right] dx = \int_0^{2/3} x\,dx = 2/9.$$

1.2.14 Note that

$$25 = Q(\mathcal{C}) = Q(C_1) + Q(C_2) - Q(C_1 \cap C_2) = 19 + 16 - Q(C_1 \cap C_2).$$

Hence, $Q(C_1 \cap C_2) = 10$.

1.2.15 By studying a Venn diagram with 3 intersecting sets, it should be true that

$$11 \ge 8 + 6 + 5 - 3 - 2 - 1 = 13.$$

It is not, and the accuracy of the report should be questioned.

1.3.3

$$P(\mathcal{C}) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1/2}{1-(1/2)} = 1.$$

1.3.6

$$P(\mathcal{C}) = \int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx = 2 \neq 1.$$

We must multiply by $1/2$.

1.3.8

$$P(C_1^c \cup C_2^c) = P[(C_1 \cap C_2)^c] = P(\mathcal{C}) = 1,$$

because $C_1 \cap C_2 = \phi$ and $\phi^c = \mathcal{C}$.

1.3.11 The probability that he does not win a prize is

$$\binom{990}{5} \Big/ \binom{1000}{5}.$$

1.3.13 Part (a): We must have 3 even or one even, 2 odd to have an even sum. Hence, the answer is

$$\frac{\binom{10}{3}\binom{10}{0}}{\binom{20}{3}} + \frac{\binom{10}{1}\binom{10}{2}}{\binom{20}{3}}.$$

1.3.14 There are 5 mutual exclusive ways this can happen: two "ones", two "twos", two "threes", two "reds", two "blues." The sum of the corresponding probabilities is

$$\frac{\binom{2}{2}\binom{6}{0} + \binom{2}{2}\binom{6}{0} + \binom{2}{2}\binom{6}{0} + \binom{5}{2}\binom{3}{0} + \binom{3}{2}\binom{5}{0}}{\binom{8}{2}}.$$

1.3.15

(a) $\quad 1 - \dfrac{\binom{48}{5}\binom{2}{0}}{\binom{50}{5}}$

(b) $\quad 1 - \dfrac{\binom{48}{n}\binom{2}{0}}{\binom{50}{n}} \geq \dfrac{1}{2}$, Solve for n.

1.3.20 Choose an integer $n_0 > \max\{a^{-1}, (1-a)^{-1}\}$. Then $\{a\} = \cap_{n=n_0}^{\infty}\left(a - \frac{1}{n}, a + \frac{1}{n}\right)$. Hence by (1.3.10),

$$P(\{a\}) = \lim_{n\to\infty} P\left[\left(a - \frac{1}{n}, a + \frac{1}{n}\right)\right] = \frac{2}{n} = 0.$$

1.3.21 Choose n_0 such that $0 < a - (1/n_0) < a < a + (1/n_0) < 1$. Let $A_n = (a - (1/n), a + (1/n))$, for $n \geq n_0$. Since $\{a\} = \cap_{n=n_0}^{\infty} A_n$, we have

$$P(\{a\}) = P\left(\cap_{n=n_0}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n) = \lim_{n\to\infty} \frac{2}{n} = 0.$$

1.4.2
$$P[(C_1 \cap C_2 \cap C_3) \cap C_4] = P[C_4|C_1 \cap C_2 \cap C_3]P(C_1 \cap C_2 \cap C_3),$$
and so forth. That is, write the last factor as
$$P[(C_1 \cap C_2) \cap C_3] = P[C_3|C_1 \cap C_2]P(C_1 \cap C_2).$$

1.4.5
$$\frac{\left[\binom{4}{3}\binom{48}{10} + \binom{4}{4}\binom{48}{9}\right] / \binom{52}{13}}{\left[\binom{4}{2}\binom{48}{11} + \binom{4}{3}\binom{48}{10} + \binom{4}{4}\binom{48}{9}\right] / \binom{52}{13}}.$$

1.4.10
$$P(C_1|C) = \frac{(2/3)(3/10)}{(2/3)(3/10) + (1/3)(8/10)} = \frac{3}{7} < \frac{2}{3} = P(C_1).$$

1.4.12 Part (c):
$$\begin{aligned} P(C_1 \cup C_2^c) &= 1 - P[(C_1 \cup C_2^c)^c] = 1 - P(C_1^* \cap C_2) \\ &= 1 - (0.4)(0.3) = 0.88. \end{aligned}$$

1.4.14 Part (d):
$$1 - (0.3)^2(0.1)(0.6).$$

1.4.16 $1 - P(TT) = 1 - (1/2)(1/2) = 3/4$, assuming independence and that H and T are equilikely.

