Chapter 2

Random Variables, Distributions, and Expectations

2.1 Discrete; continuous; continuous; discrete; discrete; continuous.

2.2 A table of sample space and assigned values of the random variable is shown next.

Sample Space	x
NNN	0
NNB	1
NBN	1
BNN	1
NBB	2
BNB	2
BBN	2
BBB	3

2.3 A table of sample space and assigned values of the random variable is shown next.

Sample Space	w
HHH	3
HHT	1
HTH	1
THH	1
HTT	−1
THT	−1
TTH	−1
TTT	−3

2.4 $S = \{HHH, THHH, HTHHH, TTHHH, TTTHHH, HTTHHH, THTHHH, HHTHHH\}$; The sample space is discrete.

2.5 (a) $c = 1/30$ since $1 = \sum_{x=0} c(x^2 + 4) = 30c$.
(b) $c = 1/10$ since

$$1 = \sum_{x=0} c\binom{2}{x}\binom{3}{3-x} = c\left[\binom{2}{0}\binom{3}{3} + \binom{2}{1}\binom{3}{2} + \binom{2}{2}\binom{3}{1}\right] = 10c.$$

2.6 (a) $P(X > 200) = \int_{200}^{\infty} \frac{20000}{(x+100)^3}\,dx = -\frac{10000}{(x+100)^2}\Big|_{200}^{\infty} = \frac{1}{9}.$

(b) $P(80 < X < 200) = \int_{80}^{120} \frac{20000}{(x+100)^3}\,dx = -\frac{10000}{(x+100)^2}\Big|_{80}^{120} = \frac{1000}{9801} = 0.1020.$

2.7 (a) $P(X < 1.2) = \int_0^1 x\,dx + \int_1^{1.2} (2 - x)\,dx = \frac{x^2}{2}\Big|_0^1 + \left(2x - \frac{x^2}{2}\right)\Big|_1^{1.2} = 0.68.$

(b) $P(0.5 < X < 1) = \int_{0.5}^{1} x\,dx = \frac{x^2}{2}\Big|_{0.5}^{1} = 0.375.$

2.8 (a) $P(0 < X < 1) = \int_0^1 \frac{2(x+2)}{5}\,dx = \frac{(x+2)^2}{5}\Big|_0^1 = 1.$

(b) $P(1/4 < X < 1/2) = \int_{1/4}^{1/2} \frac{2(x+2)}{5}\,dx = \frac{(x+2)^2}{5}\Big|_{1/4}^{1/2} = 19/80.$

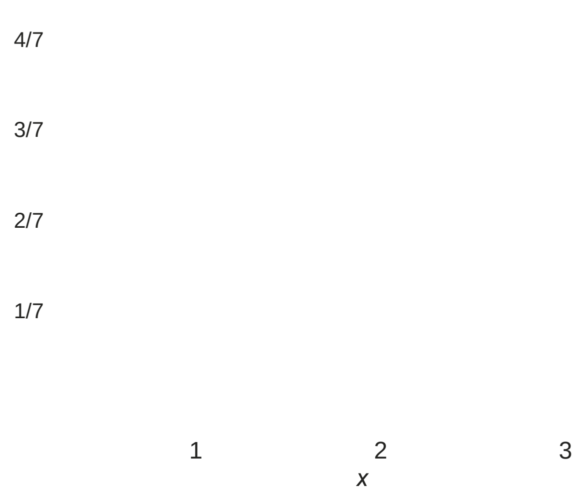
2.9 We can select x defective sets from 2, and $3 - x$ good sets from 5 in $\binom{2}{x}\binom{5}{3-x}$ ways. A random selection of 3 from 7 sets can be made in $\binom{7}{3}$ ways. Therefore,

$$f(x) = \frac{\binom{2}{x}\binom{5}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2.$$

In tabular form

x	0	1	2
$f(x)$	2/7	4/7	1/7

The following is a probability histogram:



