Instructor's Resource Manual

Differential Equations with Boundary Value Problems

EIGHTH EDITION

and

A First Course in Differential Equations

TENTH EDITION

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1 2 3 4 5 6 7 16 15 14 13 12

CONTENTS

1 INTRODUCTION TO DIFFERENTIAL EQUATIONS

1.1 Definitions and Terminology

1. Second order; linear

2. Third order; nonlinear because of $(dy/dx)^4$

3. Fourth order; linear

4. Second order; nonlinear because of $\cos(r+u)$

5. Second order; nonlinear because of $(dy/dx)^2$ or $\sqrt{1+(dy/dx)^2}$

6. Second order; nonlinear because of R^2

7. Third order; linear

8. Second order; nonlinear because of $\dot{x}^2$

9. Writing the boundary-value problem in the form $x(dy/dx)+y^2=1$, we see that it is nonlinear in y because of y^2. However, writing it in the form $(y^2-1)(dx/dy)+x=0$, we see that it is linear in x.

10. Writing the differential equation in the form $u(dv/du)+(1+u)v=ue^u$ we see that it is linear in v. However, writing it in the form $(v+uv-ue^u)(du/dv)+u=0$, we see that it is nonlinear in u.

11. From $y=e^{-x/2}$ we obtain $y'=-\frac{1}{2}e^{-x/2}$. Then $2y'+y=-e^{-x/2}+e^{-x/2}=0$.

12. From $y=\frac{6}{5}-\frac{6}{5}e^{-20t}$ we obtain $dy/dt=24e^{-20t}$, so that

$$\frac{dy}{dt}+20y=24e^{-20t}+20\left(\frac{6}{5}-\frac{6}{5}e^{-20t}\right)=24.$$

13. From $y=e^{3x}\cos 2x$ we obtain $y'=3e^{3x}\cos 2x-2e^{3x}\sin 2x$ and $y''=5e^{3x}\cos 2x-12e^{3x}\sin 2x$, so that $y''-6y'+13y=0$.

14. From $y=-\cos x\ln(\sec x+\tan x)$ we obtain $y'=-1+\sin x\ln(\sec x+\tan x)$ and

$y''=\tan x+\cos x\ln(\sec x+\tan x)$. Then $y''+y=\tan x$.

15. The domain of the function, found by solving $x+2 \geq 0$, is $[-2, \infty)$. From $y' = 1+2(x+2)^{-1/2}$ we have

$$\begin{aligned}(y-x)y' &= (y-x)[1+(2(x+2)^{-1/2}] \\ &= y-x+2(y-x)(x+2)^{-1/2} \\ &= y-x+2[x+4(x+2)^{1/2}-x](x+2)^{-1/2} \\ &= y-x+8(x+2)^{1/2}(x+2)^{-1/2} = y-x+8.\end{aligned}$$

An interval of definition for the solution of the differential equation is $(-2, \infty)$ because y' is not defined at $x=-2$.

16. Since $\tan x$ is not defined for $x = \pi/2 + n\pi$, n an integer, the domain of $y = 5\tan 5x$ is $\{x \mid 5x \neq \pi/2 + n\pi\}$ or $\{x \mid x \neq \pi/10 + n\pi/5\}$. From $y' = 25\sec^2 5x$ we have

$$y' = 25(1+\tan^2 5x) = 25 + 25\tan^2 5x = 25 + y^2.$$

An interval of definition for the solution of the differential equation is $(-\pi/10, \pi/10)$. Another interval is $(\pi/10, 3\pi/10)$, and so on.

17. The domain of the function is $\{x \mid 4-x^2 \neq 0\}$ or $\{x \mid x \neq -2 \text{ or } x \neq 2\}$. From $y' = 2x/(4-x^2)^2$ we have

$$y' = 2x\left(\frac{1}{4-x^2}\right)^2 = 2xy^2.$$

An interval of definition for the solution of the differential equation is $(-2, 2)$. Other intervals are $(-\infty, -2)$ and $(2, \infty)$.

18. The function is $y = 1/\sqrt{1-\sin x}$, whose domain is obtained from $1-\sin x \neq 0$ or $\sin x \neq 1$. Thus, the domain is $\{x \mid x \neq \pi/2 + 2n\pi\}$. From $y' = -\frac{1}{2}(1-\sin x)^{-3/2}(-\cos x)$ we have

$$2y' = (1-\sin x)^{-3/2}\cos x = [(1-\sin x)^{-1/2}]^3\cos x = y^3\cos x.$$

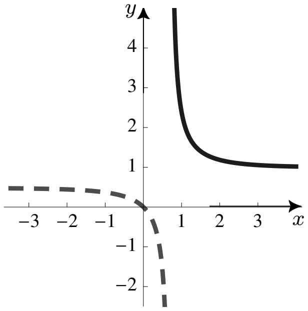
An interval of definition for the solution of the differential equation is $(\pi/2, 5\pi/2)$. Another interval is $(5\pi/2, 9\pi/2)$ and so on.