1.4.19 Let C be the complement of the event; i.e., C equals at most 3 draws to get the first spade.

(a) $P(C) = \frac{1}{4} + \frac{3}{4}\frac{1}{4} + \left(\frac{3}{4}\right)^2 \frac{1}{4}$.
(b) $P(C) = \frac{1}{4} + \frac{13}{51}\frac{39}{52} + \frac{13}{50}\frac{38}{51}\frac{39}{52}$.

1.4.22 The probability that A wins is: $1/6+5/6\times4/6\times3/6+5/6\times4/6\times3/6\times2/6\times5/6$.

1.4.26 Let Y denote the bulb is yellow and let T_1 and T_2 denote bags of the first and second types, respectively.

(a)
$$P(Y) = P(Y|T_1)P(T_1) + P(Y|T_2)P(T_2) = \frac{20}{25}.6 + \frac{10}{25}.4.$$

(b)
$$P(T_1|Y) = \frac{P(Y|T_1)P(T_1)}{P(Y)}.$$

1.4.30 Suppose without loss of generality that the prize is behind curtain 1. Condition on the event that the contestant switches. If the contestant chooses curtain 2 then she wins, (In this case Monte cannot choose curtain 1, so he must choose curtain 3 and, hence, the contestant switches to curtain 1). Likewise, in the case the contestant chooses curtain 3. If the contestant chooses curtain 1, she loses. Therefore the conditional probability that she wins is $\frac{2}{3}$.

1.4.31 (1) The probability is $1 - \left(\frac{5}{6}\right)^4$.

(2) The probability is $1 - \left[\left(\frac{5}{6}\right)^2 + \frac{10}{36}\right]^{24}$.

1.5.2 Part (a):

$$c[(2/3) + (2/3)^2 + (2/3)^3 + \cdots] = \frac{c(2/3)}{1-(2/3)} = 2c = 1,$$

so $c = 1/2$.

1.5.5 Part (a):

$$p(x) = \begin{cases} \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}} & x = 0, 1, \ldots, 5 \\ 0 & \text{elsewhere.} \end{cases}$$

1.5.9 Part (b):

$$\sum_{x=1}^{50} x/5050 = \frac{50(51)}{2(5050)} = \frac{51}{202}.$$

1.5.10 For Part (c): Let $C_n = \{X \le n\}$. Then $C_n \subset C_{n+1}$ and $\cup_n C_n = R$. Hence, $\lim_{n\to\infty} F(n) = 1$. Let $\epsilon > 0$ be given. Choose n_0 such that $n \ge n_0$ implies $1 - F(n) < \epsilon$. Then if $x \ge n_0$, $1 - F(x) \le 1 - F(n_0) < \epsilon$.

1.6.2 Part (a):

$$p(x) = \frac{\binom{9}{x-1}}{\binom{10}{x-1}} \frac{1}{11-x} = \frac{1}{10}, \quad x = 1, 2, \ldots 10.$$

1.6.3

$$\text{(a)} \quad p(x) = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right), \quad x = 1, 2, 3, \ldots$$

$$\text{(b)} \quad \sum_{x=1}^{\infty}\left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right) = \frac{1/6}{1-(25/36)} = \frac{6}{11}.$$

1.6.5 Here are R functions which compute the pmf and the cdf:

```
p165 <- function(){
    x = 0:5
    p165 = choose(20,x)*choose(80,5-x)/choose(100,5)
    return(p165)
}
cdf165 <- function(){
    pm <- p165(); dm = -1:6; dx = c(dm,dm); dy = c(rep(0,8),rep(1,8))
    plot(dy~dx,pch=" ",xlab="x",ylab=expression(F(x)));segments(-1,0,0,0)

    ct = 0; cx = -1
```

```
    for(j in 1:6){
        ct = ct + pm[j]; cx = cx + 1
        segments(cx,ct,cx+1,ct)
    }
}
```