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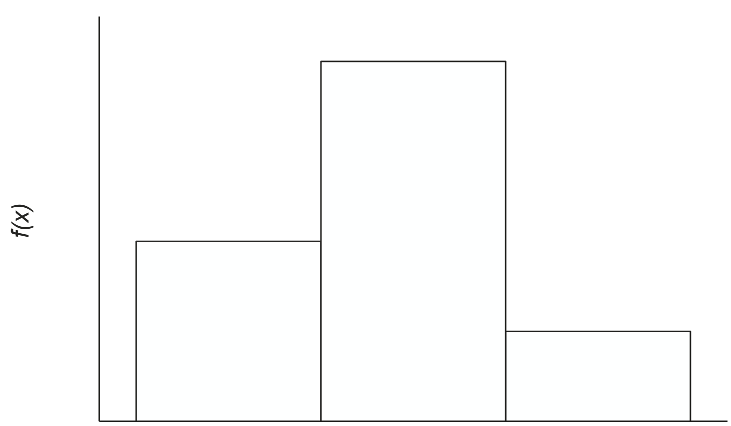


2.10 (a) $P(T = 5) = F(5) - F(4) = 3/4 - 1/2 = 1/4$.

(b) $P(T > 3) = 1 - F(3) = 1 - 1/2 = 1/2$.

(c) $P(1.4 < T < 6) = F(6) - F(1.4) = 3/4 - 1/4 = 1/2$.

(d) $P(T \le 5 | T \ge 2) = \frac{P(2 \le T \le 5)}{P(T \ge 2)} = \frac{3/4 - 1/4}{1 - 1/4} = \frac{2}{3}$.

2.11 The c.d.f. of X is

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ 0.41, & \text{for } 0 \le x < 1, \\ 0.78, & \text{for } 1 \le x < 2, \\ 0.94, & \text{for } 2 \le x < 3, \\ 0.99, & \text{for } 3 \le x < 4, \\ 1, & \text{for } x \ge 4. \end{cases}$$

2.12 (a) $P(X < 0.2) = F(0.2) = 1 - e^{-1.6} = 0.7981$;

(b) $f(x) = F'(x) = 8e^{-8x}$. Therefore, $P(X < 0.2) = 8 \int_0^{0.2} e^{-8x}\, dx = -e^{-8x}\big|_0^{0.2} = 0.7981$.

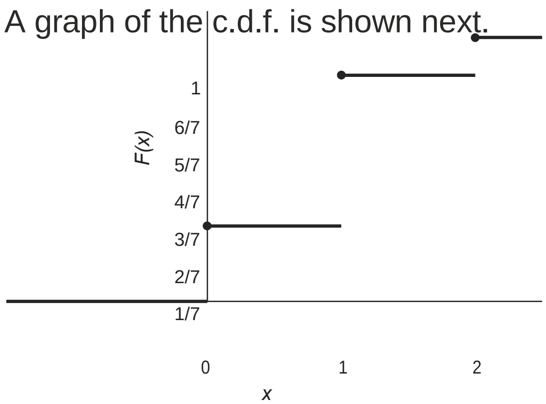
2.13 The c.d.f. of X is

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ 2/7, & \text{for } 0 \le x < 1, \\ 6/7, & \text{for } 1 \le x < 2, \\ 1, & \text{for } x \ge 2. \end{cases}$$

(a) $P(X = 1) = P(X \le 1) - P(X \le 0) = 6/7 - 2/7 = 4/7$;

(b) $P(0 < X \le 2) = P(X \le 2) - P(X \le 0) = 1 - 2/7 = 5/7$.

2.14 A graph of the c.d.f. is shown next.



2.15 (a) $1 = k \int_0^1 \sqrt{x}\, dx = \frac{2k}{3} x^{3/2} \Big|_0^1 = \frac{2k}{3}$. Therefore, $k = \frac{3}{2}$.

(b) For $0 \le x < 1$, $F(x) = \frac{3}{2}\int_0^x \sqrt{t}\, dt = t^{3/2}\Big|_0^x = x^{3/2}$.

Hence,

$$F(x) = \begin{cases} 0, & x < 0 \\ x^{3/2}, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

$P(0.3 < X < 0.6) = F(0.6) - F(0.3) = (0.6)^{3/2} - (0.3)^{3/2} = 0.3004$.