19. Writing $\ln(2X-1) - \ln(X-1) = t$ and differentiating implicitly we obtain

$$\begin{aligned}\frac{2}{2X-1}\frac{dX}{dt} - \frac{1}{X-1}\frac{dX}{dt} &= 1 \\ \left(\frac{2}{2X-1} - \frac{1}{X-1}\right)\frac{dX}{dt} &= 1 \\ \frac{2X-2-2X+1}{(2X-1)(X-1)}\frac{dX}{dt} &= 1 \\ \frac{dX}{dt} &= -(2X-1)(X-1) = (X-1)(1-2X).\end{aligned}$$

Exponentiating both sides of the implicit solution we obtain

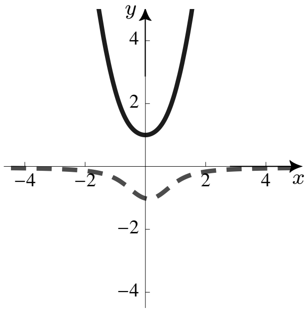
$$\begin{aligned}\frac{2X-1}{X-1} &= e^t\\ 2X-1 &= Xe^t - e^t\\ e^t - 1 &= (e^t-2)X\\ X &= \frac{e^t-1}{e^t-2}.\end{aligned}$$



Solving $e^t - 2 = 0$ we get $t = \ln 2$. Thus, the solution is defined on $(-\infty, \ln 2)$ or on $(\ln 2, \infty)$. The graph of the solution defined on $(-\infty, \ln 2)$ is dashed, and the graph of the solution defined on $(\ln 2, \infty)$ is solid.

20. Implicitly differentiating the solution, we obtain

$$\begin{aligned}-2x^2\frac{dy}{dx} - 4xy + 2y\frac{dy}{dx} &= 0\\ -x^2\,dy - 2xy\,dx + y\,dy &= 0\\ 2xy\,dx + (x^2-y)dy &= 0.\end{aligned}$$



Using the quadratic formula to solve $y^2 - 2x^2y - 1 = 0$ for y, we get $y = \left(2x^2 \pm \sqrt{4x^4+4}\right)/2 = x^2 \pm \sqrt{x^4+1}$. Thus, two explicit solutions are $y_1 = x^2 + \sqrt{x^4+1}$ and $y_2 = x^2 - \sqrt{x^4+1}$. Both solutions are defined on $(-\infty, \infty)$. The graph of $y_1(x)$ is solid and the graph of y_2 is dashed.

21. Differentiating $P = c_1e^t/\left(1+c_1e^t\right)$ we obtain

$$\begin{aligned}\frac{dP}{dt} &= \frac{\left(1+c_1e^t\right)c_1e^t - c_1e^t\cdot c_1e^t}{\left(1+c_1e^t\right)^2} = \frac{c_1e^t}{1+c_1e^t}\,\frac{\left[\left(1+c_1e^t\right)-c_1e^t\right]}{1+c_1e^t}\\ &= \frac{c_1e^t}{1+c_1e^t}\left[1-\frac{c_1e^t}{1+c_1e^t}\right] = P(1-P).\end{aligned}$$

22. Differentiating $y = e^{-x^2}\int_0^x e^{t^2}dt + c_1e^{-x^2}$ we obtain

$$y' = e^{-x^2}e^{x^2} - 2xe^{-x^2}\int_0^x e^{t^2}dt - 2c_1xe^{-x^2} = 1 - 2xe^{-x^2}\int_0^x e^{t^2}dt - 2c_1xe^{-x^2}.$$

Substituting into the differential equation, we have

$$y' + 2xy = 1 - 2xe^{-x^2}\int_0^x e^{t^2}dt - 2c_1xe^{-x^2} + 2xe^{-x^2}\int_0^x e^{t^2}dt + 2c_1xe^{-x^2} = 1.$$

23. From $y = c_1e^{2x} + c_2xe^{2x}$ we obtain $\dfrac{dy}{dx} = (2c_1+c_2)e^{2x} + 2c_2xe^{2x}$ and $\dfrac{d^2y}{dx^2} = (4c_1+4c_2)e^{2x} + 4c_2xe^{2x}$, so that

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = (4c_1+4c_2-8c_1-4c_2+4c_1)e^{2x} + (4c_2-8c_2+4c_2)xe^{2x} = 0.$$

24. From $y = c_1x^{-1} + c_2x + c_3x\ln x + 4x^2$ we obtain

$$\frac{dy}{dx} = -c_1x^{-2} + c_2 + c_3 + c_3\ln x + 8x,$$
$$\frac{d^2y}{dx^2} = 2c_1x^{-3} + c_3x^{-1} + 8,$$

and

$$\frac{d^3y}{dx^3} = -6c_1x^{-4} - c_3x^{-2},$$

so that

$$\begin{aligned} x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y &= (-6c_1 + 4c_1 + c_1 + c_1)x^{-1} + (-c_3 + 2c_3 - c_2 - c_3 + c_2)x \\ &\quad + (-c_3 + c_3)x\ln x + (16 - 8 + 4)x^2 \\ &= 12x^2. \end{aligned}$$

25. From $y = \begin{cases} -x^2, & x < 0 \\ x^2, & x \ge 0 \end{cases}$ we obtain $y' = \begin{cases} -2x, & x < 0 \\ 2x, & x \ge 0 \end{cases}$ so that $xy' - 2y = 0$.

26. The function $y(x)$ is not continuous at $x = 0$ since $\lim_{x\to 0^-} y(x) = 5$ and $\lim_{x\to 0^+} y(x) = -5$. Thus, $y'(x)$ does not exist at $x = 0$.

27. From $y = e^{mx}$ we obtain $y' = me^{mx}$. Then $y' + 2y = 0$ implies

$$me^{mx} + 2e^{mx} = (m+2)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = -2$. Thus $y = e^{-2x}$ is a solution.

28. From $y = e^{mx}$ we obtain $y' = me^{mx}$. Then $5y' = 2y$ implies

$$5me^{mx} = 2e^{mx} \quad \text{or} \quad m = \frac{2}{5}.$$

Thus $y = e^{2x/5} > 0$ is a solution.

