Solutions to Exercises

Exercises 0.1

1. (a) [BB] True (b) [BB] Not valid (c) [BB] False (0^2 is not positive.) (d) Not valid
   (e) True (f) Not valid (g) False

2. (a) [BB] True, because both $4 = 2 + 2$ and $7 < \sqrt{50}$ are true statements.
   (b) False, because one of the two statements is false.
   (c) [BB] False, because 5 is not even.
   (d) True, because $16^{-1/4} = \frac{1}{2}$.
   (e) [BB] True, since $9 = 3^2$ is true (or because $3.14 < \pi$).
   (f) True, because $(-4)^2 = 16$ is true.
   (g) [BB] True, because both hypothesis and conclusion are true.
   (h) False, because the hypothesis is true but the conclusion is false.
   (i) [BB] Not a valid mathematical statement.
   (j) True, because both statements are true.
   (k) True, because this is an implication with false hypothesis.
   (l) False, because one of the statements is false while the other is true.
   (m) False, because the hypothesis is true but the conclusion is false.
   (n) [BB] False, because the area of a circle of radius r is not $2\pi r$ and its circumference is not πr^2.
   (o) False, because the hypothesis of this implication is true, but the conclusion is false.
   (p) [BB] This is true: The hypothesis is true only when $a \geq b$ and $b \geq a$, that is, when $a = b$, and then the conclusion is also true.
   (q) This implication is true because the hypothesis is always false.

3. (a) [BB] If $x > 0$, then $\dfrac{1}{x} > 0$.
   (b) If a and b are rational numbers, then ab is a rational number.
   (c) If f is a differentiable function, then f is continuous.
   (d) [BB] If $\mathcal{G}$ is a graph, then the sum of the degrees of the vertices of $\mathcal{G}$ is even.
   (e) [BB] If A is a matrix and $A \neq 0$, then A is invertible.
   (f) If P is a parallelogram, then the diagonals of P bisect each other.
   (g) If n is an even integer, then $n < 0$.
   (h) If two vectors are orthogonal, then their dot product is 0.
   (i) If n is an integer, then $\frac{n}{n+1}$ is not an integer.
   (j) If n is a natural number, then $n + 3 > 2$.

4. (a) [BB] True (the hypothesis is false).
   (b) True (hypothesis and conclusion are both true).

(c) [BB] True (the hypothesis is false).

(d) False (hypothesis is true, conclusion is false).

(e) [BB] False (hypothesis is true, conclusion is false: $\sqrt{4} = 2$).

(f) True (g) [BB] True (h) True (i) [BB] True (the hypothesis is false: $\sqrt{x^2} = |x|$)
(j) True (k) [BB] False (l) True

5. (a) [BB] $a^2 \leq 0$ and a is a real number (more simply, $a = 0$).
   (b) x is not real or $x^2 + 1 \neq 0$ (more simply, x is any number, complex or real).
   (c) [BB] $x \neq 1$ and $x \neq -1$.
   (d) There exists an integer which is not divisible by a prime.
   (e) [BB] There exists a real number x such that $n \leq x$ for every integer n.
   (f) $(ab)c = a(bc)$ for all a, b, c.
   (g) [BB] Every planar graph can be colored with at most four colors.
   (h) Some Canadian is a fan of neither the Toronto Maple Leafs nor the Montreal Canadiens.
   (i) There exists $x > 0$ and some y such that $x^2 + y^2 \leq 0$.
   (j) $x \geq 2$ or $x \leq -2$.
   (k) [BB] There exist integers a and b such that for all integers q and r, $b \neq qa + r$.
   (l) [BB] For any infinite set, some proper subset is not finite.
   (m) For every real number x, there exists an integer n such that $x \leq n < x + 1$.
   (n) There exists an integer n such that $\frac{n}{n+1}$ is an integer.
   (o) $a > x$ or $a > y$ or $a > z$.
   (p) There exists a vector in the plane and there exists a normal to the plane such that the vector is not orthogonal to the normal.

6. (a) [BB] Converse: If $\frac{a}{c}$ is an integer, then $\frac{a}{b}$ and $\frac{b}{c}$ are also integers.
   Contrapositive: If $\frac{a}{c}$ is not an integer, then $\frac{a}{b}$ is not an integer or $\frac{b}{c}$ is not an integer.
   (b) Converse: $x = \pm 1 \rightarrow x^2 = 1$.
   Contrapositive: $x \neq 1$ and $x \neq -1 \rightarrow x^2 \neq 1$.
   (c) Converse: If $x = 1 + \sqrt{5}$ or $x = 1 - \sqrt{5}$, then $x^2 = x + 1$.
   Contrapositive: If $x \neq 1 + \sqrt{5}$ and $x \neq 1 - \sqrt{5}$, then $x^2 \neq x + 1$.
   (d) Converse: If $n^2 + n - 2$ is an even integer, then n is an odd integer.
   Contrapositive: If $n^2 + 2 - 2$ is an odd integer, then n is an even integer.
   (e) [BB] Converse: A connected graph is Eulerian.
   Contrapositive: If a graph is not connected, then it is not Eulerian.
   (f) Converse: $a = 0$ or $b = 0 \rightarrow ab = 0$.
   Contrapositive: $a \neq 0$ and $b \neq 0 \rightarrow ab \neq 0$.
   (g) Converse: A four-sided figure is a square.
   Contrapositive: If a figure does not have four sides, then it is not a square.
   (h) [BB] Converse: If $a^2 = b^2 + c^2$, then $\triangle BAC$ is a right triangle.
   Contrapositive: If $a^2 \neq b^2 + c^2$, then $\triangle BAC$ is not a right triangle.

(i) Converse: If $p(x)$ is a polynomial with at least one real root, then $p(x)$ has odd degree.
Contrapositive: If $p(x)$ is a polynomial with no real roots, then $p(x)$ has even degree.

(j) Converse: A set of at most n vectors is linearly independent.
Contrapositive: A set of more than n vectors is not linearly independent.

(k) Converse: If f is not one-to-one, then, for all real numbers x and y, $x \neq y$ and $x^2 + xy + y^2 + x + y = 0$.
Contrapositive: If f is one-to-one, then there exist real numbers x and y such that $x = y$ or $x^2 + xy + y^2 + x + y \neq 0$.

(l) [BB] Converse: If f is not one-to-one, then there exist real numbers x and y with $x \neq y$ and $x^2 + xy + y^2 + x + y = 0$.
Contrapositive: If f is one-to-one, then for all real numbers x and y either $x = y$ or $x^2 + xy + y^2 + x + y = 0$.

7. (a) [BB] There exists a continuous function which is not differentiable.
(b) $2^x \geq 0$ for all real numbers x.
(c) [BB] For every real number x, there exists a real number y such that $y > x$.
(d) For every set of primes $p_1, p_2, \ldots, p_n$, there exists a prime not in this set.
(e) [BB] For every positive integer n, there exist primes $p_1, p_2, \ldots, p_t$ such that $n = p_1 p_2 \cdots p_t$.
(f) For every real number $x > 0$, there exists a real number a such that $a^2 = x$.
(g) [BB] For every integer n, there exists an integer m such that $m < n$.
(h) For every real number $x > 0$, there exists a real number $y > 0$ such that $y < x$.
(i) [BB] There exists a polynomial p such that for every real number x, $p(x) \neq 0$.
(j) For every pair of real numbers x and y with $x < y$, there exists a rational number a such that $x < a < y$.
(k) For every polynomial $p(x)$ of degree 3, there exists a real number x such that $p(x) = 0$.
(l) There exists a matrix $A \neq 0$ such that A is not invertible.
(m) There exists a real number x such that $x \geq 0$.
(n) For any integer n, n is not both even and odd.
(o) For all integers a, b, c, $a^3 + b^3 \neq c^3$.

8. If a given implication "$\mathcal{A} \to \mathcal{B}$" is false, then $\mathcal{A}$ is true and $\mathcal{B}$ is false. The converse, "$\mathcal{B} \to \mathcal{A}$" is then true because its hypothesis, $\mathcal{B}$, is false. It is **not** possible for both an implication and its converse to be false.

9. First we remember that x **and** y is true if x and y are both true and false otherwise. Now $p \leftrightarrow q$ means $p \to q$ **and** $q \to p$. Also

A. $p \to q$ is true if p is false or if p is true and q is true.
B. $q \to p$ is true if q is false or if q is true and p is true.

If p and q are both false, both statements A and B are true, so $p \leftrightarrow q$ is true. Similarly, if both p and q are true, then statements A and B are again true, so $p \leftrightarrow q$ is true. Thus $p \leftrightarrow q$ is true if p and q have the same truth values. Suppose p and q have different truth values. To be specific, say p is true and q is false. If p is true and q is false, we see that statement A is false, so A **and** B is false. Similarly, if p is false and q is true, then statement B is false, so A **and** B is false. This verifies statement $(*)$.

Exercises 0.2

1. (a) [BB] Hypothesis: a and b are positive numbers.
   Conclusion: $a + b$ is positive.

   (b) Hypothesis: T is a right angled triangle with hypotenuse of length c and the other sides of lengths a and b.
   Conclusion: $a^2 + b^2 = c^2$.

   (c) [BB] Hypothesis: p is a prime.
   Conclusion: p is even.

   (d) Hypothesis: $n > 1$ is an integer.
   Conclusion: n is the product of prime numbers.

   (e) Hypothesis: A graph is planar.
   Conclusion: The chromatic number is 3.

2. (a) [BB] a and b are positive is sufficient for $a + b$ to be positive; $a + b$ is positive is necessary for a and b to be positive.

   (b) A right angled triangle has sides of lengths a, b, c, c the hypotenuse, is sufficient for $a^2 + b^2 = c^2$; $a^2 + b^2 = c^2$ is necessary for a right angled triangle to have sides of lengths a, b, c, c the hypotenuse.

   (c) [BB] p is a prime is sufficient for p to be even; p is even is necessary for p to be prime.

   (d) $n > 1$ an integer is sufficient for n to be the product of primes; n a product of primes is necessary for n to be an integer bigger than 1.

   (e) A graph being planar is sufficient for its chromatic number to be 3. Chromatic number 3 is necessary for a graph to be planar.

3. (a) [BB] $x = -2$ (b) $a = b = -1$ (c) [BB] $x = 4$
   (d) $8, 9, 11, 12$ (e) $\sqrt{2}$ and $\frac{1}{\sqrt{2}}$ (f) $x = 5, y = 2$

4. $\mathcal{A}$ can easily be proven false with the counterexample 0. No single counterexample can disprove a statement claiming "there exists" so we prove $\mathcal{B}$ directly. $\mathcal{B}$ is false because the square of a real number is nonnegative.

5. [BB] This statement is true. Suppose the hypothesis, x is an even integer, is true. Then $x = 2k$ for some other integer k. Then $x + 2 = 2k + 2 = 2(k + 1)$ is also twice an integer. So $x + 2$ is even. The conclusion is also true.

6. The converse is "$x + 2$ is an even integer $\rightarrow x$ is an even integer." This is true, for suppose that the hypothesis, $x + 2$ is an even integer, is true. Then $x + 2 = 2k$ for some integer k, so $x = 2k - 2 = 2(k - 1)$ is also twice an integer. The conclusion is also true.

7. This is true. Let $\mathcal{A}$ be the statement "x is an even integer" and let $\mathcal{B}$ be the statement $x + 2$ is an even integer". In Exercise 5, we showed that $\mathcal{A} \rightarrow \mathcal{B}$ is true and, in Exercise 6, that the converse $\mathcal{B} \rightarrow \mathcal{A}$ is also true. Thus $\mathcal{A} \leftrightarrow \mathcal{B}$ is also true.

8. (a) $\mathcal{A}$ is false: $n = 0$ is a counterexample.

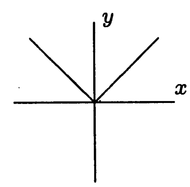
(b) Converse: If $\frac{n}{n+1}$ is not an integer, then n is an integer. This is false: $n = \frac{1}{2}$ is a counterexample ($\frac{n}{n+1} = \frac{1}{3}$).
Contrapositive: If $\frac{n}{n+1}$ is an integer, then n is not an integer. This is false: $n = 0$ is a counterexample.
Negation: There exists an integer n such that $\frac{n}{n+1}$ is an integer. This is true: Take $n = 0$.

9. (a) n prime $\rightarrow 2^n - 1$ prime.
(b) n prime is sufficient for $2^n - 1$ to be prime.
(c) $\mathcal{A}$ is false. For example, $n = 11$ is prime, but $2^{11} - 1 = 2047 = 23(89)$ is not. The integer $n = 11$ is a counterexample to $\mathcal{A}$.
(d) $2^n - 1$ prime $\rightarrow n$ prime.
(e) The converse of $\mathcal{A}$ is true. To show this, we establish the contrapositive. Thus, we assume n is not prime. Then there exists a pair of integers a and b such that $a > 1$, $b > 1$, and $n = ab$. Using the hint, we can factor $2^n - 1$ as

$$2^n - 1 = (2^a)^b - 1 = (2^a - 1)[(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1].$$

Since $a > 1$ and $b > 1$, we have $2^a - 1 > 1$ and $(2^a)^{b-1} + (2^a)^{b-2} + \cdots + 2^a + 1 > 1$, so $2^n - 1$ is the product of two integers both of which exceed one. Hence, $2^n - 1$ is not prime.

10. [BB] The converse is the statement, "A continuous function is differentiable." This is false. The absolute value function whose graph is shown to the right is continuous, but not differentiable at $x = 0$.



11. (a) $2(n-1)$ (b) n
(c) $\mathcal{A}_n \rightarrow \mathcal{A}_{n-1}, \mathcal{A}_{n-1} \rightarrow \mathcal{A}_{n-2}, \ldots, \mathcal{A}_2 \rightarrow \mathcal{A}_1, \mathcal{A}_1 \rightarrow \mathcal{A}_n$.

12. [BB] $\mathcal{A}$ is true. It expresses the fact that every real number lies between two consecutive integers. Statement $\mathcal{B}$ is most definitely false. It asserts that there is a remarkable integer n with the property that every real number lies in the unit interval between n and $n+1$.

13. $\mathcal{A}$ is false; $\mathcal{B}$ is true. There can be no y with the property described since y is not bigger than $y+1$; $x = y + 1$ provides a counterexample. To prove $\mathcal{B}$, we note that for every real number x, we have $x + 1 > x$ and so $x + 1$ is a suitable y.

14. (a) This is false. Suppose such an n exists. Then $q = \dfrac{1}{n+1}$ is rational but nq is not an integer.
(b) This is true. Given a rational number q, there exist integers m and n, $n \neq 0$, such that $q = \dfrac{m}{n}$. Then $nq = m$ is an integer.

15. (a) Since n is even, $n = 2k$ for some integer k. Thus $n^2 + 3n = 4k^2 + 6k = 2(2k^2 + 3k)$ is even too.
(b) The converse is the statement $n^2 + 3n$ even $\rightarrow n$ even. This is false and $n = 1$ is a counterexample.

16. (a) [BB] **Case 1:** a is even. In this case, we have one of the desired conclusions.
Case 2: a is odd. In this case, $a = 2m + 1$ for some integer m, so $a + 1 = 2m + 2 = 2(m + 1)$ is even, another desired result.
(b) [BB] $n^2 + n = n(n+1)$ is the product of consecutive integers one of which must be even; so $n^2 + n$ is even.

17. $n^2 - n + 5 = n(n-1) + 5$. Now either $n-1$ or n is even, since these integers are consecutive. So $n(n-1)$ is even. Since the sum of an even integer and the odd integer 5 is odd, the result follows.

18. [BB] $2x^2 - 4x + 3 = 2(x^2 - 2x) + 3 = 2[(x-1)^2 - 1] + 3 = 2(x-1)^2 + 1$ is the sum of 1 and a nonnegative number. So it is at least 1 and hence positive.

19. For $a^2 - b^2$ to be odd, it is necessary and sufficient for one of a or b to be even while the other is odd. Here's why.

    Case i: a, b even.

    In this case, $a = 2n$ and $b = 2m$ for some integers m and n, so $a^2 - b^2 = 4n^2 - 4m^2 = 4(n^2 - m^2)$ is even.

    Case ii: a, b odd.

    In this case, $a = 2n+1$ and $b = 2m+1$ for some integers m and n, so $a^2 - b^2 = (4n^2 + 4n + 1) - (4m^2 + 4m + 1) = 4(n^2 + n - m^2 - m)$ is even.

    Case iii: a even, b odd.

    In this case, $a = 2n$ and $b = 2m+1$ for some integers n and m, so $a^2 - b^2 = 4n^2 - (4m^2 + 4m + 1) = 4(n^2 - m^2 - m) - 1$ is odd.

    Case iv: a odd, b even.

    This is similar to Case iii, and the result follows.

20. [BB] ($\longrightarrow$) To prove this direction, we establish the contrapositive, that is, we prove that n odd implies n^2 odd. For this, if n is odd, then $n = 2m + 1$ for some integer m. Thus $n^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$ is odd.

    ($\longleftarrow$) Here we assume that n is even. Therefore, $n = 2m$ for some integer m. So $n^2 = (2m)^2 = 4m^2 = 2(2m^2)$ which is even, as required.

21. We assert that $x + \frac{1}{x} \geq 2$ if and only if $x > 0$.

    Proof. ($\longrightarrow$) We offer a proof by contradiction. Suppose $x + \frac{1}{x} \geq 2$ but $x > 0$ is not true; thus $x \leq 0$. If $x = 0$, $\frac{1}{x}$ is not defined, so $x < 0$. In this case, however, $x + \frac{1}{x} < 0$, a contradiction.

    ($\longleftarrow$) Conversely, assume that $x > 0$. Note that $(x-1)^2 \geq 0$ implies $x^2 - 2x + 1 \geq 0$, which in turn implies $x^2 + 1 \geq 2x$. Division by the positive number x gives $x + \frac{1}{x} \geq 2$ as required. ■

22. [BB] Since n is odd, $n = 2k + 1$ for some integer k.

    Case 1: k is even.

    In this case $k = 2m$ for some integer m, so $n = 2(2m) + 1 = 4m + 1$.

    Case 2: k is odd.

    In this case, $k = 2m + 1$ for some integer m, so $n = 2(2m+1) + 1 = 4m + 3$.

    Since each case leads to one of the desired conclusions, the result follows.

23. By Exercise 22, there exists an integer k such that $n = 4k+1$ or $n = 4k+3$.

Case 1: $n = 4k+1$.

If k is even, there exists an integer m such that $k = 2m$, so $n = 4(2m)+1 = 8m+1$, and the desired conclusion is true. If k is odd, there exists an integer m such that $k = 2m+1$, so $n = 4(2m+1)+1 = 8m+5$, and the desired conclusion is true.

Case 2: $n = 4k+3$.

If k is even, there exists an integer m such that $k = 2m$, so $n = 4(2m)+3 = 8m+3$, and the desired conclusion is true. If k is odd, there exists an integer m such that $k = 2m+1$, so $n = 4(2m+1)+3 = 8m+7$, and the desired conclusion is true. In all cases, the desired conclusion is true.

24. [BB] If the statement is false, then there does exist a smallest positive real number r. Since $\frac{1}{2}r$ is positive and smaller than r, we have reached an absurdity. So the statement must be true.

25. We give a proof by contradiction. If the result is false, then both $a > \sqrt{n}$ and $b > \sqrt{n}$. (Note that the negation of an "or" statement is an "and" statement.) But then $n = ab > \sqrt{n}\sqrt{n} = n$, which isn't true.

26. [BB] Since 0 is an eigenvalue of A, there is a nonzero vector x such that $A\mathsf{x} = 0$. Now suppose that A is invertible. Then $A^{-1}(A\mathsf{x}) = A^{-1}0 = 0$, so $\mathsf{x} = 0$, a contradiction.

27. (a) The given equation is equivalent to $(b-5)\sqrt{2} = 3-a$. If $b \neq 5$, then $\sqrt{2} = \frac{3-a}{b-5}$ is a rational number. This is false. Thus $b = 5$, so $a = 3$.

    (b) Note that $(a+b\sqrt{2})^2 = (a^2+2b^2)+2ab\sqrt{2}$. Thus, if $(a+b\sqrt{2})^2 = 3+5\sqrt{2}$, then $a^2+2b^2 = 3$ and $2ab = 5$, by part (a). The second equation says $a \neq 0$ and $b \neq 0$. Since a and b are integers, it follows that $a^2 \geq 1$ and $b^2 \geq 1$ and $a^2+2b^2 \geq 3$ with equality if and only $a = \pm 1 = b$. But then $2ab \neq 5$.

28. [BB] Observe that $(1+a)(1+b) = 1+a+b+ab = 1$. Thus $1+a$ and $1+b$ are integers whose product is 1. There are two possibilities: $1+a = 1+b = 1$, in which case $a = b = 0$, or $1+a = 1+b = -1$, in which case $a = b = -2$.

29. We offer a proof by contradiction. Suppose $\dfrac{1}{a}$ is not irrational. Then it is rational, so there exist integers m and n, $n \neq 0$, such that $\dfrac{1}{a} = \dfrac{m}{n}$. Since $\dfrac{1}{a} \neq 0$, we know also that $m \neq 0$. Now $\dfrac{1}{a} = \dfrac{m}{n}$ implies $a = \dfrac{n}{m}$ is a rational number, a contradiction.

30. We give a proof by contradiction. Assume that a is rational, b is irrational and $a+b$ is rational. Then $a+b = \frac{m}{n}$ for integers m and n, $n \neq 0$. Since a is rational, $a = \frac{k}{\ell}$ for integers k and ℓ, $\ell \neq 0$. Thus

$$b = \frac{m}{n} - a = \frac{m}{n} - \frac{k}{\ell} = \frac{m\ell - kn}{n\ell}$$

is the quotient of integers with nonzero denominator. This contradicts the fact that b is not rational.

31. [BB] We begin by assuming the negation of the desired conclusion; in other words, we assume that there exist real numbers x, y, z which simultaneously satisfy each of these three equations. Subtracting the second equation from the first we see that $x+5y-4z = -2$. Since the third equation we were given says $x+5y-4z = 0$, we have $x+5y-4z$ equal to both 0 and to -2. Thus, the original assumption has led us to a contradiction.

32. (a) [BB] False: $x = y = 0$ is a counterexample.
    (b) False: $a = 6$ is a counterexample.
    (c) [BB] False: $x = 0$ is a counterexample.
    (d) False: $a = \sqrt{2}$, $b = -\sqrt{2}$ is a counterexample.
    (e) [BB] False: $a = b = \sqrt{2}$ is a counterexample.
    (f) The roots of the polynomial $ax^2 + bx + c$ are $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If $b^2 - 4ac > 0$, $\sqrt{b^2 - 4ac}$ is real and not 0, so the formula produces two distinct real numbers $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$.
    (g) False: $x = \frac{1}{2}$ is a counterexample.
    (h) True: If n is a positive integer, then $n \geq 1$, so $n^2 = n(n) \geq n$.

33. The result is false. A square and a rectangle (which is not a square) have equal angles but not pairwise proportional sides.

34. (a) [BB] Since n^2+1 is even, n^2 is odd, so n must also be odd. Writing $n = 2k+1$, then $n^2+1 = 2m$ says $4k^2 + 4k + 2 = 2m$, so $m = 2k^2 + 2k + 1 = (k+1)^2 + k^2$ is the sum of two squares as required.
    (b) [BB] We are given that $n^2+1 = 2m$ for $n = 4373$ and $m = 9561565$. Since $n = 2(2186)+1$, our solution to (a) shows that $m = k^2 + (k+1)^2$ where $k = 2186$. Thus, $9561565 = 2186^2 + 2187^2$.

35. (a) $2^{4n+2} + 1 = 4(2^{4n}) + 1 = 4(2^n)^4 + 1$. Applying the given identity with $x = 2^n$, we get

$$\begin{aligned} 2^{4n+2} + 1 &= (2 \cdot 2^{2n} + 2^{n+1} + 1)(2 \cdot 2^{2n} - 2^{n+1} + 1) \\ &= (2^{2n+1} + 2^{n+1} + 1)(2^{2n+1} - 2^{n+1} + 1). \end{aligned}$$

    With $n = 4$, we get $2^{18} + 1 = (2^9 + 2^5 + 1)(2^9 - 2^5 + 1) = 545(481)$.
    (b) $2^{36} - 1 = (2^{18} - 1)(2^{18} + 1) = (2^9 - 1)(2^9 + 1)(545)(481)$ (using the result of part (a)) $= 511(513)(545)(481)$.

36. If the result is false, then $f(n) = a_0 + a_1 n + \cdots + a_t n^t$ for some $t \geq 1$. Since $f(0) = a_0 = p$ is prime, $f(n) = p + ng(n)$ for $g(n) = a_1 + a_2 n + \cdots + a_t n^{t-1}$. Replacing n by pn, we have $f(pn) = p + npg(pn)$. The right hand side is divisible by the prime p, hence $f(pn)$ is divisible by p. But $f(pn)$ is prime, by hypothesis, so $f(pn) = p$. This means $g(pn) = 0$, contradicting the fact that a polynomial has only finitely many roots.

37. We offer a proof by contradiction. Suppose all the digits occur just a finite number of times. Then there is a number n_1 which has the property that after n_1 digits in the decimal expansion of π, the digit 1 no longer occurs. Similarly, there is a number n_2 such that after n_2 digits, the digit 2 no longer occurs, and so on. In general, for each $k = 1, 2, \ldots, 9$, there is a number n_k such that after n_k digits, the digit k no longer occurs. Let N be the largest of the numbers $n_1, n_2, \ldots, n_k$. Then after N digits in the decimal expansion of π, the only digit which can appear is 0. This contradicts the fact that the decimal expansion of π does not terminate.

38. We have proven in the text that $\sqrt{2}$ is irrational. Thus, if $\sqrt{2}^{\sqrt{2}}$ is rational, we are done (with $a = b = \sqrt{2}$). On the other hand, if $\sqrt{2}^{\sqrt{2}}$ is irrational, then let $a = \sqrt{2}^{\sqrt{2}}$ and $b = \sqrt{2}$ in which case $a^b = \sqrt{2}^2 = 2$ is rational.

Chapter 0 Review

1. (a) This implication is true because the hypothesis is always false: $a - b > 0$ and $b - a > 0$ give $a > b$ and $b > a$, which never holds.
   (b) This implication is false: When $a = b$, the hypotheses are true while the conclusion is false.

2. (a) x is a real number and $x \leq 5$.
   (b) For every real number x, there exists an integer n such that $n \leq x$.
   (c) There exist positive integers x, y, z such that $x^3 + y^3 = z^3$.
   (d) There exists a graph with n vertices and $n + 1$ edges whose chromatic number is more than 3.
   (e) There exists an integer n such that for any rational number a, $a \neq n$.
   (f) $a \neq 0$ or $b \neq 0$.

3. (a) Converse: If ab is an integer, then a and b are integers.
   Contrapositive: If ab is not an integer, then either a or b is not an integer.
   Negation: There exist integers a and b such that ab is not an integer.
   (b) Converse: If x^2 is an even integer, then x is an even integer.
   Contrapositive: If x^2 is an odd integer, then x is an odd integer.
   Negation: There exists an even integer x such that x^2 is odd.
   (c) Converse: Every graph which can be colored with at most four colors is planar.
   Contrapositive: Every graph which cannot be colored with at most four colors is not planar.
   Negation: There exists a planar graph which cannot be colored with at most four colors.
   (d) Converse: A matrix which equals its transpose is symmetric.
   Contrapositive: If a matrix does not equal its transpose, then it is not symmetric.
   Negation: There exists a symmetric matrix which is not equal to its transpose.
   (e) Converse: A set of at least n vectors is a spanning set.
   Contrapositive: A set of less than n vectors is not a spanning set.
   Negation: There exists a spanning set containing less than n vectors.
   (f) Converse: If $x > -2$ and $x < 1$, then $x^2 + x - 2 < 0$.
   Contrapositive: If $x \leq -2$ or $x \geq 1$, then $x^2 + x - 2 \geq 0$.
   Negation: There exists an $x \leq -2$ or $x \geq 1$ such that $x^2 + x - 2 < 0$.

4. (a) A is false: $a = u = b = 1$, $v = -1$ provides a counterexample.
   (b) Converse: Given four integers a, b, u, v with $u \neq 0$, $v \neq 0$, if $a = b = 0$, then $au + bv = 0$.
   Negation: There exist integers a, b, u, v, $u \neq 0$, $v \neq 0$, with $au + bv = 0$ and $a \neq 0$ or $b \neq 0$.
   Contrapositive: Given four integers a, b, u, v with $u \neq 0$ and $v \neq 0$, if $a \neq 0$ or $b \neq 0$, then $au + bv \neq 0$.

(c) The converse is certainly true since $0u + 0v = 0$.

(d) The negation is true: Take $a = u = b = 1$ and $v = -1$.
The contrapositive of A is false since A is false.

5. (a) There exists a countable set which is infinite.
(b) For all positive integers n, $1 \leq n$.

6. (a) This is true. If x is positive, $x + 2$ is positive. In addition, if x is odd, $x + 2$ is odd.
(b) This is false. When $x = -1$, $x + 2 = +1$ is a positive odd integer, while x is not.

7. (a) This statement expresses a well-known property of the real numbers. It is true.
(b) This is false. The conclusion would have us believe that every two real numbers are equal.

8. The desired formula is $ab = \dfrac{(a+b)^2 - (a-b)^2}{4}$ which holds because $(a+b)^2 - (a-b)^2 = (a^2 + 2ab + b^2) - (a^2 - 2ab + b^2) = 4ab$.

9. ($\longrightarrow$) Assume n^3 is odd and suppose, to the contrary, that n is even. Thus $n = 2x$ for some integer x. But then $n^3 = 8x^3 = 2(4x^3)$ is even, a contradiction. This means that n must be odd.
($\longleftarrow$) Assume n is odd. This means that $n = 2x + 1$ for some integer x. Then $n^3 = (2x+1)^3 = 8x^3 + 12x^2 + 6x + 1 = 2(4x^3 + 6x^2 + 3x) + 1$ is odd.

10. (a) $a^2 - 5a + 6 = (a-2)(a-3)$ is the product of two consecutive integers, one of which must be even.
(b) The sum $(a^2 - 5b) + (b^2 - 5a)$ is $(a^2 - 5a) + (b^2 - 5b) = (a^2 - 5a + 6) + (b^2 - 5b + 6) - 12$ is the sum of three even integers [using the result of part (a)] and hence even. Thus $b^2 - 5a$ is the difference of even integers and hence even as well.

11. The sum of the angles of a triangle is 180°, so $\angle C = 45°$ and $\angle D = 75°$. Since triangles ABC and DEF are similar, $\dfrac{AC}{AB} = \dfrac{DE}{DF}$, so the length of AC is $|AB| \times \dfrac{DE}{DF} = 12 \times \dfrac{8}{6} = 16$.

12. The rectangle that remains has dimensions 1 by $\tau - 1$. These are in ratio

$$\frac{1}{\tau - 1} = \frac{\tau + 1}{(\tau - 1)(\tau + 1)} = \frac{\tau + 1}{\tau^2 - 1} = \frac{\tau + 1}{\tau} = \tau$$

using twice the fact that $\tau^2 = \tau + 1$.

13. If the result is false, then $x \geq -1$ and $x \leq 2$, so $x + 1 \geq 0$ and $x - 2 \leq 0$. But then $x^2 - x - 2 = (x+1)(x-2) \leq 0$, a contradiction.

14. Suppose, to the contrary, that $-\dfrac{x}{y}$ is the largest negative rational number, where x and y are positive integers. Then $\dfrac{x}{2y}$ is rational and $\dfrac{x}{2y} < \dfrac{x}{y}$, so $-\dfrac{x}{2y} > -\dfrac{x}{y}$. Since $-\dfrac{x}{2y}$ is a negative rational number, we have a contradiction.

15. We use a proof by contradiction which mimics that proof of the irrationality of $\sqrt{2}$ given in Problem 8. Thus, we suppose that $\sqrt{3} = \frac{a}{b}$ is rational and hence the quotient of integers a and b which have no factors in common. Squaring gives $a^2 = 3b^2$ and so $a = 3k$ is a multiple of 3. But then $9k^2 = 3b^2$, so $3k^2 = b^2$. This says that b is also a multiple of 3, contradicting our assumption that a and b have no factors in common.

16. Let the rational numbers be $\frac{a}{b}$ and $\frac{c}{d}$. We may assume that a, b, c, d are positive integers and that $\frac{a}{b} < \frac{c}{d}$. Thus $ad < bc$. The hint suggests that $\frac{a+c}{b+d}$ is between $\frac{a}{b}$ and $\frac{c}{d}$, and this is the case: $\frac{a}{b} < \frac{a+c}{b+d}$ is equivalent to $a(b+d) < b(a+c)$ and $\frac{a+c}{b+d} < \frac{c}{d}$ is equivalent to $(a+c)d < (b+d)c$, both of which are true because $ad < bc$.

