Chapter 1

Probability and Distributions

1.2.1 Part (c): $C_1 \cap C_2 = \{(x,y) : 1 < x < 2, 1 < y < 2\}$.

1.2.3 $C_1 \cap C_2 = \{\text{mary,mray}\}$.

1.2.6 $C_k = \{x : 1/k \le x \le 1 - (1/k)\}$.

1.2.7 $C_k = \{(x,y) : 0 \le x \le 1/k, 0 \le y \le 1/k\}$.

1.2.8 $\lim_{k\to\infty} C_k = \{x : 0 < x < 3\}$. Note: neither the number 0 nor the number 3 is in any of the sets C_k, $k = 1, 2, 3, \ldots$

1.2.9 Part (b): $\lim_{k\to\infty} C_k = \phi$, because no point is in all the sets C_k, $k = 1, 2, 3, \ldots$

1.2.11 Because $f(x) = 0$ when $1 \le x < 10$,

$$Q(C_3) = \int_0^{10} f(x)\,dx = \int_0^1 6x(1-x)\,dx = 1.$$

1.2.13 Part (c): Draw the region C carefully, noting that $x < 2/3$ because $3x/2 < 1$. Thus

$$Q(C) = \int_0^{2/3} \left[\int_{x/2}^{3x/2} dy \right] dx = \int_0^{2/3} x\,dx = 2/9.$$

1.2.16 Note that

$$25 = Q(\mathcal{C}) = Q(C_1) + Q(C_2) - Q(C_1 \cap C_2) = 19 + 16 - Q(C_1 \cap C_2).$$

Hence, $Q(C_1 \cap C_2) = 10$.

1.2.17 By studying a Venn diagram with 3 intersecting sets, it should be true that

$$11 \ge 8 + 6 + 5 - 3 - 2 - 1 = 13.$$

It is not, and the accuracy of the report should be questioned.

1.3.3
$$P(\mathcal{C}) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \frac{1/2}{1-(1/2)} = 1.$$

1.3.6
$$P(\mathcal{C}) = \int_{-\infty}^{\infty} e^{-|x|}\,dx = \int_{-\infty}^{0} e^{x}\,dx + \int_{0}^{\infty} e^{-x}\,dx = 2 \neq 1.$$
We must multiply by 1/2.

1.3.8
$$P(C_1^c \cup C_2^c) = P[(C_1 \cap C_2)^c] = P(\mathcal{C}) = 1,$$
because $C_1 \cap C_2 = \phi$ and $\phi^c = \mathcal{C}$.

1.3.11 The probability that he does not win a prize is
$$\binom{990}{5}\Big/\binom{1000}{5}.$$

1.3.13 Part (a): We must have 3 even or one even, 2 odd to have an even sum. Hence, the answer is
$$\frac{\binom{10}{3}\binom{10}{0}}{\binom{20}{3}} + \frac{\binom{10}{1}\binom{10}{2}}{\binom{20}{3}}.$$

1.3.14 There are 5 mutual exclusive ways this can happen: two "ones", two "twos", two "threes", two "reds", two "blues." The sum of the corresponding probabilities is
$$\frac{\binom{2}{2}\binom{6}{0} + \binom{2}{2}\binom{6}{0} + \binom{2}{2}\binom{6}{0} + \binom{5}{2}\binom{3}{0} + \binom{3}{2}\binom{5}{0}}{\binom{8}{2}}.$$

1.3.15
$$\text{(a)} \qquad 1 - \frac{\binom{48}{5}\binom{2}{0}}{\binom{50}{5}}$$
$$\text{(b)} \qquad 1 - \frac{\binom{48}{n}\binom{2}{0}}{\binom{50}{n}} \geq \frac{1}{2}, \text{ Solve for n.}$$

1.3.20 Choose an integer $n_0 > \max\{a^{-1}, (1-a)^{-1}\}$. Then $\{a\} = \cap_{n=n_0}^{\infty}\left(a - \frac{1}{n}, a + \frac{1}{n}\right)$. Hence by (1.3.10),
$$P(\{a\}) = \lim_{n\to\infty} P\left[\left(a - \frac{1}{n}, a + \frac{1}{n}\right)\right] = \frac{2}{n} = 0.$$