29. From $y = e^{mx}$ we obtain $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Then $y'' - 5y' + 6y = 0$ implies

$$m^2e^{mx} - 5me^{mx} + 6e^{mx} = (m-2)(m-3)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = 2$ and $m = 3$. Thus $y = e^{2x}$ and $y = e^{3x}$ are solutions.

30. From $y = e^{mx}$ we obtain $y' = me^{mx}$ and $y'' = m^2e^{mx}$. Then $2y'' + 7y' - 4y = 0$ implies

$$2m^2e^{mx} + 7me^{mx} - 4e^{mx} = (2m-1)(m+4)e^{mx} = 0.$$

Since $e^{mx} > 0$ for all x, $m = \frac{1}{2}$ and $m = -4$. Thus $y = e^{x/2}$ and $y = e^{-4x}$ are solutions.

31. From $y = x^m$ we obtain $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Then $xy'' + 2y' = 0$ implies

$$\begin{aligned} xm(m-1)x^{m-2} + 2mx^{m-1} &= [m(m-1) + 2m]x^{m-1} = (m^2 + m)x^{m-1} \\ &= m(m+1)x^{m-1} = 0. \end{aligned}$$

Since $x^{m-1} > 0$ for $x > 0$, $m = 0$ and $m = -1$. Thus $y = 1$ and $y = x^{-1}$ are solutions.

32. From $y = x^m$ we obtain $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$. Then $x^2y'' - 7xy' + 15y = 0$ implies

$$\begin{aligned} x^2 m(m-1)x^{m-2} - 7xmx^{m-1} + 15x^m &= [m(m-1) - 7m + 15]x^m \\ &= (m^2 - 8m + 15)x^m = (m-3)(m-5)x^m = 0. \end{aligned}$$

Since $x^m > 0$ for $x > 0$, $m = 3$ and $m = 5$. Thus $y = x^3$ and $y = x^5$ are solutions.

In Problems 33–36 we substitute $y = c$ into the differential equations and use $y' = 0$ and $y'' = 0$.

33. Solving $5c = 10$ we see that $y = 2$ is a constant solution.

34. Solving $c^2 + 2c - 3 = (c+3)(c-1) = 0$ we see that $y = -3$ and $y = 1$ are constant solutions.

35. Since $1/(c-1) = 0$ has no solutions, the differential equation has no constant solutions.

36. Solving $6c = 10$ we see that $y = 5/3$ is a constant solution.

37. From $x = e^{-2t} + 3e^{6t}$ and $y = -e^{-2t} + 5e^{6t}$ we obtain

$$\frac{dx}{dt} = -2e^{-2t} + 18e^{6t} \quad \text{and} \quad \frac{dy}{dt} = 2e^{-2t} + 30e^{6t}.$$

Then

$$x + 3y = (e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = -2e^{-2t} + 18e^{6t} = \frac{dx}{dt}$$

and

$$5x + 3y = 5(e^{-2t} + 3e^{6t}) + 3(-e^{-2t} + 5e^{6t}) = 2e^{-2t} + 30e^{6t} = \frac{dy}{dt}\,.$$

38. From $x = \cos 2t + \sin 2t + \frac{1}{5}e^t$ and $y = -\cos 2t - \sin 2t - \frac{1}{5}e^t$ we obtain

$$\frac{dx}{dt} = -2\sin 2t + 2\cos 2t + \frac{1}{5}e^t \qquad \text{or} \qquad \frac{dy}{dt} = 2\sin 2t - 2\cos 2t - \frac{1}{5}e^t$$

and

$$\frac{d^2x}{dt^2} = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t \qquad \text{or} \qquad \frac{d^2y}{dt^2} = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t.$$

Then

$$4y + e^t = 4(-\cos 2t - \sin 2t - \frac{1}{5}e^t) + e^t = -4\cos 2t - 4\sin 2t + \frac{1}{5}e^t = \frac{d^2x}{dt^2}$$

and

$$4x - e^t = 4(\cos 2t + \sin 2t + \frac{1}{5}e^t) - e^t = 4\cos 2t + 4\sin 2t - \frac{1}{5}e^t = \frac{d^2y}{dt^2}\,.$$

Discussion Problems

39. $(y')^2 + 1 = 0$ has no real solutions because $(y')^2 + 1$ is positive for all functions $y = \phi(x)$.

40. The only solution of $(y')^2 + y^2 = 0$ is $y = 0$, since, if $y \neq 0$, $y^2 > 0$ and $(y')^2 + y^2 \geq y^2 > 0$.

41. The first derivative of $f(x) = e^x$ is e^x. The first derivative of $f(x) = e^{kx}$ is $f'(x) = ke^{kx}$. The differential equations are $y' = y$ and $y' = ky$, respectively.

42. Any function of the form $y = ce^x$ or $y = ce^{-x}$ is its own second derivative. The corresponding differential equation is $y'' - y = 0$. Functions of the form $y = c\sin x$ or $y = c\cos x$ have second derivatives that are the negatives of themselves. The differential equation is $y'' + y = 0$.

43. We first note that $\sqrt{1-y^2} = \sqrt{1-\sin^2 x} = \sqrt{\cos^2 x} = |\cos x|$. This prompts us to consider values of x for which $\cos x < 0$, such as $x = \pi$. In this case

$$\left.\frac{dy}{dx}\right|_{x=\pi} = \left.\frac{d}{dx}(\sin x)\right|_{x=\pi} = \cos x\big|_{x=\pi} = \cos\pi = -1,$$

but

$$\left.\sqrt{1-y^2}\,\right|_{x=\pi} = \sqrt{1-\sin^2\pi} = \sqrt{1} = 1.$$

Thus, $y = \sin x$ will only be a solution of $y' = \sqrt{1-y^2}$ when $\cos x > 0$. An interval of definition is then $(-\pi/2, \pi/2)$. Other intervals are $(3\pi/2, 5\pi/2)$, $(7\pi/2, 9\pi/2)$, and so on.