17. On a standard checker board, there are 32 squares of one color and 32 of another. Since squares in opposite corners have the same color, the hint shows that our defective board has 32 squares of one color and 30 of the other. Since each domino covers one square of each color, the result follows.

18. (a) We leave the primality checking of $f(1), \ldots, f(39)$ to the reader, but note that $f(40) = 41^2$.

    (b) $f(k^2+40) = 40^2 + 80k^2 + k^4 + 40 + k^2 + 41 = k^4 + 81k^2 + 41^2 = (k^2+41)^2 - k^2 = (k^2+41+k)(k^2+41-k)$.

19. The answer is no, since $333333331 = 19607843 \times 17$.

Exercises 1.1

1. (a) [BB]

p	q	$\neg q$	$(\neg q) \vee p$	$p \wedge ((\neg q) \vee p)$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	F
F	F	T	T	F

(b)

p	q	$\neg p$	$(\neg p) \to q$	$p \wedge q$	$(p \wedge q) \vee ((\neg p) \to q)$
T	T	F	T	T	T
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	F	F	F

(c)

p	q	$q \vee p$	$p \wedge (q \vee p)$	$\neg (p \wedge (q \vee p))$	$\neg (p \wedge (q \vee p)) \leftrightarrow p$
T	T	T	T	F	F
T	F	T	T	F	F
F	T	T	F	T	F
F	F	F	F	T	F

(d) [BB]

p	q	r	$\neg q$	$p \vee (\neg q)$	$\neg (p \vee (\neg q))$	$\neg p$	$(\neg p) \vee r$	$(\neg (p \vee (\neg q))) \wedge ((\neg p) \vee r)$
T	T	T	F	T	F	F	T	F
T	F	T	T	T	F	F	T	F
F	T	T	F	F	T	T	T	T
F	F	T	T	T	F	T	T	F
T	T	F	F	T	F	F	F	F
T	F	F	T	T	F	F	F	F
F	T	F	F	F	T	T	T	T
F	F	F	T	T	F	T	T	F

(e)

p	q	r	$q \to r$	$p \to (q \to r)$	$p \wedge q$	$(p \wedge q) \vee r$	$(p \to (q \to r)) \to ((p \wedge q) \vee r)$
T	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T
F	T	T	T	T	F	T	T
F	F	T	T	T	F	T	T
T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	F
F	T	F	F	T	F	F	F
F	F	F	T	T	F	F	F

2. (a) If $p \to q$ is false, then necessarily p is true and q is false. (This is the only situation in which $p \to q$ is false.) We construct the relevant row of the truth table for $(p \wedge (\neg q)) \vee ((\neg p) \to q)$.

p	q	$\neg q$	$p \wedge (\neg q)$	$\neg p$	$(\neg p) \to q$	$(p \wedge (\neg q)) \vee ((\neg p) \to q)$
T	F	T	T	F	T	T

(b) [BB] There are three situations in which $p \to q$ is true. The question then is whether or not the truth value of $(p \wedge (\neg q)) \vee ((\neg p) \to q)$ is the same in each of these cases. We construct a partial truth table.

p	q	$\neg q$	$p \wedge (\neg q)$	$\neg p$	$(\neg p) \to q$	$(p \wedge (\neg q)) \vee ((\neg p) \to q)$
T	T	F	F	F	T	T
F	F	T	F	T	F	F

As shown, $(p \wedge (\neg q)) \vee ((\neg p) \to q)$ has different truth values on two occasions where $p \to q$ is true, so it is **not** possible to answer the question in this case.

3. [BB]

p	q	r	s	$\neg r$	$q \wedge (\neg r)$	$p \to [q \wedge (\neg r)]$	$\neg s$	$(\neg s) \vee q$	$r \leftrightarrow [(\neg s) \vee q]$	$[p \to (q \wedge (\neg r))] \vee [r \leftrightarrow ((\neg s) \vee q)]$
T	T	T	T	F	F	F	F	T	T	T

4.

p	q	r	s	$\neg r$	$q \wedge (\neg r)$	$p \to [q \wedge (\neg r)]$	$\neg s$	$(\neg s) \vee q$	$r \leftrightarrow [(\neg s) \vee q]$	$[p \to (q \wedge (\neg r))] \vee [r \leftrightarrow ((\neg s) \vee q)]$
F	F	F	F	T	F	T	T	T	F	T

5. (a) [BB]

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

The final column shows that $(p \wedge q) \to (p \vee q)$ is true for all values of p and q, so this statement is a tautology.

(b) [BB]

p	q	$\neg p$	$(\neg p) \wedge q$	$\neg q$	$p \vee (\neg q)$	$((\neg p) \wedge q) \wedge (p \vee (\neg q))$
T	T	F	F	F	T	F
T	F	F	F	T	T	F
F	T	T	T	F	F	F
F	F	T	F	T	T	F

The final column shows that $((\neg p) \wedge q) \wedge (p \vee (\neg q))$ is false for all values of p and q, so this statement is a contradiction.

6. (a)

p	q	$p \rightarrow q$	$q \rightarrow (p \rightarrow q)$
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	T

Since $q \rightarrow (p \rightarrow q)$ is true for all values of p and q, this statement is a tautology.

(b)

p	q	$p \wedge q$	$\neg p$	$\neg q$	$((\neg p) \vee (\neg q))$	$(p \wedge q) \wedge [(\neg p) \vee (\neg q)]$
T	T	T	F	F	F	F
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	F

Since $(p \wedge q) \wedge ((\neg p) \vee (\neg q))$ is false for all values of p and q, this statement is a contradiction.

7. (a) [BB]

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$
T	T	T	T	T	T	T
T	F	T	F	T	F	T
F	T	T	T	T	T	T
F	F	T	T	T	T	T
T	T	F	T	F	F	F
T	F	F	F	T	F	F
F	T	F	T	F	F	T
F	F	F	T	T	T	T

$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow [p \rightarrow r]$
T
T
T
T
T
T
T
T

Since $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow [p \rightarrow r]$ is true for all values of p, q, and r, this statement is a tautology.

(b) [BB] If p implies q which, in turn, implies r, then certainly p implies r.

8. We must show that the given "or" statement can be both true and false. We construct truth tables for each part of the "or" and show that certain identical values for the variables make both parts T (so that the "or" is true) and other certain identical values for the variables make both parts F (so that the "or" is false).

p	r	s	$\neg r$	$\neg s$	$(\neg r) \rightarrow (\neg s)$	$p \vee [(\neg r) \rightarrow (\neg s)]$
T	T	T	F	F	T	T
F	F	T	T	F	F	F

p	q	r	s	t	$\neg t$	$(\neg t) \vee p$	$s \rightarrow [(\neg t) \vee p]$
T	T	T	T	T	F	T	T
F	F	F	T	T	F	F	F

$\neg q$	$(\neg q) \rightarrow r$	$[s \rightarrow ((\neg t) \vee p)] \vee [(\neg q) \rightarrow r]$
F	T	T
T	F	F

9. We are given that $\mathcal{A}$ is false for any values of its variables.

   (a) [BB] An implication $p \rightarrow q$ is false only if p is true and q is false. Since $\mathcal{A}$ is always false, $\mathcal{A} \rightarrow \mathcal{B}$ is always true. So it is a tautology.

   (b) An implication $p \rightarrow q$ is false only if p is true and q is false. Since $\mathcal{A}$ is false and the tautology $\mathcal{B}$ is true for any values of the variables they contain, $\mathcal{B} \rightarrow \mathcal{A}$ is always false. So it is a contradiction.

10. (a) The tables below show that when all three variables p, q and r are false, $p \rightarrow (q \rightarrow r)$ is true, whereas $(p \rightarrow q) \rightarrow r$ is false. Thus these statements have different truth tables and hence are not logically equivalent.

p	q	r	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$
F	F	F	T	T

p	q	r	$p \rightarrow q$	$(p \rightarrow q) \rightarrow r$
F	F	F	T	F

   (b) The compound statement is false.

11. (a) [BB]

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

   (b)

p	q	$\neg p$	$(\neg p) \wedge q$	$p \veebar ((\neg p) \wedge q)$	$(p \veebar ((\neg p) \wedge q)) \vee q$
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	F	T	F	F	F

   (c) [BB]

p	q	$p \veebar q$	$p \vee q$	$(p \veebar q) \rightarrow (p \vee q)$
T	T	F	T	T
T	F	T	T	T
F	T	T	T	T
F	F	F	F	T

   The truth table shows that $(p \veebar q) \rightarrow (p \vee q)$ is true for all values of p and q, so it is a tautology.

   (d)

p	q	$p \veebar q$	$p \leftrightarrow q$	$\neg (p \leftrightarrow q)$
T	T	F	T	F
T	F	T	F	T
F	T	T	F	T
F	F	F	T	F

   Columns three and five are the same. So the truth values of $p \veebar q$ and $\neg (p \leftrightarrow q)$ are the same for all values of p and q. Thus these statements are logically equivalent.

Exercises 1.2

1. 1. [BB] (Idempotence) The truth tables at the right show that $p \vee p \iff p$ and $p \wedge p \iff p$.

p	$p \vee p$
T	T
F	F

p	$p \wedge p$
T	T
F	F

2. (Commutativity) The truth tables show that $p \vee q \iff q \vee p$ and $p \wedge q \iff q \wedge p$.

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

3. [BB] (Associativity) The equality of the fifth and seventh columns in the truth table shows that $((p \vee q) \vee r) \iff (p \vee (q \vee r))$.

p	q	r	$p \vee q$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	F	T	T	T	T	T
F	T	T	T	T	T	T
F	F	T	F	T	T	T
T	T	F	T	T	T	T
T	F	F	T	T	F	T
F	T	F	T	T	T	T
F	F	F	F	F	F	F

The equality of the fifth and seventh columns in the truth table shows that $((p \wedge q) \wedge r) \iff (p \wedge (q \wedge r))$.

p	q	r	$p \wedge q$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	F	T	F	F	F	F
F	T	T	F	F	T	F
F	F	T	F	F	F	F
T	T	F	T	F	F	F
T	F	F	F	F	F	F
F	T	F	F	F	F	F
F	F	F	F	F	F	F

4. (Distributivity) The equality of the fifth and eighth columns in the truth table shows that $p \vee (q \wedge r) \iff (p \vee q) \wedge (p \vee r))$.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	F	T	F	T	T	T	T
F	T	T	T	T	T	T	T
F	F	T	F	F	F	T	F
T	T	F	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	F	F	F	T	F	F
F	F	F	F	F	F	F	F

The equality of the fifth and eighth columns in the truth table shows that $p \wedge (q \vee r) \iff (p \wedge q) \vee (p \wedge r))$.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	F	T	T	T	F	T	T
F	T	T	T	F	F	F	F
F	F	T	T	F	F	F	F
T	T	F	T	T	T	F	T
T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	F	F	F	F	F	F

5. [BB] (Double negation) The equality of the first and third columns in the truth table shows that $p \iff \neg(\neg p)$.

p	$\neg p$	$\neg(\neg p)$
T	F	T
F	T	F

6. (The Laws of De Morgan) The equality of the fourth and seventh columns of the truth table shows that $\neg(p \vee q) \iff ((\neg p) \wedge (\neg q))$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The equality of the fourth and seventh columns of the truth table shows that $\neg(p \wedge q) \iff ((\neg p) \vee (\neg q))$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$(\neg p) \vee (\neg q)$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

7. [BB] The two truth tables show, respectively, that $p \vee \mathbf{1} \iff \mathbf{1}$ and $p \wedge \mathbf{1} \iff p$.

p	$\mathbf{1}$	$p \vee \mathbf{1}$
T	T	T
F	T	T

p	$\mathbf{1}$	$p \wedge \mathbf{1}$
T	T	T
F	T	F

8. The two truth tables show, respectively, that $p \vee \mathbf{0} \iff p$ and $p \wedge \mathbf{0} \iff \mathbf{0}$.

p	$\mathbf{0}$	$p \vee \mathbf{0}$
T	F	T
F	F	F

p	$\mathbf{0}$	$p \wedge \mathbf{0}$
T	F	F
F	F	F

9. [BB] The two truth tables show, respectively, that $(p \vee (\neg p)) \iff \mathbf{1}$ and $(p \wedge (\neg p)) \iff \mathbf{0}$.

p	$\neg p$	$p \vee (\neg p)$	$\mathbf{1}$
T	F	T	T
F	T	T	T

p	$\neg p$	$p \wedge (\neg p)$	$\mathbf{0}$
T	F	F	F
F	T	F	F

10. The truth tables show, respectively, that $(\neg \mathbf{1}) \iff \mathbf{0}$ and $(\neg \mathbf{0}) \iff \mathbf{1}$.

$\mathbf{1}$	$\neg\mathbf{1}$	$\mathbf{0}$
T	F	F

$\mathbf{0}$	$\neg\mathbf{0}$	$\mathbf{1}$
F	T	T

11. [BB] The third and sixth columns of the truth table show that $(p \to q) \iff ((\neg q) \to (\neg p))$.

p	q	$p \to q$	$\neg q$	$\neg p$	$(\neg q) \to (\neg p)$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

12. The third and sixth columns of the truth table show that $(p \leftrightarrow q) \iff ((p \to q) \wedge (q \to p))$.

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

13. [BB] The third and fifth columns of the truth table show that $(p \to q) \iff ((\neg p) \vee q)$.

p	q	$p \to q$	$\neg p$	$(\neg p) \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

2. (a) We construct a truth table. Since $p \vee [\neg(p \wedge q)]$ is true for all values of p and q, this statement is a tautology.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$p \vee [\neg(p \wedge q)]$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(b) By one of the laws of DeMorgan, the negation is $(\neg p) \wedge (p \wedge q)$. By associativity, this is logically equivalent to $[(\neg p) \wedge p] \wedge q \iff \mathbf{0} \wedge q \iff \mathbf{0}$. So the negation is a contradiction.

3. (a) [BB] Using one of the laws of De Morgan and one distributive property, we obtain

$$[(p \wedge q) \vee (\neg((\neg p) \vee q))] \iff [(p \wedge q) \vee (p \wedge (\neg q))]$$
$$\iff [p \wedge (q \vee (\neg q))] \iff (p \wedge \mathbf{1}) \iff p.$$

(b) The given compound statement is of the form $x \to (y \to z)$ which is equivalent to $x \to ((\neg y) \vee z) \iff (\neg x) \vee (\neg y) \vee z$. (By associativity, no further parentheses are required here.) So the given statement is equivalent to $(\neg(p \vee r)) \vee ((\neg q) \wedge r) \vee (p \vee r)$. By commutativity, this is $(\neg(p \vee r)) \vee (p \vee r) \vee ((\neg q) \wedge r) \iff \mathbf{1} \vee ((\neg q \wedge r) \iff \mathbf{1}$. The given statement is a tautology!

(c) Using associativity to avoid extra parentheses, the left side of the given statement is

$$[(p \to q) \vee (q \to r)] \iff (\neg p) \vee q \vee (\neg q) \vee r$$
$$\iff (\neg p) \vee \mathbf{1} \vee r \iff \mathbf{1}.$$

The given statement is equivalent to $[1 \wedge (r \to s)]$ which is logically equivalent to $r \to s$.

4. (a) [BB]

p	q	$p \wedge q$	$p \vee (p \wedge q)$
T	T	T	T
T	F	F	T
F	T	F	F
F	F	F	F

(b)

p	q	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

5. (a) [BB] Distributivity gives $[(p \vee q) \wedge (\neg p)] \iff [(p \wedge (\neg p)) \vee (q \wedge (\neg p))] \iff [\mathbf{0} \vee ((\neg p) \wedge q)] \iff [(\neg p) \wedge q]$.

(b) We have $(p \to (q \to r)) \iff (p \to ((\neg q) \vee r)) \iff (\neg p) \vee (\neg q) \vee r$
$\iff (\neg p) \vee r \vee (\neg q) \iff \neg(p \wedge (\neg r)) \vee (\neg q) \iff (p \wedge (\neg r)) \to (\neg q)$.

(c) $(\neg(p \leftrightarrow q)) \iff (\neg((p \to q) \wedge (q \to p))) \iff (\neg(((\neg p) \vee q) \wedge ((\neg q) \vee p)))$

$$\begin{aligned}
&\iff ((p \wedge (\neg q)) \vee (q \wedge (\neg p)))\\
&\iff ((p \wedge (\neg q)) \vee q) \wedge ((p \wedge (\neg q)) \vee (\neg p))\\
&\iff ((p \vee q) \wedge ((\neg q) \vee q)) \wedge ((p \vee (\neg p)) \wedge ((\neg q) \vee (\neg p)))\\
&\iff ((p \vee q) \wedge \mathbf{1}) \wedge (\mathbf{1} \wedge ((\neg q) \vee (\neg p)))\\
&\iff (p \vee q) \wedge ((\neg q) \vee (\neg p))\\
&\iff ((\neg p) \vee (\neg q)) \wedge (q \vee p)\\
&\iff (p \to (\neg q)) \wedge ((\neg q) \to p) \iff (p \leftrightarrow (\neg q)).
\end{aligned}$$

(d) [BB] $\neg[(p \leftrightarrow q) \vee (p \wedge (\neg q))] \iff [\neg(p \leftrightarrow q) \wedge \neg(p \wedge (\neg q))]$
$\iff [(p \leftrightarrow (\neg q)) \wedge ((\neg p) \vee q)]$, using Exercise 5(c).

(e) This is an immediate application of absorption law **4(b)** with $p \wedge (\neg q)$ in place of p and $q \wedge (\neg r)$ in place of q.

(f) Using property 12 and associativity, $[p \to (q \vee r)] \iff [(\neg p) \vee q \vee r] \iff [\neg(p \wedge (\neg q))] \vee r$ (by De Morgan) $\iff [p \wedge (\neg q) \to r]$ using 12 again.

(g) $\neg(p \vee q) \vee [(\neg p) \wedge q] \iff [(\neg p) \wedge (\neg q)] \vee [(\neg p) \wedge q]$ (DeMorgan) $\iff (\neg p) \wedge [(\neg q) \vee q]$ (distributivity) $\iff (\neg p) \vee \mathbf{1} \iff \neg p$

6. $[(p \wedge (\neg q)) \to q] \iff [(\neg(p \wedge (\neg q))) \vee q] \iff [((\neg p) \vee q) \vee q] \iff [(\neg p) \vee q]$.
$[(p \wedge (\neg q)) \to (\neg p)] \iff [(\neg(p \wedge (\neg q))) \vee (\neg p)] \iff [(\neg p) \vee q \vee (\neg p)] \iff [(\neg p) \vee q]$.
So these are both logically equivalent to $(\neg p) \vee q$.

7. (a) We must show that $\mathcal{A} \vee \mathcal{C}$ and $\mathcal{B} \vee \mathcal{C}$ have the same truth tables, given that $\mathcal{A}$ and $\mathcal{B}$ have the same truth tables. This requires four rows of a truth table.

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{A} \vee \mathcal{C}$	$\mathcal{B} \vee \mathcal{C}$
T	T	T	T	T
T	T	F	T	T
F	F	T	T	T
F	F	F	F	F

The last two columns establish our claim.

(b) We must show that $\mathcal{A} \wedge \mathcal{C}$ and $\mathcal{B} \wedge \mathcal{C}$ have the same truth tables, given that $\mathcal{A}$ and $\mathcal{B}$ have the same truth tables. This requires four rows of a truth table.

$\mathcal{A}$	$\mathcal{B}$	$\mathcal{C}$	$\mathcal{A} \wedge \mathcal{C}$	$\mathcal{B} \wedge \mathcal{C}$
T	T	T	T	T
T	T	F	T	T
F	F	T	F	F
F	F	F	F	F

The last two columns establish our claim.

8\. 1. [BB] The truth table shows that idempotence fails.

p	$p \veebar p$
T	F
F	F

2\. Commutativity still holds.

p	q	$p \veebar q$	$q \veebar p$
T	T	F	F
T	F	T	T
F	T	T	T
F	F	F	F

3\. [BB] Associativity holds.

p	q	r	$p \veebar q$	$(p \veebar q) \veebar r$	$q \veebar r$	$p \veebar (q \veebar r)$
T	T	T	F	T	F	T
T	F	T	T	F	T	F
F	T	T	T	F	F	F
F	F	T	F	T	T	T
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	F	T	T	T	T
F	F	F	F	F	F	F

4\. Just one distributive law holds. The table which follows shows that $p \veebar (q \wedge r)$ is not logically equivalent to $(p \veebar q) \wedge (p \veebar r)$,

p	q	r	$q \wedge r$	$p \veebar (q \wedge r)$	$p \veebar q$	$p \veebar r$	$(p \veebar q) \wedge (p \veebar r)$
T	F	T	F	T	T	F	F

On the other hand, $(p \wedge (q \veebar r)) \iff (p \wedge q) \veebar (p \wedge r)$ as shown.

p	q	r	$q \veebar r$	$p \wedge (q \veebar r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \veebar (p \wedge r)$
T	T	T	F	F	T	T	F
T	F	T	T	T	F	T	T
F	T	T	F	F	F	F	F
F	F	T	T	F	F	F	F
T	T	F	T	T	T	F	T
T	F	F	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	F	F	F	F	F	F

6\. Both laws of De Morgan fail. The table shows that $\neg(p \veebar q)$ is not logically equivalent to $(\neg p) \wedge (\neg q)$.

p	q	$p \veebar q$	$\neg(p \veebar q)$	$\neg p$	$\neg q$	$(\neg p) \wedge (\neg q)$
T	T	F	T	F	F	F

The next table shows that $\neg(p \wedge q)$ is not logically equivalent to $(\neg p) \veebar (\neg q)$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$(\neg p) \veebar (\neg q)$
F	F	F	T	T	T	F

7\. [BB] The truth table shows that $p \veebar \mathbf{1}$ is no longer $\mathbf{1}$:

p	$\mathbf{1}$	$p \veebar \mathbf{1}$
T	T	F

8. The truth table shows that $p \veebar \mathbf{0} \iff p$:

p	$\mathbf{0}$	$p \veebar \mathbf{0}$
T	F	T
F	F	F

9. [BB] The truth table shows that $(p \veebar (\neg p)) \iff \mathbf{1}$:

p	$\neg p$	$p \veebar (\neg p)$	$\mathbf{1}$
T	F	T	T
F	T	T	T

12. [BB] This is no longer true:

p	q	$p \to q$	$\neg p$	$(\neg p) \veebar q$
F	T	T	T	F

Finally, neither law of absorption holds: $p \veebar (p \wedge q)$ is not logically equivalent to p and $p \wedge (p \veebar q)$ is not logically equivalent to p.

p	q	$p \wedge q$	$p \veebar (p \wedge q)$
T	T	T	F

p	q	$p \veebar q$	$p \wedge (p \veebar q)$
T	T	F	F

9. (a) $(p \vee q) \wedge ((\neg p) \vee (\neg q))$ is not in disjunctive normal form because the terms are not joined by $\vee$.

(b) [BB] $(p \wedge q) \vee ((\neg p) \wedge (\neg q))$ is in disjunctive normal form.

(c) [BB] $p \vee ((\neg p) \wedge q)$ is not in disjunctive normal form: not all variables are included in the first term.

(d) $(p \wedge q) \vee ((\neg p) \wedge (\neg q) \wedge r)$ is not in disjunctive normal form: not all variables are included in the first term.

(e) $(p \wedge q \wedge r) \vee ((\neg p) \wedge (\neg q) \wedge (\neg r))$ is in disjunctive normal form.

10. (a) [BB] This is already in disjunctive normal form! The definition permits just a single midterm.

(b) [BB] One of the laws of De Morgan gives immediately that $(p \wedge q) \vee (\neg((\neg p) \vee q)) \iff (p \wedge q) \vee (p \wedge (\neg q))$.

(c) $(p \to q) \iff ((\neg p) \vee q) \iff [((\neg p) \wedge q) \vee ((\neg p) \wedge (\neg q)) \vee (q \wedge p) \vee (q \wedge (\neg p))]$.

(d) $[(p \vee q) \wedge ((\neg p) \vee (\neg q))] \iff [(p \wedge ((\neg p) \vee (\neg q))] \vee [q \wedge ((\neg p) \vee (\neg q))] \iff [(p \wedge (\neg p)) \vee (p \wedge ((\neg q))] \vee [(q \wedge ((\neg p)) \vee (q \wedge ((\neg q))] \iff [(p \wedge (\neg q)) \vee ((\neg p) \wedge q))]$

(e) $[(p \to q) \wedge (q \wedge r)] \iff [((\neg p) \vee q) \wedge (q \wedge r)] \iff [((\neg p) \wedge q \wedge r) \vee (q \wedge q \wedge r)] \iff [((\neg p) \wedge q \wedge r) \vee (p \wedge q \wedge r) \vee ((\neg p) \wedge q \wedge r))] \iff [((\neg p) \wedge q \wedge r) \vee (p \wedge q \wedge r)]$, omitting the repeated minterm at the final step.

(f) $\big(p \vee [q \wedge (p \vee (\neg r))]\big) \iff \big(p \vee [(q \wedge p) \vee (q \wedge (\neg r))]\big) \iff \big(p \vee (q \wedge p) \vee (q \wedge (\neg r))\big) \iff \big((p \wedge q) \vee (p \wedge (\neg q)) \vee (p \wedge q) \vee (q \wedge (\neg r))\big) \iff \big((p \wedge q) \vee (p \wedge (\neg q)) \vee (q \wedge (\neg r))\big) \iff \big((p \wedge q \wedge r) \vee (p \wedge q \wedge (\neg r)) \vee (p \wedge (\neg q) \wedge r) \vee (p \wedge (\neg q) \wedge (\neg r)) \vee (p \wedge q \vee (\neg r)) \vee ((\neg p) \wedge q \wedge (\neg r))\big) \iff \big((p \wedge q \wedge r) \vee (p \wedge q \wedge (\neg r)) \vee (p \wedge (\neg q) \wedge r) \vee (p \wedge (\neg q) \wedge (\neg r)) \vee ((\neg p) \wedge q \wedge (\neg r))\big)$, omitting the repeated minterm at the final step.

11. August De Morgan was born in India in 1806, died in England in 1871 and lived his life without the sight of his right eye, which was damaged at birth. He was the first Professor of Mathematics at University College, London, founded the London Mathematical Society and was its first president. He was apparently quite a man of principle. He refused to study for the MA degree because of a required theological exam and twice resigned his chair at University College, on matters of principle. He was

never a Fellow of the Royal Society because he refused to let his name be put forward and, similarly, refused an honorary degree from the University of Edinburgh.

De Morgan's mathematical contributions include the definition and introduction of "mathematical induction", the most important method of proof in mathematics today. See Section 5.1. His definition of a limit was the first attempt to define the idea in precise mathematical terms. Nonetheless, it is the area of mathematical logic with which De Morgan's name is most closely associated. The "Laws of De Morgan" introduced in this section, together with their set theoretical analogues—$(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$ (see Section 2.2)—are used extensively and of fundamental importance. De Morgan also developed a system of notation for symbolic logic that could denote converses and contradictions.

Exercises 1.3

1. (a) [BB] Since $[p \to (q \to r)] \iff [p \to ((\neg q) \vee r)] \iff [(\neg p) \vee (\neg q) \vee r]$, the given argument can be rewritten

$$\begin{array}{c} [(\neg p) \vee r] \vee (\neg q) \\ q \\ \hline (\neg p) \vee r \end{array}$$

which is valid by disjunctive syllogism.

(b) [BB] We analyze with a truth table. In row one, the premises are true but the conclusion is not. The argument is not valid.

p	q	r	$p \to q$	$q \vee r$	$\neg q$	$r \to (\neg q)$	
T	T	T	T	T	F	F	$\star$
T	F	T	F	T	T	T	
F	T	T	T	T	F	F	$\star$
F	F	T	T	T	T	T	$\star$
T	T	F	T	T	F	T	$\star$
T	F	F	F	F	T	T	
F	T	F	T	T	F	T	$\star$
F	F	F	T	F	T	T	

(c) We analyze with a truth table. In row three, the premises are true but the conclusion is not. The argument is not valid.

p	q	r	$p \to q$	$r \to q$	$r \to p$	
T	T	T	T	T	T	$\star$
T	F	T	F	F	T	
F	T	T	T	T	F	$\star$
F	F	T	T	F	F	
T	T	F	T	T	T	$\star$
T	F	F	F	T	T	
F	T	F	T	T	T	$\star$
F	F	F	T	T	T	$\star$

(d) This can be solved using a truth table with 16 rows. Alternatively, we can proceed as follows. Assume that the argument is not valid. This means that we can find truth values for p, q, r and s such that the premises are true but the conclusion is false. Since $s \to (r \vee q)$ is false, we must have s true and $r \vee q$ false. But this means both r and q are false. Since $p \to q$ is true and q is false, p must be false. But then $q \vee (\neg r)$ is true and $p \wedge s$ is false, contradicting the truth of $(q \vee (\neg r)) \to (p \wedge s)$. Hence we have a contradiction, so the argument is valid.

(e) This argument is valid. The second and third premises give q by Modus Ponens. Together with the first premise (double negation and dijunctive syllogism), we get $\neg p$.

(f) This argument is valid. The first and third premises give $\neg p$ by modus tollens. Together with the second premise (double negation and a second application of modus tollens), we get r.

2. (a) If p is true and p implies q, then q is true.

(b) This was PAUSE 5.

(c) If p is true or q is true and p is not true, then q must be true.

(d) We do this by contradiction. So assume the premises are true but the conclusion is false. Since $p \to r$ is false, we must have p true and r false. Since $q \to r$ is true and r is false, q must be false. Then, since $p \to q$ is true, p must be false, giving a contradiction.

(e) We prove this by contradiction. Assume $p \vee q$ is false. This means both p and q are false. Since $p \vee r$ is true and p is false, r must be true. Since $q \vee (\neg r)$ is true and q is false, r must be false, giving a contradiction.

3. (a) [BB] We analyze with a truth table. There are five rows when the premises are all true and in each case the conclusion is also true. The argument is valid.

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$(p \vee q) \to r$	
T	T	T	T	T	T	T	⋆
T	F	T	T	T	T	T	⋆
F	T	T	T	T	T	T	⋆
F	F	T	F	T	T	T	⋆
T	T	F	T	F	F	F	
T	F	F	T	F	T	F	
F	T	F	T	T	F	F	
F	F	F	F	T	T	T	⋆

(b) We analyze with a partial truth table showing the nine situations in which both the premises are true. In every case, the conclusion is true. The argument is valid.

p	r	q	s	$p \wedge q$	$r \wedge s$	$(p \wedge q) \to (r \wedge s)$
T	T	T	T	T	T	T
T	T	F	T	F	T	T
T	T	F	F	F	F	T
F	T	T	T	F	T	T
F	T	F	T	F	T	T
F	T	F	F	F	F	T
F	F	T	T	F	F	T
F	F	F	T	F	F	T
F	F	F	F	F	F	T

(c) The second and third premises are $p \to r$ and $r \to s$ which together imply $p \to s$ by the chain rule. Thus the argument becomes

$$\begin{array}{c} p \vee q \\ p \to s \\ \hline q \vee s \end{array}$$

which we check with a truth table.

p	q	s	$p \vee q$	$p \to s$	$q \vee s$	
T	T	T	T	T	T	$\star$
T	F	T	T	T	T	$\star$
F	T	T	T	T	T	$\star$
F	F	T	F	T	T	
T	T	F	T	F	T	
T	F	F	T	F	F	
F	T	F	T	T	T	$\star$
F	F	F	F	T	F	

There are four rows when the premises are true and, in each case, the conclusion is also true. The argument is valid.

(d) Since $((\neg q) \wedge r) \iff \neg(q \vee (\neg r))$ and $\neg(p \wedge s) \iff [(\neg p) \vee (\neg s)] \iff (p \to (\neg s))$, the premises become

$$(q \vee (\neg r)) \to p$$
$$p \to (\neg s)$$

so the chain rule gives $(q \vee (\neg r)) \to (\neg s)$, which is logically equivalent to $(\neg(q \vee (\neg r))) \vee (\neg s)$, which is the desired conclusion by a law of De Morgan.

4. (a) [BB] Since $[(\neg r) \vee (\neg q)] \iff [q \to (\neg r)]$, the first two premises give $p \to (\neg r)$ by the chain rule. Now $\neg p$ follows by modus tollens.

(b) This argument is not valid. If q and r are false, p and s are true, and t takes on any truth value, then all premises are true, yet the conclusion is false.

