1 PRECALCULUS REVIEW

1.1 Real Numbers, Functions, and Graphs

Preliminary Questions

1. Give an example of numbers a and b such that $a < b$ and $|a| > |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then $a < b$ but $|a| = 3 > 1 = |b|$.

2. Which numbers satisfy $|a| = a$? Which satisfy $|a| = -a$? What about $|-a| = a$?

SOLUTION The numbers $a \geq 0$ satisfy $|a| = a$ and $|-a| = a$. The numbers $a \leq 0$ satisfy $|a| = -a$.

3. Give an example of numbers a and b such that $|a + b| < |a| + |b|$.

SOLUTION Take $a = -3$ and $b = 1$. Then

$$|a + b| = |-3 + 1| = |-2| = 2, \qquad \text{but} \qquad |a| + |b| = |-3| + |1| = 3 + 1 = 4$$

Thus, $|a + b| < |a| + |b|$.

4. Are there numbers a and b such that $|a + b| > |a| + |b|$?

SOLUTION No. By the triangle inequality, $|a + b| \leq |a| + |b|$ for all real numbers a and b.

5. What are the coordinates of the point lying at the intersection of the lines $x = 9$ and $y = -4$?

SOLUTION The point $(9, -4)$ lies at the intersection of the lines $x = 9$ and $y = -4$.

6. In which quadrant do the following points lie?

(a) $(1, 4)$ **(b)** $(-3, 2)$ **(c)** $(4, -3)$ **(d)** $(-4, -1)$

SOLUTION

(a) Because both the x- and y-coordinates of the point $(1, 4)$ are positive, the point $(1, 4)$ lies in the first quadrant.

(b) Because the x-coordinate of the point $(-3, 2)$ is negative but the y-coordinate is positive, the point $(-3, 2)$ lies in the second quadrant.

(c) Because the x-coordinate of the point $(4, -3)$ is positive but the y-coordinate is negative, the point $(4, -3)$ lies in the fourth quadrant.

(d) Because both the x- and y-coordinates of the point $(-4, -1)$ are negative, the point $(-4, -1)$ lies in the third quadrant.

7. What is the radius of the circle with equation $(x - 7)^2 + (y - 8)^2 = 9$?

SOLUTION The circle with equation $(x - 7)^2 + (y - 8)^2 = 9$ has radius 3.

8. The equation $f(x) = 5$ has a solution if (choose one):

(a) 5 belongs to the domain of f.

(b) 5 belongs to the range of f.

SOLUTION The correct response is **(b)**: the equation $f(x) = 5$ has a solution if 5 belongs to the range of f.

9. What kind of symmetry does the graph have if $f(-x) = -f(x)$?

SOLUTION If $f(-x) = -f(x)$, then the graph of f is symmetric with respect to the origin.

10. Is there a function that is both even and odd?

SOLUTION Yes. The constant function $f(x) = 0$ for all real numbers x is both even and odd because

$$f(-x) = 0 = f(x)$$

and

$$f(-x) = 0 = -0 = -f(x)$$

for all real numbers x.

Exercises

1. Which of the following equations is incorrect?

(a) $3^2 \cdot 3^5 = 3^7$

(b) $(\sqrt{5})^{4/3} = 5^{2/3}$

(c) $3^2 \cdot 2^3 = 1$

(d) $(2^{-2})^{-2} = 16$

SOLUTION

(a) This equation is correct: $3^2 \cdot 3^5 = 3^{2+5} = 3^7$.

(b) This equation is correct: $(\sqrt{5})^{4/3} = (5^{1/2})^{4/3} = 5^{(1/2)\cdot(4/3)} = 5^{2/3}$.

(c) This equation is incorrect: $3^2 \cdot 2^3 = 9 \cdot 8 = 72 \neq 1$.

(d) This equation is correct: $(2^{-2})^{-2} = 2^{(-2)\cdot(-2)} = 2^4 = 16$.