2.16 Denote by X the number of spades int he three draws. Let S and N stand for a spade and not a spade, respectively. Then

$P(X = 0) = P(NNN) = (39/52)(38/51)(37/50) = 703/1700$,

$P(X = 1) = P(SNN) + P(NSN) + P(NNS) = 3(13/52)(39/51)(38/50) = 741/1700$, $P(X = 3) = P(SSS) = (13/52)(12/51)(11/50) = 11/850$, and

$P(X = 2) = 1 - 703/1700 - 741/1700 - 11/850 = 117/850$.

The probability mass function for X is then

x	0	1	2	3
$f(x)$	703/1700	741/1700	117/850	11/850

2.17 Let T be the total value of the three coins. Let D and N stand for a dime and nickel, respectively. Since we are selecting without replacement, the sample space containing elements for which $t = 20$, 25, and 30 cents corresponding to the selecting of 2 nickels and 1 dime, 1 nickel and 2 dimes, and 3 dimes. Therefore, $P(T = 20) = \frac{\binom{2}{2}\binom{4}{1}}{\binom{6}{3}} = \frac{1}{5}$,

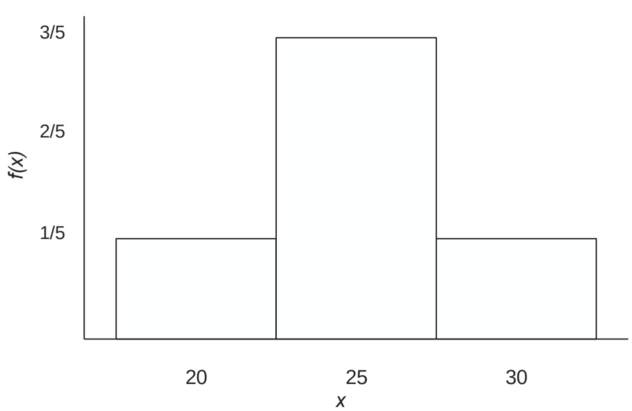
$P(T = 25) = \frac{\binom{2}{1}\binom{4}{2}}{\binom{6}{3}} = \frac{3}{5}$,

$P(T = 30) = \frac{\binom{4}{3}}{\binom{6}{3}} = \frac{1}{5}$,

and the probability distribution in tabular form is

t	20	25	30
$P(T = t)$	1/5	3/5	1/5

As a probability histogram



2.18 There are $\binom{10}{4}$ ways of selecting any 4 CDs from 10. We can select x jazz CDs from 5 and $4-x$ from the remaining CDs in $\binom{5}{x}\binom{5}{4-x}$ ways. Hence

$$f(x) = \frac{\binom{5}{x}\binom{5}{4-x}}{\binom{10}{4}}, \qquad x = 0, 1, 2, 3, 4.$$

2.19 (a) For $x \geq 0$, $F(x) = \int_0^x \frac{1}{2000}\exp(-t/2000)\,dt = -\exp(-t/2000)\big|_0^x$ $= 1 - \exp(-x/2000)$. So

$$F(x) = \begin{cases} 0, & x < 0, \\ 1 - \exp(-x/2000), & x \geq 0. \end{cases}$$

(b) $P(X > 1000) = 1 - F(1000) = 1 - [1 - \exp(-1000/2000)] = 0.6065$.

(c) $P(X < 2000) = F(2000) = 1 - \exp(-2000/2000) = 0.6321$.

2.20 (a) $f(x) \geq 0$ and $\int_{23.75}^{26.25} \frac{2}{5}\,dx = \frac{2t}{5}\Big|_{23.75}^{26.25} = \frac{2.5}{2.5} = 1$.

(b) $P(X < 24) = \int_{23.75}^{24} \frac{2}{5}\,dx = \frac{2}{5}(24 - 23.75) = 0.1$.

(c) $P(X > 26) = \int_{26}^{26.25} \frac{2}{5}\,dx = \frac{2}{5}(26.25 - 26) = 0.1$. It is not extremely rare.

2.21 (a) $f(x) \geq 0$ and $\int_1^\infty 3x^{-4}\,dx = -3\frac{x^{-3}}{3}\Big|_1^\infty = 1$. So, this is a valid density function.

(b) For $x \geq 1$, $F(x) = \int_1^x 3t^{-4}\,dt = 1 - x^{-3}$. So,

$$F(x) = \begin{cases} 0, & x < 1, \\ 1 - x^{-3}, & x \geq 1. \end{cases}$$

(c) $P(X > 4) = 1 - F(4) = 4^{-3} = 0.0156$.