(c) [BB] This argument is valid. Since $p \leftrightarrow (t \vee s)$, we can replace $t \vee s$ with p so that the premises become $p \vee (\neg q)$, $p \to (p \vee r)$, $(\neg r) \vee p$. The first of these is logically equivalent to $q \to p$ while the third is $r \to p$. Using Exercise 3(**a**), we get $q \vee r \to p$, so certainly $q \vee r \to p \vee r$.

(d) This argument is valid. To see this, note that the second premise is $r \to (t \vee s)$, so the chain rule gives $r \to p$. Also, the first premise is $q \to p$. Exercise 3(**a**) tells us that $(q \vee r) \to p$, so certainly $(q \vee r) \to (p \vee r)$.

(e) This argument is not valid. If p and r are false while q and s are true, all premises are true, yet the conclusion is false.

(f) [BB] This is valid. The first premise is $q \vee [(\neg p) \vee s]$ while the second is $(\neg q) \vee r$. Now $(\neg p) \vee s \vee r$ follows by resolution, and this is the conclusion.

(g) This argument is valid. The first premise is $p \vee [(\neg q) \vee r]$, which is $(\neg q) \vee p \vee r$. Since $q \vee r$, resolution gives $p \vee r$. Using $r \to p$ and the result of part (f), we get $p \vee p$, which is p.

(h) This argument is valid. Using (g) twice, we have $p \to [s \vee (\neg p)]$ which is $(\neg p) \vee s \vee (\neg p)$, which is $(\neg p) \vee s$, which is the conclusion.

5. (a) [BB] Let p and q be the statements
p: I stay up late at night
q: I am tired in the morning.

The given argument is
$$\begin{array}{c} p \to q \\ \underline{\quad p \quad} \\ q \end{array}.$$

This is valid by modus ponens.

(b) [BB] Let p and q be the statements p: I stay up late at night
q: I am tired in the morning.

The given argument is $\begin{array}{c} p \to q \\ \underline{\quad q \quad} \\ p \end{array}$.

This is not valid, as the truth table shows. In row three, the two premises are true but the conclusion is false.

p	q	$p \to q$	
T	T	T	⋆
T	F	F	
F	T	T	⋆
F	F	T	

(c) Let p and q be the statements p: I stay up late at night
q: I am tired in the morning.

The given argument is $\begin{array}{c} p \to q \\ \underline{\quad \neg q \quad} \\ \neg p \end{array}$.

This is valid by modus tollens.

(d) Let p and q be the statements p: I stay up late at night
q: I am tired in the morning.

The given argument is $\begin{array}{c} p \to q \\ \underline{\quad \neg p \quad} \\ \neg q \end{array}$.

This is not valid, as the truth table shows. In row three, the two premises are true but the conclusion is false.

p	q	$p \to q$	$\neg p$	$\neg q$	
T	T	T	F	F	
T	F	F	F	T	
F	T	T	T	F	⋆
F	F	T	T	T	⋆

(e) [BB] Let p and q be the statements p: I wear a red tie
q: I wear blue socks.

The given argument is $\begin{array}{c} p \vee q \\ \underline{\quad \neg q \quad} \\ p \end{array}$.

This is valid by disjunctive syllogism.

(f) Let p and q be the statements p: I wear a red tie
q: I wear blue socks.

The given argument is $\begin{array}{c} p \vee q \\ \underline{\quad q \quad} \\ \neg p \end{array}$.

This is not valid, as the truth table shows. In row one, the two premises are true but the conclusion is false.

p	q	$p \vee q$	$\neg p$	
T	T	T	F	⋆
T	F	T	F	
F	T	T	T	⋆
F	F	F	T	

(g) [BB] Let p, q, and r be the statements p: I work hard
q: I earn lots of money
r: I pay high taxes.

The given argument is $\begin{array}{c} p \to q \\ q \to r \\ \hline r \to p \end{array}$.

This is not valid, as the truth table shows. In row five, the two premises are true but the conclusion is false.

p	q	r	$p \to q$	$q \to r$	$r \to p$	
T	T	T	T	T	T	⋆
T	F	T	F	T	T	
F	T	T	T	T	F	⋆
F	F	T	T	T	F	⋆
T	T	F	T	F	T	
T	F	F	F	T	T	
F	T	F	T	F	T	
F	F	F	T	T	T	⋆

(h) Let p, q, and r be the statements
p: I work hard
q: I earn lots of money
r: I pay high taxes.

The given argument is $\begin{array}{c} p \to q \\ q \to r \\ \hline p \to r \end{array}$.

This is valid by the chain rule.

(i) Let p, q, and r be the statements
p: I work hard
q: I earn lots of money
r: I pay high taxes.

The given argument is $\begin{array}{c} p \to q \\ q \to r \\ \hline \neg p \to \neg r \end{array}$.

The conclusion is logically equivalent to $r \to p$, so this is the same as Exercise 5(g), hence not valid.

(j) Let p, q, and r be the statements
p: I like mathematics
q: I study
r: I like football.

The given argument is $\begin{array}{c} p \to q \\ \neg q \\ p \vee r \\ \hline r \end{array}$.

The first two premises give $\neg p$ by modus tollens, so, since $p \vee r$ is true, the conclusion follows by disjunctive syllogism.

(k) Let p, q, and r be the statements
p: I like mathematics
q: I study
r: I like football.

The given argument is $\begin{array}{c} q \vee r \\ r \to p \\ \hline (\neg q) \to p \end{array}$.

This is the same as $\begin{array}{c} q \vee r \\ (\neg r) \vee p \\ \hline q \vee p \end{array}$ and hence valid by resolution.

(l) [BB] Let p, q, and r be the statements

p: I like mathematics
q: I study
r: I pass mathematics
s: I graduate.

The given argument is
$$\begin{array}{c} p \to q \\ (\neg q) \vee r \\ (\neg s) \to (\neg r) \\ \hline s \to q \end{array}.$$

This is the same as
$$\begin{array}{c} p \to q \\ q \to r \\ r \to s \\ \hline s \to q \end{array}$$

which is certainly not valid, as the following partial truth table shows.

p	q	r	s	$p \to q$	$q \to r$	$r \to s$	$s \to q$
F	F	F	T	T	T	T	F

(m) Let p, q, and r be the statements

p: I like mathematics
q: I study
r: I pass mathematics
s: I graduate.

The given argument is
$$\begin{array}{c} p \to q \\ (\neg q) \vee r \\ (\neg s) \to (\neg r) \\ \hline p \to s \end{array}$$

which is the same as
$$\begin{array}{c} p \to q \\ q \to r \\ r \to s \\ \hline p \to s \end{array}$$

which is valid by two applications of the chain rule.

6. [BB] $r \vee q$ is logically equivalent to $[\neg(\neg r) \vee q] \iff [(\neg r) \to q]$ so, with $p \to \neg r$, we get $p \to q$ by the chain rule.

7. We will prove by contradiction that no such conclusion is possible. Say to the contrary that there is such a conclusion $\mathcal{C}$. Since $\mathcal{C}$ is not a tautology, some set of truth values for p and q must make $\mathcal{C}$ false. But if r is true, then both the premises $(\neg p) \to r$ and $r \vee q$ are true regardless of the values of p and q. This contradicts $\mathcal{C}$ being a valid conclusion for this argument.

8. (a) [BB] $p \wedge q$ is true precisely when p and q are both true.

   (b) By 8(a), we can replace $p \wedge q$ by the two premises p and q. Using modus ponens, p and $p \to r$ lead to the conclusion r. Using modus tollens, q and $s \to (\neg q))$ lead to the conclusion $\neg s$. Finally, 8(a) says we can replace $\neg s$ and r with $(\neg s) \wedge r$.

9. By Exercise 8(a), the final premise is equivalent to the list of premises $q_1, q_2, \ldots, q_n$. Now

$$[p_1 \to (q_1 \to r_1)] \iff [p_1 \to ((\neg q_1) \vee r_1)] \iff [(\neg p_1) \vee (\neg q_1) \vee r_1].$$

Together with q_1, disjunctive syllogism gives $(\neg p_1) \vee r_1$ which is logically equivalent to $p_1 \rightarrow r_1$. Thus the given premises imply

$$\begin{gathered} p_1 \rightarrow r_1 \\ p_2 \rightarrow r_2 \\ \vdots \\ p_n \rightarrow r_n \end{gathered}$$

which, again using **8(a)**, are logically equivalent to the single premise

$$(p_1 \rightarrow r_1) \wedge (p_2 \rightarrow r_2) \wedge \cdots \wedge (p_n \rightarrow r_n).$$

10. [BB] In Latin, *modus ponens* means "method of affirming" and *modus tollens* means "method of denying". This is a reflection of the fact that *modus tollens* has a negative $\neg p$ as its conclusion, while *modus ponens* affirms the truth of a statement q.

Chapter 1 Review

1.

p	q	r	$\neg r$	$q \rightarrow (\neg r)$	$p \wedge (q \rightarrow (\neg r))$	$\neg q$	$(\neg q) \vee r$	$(p \wedge (q \rightarrow (\neg r))) \rightarrow ((\neg q) \vee r)$
T	T	T	F	F	F	F	T	T
T	T	F	T	T	T	F	F	F
T	F	T	F	T	T	T	T	T
F	T	T	F	F	F	F	T	T
T	F	F	T	T	T	T	T	T
F	T	F	T	T	F	F	F	T
F	F	T	F	T	F	T	T	T
F	F	F	T	T	F	T	T	T

2. We have $\neg r$ is F, so $(\neg r) \wedge s$ is F. Since q is T, this means $q \rightarrow ((\neg r) \wedge s)$ is F. But p is T, so $p \vee (q \rightarrow ((\neg r) \wedge s))$ is T. Also $r \wedge t$ is T. Hence the entire statement has truth value T.

3. (a) A truth table shows this is a contradiction.

p	q	$\neg q$	$\neg p$	$p \wedge (\neg q)$	$(\neg p) \vee q$	$[p \wedge (\neg q)] \wedge [(\neg p) \vee q]$
T	T	F	F	F	T	F
T	F	T	F	T	F	F
F	T	F	T	F	T	F
F	F	T	T	F	T	F

(b) A truth table shows that this is neither a tautology nor a contradiction.

p	q	$p \to q$	$p \vee q$	$(p \to q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	F

(c) A truth table shows this is a tautology.

p	q	r	$\neg q$	$p \wedge (\neg q)$	$[p \wedge (\neg q)] \to r$	$p \vee [(p \wedge (\neg q)) \to r]$
T	T	T	F	F	T	T
T	F	T	T	T	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	F	F	F	T	T
F	F	F	T	F	T	T

(d) This is neither a tautology nor a contradiction.

p	q	r	$\neg q$	$(\neg q) \wedge r$	$p \vee q$
T	T	T	F	F	T
T	F	T	T	T	T
F	T	T	F	F	T
F	F	T	T	T	F
T	T	F	F	F	T
T	F	F	T	F	T
F	T	F	F	F	T
F	F	F	T	F	F

$[(\neg q) \wedge r] \to (p \vee q)$	$(p \vee q) \to [(\neg q) \wedge r]$	$(p \vee q) \leftrightarrow [(\neg q) \wedge r]$
T	F	F
T	T	T
T	F	F
F	T	F
T	F	F
T	F	F
T	F	F
F	T	F

4. Assume that some set of truth values on the variables makes $\mathcal{A}$ true. If $\mathcal{B}$ were false, this would make $\mathcal{A} \to \mathcal{B}$ false and $\mathcal{B} \to \mathcal{A}$ true, contradicting logical equivalence. So $\mathcal{B}$ must be true also. Similarly, if $\mathcal{B}$ is true, then $\mathcal{A}$ must also be true. We conclude that $\mathcal{A}$ is true if and only if $\mathcal{B}$ is true. This means $\mathcal{A}$ and $\mathcal{B}$ are logically equivalent.

5. (a) Since $\mathcal{A} \iff \mathcal{B}$, we know that $\mathcal{A}$ is true precisely when $\mathcal{B}$ is true. Since $\mathcal{B} \iff \mathcal{C}$, $\mathcal{B}$ is true precisely when $\mathcal{C}$ is true. Hence $\mathcal{A}$ is true if and only if $\mathcal{C}$ is true, that is, $\mathcal{A} \iff \mathcal{C}$.

   (b) Property 12 says $(p \to q) \iff ((\neg p) \vee q)$. Clearly, $((\neg p) \vee q) \iff (q \vee (\neg p))$, so part (a) tells us $(p \to q) \iff (q \vee (\neg p))$. But Property 12 also says $((\neg q) \to (\neg p)) \iff (\neg(\neg q) \vee (\neg p))$ and clearly $(\neg(\neg q) \vee (\neg p)) \iff (q \vee (\neg p))$. So we have $(p \to q) \iff (q \vee (\neg p))$ and also

$(q \vee (\neg p)) \iff ((\neg q) \to (\neg p))$. Hence part (a) again gives $(p \to q) \iff ((\neg q) \to (\neg p))$, which is Property 11.

6. (a) $((p \to q) \to r) \iff (((\neg p) \vee q) \to r) \iff ((\neg((\neg p) \vee q)) \vee r) \iff ((p \wedge (\neg q)) \vee r) \iff ((p \vee r) \wedge ((\neg q) \vee r)) \iff ((p \vee r) \wedge (\neg(q \wedge (\neg r))))$.

   (b) $[p \to (q \vee r)] \iff (\neg p) \vee (q \vee r) \iff \neg p \vee q \vee r$ (associativity) $\iff [\neg(p \wedge (\neg q)] \vee r$ (De Morgan) $\iff [p \wedge (\neg q)] \to r$.

7. (a) $((p \vee q) \wedge r) \vee ((p \vee q) \wedge (\neg p))$

$$
\begin{aligned}
&\iff ((p \wedge r) \vee (q \wedge r)) \vee ((p \wedge (\neg p)) \vee (q \wedge (\neg p)))\\
&\iff ((p \wedge r) \vee (q \wedge r)) \vee (0 \vee (q \wedge (\neg p)))\\
&\iff ((p \wedge r) \vee (q \wedge r)) \vee (q \wedge (\neg p))\\
&\iff (p \wedge r \wedge q) \vee (p \wedge r \wedge (\neg q)) \vee (q \wedge r \wedge p) \vee (q \wedge r \wedge (\neg p))\\
&\qquad \vee (q \wedge (\neg p) \wedge (\neg r))\\
&\iff (p \wedge q \wedge r) \vee (p \wedge (\neg q) \wedge r) \vee ((\neg p) \wedge q \wedge r)\\
&\qquad \vee ((\neg p) \wedge q \wedge (\neg r))
\end{aligned}
$$

   (b) $[p \vee (q \wedge (\neg r))] \wedge \neg(q \wedge r)$

$$
\begin{aligned}
&\iff [p \vee (q \wedge (\neg r))] \wedge [(\neg q) \vee (\neg r)]\\
&\iff \Big([p \vee (q \wedge (\neg r))] \wedge (\neg q)\Big) \vee \Big([p \vee (q \wedge (\neg r))] \wedge (\neg r)\Big)\\
&\iff (p \wedge (\neg q)) \vee (q \wedge (\neg r) \wedge (\neg q)) \vee (p \wedge (\neg r)) \vee (q \wedge (\neg r) \wedge (\neg r))\\
&\iff (p \wedge (\neg q) \wedge r) \vee (p \wedge (\neg q) \wedge (\neg r)) \vee \mathbf{0} \vee (p \wedge q \wedge (\neg r))\\
&\qquad \vee (p \wedge (\neg q) \wedge (\neg r)) \vee (p \wedge q \wedge (\neg r) \vee ((\neg p) \wedge q \wedge (\neg r))\\
&\iff (p \wedge (\neg q) \wedge r) \vee (p \wedge (\neg q) \wedge (\neg r)) \vee (p \wedge q \wedge (\neg r))\\
&\qquad \vee (p \wedge q \wedge (\neg r) \vee ((\neg p) \wedge q \wedge (\neg r))
\end{aligned}
$$

8. (a) This is not valid. If p is false and q is true, the premises are true but the conclusion is not.

   (b) The first hypothesis can be rewritten as $p \vee (\neg q)$, which is the same as $q \to p$. The second hypothesis is $(\neg p) \vee (\neg r)$, which is $p \to (\neg r)$. The third hypothesis is $(\neg r) \to s$. So the given argument is

$$
\begin{array}{c}
q \to p\\
p \to (\neg r)\\
\underline{(\neg r) \to s}\\
q \to s
\end{array}
$$

   Two applications of the chain rule tell us this is valid.

   (c) This argument is not valid. If p, r and t are true while q is false (and s takes on either truth value), the hypotheses are true while the conclusion is false.

9. This argument is valid. The hypotheses can never both be true at the same time, so there can be no case when the hypotheses are true while the conclusion is false.

10. (a) This argument, an example of resolution, is valid.

   (b) This argument is not valid. We can write it as shown. When s and r are true and p is false, the hypotheses are true, while the conclusion is false.

$$
\begin{array}{c}
p \to q\\
(\neg q) \vee r\\
\underline{(\neg r) \to (\neg s)}\\
s \to p.
\end{array}
$$

Exercises 2.1

1. (a) [BB] $\{-\sqrt{5}, \sqrt{5}\}$
   (b) $\{1, 3, 5, 15, -1, -3, -5, -15\}$
   (c) [BB] $\{0, -\frac{3}{2}\}$ (Although $\pm\sqrt{2}$ are solutions to the equation, they are not rational.)
   (d) $\{-1, 0, 1, 2, 3\}$
   (e) This is the empty set. There are no numbers less than -4 **and** bigger than $+4$.

2. (a) [BB] For example, $1 + i$, $1 + 2i$, $1 + 3i$, $-8 - 5i$ and $17 - 43i$.
   (b) For example, $\{1 - 2\sqrt{2}, 1 - 5\sqrt{2}, 1 - 7\sqrt{2}, 316 - 2\sqrt{2}$ and $394 - 7\sqrt{2}\}$.
   (c) If $x = 0$, $y = \pm 5$ and $x/y = 0$. If $x = 1$, $y = \pm\sqrt{24}$, and $x/y = \pm 1/\sqrt{24} = \pm\sqrt{24}/24$. If $x = 2$, $y = \pm\sqrt{21}$ and $x/y = \pm 2\sqrt{21}/21$. Five elements of the given set are 0, $\sqrt{24}/24$, $-\sqrt{24}/24$, $2\sqrt{21}/21$ and $-2\sqrt{21}/21$.
   (d) $\{2, 3, 5, 6, 8\}$.

3. (a) [BB] $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 3, 4\}$
   (b) $\emptyset$, $\{1\}$, $\{2\}$, $\{1, 2\}$
   (c) $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$
   (d) $\{3\}$, $\{4\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3\}$, $\{2, 4\}$, $\{3, 4\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, $\{1, 2, 3, 4\}$
   (e) $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 2, 3, 4\}$
   (f) $\emptyset$, $\{1\}$, $\{2\}$.

4. [BB] Only (c) is true. The set A contains one element, $\{a, b\}$.

5. (a) [BB] True. 3 belongs to the set $\{1, 3, 5\}$.
   (b) False. $\{3\}$ is a subset of $\{1, 3, 5\}$ but not a member of this set.
   (c) True. $\{3\}$ is a proper subset of $\{1, 3, 5\}$.
   (d) [BB] False. $\{3, 5\}$ is a subset of $\{1, 3, 5\}$.
   (e) False. Although $\{1, 3, 5\}$ is a subset of itself, it is not a **proper** subset.
   (f) False. If $a + 2b$ is in the given set, a is even, so $a + 2b$ is even and can't equal 1.
   (g) False. If $a + b\sqrt{2} = 0$ and $b \neq 0$, then $\sqrt{2} = -\frac{a}{b}$ is the quotient of rational numbers and, hence, rational. But this is not true.

6. (a) [BB] $\{\emptyset\}$; (b) $\{\emptyset, \{\emptyset\}\}$; (c) $\{\emptyset, \{\emptyset, \{\emptyset\}\}, \{\emptyset\}, \{\{\emptyset\}\}\}$.

7. (a) [BB] True. The empty set is a subset of every set.
   (b) True. The empty set is a subset of every set.
   (c) False. The empty set does not contain any elements.
   (d) True. $\{\emptyset\}$ is a set containing one element, namely, $\emptyset$.
   (e) [BB] False. $\{1, 2\}$ is a subset of $\{1, 2, 3, \{1, 2, 3\}\}$.
   (f) False. $\{1, 2\}$ is not an element of $\{1, 2, 3, \{1, 2, 3\}\}$.
   (g) True. $\{1, 2\}$ is a proper subset of $\{1, 2, \{\{1, 2\}\}\}$.
   (h) [BB] False. $\{1, 2\}$ is not an element of $\{1, 2, \{\{1, 2\}\}\}$.

(i) True. $\{\{1,2\}\}$ contains just one element, $\{1,2\}$, and this is an element of $\{1,2,\{1,2\}\}$.

8. [BB] Yes it is; for example, let $x = \{1\}$ and $A = \{1, \{1\}\}$.

9. (a) i. $\{a,b,c,d\}$ ii. [BB] $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$
iii. $\{a,b\}$, $\{a,c\}$, $\{a,d\}$, $\{b,c\}$, $\{b,d\}$, $\{c,d\}$ iv. $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$ v. $\emptyset$
(b) 16

10. (a) If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$ is a set containing one element, so **its** power set contains two elements.
(b) $\mathcal{P}(A)$ contains two elements; $\mathcal{P}(\mathcal{P}(A))$ has four elements.

11. (a) [BB] 4; (b) [BB] 8.
(c) [BB] There are 2^n subsets of a set of n elements. (See Exercise 15 in Section 5.1 for a proof.)

12. (a) [BB] False. Let $A = \{2\}$, $B = \{\{2\}\}$, $C = \{\{\{2\}\}\}$. Then A is an element of B (that is, $A \in B$) and B is an element of C ($B \in C$), but A is not an element of C (since B is C's only element).
(b) True. If $x \in A$, then $x \in B$ since $A \subseteq B$. But since $x \in B$, then $x \in C$ since $B \subseteq C$.
(c) True. As in the previous part, we know that $A \subseteq C$. To prove $A \neq C$, we note that there is some $x \in C$ such that $x \notin B$ (since $B \subsetneqq C$). Then, since $x \notin B$, $x \notin A$. Therefore, x is an element of C which is not in A, proving $A \neq C$.
(d) [BB] True. $A \in B$ means that A belongs to the set B. Since B is a subset of C, any element of B also belongs to C. Hence, $A \in C$.
(e) False. For example, let $A = \{1\}$, $B = \{\{1\}, 2\}$ and $C = \{\{1\}, 2, 3\}$. Then $A \in B$, $B \subseteq C$, but $A \not\subseteq C$.
(f) False. Let $A = \{1\}$, $B = \{1,2\}$, $C = \{\{1,2\}, 3\}$. Then $A \subseteq B$ and $B \in C$, but $A \notin C$.
(g) False. Same example as 12(f) where $A \not\subseteq C$.

13. (a) This is false. As a counter-example, consider $A = \{1\}$, $B = \{2\}$. Then A is not a subset of B and B is not a proper subset of A.
(b) The converse of the implication in (a) is the implication $B \subsetneqq A \rightarrow A \not\subseteq B$. This is true. Since $B \subsetneqq A$, there exists some element $a \in A$ which is not in B. Thus A is not a subset of B.

14. (a) [BB] True. ($\longrightarrow$) If $C \in \mathcal{P}(A)$, then by definition of "power set," C is a subset of A; that is, $C \subseteq A$.
($\longleftarrow$) If $C \subseteq A$, then C is a subset of A and so, again by definition of "power set," $C \in \mathcal{P}(A)$.
(b) True. ($\longrightarrow$) Suppose $A \subseteq B$. We prove $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. For this, let $X \in \mathcal{P}(A)$. Therefore, X is a subset of A; that is, every element of X is an element of B. Since $A \subseteq B$, every element of X must be an element of B. So $X \subseteq B$; hence, $X \in \mathcal{P}(B)$.
($\longleftarrow$) Conversely, assume $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. We must prove $A \subseteq B$. For any set A, we know that $A \subseteq A$ and, hence, $A \in \mathcal{P}(A)$. Here, with $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, we have, therefore, $A \in \mathcal{P}(B)$; that is, $A \subseteq B$, as desired.
(c) The double implication here is false because the implication $\longrightarrow$ is false. If $A = \emptyset$, then $\mathcal{P}(A) = \{\emptyset\}$ and $\{\emptyset\} \neq \emptyset$.

Exercises 2.2

1. (a) [BB] $A = \{1,2,3,4,5,6\}$, $B = \{-1,0,1,2,3,4,5\}$, $C = \{0,2,-2\}$.
   (b) $A \cup C = \{-2,0,1,2,3,4,5,6\}$, $B \cap C = \{0,2\}$,
   $B \setminus C = \{-1,1,3,4,5\}$, $A \oplus B = \{-1,6,0\}$,
   $C \times (B \cap C) = \{(0,0),(0,2),(2,0),(2,2),(-2,0),(-2,2)\}$,
   $(A \setminus B) \setminus C = \{6\}$, $A \setminus (B \setminus C) = \{2,6\}$,
   $(B \cup \emptyset) \cap \{\emptyset\} = \emptyset$.
   (c) $S = \{(1,-1),(2,0),(3,1),(4,2),(5,3),(6,4)\}$; $T = \{(1,2),(2,2)\}$.

2. (a) [BB] $S \cap T = \{\sqrt{2}, 25\}$, $S \cup T = \{2,5,\sqrt{2},25,\pi,\frac{5}{2},4,6,\frac{3}{2}\}$,
   $T \times (S \cap T) = \{(4,\sqrt{2}),(4,25),(25,\sqrt{2}),(25,25),(\sqrt{2},\sqrt{2}),$
   $(\sqrt{2},25),(6,\sqrt{2}),(6,25),(\frac{3}{2},\sqrt{2}),(\frac{3}{2},25)\}$.
   (b) [BB] $\mathsf{Z} \cup S = \{\sqrt{2},\pi,\frac{5}{2},0,1,-1,2,-2,\ldots\}$; $\mathsf{Z} \cap S = \{2,5,25\}$;
   $\mathsf{Z} \cup T = \{\sqrt{2},\frac{3}{2},0,1,-1,2,-2,\ldots\}$; $\mathsf{Z} \cap T = \{4,25,6\}$.
   (c) $\mathsf{Z} \cap (S \cup T) = \{2,5,25,4,6\} = (\mathsf{Z} \cap S) \cup (\mathsf{Z} \cap T)$. The two sets are equal.
   (d) $\mathsf{Z} \cup (S \cap T) = \{\sqrt{2},0,1,-1,2,-2,\ldots\} = \mathsf{Z} \cup \{\sqrt{2}\} = (\mathsf{Z} \cup S) \cap (\mathsf{Z} \cup T)$.
   The two sets are equal.

3. (a) [BB] $\{1,9,0,6,7\}$; (b) $\{4,6,5\}$; (c) $\{0,1\}$.

4. $A = \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\}$ and $B = \{\pm\frac{1}{2},\pm 1,\pm 2\}$.

5. (a) [BB] $\{c,\{a,b\}\}$; (b) $\{\emptyset\}$; (c) A; (d) $\emptyset$;
   (e) [BB] $\emptyset$; (f) $\{A\}$.

6. (a) [BB] $A^c = (-2,1]$; (b) $A^c = (-\infty,-3] \cup (4,\infty)$; (c) $A^c = \mathsf{R}$.

7. (a) $Y \cap Z = \{3,4,5\}$, so $X \oplus (Y \cap Z) = \{1,2,5\}$.
   (b) $(X^c \cup Y)^c = X \cap Y^c = X \setminus Y = \{1\}$.

8. (a) [BB] The subsets of A containing $\{1,2\}$ are obtained by taking the union of $\{1,2\}$ with a subset of $\{3,4,5,\ldots,n\}$. Their number is the number of subsets of $\{3,4,5,\ldots,n\}$ which is 2^{n-2}. (See Exercise 11 of Section 2.1.)
   (b) The subsets B which have the property that $B \cap \{1,2\} = \emptyset$ are exactly the subsets of $\{3,4,5,\ldots,n\}$ and these number 2^{n-2}.
   (c) The subsets B which have the property that $B \cup \{1,2\} = A$ are precisely those subsets which contain $\{3,4,5,\ldots,n\}$ and these correspond, as in (a), to the subsets of $\{1,2\}$. There are four.

9. [BB] $(a,b)^c = (-\infty,a] \cup [b,\infty)$, $[a,b)^c = (-\infty,a) \cup [b,\infty)$, $(a,\infty)^c = (-\infty,a]$,
   $(-\infty,b]^c = (b,\infty)$.

10. (a) [BB] $CS \subseteq T$; (b) [BB] $M \cap P = \emptyset$; (c) $M \not\subseteq P$; (d) $CS \setminus T \subseteq P$;
    (e) $(M \cup CS) \cap P \subseteq T^c$.

11. (a) Negation: $CS \not\subseteq T$; Converse: $T \subseteq CS$.
    (b) Negation: $M \cap P \neq \emptyset$; Converse: $P \subseteq M^c$ or $P \cap M = \emptyset$,

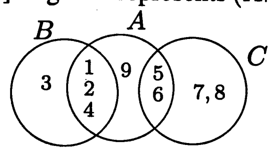
(c) Negation: $M \subseteq P$; No converse since the statement is not an implication.
(d) Negation: $(M \cup CS) \cap P \not\subseteq T^c$; Converse: $T^c \subseteq (M \cup CS) \cap P$.
(e) Negation: $(M \cup CS) \cap P \cap T \neq \emptyset$; Converse $T^c \subseteq (M \cup CS) \cap P$.

12. (a) [BB] $\mathsf{E} \cap \mathsf{P} \neq \emptyset$ (b) $0 \in \mathsf{Z} \setminus \mathsf{N}$ (c) $\mathsf{N} \subseteq \mathsf{Z}$
(d) $\mathsf{Z} \not\subseteq \mathsf{N}$ (e) $(\mathsf{P} \setminus \{2\}) \subseteq \mathsf{E}^c$ (f) $2 \in \mathsf{E} \cap \mathsf{P}$
(g) $\mathsf{E} \cap \mathsf{P} = \{2\}$

13. (a) [BB] Since $A_{-3} \subseteq A_3$, $A_3 \cup A_{-3} = A_3$.
(b) Since $A_{-3} \subseteq A_3$, $A_3 \cap A_{-3} = A_{-3}$.
(c) $A_3 \cap (A_{-3})^c = \{a \in \mathsf{Z} \mid -3 < a \leq 3\} = \{-2, -1, 0, 1, 2, 3\}$.
(d) Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq A_4$, we have $\bigcap_{i=0}^{4} A_i = A_0$.

14. [BB] Region 2 represents $(A \cap C) \setminus B$. Region 3 represents $A \cap B \cap C$; region 4 represents $(A \cap B) \setminus C$.

15. (a)

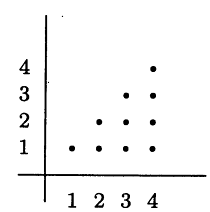


(b) i. $(A \cup B) \cap C = \{5, 6\}$
ii. $A \setminus (B \setminus A) = A = \{1, 2, 4, 7, 8, 9\}$
iii. $(A \cup B) \setminus (A \cap C) = \{1, 2, 3, 4, 9\}$
iv. $A \oplus C = \{1, 2, 4, 7, 8, 9\}$
v. $(A \cap C) \times (A \cap B) = \{(5, 1), (5, 2), (5, 4), (6, 1), (6, 2), (6, 4)\}$

16. (a) [BB] $A \subseteq B$, by Problem 7; (b) $B \subseteq A$, by PAUSE 4 with A and B reversed.

17. [BB] Think of listing the elements of the given set. There are n pairs of the form $(1, b)$, $n - 1$ pairs of the form $(2, b)$, $n - 2$ pairs of the form $(3, b)$, and so on until finally we list the only pair of the form (n, b). The answer is $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$.

18. Since $(1, 1) \in A$, $(2, 1)$ and $(2, 2)$ are in A. Since $(2, 1) \in A$, we get $(3, 1)$ and $(3, 2)$ in A and since $(2, 2) \in A$, $(3, 3) \in A$. Now $(3, 1) \in A \to (4, 1)$ and $(4, 2) \in A$; $(3, 2) \in A \to (4, 3) \in A$ and $(3, 3) \in A \to (4, 4) \in A$. The points shown so far which belong to A are plotted in the picture to the right and this makes it seem very plausible that A contains the set $\{(m, n) \in \mathsf{N} \times \mathsf{N} \mid m \geq n\}$.