1.4.2
$$P[(C_1 \cap C_2 \cap C_3) \cap C_4] = P[C_4 | C_1 \cap C_2 \cap C_3] P(C_1 \cap C_2 \cap C_3),$$
and so forth. That is, write the last factor as
$$P[(C_1 \cap C_2) \cap C_3] = P[C_3 | C_1 \cap C_2] P(C_1 \cap C_2).$$

1.4.5
$$\frac{\left[\binom{4}{3}\binom{48}{10}+\binom{4}{4}\binom{48}{9}\right]/\binom{52}{13}}{\left[\binom{4}{2}\binom{48}{11}+\binom{4}{3}\binom{48}{10}+\binom{4}{4}\binom{48}{9}\right]/\binom{52}{13}}.$$

1.4.10
$$P(C_1|C)=\frac{(2/3)(3/10)}{(2/3)(3/10)+(1/3)(8/10)}=\frac{3}{7}<\frac{2}{3}=P(C_1).$$

1.4.12 Part (c):
$$\begin{aligned} P(C_1\cup C_2^c) &= 1-P[(C_1\cup C_2^c)^c]=1-P(C_1^*\cap C_2)\\ &= 1-(0.4)(0.3)=0.88.\end{aligned}$$

1.4.14 Part (d):
$$1-(0.3)^2(0.1)(0.6).$$

1.4.16 $1-P(TT)=1-(1/2)(1/2)=3/4$, assuming independence and that H and T are equilikely.

1.4.19 Let C be the complement of the event; i.e., C equals at most 3 draws to get the first spade.

(a) $P(C)=\frac{1}{4}+\frac{3}{4}\frac{1}{4}+\left(\frac{3}{4}\right)^2\frac{1}{4}$.
(b) $P(C)=\frac{1}{4}+\frac{13}{51}\frac{39}{52}+\frac{13}{50}\frac{38}{51}\frac{39}{52}$.

1.4.22 The probability that A wins is $\sum_{n=0}^{\infty}\left(\frac{5}{6}\frac{4}{6}\right)^n\frac{1}{6}=\frac{3}{8}$.

1.4.27 Let Y denote the bulb is yellow and let T_1 and T_2 denote bags of the first and second types, respectively.

(a)
$$P(Y)=P(Y|T_1)P(T_1)+P(Y|T_2)P(T_2)=\frac{20}{25}.6+\frac{10}{25}.4.$$

(b)
$$P(T_1|Y)=\frac{P(Y|T_1)P(T_1)}{P(Y)}.$$

1.4.30 Suppose without loss of generality that the prize is behind curtain 1. Condition on the event that the contestant switches. If the contestant chooses curtain 2 then she wins, (In this case Monte cannot choose curtain 1, so he must choose curtain 3 and, hence, the contestant switches to curtain 1). Likewise, in the case the contestant chooses curtain 3. If the contestant chooses curtain 1, she loses. Therefore the conditional probability that she wins is $\frac{2}{3}$.

1.4.31 (1) The probability is $1-\left(\frac{5}{6}\right)^4$.

(2) The probability is $1-\left[\left(\frac{5}{6}\right)^2+\frac{10}{36}\right]^{24}$.

1.5.2 Part (a):

$$c[(2/3)+(2/3)^2+(2/3)^3+\cdots] = \frac{c(2/3)}{1-(2/3)} = 2c = 1,$$

so $c = 1/2$.

1.5.5 Part (a):

$$p(x) = \begin{cases} \frac{\binom{13}{x}\binom{39}{5-x}}{\binom{52}{5}} & x = 0,1,\ldots,5 \\ 0 & \text{elsewhere.} \end{cases}$$

1.5.9 Part (b):

$$\sum_{x=1}^{50} x/5050 = \frac{50(51)}{2(5050)} = \frac{51}{202}.$$

1.5.10 For Part (c): Let $C_n = \{X \le n\}$. Then $C_n \subset C_{n+1}$ and $\cup_n C_n = R$. Hence, $\lim_{n\to\infty} F(n) = 1$. Let $\epsilon > 0$ be given. Choose n_0 such that $n \ge n_0$ implies $1 - F(n) < \epsilon$. Then if $x \ge n_0$, $1 - F(x) \le 1 - F(n_0) < \epsilon$.