2.22 (a) $1 = k\int_{-1}^{1}(3 - x^2)\,dx = k\left(3x - \frac{x^3}{3}\right)\Big|_{-1}^{1} = \frac{16}{3}k$. So, $k = \frac{3}{16}$.

(b) For $-1 \leq x < 1$, $F(x) = \frac{3}{16}\int_{-1}^{x}(3 - t^2)\,dt = \left(3t - \frac{1}{3}t^3\right)\Big|_{-1}^{x} = \frac{1}{2} + \frac{9}{16}x - \frac{1}{16}x^3$.

So, $P\left(X < \frac{1}{2}\right) = \frac{1}{2} + \left(\frac{9}{16}\right)\left(\frac{1}{2}\right) - \frac{1}{16}\left(\frac{1}{2}\right)^3 = \frac{99}{128}$.

(c) $P(|X| < 0.8) = P(X < -0.8) + P(X > 0.8) = F(-0.8) + 1 - F(0.8)$
$= 1 + \left(\frac{1}{2} - \frac{9}{16}0.8 + \frac{1}{16}0.8^3\right) - \left(\frac{1}{2} + \frac{9}{16}0.8 - \frac{1}{16}0.8^3\right) = 0.164$.

2.23 (a) For $y \geq 0$, $F(y) = \frac{1}{4}\int_0^y e^{-t/4}\,dt = 1 - e^{-y/4}$. So, $P(Y > 6) = e^{-6/4} = 0.2231$. This probability certainly cannot be considered as "unlikely."

(b) $P(Y \leq 1) = 1 - e^{-1/4} = 0.2212$, which is not so small either.

2.24 (a) $f(y) \geq 0$ and $\int_0^1 5(1-y)^4\,dy = -(1-y)^5\big|_0^1 = 1$. So, this is a valid density function.

(b) $P(Y < 0.1) = -(1-y)^5\big|_0^{0.1} = 1 - (1-0.1)^5 = 0.4095$.

(c) $P(Y > 0.5) = (1-0.5)^5 = 0.03125$.

2.25 (a) Using integral by parts and setting $1 = k\int_0^1 y^4(1-y)^3\,dy$, we obtain $k = 280$.

(b) For $0 \le y < 1$, $F(y) = 56y^5(1-y)^3 + 28y^6(1-y)^2 + 8y^7(1-y) + y^8$. So, $P(Y \le 0.5) = 0.3633$.

(c) Using the cdf in (b), $P(Y > 0.8) = 0.0563$.

2.26 (a) The event $Y = y$ means that among 5 selected, exactly y tubes meet the specification (M) and $5-y$ (M') does not. The probability for one combination of such a situation is $(0.99)^y(1-0.99)^{5-y}$ if we assume independence among the tubes. Since there are $\frac{5!}{y!(5-y)!}$ permutations of getting y Ms and $5-y$ M's, the probability of this event ($Y = y$) would be what it is specified in the problem.

(b) Three out of 5 is outside of specification means that $Y = 2$. $P(Y = 2) = 9.8\times10^{-6}$ which is extremely small. So, the conjecture is false.

2.27 (a) $P(X > 8) = 1 - P(X \le 8) = \sum_{x=0}^{8} e^{-6}\frac{6^x}{x!} = 1 - e^{-6}\left(\frac{6^0}{0!} + \frac{6^1}{1!} + \cdots + \frac{6^8}{8!}\right) = 0.1528$.

(b) $P(X = 2) = e^{-6}\frac{6^2}{2!} = 0.0446$.

2.28 For $0 < x < 1$, $F(x) = 2\int_0^x (1-t)\,dt = -(1-t)^2\big|_0^x = 1-(1-x)^2$.

(a) $P(X \le 1/3) = 1 - (1-1/3)^2 = 5/9$.

(b) $P(X > 0.5) = (1-1/2)^2 = 1/4$.

(c) $P(X < 0.75 \mid X \ge 0.5) = \frac{P(0.5 \le X < 0.75)}{P(X \ge 0.5)} = \frac{(1-0.5)^2 - (1-0.75)^2}{(1-0.5)^2} = \frac{3}{4}$.