2. Rewrite as a whole number (without using a calculator):

(a) 7^0

(b) $10^2(2^{-2} + 5^{-2})$

(c) $\frac{(4^3)^5}{(4^5)^3}$

(d) $27^{4/3}$

(e) $8^{-1/3} \cdot 8^{5/3}$

(f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2}$

SOLUTION

(a) $7^0 = 1$

(b) $10^2(2^{-2} + 5^{-2}) = 100(1/4 + 1/25) = 25 + 4 = 29$

(c) $(4^3)^5/(4^5)^3 = 4^{15}/4^{15} = 1$

(d) $(27)^{4/3} = (27^{1/3})^4 = 3^4 = 81$

(e) $8^{-1/3} \cdot 8^{5/3} = (8^{1/3})^5/8^{1/3} = 2^5/2 = 2^4 = 16$

(f) $3 \cdot 4^{1/4} - 12 \cdot 2^{-3/2} = 3 \cdot 2^{1/2} - 3 \cdot 2^2 \cdot 2^{-3/2} = 0$

3. Use the binomial expansion formula to expand $(2 - x)^7$.

SOLUTION Using the binomial expansion formula,

$$\begin{aligned}(2-x)^7 &= \frac{7!}{7!0!}2^7(-x)^0 + \frac{7!}{6!1!}2^6(-x) + \frac{7!}{5!2!}2^5(-x)^2 + \frac{7!}{4!3!}2^4(-x)^3 + \frac{7!}{3!4!}2^3(-x)^4 \\ &\quad + \frac{7!}{2!5!}2^2(-x)^5 + \frac{7!}{1!6!}2(-x)^6 + \frac{7!}{0!7!}2^0(-x)^7 \\ &= 128 - 448x + 672x^2 - 560x^3 + 280x^4 - 84x^5 + 14x^6 - x^7\end{aligned}$$

4. Use the binomial expansion formula to expand $(x + 1)^9$.

SOLUTION Using the binomial expansion formula,

$$\begin{aligned}(x+1)^9 &= \frac{9!}{9!0!}x^9 + \frac{9!}{8!1!}x^8 + \frac{9!}{7!2!}x^7 + \frac{9!}{6!3!}x^6 + \frac{9!}{5!4!}x^5 + \frac{9!}{4!5!}x^4 + \frac{9!}{3!6!}x^3 + \frac{9!}{2!7!}x^2 + \frac{9!}{1!8!}x + \frac{9!}{0!9!} \\ &= x^9 + 9x^8 + 36x^7 + 84x^6 + 126x^5 + 126x^4 + 84x^3 + 36x^2 + 9x + 1\end{aligned}$$

5. Which of (a)–(d) are true for $a = 4$ and $b = -5$?

(a) $-2a < -2b$

(b) $|a| < -|b|$

(c) $ab < 0$

(d) $\frac{1}{a} < \frac{1}{b}$

SOLUTION

(a) True

(b) False; $|a| = 4 > -5 = -|b|$

(c) True

(d) False; $\frac{1}{a} = \frac{1}{4} > -\frac{1}{5} = \frac{1}{b}$

6. Which of (a)–(d) are true for $a = -3$ and $b = 2$?

(a) $a < b$

(b) $|a| < |b|$

(c) $ab > 0$

(d) $3a < 3b$

SOLUTION

(a) True

(b) False; $|a| = 3 > 2 = |b|$

(c) False; $(-3)(2) = -6 < 0$

(d) True

In Exercises 7–12, express the interval in terms of an inequality involving absolute value.

7. $[-2, 2]$

SOLUTION $|x| \le 2$

8. $(-4, 4)$

SOLUTION $|x| < 4$

9. $(0, 4)$

SOLUTION The midpoint of the interval is $c = (0 + 4)/2 = 2$, and the radius is $r = (4 - 0)/2 = 2$; therefore, $(0, 4)$ can be expressed as $|x - 2| < 2$.

10. $[-4, 0]$

SOLUTION The midpoint of the interval is $c = (-4 + 0)/2 = -2$, and the radius is $r = (0 - (-4))/2 = 2$; therefore, the interval $[-4, 0]$ can be expressed as $|x + 2| \le 2$.