1.6.8 $\mathcal{D}_y = \{1, 2^3, 3^3, \ldots\}$. The pmf of Y is

$$p(y) = \left(\frac{1}{2}\right)^{y^{1/3}}, \quad y \in \mathcal{D}_y.$$

1.7.1 If $\sqrt{x} < 10$ then

$$F(x) = P[X(c) = c^2 \le x] = P(c \le \sqrt{x}) = \int_0^{\sqrt{x}} \frac{1}{10}\, dz = \frac{\sqrt{x}}{10}.$$

Thus

$$f(x) = F'(x) = \begin{cases} \frac{1}{20\sqrt{x}} & 0 < x < 100 \\ 0 & \text{elsewhere.} \end{cases}$$

1.7.2

$$C_2 \subset C_1^c \Rightarrow P(C_2) \le P(C_1^c) = 1 - (3/8) = 5/8.$$

1.7.4 Among other characteristics,

$$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\, dx = \frac{1}{\pi} \text{arctan} x \Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1.$$

1.7.6 Part (b):

$$\begin{aligned} P(X^2 < 9) &= P(-3 < X < 3) = \int_{-2}^{3} \frac{x+2}{19}\, dx \\ &= \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-2}^{3} = \frac{1}{18}\left[\frac{21}{2} - (-2)\right] = \frac{25}{36}. \end{aligned}$$

1.7.8 Part (c):

$$f'(x) = xe^{-x} - \frac{1}{2}x^2 e^{-x} = \frac{1}{2}xe^{-x}(2-x) = 0;$$

hence, $x = 2$ is the mode because it maximizes $f(x)$.

1.7.9 Part (b):

$$\int_0^m 3x^2\, dx = \frac{1}{2};$$

hence, $m^3 = 2^{-1}$ and $m = (1/2)^{1/3}$.

1.7.10

$$\int_0^{\xi_{0.2}} 4x^3\,dx = 0.2:$$

hence, $\xi_{0.2}^4 = 0.2$ and $\xi_{0.2} = 0.2^{1/4}$.

1.7.13 $x = 1$ is the mode because for $0 < x < \infty$ because

$$\begin{aligned} f(x) &= F'(x) = e^{-x} - e^{-x} + xe^{-x} = xe^{-x} \\ f'(x) &= -xe^{-x} + e^{-x} = 0, \end{aligned}$$

and $f'(1) = 0$.

1.7.16 Since $\Delta > 0$

$$X > z \Rightarrow Y = X + \Delta > z.$$

Hence, $P(X > z) \leq P(Y > z)$.

1.7.19 Since $f(x)$ is symmetric about 0, $\xi_{.25} < 0$. So we need to solve,

$$\int_{-2}^{\xi_{.25}} \left(-\frac{x}{4}\right)\,dx = .25.$$

The solution is $\xi_{.25} = -\sqrt{2}$.

1.7.22 For $0 < y < 27$,

$$\begin{aligned} x &= y^{1/3}, \quad \frac{dx}{dy} = \frac{1}{3}y^{-2/3} \\ g(y) = &= \frac{1}{3y^{2/3}}\frac{y^{2/3}}{9} = \frac{1}{27}. \end{aligned}$$

1.7.24

$$\begin{aligned} f(x) &= \frac{1}{\pi}, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}. \\ x &= \arctan y, \quad \frac{dx}{dy} = \frac{1}{1+y^2}, \quad -\infty < y < \infty. \\ g(y) &= \frac{1}{\pi}\frac{1}{1+y^2}, \quad -\infty < y < \infty. \end{aligned}$$

1.7.25

$$\begin{aligned} G(y) &= P(-2\log X^4 \leq y) = P(X \geq e^{-y/8}) = \int_{e^{-y/8}}^{1} 4x^3\,dx = 1 - e^{-y/2}, \quad 0 < y < \infty \\ g(y) &= G'(y) = \begin{cases} e^{-y/2} & 0 < y < \infty \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