2.29 (a) $\sum_{x=0}^{3}\sum_{y=0}^{3} f(x,y) = c\sum_{x=0}^{3}\sum_{y=0}^{3} xy = 36c = 1$. Hence $c = 1/36$.

(b) $\sum_x\sum_y f(x,y) = c\sum_x\sum_y |x-y| = 15c = 1$. Hence $c = 1/15$.

2.30 The joint probability distribution of (X, Y) is

$f(x,y)$		x = 0	1	2	3
y	0	0	1/30	2/30	3/30
	1	1/30	2/30	3/30	4/30
	2	2/30	3/30	4/30	5/30

(a) $P(X \le 2, Y = 1) = f(0,1) + f(1,1) + f(2,1) = 1/30 + 2/30 + 3/30 = 1/5$.

(b) $P(X > 2, Y \le 1) = f(3,0) + f(3,1) = 3/30 + 4/30 = 7/30$.

(c) $P(X > Y) = f(1,0) + f(2,0) + f(3,0) + f(2,1) + f(3,1) + f(3,2)$
$= 1/30 + 2/30 + 3/30 + 3/30 + 4/30 + 5/30 = 3/5$.

(d) $P(X+Y=4) = f(2,2) + f(3,1) = 4/30 + 4/30 = 4/15$.

(e) The possible outcomes of X are 0, 1, 2, and 3, and the possible outcomes of Y are 0, 1, and 2. The marginal distribution of X can be calculated such as $f_X(0) = 1/30 + 2/30 = 1/10$. Finally, we have the distribution tables.

x	0	1	2	3
$f_X(x)$	1/10	1/5	3/10	4/10

y	0	1	2
$f_Y(y)$	1/5	1/3	7/15

2.31 (a) We can select x oranges from 3, y apples from 2, and $4-x-y$ bananas from 3 in $\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}$ ways. A random selection of 4 pieces of fruit can be made in $\binom{8}{4}$ ways. Therefore,

$$f(x,y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{4-x-y}}{\binom{8}{4}}, \quad x=0,1,2,3;\ y=0,1,2;\quad 1 \le x+y \le 4.$$

Hence, we have the following joint probability table with the marginal distributions on the last row and last column.

$f(x,y)$		x=0	1	2	3	$f_Y(y)$
	0	0	3/70	9/70	3/70	3/14
y	1	2/70	18/70	18/70	2/70	8/14
	2	3/70	9/70	3/70	0	3/14
$f_X(x)$		1/14	6/14	6/14	1/14	

(b) $P[(X,Y) \in A] = P(X+Y \le 2) = f(1,0) + f(2,0) + f(0,1) + f(1,1) + f(0,2)$
$= 3/70 + 9/70 + 2/70 + 18/70 + 3/70 = 1/2$.

(c) $P(Y=0|X=2) = \frac{P(X=2, Y=0)}{P(X=2)} = \frac{9/70}{6/14} = \frac{3}{10}$.

(d) We know from (c) that $P(Y=0|X=2) = 3/10$, and we can calculate

$$P(Y=1|X=2) = \frac{18/70}{6/14} = \frac{3}{5}, \text{ and } P(Y=2|X=2) = \frac{3/70}{6/14} = \frac{1}{10}.$$

2.32 (a) $g(x) = \frac{2}{3}\int_0^1 (x+2y)\,dy = \frac{2}{3}(x+1)$, for $0 \le x \le 1$.

(b) $h(y) = \frac{2}{3}\int_0^1 (x+2y)\,dx = \frac{1}{3}(1+4y)$, for $0 \le y \le 1$.

(c) $P(X < 1/2) = \frac{2}{3}\int_0^{1/2} (x+1)\,dx = \frac{5}{12}$.

2.33 (a) $P(X+Y \le 1/2) = \int_0^{1/2}\int_0^{1/2-y} 24xy\,dx\,dy = 12\int_0^{1/2}\left(\frac{1}{2}-y\right)^2 y\,dy = \frac{1}{16}$.

(b) $g(x) = \int_0^{1-x} 24xy\,dy = 12x(1-x)^2$, for $0 \le x < 1$.