11. $[-1, 8]$

SOLUTION The midpoint of the interval is $c = (-1 + 8)/2 = \frac{7}{2}$, and the radius is $r = (8 - (-1))/2 = \frac{9}{2}$; therefore, the interval $[-1, 8]$ can be expressed as $\left|x - \frac{7}{2}\right| \le \frac{9}{2}$.

12. $(-2.4, 1.9)$

SOLUTION The midpoint of the interval is $c = (-2.4 + 1.9)/2 = -0.25$, and the radius is $r = (1.9 - (-2.4))/2 = 2.15$; therefore, the interval $(-2.4, 1.9)$ can be expressed as $|x + 0.25| < 2.15$.

In Exercises 13–16, write the inequality in the form $a < x < b$.

13. $|x| < 8$

SOLUTION $-8 < x < 8$

14. $|x - 12| < 8$

SOLUTION $-8 < x - 12 < 8$ so $4 < x < 20$

15. $|2x + 1| < 5$

SOLUTION $-5 < 2x + 1 < 5$ so $-6 < 2x < 4$ and $-3 < x < 2$

16. $|3x - 4| < 2$

SOLUTION $-2 < 3x - 4 < 2$ so $2 < 3x < 6$ and $\frac{2}{3} < x < 2$

In Exercises 17–22, express the set of numbers x satisfying the given condition as an interval.

17. $|x| < 4$

SOLUTION $(-4, 4)$

18. $|x| \le 9$

SOLUTION $[-9, 9]$

19. $|x - 4| < 2$

SOLUTION The expression $|x - 4| < 2$ is equivalent to $-2 < x - 4 < 2$. Therefore, $2 < x < 6$, which represents the interval $(2, 6)$.

20. $|x + 7| < 2$

SOLUTION The expression $|x + 7| < 2$ is equivalent to $-2 < x + 7 < 2$. Therefore, $-9 < x < -5$, which represents the interval $(-9, -5)$.

21. $|4x - 1| \le 8$

SOLUTION The expression $|4x - 1| \le 8$ is equivalent to $-8 \le 4x - 1 \le 8$ or $-7 \le 4x \le 9$. Therefore, $-\frac{7}{4} \le x \le \frac{9}{4}$, which represents the interval $[-\frac{7}{4}, \frac{9}{4}]$.

22. $|3x + 5| < 1$

SOLUTION The expression $|3x + 5| < 1$ is equivalent to $-1 < 3x + 5 < 1$ or $-6 < 3x < -4$. Therefore, $-2 < x < -\frac{4}{3}$, which represents the interval $(-2, -\frac{4}{3})$.

In Exercises 23–26, describe the set as a union of finite or infinite intervals.

23. $\{x : |x - 4| > 2\}$

SOLUTION $x - 4 > 2$ or $x - 4 < -2 \Rightarrow x > 6$ or $x < 2 \Rightarrow (-\infty, 2) \cup (6, \infty)$

24. $\{x : |2x + 4| > 3\}$

SOLUTION $2x + 4 > 3$ or $2x + 4 < -3 \Rightarrow 2x > -1$ or $2x < -7 \Rightarrow (-\infty, -\frac{7}{2}) \cup (-\frac{1}{2}, \infty)$

25. $\{x : |x^2 - 1| > 2\}$

SOLUTION $x^2 - 1 > 2$ or $x^2 - 1 < -2 \Rightarrow x^2 > 3$ or $x^2 < -1$ (this will never happen) $\Rightarrow x > \sqrt{3}$ or $x < -\sqrt{3} \Rightarrow (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

26. $\{x : |x^2 + 2x| > 2\}$

SOLUTION $x^2 + 2x > 2$ or $x^2 + 2x < -2 \Rightarrow x^2 + 2x - 2 > 0$ or $x^2 + 2x + 2 < 0$. For the first case, the zeroes are

$$x = -1 \pm \sqrt{3} \Rightarrow (-\infty, -1 - \sqrt{3}) \cup (-1 + \sqrt{3}, \infty).$$

For the second case, note there are no real zeros. Because the parabola opens upward and its vertex is located above the x-axis, there are no values of x for which $x^2 + 2x + 2 < 0$. Hence, the solution set is $(-\infty, -1 - \sqrt{3}) \cup (-1 + \sqrt{3}, \infty)$.