1.7.26

$$
\begin{aligned}
G(y) &= P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= \begin{cases} \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\, dx = \frac{2\sqrt{y}}{3} & 0 \leq y < 1 \\ \int_{-1}^{\sqrt{y}} \frac{1}{3}\, dx = \frac{\sqrt{y}}{3} + \frac{1}{3} & 1 \leq y < 4 \end{cases} \\
g(y) &= \begin{cases} \frac{1}{3\sqrt{y}} & 0 \leq y < 1 \\ \frac{1}{6\sqrt{y}} & 1 \leq y < 4 \\ 0 & \text{elsewhere.} \end{cases}
\end{aligned}
$$

1.8.5

$$E(1/X) = \sum_{x=51}^{100} \frac{1}{x}\frac{1}{50}.$$

The latter sum is bounded by the two integrals

$$\int_{51}^{101} \tfrac{1}{x}\, dx \text{ and } \int_{50}^{100} \tfrac{1}{x}\, dx.$$

An appropriate approximation might be

$$\frac{1}{50}\int_{50.5}^{101.5} \frac{1}{x}\, dx = \frac{1}{50}(\log 100.5 - \log 50.5).$$

1.8.7

$$E[X(1-X)] = \int_0^1 x(1-x)3x^2\, dx.$$

1.8.9 When $1 < y < \infty$

$$
\begin{aligned}
G(y) &= P(1/X \leq y) = P(X \geq 1/y) = \int_{1/y}^1 2x\, dx = 1 - \frac{1}{y^2} \\
g(y) &= \frac{2}{y^3} \\
E(Y) &= \int_1^\infty y\frac{2}{y^3}\, dy = 2, \text{ which equals } \int_0^1 (1/x)2x\, dx.
\end{aligned}
$$

1.8.11 The expectation of X does not exist because

$$E(|X|) = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}\, dx = \frac{1}{\pi}\int_1^\infty \frac{1}{u}\, du = \infty,$$

where the change of variable $u = 1 + x^2$ was used.

1.8.14 Here is the pmf of G:

$-p_0+2$	$-p_0+5$	$-p_0+8$
$\frac{\binom{3}{2}}{\binom{5}{2}}$	$\frac{\binom{3}{1}\binom{2}{1}}{\binom{5}{2}}$	$\frac{\binom{2}{2}}{\binom{5}{2}}$

It follows that $E(G) = -p_0 + 4.4$; so for a fair game take $p_0 = \$4.40$. Here is an R function which simulates the game.

```
game1814 <- function(p0,nsims){
    collG = c()

    for(i in 1:nsims){
        p = sample(c(1,1,1,4,4),2)
        collG = c(collG,-p0 + sum(p))
    }
    game1814 = mean(collG)
    return(game1814)
}
```

1.9.2

$$M(t) = \sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x = \frac{e^t/2}{1-(e^t/2)}, \quad e^t/2 < 1.$$

Find $E(X) = M'(0)$ and $\text{Var}(X) = M''(0) - [M'(0)]^2$.

1.9.4

$$0 \le \text{var}(X) = E(X^2) - [E(X)]^2.$$

1.9.6

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \frac{1}{\sigma^2}\sigma^2 = 1.$$

1.9.8

$$\begin{aligned} K(b) &= E[(X-b)^2] = E(X^2) - 2bE(X) + b^2 \\ K'(b) &= -2E(X) + 2b = 0 \Rightarrow b = E(X). \end{aligned}$$

1.9.11 For a continuous type random variable,

$$\begin{aligned} K(t) &= \int_{-\infty}^{\infty} t^x f(x)\,dx. \\ K'(t) &= \int_{-\infty}^{\infty} xt^{x-1} f(x)\,dx \Rightarrow K'(1) = E(X). \\ K''(t) &= \int_{-\infty}^{\infty} x(x-1)t^{x-2} f(x)\,dx \Rightarrow K''(1) = E[X(X_1)]; \end{aligned}$$

and so forth.