19. (a) Let $x \in B$. Certainly x is also in A or in A^c. This suggests cases.
Case 1: If $x \in A$, then $x \in A \cap B$, so $x \in C$.
Case 2: If $x \notin A$, then $x \in A^c \cap B$, so $x \in C$.
In either case, $x \in C$, so $B \subseteq C$.
(b) [BB] Yes. Given $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, certainly we have $A \cap B \subseteq C$ and $A^c \cap B \subseteq C$ so, from (a), we have that $B \subseteq C$. Reversing the roles of B and C in (a), we can also conclude that $C \subseteq B$; hence, $B = C$.

20. (a) The Venn diagram shown in Fig. 2.1 suggests the following counterexample: Let $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$ and $C = \{2, 3, 5, 7\}$. Then $A \cup (B \cap C) = A \cup \{3, 5\} = \{1, 2, 3, 4, 5\}$ whereas $(A \cup B) \cap C = \{1, 2, 3, 4, 5, 6\} \cap C = \{2, 3, 5\}$.

(b) First, we prove $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.

So let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$, then $x \in A \cup B$ and $x \in A \cup C$, so $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in B$ and $x \in C$ so $x \in A \cup B$ and $x \in A \cup C$; that is, $x \in (A \cup B) \cap (A \cup C)$. In either case, $x \in (A \cup B) \cap (A \cup C)$ giving the desired inclusion. Second, we prove $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

So let $x \in (A \cup B) \cap (A \cup C)$. Thus, $x \in A \cup B$ and $x \in A \cup C$. If $x \in A$, then $x \in A \cup (B \cap C)$. If $x \notin A$, then we must have $x \in B$ and $x \in C$; that is, $x \in B \cap C$, so $x \in A \cup (B \cap C)$. In either case, $x \in A \cup (B \cap C)$ giving the desired inclusion and equality.

21. We use the fact that $(X^c)^c = X$ for any set X.

Let $X = A^c$ and $Y = B^c$. Then $A = X^c$ and $B = Y^c$, so $(A \cap B)^c = [X^c \cap Y^c]^c = [(X \cup Y)^c]^c$ (by the first law of De Morgan) $= X \cup Y = A^c \cup B^c$, as required.

22. [BB] Using the fact that $X \setminus Y = X \cap Y^c$, we have

$$(A \setminus B) \setminus C = (A \cap B^c) \cap C^c = A \cap (B^c \cap C^c) = A \cap (B \cup C)^c = A \setminus (B \cup C).$$

23. We use the laws of De Morgan and the facts that $(X^c)^c = X$ and $X \cap X^c = \emptyset$ for any set X. We have $[(A \cup B)^c \cap (A^c \cup C)^c]^c \setminus D^c = [(A^c \cap B^c) \cap (A \cap C^c)]^c \setminus D^c = \emptyset^c \setminus D^c = U \setminus D^c = U \cap (D^c)^c = U \cap D = D$.

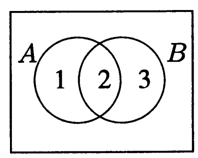
24. $A \setminus (B \setminus C) = A \setminus (B \cap C^c) = A \cap (B \cap C^c)^c = A \cap (B^c \cup C) = (A \cap B^c) \cup (A \cap C) = (A \cap B^c) \cup (A \cap (C^c)^c) = (A \setminus B) \cup (A \setminus C^c)$.

25. (a) [BB] $(A \cup B \cup C)^c = [A \cup (B \cup C)]^c = A^c \cap (B \cup C)^c = A^c \cap (B^c \cap C^c) = A^c \cap B^c \cap C^c$.

$(A \cap B \cap C)^c = [A \cap (B \cap C)]^c = A^c \cup (B \cap C)^c = A^c \cup (B^c \cup C^c) = A^c \cup B^c \cup C^c$.

(b)
$$\begin{aligned}(A \cap (B \setminus C))^c \cap A &= (A \cap B \cap C^c)^c \cap A = (A^c \cup B^c \cup C) \cap A \\ &= (A \cap (A^c \cup B^c)) \cup (A \cap C) = (A \cap A^c) \cup (A \cap B^c) \cup (A \cap C) \\ &= \emptyset \cup (A \cap B^c) \cup (A \cap C) = (A \cap B^c) \cup (A \cap C) \\ &= (A \setminus B) \cup (A \cap C)\end{aligned}$$

26. (a) [BB] Looking at the Venn diagram at the right, $A \oplus B$ consists of the points in regions 1 and 3. To have $A \oplus B = A$, we must have both regions 2 and 3 empty; that is, $B = \emptyset$. On the other hand, since $A \oplus \emptyset = A$, this condition is necessary and sufficient.



(b) Looking at the Venn diagram, $A \cap B$ is the set of points in region 2 while $A \cup B$ is the set of points in regions 1, 2 and 3. Hence, $A \cap B = A \cup B$ if and only if regions 1 and 3 are both empty; that is, if and only if $A = B$.

27. (a) [BB] This does not imply $B = C$. For example, let $A = \{1, 2\}$, $B = \{1\}$, $C = \{2\}$. Then $A \cup B = A \cup C$, but $B \neq C$.

(b) This does not imply $B = C$. For example, let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 3\}$. Then $A \cap B = A \cap C = A$, but $B \neq C$.

(c) This does imply $B = C$, and here is a proof. First let $b \in B$. Then, in addition, either $b \in A$ or $b \notin A$.

Case 1: $b \notin A$

In this case, $b \in A \oplus B$, so $b \in A \oplus C$ and since $b \notin A$ it follows that $b \in C$.

Case 2: $b \in A$. Here we have $b \in B \cap A$ and, hence, $b \notin A \oplus B$, so $b \notin A \oplus C$. Since $b \in A$, we must have $b \in C$ (otherwise, $b \in A \setminus C \subseteq A \oplus C$).

In either case, we obtain $b \in C$. It follows that $B \subseteq C$. A similar argument shows $C \subseteq B$ and, hence, $C = B$.

(d) This is false since for $A = \emptyset$, $A \times B = A \times C = \emptyset$ regardless of B and C.

28. (a) True. Let $(a, b) \in A \times B$. Since $a \in A$ and $A \subseteq C$, we have $a \in C$. Since $b \in B$ and $B \subseteq D$, $b \in D$. Thus, $(a, b) \in C \times D$ and $A \times B \subseteq C \times D$.

(b) False: Consider $A = \{1\}$, $B = \{2, 3\}$, $C = \{1, 2, 3\}$.

(c) False. Let $A = \{1\}$, $B = \emptyset$, $C = \{2\}$, $D = \{3\}$. Then $A \times B = \emptyset \subseteq \{(2, 3)\} = C \times D$, but $A \not\subseteq C$.

(d) False since, by (b), the implication $\leftarrow$ is false.

(e) [BB] True. Let $x \in A$. Then $x \in A \cup B$, so $x \in A \cap B$ and, in particular, $x \in B$. Thus, $A \subseteq B$. Similarly, we have $B \subseteq A$, so $A = B$.

29. Let $(x, y) \in (A \cap B) \times C$. This means $x \in A \cap B$ and $y \in C$. Hence, $x \in A$, $x \in B$, $y \in C$. Thus, $(x, y) \in A \times C$ and $(x, y) \in B \times C$; i.e., $(x, y) \in (A \times C) \cap (B \times C)$. Therefore, $(A \cap B) \times C \subseteq (A \times C) \cap (B \times C)$.

Now let $(x, y) \in (A \times C) \cap (B \times C)$. This means that $(x, y) \in A \times C$ and $(x, y) \in B \times C$); that is, $x \in A$, $x \in B$, $y \in C$, so $x \in A \cap B$ and $y \in C$. Hence, $(x, y) \in (A \cap B) \times C$. Therefore, $(A \times C) \cap (B \times C) \subseteq (A \cap B) \times C$ and we have equality, as desired.

30. (a) False. For example, let $A = \{1, 2\}$, $B = \{1\}$ and $C = \{2\}$. Then $A \setminus (B \cup C) = \{1, 2\} \setminus \{1, 2\} = \emptyset$, but $(A \setminus B) \cup (A \setminus C) = \{2\} \cup \{1\} = \{1, 2\}$.

(b) True. Let $(x, y) \in (A \setminus B) \times C$. This means that $x \in A \setminus B$ and $y \in C$; that is, $x \in A$, $x \notin B$, $y \in C$. Hence, $(x, y) \in A \times C$, but $(x, y) \notin B \times C$, so $(x, y) \in (A \times C) \setminus (B \times C)$. Therefore, $(A \setminus B) \times C \subseteq (A \times C) \setminus (B \times C)$.

Now let $(x, y) \in (A \times C) \setminus (B \times C)$. This means that $(x, y) \in A \times C$, but $(x, y) \notin B \times C$. Since $(x, y) \in A \times C$, we have $x \in A$, $y \in C$. Since $y \in C$ and $(x, y) \notin B \times C$, we must have $x \notin B$; that is, $x \in A \setminus B$, $y \in C$, so $(x, y) \in (A \setminus B) \times C$. Therefore, $(A \times C) \setminus (B \times C) \subseteq (A \setminus B) \times C$ and we have equality as claimed.

(c) [BB] True. Let $(x, y) \in (A \oplus B) \times C$. This means that $x \in A \oplus B$ and $y \in C$; that is, $x \in A \cup B$, $x \notin A \cap B$, $y \in C$. If $x \in A$, then $x \notin B$, so $(x, y) \in (A \times C) \setminus (B \times C)$. If $x \in B$, then $x \notin A$, so $(x, y) \in (B \times C) \setminus (A \times C)$. In either case, $(x, y) \in (A \times C) \oplus (B \times C)$. So $(A \oplus B) \times C \subseteq (A \times C) \oplus (B \times C)$.

Now, let $(x, y) \in (A \times C) \oplus (B \times C)$. This means that $(x, y) \in (A \times C) \cup (B \times C)$, but $(x, y) \notin (A \times C) \cap (B \times C)$. If $(x, y) \in A \times C$, then $(x, y) \notin B \times C$, so $x \in A$, $y \in C$ and, therefore, $x \notin B$. If $(x, y) \in B \times C$, then $(x, y) \notin A \times C$, so $x \in B$, $y \in C$ and, therefore, $x \notin A$. In either case, $x \in A \oplus B$ and $y \in C$, so $(x, y) \in (A \oplus B) \times C$. Therefore, $(A \times C) \oplus (B \times C) \subseteq (A \oplus B) \times C$ and we have equality, as claimed.

(d) False. Let $A = \{1\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$. Then $1 \in A \cup B$, $4 \in C \cup D$, so $(1, 4) \in (A \cup B) \times (C \cup D)$. But $(1, 4) \notin A \times C$ and $(1, 4) \notin B \times D$, so $(1, 4) \notin (A \times C) \cup (B \times D)$.

(e) False. Let $A = \{1, 2\}$, $B = \{2\}$, $C = \{3\}$, $D = \{4\}$. Then, since $3 \notin D$, $(2, 3) \in (A \times C) \setminus (B \times D)$. However, because $2 \in B$, $2 \notin A \setminus B$, so $(2, 3) \notin (A \setminus B) \times (C \setminus D)$.

31. George Boole (1815-1864) was one of the greatest mathematicians of the nineteenth century. He was the first Professor of Mathematics at University College Cork (then called Queen's College) and is best known today as the inventor of a subject called *mathematical logic*. Indeed he introduced much of the symbolic language and notation we use today. Like Charles Babbage and Alan Turing, Boole also had a great impact in computer science, long before the computer was even a dream. He invented an *algebra of logic* known as Boolean Algebra, which is used widely today and forms the basis of much of the internal logic of computers. His books, "The Mathematical Analysis of Logic" and "An Investigation of the Laws of Thought" form the basis of present-day computer science.

Exercises 2.3

1. [BB] $S \times B$ is the set of ordered pairs (s, b), where s is a student and b is a book; thus, $S \times B$ represents all possible pairs of students and books. One sensible example of a binary relation is $\{(s, b) \mid s \text{ has used book } b\}$.

2. $A \times B$ is the set of all ordered pairs (a, b) where a is a street and b is a person. One binary relation would be $\{(a, b) \mid b \text{ lives on street } a\}$.

3. (a) [BB] not reflexive, not symmetric, not transitive.
   (b) (in most cases) reflexive, (in somewhat fewer cases) symmetric, certainly not transitive!
   (c) [BB] not reflexive, not symmetric, but it is transitive.
   (d) reflexive, symmetric, transitive.
   (e) not reflexive, not symmetric, not transitive

4. (a) [BB]

	a	b	c	d
a	×	×		
b	×	×		
c			×	
d				×

(b)

	a	b	c	d
a	×	×	×	
b		×		
c	×		×	
d				

(c)

	a	b	c	d
a	×	×	×	
b				
c			×	×
d				×

(d)

	a	b	c	d
a	×	×	×	×
b		×	×	×
c			×	
d			×	×

5. (a) [BB] $\{(1,1),(1,2),(2,3)\}$; (b) $\{(1,1),(2,2),(3,3),(1,2),(2,3)\}$;
   (c) $\{(1,2),(2,3),(2,1),(3,2)\}$; (d) $\{(1,2),(1,3),(2,3)\}$;
   (e) $\{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}$;
   (f) $\{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$; (g) [BB] $\{(1,2),(2,1),(1,1),(2,2)\}$;
   (h) $\{(1,1),(2,2),(3,3),(1,2),(2,3),(1,3),(2,1),(3,2),(3,1)\}$.

6. The answer is yes and the only such binary relations are subsets of the equality binary relation. To see why, let $\mathcal{R}$ be a binary relation on a set A which is both symmetric and antisymmetric. Let $(a, b) \in \mathcal{R}$. Then $(b, a) \in \mathcal{R}$ by symmetry, so $a = b$ by antisymmetry.

7. [BB] The argument assumes that for $a \in \mathcal{R}$ there exists a b such that $(a, b) \in \mathcal{R}$. This need not be the case: See Exercise 5(g).

8. (a) [BB] **Reflexive**: Every word has at least one letter in common with itself.
   Symmetric: If a and b have at least one letter in common, then so do b and a.
   Not antisymmetric: (cat, dot) and (dot, cat) are both in the relation but dot $\neq$ cat!!
   Not transitive: (cat, dot) and (dot, mouse) are both in the relation but (cat, mouse) is not.

   (b) **Reflexive**: Let a be a person. If a is not enrolled at Miskatonic University, then $(a, a) \in \mathcal{R}$. On the other hand, if a is enrolled at MU, then a is taking at least one course with himself, so again $(a, a) \in \mathcal{R}$.
   Symmetric: If $(a, b) \in \mathcal{R}$, then either it is the case that neither a nor b is enrolled at MU (so neither is b or a, hence, $(b, a) \in \mathcal{R}$) or it is the case that a and b are both enrolled and are taking at least one course together (in which case b and a are enrolled and taking a common course, so $(b, a) \in \mathcal{R}$). In any case, if $(a, b) \in \mathcal{R}$, then $(b, a) \in \mathcal{R}$.
   Not antisymmetric: If a and b are two different students in the same class at Miskatonic University, then $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$, but $a \neq b$.
   At most universities, this is not a transitive relation. Let a, b and c be three students enrolled at MU such that a and b are enrolled in some course together and b and c are enrolled in some (other) course together, but a and c are taking no courses together. Then (a, b) and (b, c) are in $\mathcal{R}$ but $(a, c) \notin \mathcal{R}$.

9. (a) **Not reflexive**: $(1, 1) \notin \mathcal{R}$.
   Not symmetric: $(1, 2) \in \mathcal{R}$ but $(2, 1) \notin \mathcal{R}$.
   Antisymmetric: It is never the case that for two different elements a and b in A we have both (a, b) and (b, a) in $\mathcal{R}$.
   Transitive vacuously; that is, there exists no counterexample to disprove transitivity: The situation $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$ never occurs.

   (b) [BB] **Not reflexive**: $(2, 2) \notin \mathcal{R}$.
   Not symmetric: $(3, 4) \in \mathcal{R}$ but $(4, 3) \notin \mathcal{R}$.
   Not antisymmetric: $(1, 2)$ and $(2, 1)$ are both in $\mathcal{R}$ but $1 \neq 2$.
   Not transitive: $(2, 1)$ and $(1, 2)$ are in $\mathcal{R}$ but $(2, 2)$ is not.

   (c) [BB] **Reflexive**: For any $a \in \mathsf{Z}$, it is true that $a^2 \geq 0$. Thus, $(a, a) \in \mathcal{R}$.
   Symmetric: If $(a, b) \in \mathcal{R}$, then $ab \geq 0$, so $ba \geq 0$ and hence, $(b, a) \in \mathcal{R}$.
   Not antisymmetric: $(5, 2) \in \mathcal{R}$ because $5(2) = 10 \geq 0$ and similarly $(2, 5) \in \mathcal{R}$, but $5 \neq 2$.
   Not transitive: $(5, 0) \in \mathcal{R}$ because $5(0) = 0 \geq 0$ and similarly, $(0, -6) \in \mathcal{R}$; however, $(5, -6) \notin \mathcal{R}$ because $5(-6) \not\geq 0$.

   (d) **Reflexive**: For any $a \in \mathsf{R}$, $a^2 = a^2$, so $(a, a) \in \mathcal{R}$.
   Symmetric: If $(a, b) \in \mathcal{R}$ then $a^2 = b^2$, so $b^2 = a^2$ which says that $(b, a) \in \mathcal{R}$.
   Not antisymmetric: $(1, -1) \in \mathcal{R}$ and $(-1, 1) \in \mathcal{R}$ but $1 \neq -1$.
   Transitive: If (a, b) and (b, c) are both in $\mathcal{R}$, then $a^2 = b^2$ and $b^2 = c^2$, so $a^2 = c^2$ which says $(a, c) \in \mathcal{R}$.

   (e) **Reflexive**: For any $a \in \mathsf{R}$, $a - a = 0 \leq 3$ and so $(a, a) \in \mathcal{R}$.
   Not symmetric: For example, $(0, 7) \in \mathcal{R}$ because $0 - 7 = -7 \leq 3$, but $(7, 0) \notin \mathcal{R}$ because $7 - 0 = 7 \not\leq 3$.

Not antisymmetric: $(2,1) \in \mathcal{R}$ because $2-1=1 \leq 3$ and $(1,2) \in \mathcal{R}$ because $1-2=-1 \leq 3$, but $1 \neq 2$.

Not transitive: $(5,3) \in \mathcal{R}$ because $5-3=2 \leq 3$ and $(3,1) \in \mathcal{R}$ because $3-1=2 \leq 3$, but $(5,1) \notin \mathcal{R}$ because $5-1=4 \not\leq 3$.

(f) **Reflexive**: For any $(a,b) \in A$, $a-a=b-b$; thus, $((a,b),(a,b)) \in \mathcal{R}$.

Symmetric: If $((a,b),(c,d)) \in \mathcal{R}$, then $a-c=b-d$, so $c-a=d-b$ and, hence, $((c,d),(a,b)) \in \mathcal{R}$.

Not antisymmetric: $((5,2),(15,12)) \in \mathcal{R}$ because $5-15=2-12$ and similarly, $((15,12),(5,2)) \in \mathcal{R}$; however, $(15,12) \neq (5,2)$.

If $((a,b),(c,d)) \in \mathcal{R}$ and $((c,d),(e,f)) \in \mathcal{R}$ then $a-c=b-d$ and $c-e=d-f$. Thus, $a-e=(a-c)+(c-e)=(b-d)+(d-f)=b-f$ and so $((a,b),(e,f)) \in \mathcal{R}$.

(g) **Not reflexive**: If $n \in \mathsf{N}$, then $n \neq n$ is not true.

Symmetric: If $n_1 \neq n_2$, then $n_2 \neq n_1$.

Not antisymmetric: $1 \neq 2$ and $2 \neq 1$ so both $(1,2)$ and $(2,1)$ are in $\mathcal{R}$, yet $1 \neq 2$.

Not transitive: $1 \neq 2$, $2 \neq 1$, but $1 = 1$.

(h) **Not reflexive**: $(2,2) \notin \mathcal{R}$ because $2+2 \neq 10$.

Symmetric: If $(x,y) \in \mathcal{R}$, then $x+y=10$, so $y+x=10$, and hence, $(y,x) \in \mathcal{R}$.

Not antisymmetric: $(6,4) \in \mathcal{R}$ because $6+4=10$ and similarly, $(4,6) \in \mathcal{R}$, but $6 \neq 4$.

Not transitive: $(6,4) \in \mathcal{R}$ because $6+4=10$ and similarly, $(4,6) \in \mathcal{R}$, but $(6,6) \notin \mathcal{R}$ because $6+6 \neq 10$.

(i) [BB] **Reflexive**: If $(x,y) \in \mathsf{R}^2$, then $x+y \leq x+y$, so $((x,y),(x,y)) \in \mathcal{R}$.

Not symmetric: $((1,2),(3,4)) \in \mathcal{R}$ since $1+2 \leq 3+4$, but

$((3,4),(1,2)) \notin \mathcal{R}$ since $3+4 \not\leq 1+2$.

Not antisymmetric: $((1,2),(0,3)) \in \mathcal{R}$ since $1+2 \leq 0+3$

and $((0,3),(1,2)) \in \mathcal{R}$ since $0+3 \leq 1+2$, but $(1,2) \neq (0,3)$.

Transitive: If $((a,b),(c,d))$ and $((c,d),(e,f))$ are both in $\mathcal{R}$, then $a+b \leq c+d$ and $c+d \leq e+f$, so $a+b \leq e+f$ (by transitivity of $\leq$) which says $((a,b),(e,f)) \in \mathcal{R}$.

(j) **Reflexive**: $\frac{a}{a} = 1 \in \mathsf{N}$ for any $a \in \mathsf{N}$.

Not symmetric: $(4,2) \in \mathcal{R}$ but $(2,4) \notin \mathcal{R}$.

Antisymmetric: If $\frac{a}{b} = n$ and $\frac{b}{a} = m$ are integers then $nm = 1$ so $n, m \in \{\pm 1\}$. Since a and b are positive, so are n and m. Therefore, $n = m = 1$ and $a = b$.

Transitive: The argument given in Example 24 for Z works the same way for N.

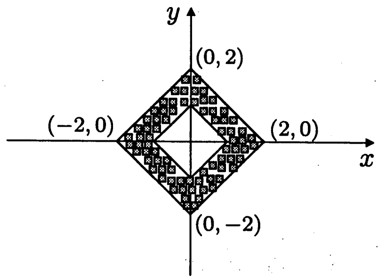
(k) **Not reflexive**: $\frac{0}{0}$ is not defined, let alone an integer!

Not symmetric: As before.

Not antisymmetric: $(4,-4)$ and $(-4,4)$ are both in $\mathcal{R}$.

Transitive: As shown in Example 24.

10. (a)



(b) The relation is not reflexive because, for example, $(2,2) \notin \mathcal{R}$. It is not transitive because, for example, $(2,0) \in \mathcal{R}$ and $(0,1) \in \mathcal{R}$ but $(2,1) \notin \mathcal{R}$.

(c) The relation is symmetric since if $(x,y) \in \mathcal{R}$, then $1 \leq |x| + |y| \leq 2$, so $1 \leq |y| + |x| \leq 2$, so $(y,x) \in \mathcal{R}$. It is not antisymmetric since, for example, $(0,1) \in \mathcal{R}$ and $(1,0) \in \mathcal{R}$, but $0 \neq 1$.

11. (a) [BB] **Reflexive**: For any set X, we have $X \subseteq X$.
Not symmetric: Let $a,b \in S$. Then $\{a\} \subseteq \{a,b\}$ but $\{a,b\} \not\subseteq \{a\}$.
Antisymmetric: If $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$.
Transitive: If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

(b) **Not reflexive**: For no set X is it true that $X \subsetneq X$.
Not symmetric: As before.
Antisymmetric "vacuously": It is impossible for $X \subsetneq Y$ and $Y \subsetneq X$. (Recall that an implication is false only when the hypothesis is true and the conclusion is false.)
Transitive: As before.

(c) **Not reflexive**: Since $S \neq \emptyset$, there is some element $a \in S$, and so some set $X = \{a\} \neq \emptyset \in \mathcal{P}(S)$. For this X, however, $X \cap X = X \neq \emptyset$, so $(X,X) \notin \mathcal{R}$.
Symmetric: If $(X,Y) \in \mathcal{R}$, then $X \cap Y = \emptyset$, so $Y \cap X = \emptyset$, hence, $(Y,X) \in \mathcal{R}$.
Not antisymmetric: Let a,b be two elements in S and let $X = \{a\}$, $Y = \{b\}$. Then $(X,Y) \in \mathcal{R}$ and $(Y,X) \in \mathcal{R}$, but $X \neq Y$.
Not transitive: Let a,b be two elements in S and let $X = \{a\}$, $Y = \{b\}$, $Z = \{a\}$. Then $(X,Y) \in \mathcal{R}$, $(Y,Z) \in \mathcal{R}$, but $(X,Z) \notin \mathcal{R}$.

12. (a) [BB] **Reflexive**: Any book has price $\geq$ its own price and length $\geq$ its own length, so $(a,a) \in \mathcal{R}$ for any book a.
Not symmetric: $(Y,Z) \in \mathcal{R}$ because the price of Y is greater than the price of Z and the length of Y is greater than the length of Z, but for these same reasons, $(Z,Y) \notin \mathcal{R}$.
Antisymmetric: If (a,b) and (b,a) are both in $\mathcal{R}$, then a and b must have the same price and length. This is not the case here unless $a = b$.
Transitive: If (a,b) and (b,c) are in $\mathcal{R}$, then the price of a is $\geq$ the price of b and the price of b is $\geq$ the price of c, so the price of a is $\geq$ the price of c. Also the length of a is $\geq$ the length of b and the length of b is $\geq$ the length of c, so the length of a is $\geq$ the length of c. Hence, $(a,c) \in \mathcal{R}$.

(b) **Reflexive**: For any book a, the price of a is $\geq$ the price of a so $(a,a) \in \mathcal{R}$. (One could also use a similar argument concerning length.)
Not symmetric: As in part (a), $(Y,Z) \in \mathcal{R}$, but $(Z,Y) \notin \mathcal{R}$.
Not antisymmetric: $(W,X) \in \mathcal{R}$ because the price of W is greater than or equal to the price of X, and $(X,W) \in \mathcal{R}$ because the length of W is greater than or equal to the length of X, but $W \neq X$.
Not transitive: $(Z,U) \in \mathcal{R}$ because the length of Z is $\geq$ the length of U and $(U,Y) \in \mathcal{R}$ because the price of U is $\geq$ the price of Y, but $(Z,Y) \notin \mathcal{R}$ because neither is the price of $Z \geq$ the price of Y nor is the length of $Z \geq$ the length of Y.

13. Now the second binary relation would have an extra term, {Mike, 120}, and the third would have the extra term, {Pippy Park, 120}. But, in addition, the entry {Pippy Park, 74} would be deleted. So Mike is now clearly identified as the one who shot 120, and Pippy Park is where that occurred. Hence, Mike's round of 74 was at Clovelly. Since Edgar has only one entry in binary relation two, he must have shot 72 at both courses. Finally, Bruce's 74 must have been at Clovelly and hence his 72 was at Pippy Park. All information has been retrieved in this case.

Exercises 2.4

1. **Reflexive:** For any citizen a of New York City, either a does not own a cell phone (in which case $a \sim a$) or a has a cell phone and a's exchange is the same as a's exchange (in which case again $a \sim a$).

   Symmetric: If $a \sim b$ and a does not have a cell phone, then neither does b, so $b \sim a$; on the other hand, if a does have a cell phone, then so does b and their exchanges are the same, so again, $b \sim a$.

   Transitive: Suppose $a \sim b$ and $b \sim c$. If a does not have a cell phone, then neither does b and, since $b \sim c$, neither does c, so $a \sim c$. On the other hand, if a does have a cell phone then so does b and a's and b's exchanges are the same. Since $b \sim c$, c has a cell phone with the same exchange as b. It follows that a and c have the same exchange and so, in this case as well, $a \sim c$.

   There is one equivalence class consisting of all residents of New York who do not own a cell phone and one equivalence class for each New York City exchange consisting of all residents who have cell phones in that exchange.

2. (a) [BB] This is not reflexive: $(2,2) \notin \mathcal{R}$.

   (b) This is not symmetric: $(2,3) \in \mathcal{R}$ but $(3,2) \notin \mathcal{R}$.
   It would also be acceptable to not that $\mathcal{R}$ is not transitive. $(3,1) \in \mathcal{R}$ and $(1,2) \in \mathcal{R}$, but $(3,2) \notin \mathcal{R}$.

   (c) This is not symmetric: $(1,3)$ is in the relation but $(3,1)$ is not.
   It would also be acceptable to note that this relation is not transitive: $(2,1) \in \mathcal{R}$, $(1,3) \in \mathcal{R}$, but $(2,3) \notin \mathcal{R}$.

3. [BB] Equality! The equivalence classes specify that $x \sim y$ if and only if $x = y$.

4. (a) **Reflexive**: If $a \in S$, then a and a have the same number of elements, so $a \sim a$.
   Symmetric: If $a \sim b$, then a and b have the same number of elements, so b and a have the same number of elements. Thus $b \sim a$.
   Transitive: If $a \sim b$ and $b \sim c$, then a and b have the same number of elements, and b and c have the same number of elements, so a and c have the same number of elements. Thus $a \sim c$.

   (b) There are **seven** equivalence classes, represented by $\emptyset$, $\{1\}$, $\{1,2\}$, $\{1,2,3\}$, $\{1,2,3,4\}$, $\{1,2,3,4,5\}$, $\{1,2,3,4,5,6\}$.

5. (a) [BB] **Reflexive:** If $a \in \mathsf{R} \setminus \{0\}$, then $a \sim a$ because $\frac{a}{a} = 1 \in \mathsf{Q}$.
   Symmetric: If $a \sim b$, then $\frac{a}{b} \in \mathsf{Q}$ and this fraction is not zero (because $0 \notin A$). So it can be inverted and we see that $\frac{b}{a} = 1/\frac{a}{b} \in \mathsf{Q}$ too. Therefore, $b \sim a$.
   Transitive: If $a \sim b$ and $b \sim c$, then $\frac{a}{b} \in \mathsf{Q}$ and $\frac{b}{c} \in \mathsf{Q}$. Since the product of rational numbers is rational, $\frac{a}{c} = \frac{a}{b}\frac{b}{c}$ is in Q, so $a \sim c$.

(b) [BB] $\overline{1} = \{a \mid a \sim 1\} = \{a \mid \frac{a}{1} \in \mathsf{Q}\} = \{a \mid a \in \mathsf{Q}\} = \mathsf{Q} \setminus \{0\}$.

(c) [BB] $\frac{\sqrt{12}}{\sqrt{3}} = \frac{2\sqrt{3}}{\sqrt{3}} = 2 \in \mathsf{Q}$, so $\sqrt{3} \sim \sqrt{12}$ and hence $\overline{\sqrt{3}} = \overline{\sqrt{12}}$.

6. **Reflexive:** For any $a \in \mathsf{N}$, $a \sim a$ since $a^2 + a = a(a+1)$ is even, as the product of consecutive natural numbers.

Symmetric: If $a \sim b$, then $a^2 + b$ is even. It follows that either a and b are both even or both are odd. If they are both even, $b^2 + a$ is the sum of even numbers, hence, even. If they are both odd, $b^2 + a$ is the sum of odd numbers and, hence, again, even. In both cases $b^2 + a$ is even, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $a^2 + b$ and $b^2 + c$ are even, so $(a^2 + b) + (b^2 + c)$ is even; in other words, $(a^2 + c) + (b^2 + b)$ is even. Since $b^2 + b$ is even, $a^2 + c$ is even too; therefore, $a \sim c$.

The quotient set is the set of equivalence classes. Now

$$\overline{a} = \{x \mid x^2 + a \text{ is even}\} = \begin{cases} \text{evens} & \text{if } a \text{ is even} \\ \text{odds} & \text{if } a \text{ is odd} \end{cases}$$

So $A/\!\sim = \{2\mathsf{Z}, 2\mathsf{Z}+1\}$.

7. (a) [BB] **Reflexive:** For any $a \in \mathsf{R}$, $a \sim a$ because $a - a = 0 \in \mathsf{Z}$.
Symmetric: If $a \sim b$, then $a - b \in \mathsf{Z}$, so $b - a \in \mathsf{Z}$ (because $b - a = -(a-b)$) and, hence, $b \sim a$.
Transitive: If $a \sim b$ and $b \sim c$, then both $a - b$ and $b - c$ are integers; hence, so is their sum, $(a - b) + (b - c) = a - c$. Thus, $a \sim c$.