27. Match (a)–(f) with (i)–(vi).

(a) $a > 3$

(b) $|a - 5| < \frac{1}{3}$

(c) $\left|a - \frac{1}{3}\right| < 5$

(d) $|a| > 5$

(e) $|a - 4| < 3$

(f) $1 \le a \le 5$

(i) a lies to the right of 3.
(ii) a lies between 1 and 7.
(iii) The distance from a to 5 is less than $\frac{1}{3}$.
(iv) The distance from a to 3 is at most 2.
(v) a is less than 5 units from $\frac{1}{3}$.
(vi) a lies either to the left of -5 or to the right of 5.

SOLUTION

(a) On the number line, numbers greater than 3 appear to the right; hence, $a > 3$ is equivalent to the numbers to the right of 3: **(i)**.

(b) $|a - 5|$ measures the distance from a to 5; hence, $|a - 5| < \frac{1}{3}$ is satisfied by those numbers less than $\frac{1}{3}$ of a unit from 5: **(iii)**.

(c) $|a - \frac{1}{3}|$ measures the distance from a to $\frac{1}{3}$; hence, $|a - \frac{1}{3}| < 5$ is satisfied by those numbers less than 5 units from $\frac{1}{3}$: **(v)**.

(d) The inequality $|a| > 5$ is equivalent to $a > 5$ or $a < -5$; that is, either a lies to the right of 5 or to the left of -5: **(vi)**.

(e) The interval described by the inequality $|a - 4| < 3$ has a center at 4 and a radius of 3; that is, the interval consists of those numbers between 1 and 7: **(ii)**.

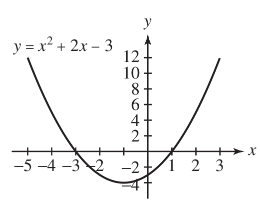
(f) The interval described by the inequality $1 < x < 5$ has a center at 3 and a radius of 2; that is, the interval consists of those numbers less than 2 units from 3: **(iv)**.

28. Describe $\left\{x : \frac{x}{x+1} < 0\right\}$ as an interval. *Hint:* Consider the sign of x and $x + 1$ individually.

SOLUTION Case 1: $x < 0$ and $x + 1 > 0$. This implies that $x < 0$ and $x > -1 \Rightarrow -1 < x < 0$.
Case 2: $x > 0$ and $x < -1$ for which there is no such x. Thus, solution set is therefore $(-1, 0)$.

29. Describe $\{x : x^2 + 2x < 3\}$ as an interval. *Hint:* Plot $y = x^2 + 2x - 3$.

SOLUTION The inequality $x^2 + 2x < 3$ is equivalent to $x^2 + 2x - 3 < 0$. The graph of $y = x^2 + 2x - 3$ is shown here. From this graph, it follows that $x^2 + 2x - 3 < 0$ for $-3 < x < 1$. Thus, the set $\{x : x^2 + 2x < 3\}$ is equivalent to the interval $(-3, 1)$.



30. Describe the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ as a half-infinite interval.

SOLUTION Case 1: If $x \ge 3$, then $|x - 3| = x - 3$, $|x - 2| = x - 2$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $x - 3 = x - 2 + 1$ or $-3 = -1$. As this is never true, the given equation has no solution for $x \ge 3$.

Case 2: If $2 \le x < 3$, then $|x - 3| = -(x - 3) = 3 - x$, $|x - 2| = x - 2$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $3 - x = x - 2 + 1$ or $x = 2$.

Case 3: If $x < 2$, then $|x - 3| = -(x - 3) = 3 - x$, $|x - 2| = -(x - 2) = 2 - x$, and the equation $|x - 3| = |x - 2| + 1$ reduces to $3 - x = 2 - x + 1$ or $1 = 1$. As this is always true, the given equation holds for all $x < 2$.