1.9.12

$$\begin{aligned} 3 &= E(X-7) \Rightarrow E(X) = 10 = \mu. \\ 11 &= E[(X-7)^2] = E(X^2) - 14E(X) + 49 = E(X^2) - 91 \\ &\Rightarrow E(X^2) = 102 \text{ and } \operatorname{var}(X) = 102 - 100 = 2. \\ 15 &= E[(X-7)^3].\ \text{Expand } (X-7)^3 \text{ and continue.} \end{aligned}$$

1.9.16

$$\begin{aligned} E(X) &= 0 \Rightarrow \operatorname{var}(X) = E(X^2) = 2p. \\ E(X^4) &= 2p \Rightarrow \text{kurtosis} = 2p/4p^2 = 1/2p. \end{aligned}$$

1.9.17

$$\begin{aligned} \psi'(t) &= M'(t)/M(t) \Rightarrow \psi'(0) = M'(0)/M(0) = E(X). \\ \psi''(t) &= \frac{M(t)M''(t) - M'(t)M'(t)}{[M(t)^2]} \\ &\Rightarrow \psi''(0) = \frac{M(0)M''(0) - M'(0)M'(0)}{[M(0)^2]} = M''(0) - [M'(0)]^2 = \operatorname{var}(X). \end{aligned}$$

1.9.19

$$M(t) = (1-t)^{-3} = 1 + 3t + 3 \cdot 4\frac{t^2}{2!} + 3 \cdot 4 \cdot 5\frac{t^3}{3!} + \cdots$$

Considering the coefficient of $t^r/r!$, we have

$$E(X^r) = 3 \cdot 4 \cdot 5 \cdots (r+2), \quad r = 1, 2, 3 \ldots.$$

1.9.21 Integrating the parts with $u = 1 - F(x)$, $dv = dx$, we get

$$\{[1 - F(x)]x\}_0^b - \int_0^b x[-f(x)]\,dx = \int_0^b xf(x)\,dx = E(X).$$

1.9.24

$$\begin{aligned} E(X) &= \int_0^1 x\frac{1}{4}\,dx + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} = \frac{5}{8}. \\ E(X^2) &= \int_0^1 x^2\frac{1}{4}\,dx + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} = \frac{7}{12}. \\ \operatorname{var}(X) &= \frac{7}{12} - \left(\frac{5}{8}\right)^2 = \frac{37}{192}. \end{aligned}$$

1.9.25

$$E(X) = \int_{-\infty}^{\infty} x[c_1 f_1(x) + \cdots + c_k f_k(x)]\,dx = \sum_{i=1}^{k} c_i \mu_i = \mu.$$

Because $\int_{-\infty}^{\infty}(x-\mu)^2 f_i(x)\,dx = \sigma_i^2 + (\mu_i - \mu)^2$, we have

$$E[(X-\mu)^2] = \sum_{i=1}^{k} c_i[\sigma_i^2 + (\mu_i - \mu)^2].$$

1.10.2
$$\mu = \int_0^{\infty} xf(x)\,dx \geq \int_{2\mu}^{\infty} 2\mu f(x)\,dx = 2\mu P(X > 2\mu).$$

Thus $\frac{1}{2} \geq P(X > 2\mu)$.

1.10.5 If, in Theorem 1.10.2, we take $u(X) = \exp\{tX\}$ and $c = \exp\{ta\}$, we have

$$P(\exp\{tX\} \geq \exp\{ta\}] \leq M(t)\exp\{-ta\}.$$

If $t > 0$, the events $\exp\{tX\} \geq \exp\{ta\}$ and $X \geq a$ are equivalent. If $t < 0$, the events $\exp\{tX\} \geq \exp\{ta\}$ and $X \leq a$ are equivalent.

1.10.6 We have $P(X \geq 1) \leq [1 - \exp\{-2t\}]/2t$ for all $0 < t < \infty$, and $P(X \leq -1) \leq [\exp\{2t\} - 1]/2t$ for all $-\infty < t < 0$. Each of these bounds has the limit 0 as $t \to \infty$ and $t \to -\infty$, respectively.

Chapter 2

Multivariate Distributions

2.1.2

$$P(A_5) = \frac{7}{8} - \frac{4}{8} - \frac{3}{8} + \frac{2}{8} = \frac{2}{8}.$$

2.1.5

$$\begin{aligned}\int_0^\infty \int_0^\infty \left[2g(\sqrt{x_1^2+x_2^2})/\pi\sqrt{x_1^2+x_2^2}\right] dx_1 dx_2 &= \int_0^\infty \int_0^{\pi/2} [2g(\rho)/\pi\rho]\rho\, d\theta d\rho \\ &= \int_0^\infty g(\rho)\, d\rho = 1.\end{aligned}$$

2.1.6 We can write the double integration as

$$P(a < X < b, c < Y < d) = \int_a^b 2xe^{-x^2}\, dx \cdot \int_c^d 2ye^{-y^2}\, dy.$$

Since $a, c > 0$, the one-to-one transformations $z = x^2$ and $w = y^2$, lead to the answer.