(b) [BB] The equivalence class of 5 is $\overline{5} = \{x \in \mathsf{R} \mid x \sim 5\} = \{x \mid x - 5 \in \mathsf{Z}\} = \mathsf{Z}$, because $x - 5 \in \mathsf{Z}$ implies $x \in \mathsf{Z}$.

$$\begin{aligned} \overline{5\tfrac{1}{2}} &= \{x \in \mathsf{R} \mid x \sim 5\tfrac{1}{2}\} \\ &= \{x \mid x - 5\tfrac{1}{2} \in \mathsf{Z}\} \\ &= \{x \mid x = 5\tfrac{1}{2} + k, \text{ for some } k \in \mathsf{Z}\} \\ &= \{x \mid x = 5 + k + \tfrac{1}{2}, \text{ for some } k \in \mathsf{Z}\} \\ &= \{x \mid x = n + \tfrac{1}{2}, \text{ for some } n \in \mathsf{Z}\} \end{aligned}$$

(c) [BB] For each $a \in \mathsf{R}$, $0 \le a < 1$, there is one equivalence class,

$$\overline{a} = \{x \in \mathsf{R} \mid x = a + n \text{ for some integer } n\}.$$

The quotient set is $\{\overline{a} \mid 0 \le a < 1\}$.

8. [BB] **Reflexive:** For any $a \in \mathsf{Z}$, $a \sim a$ because $2a + 3a = 5a$.

Symmetric: If $a \sim b$, then $2a + 3b = 5n$ for some integer n. So $2b + 3a = (5a + 5b) - (2a + 3b) = 5(a + b) - 5n = 5(a + b - n)$. Since $a + b - n$ is an integer, $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $2a + 3b = 5n$ and $2b + 3c = 5m$ for integers n and m. Therefore, $(2a + 3b) + (2b + 3c) = 5(n + m)$ and $2a + 3c = 5(n + m) - 5b = 5(n + m - b)$. Since $n + m - b$ is an integer, $a \sim c$.

9. (a) **Reflexive:** For any $a \in \mathsf{Z}$, $3a + a = 4a$ is a multiple of 4, so $a \sim a$.
Symmetric: If $a \sim b$, then $3a + b = 4k$ for some integer k. Since $(3a + b) + (3b + a) = 4(a + b)$, we see that $3b + a = 4(a + b) - 4k$ is a multiple of 4, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $3a + b = 4k$ for some integer k and $3b + c = 4\ell$ for some integer ℓ. Since $4(k+\ell) = (3a+b)+(3b+c) = (3a+c)+4b$, we see that $3a+c = 4(k+\ell)-4b$ is a multiple of 4 and, hence, that $a \sim c$.

(b) $\overline{0} = \{x \in \mathsf{Z} \mid x \sim 0\} = \{x \mid 3x = 4k \text{ for some integer } k\}$. Now if $3x = 4k$, k must be a multiple of 3. So $3x = 12\ell$ for some $\ell \in \mathsf{Z}$ and $x = 4\ell$. $\overline{0} = 4\mathsf{Z}$.

(c) $\overline{2} = \{x \in \mathsf{Z} \mid x \sim 2\} = \{x \mid 3x + 2 = 4k \text{ for some integer } k\} = \{x \mid 3x = 4k - 2 \text{ for some integer } k\}$ Now if $3x = 4k - 2$, then $3x = 3k + k - 2$ and so $k - 2$ is a multiple of 3. Therefore, $k = 3\ell + 2$ for some integer ℓ, $3x = 4(3\ell + 2) - 2 = 12\ell + 6$ and $x = 4\ell + 2$. So $\overline{2} = 4\mathsf{Z} + 2$.

(d) The quotient set is $\{4\mathsf{Z}, 4\mathsf{Z}+1, 4\mathsf{Z}+2, 4\mathsf{Z}+3\}$.

10. (a) **Reflexive:** For any $a \in \mathsf{Z}$, $a \sim a$ because $3a + 4a = 7a$ and a is an integer.

Symmetric: If $a, b \in \mathsf{Z}$ and $a \sim b$, then $3a + 4b = 7n$ for some integer n. Then $3b + 4a = (7a + 7b) - (3a + 4b) = 7a + 7b - 7n = 7(a + b - n)$ and $a + b - n$ is an integer. Thus $b \sim a$.

Transitive: Suppose $a, b, c \in \mathsf{Z}$ with $a \sim b$ and $b \sim c$. Then $3a + 4b = 7n$ and $3b + 4c = 7m$ for some integers n and m. Then $7n + 7m = (3a + 4b) + (3b + 4c) = (3a + 4c) + 7b$, so $3a + 4c = 7n + 7m - 7b = 7(n + m - b)$ and $n + m - b$ is an integer. Thus $a \sim c$.

(b) $\overline{0} = \{x \in \mathsf{Z} \mid x \sim 0\} = \{x \in \mathsf{Z} \mid 3x = 7n \text{ for some integer } n\}$. Now if $3x = 7n$, n must be a multiple of 3. So $3x = 21k$ for some $k \in \mathsf{Z}$ and $x = 7k$. We conclude that $\overline{0} = 7\mathsf{Z}$.

11. (a) [BB] **Reflexive:** If $a \in \mathsf{Z} \setminus \{0\}$, then $aa = a^2 > 0$, so $a \sim a$.

Symmetric: If $a \sim b$, then $ab > 0$. So $ba > 0$ and $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $ab > 0$ and $bc > 0$. Also $b^2 > 0$ since $b \neq 0$. Hence,

$$ac = \frac{(ac)b^2}{b^2} = \frac{(ab)(bc)}{b^2} > 0$$

since $ab > 0$, $bc > 0$. Hence, $a \sim c$.

(b) [BB]
$$\begin{aligned} \overline{5} &= \{x \in \mathsf{Z} \setminus \{0\} \mid x \sim 5\} = \{x \mid 5x > 0\} = \{x \mid x > 0\} \\ \overline{-5} &= \{x \in \mathsf{Z} \setminus \{0\} \mid x \sim -5\} = \{x \mid -5x > 0\} = \{x \mid x < 0\} \end{aligned}$$

(c) [BB] This equivalence relation partitions $\mathsf{Z} \setminus \{0\}$ into the positive and the negative integers.

12. (a) [BB] **Reflexive:** For any $a \in \mathsf{Z}$, $a^2 - a^2 = 0$ is divisible by 3, so $a \sim a$.

Symmetric: If $a \sim b$, then $a^2 - b^2$ is divisible by 3, so $b^2 - a^2$ is divisible by 3, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $a^2 - b^2$ is divisible by 3 and $b^2 - c^2$ is divisible by 3, so $a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2)$ is divisible by 3.

(b)
$$\begin{aligned} \overline{0} &= \{x \in \mathsf{Z} \mid x \sim 0\} \\ &= \{x \in \mathsf{Z} \mid x^2 \text{ is divisible by } 3\} \\ &= \{x \in \mathsf{Z} \mid x \text{ is divisible by } 3\} = 3\mathsf{Z} \end{aligned}$$

$$\begin{aligned} \overline{1} &= \{x \in \mathsf{Z} \mid x \sim 1\} \\ &= \{x \mid x^2 - 1 \text{ is divisible by } 3\} \\ &= \{x \mid (x-1)(x+1) \text{ is divisible by } 3\} \\ &= \{x \mid x - 1 \text{ or } x + 1 \text{ is divisible by } 3\} \\ &= 3\mathsf{Z} + 1 \cup 3\mathsf{Z} + 2 \end{aligned}$$

(c) This equivalence relation partitions the integers into the two disjoint sets $3\mathsf{Z}$ and $(3\mathsf{Z}+1) \cup (3\mathsf{Z}+2)$.

13. (a) [BB] Yes, this is an equivalence relation.

Reflexive: Note that if a is any triangle, $a \sim a$ because a is congruent to itself.

Symmetric: Assume $a \sim b$. Then a and b are congruent. Therefore, b and a are congruent, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then a and b are congruent and b and c are congruent, so a and c are congruent. Thus, $a \sim c$.

(b) Yes, this is an equivalence relation.

Reflexive: If a is a circle, then $a \sim a$ because a has the same center as itself.

Symmetric: Assume $a \sim b$. Then a and b have the same center. Thus, b and a have the same center, so $b \sim a$.

Transitive: Assume $a \sim b$ and $b \sim c$. Then a and b have the same center and b and c have the same center, so a and c have the same center. Thus, $a \sim c$.

(c) Yes, this is an equivalence relation.

Reflexive: If a is a line, then a is parallel to itself, so $a \sim a$.

Symmetric: If $a \sim b$, then a is parallel to b. Thus, b is parallel to a. Hence, $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then a is parallel to b and b is parallel to c, so a is parallel to c. Thus, $a \sim c$.

(d) No, this is not an equivalence relation. The reflexive property does not hold because no line is perpendicular to itself. Neither is this relation transitive; if ℓ_1 is perpendicular to ℓ_2 and ℓ_2 is perpendicular to ℓ_3, then ℓ_1 and ℓ_3 are **parallel**, not perpendicular to one another.

14. (a) [BB] $\mathcal{R} = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,4),(4,5),(5,5),(4,3),(5,3),(5,4)\}$

(b) $\mathcal{R} = \{((1,1),(2,2),(3,3),(3,4),(4,3),(4,4),(5,5)\}$

(c) $\mathcal{R} = \{(1,1),(1,2),(1,3),(1,4),(1,5),(2,1),(2,2),(2,3),(2,4),(2,5),(3,1),(3,2),(3,3),(3,4),(3,5),(4,1),(4,2),(4,3),(4,4),(4,5),(5,1),(5,2),(5,3),(5,4),(5,5)\}$

15. (a) As suggested in the text, we list the partitions of $\{a\}$. There is only one; namely, $\{a\}$.

(b) As suggested in the text, we list the partitions of $\{a, b\}$. There are two; namely, $\{a, b\}$ and $\{a\}, \{b\}$.

(c) [BB] As suggested in the text, a good way to list the equivalence relations on $\{a, b, c\}$ is to list the partitions of this set. Here they are:

$\{ \{a\}, \{b\}, \{c\} \}$;
$\{ \{a, b, c\} \}$;
$\{ \{a, b\}, \{c\} \}$; $\{ \{a, c\}, \{b\} \}$; $\{ \{b, c\}, \{a\} \}$
There are five in all.

(d) As suggested in the text, we list the partitions of $\{a, b, c, d\}$.

$\{ \{a\}, \{b\}, \{c\}, \{d\} \}$
$\{ \{a, b, c, d\} \}$
$\{ \{a, b\}, \{c, d\} \}$; $\{ \{a, c\}, \{b, d\} \}$; $\{ \{a, d\}, \{b, c\} \}$
$\{ \{a, b\}, \{c\}, \{d\} \}$; $\{ \{a, c\}, \{b\}, \{d\} \}$; $\{ \{a, d\}, \{b\}, \{c\} \}$; $\{ \{b, c\}, \{a\}, \{d\} \}$; $\{ \{b, d\}, \{a\}, \{c\} \}$; $\{ \{c, d\}, \{a\}, \{b\} \}$
$\{ \{a, b, c\}, \{d\} \}$; $\{ \{a, b, d\}, \{c\} \}$; $\{ \{a, c, d\}, \{b\} \}$; $\{ \{b, c, d\}, \{a\} \}$
There are 15 in all.

16. (a) [BB] The given statement is an implication which concludes "$x - y = x - y$," whereas what is required is a logical argument which concludes "so $\sim$ is reflexive."
A correct argument is this: For any $(x, y) \in \mathsf{R}^2$, $x - y = x - y$; thus, $(x, y) \sim (x, y)$. Therefore, $\sim$ is reflexive.
(b) There is confusion between the elements of a binary relation on a set A (which are ordered pairs) and the elements of A which are themselves ordered pairs in this situation. The given statement is correct *provided* each of x and y is understood to be an ordered pair of real numbers, and we understand $\mathcal{R} = \{(x, y) \mid x \sim y\}$ but this is very misleading. Much better is to state symmetry like this:

if $(x, y) \sim (u, v)$, then $(u, v) \sim (x, y)$.

(c) The first statement asserts the implication "$x - y = u - v \rightarrow (x, y) \sim (u, v)$" which is the converse of what should have been said. Here is the correct argument:

If $(x, y) \sim (u, v)$, then $x - y = u - v$, so $u - v = x - y$ and, hence, $(u, v) \sim (x, y)$.

(d) This suggested answer is utterly confusing. Logical arguments consist of a sequence of implications but here it is not clear where these implications start. Certainly the first sentence is not an implication.

If $(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$ then $x - y = u - v$ and $u - v = w - z$. So $x - y = w - z$ and, hence, $(x, y) \sim (w, z)$.

(e) $\sim$ defines an equivalence relation on R^2 because it is a reflexive, symmetric and transitive binary relation on R^2.
(f) The equivalence class of $(0, 0)$ is

$$\{(x, y) \mid (x, y) \sim (0, 0)\} = \{(x, y) \mid x - y = 0 - 0\} = \{(x, y) \mid y = x\}$$

which is a straight line of slope 1 in the Cartesian plane passing through the origin. The equivalence class of $(2, 3)$ is

$$\{(x, y) \mid (x, y) \sim (2, 3)\} = \{(x, y) \mid x - y = 2 - 3 = -1\} = \{(x, y) \mid y = x + 1\}$$

which is a straight line of slope 1 passing through the point $(2, 3)$.

17. [BB] **Reflexive:** If $(x, y) \in \mathsf{R}^2$, then $x^2 - y^2 = x^2 - y^2$, so $(x, y) \sim (x, y)$.
Symmetric: If $(x, y) \sim (u, v)$, then $x^2 - y^2 = u^2 - v^2$, so $u^2 - v^2 = x^2 - y^2$ and $(u, v) \sim (x, y)$.
Transitive: If $(x, y) \sim (u, v)$ and $(u, v) \sim (w, z)$, then $x^2 - y^2 = u^2 - v^2$ and $u^2 - v^2 = w^2 - z^2$, so $x^2 - y^2 = u^2 - v^2 = w^2 - z^2$; $x^2 - y^2 = w^2 - z^2$ and $(x, y) \sim (w, z)$.

$$\overline{(0,0)} = \{(x, y) \mid (x, y) \sim (0, 0)\} = \{(x, y) \mid x^2 - y^2 = 0^2 - 0^2 = 0\} = \{(x, y) \mid y = \pm x\}$$

Thus, the equivalence class of $(0, 0)$ is the pair of lines with equations $y = x$, $y = -x$.

$$\overline{(1,0)} = \{(x, y) \mid (x, y) \sim (1, 0)\} = \{(x, y) \mid x^2 - y^2 = 1^2 - 0^2 = 1\}.$$

Thus, the equivalence class of $(1, 0)$ is the hyperbola whose equation is $x^2 - y^2 = 1$.

18. (a) This is an equivalence relation.
Reflexive: If $(a, b) \in \mathsf{R}^2$, then $a + 2b = a + 2b$, so $(a, b) \sim (a, b)$.
Symmetric: If $(a, b) \sim (c, d)$, then $a + 2b = c + 2d$, so $c + 2d = a + 2b$ and $(c, d) \sim (a, b)$.

Transitive: If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $a+2b = c+2d$ and $c+2d = e+2f$, so $a+2b = e+2f$ and $(a,b) \sim (e,f)$.

The quotient set is the set of equivalence classes. We have

$$\begin{aligned}\overline{(a,b)} &= \{(x,y) \mid (x,y) \sim (a,b)\} = \{(x,y) \mid x+2y = a+2b\} \\ &= \{(x,y) \mid y-b = -\tfrac{1}{2}(x-a)\}\end{aligned}$$

which describes the line through (a,b) with slope $-\frac{1}{2}$. The quotient set is the set of lines with slope $-\frac{1}{2}$.

(b) This is an equivalence relation.

Reflexive: If $(a,b) \in \mathsf{R}^2$, then $ab = ab$, so $(a,b) \sim (a,b)$.

Symmetric: If $(a,b) \sim (c,d)$, then $ab = cd$ so $cd = ab$ and $(c,d) \sim (a,b)$.

Transitive: If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $ab = cd$ and $cd = ef$, so $ab = cd = ef$, $ab = ef$ and $(a,b) \sim (e,f)$.

The quotient set is the set of equivalence classes. We have

$$\overline{(a,b)} = \{(x,y) \mid (x,y) \sim (a,b)\} = \{(x,y) \mid xy = ab\}$$

and consider two cases. If either $a = 0$ or $b = 0$, then $\overline{(a,b)} = \{(x,y) \mid xy = 0\}$; that is, $\{(x,y) \mid x = 0 \text{ or } y = 0\}$. Hence, $\overline{(a,b)}$ is the union of the x-axis and the y-axis. On the other hand, if $a \neq 0$ and $b \neq 0$, then

$$\overline{(a,b)} = \{(x,y) \mid xy = ab\} = \{(x,y) \mid y = \frac{ab}{x}\}$$

since $x \neq 0$ in this case. This time, $\overline{(a,b)}$ is the hyperbola whose equation is $y = ab/x$.

(c) This is not an equivalence relation. We have $(0,2) \sim (1,1)$ because $0^2 + 2 = 2 = 1 + 1^2$; however, $(1,1) \not\sim (0,2)$ because $1^2 + 1 = 2 \neq 4 = 0 + 2^2$. The relation is not symmetric.

(d) **Reflexive:** For any $(a,b) \in \mathsf{R}^2$, $a = a$, so $(a,b) \sim (a,b)$.

Symmetric: If $(a,b) \sim (c,d)$, then $a = c$, so $c = a$ and, hence, $(c,d) \sim (a,b)$.

Transitive: If $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$, then $a = c$ and $c = e$; hence, $a = e$ and so $(a,b) \sim (e,f)$.

Since the relation is reflexive, symmetric and transitive, it is an equivalence relation. The quotient set is the set of equivalence classes. The equivalence class of (a,b) is

$$\{(x,y) \in \mathsf{R}^2 \mid (x,y) \sim (a,b)\} = \{(x,y) \in \mathsf{R}^2 \mid x = a\}.$$

Geometrically, this set is the vertical line with equation $x = a$. The quotient set is the set of vertical lines.

(e) This is not an equivalence relation. For example, it is not reflexive: $(1,2) \not\sim (1,2)$ because $1(2) = 2 \neq 1 = 1^2$.

19. (a) "If $\overline{a} \cap \overline{b} = \emptyset$, then $\overline{a} \neq \overline{b}$."

(b) The converse is true. If $\overline{a} \cap \overline{b} = \emptyset$, then $a \in \overline{a}$ but $a \notin \overline{b}$, so $\overline{a} \neq \overline{b}$.

20. Remembering that $\overline{x}$ is just the set of elements equivalent to x, we are given that $a \sim b$, $c \sim d$ and $d \sim b$. By Proposition 2.4.3, $\overline{a} = \overline{b}$, $\overline{c} = \overline{d}$ and $\overline{d} = \overline{b}$. Thus $\overline{a} = \overline{b} = \overline{d} = \overline{c}$.

21. (a) [BB] The ordered pairs defined by $\sim$ are $(1,1)$, $(1,4)$, $(1,9)$, $(2,2)$, $(2,8)$, $(3,3)$, $(4,1)$, $(4,4)$, $(4,9)$, $(5,5)$, $(6,6)$, $(7,7)$, $(8,2)$, $(8,8)$, $(9,1)$, $(9,4)$, $(9,9)$.

(b) [BB] $\overline{1} = \{1,4,9\} = \overline{4} = \overline{9}$; $\overline{2} = \{2,8\} = \overline{8}$; $\overline{3} = \{3\}$; $\overline{5} = \{5\}$; $\overline{6} = \{6\}$; $\overline{7} = \{7\}$.

(c) [BB] Since the sets $\{1,4,9\}$, $\{2,8\}$, $\{3\}$, $\{5\}$, $\{6\}$ and $\{7\}$ partition A, they determine an equivalence relation, namely, that equivalence relation in which $a \sim b$ if and only if a and b belong to the same one of these sets. This is the given relation.

22. [BB] **Reflexive:** If $a \in A$, then a^2 is a perfect square, so $a \sim a$.

Symmetric: If $a \sim b$, then ab is a perfect square. Since $ba = ab$, ba is also a perfect square, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then ab and bc are each perfect squares. Thus $ab = x^2$ and $bc = y^2$ for integers x and y. Now $ab^2c = x^2y^2$, so $ac = \dfrac{x^2y^2}{b^2} = \left(\dfrac{xy}{b}\right)^2$. Because ac is an integer, so also $\dfrac{xy}{b}$ is an integer. Therefore, $a \sim c$.

23. (a) The order pairs of $\sim$ are $(1,1)$, $(2,2)$, $(3,3)$, $(4,4)$, $(5,5)$, $(6,6)$, $(7,7)$, $(1,2)$, $(2,1)$, $(1,4)$, $(4,1)$, $(2,4)$, $(4,2)$, $(3,6)$, $(6,3)$.

(b) $\overline{1} = \{1,2,4\} = \overline{2} = \overline{4}$; $\overline{3} = \{3,6\} = \overline{6}$; $\overline{5} = \{5\}$; $\overline{7} = \{7\}$.

(c) The sets $\{1,2,4\}$, $\{3,6\}$, $\{5\}$, $\{7\}$ partition A, so the given relation is an equivalence relation.

24. **Reflexive:** If $a \in A$, then $\dfrac{a}{a} = 1 = 2^0$ is a power of 2, so $a \sim a$.

Symmetric: If $a \sim b$, then $\dfrac{a}{b} = 2^t$, so $\dfrac{b}{a} = 2^{-t}$. Since $-t$ is an integer, $\dfrac{b}{a}$ is also a power of 2, so $b \sim a$.

Transitive: If $a \sim b$ and $b \sim c$, then $\dfrac{a}{b} = 2^t$ and $\dfrac{b}{c} = 2^s$ for integers t and s. Thus $\dfrac{a}{c} = \dfrac{a}{b}\dfrac{b}{c} = 2^{t+s}$, showing that $a \sim c$.

25. We have to prove that the given sets are disjoint and have union S. For the latter, we note that since $\mathcal{R}$ is reflexive, for any $a \in S$, $(a,a) \in \mathcal{R}$ and so a and a are elements of the same set S_i; that is, $a \in S_i$ for some i. To prove that the sets are disjoint, suppose there is some $x \in S_k \cap S_\ell$. Since $S_k \not\subseteq \bigcup_{j \neq k} S_j$, there exists $y \in S_k$ such that $y \notin S_j$ for any $j \neq k$. Similarly, there exists $z \in S_\ell$ such that $z \notin S_j$ if $j \neq \ell$. Now if $y, x \in S_k$, then $(y,x) \in \mathcal{R}$ and $x, z \in S_\ell$ implies $(x,z) \in \mathcal{R}$. By transitivity, $(y,z) \in \mathcal{R}$, hence, y and z belong to the same set. But the only set to which y belongs is S_k. Since z does not belong to S_k, we have a contradiction: No $x \in S_k \cap S_\ell$ exists.

Exercises 2.5

1. (a) [BB] This defines a partial order.

Reflexive: For any $a \in \mathsf{R}$, $a \geq a$.

Antisymmetric: If $a, b \in \mathsf{R}$, $a \geq b$ and $b \geq a$, then $a = b$.

Transitive: If $a, b, c \in \mathsf{R}$, $a \geq b$ and $b \geq c$, then $a \geq c$.

This partial order is a total order because for any $a, b \in \mathsf{R}$, either $a \geq b$ or $b \geq a$.

(b) [BB] This is not a partial order because the relation is not reflexive; for example, $1 < 1$ is not true.

(c) This is not a partial order because the relation is not antisymmetric; for example, $-3 \preceq 3$ because $(-3)^2 \leq 3^2$ and $3 \preceq -3$ because $3^2 \leq (-3)^2$ but $-3 \neq 3$.

(d) This is not a partial order because the relation is not antisymmetric; for example, $(1,4) \preceq (1,8)$ because $1 \leq 1$ and similarly, $(1,8) \preceq (1,4)$, but $(1,4) \neq (1,8)$.

(e) This is a partial order.

Reflexive: For any $(a,b) \in \mathsf{N} \times \mathsf{N}$, $(a,b) \preceq (a,b)$ because $a \leq a$ and $b \geq b$.

Antisymmetric: If $(a,b),(c,d) \in \mathsf{N} \times \mathsf{N}$, $(a,b) \preceq (c,d)$ and $(c,d) \preceq (a,b)$, then $a \leq c$, $b \geq d$, $c \leq a$ and $d \geq b$. So $a = c$, $b = d$ and, hence, $(a,b) = (c,d)$.

Transitive: If $(a,b),(c,d),(e,f) \in \mathsf{N} \times \mathsf{N}$, $(a,b) \preceq (c,d)$ and $(c,d) \preceq (e,f)$, then $a \leq c$, $b \geq d$, $c \leq e$ and $d \geq f$. So $a \leq e$ (because $a \leq c \leq e$) and $b \geq f$ (because $b \geq d \geq f$) and, therefore, $(a,b) \preceq (e,f)$.

This is not a total order; for example, $(1,4)$ and $(2,5)$ are incomparable.

(f) This is reflexive and transitive but not antisymmetric and, hence, not a partial order. For example, *cat* $\preceq$ *dog* and *dog* $\preceq$ *cat* but *dog* $\neq$ *cat*.

2. (a) [BB] 1, 10, 100, 1000, 1001, 101, 1010, 11, 110, 111

(b) 1, 11, 111, 110, 10, 101, 1010, 100, 1001, 1000

3. (a) [BB] $(a,b),(a,c),(a,d),(b,c),(b,d),(c,d)$; (b) [BB] $(a,b),(c,d)$;
(c) $(a,b),(a,d),(c,d)$; (d) $(a,b),(a,c),(a,d)$;
(e) $(a,d),(a,e),(b,e),(b,c),(b,f),(c,f),(d,e)$;
(f) $(a,f),(b,c),(d,b),(d,c),(d,h),(d,i),(e,c),(e,i),(g,f),(h,i)$.

4. (a) [BB] a is minimal and minimum; d is maximal and maximum.

(b) [BB] a and c are minimal; b and d are maximal; there are no minimum nor maximum elements.

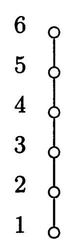
(c) a and c are minimal; b and d are maximal; there are no minimum nor maximum elements.

(d) a is minimal and minimum; b, c and d and maximal; there is no maximum element.

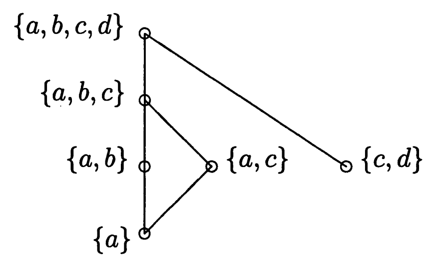
(e) a and b are minimal; e and f and maximal; there are no minimum nor maximum elements.

(f) a, d, e and g are minimal; c, f and i are maximal; there are no minimum nor maximum elements.

5. (a)



(b)



6. (a) 1 is minimal and minimum; 6 is maximal and maximum.

(b) $\{a\}$ and $\{c,d\}$ are minimal; there is no minimum.
The set $\{a,b,c,d\}$ is maximal and maximum.

7. [BB] $A \subsetneqq B$ and the set B contains exactly one more element than A.

8. Helmut Hasse (1898–1979) was one of the more important mathematicians of the twentieth century. He grew up in Berlin and was a member of Germany's navy during the first World War. He received his PhD from the University of Göttingen in 1921 for a thesis in number theory, which was to be the subject of his life's work. He is known for his research with Richard Brauer and Emmy Noether on simple algebras, his proof of the Riemann Hypothesis (one of today's most famous open problems) for zeta functions on elliptic curves, and his work on the arithmetical properties of abelian number fields. Hasse's career started at Kiel and continued at Halle and Marburg. When the Nazis came to power in 1933, all Jewish mathematicians, including eighteen at the University of Göttingen, were summarily dismissed from their jobs. It is hard to know the degree of ambivalence Hasse may have had when he received an offer of employment at Göttingen around this time, but he accepted the position. While some of Hasse's closest research collaborators were Jewish, he nonetheless made no secret of his support for Hitler's policies. In 1945, he was dismissed by the British, lost his right to teach and eventually moved to Berlin. In May 1949, he was appointed professor at the Humboldt University in East Berlin but he moved to Hamburg the next year and worked there until his retirement in 1966.

9. (a) [BB] Let $(A, \preceq)$ be a finite poset and let $a \in A$. If a is not maximal, there is an element a_1 such that $a_1 \succ a$. If a_1 is not maximal, there is an element a_2 such that $a_2 \succ a_1$. Continue. Since A is finite, eventually this process must stop, and it stops at a maximal element. A similar argument shows that $(A, \preceq)$ must also contain minimal elements.

   (b) The result of (a) is not true in general. For example, $(\mathsf{R}, \leq)$ is a poset without maximal elements and without minimal elements.

10. (a) **Reflexive:** For any $\mathsf{a} = (a_1, a_2) \in A$, $\mathsf{a} \leq \mathsf{a}$ because $a_1 \leq a_1$ and $a_1 + a_2 \leq a_1 + a_2$.

    Antisymmetric: If $\mathsf{a} = (a_1, a_2)$ and $\mathsf{b} = (b_1, b_2)$ are in A with $\mathsf{a} \preceq \mathsf{b}$ and $\mathsf{b} \preceq \mathsf{a}$, then $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2$, and $b_1 \leq a_1$, $b_1 + b_2 \leq a_1 + a_2$. Since $a_1 \leq b_1$ and $b_1 \leq a_1$, we have $a_1 = b_1$. Since $a_1 + a_2 \leq b_1 + b_2$ and $a_1 = b_1$, we have $a_2 \leq b_2$. Similarly, $b_2 \leq a_2$, so $a_2 = b_2$. Thus $\mathsf{a} = \mathsf{b}$.

    Transitive: If $\mathsf{a} = (a_1, a_2)$, $\mathsf{b} = (b_1, b_2)$ and $\mathsf{c} = (c_1, c_2)$ are elements of A with $\mathsf{a} \preceq \mathsf{b}$ and $\mathsf{b} \preceq \mathsf{c}$, then $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2$, and also $b_1 \leq c_1$, $b_1 + b_2 \leq c_1 + c_2$. Since $a_1 \leq b_1$ and $b_1 \leq c_1$, we have $a_1 \leq c_1$. Since $a_1 + a_2 \leq b_1 + b_2$ and $b_1 + b_2 \leq c_1 + c_2$, we have $a_1 + a_2 \leq c_1 + c_2$, so $\mathsf{a} \preceq \mathsf{c}$.

    This partial order is not a total order: for example, $\mathsf{a} = (0, 0)$ and $\mathsf{b} = (-1, -2)$ are not comparable.

    (b) Let $A = \mathsf{Z}^n$ and for $\mathsf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathsf{b} = (b_1, b_2, \ldots, b_n)$ in A, define $\mathsf{a} \preceq \mathsf{b}$ if and only if $a_1 \leq b_1$, $a_1 + a_2 \leq b_1 + b_2$, $a_1 + a_2 + a_3 \leq b_1 + b_2 + b_3$, $a_1 + a_2 + a_3 + a_4 \leq b_1 + b_2 + b_3 + b_4$, $\ldots$, $a_1 + a_2 + \cdots + a_n \leq b_1 + b_2 + \cdots + b_n$. Then $\preceq$ is a partial order on A.

11. (a) [BB] Suppose that a and b are two maximum elements in a poset $(A, \preceq)$. Then $a \preceq b$ because b is maximum and $b \preceq a$ because a is maximum, so $a = b$ by antisymmetry.

    (b) Suppose that a and b are two minimum elements in a poset $(A, \preceq)$. Then $a \preceq b$ because a is minimum and $b \preceq a$ because b is minimum, so $a = b$ by antisymmetry.

12. (a) [BB] Assuming it exists, the greatest lower bound G of A and B has two properties:

    (1) $G \subseteq A, G \subseteq B$;

    (2) if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq G$.

    We must prove that $A \cap B$ has these properties. Note first that $A \cap B \subseteq A$ and $A \cap B \subseteq B$, so $A \cap B$ satisfies (1). Also, if $C \subseteq A$ and $C \subseteq B$, then $C \subseteq A \cap B$, so $A \cap B$ satisfies (2) and $A \cap B = A \wedge B$.

(b) Assuming it exists, the least upper bound of A and B has two properties:

(1) $A \subseteq L, B \subseteq L$;

(2) if $A \subseteq C$ and $B \subseteq C$, then $L \subseteq C$.

We must prove that $A \cup B$ has these properties. Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, $A \cup B$ satisfies (1). Also, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$, so $A \cup B$ satisfies (2) and $A \cup B = A \vee B$.

13. (a) [BB] $a \vee b = b$ and here is why. We are given $a \preceq b$ and have $b \preceq b$ by reflexivity. Thus b is an upper bound for a and b. It is least because if c is any other upper bound, then $a \preceq c$, $b \preceq c$; in particular, $b \preceq c$.