Combining the results from all three cases, it follows that the set of real numbers satisfying $|x - 3| = |x - 2| + 1$ is equivalent to the half-infinite interval $(-\infty, 2]$.

31. Show that if $a > b$, and $a, b \neq 0$, then $b^{-1} > a^{-1}$, provided that a and b have the same sign. What happens if $a > 0$ and $b < 0$?

SOLUTION Case 1a: If a and b are both positive, then $a > b \Rightarrow 1 > \frac{b}{a} \Rightarrow \frac{1}{b} > \frac{1}{a}$.
Case 1b: If a and b are both negative, then $a > b \Rightarrow 1 < \frac{b}{a}$ (since a is negative) $\Rightarrow \frac{1}{b} > \frac{1}{a}$ (again, since b is negative).
Case 2: If $a > 0$ and $b < 0$, then $\frac{1}{a} > 0$ and $\frac{1}{b} < 0$ so $\frac{1}{b} < \frac{1}{a}$. (See Exercise 6f for an example of this.)

32. Which x satisfies both $|x - 3| < 2$ and $|x - 5| < 1$?

SOLUTION $|x - 3| < 2 \Rightarrow -2 < x - 3 < 2 \Rightarrow 1 < x < 5$. Also $|x - 5| < 1 \Rightarrow 4 < x < 6$. Since we want an x that satisfies both of these, we need the intersection of the two solution sets, that is, $4 < x < 5$.

33. Show that if $|a - 5| < \frac{1}{2}$ and $|b - 8| < \frac{1}{2}$, then
$|(a + b) - 13| < 1$. *Hint:* Use the triangle inequality ($|a + b| \leq |a| + |b|$).

SOLUTION

$$\begin{aligned} |a + b - 13| &= |(a - 5) + (b - 8)| \\ &\leq |a - 5| + |b - 8| \quad \text{(by the triangle inequality)} \\ &< \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

34. Suppose that $|x - 4| \leq 1$.
(a) What is the maximum possible value of $|x + 4|$?
(b) Show that $|x^2 - 16| \leq 9$.

SOLUTION
(a) $|x - 4| \leq 1$ guarantees $3 \leq x \leq 5$. Thus, $7 \leq x + 4 \leq 9$, so $|x + 4| \leq 9$.
(b) $|x^2 - 16| = |x - 4| \cdot |x + 4| \leq 1 \cdot 9 = 9$

35. Suppose that $|a - 6| \leq 2$ and $|b| \leq 3$.
(a) What is the largest possible value of $|a + b|$?
(b) What is the smallest possible value of $|a + b|$?

SOLUTION $|a - 6| \leq 2$ guarantees $4 \leq a \leq 8$, and $|b| \leq 3$ guarantees $-3 \leq b \leq 3$, so $1 \leq a + b \leq 11$. Based on this information,
(a) the largest possible value of $|a + b|$ is 11; and
(b) the smallest possible value of $|a + b|$ is 1.

36. Prove that $|x| - |y| \leq |x - y|$. *Hint:* Apply the triangle inequality to y and $x - y$.

SOLUTION First note

$$|x| = |x - y + y| \leq |x - y| + |y|$$

by the triangle inequality. Subtracting $|y|$ from both sides of this inequality yields

$$|x| - |y| \leq |x - y|$$

37. Express $r_1 = 0.\overline{27}$ as a fraction. *Hint:* $100r_1 - r_1$ is an integer. Then express $r_2 = 0.2666\ldots$ as a fraction.