2.1.7

$$\begin{aligned}G(z) &= P(X+Y \le z) = \int_0^z \int_0^{z-x} e^{-x-y}\, dydx \\ &= \int_0^z [1 - e^{-(z-x)}]e^{-x}\, dx = 1 - e^{-z} - ze^{-z}. \\ g(z) &= G'(z) = \begin{cases} ze^{-z} & 0 < z < \infty \\ 0 & \text{elsewhere.}\end{cases}\end{aligned}$$

2.1.8

$$\begin{aligned} G(z) &= P(XY \le z) = 1 - \int_z^1 \int_{z/x}^1 dy dx \\ &= 1 - \int_z^1 \left(1 - \frac{z}{x}\right) dx = z - z \log z \\ g(z) &= G'(z) = \begin{cases} -\log z & 0 < z < 1 \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

Why is $-\log z > 0$?

2.1.9

$$f(x,y) = \begin{cases} \frac{\binom{13}{x}\binom{13}{y}\binom{26}{13-x-y}}{\binom{52}{13}} & x \ge 0, y \ge 0, x+y \le 13, x \text{ and } y \text{ integers} \\ 0 & \text{elsewhere.} \end{cases}$$

2.1.11

$$P(X_1 + X_2 \le 1) = 15 \int_0^{1/2} x_1^2 \left[\int_{x_1}^{1-x_1} x_2 \, dx_2\right] dx_1.$$

2.1.15

$$\begin{aligned} E\left[e^{t_1X_1+t_2X_2}\right] &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} e^{t_1 i + t_2 j}\left(\frac{1}{2}\right)^{i+j} \\ &= \sum_{i=1}^{\infty}\left(e^{t_1}\frac{1}{2}\right)^i \sum_{j=1}^{\infty}\left(e^{t_1}\frac{1}{2}\right)^j \\ &= \left[\frac{1}{1-2^{-1}e^{t_1}} - 1\right]\left[\frac{1}{1-2^{-1}e^{t_2}} - 1\right], \end{aligned}$$

provided $t_i < \log 2$, $i = 1, 2$.

2.2.1

$$p(y_1, y_2) = \begin{cases} \left(\frac{2}{3}\right)^{y_2}\left(\frac{1}{3}\right)^{2-y_2} & (y_1, y_2) = (0,0), (-1,1), (1,1), (0,2) \\ 0 & \text{elsewhere.} \end{cases}$$

2.2.2

$$p(y_1, y_2) = \begin{cases} y_1/36 & y_1 = y_2, 2y_2, 3y_2; y_2 = 1, 2, 3 \\ 0 & \text{elsewhere.} \end{cases}$$

y_1	1	2	3	4	6	9
$p(y_1)$	1/36	4/36	6/36	4/36	12/36	9/36

2.2.4 The inverse transformation is given by $x_1 = y_1 y_2$ and $x_2 = y_2$ with Jacobian $J = y_2$. By noting what the boundaries of the space $\mathcal{S}(X_1, X_2)$ map into, it follows that the space $\mathcal{T}(Y_1, Y_2) = \{(y_1, y_2) : 0 < y_i < 1, i = 1, 2\}$. The pdf of (Y_1, Y_2) is $f_{Y_1,Y_2}(y_1, y_2) = 8y_1y_2^3$.

2.2.5 The inverse transformation is $x_1 = y_1 - y_2$ and $x_2 = y_2$ with Jacobian $J = 1$. The space of (Y_1, Y_2) is $\mathcal{T} = \{(y_1, y_2) : -\infty < y_i < \infty, i = 1, 2\}$. Thus the joint pdf of (Y_1, Y_2) is

$$f_{Y_1,Y_2}(y_1, y_2) = f_{X_1,X_2}(y_1 - y_2, y_2),$$

which leads to formula (2.2.1).