44. Since the first and second derivatives of $\sin t$ and $\cos t$ involve $\sin t$ and $\cos t$, it is plausible that a linear combination of these functions, $A\sin t + B\cos t$, could be a solution of the differential equation. Using $y' = A\cos t - B\sin t$ and $y'' = -A\sin t - B\cos t$ and substituting into the differential equation we get

$$\begin{aligned} y'' + 2y' + 4y &= -A\sin t - B\cos t + 2A\cos t - 2B\sin t + 4A\sin t + 4B\cos t \\ &= (3A - 2B)\sin t + (2A + 3B)\cos t = 5\sin t. \end{aligned}$$

Thus $3A - 2B = 5$ and $2A + 3B = 0$. Solving these simultaneous equations we find $A = \frac{15}{13}$ and $B = -\frac{10}{13}$. A particular solution is $y = \frac{15}{13}\sin t - \frac{10}{13}\cos t$.

45. One solution is given by the upper portion of the graph with domain approximately $(0, 2.6)$. The other solution is given by the lower portion of the graph, also with domain approximately $(0, 2.6)$.

46. One solution, with domain approximately $(-\infty, 1.6)$ is the portion of the graph in the second quadrant together with the lower part of the graph in the first quadrant. A second solution, with domain approximately $(0, 1.6)$ is the upper part of the graph in the first quadrant. The third solution, with domain $(0, \infty)$, is the part of the graph in the fourth quadrant.

47. Differentiating $(x^3 + y^3)/xy = 3c$ we obtain

$$\begin{aligned} \frac{xy(3x^2 + 3y^2 y') - (x^3 + y^3)(xy' + y)}{x^2 y^2} &= 0 \\ 3x^3 y + 3xy^3 y' - x^4 y' - x^3 y - xy^3 y' - y^4 &= 0 \\ (3xy^3 - x^4 - xy^3)y' &= -3x^3 y + x^3 y + y^4 \\ y' &= \frac{y^4 - 2x^3 y}{2xy^3 - x^4} = \frac{y(y^3 - 2x^3)}{x(2y^3 - x^3)}. \end{aligned}$$

48. A tangent line will be vertical where y' is undefined, or in this case, where $x(2y^3 - x^3) = 0$. This gives $x = 0$ and $2y^3 = x^3$. Substituting $y^3 = x^3/2$ into $x^3 + y^3 = 3xy$ we get

$$\begin{aligned} x^3 + \frac{1}{2}x^3 &= 3x\left(\frac{1}{2^{1/3}}\,x\right) \\ \frac{3}{2}x^3 &= \frac{3}{2^{1/3}}x^2 \\ x^3 &= 2^{2/3}x^2 \\ x^2(x - 2^{2/3}) &= 0. \end{aligned}$$

Thus, there are vertical tangent lines at $x = 0$ and $x = 2^{2/3}$, or at $(0, 0)$ and $(2^{2/3}, 2^{1/3})$. Since $2^{2/3} \approx 1.59$, the estimates of the domains in Problem 46 were close.

49. The derivatives of the functions are $\phi_1'(x) = -x/\sqrt{25 - x^2}$ and $\phi_2'(x) = x/\sqrt{25 - x^2}$, neither of which is defined at $x = \pm 5$.

50. To determine if a solution curve passes through $(0, 3)$ we let $t = 0$ and $P = 3$ in the equation $P = c_1 e^t/(1 + c_1 e^t)$. This gives $3 = c_1/(1 + c_1)$ or $c_1 = -\frac{3}{2}$. Thus, the solution curve

$$P = \frac{(-3/2)e^t}{1 - (3/2)e^t} = \frac{-3e^t}{2 - 3e^t}$$

passes through the point $(0, 3)$. Similarly, letting $t = 0$ and $P = 1$ in the equation for the one-parameter family of solutions gives $1 = c_1/(1 + c_1)$ or $c_1 = 1 + c_1$. Since this equation has no solution, no solution curve passes through $(0, 1)$.

51. For the first-order differential equation integrate $f(x)$. For the second-order differential equation integrate twice. In the latter case we get $y = \int(\int f(x)dx)dx + c_1 x + c_2$.

52. Solving for y' using the quadratic formula we obtain the two differential equations

$$y' = \frac{1}{x}\left(2 + 2\sqrt{1 + 3x^6}\right) \quad \text{and} \quad y' = \frac{1}{x}\left(2 - 2\sqrt{1 + 3x^6}\right),$$

so the differential equation cannot be put in the form $dy/dx = f(x, y)$.

53. The differential equation $yy' - xy = 0$ has normal form $dy/dx = x$. These are not equivalent because $y = 0$ is a solution of the first differential equation but not a solution of the second.

54. Differentiating $y = c_1x + c_2x^2$ we get $y' = c_1 + 2c_2x$ and $y'' = 2c_2$. Then $c_2 = \frac{1}{2}y''$ and $c_1 = y' - xy''$, so

$$y = c_1x + c_2x^2 = (y' - xy'')x + \frac{1}{2}y''x^2 = xy' - \frac{1}{2}x^2y''.$$

The differential equation is $\frac{1}{2}x^2y'' - xy' + y = 0$ or $x^2y'' - 2xy' + 2y = 0$.

55. **(a)** Since e^{-x^2} is positive for all values of x, $dy/dx > 0$ for all x, and a solution, $y(x)$, of the differential equation must be increasing on any interval.