SOLUTION Let $r_1 = 0.\overline{27}$. We observe that $100r_1 = 27.\overline{27}$. Therefore, $100r_1 - r_1 = 27.\overline{27} - 0.\overline{27} = 27$ and

$$r_1 = \frac{27}{99} = \frac{3}{11}$$

Now, let $r_2 = 0.2\overline{666}$. Then $10r_2 = 2.\overline{666}$ and $100r_2 = 26.\overline{666}$. Therefore, $100r_2 - 10r_2 = 26.\overline{666} - 2.\overline{666} = 24$ and

$$r_2 = \frac{24}{90} = \frac{4}{15}$$

38. Represent $1/7$ and $4/27$ as repeating decimals.

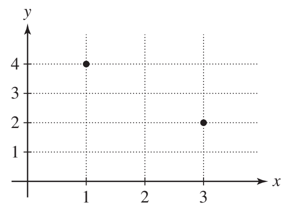
SOLUTION $\frac{1}{7} = 0.\overline{142857}$; $\frac{4}{27} = 0.\overline{148}$

39. Plot each pair of points and compute the distance between them:
(a) $(1, 4)$ and $(3, 2)$
(b) $(2, 1)$ and $(2, 4)$

SOLUTION

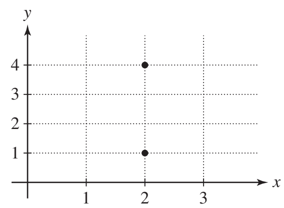
(a) The points (1, 4) and (3, 2) are plotted in the figure. The distance between the points is

$$d = \sqrt{(3-1)^2 + (2-4)^2} = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$



(b) The points (2, 1) and (2, 4) are plotted in the figure. The distance between the points is

$$d = \sqrt{(2-2)^2 + (4-1)^2} = \sqrt{9} = 3$$



40. Plot each pair of points and compute the distance between them:

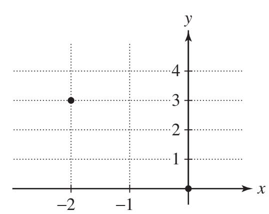
(a) (0, 0) and (−2, 3)

(b) (−3, −3) and (−2, 3)

SOLUTION

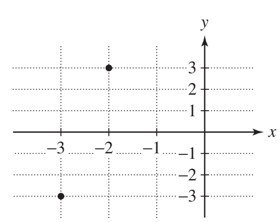
(a) The points (0, 0) and (−2, 3) are plotted in the figure. The distance between the points is

$$d = \sqrt{(-2-0)^2 + (3-0)^2} = \sqrt{4+9} = \sqrt{13}$$



(b) The points (−3, −3) and (−2, 3) are plotted in the figure. The distance between the points is

$$d = \sqrt{(-3-(-2))^2 + (-3-3)^2} = \sqrt{1+36} = \sqrt{37}$$



41. Find the equation of the circle with center (2, 4):

(a) With radius $r = 3$

(b) That passes through (1, −1)

SOLUTION (a) The equation of the indicated circle is $(x-2)^2 + (y-4)^2 = 3^2 = 9$.

(b) First, determine the radius as the distance from the center to the indicated point on the circle:

$$r = \sqrt{(2-1)^2 + (4-(-1))^2} = \sqrt{26}$$

Thus, the equation of the circle is $(x-2)^2 + (y-4)^2 = 26$.

42. Find all points in the xy-plane with integer coordinates located at a distance 5 from the origin. Then find all points with integer coordinates located at a distance 5 from (2, 3).

SOLUTION

- To be located a distance 5 from the origin, the points must lie on the circle $x^2 + y^2 = 25$. This leads to 12 points with integer coordinates:

$$\begin{array}{cccc} (5,0) & (-5,0) & (0,5) & (0,-5) \\ (3,4) & (-3,4) & (3,-4) & (-3,-4) \\ (4,3) & (-4,3) & (4,-3) & (-4,-3) \end{array}$$

- To be located a distance 5 from the point $(2,3)$, the points must lie on the circle $(x-2)^2 + (y-3)^2 = 25$, which implies that we must shift the points listed 2 units to the right and 3 units up. This gives the 12 points

$$\begin{array}{cccc} (7,3) & (-3,3) & (2,8) & (2,-2) \\ (5,7) & (-1,7) & (5,-1) & (-1,-1) \\ (6,6) & (-2,6) & (6,0) & (-2,0) \end{array}$$

43. Determine the domain and range of the function

$$f : \{r,s,t,u\} \to \{A,B,C,D,E\}$$

defined by $f(r) = A$, $f(s) = B$, $f(t) = B$, $f(u) = E$.