1.6.2 Part (a):

$$p(x) = \frac{\binom{9}{x-1}}{\binom{10}{x-1}} \frac{1}{11-x} = \frac{1}{10}, \quad x = 1,2,\ldots 10.$$

1.6.3

$$\text{(a)} \quad p(x) = \left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right), \quad x = 1,2,3,\ldots$$

$$\text{(b)} \quad \sum_{x=1}^{\infty}\left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right) = \frac{1/6}{1-(25/36)} = \frac{6}{11}.$$

1.6.8 $\mathcal{D}_y = \{1, 2^3, 3^3, \ldots\}$. The pmf of Y is

$$p(y) = \left(\frac{1}{2}\right)^{y^{1/3}}, \quad y \in \mathcal{D}_y.$$

1.7.1 If $\sqrt{x} < 10$ then

$$F(x) = P[X(c) = c^2 \le x] = P(c \le \sqrt{x}) = \int_0^{\sqrt{x}} \frac{1}{10}\,dz = \frac{\sqrt{x}}{10}.$$

Thus

$$f(x) = F'(x) = \begin{cases} \frac{1}{20\sqrt{x}} & 0 < x < 100 \\ 0 & \text{elsewhere.} \end{cases}$$

1.7.2

$$C_2 \subset C_1^c \Rightarrow P(C_2) \le P(C_1^c) = 1 - (3/8) = 5/8.$$

1.7.4 Among other characteristics,

$$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)}\,dx = \frac{1}{\pi}\text{arctan}x\Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1.$$

1.7.6 Part (b):

$$\begin{aligned} P(X^2 < 9) &= P(-3 < X < 3) = \int_{-2}^{3} \frac{x+2}{19}\,dx \\ &= \frac{1}{18}\left[\frac{x^2}{2} + 2x\right]_{-2}^{3} = \frac{1}{18}\left[\frac{21}{2} - (-2)\right] = \frac{25}{36}. \end{aligned}$$

1.7.8 Part (c):

$$f'(x) = \frac{1}{2}2xe^{-x} = 0;$$

hence, $x = 2$ is the mode because it maximizes $f(x)$.

1.7.9 Part (b):

$$\int_0^m 3x^2\,dx = \frac{1}{2};$$

hence, $m^3 = 2^{-1}$ and $m = (1/2)^{1/3}$.

1.7.10

$$\int_0^{\xi_{0.2}} 4x^3\,dx = 0.2:$$

hence, $\xi_{0.2}^4 = 0.2$ and $\xi_{0.2} = 0.2^{1/4}$.

1.7.13 $x = 1$ is the mode because for $0 < x < \infty$ because

$$\begin{aligned} f(x) &= F'(x) = e^{-x} - e^{-x} + xe^{-x} = xe^{-x} \\ f'(x) &= -xe^{-x} + e^{-x} = 0, \end{aligned}$$

and $f'(1) = 0$.

1.7.16 Since $\Delta > 0$

$$X > z \Rightarrow Y = X + \Delta > z.$$

Hence, $P(X > z) \leq P(Y > z)$.

1.7.19 Since $f(x)$ is symmetric about 0, $\xi_{.25} < 0$. So we need to solve,

$$\int_{-2}^{\xi_{.25}} \left(-\frac{x}{4}\right)\,dx = .25.$$

The solution is $\xi_{.25} = -\sqrt{2}$.