(b) $a \wedge b = a$ and here is why. We are given $a \preceq b$ and have $a \preceq a$ by reflexivity. Thus a is a lower bound for a and b. It is greatest because if c is any other lower bound, then $c \preceq a$, $c \preceq b$; in particular, $c \preceq a$.

14. (a) [BB] Suppose x and y are each glbs of two elements a and b. Then $x \preceq a$, $x \preceq b$ implies $x \preceq y$ because y is a **greatest** lower bound, and $y \preceq a$, $y \preceq b$ implies $y \preceq x$ because x is greatest. So, by antisymmetry, $x = y$.

(b) Suppose x and y are each lubs of two elements a and b. Then $a \preceq x$, $b \preceq x$ implies $y \preceq x$ because y is a **least** upper bound, and $a \preceq y$, $b \preceq y$ implies $x \preceq y$ because x is least. So, by antisymmetry, $x = y$.

15. (a) In a totally ordered set, every two elements are comparable. So given a and b, either $a \preceq b$ or $b \preceq a$; hence, the elements $\max(a,b)$ and $\min(a,b)$ always exist. In a poset which is not totally ordered, they don't necessarily, however. In the two element poset $\{a\}, \{b\}$ with the relation $\subseteq$, for example, $\max(\{a\},\{b\})$ does not exist because there is no element in the poset containing both $\{a\}$ and $\{b\}$. (Similarly, $\min(\{a\},\{b\})$ does not exist.)

(b) To prove that a totally ordered set $(A, \preceq)$ is a lattice we must prove that every pair of elements has a glb and lub. We claim that $\operatorname{glb}(a,b) = \min(a,b)$ and $\operatorname{lub}(a,b) = \max(a,b)$.

We show that $\operatorname{glb}(a,b) = \min(a,b)$. (The argument to show that $\operatorname{lub}(a,b) = \max(a,b)$ is very similar.) Let $m = \min(a,b)$. (Note that $m = a$ or $m = b$). Certainly we have $m \preceq a$ and $m \preceq b$ so m is a lower bound. Also, if for some element c we have $c \preceq a$ and $c \preceq b$, then $c \preceq m$ if $m = a$, and $c \preceq m$ if $m = b$. In either case, we have $c \preceq m$, so $\operatorname{glb}(a,b)$ is $\min(a,b)$ as required.

16. (a) [BB] $(\mathcal{P}(S), \subseteq)$ is not totally ordered provided $|S| \geq 2$ (since $\{a\}$ and $\{b\}$ are not comparable if $a \neq b$). But $\emptyset$ is a minimum because $\emptyset$ is a subset of any set and the set S itself is a maximum because any of its subsets is contained in it.

(b) $(\mathsf{Z}, \leq)$ or $(\mathsf{R}, \leq)$ are obvious examples.

17. Suppose a is maximal in a totally ordered set $(A, \preceq)$ and let b be any other element of A. Since A is totally ordered, either $a \preceq b$ or $b \preceq a$. In the first case, $a = b$ because a is maximal so in either case, $b \preceq a$. Thus, a is a maximum.

18. (a) [BB] We have to prove that if $b \preceq a$, then $b = a$. So suppose $b \preceq a$. Since a is minimum, we have also $a \preceq b$. By antisymmetry, $b = a$.

(b) Let b be a minimal element. We claim $b = a$. To see why, note that a minimum implies $a \preceq b$. Then minimality of b says $a = b$.

Chapter 2 Review

1. Since $A = \{1,2,3,4,5,6\}$ and $B = \{3,4,5,6,7\}$, we have $A \oplus B = \{1,2,7\}$ and $(A \oplus B) \setminus C = \{1,7\}$.

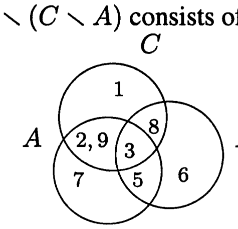
2. (a) $A = \{-1,0,1,2\}$, $B = \{-5,-3,-1,1\}$, $C = \{-\frac{2}{3}, -\frac{2}{5}, 0, \pm 1, \pm 2, \pm\frac{1}{5}, \pm\frac{1}{3}\}$;
   (b) $A \cap B = \{\pm 1\}$, so $(A \cap B) \times B = \{(1,-5),(1,-3),(1,-1),(1,1),(-1,-5),(-1,-3),(-1,-1),(-1,1)\}$;
   (c) $B \setminus C = \{-5,-3\}$; (d) $A \oplus C = \{-\frac{2}{3}, -\frac{2}{5}, -2, \pm\frac{1}{5}, \pm\frac{1}{3}\}$.

3. (a) True. ($\longrightarrow$) Suppose $A \cap B = A$ and let $a \in A$. Then $a \in A \cap B$, so $a \in B$. Thus $A \subseteq B$.
   ($\longleftarrow$) Suppose $A \subseteq B$. To prove $A \cap B = A$, we prove each of the two sets, $A \cap B$ and A, is a subset of the other. Let $x \in A \cap B$. By definition of $\cap$, x is in both A and B, in particular, $x \in A$. Thus $A \cap B \subseteq A$. Conversely, let $x \in A$. Since $A \subseteq B$, $x \in B$. Since x is in both A and B, $x \in A \cap B$. Thus $A \subseteq A \cap B$.
   (b) This is false. If $A = \emptyset$, $(A \cap B) \cup C = C$ while $A \cap (B \cup C) = \emptyset$, so any $C \neq \emptyset$, any B, and $A = \emptyset$ provides a counterexample.
   (c) False. Take $A = B = \emptyset$.

4. Let $b \in B$ and let a be any element of A. Then $(a,b) \in A \times B$, so $(a,b) \in A \times C$. Thus $b \in C$. This shows tht $B \subseteq C$ and a similar argument shows $C \subseteq B$, so $B = C$.

5. (a) Region 2: $(A \cap C) \setminus B$ Region 3: $A \cap B \cap C$ Region 4: $(A \cap B) \setminus C$
   Region 5: $(B \cap C) \setminus A$ Region 6: $B \setminus (A \cup C)$ Region 7: $C \setminus (A \cup B)$
   (b) Region 2, 3, 4, 5, 7 is $(A \cap B) \cup C$; region 2, 3, 4 is $A \cap (B \cup C)$
   (c) $B \setminus (C \setminus A)$ consists of regions 3, 4, and 6. $(B \setminus C) \setminus A$ consists of region 6.

6. (a)



   (b) i. $(A \cup B) \cap C = \{2,3,8,9\}$
   ii. $A \setminus (B \setminus C) = \{2,3,7,9\}$
   iii. $A \oplus B = \{2,6,7,8,9\}$
   iv. $(A \setminus B) \times (B \cap C) = \{(2,3),(2,8),(7,3),(7,8),(9,3),(9,8)\}$

7. $\mathcal{P}(A) = \{\emptyset, A\}$, so $\mathcal{P}(\mathcal{P}(A)) = \{\emptyset, \{\emptyset\}, \{A\}, \{\emptyset, A\}\}$.

8. (a) Take $A = B = C = \{3\}$. Then $B \setminus C = \emptyset$ so $A \oplus (B \setminus C) = A$. On the other hand, $A \oplus B = \emptyset$ so $(A \oplus B) \setminus C = \emptyset \neq A$.
   (b) Let $(a,b) \in A \times B$. Then $a \in A$ and $b \in B$. Since $a \in A$ and $A \subseteq C$, $a \in C$. Since $b \in B$ and $B \subseteq D$, $b \in D$. So $(a,b) \in C \times D$. Hence $A \times B \subseteq C \times D$.

9. Take $B = C = \emptyset$, $A = \{1\} = D$. Then $A \times B = \emptyset = C \times D$, so $A \times B \subseteq C \times D$. On the other hand, $A \not\subseteq C$.

10. This follows quickly from one of the laws of De Morgan and the identity $X \setminus Y = X \cap Y^c$.

$$A \setminus (B \cap C) = A \cap (B \cap C)^c = A \cap (B^c \cup C^c) \overset{\downarrow}{=} (A \cap B^c) \cup (A \cap C^c) = (A \setminus B) \cup (A \setminus C),$$

using (3), p. 62 at the spot marked with the arrow.

11. (a) A *binary relation* on A is a subset of $A \times A$.

    (b) If A has 10 elements, $A \times A$ has 100 elements, so there are 2^{100} binary relations on A.

12. **Reflexive**: For any a with $|a| \leq 1$, we have $a^2 = |a^2| = |a||a| \leq |a|$, thus $(a, a) \in \mathcal{R}$.

    Symmetric by definition.

    Not antisymmetric because $(\frac{1}{2}, \frac{1}{4})$ is in $\mathcal{R}$ $((\frac{1}{2})^2 \leq \frac{1}{4}$ and $(\frac{1}{4})^2 \leq \frac{1}{2})$ but $\frac{1}{2} \neq \frac{1}{4}$.

    Not transitive. We have $(\frac{1}{2}, \frac{1}{3}) \in \mathcal{R}$ because $(\frac{1}{2})^2 \leq \frac{1}{3}$ and $(\frac{1}{3})^2 \leq \frac{1}{2}$ and $(\frac{1}{3}, \frac{1}{5}) \in \mathcal{R}$ because $(\frac{1}{3})^2 \leq \frac{1}{5}$ and $(\frac{1}{5})^2 \leq \frac{1}{3}$, but $(\frac{1}{2}, \frac{1}{5})$ is not in $\mathcal{R}$ because $(\frac{1}{2})^2 \not\leq \frac{1}{5}$.

13. (a) **Reflexive:** For any natural number a, we have $a \leq 2a$, so $a \sim a$.

    Not symmetric: $2 \sim 5$ because $2 \leq 2(5)$, but $5 \not\sim 2$ because $5 \not\leq 2(2)$.

    Not antisymmetric: Let $a = 1$ and $b = 2$. Then $a \sim b$ because $1 \leq 2(2)$ and also $b \sim a$ because $2 \leq 2(1)$.

    Not transitive: Let $a = 3$, $b = 2$, $c = 1$. Then $a \sim b$ because $3 \leq 2(2)$ and $b \sim c$ because $2 \leq 2(1)$. However, $a \not\sim c$ because $3 \not\leq 2(1)$.

    Since the relation is not transitive, it is not an equivalence relation and it is not a partial order.

    (b) **Not reflexive:** $(1, 2) \not\sim (1, 2)$ because $1 \leq 2$ and $2 \not\leq 1$.

    Not symmetric: $(1, 2) \sim (4, 3)$ because $1 \leq 2$ and $3 \leq 4$, but $(4, 3) \not\sim (1, 2)$ because $4 \not\leq 3$.

    Not antisymmetric: $(1, 1) \sim (2, 2)$ because $1 \leq 1$ and $2 \leq 2$, and $(2, 2) \sim (1, 1)$ for the same reason, but $(1, 1) \neq (2, 2)$.

    Transitive: if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $a \leq b$, $d \leq c$, $c \leq d$ and $f \leq e$, so $a \leq b$ and $f \leq e$ which implies $(a, b) \sim (e, f)$.

    This is not an equivalence relation because it's not reflexive (or symmetric).

    This is not a partial order because it's not reflexive (or antisymmetric).

14. We must determine whether or not $a < b$ and $b < a$ implies $a = b$. Since the hypothesis is always false, this implication is true. The relation is antisymmetric.

15. **Reflexive:** For any $a \in Z$, $4a + a = 5a$ is a multiple of 5.

    Symmetric: If $a\mathcal{R}b$, then $4a + b$ is a multiple of 5, so $4b + a = 5(a + b) - (4a + b)$ is also a multiple of 5, that is, $b\mathcal{R}a$.

    Transitive: If $a\mathcal{R}b$ and $b\mathcal{R}c$, then both $4a + b$ and $4b + c$ are multiples of 5, hence so is their sum, $4a + 5b + c$. It follows that $(4a + 5b + c) - 5b = 4a + c$ is also a multiple of 5, so $a\mathcal{R}c$.

16. (a) **Reflexive:** For any $a \in Z$, $a\mathcal{R}a$ because $2a + 5a = 7a$ is a multiple of 7.

    Symmetric: If $a, b \in Z$ and $a\mathcal{R}b$, then $2a + 5b = 7k$ for some integer k, so $5a + 2b = 7(a + b) - (2a + 5b)$ is the difference of multiples of 7, hence also a multiple of 7. Thus $b\mathcal{R}a$.

    Transitive: If $a, b, c \in Z$ with $a\mathcal{R}b$ and $b\mathcal{R}c$, then $2a + 5b = 7k$ for some integer k and $2b + 5c = 7\ell$ for some integer ℓ. Thus $(2a + 5b) + (2b + 5c) = 2a + 7b + 5c = 7(k + \ell)$ and $2a + 5c = 7(k + \ell) - 7b$ is the difference of multiples of 7, hence a multiple of 7. Thus $a\mathcal{R}c$.

(b) We have $0\mathcal{R}7$ and $7\mathcal{R}0$, yet $0 \neq 7$. The relation is not antisymmetric, so it is not a partial order.

17. ($\longrightarrow$) Assume $x \sim a$. Let $t \in \overline{x}$. Then $t \sim x$ so $t \sim a$ by transitivity. Thus $t \in \overline{a}$. This proves $\overline{x} \subseteq \overline{a}$. Similarly, $\overline{a} \subseteq \overline{x}$, so $\overline{x} = \overline{a}$.

($\longleftarrow$) Assume $\overline{x} = \overline{a}$. Since $x \in \overline{x}$, by symmetry, $x \in \overline{a}$. Thus $x \sim a$.

18. Since $a \in \overline{b}$, $a \sim b$ and hence $\overline{a} = \overline{b}$ by Proposition 2.4.3. Similarly $d \in \overline{b}$ implies $\overline{d} = \overline{b}$, hence $\overline{a} = \overline{d}$. Now $d \notin \overline{c}$ implies $\overline{d} \neq \overline{c}$, so $\overline{c} \cap \overline{d} = \emptyset$ by Proposition 2.4.4. Since $\overline{a} = \overline{d}$, so also $\overline{a} \cap \overline{c} = \emptyset$.

19. (a) We must show that $\preceq$ is reflexive, antisymmetric and transitive on A. The relation is reflexive: For any $(a, b) \in A$, $(a, b) \sim (a, b)$ because $a \leq a$ and $b \leq b$.

It is antisymmetric: If $(a, b), (c, d) \in A$, with $(a, b) \sim (c, d)$ and $(c, d) \sim (a, b)$, then $a \leq c$, $d \leq b$, $c \leq a$ and $b \leq d$. So $a = c$, $b = d$, hence, $(a, b) = (c, d)$.

It is transitive: If $(a, b), (c, d), (e, f) \in A$ with $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $a \leq c$, $d \leq b$, $c \leq e$ and $f \leq d$. So $a \leq e$ (because $a \leq c \leq e$) and $f \leq b$ (because $f \leq d \leq b$), so $(a, b) \sim (e, f)$.

(b) $(A, \preceq)$ is not totally ordered since, for example, $(1, 4)$ and $(2, 5)$ are not comparable: $(1, 4) \not\sim (2, 5)$ because $5 \not\leq 4$ and $(2, 5) \not\sim (1, 4)$ because $2 \not\leq 1$.

20. (a) **Reflexive:** For any $p \in A$, $p \sim p$ since $p = p$.

Symmetric: If $p \sim q$ then either $p = q$ (so $q = p$) or the line through p and q passes through the origin (in which case the line through q and p passes through the origin). Thus $q \sim p$.

Transitive: Suppose $p \sim q$ and $q \sim r$. If the points p, q, r are different, then the line through p and q passes through the origin, as does the line through q and r. Since the line through the origin and q is unique, p and r lie on this line, $p \sim r$. If $p = r$, then $p \sim r$. If $p = q \neq r$, then the line through q and r passes through the origin, so the line through p and r passes through the origin; thus $p \sim r$. The remaining case, $p \neq q = r$ is similar.

(b) The equivalence class of a point p is the line through the origin and p. The equivalence classes are lines through the origin.

21. **Reflexive:** For any $A \in \mathcal{P}(\mathsf{Z})$, $A \subseteq A$.

Antisymmetric: If $A, B \subseteq \mathcal{P}(\mathsf{Z})$ with $A \subseteq B$ and $B \subseteq A$, then $A = B$.

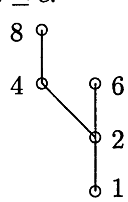
Transitive: If $A, B, C \subseteq \mathcal{P}(\mathsf{Z})$ with $A \subseteq B$, $B \subseteq C$, then $A \subseteq C$.

22. (a) **Reflexive:** For any $a \in A$, $\dfrac{a}{a} = 1$ is an integer, so $a \preceq a$.

Antisymmetric: If $a \preceq b$ and $b \preceq a$, then both $\dfrac{b}{a}$ and $\dfrac{a}{b}$ are (necessarily positive) integers. The only positive integer whose reciprocal is also an integer is 1, so $a = b$.

Transitive: If $a \preceq b$ and $b \preceq c$, then $\dfrac{b}{a}$ and $\dfrac{c}{b}$ are both integers. Thus $\dfrac{b}{a}\dfrac{c}{b} = \dfrac{c}{a}$ is an integer. So $a \preceq c$.

(b)



(c) 1 is minimal and minimum; 6 and 8 are maximal. There is no maximum element.

(d) $(A, \preceq)$ is not totally ordered; for example, 4 and 6 are not comparable.

23. Two elements of a poset can have at most one least upper bound and here's why. Let ℓ_1, ℓ_2 each be least upper bounds for elements a and b. Then ℓ_1 is an upper bound for a and b, so $\ell_2 \preceq \ell_1$ because ℓ_2 is **least**. Interchanging ℓ_1, ℓ_2 in the preceding statement gives $\ell_1 \preceq \ell_2$. So $\ell_1 = \ell_2$ by antisymmetry.

Exercises 3.1

1. (a) [BB] Not a function; f contains two different pairs of the form $(3, -)$.
   (b) Not a function **with domain** $\{1, 2, 3, 4\}$. There's no pair of the form $(3, -)$.
   (c) [BB] This is a function.
   (d) Certainly not a function. There is more than one pair of the form $(1, -)$: In fact, there are four!
   (e) This is a function.

2. (a) [BB] This is not a function unless each student at the University of Calgary has just one professor, for if student a is taking courses from professors b_1 and b_2, the given set contains (a, b_1) and (a, b_2).
   (b) Assuming that each student currently registered at the University of Calgary is taking at least one course, then this is a function.
   (c) Assuming some student a has no classes on Friday afternoon, then this will not be a function since the set would contain no pair of the form $(a, -)$.

3. [BB] $A \times B$ is a function $A \to B$ if and only if B contains exactly one element.

   To see why, first note that if $B = \{b\}$, then $A \times B = \{(a, b) \mid a \in A\}$ is certainly a function.

   Conversely, if $A \times B$ is a function but B contains two elements b_1, b_2, then for any $a \in A$, (a, b_1) and (a, b_2) are both in $A \times B$, so $A \times B$ is not a function.

4. (a) [BB] the function defined by $f(n) = 2n$, for example.
   (b) the function defined by $f(1) = 1$ and for $n > 1$, $f(n) = n - 1$, for example.
   (c) the constant function $f(n) = 107$ for all n, for example.
   (d) the identify function, ι_{N}, for example: $\iota(n) = n$, for all n.

5. (a) [BB] If $x \in X$, then x is a country in the British Commonwealth with a uniquely determined Prime Minister y who lives in that country; that is, $y \in Y$.

   If $y \in Y$, then y is a person living in one of the countries in the British Commonwealth. Thus, the domicile of y is a uniquely determined element $x \in X$.

   (b) If $x_1 \neq x_2$ are two different countries, then their Prime Ministers are different individuals, so Prime Minister is one-to-one. On the other hand, Prime Minister is not onto since there are people in a country who are not the Prime Minister.
   (c) If $x \in X$, then x is a country. If y is any person living in that country, then Domicile: $y \mapsto x$, so Domicile is onto. On the other hand, if y_1 and y_2 are different people living in country x, then Domicile: $y_1 \mapsto x$ and Domicile: $y_2 \mapsto x$, so Domicile is not one-to-one.

6. [BB] Parking rates, bus fares, admission prices are several common examples.

7. (a) Jumps in the graph of $y = \lfloor 2x - 3 \rfloor$ occur whenever $2x - 3$ is an integer, that is, at $\{\frac{n}{2} \mid n \in \mathsf{Z}\}$.
   (b) Jumps in the graph of $y = \lceil \frac{1}{4}x + 7 \rceil$ occur whenever $\frac{1}{4}x + 7$ is an integer, that is, at $\{4n \mid n \in \mathsf{Z}\}$.
   (c) Jumps in the graph of $y = \lfloor \frac{x+3}{5} \rfloor$ occur whenever $\frac{1}{5}(x+3)$ is an integer, that is, at $\{x \mid x = 5n - 3, n \in \mathsf{Z}\}$.

8. (a) [BB] The answer is "yes." By definition of "floor", we know that kx is the unique integer satisfying $n - 1 < kx \leq n$. Thus $\frac{n}{k} - \frac{1}{k} < x \leq \frac{n}{k}$. Since $\frac{n}{k} - 1 \leq \frac{n}{k} - \frac{1}{k}$, we have $\frac{n}{k} - 1 < x \leq \frac{n}{k}$. Thus $x = \lfloor \frac{n}{k} \rfloor$.
   (b) The multiples of 3 in the indicated interval are $3, 6, 9, \ldots, 3k$, where $3k = \lfloor 50000 \rfloor$. Thus $k = \lfloor \frac{50000}{3} \rfloor = 16666$.

9. [BB] We know that $\lfloor x \rfloor$ is an integer less than or equal to x and that $\lfloor y \rfloor$ is an integer less than or equal to y. Thus $\lfloor x \rfloor + \lfloor y \rfloor$ is an integer less than or equal to $x + y$. Since $\lfloor x + y \rfloor$ is the **greatest** such integer, we get the desired result.

10. We know that $\lceil x \rceil$ is an integer greater than or equal to x and that $\lceil y \rceil$ is an integer greater than or equal to y. Thus $\lceil x \rceil + \lceil y \rceil$ is an integer greater than or equal to $x + y$. Since $\lceil x + y \rceil$ is the **smallest** such integer, we get the desired result.

11. $\lfloor x \rfloor$ is the unique integer which satisfies $x - 1 < \lfloor x \rfloor \leq x$. This implies $(x+n) - 1 < \lfloor x \rfloor + n \leq x + n$. But there is exactly one integer a in the range $(x+n) - 1 < a \leq x + n$ and that is $\lfloor x + n \rfloor$. This gives the result.

12. We have $f(1) = 2(1) - 5 = -3$, $f(2) = 2^2 + 1 = 5$, $f(3) = 1$, $f(4) = 17$, $f(5) = 5$, and so, as a subset of $S \times \mathsf{Z}$, $f = \{(1, -3), (2, 5), (3, 1), (4, 17), (5, 5)\}$.

    No, f is not 1–1 because $f(2) = f(5)$ but $2 \neq 5$ (equivalently, $(2, 5)$ and $(5, 5)$ are both in f).

13. (a) [BB] add is not one-to-one since, for example, $\mathsf{add}(1,1) = \mathsf{add}(0,2)$ while $(1,1) \neq (0,2)$. It is onto, however, because, for any $y \in \mathsf{R}$, the equation $y = \mathsf{add}(x)$ has the solution $x = (y, 0)$.
    (b) mult is not one-to-one since, for example, $\mathsf{mult}(1,4) = \mathsf{mult}(2,2)$ while $(1,4) \neq (2,2)$. It is onto, though, since for any $y \in \mathsf{R}$, the equation $y = \mathsf{mult}(x)$ has the solution $x = (y, 1)$.

14. (a) [BB] g is not one-to-one since, for example, $g(1) = g(-1) = 2$. Neither is g onto: For any $x \in \mathsf{Z}$, $|x| \geq 0$, so $|x| + 1 \geq 1$. Thus, for example, $0 \notin \operatorname{rng} g$.
    (b) [BB] g is not one-to-one as in (a). This time it is onto, however, because for $n \in \mathsf{N}$, the equation $n = g(x)$ has the solution $x = n - 1$. (Note that $n > 0$ for $n \in \mathsf{N}$, so $|n - 1| = n - 1$.)

15. (a) [BB] If $f(x_1) = f(x_2)$, then $3x_1 + 5 = 3x_2 + 5$, so $x_1 = x_2$ which proves that f is one-to-one. Also f is onto, since given $y \in \mathsf{Q}$, $y = f(\frac{1}{3}(y - 5))$ with $\frac{1}{3}(y-5) \in \mathsf{Q}$.
    (b) This function is one-to-one, as in (a), but it is not onto because $f(n) = 3n + 5 > 8$ for any $n \in \mathsf{N}$ and so, for example, 1 is not in the range of f.

16. (a) h is not one-to-one since, for example, $h(1) = 3 = h(-1)$ and $1 \neq -1$.

    h is not onto since $1 \notin \operatorname{rng} h$: $1 \in \mathsf{Z}$, but if we try to solve $x^2 + 2 = 1$, we obtain $x^2 = -1$ which has no solutions in Z.
    (b) h is one-to-one since if $h(x_1) = h(x_2)$, then $x_1^2 + 2 = x_2^2 + 2$, so $x_1^2 = x_2^2$ and $x_1 = \pm x_2$. Since x_1 and x_2 are both in N, we must have $x_1 = x_2$.

    h is not onto; as in (a), 1 is not in the range of h.

17. (a) The function is not onto since, for example, $g(x) = 0$ has no integer solutions (by the quadratic formula), but it is one-to-one. To see why, suppose $g(x_1) = g(x_2)$. Then $3x_1^2 + 14x_1 - 51 = 3x_2^2 + 14x_2 - 51$, so $3(x_1 - x_2)(x_1 + x_2) = 14(x_2 - x_1)$. Thus $x_1 = x_2$ or $x_1 + x_2 = -\frac{14}{3}$. Since x_1 and x_2 are integers, $x_1 + x_2 = -\frac{14}{3}$ is impossible. Thus, $x_1 = x_2$, so g is one-to-one.

(b) Since the graph of g is a parabola, g is neither one-to-one nor onto.

18. (a) [BB] Note that $f(x) = (x+7)^2 - 100$. If $f(x_1) = f(x_2)$ it follows that $(x_1+7)^2 - 100 = (x_2+7)^2 - 100$, so $(x_1+7)^2 = (x_2+7)^2$ and, taking square roots, $|x_1 + 7| = |x_2 + 7|$. Since $x_1, x_2 \in \mathsf{N}$, we know $x_1 + 7 > 0$ and $x_2 + 7 > 0$. Thus, $x_1 + 7 = x_2 + 7$ and $x_1 = x_2$. Thus, f is one-to-one, but it is not onto: For example, $1 \in B$, but there is no $x \in \mathsf{N}$ with $f(x) = 1$ since $(x+7)^2 - 100 = 1$ implies $(x+7)^2 = 101$ and this equation has no solution in the natural numbers.

(b) This function is not one-to-one since, for example, $f(0) = f(-14)$ $(= -51)$ and it is not onto, as in (a).

(c) This function is not one-to-one; as in (b), $f(0) = f(-14)$. But it is onto since for any $y \geq -100$, $x = \sqrt{100 + y} - 7$ is a solution to $y = f(x)$.

19. (a) [BB] The domain of f is R. Its range is also R because every $y \in \mathsf{R}$ can be written $y = f(x)$ for some x; namely, $x = y^{1/3}$. The function is, therefore, onto. It is also one-to-one: If $f(x_1) = f(x_2)$, then $x_1^3 = x_2^3$ and this implies $x_1 = x_2$.

(b) The domain of g is R and so is the range. For the latter, note that

$$g(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

so that if $y \geq 0$, then $y = g(\sqrt{y})$ while if $y < 0$, $y = g(-\sqrt{-y})$. Since $\operatorname{rng} g = \mathsf{R}$, g is onto. To see that g is 1–1, either inspect the graph or assume $g(x_1) = g(x_2)$ and consider the cases.

Case 1: Both x_1 and x_2 are nonnegative.
In this case, $x_1^2 = x_2^2$, so $\sqrt{x_1^2} = \sqrt{x_2^2}$; hence, $|x_1| = |x_2|$ and $x_1 = x_2$.

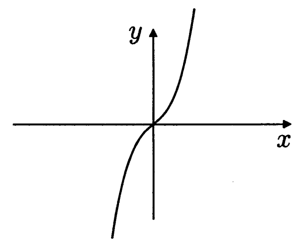
Case 2: Both x_1 and x_2 are negative.
In this case, $-x_1^2 = -x_2^2$, so $x_1^2 = x_2^2$, $\sqrt{x_1^2} = \sqrt{x_2^2}$, $|x_1| = |x_2|$, $-x_1 = -x_2$ and $x_1 = x_2$.

Case 3: One of x_1, x_2 is nonnegative and the other is negative.
Here we may assume that x_1 is nonnegative, so $x_1^2 = -x_2^2$. Since the left side is at least 0 and the right side is less than 0, the equation cannot be true.
In all cases $x_1 = x_2$, so g is 1–1.

(c) The domain of β is $(\frac{4}{3}, \infty)$ as given. Its range is R because for any $y \in \mathsf{R}$, $y = \beta(x)$ for $x = \frac{1}{3}(2^y + 4)$. Since $\operatorname{rng} \beta = \mathsf{R}$, β is onto. It is also one-to-one: If $\beta(x_1) = \beta(x_2)$, then $2^{\log_2(3x_1 - 4)} = 2^{\log_2(3x_2 - 4)}$, so $3x_1 - 4 = 3x_2 - 4$ and $x_1 = x_2$.

(d) The domain of f is R as given. To find the range, we must remember that $2^t > 0$ for all t and thus, $f(x) > 3$ for all x. In fact $\operatorname{rng} f = (3, \infty)$ since, for any $y > 3$, $y = f(x)$ for $x = 1 + \log_2(y-3)$. Since $\operatorname{rng} f \neq \mathsf{R}$, f is not onto. It is one-to-one, however, for if $f(x_1) = f(x_2)$, then $2^{x_1 - 1} + 3 = 2^{x_2 - 1} + 3$, so $2^{x_1 - 1} = 2^{x_2 - 1}$. Taking $\log_2$ of each side, $x_1 - 1 = x_2 - 1$, so $x_1 = x_2$.

20. (a) [BB] The graph of f shown at the right makes it clear that f is one-to-one and onto.



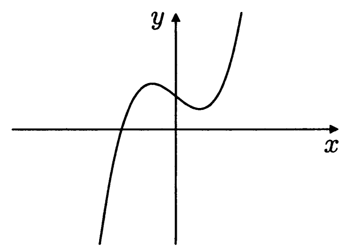
(b) [BB] **Solution 1.** Since $f: \mathsf{R} \to \mathsf{R}$ is one-to-one by part (a), it is also one-to-one as a function with domain Z. Here, however, f is not onto for we note that $f(0) = 0$, $f(1) = 4$ and f is increasing, so 1 is not in the range of f.

Solution 2. (This solution mimics that given in Problem 8 in our discussion of discrete functions in this section.)

If $f(x_1) = f(x_2)$, then $3x_1^3+x_1 = 3x_2^3+x_2$, so $3(x_1^3-x_2^3) = x_2-x_1$ and $3(x_1-x_2)(x_1^2+x_1x_2+x_2^2) = x_2 - x_1$. If $x_1 \neq x_2$, we must have $x_1^2 + x_1x_2 + x_2^2 = -\frac{1}{3}$, which is impossible for integers x_1, x_2. Thus, $x_1 = x_2$ and f is one-to-one.

On the other hand, f is not onto. In particular, 1 is not in the range of f since $1 = f(a)$ for some a implies $3a^3 + a = 1$; that is, $a(3a^2 + 1) = 1$. But the only pairs of integers whose product is 1 are the pairs $1, 1$ and $-1, -1$. So here, we'd have to have $a = 3a^2 + 1 = 1$ or $a = 3a^2 + 1 = -1$, neither of which is possible.

21. (a) The graph of g shown at the right makes it clear that g is onto but not one-to-one.



(b) Since $g(0) = g(1)$, g is not one-to-one. Furthermore, g is not onto; for example, $2 = g(x)$ has no **integer** solution since $2 = x^3 - x + 1$ implies $x(x^2 - 1) = 1$ (for an integer x), so $x = x^2 - 1 = 1$ or $x = x^2 - 1 = -1$, neither of which is possible.

(c) The argument given in (b) applies to this case as well, so g is not onto. To determine whether or not g is one-to-one, suppose $g(x_1) = g(x_2)$. Then $x_1^3 - x_1 + 1 = x_2^3 - x_2 + 1$, so $(x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1 - x_2$. If $x_1 \neq x_2$, we must have $x_1^2 + x_1x_2 + x_2^2 = 1$. For natural numbers x_1, x_2, this is not possible because $x_1^2 + x_1x_2 + x_2^2 \geq 3$. Thus, $x_1 = x_2$ and g is one-to-one.