SOLUTION The domain is the set $D = \{r,s,t,u\}$; the range is the set $R = \{A,B,E\}$.

44. Give an example of a function whose domain D has three elements and whose range R has two elements. Does a function exist whose domain D has two elements and whose range R has three elements?

SOLUTION Define f by $f : \{a,b,c\} \to \{1,2\}$, where $f(a) = 1$, $f(b) = 1$, $f(c) = 2$.

There is no function whose domain has two elements and range has three elements. If that happened, one of the domain elements would get assigned to more than one element of the range, which would contradict the definition of a function.

In Exercises 45–52, find the domain and range of the function.

45. $f(x) = -x$

SOLUTION D: all reals; R: all reals

46. $g(t) = t^4$

SOLUTION D: all reals; R: $\{y : y \ge 0\}$

47. $f(x) = x^3$

SOLUTION D: all reals; R: all reals

48. $g(t) = \sqrt{2-t}$

SOLUTION D: $\{t : t \le 2\}$; R: $\{y : y \ge 0\}$

49. $f(x) = |x|$

SOLUTION D: all reals; R: $\{y : y \ge 0\}$

50. $h(s) = \frac{1}{s}$

SOLUTION D: $\{s : s \ne 0\}$; R: $\{y : y \ne 0\}$

51. $f(x) = \frac{1}{x^2}$

SOLUTION D: $\{x : x \ne 0\}$; R: $\{y : y > 0\}$

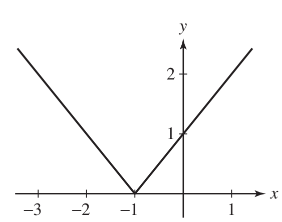
52. $g(t) = \frac{1}{\sqrt{1-t}}$

SOLUTION D: $\{t : t < 1\}$; R: $\{y : y > 0\}$

In Exercises 53–56, determine where f is increasing.

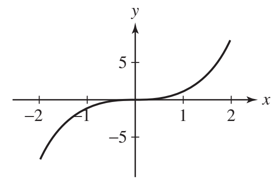
53. $f(x) = |x+1|$

SOLUTION A graph of the function $y = |x+1|$ is shown. From the graph, we see that the function is increasing on the interval $(-1, \infty)$.



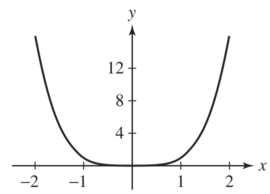
54. $f(x) = x^3$

SOLUTION A graph of the function $y = x^3$ is shown. From the graph, we see that the function is increasing for all real numbers.



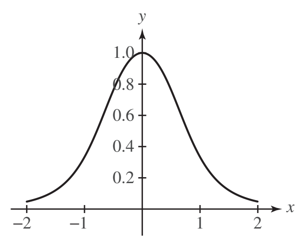
55. $f(x) = x^4$

SOLUTION A graph of the function $y = x^4$ is shown. From the graph, we see that the function is increasing on the interval $(0, \infty)$.



56. $f(x) = \frac{1}{x^4 + x^2 + 1}$

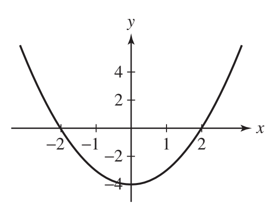
SOLUTION A graph of the function $y = \frac{1}{x^4 + x^2 + 1}$ is shown. From the graph, we see that the function is increasing on the interval $(-\infty, 0)$.



In Exercises 57–62, find the zeros of f and sketch its graph by plotting points. Use symmetry and increase/decrease information where appropriate.