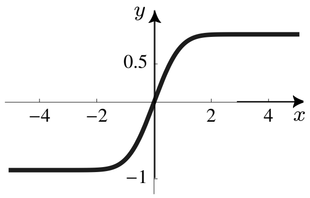
(b) $\displaystyle\lim_{x\to-\infty}\frac{dy}{dx} = \lim_{x\to-\infty}e^{-x^2} = 0$ and $\displaystyle\lim_{x\to\infty}\frac{dy}{dx} = \lim_{x\to\infty}e^{-x^2} = 0$. Since $\dfrac{dy}{dx}$ approaches 0 as x approaches $-\infty$ and ∞, the solution curve has horizontal asymptotes to the left and to the right.

(c) To test concavity we consider the second derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(e^{-x^2}\right) = -2xe^{-x^2}.$$

Since the second derivative is positive for $x < 0$ and negative for $x > 0$, the solution curve is concave up on $(-\infty, 0)$ and concave down on $(0, \infty)$. x

(d)



56. **(a)** The derivative of a constant solution $y = c$ is 0, so solving $5 - c = 0$ we see that $c = 5$ and so $y = 5$ is a constant solution.

(b) A solution is increasing where $dy/dx = 5 - y > 0$ or $y < 5$. A solution is decreasing where $dy/dx = 5 - y < 0$ or $y > 5$.

57. **(a)** The derivative of a constant solution is 0, so solving $y(a - by) = 0$ we see that $y = 0$ and $y = a/b$ are constant solutions.

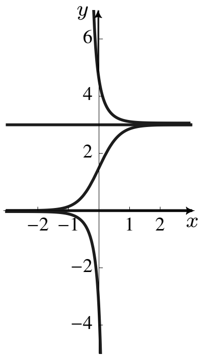
(b) A solution is increasing where $dy/dx = y(a - by) = by(a/b - y) > 0$ or $0 < y < a/b$. A solution is decreasing where $dy/dx = by(a/b - y) < 0$ or $y < 0$ or $y > a/b$.

(c) Using implicit differentiation we compute

$$\frac{d^2y}{dx^2} = y(-by') + y'(a - by) = y'(a - 2by).$$

Solving $d^2y/dx^2 = 0$ we obtain $y = a/2b$. Since $d^2y/dx^2 > 0$ for $0 < y < a/2b$ and $d^2y/dx^2 < 0$ for $a/2b < y < a/b$, the graph of $y = \phi(x)$ has a point of inflection at $y = a/2b$.

(d)

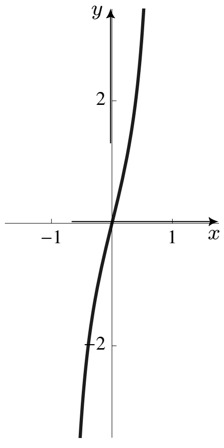


58. **(a)** If $y = c$ is a constant solution then $y' = 0$, but $c^2 + 4$ is never 0 for any real value of c.

(b) Since $y' = y^2 + 4 > 0$ for all x where a solution $y = \phi(x)$ is defined, any solution must be increasing on any interval on which it is defined. Thus it cannot have any relative extrema.

(c) Using implicit differentiation we compute $d^2y/dx^2 = 2yy' = 2y(y^2+4)$. Setting $d^2y/dx^2 = 0$ we see that $y = 0$ corresponds to the only possible point of inflection. Since $d^2y/dx^2 < 0$ for $y < 0$ and $d^2y/dx^2 > 0$ for $y > 0$, there is a point of inflection where $y = 0$.

(d)



Computer Lab Assignments

59. In *Mathematica* use

Clear[y]

y[x_]:= x **Exp**[5x] **Cos**[2x]

y[x]

y''''[x] - 20 y'''[x] + 158 y''[x] - 580 y'[x] + 841 y[x] // **Simplify**

The output will show $y(x) = e^{5x}x\cos 2x$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

60. In *Mathematica* use

```
Clear[y]
y[x_]:= 20 Cos[5 Log[x]]/x - 3 Sin[5 Log[x]]/x
y[x]
x^3 y'''[x] + 2 x^2 y''[x] + 20 x y'[x] - 78 y[x] // Simplify
```

The output will show $y(x) = \dfrac{20\cos(5\ln x)}{x} - \dfrac{3\sin(5\ln x)}{x}$, which verifies that the correct function was entered, and 0, which verifies that this function is a solution of the differential equation.

1.2 Initial-Value Problems

1. Solving $-1/3 = 1/(1+c_1)$ we get $c_1 = -4$. The solution is $y = 1/(1-4e^{-x})$.

2. Solving $2 = 1/(1+c_1e)$ we get $c_1 = -(1/2)e^{-1}$. The solution is $y = 2/(2-e^{-(x+1)})$.

3. Letting $x = 2$ and solving $1/3 = 1/(4+c)$ we get $c = -1$. The solution is $y = 1/(x^2-1)$. This solution is defined on the interval $(1,\infty)$.

4. Letting $x = -2$ and solving $1/2 = 1/(4+c)$ we get $c = -2$. The solution is $y = 1/(x^2-2)$. This solution is defined on the interval $(-\infty, -\sqrt{2}\,)$.

5. Letting $x = 0$ and solving $1 = 1/c$ we get $c = 1$. The solution is $y = 1/(x^2+1)$. This solution is defined on the interval $(-\infty,\infty)$.

6. Letting $x = 1/2$ and solving $-4 = 1/(1/4+c)$ we get $c = -1/2$. The solution is $y = 1/(x^2-1/2) = 2/(2x^2-1)$. This solution is defined on the interval $(-1/\sqrt{2}\,, 1/\sqrt{2}\,)$.