1.7.20 For $0 < y < 27$,

$$\begin{aligned} x &= y^{1/3}, \quad \frac{dx}{dy} = \frac{1}{3}y^{-2/3} \\ g(y) = &= \frac{1}{3y^{2/3}}\frac{y^{2/3}}{9} = \frac{1}{27}. \end{aligned}$$

1.7.22

$$\begin{aligned} f(x) &= \frac{1}{\pi}, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}. \\ x &= \arctan y, \quad \frac{dx}{dy} = \frac{1}{1+y^2}, \quad -\infty < y < \infty. \\ g(y) &= \frac{1}{\pi}\frac{1}{1+y^2}, \quad -\infty < y < \infty. \end{aligned}$$

1.7.23

$$\begin{aligned} G(y) &= P(-2\log X^4 \le y) = P(X \ge e^{-y/8}) = \int_{e^{-y/8}}^{1} 4x^3\,dx = 1 - e^{-y/2}, \quad 0 < y < \infty \\ g(y) &= G'(y) = \begin{cases} e^{-y/2} & 0 < y < \infty \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

1.7.24

$$\begin{aligned} G(y) &= P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) \\ &= \begin{cases} \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{3}\,dx = \frac{2\sqrt{y}}{3} & 0 \le y < 1 \\ \int_{-1}^{\sqrt{y}} \frac{1}{3}\,dx = \frac{\sqrt{y}}{3} + \frac{1}{3} & 1 \le y < 4 \end{cases} \\ g(y) &= \begin{cases} \frac{1}{3\sqrt{y}} & 0 \le y < 1 \\ \frac{1}{6\sqrt{y}} & 1 \le y < 4 \\ 0 & \text{elsewhere.} \end{cases} \end{aligned}$$

1.8.4

$$E(1/X) = \sum_{x=51}^{100} \frac{1}{x}\frac{1}{50}.$$

The latter sum is bounded by the two integrals

$$\int_{51}^{101} \frac{1}{x}\,dx \text{ and } \int_{50}^{100} \frac{1}{x}\,dx.$$

An appropriate approximation might be

$$\frac{1}{50}\int_{50.5}^{101.5} \frac{1}{x}\,dx = \frac{1}{50}(\log 100.5 - \log 50.5).$$

1.8.6

$$E[X(1-X)] = \int_0^1 x(1-x)3x^2\,dx.$$

1.8.8 When $1 < y < \infty$

$$\begin{aligned}
G(y) &= P(1/X \le y) = P(X \ge 1/y) = \int_{1/y}^1 2x\,dx = 1 - \frac{1}{y^2} \\
g(y) &= \frac{2}{y^3} \\
E(Y) &= \int_1^\infty y\frac{2}{y^3}\,dy = 2, \text{ which equals } \int_0^1 (1/x)2x\,dx.
\end{aligned}$$

1.8.10 The expectation of X does not exist because

$$E(|X|) = \frac{2}{\pi}\int_0^\infty \frac{x}{1+x^2}\,dx = \frac{1}{\pi}\int_1^\infty \frac{1}{u}\,du = \infty,$$

where the change of variable $u = 1 + x^2$ was used.

1.9.2

$$M(t) = \sum_{x=1}^\infty \left(\frac{e^t}{2}\right)^x = \frac{e^t/2}{1-(e^t/2)}, \quad e^t/2 < 1.$$

Find $E(X) = M'(0)$ and $\text{Var}(X) = M''(0) - [M'(0)]^2$.

1.9.4

$$0 \le \text{var}(X) = E(X^2) - [E(X)]^2.$$

1.9.6

$$E\left[\left(\frac{X-\mu}{\sigma}\right)^2\right] = \frac{1}{\sigma^2}\sigma^2 = 1.$$

1.9.8

$$\begin{aligned}
K(b) &= E[(X-b)^2] = E(X^2) - 2bE(X) + b^2 \\
K'(b) &= -2E(X) + 2b = 0 \Rightarrow b = E(X).
\end{aligned}$$

1.9.11 For a continuous type random variable,

$$\begin{aligned}
K(t) &= \int_{-\infty}^\infty t^x f(x)\,dx. \\
K'(t) &= \int_{-\infty}^\infty xt^{x-1} f(x)\,dx \Rightarrow K'(1) = E(X). \\
K''(t) &= \int_{-\infty}^\infty x(x-1)t^{x-2} f(x)\,dx \Rightarrow K''(1) = E[X(X_1)];
\end{aligned}$$

and so forth.