2.3.2

$$(a) \qquad c_1 \int_0^{x_2} x_1/x_2^2 \, dx_1 = \frac{c_1}{2} = 1 \Rightarrow c_1 = 2 \text{ and } c_2 = 5.$$

$$(b) \qquad 10x_1x_2^2, 0 < x_1 < x_2 < 1; \text{zero elsewhere}$$

$$(c) \qquad \int_{1/4}^{1/2} 2x_1/(5/8)^2 \, dx = \frac{64}{25}\left(\frac{1}{4} - \frac{1}{16}\right) = \frac{12}{25}.$$

$$(d) \qquad \int_{1/4}^{1/2} \int_{x_1}^{1} 10x_1x_2^2 \, dx_2 dx_1 = \int_{1/4}^{1/2} \frac{10}{3} x_1(1 - x_1^3) \, dx_1 = \frac{135}{512}.$$

2.3.3

$$\begin{aligned}
f_2(x_2) &= \int_0^{x_2} 21x_1^2x_2^3 \, dx_1 = 7x_2^6, \quad 0 < x_2 < 1. \\
f_{1|2}(x_1|x_2) &= 21x_1^2x_2^3/7x_2^6 = 3x_1^2/x_2^3, \quad 0 < x_1 < x_2. \\
E(X_1|x_2) &= \int_0^{x_2} x_1(3x_1^2/x_2^3) \, dx_1 = \frac{3}{4}x_2. \\
G(y) &= P\left(\frac{3}{4}X_2 \le y\right) = \int_0^{4y/3} 7x_2^6 \, dx_2 = \left(\frac{4y}{3}\right)^7, \quad 0 < y < \frac{3}{4} \\
g(y) &= \begin{cases} 7\left(\frac{4}{3}\right)^7 y^6 & 0 < y < \frac{3}{4} \\ 0 & \text{elsewhere.} \end{cases} \\
E(Y) &= \frac{7}{8}\frac{3}{4} = \frac{21}{32}. \\
\text{Var}(Y) &= \frac{7}{1024}. \\
E(X_1) &= \frac{21}{32}. \\
\text{Var}(X_1) &= \frac{553}{15360} > \frac{7}{1024}.
\end{aligned}$$

2.3.8 The marginal pdf of X is

$$f_X(x) = 2\int_x^\infty e^{-x}e^{-y} \, dy = 2e^{-2x}, \quad 0 < x < \infty.$$

Hence, the conditional pdf of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{2e^{-x}e^{-y}}{2e^{-2x}} = e^{-(y-x)}, \quad 0 < x < y < \infty,$$

with conditional mean

$$E(Y|X=x) = \int_x^\infty ye^{-(y-x)}\,dy = x+1, \quad x>0.$$

2.3.9 For Part (c):

$$\binom{13}{x_2}\binom{13}{x_3}\binom{13}{2-x_2-x_3}\Big/\binom{39}{2}, \quad \text{where integers } x_2, x_3 \geq 0 \text{ and } x_2+x_3 \leq 2.$$

2.3.11

$$(a) \qquad f_1(x_1)f_{2|1}(x_2|x_1) = 1\cdot\frac{1}{x_1}, \quad 0<x_2<x_1<1.$$

$$(b) \qquad \int_{1/2}^1\int_{1-x_1}^{x_1}\frac{1}{x_1}\,dx_2dx_1 = \int_{1/2}^1\frac{2x_1-1}{x_1}\,dx_1 = 2(1/2)+\log(1/2) = 1-\log 2.$$

2.3.12

$$(b) \qquad \int_2^\infty e^{-x}\,dx\Big/\int_1^\infty e^{-x}\,dx = e^{-2}/e^{-1} = e^{-1}.$$

2.4.2 X_1 and X_2 are dependent because $0<x_1<x_2<\infty$ is not a product space.

2.4.4 Because X_1 and X_2 are independent, the probability equals

$$\left[\int_0^{1/3}2x_1\,dx_1\right]\left[\int_0^{1/3}2(1-x_2)\,dx_2\right] = (1/3)^2[1-(2/3)^2] = 5/81.$$

2.4.7 The marginal pdf of X_1 is given by

$$f_{X_1}(x_1) = \int_{-2-\sqrt{1-(x_1-1)^2}}^{-2+\sqrt{1-(x_1-1)^2}}\frac{1}{\pi}\,dx_2 = \frac{2}{\pi}\sqrt{1-(x_1-1)^2}, \quad 0<x<2.$$

The random variables X_1 and X_2 are not independent.