22. (a) [BB] This is not one-to-one since, for example, $f(1,3) = f(4,1) = 11$ while $(1,3) \neq (4,1)$. The function is not onto since, for example, the equation $f(x) = 1$ has no solution $x = (n,m)$ (because $2n + 3m \geq 5$ for every $n, m \in \mathsf{N}$).

(b) [BB] This is not one-to-one since, for example, $f(1,3) = f(4,1) = 11$ while $(1,3) \neq (4,1)$. The function is onto, however, since for $k \in \mathsf{Z}$, the equation $k = f(x)$ has the solution $x = (-k, k)$.

(c) This is not one-to-one since, for example, $f(12,1) = f(1,8)$ $(= 190)$ but $(12,1) \neq (1,8)$. It's not onto because $f(n,m)$ is even for any (n,m) so, for example, the equation $f(n,m) = 1$ cannot be solved.

(d) This is not one-to-one since, for example, $f(246,0) = f(0,89)$. It's onto since $(-17)(246) + 47(89) = 1$ implies that for any $k \in \mathsf{Z}$, $k = (-17k)(246) + (47k)(89) = f(47k, -17k)$.

(e) This is not one-to-one because, for example, $f(7,8) = f(8,7)$. It's not onto because, for example, the equation $f(x) = 4$ has no solution $x = (n,m)$: $n^2 + m^2 + 1 \neq 4$ for $n, m \in \mathsf{Z}$.

(f) This is not one-to-one because, for example, $f(2,2) = f(3,2)$ $(= 2)$. It is onto, however, because $1 = f(1,2)$ and, for $n \geq 2$, we have $n = f(n-1, 1)$.

23. ($\rightarrow$) Suppose f is one-to-one and $x_1 \neq x_2$. We must prove that $x_1^2+x_1x_2+x_2^2+a(x_1+x_2)+b \neq 0$. Since f is one-to-one and $x_1 \neq x_2$, we know that $f(x_1) \neq f(x_2)$; thus, $x_1^3 + ax_1^2 + bx_1 + c \neq x_2^3 + ax_2^2 + bx_2 + c$; equivalently,

$$(x_1^3 - x_2^3) + a(x_1^2 - x_2^2) + b(x_1 - x_2) \neq 0.$$

Since $x_1 - x_2 \neq 0$, we may divide this last inequality by $x_1 - x_2$. Since $x_1^3 - x_2^3 = (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2)$ and $x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2)$, we obtain $(x_1^2 + x_1x_2 + x_2^2) + a(x_1 + x_2) + b \neq 0$ as required.

($\leftarrow$) Suppose that $x_1 \neq x_2$ implies $x_1^2 + x_1x_2 + x_2^2 + a(x_1 + x_2) + b \neq 0$. We must prove that f is one-to-one. Thus, we assume $f(x_1) = f(x_2)$ and prove $x_1 = x_2$. The equation $f(x_1) = f(x_2)$ says $x_1^3 + ax_1^2 + bx_1 + c = x_2^3 + ax_2^2 + bx_2 + c$; or, equivalently, as before,

$$(x_1^3 - x_2^3) + a(x_1^2 - x_2^2) + b(x_1 - x_2) = 0.$$

If $x_1 \neq x_2$, then we divide by $x_1 - x_2$ as before, obtaining $x_1^2 + x_1x_2 + x_2^2 + a(x_1 + x_2) + b = 0$, a contradiction. Thus $x_1 = x_2$.

24. (a) [BB] We require $x \neq 3$, so we take $A = \{x \in \mathsf{R} \mid x \neq 3\}$. Then $\operatorname{rng} f = \{y \mid y \neq 0\}$ because if $y \neq 0$, $y = f(x)$ for $x = 3 + \frac{1}{y} \in \operatorname{dom} f$.

    (b) We require $1 - x > 0$, so we take $A = \{x \in \mathsf{R} \mid x < 1\}$. Then $\operatorname{rng} f = \{y \mid y > 0\}$ because for any $y > 0$, $y = f(x)$ for $x = 1 - \frac{1}{y^2} \in \operatorname{dom} f$.

25. (a) The function is not one-to-one because (a, α) and (d, α) are both in f, but $a \neq d$. Restricting the domain to $\{a, b, c\}$, for instance, we obtain the one-to-one function $\{(a, \alpha), (b, \beta), (c, \gamma)\}$.

    (b) [BB] Note that $f(x) = -(2x - 3)^2$. This function is not one-to-one since, for example, $f(1) = f(2)$ $(= -1)$. Restrict the domain to $\{x \mid x \geq 3/2\}$ (or to $\{x \mid x \leq \frac{3}{2}\}$).

    (c) The sine function is not one-to-one since, for example, $\sin 0 = \sin \pi$. The largest interval on which a one-to-one function can be defined has length π; for example, $\{x \mid -\pi/2 \leq x \leq \pi/2\}$ is an interval on which sine is one-to-one.

26. (a) **Reflexive:** If $f \in A$, then $f(5) = f(5)$, so $f \sim f$.
    Symmetric: If $f \sim g$, then $f(5) = g(5)$, so $g(5) = f(5)$ and hence, $g \sim f$.
    Transitive: If $f \sim g$ and $g \sim h$, then $f(5) = g(5)$ and $g(5) = h(5)$, so $f(5) = g(5) = h(5)$ and $f \sim h$.

    (b) $\overline{f} = \{g\colon S \to S \mid g(5) = f(5) = a\}$. Since there are three possible images for each of $g(a)$ and $g(b)$, there are altogether 9 functions in $\overline{f}$; namely,

$$\begin{aligned} f_1 &= \{(5,a),(a,5),(b,5)\} & f_2 &= \{(5,a),(a,5),(b,a)\} \\ f_3 &= \{(5,a),(a,5),(b,b)\} & f_4 &= \{(5,a),(a,a),(b,5)\} \\ f_5 &= \{(5,a),(a,a),(b,a)\} & f_6 &= \{(5,a),(a,a),(b,b)\} \\ f_7 &= \{(5,a),(a,b),(b,5)\} & f_8 &= \{(5,a),(a,b),(b,a)\} \\ f_9 &= \{(5,a),(a,b),(b,b)\} \end{aligned}$$

27. (a) **Reflexive:** If $a \in A$, then $a \sim a$ because $f(a) = f(a)$.
    Symmetric: If $a, b \in A$ and $a \sim b$, then $f(a) = f(b)$, so $f(b) = f(a)$; hence, $b \sim a$.
    Transitive: If $a, b, c \in A$, $a \sim b$ and $b \sim c$, then $f(a) = f(b)$ and $f(b) = f(c)$, so $f(a) = f(c)$, implying $a \sim c$.

(b) $\overline{0} = \{x \in \mathsf{R} \mid f(x) = f(0) = 0\} = [0,1)$; $\overline{7/5} = [1,2)$ since $\lfloor 7/5 \rfloor = 1$; $\overline{-3/4} = [-1,0)$ since $\lfloor -3/4 \rfloor = -1$.

(c) $\overline{1} = \{1\}$; $\overline{2} = \{2,3,6\}$; $\overline{4} = \{4\}$; $\overline{5} = \{5\}$.

28. (a) [BB] Here are the functions $X \to Y$:

$$\begin{array}{lll} \{(a,1),(b,1)\} & \{(a,1),(b,2)\} & \{(a,1),(b,3)\} \\ \{(a,2),(b,1)\} & \{(a,2),(b,2)\} & \{(a,2),(b,3)\} \\ \{(a,3),(b,1)\} & \{(a,3),(b,2)\} & \{(a,3),(b,3)\}. \end{array}$$

Here are the functions $Y \to X$:

$$\begin{array}{ll} \{(1,a),(2,a),(3,a)\} & \{(1,a),(2,a),(3,b)\} \\ \{(1,a),(2,b),(3,a)\} & \{(1,a),(2,b),(3,b)\} \\ \{(1,b),(2,a),(3,a)\} & \{(1,b),(2,a),(3,b)\} \\ \{(1,b),(2,b),(3,a)\} & \{(1,b),(2,b),(3,b)\}. \end{array}$$

(b) [BB] There are no one-to-one functions $Y \to X$. The one-to-one functions from $X \to Y$ are

$$\begin{array}{lll} \{(a,1),(b,2)\} & \{(a,1),(b,3)\} & \{(a,2),(b,1)\} \\ \{(a,2),(b,3)\} & \{(a,3),(b,1)\} & \{(a,3),(b,2)\}. \end{array}$$

(c) [BB] There are no onto functions $X \to Y$. The onto functions $Y \to X$ are

$$\begin{array}{ll} \{(1,a),(2,a),(3,b)\} & \{(1,a),(2,b),(3,a)\} \\ \{(1,a),(2,b),(3,b)\} & \{(1,b),(2,a),(3,a)\} \\ \{(1,b),(2,a),(3,b)\} & \{(1,b),(2,b),(3,a)\}. \end{array}$$

29. (a) There are no one-to-one functions $Y \to X$. Here are the 24 one-to-one functions $X \to Y$.

$$\begin{array}{lll} \{(a,1),(b,2),(c,3)\} & \{(a,1),(b,2),(c,4)\} & \{(a,1),(b,3),(c,2)\} \\ \{(a,1),(b,3),(c,4)\} & \{(a,1),(b,4),(c,2)\} & \{(a,1),(b,4),(c,3)\} \\ \{(a,2),(b,3),(c,1)\} & \{(a,2),(b,3),(c,4)\} & \{(a,2),(b,4),(c,1)\} \\ \{(a,2),(b,4),(c,3)\} & \{(a,2),(b,1),(c,3)\} & \{(a,2),(b,1),(c,4)\} \\ \{(a,3),(b,1),(c,2)\} & \{(a,3),(b,1),(c,4)\} & \{(a,4),(b,1),(c,2)\} \\ \{(a,4),(b,1),(c,3)\} & \{(a,3),(b,2),(c,1)\} & \{(a,3),(b,2),(c,4)\} \\ \{(a,4),(b,2),(c,1)\} & \{(a,4),(b,2),(c,3)\} & \{(a,3),(b,4),(c,1)\} \\ \{(a,3),(b,4),(c,2)\} & \{(a,4),(b,3),(c,1)\} & \{(a,4),(b,3),(c,2)\} \end{array}$$

(b) There are no onto functions $X \to Y$. Here are the 36 onto functions $Y \to X$.

$$\begin{array}{lll} \{(1,a),(2,a),(3,b),(4,c)\} & \{(1,a),(2,a),(3,c),(4,b)\} & \{(1,b),(2,b),(3,a),(4,c)\} \\ \{(1,b),(2,b),(3,c),(4,a)\} & \{(1,c),(2,c),(3,a),(4,b)\} & \{(1,c),(2,c),(3,b),(4,a)\} \\ \{(1,a),(2,b),(3,a),(4,c)\} & \{(1,a),(2,c),(3,a),(4,b)\} & \{(1,b),(2,a),(3,b),(4,c)\} \\ \{(1,b),(2,c),(3,b),(4,a)\} & \{(1,c),(2,a),(3,c),(4,b)\} & \{(1,c),(2,b),(3,c),(4,a)\} \\ \{(1,a),(2,b),(3,c),(4,a)\} & \{(1,a),(2,c),(3,b),(4,a)\} & \{(1,b),(2,a),(3,c),(4,b)\} \\ \{(1,b),(2,c),(3,a),(4,b)\} & \{(1,c),(2,a),(3,b),(4,c)\} & \{(1,c),(2,b),(3,a),(4,c)\} \\ \{(1,b),(2,a),(3,a),(4,c)\} & \{(1,c),(2,a),(3,a),(4,b)\} & \{(1,a),(2,b),(3,b),(4,c)\} \\ \{(1,c),(2,b),(3,b),(4,a)\} & \{(1,a),(2,c),(3,c),(4,b)\} & \{(1,b),(2,c),(3,c),(4,a)\} \\ \{(1,b),(2,a),(3,c),(4,a)\} & \{(1,c),(2,a),(3,b),(4,a)\} & \{(1,a),(2,b),(3,c),(4,b)\} \\ \{(1,c),(2,b),(3,a),(4,b)\} & \{(1,a),(2,c),(3,b),(4,c)\} & \{(1,b),(2,c),(3,a),(4,c)\} \\ \{(1,b),(2,c),(3,a),(4,a)\} & \{(1,c),(2,b),(3,a),(4,a)\} & \{(1,a),(2,c),(3,b),(4,b)\} \\ \{(1,c),(2,a),(3,b),(4,b)\} & \{(1,a),(2,b),(3,c),(4,c)\} & \{(1,b),(2,a),(3,c),(4,c)\} \end{array}$$

30. [BB]

$m \backslash n$	1	2	3	4
1	1	2	3	4
2	1	4	9	16
3	1	8	27	64
4	1	16	81	256

We guess that the number of functions $X \to Y$ is n^m.

31. (a) Let $A = \{a_1, a_2, \ldots, a_n\}$.
If $f\colon A \to B$ were one-to-one, then $f(a_1), f(a_2), \ldots, f(a_n)$ would be n different elements in B contradicting the fact that B has only $m < n$ elements.

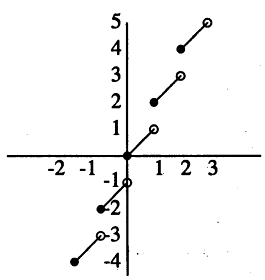
(b) Let $B = \{b_1, b_2, \ldots, b_m\}$. If $f\colon A \to B$ were onto, then there would exist elements $a_1, a_2, \ldots, a_m$ in A such that $f(a_i) = b_i$, distinct by definition of "function." But this contradicts the fact that A has $n < m$ elements.

32. (a) [BB ($\longrightarrow$)] Suppose A and B each contain n elements. Assume that $f\colon A \to B$ is one-to-one and let $C = \{f(a) \mid a \in A\}$. Since $f(a_1) \neq f(a_2)$ if $a_1 \neq a_2$, C is a subset of B containing n elements; so $C = B$. Therefore, f is onto.
Conversely, suppose f is onto. Then $\{f(a) \mid a \in A\} = B$ and so the set on the left here contains n distinct elements. Since A contains only n elements, we cannot have $f(a_1) = f(a_2)$ for distinct a_1, a_2; thus, f is one-to-one.

(b) $f(a) = 2a$ for all $a \in \mathsf{N}$. This does not contradict (a) because N is not finite.

(c) $f(1) = 1$, $f(n) = n - 1$ for $n \geq 2$. Again there is no contradiction since N is not finite.

33. (a) [BB]



(b) [BB] The domain of f is R. The range is the set of all real numbers of the form $2n + a$, where $n \in \mathsf{Z}$ and $0 \leq a < 1$. To prove this algebraically, we first note that any $y = 2n + a$ is $f(x)$ for $x = n + a$. On the other hand, suppose $y = f(x)$ for some x. Writing $x = n + a$, $n = \lfloor x \rfloor$, $0 \leq a < 1$, then $y = x + \lfloor x \rfloor = (n + a) + n = 2n + a$ as required.

34. s is not one-to-one since, for example, $s(\frac{1}{2}) = s(1\frac{1}{2})$ $(= \frac{1}{2})$. Neither is s onto since, for any x, $x - \lfloor x \rfloor$ is in the interval $[0, 1) = \{x \in \mathsf{R} \mid 0 \leq x < 1\}$, so, for example, $1 \neq s(x)$ for any x.

Exercises 3.2

1. [BB]

(a) $f^{-1} = \{(1,5), (2,1), (3,2), (4,3), (5,4)\}$

(b) $f^{-1} = \{(1,4), (2,1), (3,3), (4,2), (5,5)\}$

2. [BB] Let $y = f^{-1}(x)$. Then $x = f(y) = -y^2$, so $y^2 = -x$ and $y = \pm\sqrt{-x}$. Since $y \in \operatorname{dom} f$, $y \geq 0$, thus $y = +\sqrt{-x} = f^{-1}(x)$.

3. Let $y = f^{-1}(x)$. Then $x = f(y) = y^2$, so $y = \pm\sqrt{x}$. Since $y \in \operatorname{dom} f$, $y \leq 0$, so $y = -\sqrt{x} = f^{-1}(x)$.

4. Let $y = f^{-1}(x)$. Then $x = f(y) = -\sqrt{y}$, so $y = x^2 = f^{-1}(x)$.

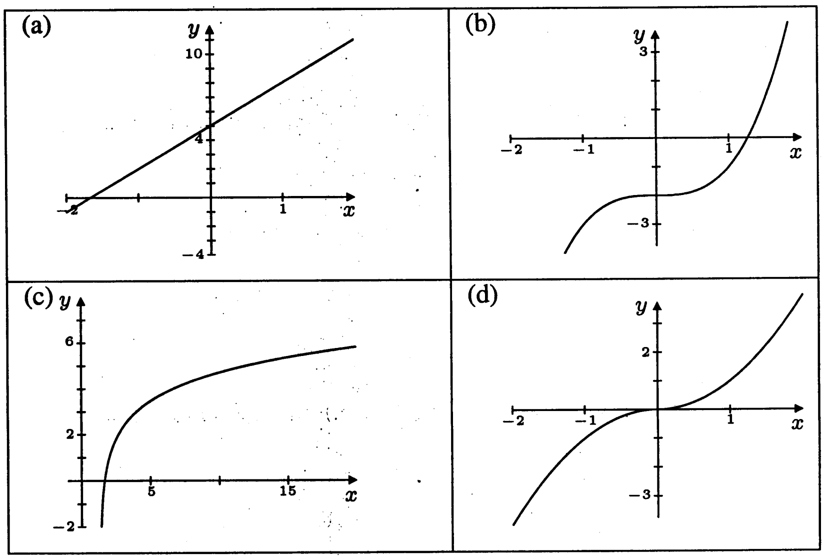
5. Since $g(x)$ is an integer, $f \circ g(x) = g(x)$. Similarly, $g \circ f(x) = f(x)$.

6. (a) [BB] $f^{-1}\colon \mathsf{R} \to \mathsf{R}$ is defined by $f^{-1}(x) = \frac{1}{3}(x-5)$.

   (b) $f^{-1}\colon \mathsf{R} \to \mathsf{R}$ is defined by $f^{-1}(x) = (x+2)^{1/3}$.

   (c) $\beta^{-1}\colon \mathsf{R} \to (\frac{4}{3}, \infty)$ is given by $\beta^{-1}(x) = \frac{1}{3}(2^x + 4)$.

   (d) $g^{-1}\colon \mathsf{R} \to \mathsf{R}$ is defined by $g^{-1}(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0. \end{cases}$



7. (a) [BB] If $f(x_1) = f(x_2)$, then $1 + \dfrac{1}{x_1 - 4} = 1 + \dfrac{1}{x_2 - 4}$, $\dfrac{1}{x_1 - 4} = \dfrac{1}{x_2 - 4}$ and $x_1 - 4 = x_2 - 4$. Thus $x_1 = x_2$ and f is one-to-one. Next

$$\begin{aligned} y \in \operatorname{rng} f &\leftrightarrow y = f(x) \quad \text{for some } x \in A \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y = 1 + \frac{1}{x-4} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y - 1 = \frac{1}{x-4} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } (y-1)(x-4) = 1 \\ &\leftrightarrow y \neq 1. \end{aligned}$$

Thus $\operatorname{rng} f = B = \{y \in \mathsf{R} \mid y \neq 1\}$ and f has an inverse $B \to A$. To find a formula for $f^{-1}(x)$, let $y = f^{-1}(x)$, $x \in B$. Then $x = f(y) = 1 + \dfrac{1}{y-4}$, so $x - 1 = \dfrac{1}{y-4}$, $(x-1)(y-4) = 1$ and, since $x \neq 1$, $y - 4 = \dfrac{1}{x-1}$ and $f^{-1}(x) = y = 4 + \dfrac{1}{x-1}$.

(b) Suppose $f(x_1) = f(x_2)$. Then $5 - \dfrac{1}{1+x_1} = 5 - \dfrac{1}{1+x_2}$, so $\dfrac{1}{1+x_1} = \dfrac{1}{1+x_2}$, $1 + x_1 = 1 + x_2$

and $x_1 = x_2$. Thus f is one-to-one. Next

$$\begin{aligned} y \in \operatorname{rng} f &\leftrightarrow y = f(x) \quad \text{for some } x \in A \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y = 5 - \frac{1}{1+x} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y - 5 = -\frac{1}{1+x} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } (y-5)(1+x) = -1 \\ &\leftrightarrow y \neq 5. \end{aligned}$$

Thus $\operatorname{rng} f = B = \{y \in \mathsf{R} \mid y \neq 5\}$ and f has an inverse $B \to A$. To find a formula for $f^{-1}(x)$, let $y = f^{-1}(x)$, $x \in B$. Then $x = f(y) = 5 - \dfrac{1}{1+y}$, so $x - 5 = -\dfrac{1}{1+y}$, $(x-5)(1+y) = -1$ and, since $x \neq 5$, $1 + y = -\dfrac{1}{x-5}$ and $f^{-1}(x) = y = -1 - \dfrac{1}{x-5}$.

(c) Suppose $f(x_1) = f(x_2)$. Then $\dfrac{3x_1}{2x_1+1} = \dfrac{3x_2}{2x_2+1}$, so $6x_1x_2 + 3x_1 = 6x_1x_2 + 3x_2$ and $x_1 = x_2$. Thus f is one-to-one. Now

$$\begin{aligned} y \in \operatorname{rng} f &\leftrightarrow y = f(x) \quad \text{for some } x \in A \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y = \frac{3x}{2x+1} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } 2xy + y = 3x \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } x(2y-3) = -y \\ &\leftrightarrow y \neq \frac{3}{2}. \end{aligned}$$

Thus $\operatorname{rng} f = B = \{y \in \mathsf{R} \mid y \neq \frac{3}{2}\}$ and f has an inverse $B \to A$. To find a formula for $f^{-1}(x)$, let $y = f^{-1}(x)$, $x \in B$. Then $x = f(y) = \dfrac{3y}{2y+1}$, so $2xy + x = 3y$ and $y(2x-3) = -x$. Since $x \in B$, we know $2x - 3 \neq 0$; thus, $y = \dfrac{-x}{2x-3} = f^{-1}(x)$.

(d) Suppose $f(x_1) = f(x_2)$. Then $\dfrac{x_1-3}{x_1+3} = \dfrac{x_2-3}{x_2+3}$, so $x_1x_2 + 3x_1 - 3x_2 - 9 = x_1x_2 - 3x_1 + 3x_2 - 9$, $6x_1 = 6x_2$ and $x_1 = x_2$. Thus f is one-to-one. Next

$$\begin{aligned} y \in \operatorname{rng} f &\leftrightarrow y = f(x) \quad \text{for some } x \in A \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y = \frac{x-3}{x+3} \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } y(x+3) = x - 3 \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } x - yx = 3y + 3 \\ &\leftrightarrow \text{there is an } x \in A \text{ such that } x(1-y) = 3(y+1) \\ &\leftrightarrow y \neq 1. \end{aligned}$$

Thus $\operatorname{rng} f = B = \{y \in \mathsf{R} \mid y \neq 1\}$ and f has an inverse $B \to A$. Let $y = f^{-1}(x)$ with $x \in B$. Then $x = f(y) = \dfrac{y-3}{y+3}$, so $x(y+3) = y - 3$, $xy + 3x = y - 3$, $y(1-x) = 3(x+1)$. Since $x \neq 1$, $f^{-1}(x) = y = \dfrac{3(1+x)}{1-x}$.

8. (a) First we show that f is one-to-one. Suppose then that $f(x_1) = f(x_2)$. If x_1 and x_2 are both negative, then $2|x_1| = 2|x_2|$. Since $|x| = -x$ for $x < 0$, we have $-2x_1 = -2x_2$ and hence, $x_1 = x_2$. If x_1 and x_2 are both ≥ 0, we have $2x_1 + 1 = 2x_2 + 1$, so $2x_1 = 2x_2$ and again, $x_1 = x_2$. Finally, we note that it is impossible for $f(x_1) = f(x_2)$ with $x_1 < 0$ and $x_2 \geq 0$ (or vice versa) since in this case, one of $f(x_1), f(x_2)$ is an even integer while the other is odd.
   Next we show that f is onto. Suppose that $n \in \mathsf{N}$. If $n = 2k$ is even, then $n = 2|-k| = f(-k)$ since $-k < 0$, while if $n = 2k + 1$ is odd, then $n = f(k)$ since $k \geq 0$. Since f is one-to-one and onto, it has an inverse.
   (b) Let $f^{-1}(2586) = a$. We must solve the equation $2586 = f(a)$. Since 2586 is even, we see that $2586 = f(-2586/2) = f(-1293)$.

9. (a) [BB] maternal grandmother (b) paternal grandmother (c) maternal grandfather
   (d) mother-in-law (e) father (f) father-in-law
   (g) you (h) maternal grandmother (i) maternal grandmother

10. (a) [BB] $f \circ g = \{(1,1),(3,8),(2,1),(4,9),(5,1)\}$;
    $g \circ f$ is not defined because $\operatorname{rng} f = \{1,2,3,8,9\} \not\subseteq \operatorname{dom} g = \{1,2,3,4,5\}$.
    $f \circ f$ is not defined because $\operatorname{rng} f = \{1,2,3,8,9\} \not\subseteq \operatorname{dom} f = \{1,2,3,4,5\}$.
    $g \circ g = \{(1,2),(2,2),(3,2),(4,1),(5,2)\}$.
    (b) f is one-to-one and onto;
    g is not one-to-one: It contains both $(1,2)$ and $(2,2)$;
    g is not onto: 4 is not in the range of g.
    (c) Since f is one-to-one and onto, it has an inverse: $f^{-1} = \{(8,1),(9,3),(3,4),(1,2),(2,5)\}$.
    (d) Since g is not one-to-one (or since g is not onto), g does not have an inverse.

11. (a) [BB] $g^{-1} \circ f \circ g = \{(1,3),(2,1),(3,2),(4,4)\}$; (b) $f \circ g^{-1} \circ g = f$;
    (c) $g \circ f \circ g^{-1} = \{(1,2),(2,4),(3,3),(4,1)\}$; (d) $g \circ g^{-1} \circ f = f$;
    (e) $f^{-1} \circ g^{-1} \circ f \circ g = \{(1,1),(2,4),(3,2),(4,3)\}$.

12. [BB] $g \circ f(x) = \dfrac{1}{(x+2)^2+1} = \dfrac{1}{x^2+4x+5}$; $f \circ g(x) = \dfrac{1}{x^2+1} + 2 = \dfrac{1+2(x^2+1)}{x^2+1} = \dfrac{2x^2+3}{x^2+1}$;
    $h \circ g \circ f(x) = 3$; $g \circ h \circ f(x) = g(3) = \frac{1}{10}$.
    Since $f^{-1} \circ f(x) = x$, we have $g \circ f^{-1} \circ f(x) = g(x) = \dfrac{1}{x^2+2}$.
    Since $f^{-1}(x) = x - 2$, we have

$$f^{-1} \circ g \circ f(x) = f^{-1}\Big(\frac{1}{(x+2)^2+1}\Big) = \frac{1}{(x+2)^2+1} - 2 = \frac{-2(x+2)^2-1}{(x+2)^2+1}.$$

13. $g \circ f(x) = \dfrac{1}{\frac{x}{x+1}} = \dfrac{x+1}{x}$; $\quad f \circ g(x) = \dfrac{\frac{1}{x}}{\frac{1}{x}+1} = \dfrac{1}{x+1}$;

$$h \circ g \circ f(x) = h\Big(\frac{x+1}{x}\Big) = \frac{x+1}{x} + 1 = \frac{2x+1}{x}; \quad f \circ g \circ h(x) = f\Big(\frac{1}{x+1}\Big) = \frac{\frac{1}{x+1}}{\frac{1}{x+1}+1} = \frac{1}{x+2}.$$

14. [BB]$(g \circ f)(x) = g(f(x)) = f(x) - c$. Thus the graph of $g \circ f$ is the graph of f translated vertically c units down if $c > 0$ and $-c$ units up if $c < 0$. The graphs are identical if $c = 0$.

15. $(f \circ g)(x) = f(g(x)) = f(x-c)$. Thus the graph of $f \circ g$ is the graph of f, but translated horizontally c units to the right if $c > 0$ and $-c$ units to the left if $c < 0$. The graphs are identical if $c = 0$.

16. (a) [BB] Since $-|x| = \begin{cases} -x & \text{if } x \ge 0 \\ x & \text{if } x < 0 \end{cases}$ we have $f \circ g(x) = f(-|x|) = \begin{cases} f(-x) & \text{if } x \ge 0 \\ f(x) & \text{if } x < 0. \end{cases}$
So the graph of $f \circ g$ is the same as the graph of f to the left of the y-axis (where $x < 0$) while to the right of the y-axis, the graph of $f \circ g$ is the reflection (mirror image) of the left half of the graph of f in the y-axis. We call $f \circ g$ an *even* function since it is symmetric with respect to the y-axis: $f \circ g(-x) = f \circ g(x)$.

(b) Since $g \circ f(x) = -|f(x)| = \begin{cases} -f(x) & \text{if } f(x) \ge 0 \\ f(x) & \text{if } f(x) < 0 \end{cases}$ the graph of $g \circ f$ is the same as the graph of f wherever the graph of f is below the x-axis ($y < 0$) while it is the reflection (mirror image) of the graph of f in the x-axis wherever the graph of f is above the x-axis. (In particular, the graph of $g \circ f$ lies entirely on or below the x-axis.)

17. (a) $f \circ g(x) = f(g(x)) = f(\frac{1}{1-x}) = 1 - \dfrac{1}{\frac{1}{1-x}} = 1 - (1-x) = \iota(x)$.

$g \circ r(x) = g(r(x)) = g(\frac{x}{x-1}) = \dfrac{1}{1 - \frac{x}{x-1}} = \frac{x-1}{(x-1)-x} = \frac{x-1}{-1} = 1 - x = s(x)$.

$\circ$	ι	f	g	h	r	s
ι	ι	f	g	h	r	s
f	f	g	ι	s	h	r
g	g	ι	f	r	s	h
h	h	r	s	ι	f	g
r	r	s	h	g	ι	f
s	s	h	r	f	g	ι

(b) All these functions have inverses.

function	ι	f	g	h	r	s
inverse	ι	g	f	h	r	s

18. (a)

$\circ$	ι	f_1	f_2	f_3	f_4	f_5
ι	ι	f_1	f_2	f_3	f_4	f_5
f_1	f_1	ι	f_4	f_5	f_2	f_3
f_2	f_2	f_5	ι	f_4	f_3	f_1
f_3	f_3	f_4	f_5	ι	f_1	f_2
f_4	f_4	f_3	f_1	f_2	f_5	ι
f_5	f_5	f_2	f_3	f_1	ι	f_4

(b) All these functions have inverses.

function	ι	f_1	f_2	f_3	f_4	f_5
inverse	ι	f_1	f_2	f_3	f_5	f_4

19. (a) [BB] $f \circ g = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$; $g \circ f = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$
Clearly, $f \circ g \ne g \circ f$.

(b) $f^{-1} = \{(1,2),(2,1),(3,5),(4,3),(5,4)\}$; $g^{-1} = \{(1,3),(2,4),(3,1),(4,5),(5,2)\}$
Functions f and g have inverses because they are one-to-one and onto while h does not have an inverse because it is not one-to-one (equally because it is not onto).

(c) $(f \circ g)^{-1} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$
$g^{-1} \circ f^{-1} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\} = (f \circ g)^{-1}$.
$f^{-1} \circ g^{-1} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\} \neq (f \circ g)^{-1}$.

20. (a) [BB] For $x \in B$, $(f \circ g)(x) = f(\frac{2x}{x-1}) = \dfrac{\frac{2x}{x-1}}{\frac{2x}{x-1} - 2} = \dfrac{2x}{2x - 2(x-1)} = \dfrac{2x}{2} = x$.

(b) [BB] For $x \in A$, $(g \circ f)(x) = g(\frac{x}{x-2}) = \dfrac{2(\frac{x}{x-2})}{\frac{x}{x-2} - 1} = \dfrac{2x}{x-(x-2)} = \dfrac{2x}{2} = x$ and so, by Proposition 3.2.7, f and g are inverses.

21. (a) Suppose $0 \in A$ and let $a = f(0)$. Then $0 = f^{-1}(a) = \dfrac{1}{f(a)}$ which is not possible for any real number $f(a)$.