In Problems 7–10 we use $x = c_1\cos t + c_2\sin t$ and $x' = -c_1\sin t + c_2\cos t$ to obtain a system of two equations in the two unknowns c_1 and c_2.

7. From the initial conditions we obtain the system

$$c_1 = -1$$
$$c_2 = 8.$$

The solution of the initial-value problem is $x = -\cos t + 8\sin t$.

8. From the initial conditions we obtain the system

$$c_2 = 0$$
$$-c_1 = 1.$$

The solution of the initial-value problem is $x = -\cos t$.

9. From the initial conditions we obtain

$$\frac{\sqrt{3}}{2}\,c_1 + \frac{1}{2}\,c_2 = \frac{1}{2}$$

$$-\frac{1}{2}\,c_1 + \frac{\sqrt{3}}{2}\,c_2 = 0.$$

Solving, we find $c_1 = \sqrt{3}/4$ and $c_2 = 1/4$. The solution of the initial-value problem is

$$x = (\sqrt{3}/4)\cos t + (1/4)\sin t.$$

10. From the initial conditions we obtain

$$\frac{\sqrt{2}}{2}\,c_1 + \frac{\sqrt{2}}{2}\,c_2 = \sqrt{2}$$

$$-\frac{\sqrt{2}}{2}\,c_1 + \frac{\sqrt{2}}{2}\,c_2 = 2\sqrt{2}\,.$$

Solving, we find $c_1 = -1$ and $c_2 = 3$. The solution of the initial-value problem is

$$x = -\cos t + 3\sin t.$$

In Problems 11–14 we use $y = c_1e^x + c_2e^{-x}$ and $y' = c_1e^x - c_2e^{-x}$ to obtain a system of two equations in the two unknowns c_1 and c_2.

11. From the initial conditions we obtain

$$c_1 + c_2 = 1$$
$$c_1 - c_2 = 2.$$

Solving, we find $c_1 = \frac{3}{2}$ and $c_2 = -\frac{1}{2}$. The solution of the initial-value problem is

$$y = \frac{3}{2}\,e^x - \frac{1}{2}\,e^{-x}.$$

12. From the initial conditions we obtain

$$ec_1 + e^{-1}c_2 = 0$$
$$ec_1 - e^{-1}c_2 = e.$$

Solving, we find $c_1 = \frac{1}{2}$ and $c_2 = -\frac{1}{2}e^2$. The solution of the initial-value problem is

$$y = \frac{1}{2}\,e^x - \frac{1}{2}\,e^2e^{-x} = \frac{1}{2}\,e^x - \frac{1}{2}\,e^{2-x}.$$

13. From the initial conditions we obtain

$$e^{-1}c_1 + ec_2 = 5$$
$$e^{-1}c_1 - ec_2 = -5.$$

Solving, we find $c_1 = 0$ and $c_2 = 5e^{-1}$. The solution of the initial-value problem is

$$y = 5e^{-1}e^{-x} = 5e^{-1-x}.$$

14. From the initial conditions we obtain

$$c_1 + c_2 = 0$$
$$c_1 - c_2 = 0.$$

Solving, we find $c_1 = c_2 = 0$. The solution of the initial-value problem is $y = 0$.

15. Two solutions are $y = 0$ and $y = x^3$.

16. Two solutions are $y = 0$ and $y = x^2$. A lso, any constant multiple of x^2 is a solution.

17. For $f(x, y) = y^{2/3}$ we have Thus, the differential equation will have a unique solution in any rectangular region of the plane where $y \neq 0$.

18. For $f(x, y) = \sqrt{xy}$ we have $\partial f/\partial y = \frac{1}{2}\sqrt{x/y}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ and $y > 0$ or where $x < 0$ and $y < 0$.

19. For $f(x, y) = \dfrac{y}{x}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{1}{x}$. Thus, the differential equation will have a unique solution in any region where $x > 0$ or where $x < 0$.

20. For $f(x, y) = x + y$ we have $\dfrac{\partial f}{\partial y} = 1$. Thus, the differential equation will have a unique solution in the entire plane.

21. For $f(x, y) = x^2/(4 - y^2)$ we have $\partial f/\partial y = 2x^2y/(4 - y^2)^2$. Thus the differential equation will have a unique solution in any region where $y < -2$, $-2 < y < 2$, or $y > 2$.

22. For $f(x, y) = \dfrac{x^2}{1 + y^3}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{-3x^2y^2}{(1 + y^3)^2}$. Thus, the differential equation will have a unique solution in any region where $y \neq -1$.

23. For $f(x, y) = \dfrac{y^2}{x^2 + y^2}$ we have $\dfrac{\partial f}{\partial y} = \dfrac{2x^2y}{(x^2 + y^2)^2}$. Thus, the differential equation will have a unique solution in any region not containing $(0, 0)$.

24. For $f(x, y) = (y + x)/(y - x)$ we have $\partial f/\partial y = -2x/(y - x)^2$. Thus the differential equation will have a unique solution in any region where $y < x$ or where $y > x$.

In Problems 25–28 we identify $f(x,y)=\sqrt{y^2-9}$ *and* $\partial f/\partial y = y/\sqrt{y^2-9}$. *We see that* f *and* $\partial f/\partial y$ *are both continuous in the regions of the plane determined by* $y<-3$ *and* $y>3$ *with no restrictions on* x.

25. Since $4>3$, $(1,4)$ is in the region defined by $y>3$ and the differential equation has a unique solution through $(1,4)$.