2.4.8 X and Y are dependent because $0<y<x<1$ is not a product space.

$$E(X|y) = \int_y^1 x[2x/(1-y^2)]\,dx = \frac{2(1-y^2)}{3(1-y^2)}.$$

2.4.9

$$\begin{aligned} P(X+Y\leq 60) &= P(X\leq 10)+\int_{10}^{20}\int_{40}^{60-x}\frac{1}{300}\,dy\,dx \\ &= \frac{1}{3}+\int_{10}^{20}(20-x)/300\,dx = \frac{1}{3}+\frac{1}{6} = \frac{1}{2}. \end{aligned}$$

2.4.12

$$\begin{aligned} P(|X_1 - X_2| = 1) &= P(X_1 = 0, X_2 = 1) + P(X_1 = 1, X_2 = 0) \\ &= P(X_1 = 0)P(X_2 = 1) + P(X_1 = 1)P(X_2 = 0) = \frac{1}{3}. \end{aligned}$$

2.5.1 For Part (c):

$$\text{cov} = (0)(0)(1/3) + (1)(1)(1/3) + (2)(0)(1/3) - (1)(1/3) = 0.$$

Thus $\rho = 0$ and yet X and Y are dependent.

2.5.3

$$\rho^2 = (1/2)(1/2) = 1/4 \Rightarrow \rho = 1/2.$$

2.5.11 Let $Y = (X_1 - \mu_1) + (X_2 - \mu_2)$. Then the mean of Y is 0 and its variance is

$$\text{Var}(Y) = \text{Var}(X_1 + X_2) = \sigma^2 + \sigma^2 + 2\rho\sigma^2 = 2\sigma^2(1 - \rho).$$

Use Chebyshev's inequality to obtain the result.

2.6.1 For Part (g):

$$E(X|y,z) = \int_0^1 x\frac{3(x+y+z)/2}{3((1/2)+y+z)/2}\,dx = \frac{(1/3)+(y/2)+(z/2)}{(1/2)+y+z}.$$

2.6.3

$$\begin{aligned} G(y) &= 1 - P(y < X_i, i = 1,2,3,4) = 1 - [(1-y)^3]^4 = 1 - (1-y)^{12} \\ g(y) &= G'(y) = \begin{cases} 12(1-y)^{11} & 0 < y < 1 \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

2.6.6 Multiply both members of $E[X_1 - \mu_1|x_2, x_3] = b_2(x_2 - \mu_2) + b_3(x_3 - \mu_3)$ by the joint pdf of X_2 and X_3 and denote the result by (1). Multiply both members of (1) by $(x_2 - \mu_2)$ and integrate (or sum) on x_2 and x_3. This gives (2), $\rho_{12}\sigma_1\sigma_2 = b_2\sigma_2^2 + 3\rho_{23}\sigma_1\sigma_2$. Return to (1) and multiply each member by $(x_3 - \mu_3)$ and integrate (or sum) on x_2 and x_3. This yields (3) $\rho_{13}\sigma_1\sigma_3 = b_2\rho_{23}\sigma_2\sigma_3 + b_3\sigma_3^2$. Solve (2) and (3) for b_2 and b_3.

2.6.9

$$\begin{aligned} (a) \quad & \int_0^\infty \int_{x_1}^\infty e^{-x_1-x_2}\,dx_2\,dx_1 \Big/ \int_0^\infty \int_{x_1/2}^\infty e^{-x_1-x_2}\,dx_2\,dx_1 \\ & + \int_0^\infty e^{-2x_1}\,dx_1 \Big/ \int_0^\infty e^{-3x_1/2}\,dx_1 = \frac{1}{2}\frac{2}{3} = \frac{3}{4}. \end{aligned}$$

2.7.1

$$x_1 = y_1y_2y_3,\ x_2 = y_2y_3 - y_1y_2y_3,\ x_3 = y_3 - y_2y_3.$$

with $J = y_2y_3^2$, and $0 < y_1 < 1, 0 < y_2 < 1, 0 < y_3 < \infty$. This yields

$$g(y_1,y_2,y_3) = y_2y_3^2e^{-y_3} = (1)(2y_2)(y_3^2e^{-y_3}/2) = g_1(y_1)g_2(y_2)g_3(y_3).$$