(b) Since f and f^4 each have domain A, we have only to prove that $f^4(a) = a$ for all $a \in A$. Let then a be some element of A. Let $a_1 = f(a)$, $a_2 = f(a_1) = f^2(a)$, $a_3 = f(a_2) = f^3(a)$ and $a_4 = f(a_3) = f^4(a)$. We must show that $a_4 = a$. From $a_1 = f(a)$, we have $a = f^{-1}(a_1) = \dfrac{1}{f(a_1)}$ and so $a_2 = f(a_1) = \dfrac{1}{a}$. From $a_3 = f(a_2)$, we obtain $a_2 = f^{-1}(a_3) = \dfrac{1}{f(a_3)}$ and so $a_4 = f(a_3) = \dfrac{1}{a_2} = \dfrac{1}{1/a} = a$ as desired.

22. (a) [BB] Suppose $g(b_1) = g(b_2)$ for $b_1, b_2 \in B$. Since f is onto, $b_1 = f(a_1)$ and $b_2 = f(a_2)$ for some $a_1, a_2 \in A$. Thus, $g(f(a_1)) = g(f(a_2))$; that is, $g \circ f(a_1) = g \circ f(a_2)$. Since $g \circ f$ is one-to-one, $a_1 = a_2$. Therefore, $b_1 = f(a_1) = f(a_2) = b_2$ proving that g is one-to-one.

(b) Given $b \in B$, we must find $a \in A$ such that $f(a) = b$. Consider $g(b) \in C$. Since $g \circ f \colon A \to C$ is onto, there is some $a \in A$ with $g \circ f(a) = g(b)$; that is, $g(f(a)) = g(b)$. But g one-to-one implies $f(a) = b$, so we have the desired element a.

23. (a) [BB] Suppose $f \colon A \to B$ and $g \colon B \to C$ are one-to-one. We prove that $g \circ f \colon A \to C$ is one-to-one. For this, suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$ for some $a_1, a_2 \in A$. Then $g(f(a_1)) = g(f(a_2))$ [an equation of the form $g(b_1) = g(b_2)$]. Since g is one-to-one, we conclude that $f(a_1) = f(a_2)$, and then, since f is one-to-one, that $a_1 = a_2$.

(b) Let $f = \{(1,1),(2,2)\}$ and $g = \{(-1,5),(1,5),(2,10)\}$. Then g is not 1–1, but the composition $g \circ f = \{(1,5),(2,10)\}$ is.

(c) Let f and g be functions, $f \colon A \to B$, $g \colon B \to C$ and suppose $g \circ f \colon A \to C$ is 1–1. Then f must be 1–1 since $f(a_1) = f(a_2)$ implies $g(f(a_1)) = g(f(a_2))$, that is, $g \circ f(a_1) = g \circ f(a_2)$, from which we conclude that $a_1 = a_2$, because $g \circ f$ is 1–1.

24. (a) Suppose $f \colon A \to B$ and $g \colon B \to C$ are onto. We prove that $g \circ f \colon A \to C$ is onto. Let then $c \in C$. We must find $a \in A$ such that $g \circ f(a) = c$; that is, $g(f(a)) = c$ [an equation of the form $g(b) = c$]. Since g is onto, there is $b \in B$ such that $g(b) = c$. Also, since f is onto, we know there is $a \in A$ such that $b = f(a)$. Thus, $g(f(a)) = g(b) = c$ as desired.

(b) [BB] With $A = \{1,2\}$, $B = \{-1,1,2\}$, $C = \{5,10\}$, $f = \{(1,1),(2,2)\}$ and $g = \{(-1,5),(1,5),(2,10)\}$, we have $g \circ f = \{(1,5),(2,10\}$. Then $g \circ f \colon A \to C$ is onto but f is not.

(c) Suppose $f \colon A \to B$, $g \colon B \to C$ and $g \circ f \colon A \to C$ is onto. Then g must be onto because if $c \in C$, there exists $a \in A$ with $g \circ f(a) = c$ ($g \circ f$ is onto), so $g(f(a)) = c$. Thus, there exists $b \in B$ with $g(b) = c$ ($b = f(a)$).

25. [BB] Since a bijective function is, by definition, a one-to-one onto function, we conclude, by the results of part (a) of the previous two exercises, that indeed the composition of bijective functions is bijective.

26. (a) $f(1000) = 998$; $f(999) = f(f(1003)) = f(1001) = 999$; $f(998) = f(f(1002)) = f(1000) = 998$; $f(997) = f(f(1001)) = f(999) = 999$.

(b) We guess that $f(n) = \begin{cases} 998 & \text{if } n \text{ is even} \\ 999 & \text{if } n \text{ is odd.} \end{cases}$ (c) We guess $\operatorname{rng} f = \{998, 999\}$.

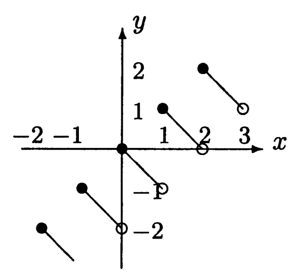
27. Suppose $f(x_1) = f(x_2)$. Then $\frac{x_1}{\sqrt{x_1^2+2}} = \frac{x_2}{\sqrt{x_2^2+2}}$, so $\frac{x_1^2}{x_1^2+2} = \frac{x_2^2}{x_2^2+2}$, $x_1^2x_2^2 + 2x_1^2 = x_1^2x_2^2 + 2x_2^2$, $2x_1^2 = 2x^2$ and $x_1 = \pm x_2$. Note that $x_2 = -x_1$ is not possible (unless $x_1 = x_2 = 0$) since $f(-x_1) = \frac{-x_1}{\sqrt{x_1^2+2}} \neq \frac{x_1}{\sqrt{x_1^2+2}} = f(x_1)$. Thus f is one-to-one. Next we note that

$$y \in \operatorname{rng} f \leftrightarrow y = f(x) \text{ for some } x \in \mathsf{R} \leftrightarrow y = \frac{x}{\sqrt{x^2+2}} \text{ for some } x \in \mathsf{R}.$$

Now, if $y = \frac{x}{\sqrt{x^2+2}}$, then $y^2(x^2+2) = x^2$, so $x^2(y^2-1) = -2y^2$ and this implies that $y^2 \neq 1$ (otherwise we have the equation $0 = 1$) and also $y^2 - 1 \leq 0$ because $x^2 \geq 0$, $-2y^2 \leq 0$ and the product of two nonnegative numbers is nonnegative. If $y \in \operatorname{rng} f$, then $y^2 - 1 < 0$; that is, $-1 < y < 1$. On the other hand, if $-1 < y < 1$, then $y^2 - 1 < 0$, $\frac{-2y^2}{y^2-1} \geq 0$, and so this element is x^2 for some x; hence $y \in \operatorname{rng} f$. This shows that $\operatorname{rng} f = B = (-1, 1)$. To find a formula for $f^{-1}(x)$, let $y = f^{-1}(x)$ for $x \in B$. Then $x = f(y) = \frac{y}{\sqrt{y^2+2}}$, so $x^2(y^2+2) = y^2$ and $y^2(x^2-1) = -2x^2$. Since $x^2 - 1 \neq 0$, $y^2 = \frac{2x^2}{1-x^2}$ and since the fraction on the right here is nonnegative, there are (apparently) two solutions $y = \pm\sqrt{\frac{2x^2}{1-x^2}}$. The equation $x = \frac{y}{\sqrt{y^2+2}}$, however, shows that x and y have the same sign. Thus

$$y = f^{-1}(x) = \begin{cases} \sqrt{\dfrac{2x^2}{1-x^2}} & \text{if } x > 0 \\ -\sqrt{\dfrac{2x^2}{1-x^2}} & \text{if } x < 0. \end{cases}$$

28. (a) From the graph, we see immediately that t is one-to-one and onto, so t has an inverse.



(b) [BB] Writing $x = \lfloor x \rfloor + a$, $0 \leq a < 1$, we have $t(x) = \lfloor x \rfloor - a$. Now it is straightforward to see that t is one-to-one: Suppose $t(x_1) = t(x_2)$, where $x_1 = \lfloor x_1 \rfloor + a_1$, $x_2 = \lfloor x_2 \rfloor + a_2$ and $0 \leq a_1, a_2 < 1$. Then $\lfloor x_1 \rfloor - a_1 = \lfloor x_2 \rfloor - a_2$, so $\lfloor x_1 \rfloor - \lfloor x_2 \rfloor = a_1 - a_2$. The left side is an integer; hence, so is the right. Because of the restrictions on a_1 and a_2, the only possibility is $a_1 = a_2$. Hence, also $\lfloor x_1 \rfloor = \lfloor x_2 \rfloor$, so $x_1 = x_2$.

(c) To see that t is onto, let $y \in \mathsf{R}$. Then we have $y = \lfloor y \rfloor + b$ for $0 \le b < 1$. If $b = 0$, then $t(y) = y$. Otherwise, $0 < 1 - b < 1$. So, with $x = \lfloor y \rfloor + 1 + (1 - b)$, we have $\lfloor x \rfloor = \lfloor y \rfloor + 1$. Thus, $t(x) = \lfloor x \rfloor - (1 - b) = (\lfloor y \rfloor + 1) - (1 - b) = \lfloor y \rfloor + b = y$. In any case, we have an x such that $t(x) = y$.

(d) $t^{-1} \colon \mathsf{R} \to \mathsf{R}$ is given by $t^{-1}(x) = \begin{cases} x & \text{if } x \in \mathsf{Z} \\ n + 1 + (1 - b) & \text{if } x = n + b, n \in \mathsf{Z}, 0 < b < 1 \end{cases}$

Exercises 3.3

1. [BB] Ask everyone to find a seat.

2. [BB] The two lists $1^2, 2^2, 3^2, 4^2, \ldots$ and $1, 2, 3, 4, \ldots$ obviously have the same length; $a^2 \mapsto a$ is a one-to-one correspondence between the set of perfect squares and N.

3. (a) $x \leftrightarrow 14$, $y \leftrightarrow -3$, $\{a, b, c\} \leftrightarrow t$.

   (b) The function $f \colon 2\mathsf{Z} \to 17\mathsf{Z}$ defined by $f(2k) = 17k$ for $2k \in 2\mathsf{Z}$.

   (c) [BB] The function $f \colon \mathsf{N} \times \mathsf{N} \to \mathsf{C}$ defined by $f(m, n) = m + ni$ for all $m, n \in \mathsf{N}$.

   (d) The function $f \colon \mathsf{N} \to \mathsf{Q}$ defined by $f(n) = n/2$.

4. $f^{-1}(z) = \begin{cases} 2z & \text{if } z > 0 \\ 1 - 2z & \text{if } z \le 0 \end{cases}$

5. (a) If $g(m_1, n_1) = g(m_2, n_2)$, then $(m_1, f(n_1)) = (m_2, f(n_2))$, so $m_1 = m_2$ and $f(n_1) = f(n_2)$. Since f is one-to-one, $n_1 = n_2$. Thus $(m_1, n_1) = (m_2, n_2)$, so g is one-to-one. To show that g is onto, let $(a, b) \in \mathsf{N} \times \mathsf{Z}$. Since f is onto, there exists $n \in \mathsf{N}$ such that $f(n) = b$. Then $g(a, n) = (a, f(n)) = (a, b)$, so g is onto.

   (b) Let $f \colon \mathsf{N} \to \mathsf{Z}$ be any one-to-one onto function (for example, the function defined in Problem 29, of this section). By part (a), the function $g \colon \mathsf{N} \times \mathsf{N} \to \mathsf{N} \times \mathsf{Z}$ defined by $g(m, n) = (m, f(n))$ is a one-to-one correspondence $\mathsf{N} \times \mathsf{N} \to \mathsf{N} \times \mathsf{Z}$.

6. [BB] This is false. For example, $|\mathsf{N}| = |\mathsf{N} \cup \{0\}|$, as shown in the text.

7. This is not a partial order because it is not antisymmetric. If $a, b \in S$ and $a \neq b$, then $\{a\} \preceq \{b\}$ because $|\{a\}| \le |\{b\}|$ and for the same reason, $\{b\} \preceq \{a\}$; however $\{a\} \neq \{b\}$.

8. [BB] In this case, either $S = \emptyset$ or $|S| = 1$. This time, in each case, $(S, \preceq)$ is a partial order.

   Case 1: If $S = \emptyset$, then $\mathcal{P}(S)$ contains a single element, $\emptyset$.

   Reflexive: Certainly $A \preceq A$ for all $A \in \mathcal{P}(S)$ since $0 = |\emptyset| \le |\emptyset|$.

   Antisymmetric: If $A \preceq B$ and $B \preceq A$, then $A = B$ since there is only one set in $\mathcal{P}(S)$.

   Transitive: If $A \preceq B$ and $B \preceq C$, then $A \preceq C$ since necessarily $A = B = C$ and for the single set A in $\mathcal{P}(S)$, $A \preceq A$.

   Case 2: If S contains one element, then $\mathcal{P}(S) = \{\emptyset, S\}$ contains two elements.

   Reflexive: $A \preceq A$ for each $A \in \mathcal{P}(S)$ because $|A| \le |A|$.

Antisymmetric: If $A, B \in \mathcal{P}(S)$, $A \preceq B$ and $B \preceq A$, then we have $|A| \leq |B|$ and $|B| \leq |A|$, so $|A| = |B|$. Since $\mathcal{P}(S)$ does not contain different sets of the same cardinality, it follows that $A = B$.

Transitive: Suppose $A \preceq B$ and $B \preceq C$. If $A = \emptyset$, then $|A| = 0 \leq |C|$ no matter what C is, so we'd have $A \preceq C$. If $A = S$, then $A \preceq B$ means $B = S$ and $B \preceq C$ means $C = S$, so $A = B = C = S$ and $A \preceq C$.

9. $f\colon A \times B \to B \times A$ defined by $f(a,b) = (b,a)$ is a one-to-one onto function.

10. (a) [BB] False. Let $X = \{1\}, Y = \{2\}, Z = \{3\}$. Then $((1,2),3) \in (X \times Y) \times Z$ but $((1,2),3) \notin X \times (Y \times Z)$ (a set whose **second** coordinates are ordered pairs).

    (b) [BB] Define $f\colon (X \times Y) \times Z \to X \times (Y \times Z)$ by $f((x,y),z) = (x,(y,z))$.

11. (a) **Reflexivity:** For any set A, ι_A is a one-to-one onto function $A \to A$, so A has the same cardinality as itself.

    Symmetry: If A and B have the same cardinality, then there is a one-to-one onto function $f\colon A \to B$. Such a function has an inverse $f^{-1}\colon B \to A$ which is one-to-one and onto (because it has an inverse), so B and A have the same cardinality.

    Transitivity: Suppose A, B and C are sets such that A and B have the same cardinality and B and C have the same cardinality. Then there is a one-to-one onto function $f\colon A \to B$ and a one-to-one onto function $g\colon B \to C$. Since the composition of one-to-one functions is one-to-one and the composition of onto functions is onto (Exercises 23 and 24 of Section 3.2), $g \circ f\colon A \to C$ is one-to-one and onto. Thus, A and C have the same cardinality.

    (b) [BB] By Problem 27, $(0,1)$ and $(1,\infty)$ have the same cardinality and $(0,1)$ and $(3,\infty)$ have the same cardinality. By transitivity of "same cardinality", $(1,\infty)$ and $(3,\infty)$ have the same cardinality too.

    (c) [BB] To find an explicit one-to-one onto function $(1,\infty) \to (3,\infty)$, we use the function f defined by $f(x) = \frac{1}{x} - 1 + 3 = \frac{1}{x} + 2$ from $(0,1)$ to $(3,\infty)$ exhibited in Problem 27 and the function g defined by $g(x) = \frac{1}{x+1-1} = \frac{1}{x}$ from $(1,\infty)$ to $(0,1)$, which is the inverse of the function exhibited in Problem 27—see line (3). The function we seek is the composition $f \circ g$, which is defined by $f \circ g(x) = f(g(x)) = \frac{1}{\frac{1}{x}} + 2 = x + 2$, a rather obvious choice!

    (d) $f(x) = x + b - a$

12. (a) [BB] As suggested just before Problem 28, we look for a function defined with a rule like $f(x) = kx + \ell$ and discover (the rather obvious) $f(x) = x + 1$.

    (b) The function defined by $f(x) = 2x + 4$ is a one-to-one correspondence between $(0,1)$ and $(4,6)$.

    (c) The function $f\colon (0,1) \to (a,b)$ defined by $f(x) = a + (b-a)x$ is a one-to-one correspondence, and the function $g\colon (0,1) \to (c,d)$ given by $g(x) = c + (d-c)x$ is also a one-to-one correspondence. Thus $g \circ f^{-1}$ provides a one-to-one correspondence $(a,b) \to (c,d)$. The reader may check that $g \circ f^{-1}(x) = c + \frac{d-c}{b-a}(x-a)$.

13. (a) [BB] Using the result of Problem 27, we obtain the function defined by $g(x) = \frac{1}{x} + 9$ as a one-to-one correspondence between $(0,1)$ and $(10,\infty)$.

(b) [BB] The function $f\colon (2,5) \to (0,1)$ defined by $f(x) = \frac{1}{3}x - \frac{2}{3}$ is a one-to-one correspondence, as is the function $g\colon (0,1) \to (10,\infty)$ defined by $g(x) = \dfrac{1}{x} + 9$. Thus, the composition $g \circ f\colon (2,5) \to (10,\infty)$ is also a one-to-one correspondence. Note that $(g \circ f)(x) = \dfrac{3}{x-2} + 9$.

(c) The function $f\colon (a,b) \to (0,1)$ defined by $f(x) = \dfrac{x-a}{b-a}$ is a one-to-one correspondence, as is the function $g\colon (0,1) \to (c,\infty)$ defined by $g(x) = \dfrac{1}{x} - 1 + c$. Thus a one-to-one correspondence $(a,b) \to (c,\infty)$ is the composition $g \circ f$. Note that $(g \circ f)(x) = \dfrac{b-a}{x-a} - 1 + c$.

14. In Problem 27, p. 129, we saw that the function $g\colon (0,1) \to (0,\infty)$ defined by $g(x) = \dfrac{1}{x} - 1$ is a one-to-one correspondence. Furthermore, the function $f\colon (a,b) \to (0,1)$ defined by $f(x) = \dfrac{x-a}{b-a}$ is a one-to-one correspondence. (See the remarks preceding Problem 28.) Thus $g \circ f\colon (a,b) \to (0,\infty)$ is a one-to-one correspondence. Note that $(g \circ f)(x) = \dfrac{b-a}{x-a} - 1$.

15. [BB] f is certainly a function from R to R$^+$ since $2^x > 0$ for all $x \in$ R. If $2^x = 2^y$, then $x \log 2 = y \log 2$ (any base), so $x = y$. Thus, f is one-to-one. If $r \in$ R$^+$, then $2^{\log_2 r} = r$, so f is onto. We conclude that R and R$^+$ have the same cardinality.

16. [BB] Since (a,b) has the same cardinality as R$^+$ by Exercise 14 and since R$^+$ and R have the same cardinality by Exercise 15, the result follows by transitivity of the notion of "same cardinality"—see Exercise 11.

17. f is certainly defined on $(0,1)$ and takes values in R. To show that f is one-to-one, assume $f(x_1) = f(x_2)$; that is, assume

$$\frac{x_1 - \frac{1}{2}}{x_1(x_1-1)} = \frac{x_2 - \frac{1}{2}}{x_2(x_2-1)}.$$

Then $x_2(x_2-1)(x_1 - \frac{1}{2}) = x_1(x_1-1)(x_2 - \frac{1}{2})$, so

$$x_1x_2^2 - \tfrac{1}{2}x_2^2 - x_2x_1 + \tfrac{1}{2}x_2 = x_2x_1^2 - \tfrac{1}{2}x_1^2 - x_1x_2 + \tfrac{1}{2}x_1.$$

Therefore,

$$x_1x_2^2 - x_2x_1^2 - \tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2 - \tfrac{1}{2}x_1 = 0$$

$$x_1x_2(x_2 - x_1) - \tfrac{1}{2}(x_2 + x_1)(x_2 - x_1) + \tfrac{1}{2}(x_2 - x_1) = 0$$

$$(x_2 - x_1)(2x_1x_2 - x_2 - x_1 + 1) = 0$$

and so $x_2 = x_1$ or $2x_1x_2 - x_2 - x_1 + 1 = 0$. The second possibility is the same as $x_1x_2 + (x_1 - 1)(x_2 - 1) = 0$ and, since for $0 < x_1, x_2 < 1$ both x_1x_2 and $(x_1-1)(x_2-1)$ are nonnegative, this case is impossible. We conclude that $x_1 = x_2$; thus, f is one-to-one.

To show that f is onto, let $r \in$ R. We wish to find an x such that $f(x) = r$; that is, we wish to find an x such that $\dfrac{x - \frac{1}{2}}{x(x-1)} = r$. Hence, we wish to solve $rx^2 - (r+1)x + \frac{1}{2} = 0$. If $r = 0$, the solution is

$x = \frac{1}{2}$. If $r \neq 0$, the quadratic formula tells us that this equation has solutions

$$x = \frac{r+1 \pm \sqrt{(r+1)^2 - 2r}}{2r} = \frac{r+1 \pm \sqrt{r^2+1}}{2r}.$$

We claim that the solution

$$x = \frac{r+1-\sqrt{r^2+1}}{2r}$$

is always in $(0,1)$. First note that if $r > 0$, then $r+1 > \sqrt{r^2+1}$ since $(r+1)^2 > r^2+1$. On the other hand, if $r < 0$, then $r+1 < 1 < \sqrt{r^2+1}$. In either case, we have shown that

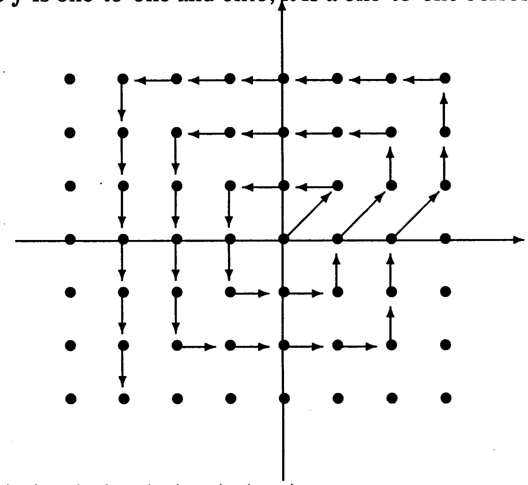
$$\frac{r+1-\sqrt{r^2+1}}{2r} > 0.$$

To show that this expression is also less than 1, we will prove the equivalent inequality

$$\frac{1-r-\sqrt{r^2+1}}{2r} < 0.$$

If $r > 0$, then $1 - r < 1 < \sqrt{r^2+1}$ giving the result. If $r < 0$, then $1 - r > \sqrt{r^2+1}$ because $(1-r)^2 > r^2+1$, again giving the result. Since f is one-to-one and onto, it is a one-to-one correspondence.

18. Start at $(0,0)$ and move as illustrated.



19. (a) $2, -2, 4, -4, 8, -8, 16, -16, \dots$
    (b) $1 = 2^0, 2, \frac{1}{2}, 4, \frac{1}{4}, 8, \frac{1}{8}, \dots$
    (c) $1, 4, 7, 10, 13, 16, 19, \dots$
    (d) $(1,1), (1,2), (1,3), (2,1), (2,2), (3,3), (3,1), (3,2), (3,3), (4,1), \dots$
    (e) [BB] Follow the procedure given in the text for all rational numbers and omit those with even denominators. The listing starts $1, 2, \frac{1}{3}, 3, 4, \frac{2}{3}, \frac{1}{5}, 5, 6, \frac{4}{3}, \frac{2}{5}, \frac{1}{7}, \frac{3}{5}, \frac{5}{3}, 7, \dots$.
    (f) Follow the procedure given in the text for N × N but with two modifications. We include an extra row across the bottom as follows: $(1,0)$ $(2,0)$ $(3,0)$ $(4,0)$ $\dots$. Then, when doing the listing, follow immediately every pair (a,b), $b > 0$, with the pair $(a,-b)$. The first few terms would be $(1,0)$, $(2,0)$, $(1,1)$, $(1,-1)$, $(1,2)$, $(1,-2)$, $(2,1)$, $(2,-1)$, $(3,0)$, $(4,0)$, $(3,1)$, $(3,-1)$, $(2,2)$, $(2,-2)$, ...
    (g) First of all, enumerate $\mathsf{N} \cup \{0\} \times \mathsf{N} \cup \{0\}$ by the diagonal procedure given in the text for N × N. The first few terms would be $(0,0)$, $(1,0)$, $(0,1)$, $(0,2)$, $(1,1)$, $(2,0)$, $(3,0)$, $(2,1)$, $(1,2)$, $(0,3)$. Then enumerate Z × Z by taking this list, replacing every element $(a,0)$, $a > 0$, with $(a,0), (-a,0)$; every element $(0,a)$, $a > 0$, with $(0,a), (0,-a)$, and every pair (a,b), $a,b > 0$ with (a,b), $(a,-b)$, $(-a,b)$, $(-a,-b)$.

20. (a) [BB] This set is uncountable. The function defined by $f(x) = x - 1$ gives a one-to-one correspondence between it and $(0, 1)$, which we showed in the text to be uncountable.

(b) This set is countably infinite. Just follow the sequence given in the text for the set of all positive rationals, but omit any rational number not in $(1, 2)$.

(c) This set is finite. In fact, it contains at most 99^2 elements since there are 99 possible numerators and, for each numerator, 99 possible denominators.

(d) This set is countably infinite. List the elements as follows (deleting any repetitions such as $5/100 = 1/20$): $\frac{99}{6}, \frac{99}{7}, \ldots, \frac{99}{104}, \frac{98}{6}, \frac{98}{7}, \ldots, \frac{98}{104}, \ldots$.

(e) [BB] This set is countably infinite. In Exercise 3(c) we showed it is in one-to-one correspondence with $\mathsf{N} \times \mathsf{N}$.

(f) This set is countably infinite because it is in one-to-one correspondence with Q via the function f defined by $f(a, b) = a$.

(g) This set is uncountable. It is in one-to-one correspondence with $[-1, 1] = \{x \mid -1 \leq x \leq 1\}$ via the function f defined by $f(a, b) = a$. The interval $[-1, 1]$ contains $(0, 1)$ which we showed in the text to be uncountable.

21. (a) Finite! There is some minimum volume V which a grain of sand must occupy. On the other hand, there is a finite value M for the volume of all the sand. So the number of grains is $\leq M/V$.

(b) Countably infinite. Here is a listing: $3^0, 3^1, 3^{-1}, 3^2, 3^{-2}, \ldots$.

(c) The set of sentences in the English language is certainly not finite for if it were, and S were the sentence with the most words, then "S and roses are red." would be a longer sentence. So we have to decide if the set of sentences is countably infinite or uncountable. In fact, it is countably infinite. To see this, first note that since there are only finitely many words (see part (c)) there are only finitely many sentences k words long for each $k = 1, 2, \ldots$. If $a_{k,1}, a_{k,2}, a_{k,3}, \ldots$ is a list of the sentences which are k words long (extended to a countably infinite list by defining all $a_{k,i} =$ "Jqrzx" after we run out of sentences), then the set of all sentences can be listed by the diagonal scheme that we used for $\mathsf{N} \times \mathsf{N}$: $a_{1,1}, a_{2,1}, a_{1,2}, a_{1,3}, a_{2,2} \ldots$, where we omit all occurrences of "Jqrzx."

22. (a) [BB] Impossible. To the contrary, suppose that the union were a finite set S. Since S has only finitely many subsets (the precise number is $2^{|S|}$), there could not have been infinitely many sets at the outset.

(b) Impossible. The maximum number of elements in the union of sets $A_1, \ldots, A_n$ occurs when each intersection $A_i \cap A_j$ is empty. Thus, $|A_1 \cup A_2 \cup \cdots \cup A_n| \leq |A_1| + |A_2| + \cdots + |A_n|$, which is finite.

(c) [BB] Impossible. If even one infinite set is contained in a **union**, then the union must be infinite.

23. [BB] Imagine S_1 sitting inside S_2, both spheres with the same center. Rays emanating from this common point establish a one-to-one correspondence between the points on S_1 and the points on S_2.

24. (a) [BB] Let $s_1, s_2, s_3, \ldots$ be a countably infinite subset of S. Define $f\colon S \to S \cup \{x\}$ by

$$\begin{aligned} f(s_1) &= x \\ f(s_{k+1}) &= s_k \quad \text{for } k \geq 1 \\ f(s) &= s \quad \text{if } s \notin \{s_1, s_2, s_3, \ldots\}. \end{aligned}$$

Then f is a one-to-one correspondence.

(b) This follows immediately from part (a) since $(0,1] = (0,1) \cup \{1\}$.

25. We employ a concept known as *stereographic projection.* Imagine the sphere sitting on the Cartesian plane with south pole at the origin. Any line from the north pole to the plane punctures the sphere at a unique point and the collection of such lines establishes a one-to-one correspondence between the points of the plane and the sphere except for the north pole. A small modification of this correspondence finishes the job. Suppose $p_0, p_1, p_2, \ldots$ are the points of the sphere which correspond to the points $(0,0), (1,0), (2,0), \ldots$ in the plane; thus, the line from the north pole to $(n,0)$ punctures the sphere at p_n (in particular, $p_0 = (0,0)$). Map the north pole to $(0,0)$, the origin to $(1,0)$, p_1 to $(2,0)$, and so forth and let all other points of the sphere go to the same points as before.

26. [BB] We are given that $A = \{a_1, a_2, \ldots, a_n\}$ for some $n \in \mathsf{N}$ and that $B = \{b_1, b_2, b_3, \ldots\}$. Then $A \cup B$ is countably infinite because it is infinite and its elements can be listed as follows:

$$a_1, a_2, \ldots, a_n, b_1, b_2, b_3, \ldots$$

The function $f \colon \mathsf{N} \to A \cup B$ corresponding to this listing is defined by $f(i) = \begin{cases} a_i & \text{if } i \leq n \\ b_{i-n} & \text{if } i > n. \end{cases}$

27. (a) We will do this by exhibiting a one-to-one correspondence between $A \times B$ and $\mathsf{N} \times \mathsf{N}$, which was shown in the text to be countable. Since A is countable, we can list its elements: $a_1, a_2, a_3, \ldots$. Since B is countable, we can list its elements: $b_1, b_2, b_3, \ldots$. Define $f \colon A \times B \to \mathsf{N} \times \mathsf{N}$ by $f(a_i, b_j) = (i,j)$. If $f(a_i, b_j) = f(a_k, b_\ell)$, then $(i,j) = (k,\ell)$ so $i = k, j = \ell$ and $(a_i, b_j) = (a_k, b_\ell)$. Thus, f is one-to-one. Also, for any $(m,n) \in \mathsf{N} \times \mathsf{N}$, we have $f(a_m, b_n) = (m,n)$, so f is onto. Thus, f is a one-to-one correspondence.

    (b) A polynomial of degree at most one with integer coefficients is an expression of the form $a + bx$, where $a, b \in \mathsf{Z}$. The function $a + bx \mapsto (a,b)$ is a one-to-one correspondence between these polynomials and the set $\mathsf{Z} \times \mathsf{Z}$, which is countable by Exercise 27(a). Thus the given set of polynomials is countable too.

28. We offer a proof by contradiction. If $|X| = |\mathcal{P}(X)|$, then there is a one-to-one onto function $f \colon X \to \mathcal{P}(X)$. Thus, for each $x \in X$, $f(x)$ is a subset of X. Define a subset Y of X as follows: for each $x \in X$, put $x \in Y$ if and only if $x \notin f(x)$. Since f is onto, $Y = f(y)$ for some $y \in X$. Notice that if $y \in Y$, then $y \in f(y)$, so $y \notin Y = f(y)$, by definition. On the other hand, if $y \notin f(y) = Y$, then $y \in Y$, again by definition. We reach the absurd situation that y is neither in or not in Y. It follows that there can exist no onto function $f \colon X \to \mathcal{P}(X)$.

29. Assume, to the contrary, that S is countable and, as in Problem 31, write each of its elements in a list as

$$\begin{aligned} a_1 &= 0.a_{11}a_{12}a_{13}a_{14}\cdots \\ a_2 &= 0.a_{21}a_{22}a_{23}a_{24}\cdots \\ a_3 &= 0.a_{31}a_{32}a_{33}a_{34}\cdots \\ &\vdots \end{aligned}$$

where each a_i is 3 or 4. Define the sequence $b_1, b_2, b_3, \ldots$ by $b_i = \begin{cases} 4 & \text{if } a_{ii} = 3 \\ 3 & \text{if } a_{ii} = 4. \end{cases}$

Then $b = 0.b_1b_2b_3\ldots$ is in S, yet it is different from each a_i because $b_i \neq a_{ii}$ for each i. This contradiction gives the result.