26. Since $(5,3)$ is not in either of the regions defined by $y<-3$ or $y>3$, there is no guarantee of a unique solution through $(5,3)$.

27. Since $(2,-3)$ is not in either of the regions defined by $y<-3$ or $y>3$, there is no guarantee of a unique solution through $(2,-3)$.

28. Since $(-1,1)$ is not in either of the regions defined by $y<-3$ or $y>3$, there is no guarantee of a unique solution through $(-1,1)$.

29. **(a)** A one-parameter family of solutions is $y=cx$. Since $y'=c$, $xy'=xc=y$ and $y(0)=c\cdot 0=0$.

(b) Writing the equation in the form $y'=y/x$, we see that R cannot contain any point on the y-axis. Thus, any rectangular region disjoint from the y-axis and containing (x_0,y_0) will determine an interval around x_0 and a unique solution through (x_0,y_0). Since $x_0=0$ in part (a), we are not guaranteed a unique solution through $(0,0)$.

(c) The piecewise-defined function which satisfies $y(0)=0$ is not a solution since it is not differentiable at $x=0$.

30. **(a)** Since $\dfrac{d}{dx}\tan(x+c)=\sec^2(x+c)=1+\tan^2(x+c)$, we see that $y=\tan(x+c)$ satisfies the differential equation.

(b) Solving $y(0)=\tan c=0$ we obtain $c=0$ and $y=\tan x$. Since $\tan x$ is discontinuous at $x=\pm\pi/2$, the solution is not defined on $(-2,2)$ because it contains $\pm\pi/2$.

(c) The largest interval on which the solution can exist is $(-\pi/2,\pi/2)$.

31. **(a)** Since $\dfrac{d}{dx}\left(-\dfrac{1}{x+c}\right)=\dfrac{1}{(x+c)^2}=y^2$, we see that $y=-\dfrac{1}{x+c}$ is a solution of the differential equation.

(b) Solving $y(0)=-1/c=1$ we obtain $c=-1$ and $y=1/(1-x)$. Solving $y(0)=-1/c=-1$ we obtain $c=1$ and $y=-1/(1+x)$. Being sure to include $x=0$, we see that the interval of existence of $y=1/(1-x)$ is $(-\infty,1)$, while the interval of existence of $y=-1/(1+x)$ is $(-1,\infty)$.

(c) By inspection we see that $y=0$ is a solution on $(-\infty,\infty)$.

32. (a) Applying $y(1)=1$ to $y=-1/(x+c)$ gives

$$1=-\frac{1}{1+c} \quad \text{or} \quad 1+c=-1.$$

Thus $c=-2$ and

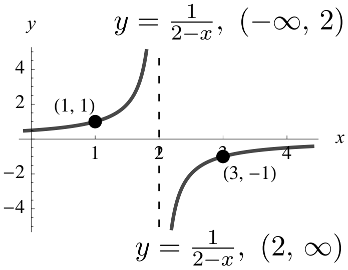
$$y=-\frac{1}{x-2}=\frac{1}{2-x}.$$

(b) Applying $y(3)=-1$ to $y=-1/(x+c)$ gives

$$-1=-\frac{1}{3+c} \quad \text{or} \quad 3+c=1.$$

Thus $c=-2$ and

$$y=-\frac{1}{x-2}=\frac{1}{2-x}.$$

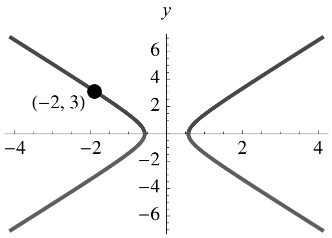


(c) No, they are not the same solution. The interval I of definition for the solution in part (a) is $(-\infty,2)$; whereas the interval I of definition for the solution in part (b) is $(2,\infty)$. See the figure.

33. (a) Differentiating $3x^2-y^2=c$ we get $6x-2yy'=0$ or $yy'=3x$.

(b) Solving $3x^2-y^2=3$ for y we get

$$\begin{aligned}
y&=\phi_1(x)=\sqrt{3(x^2-1)}, && 1<x<\infty,\\
y&=\phi_2(x)=-\sqrt{3(x^2-1)}, && 1<x<\infty,\\
y&=\phi_3(x)=\sqrt{3(x^2-1)}, && -\infty<x<-1,\\
y&=\phi_4(x)=-\sqrt{3(x^2-1)}, && -\infty<x<-1.
\end{aligned}$$

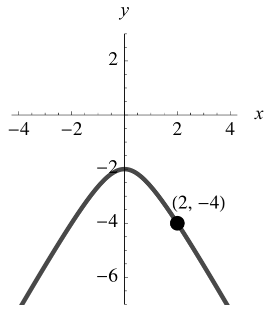


(c) Only $y=\phi_3(x)$ satisfies $y(-2)=3$.

34. (a) Setting $x=2$ and $y=-4$ in $3x^2-y^2=c$ we get $12-16=-4=c$, so the explicit solution is

$$y=-\sqrt{3x^2+4}, \quad -\infty<x<\infty.$$

(b) Setting $c=0$ we have $y=\sqrt{3}x$ and $y=-\sqrt{3}x$, both defined on $(-\infty,\infty)$ and both passing through the origin.



In Problems 35–38 we consider the points on the graphs with x-coordinates $x_0=-1$, $x_0=0$, and $x_0=1$. The slopes of the tangent lines at these points are compared with the slopes given by $y'(x_0)$ in (a) through (f).