1.9.12

$$\begin{aligned}
3 &= E(X-7) \Rightarrow E(X) = 10 = \mu. \\
11 &= E[(X-7)^2] = E(X^2) - 14E(X) + 49 = E(X^2) - 91 \\
&\Rightarrow E(X^2) = 102 \text{ and } \operatorname{var}(X) = 102 - 100 = 2. \\
15 &= E[(X-7)^3]. \text{ Expand } (X-7)^3 \text{ and continue.}
\end{aligned}$$

1.9.16

$$\begin{aligned}
E(X) &= 0 \Rightarrow \operatorname{var}(X) = E(X^2) = 2p. \\
E(X^4) &= 2p \Rightarrow \text{kurtosis} = 2p/4p^2 = 1/2p.
\end{aligned}$$

1.9.17

$$\begin{aligned}
\psi'(t) &= M'(t)/M(t) \Rightarrow \psi'(0) = M'(0)/M(0) = E(X). \\
\psi''(t) &= \frac{M(t)M''(t) - M'(t)M'(t)}{[M(t)^2]} \\
&\Rightarrow \psi''(0) = \frac{M(0)M''(0) - M'(0)M'(0)}{[M(0)^2]} = M''(0) - [M'(0)]^2 = \operatorname{var}(X).
\end{aligned}$$

1.9.19

$$M(t) = (1-t)^{-3} = 1 + 3t + 3 \cdot 4\frac{t^2}{2!} + 3 \cdot 4 \cdot 5\frac{t^3}{3!} + \cdots$$

Considering the coefficient of $t^r/r!$, we have

$$E(X^r) = 3 \cdot 4 \cdot 5 \cdots (r+2), \quad r = 1, 2, 3 \ldots.$$

1.9.20 Integrating the parts with $u = 1 - F(x)$, $dv = dx$, we get

$$\{[1 - F(x)]x\}_0^b - \int_0^b x[-f(x)]\,dx = \int_0^b xf(x)\,dx = E(X).$$

1.9.23

$$\begin{aligned}
E(X) &= \int_0^1 x\frac{1}{4}\,dx + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} = \frac{5}{8}. \\
E(X^2) &= \int_0^1 x^2\frac{1}{4}\,dx + 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} = \frac{7}{12}. \\
\operatorname{var}(X) &= \frac{7}{12} - \left(\frac{5}{8}\right)^2 = \frac{37}{192}.
\end{aligned}$$

1.9.24

$$E(X) = \int_{-\infty}^{\infty} x[c_1 f_1(x) + \cdots + c_k f_k(x)]\,dx = \sum_{i=1}^{k} c_i \mu_i = \mu.$$

Because $\int_{-\infty}^{\infty}(x-\mu)^2 f_i(x)\,dx = \sigma_i^2 + (\mu_i - \mu)^2$, we have

$$E[(X-\mu)^2] = \sum_{i=1}^{k} c_i[\sigma_i^2 + (\mu_i - \mu)^2].$$

1.10.2

$$\mu = \int_0^{\infty} x f(x)\,dx \geq \int_{2\mu}^{\infty} 2\mu f(x)\,dx = 2\mu P(X > 2\mu).$$

Thus $\frac{1}{2} \geq P(X > 2\mu)$.

1.10.4 If, in Theorem 1.10.2, we take $u(X) = \exp\{tX\}$ and $c = \exp\{ta\}$, we have

$$P(\exp\{tX\} \geq \exp\{ta\}] \leq M(t)\exp\{-ta\}.$$

If $t > 0$, the events $\exp\{tX\} \geq \exp\{ta\}$ and $X \geq a$ are equivalent. If $t < 0$, the events $\exp\{tX\} \geq \exp\{ta\}$ and $X \leq a$ are equivalent.

1.10.5 We have $P(X \geq 1) \leq [1 - \exp\{-2t\}]/2t$ for all $0 < t < \infty$, and $P(X \leq -1) \leq [\exp\{2t\} - 1]/2t$ for all $-\infty < t < 0$. Each of these bounds has the limit 0 as $t \to \infty$ and $t \to -\infty$, respectively.