35. The graph satisfies the conditions in (b) and (f).

36. The graph satisfies the conditions in (e).

37. The graph satisfies the conditions in (c) and (d).

38. The graph satisfies the conditions in (a).

In Problems 39-44 $y = c_1 \cos 2x + c_2 \sin 2x$ *is a two parameter family of solutions of the second-order differential equation* $y'' + 4y = 0$. *In some of the problems we will use the fact that* $y' = -2c_1 \sin 2x + 2c_2 \cos 2x$.

39. From the boundary conditions $y(0) = 0$ and $y\left(\frac{\pi}{4}\right) = 3$ we obtain

$$y(0) = c_1 = 0$$

$$y\left(\frac{\pi}{4}\right) = c_1 \cos\left(\frac{\pi}{2}\right) + c_2 \sin\left(\frac{\pi}{2}\right) = c_2 = 3.$$

Thus, $c_1 = 0$, $c_2 = 3$, and the solution of the boundary-value problem is $y = 3\sin 2x$.

40. From the boundary conditions $y(0) = 0$ and $y(\pi) = 0$ we obtain

$$y(0) = c_1 = 0$$

$$y(\pi) = c_1 = 0.$$

Thus, $c_1 = 0$, c_2 is unrestricted, and the solution of the boundary-value problem is $y = c_2 \sin 2x$, where c_2 is any real number.

41. From the boundary conditions $y'(0) = 0$ and $y'\left(\frac{\pi}{6}\right) = 0$ we obtain

$$y'(0) = 2c_2 = 0$$

$$y'\left(\frac{\pi}{6}\right) = -2c_1 \sin\left(\frac{\pi}{3}\right) = -\sqrt{3}\, c_1 = 0.$$

Thus, $c_2 = 0$, $c_1 = 0$, and the solution of the boundary-value problem is $y = 0$.

42. From the boundary conditions $y(0) = 1$ and $y'(\pi) = 5$ we obtain

$$y(0) = c_1 = 1$$

$$y'(\pi) = 2c_2 = 5.$$

Thus, $c_1 = 1$, $c_2 = \frac{5}{2}$, and the solution of the boundary-value problem is $y = \cos 2x + \frac{5}{2}\sin 2x$.

43. From the boundary conditions $y(0) = 0$ and $y(\pi) = 2$ we obtain

$$y(0) = c_1 = 0$$

$$y(\pi) = c_1 = 2.$$

Since $0 \neq 2$, this is not possible and there is no solution.

44. From the boundary conditions $y' = \left(\frac{\pi}{2}\right) = 1$ and $y'(\pi) = 0$ we obtain

$$y'\left(\frac{\pi}{2}\right) = -2c_2 = 1$$

$$y'(\pi) = 2c_2 = 0.$$

Since $0 \neq -1$, this is not possible and there is no solution.

Discussion Problems

45. Integrating $y' = 8e^{2x} + 6x$ we obtain

$$y = \int (8e^{2x} + 6x)dx = 4e^{2x} + 3x^2 + c.$$

Setting $x = 0$ and $y = 9$ we have $9 = 4 + c$ so $c = 5$ and $y = 4e^{2x} + 3x^2 + 5$.

46. Integrating $y'' = 12x - 2$ we obtain

$$y' = \int (12x - 2)dx = 6x^2 - 2x + c_1.$$

Then, integrating y' we obtain

$$y = \int (6x^2 - 2x + c_1)dx = 2x^3 - x^2 + c_1 x + c_2.$$

At $x = 1$ the y-coordinate of the point of tangency is $y = -1 + 5 = 4$. This gives the initial condition $y(1) = 4$. The slope of the tangent line at $x = 1$ is $y'(1) = -1$. From the initial conditions we obtain

$$2 - 1 + c_1 + c_2 = 4 \quad \text{or} \quad c_1 + c_2 = 3$$

and

$$6 - 2 + c_1 = -1 \quad \text{or} \quad c_1 = -5.$$

Thus, $c_1 = -5$ and $c_2 = 8$, so $y = 2x^3 - x^2 - 5x + 8$.

47. When $x = 0$ and $y = \frac{1}{2}$, $y' = -1$, so the only plausible solution curve is the one with negative slope at $(0, \frac{1}{2})$, or the red curve.

48. If the solution is tangent to the x-axis at $(x_0, 0)$, then $y' = 0$ when $x = x_0$ and $y = 0$. Substituting these values into $y' + 2y = 3x - 6$ we get $0 + 0 = 3x_0 - 6$ or $x_0 = 2$.

49. The theorem guarantees a unique (meaning single) solution through any point. Thus, there cannot be two distinct solutions through any point.

50. When $y = \frac{1}{16}x^4$, $y' = \frac{1}{4}x^3 = x(\frac{1}{4}x^2) = xy^{1/2}$, and $y(2) = \frac{1}{16}(16) = 1$. When

$$y = \begin{cases} 0, & x < 0 \\ \frac{1}{16}x^4, & x \geq 0 \end{cases}$$

we have

$$y' = \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^3, & x \geq 0 \end{cases} = x \begin{cases} 0, & x < 0 \\ \frac{1}{4}x^2, & x \geq 0 \end{cases} = xy^{1/2},$$

and $y(2) = \frac{1}{16}(16) = 1$. The two different solutions are the same on the interval $(0, \infty)$, which is all that is required by Theorem 1.2.1.

51. At $t = 0$, $dP/dt = 0.15P(0) + 20 = 0.15(100) + 20 = 35$. Thus, the population is increasing at a rate of 3,500 individuals per year. If the population is 500 at time $t = T$ then

$$\left.\frac{dP}{dt}\right|_{t=T} = 0.15P(T) + 20 = 0.15(500) + 20 = 95.$$

Thus, at this time, the population is increasing at a rate of 9,500 individuals per year.