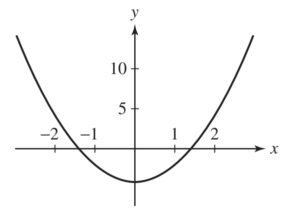
57. $f(x) = x^2 - 4$

SOLUTION Zeros: ± 2
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function); so, y-axis symmetry



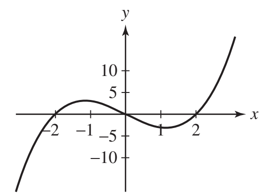
58. $f(x) = 2x^2 - 4$

SOLUTION Zeros: $\pm\sqrt{2}$
Increasing: $x > 0$
Decreasing: $x < 0$
Symmetry: $f(-x) = f(x)$ (even function); so, y-axis symmetry



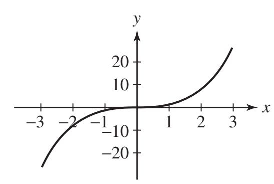
59. $f(x) = x^3 - 4x$

SOLUTION Zeros: $0, \pm 2$; symmetry: $f(-x) = -f(x)$ (odd function); so, origin symmetry



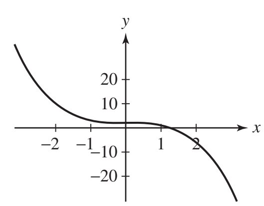
60. $f(x) = x^3$

SOLUTION Zeros: 0; increasing for all x; symmetry: $f(-x) = -f(x)$ (odd function); so, origin symmetry



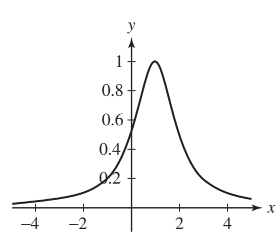
61. $f(x) = 2 - x^3$

SOLUTION This is an x-axis reflection of x^3 translated up 2 units. There is one zero at $x = \sqrt[3]{2}$.



62. $f(x) = \frac{1}{(x-1)^2 + 1}$

SOLUTION This is the graph of $\frac{1}{x^2+1}$ translated to the right 1 unit. The function has no zeros.



63. Which of the curves in Figure 27 is the graph of a function of x?

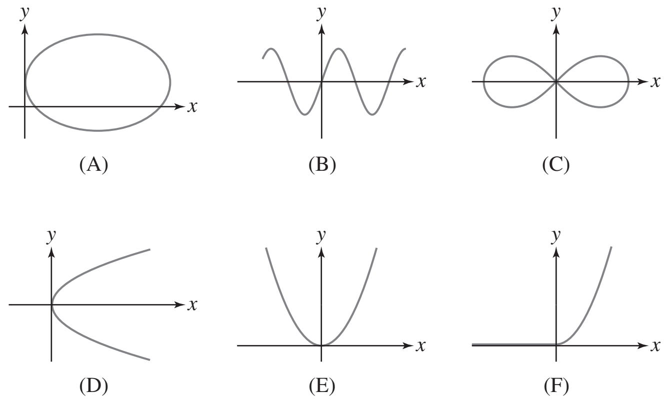


FIGURE 27

SOLUTION (B), (E), and (F) are graphs of functions. (A), (C), and (D) all fail the vertical line test.

64. Of the curves in Figure 27 that are graphs of functions, which is the graph of an odd function? Of an even function?

SOLUTION (B) is the graph of an odd function because the graph is symmetric about the origin; (E) is the graph of an even function because the graph is symmetric about the y-axis.

65. Determine whether the function is even, odd, or neither.

(a) $f(x) = x^5$ **(b)** $g(t) = t^3 - t^2$ **(c)** $F(t) = \frac{1}{t^4 + t^2}$

SOLUTION

(a) Because $f(-x) = (-x)^5 = -x^5 = -f(x)$, $f(x) = x^5$ is an odd function.

(b) Because $g(-t) = (-t)^3 - (-t)^2 = -t^3 - t^2$ equals neither $g(t)$ nor $-g(t)$, $g(t) = t^3 - t^2$ is neither an even function nor an odd function.

(c) Because $F(-t) = \frac{1}{(-t)^4 + (-t)^2} = \frac{1}{t^4 + t^2} = F(t)$, $F(t) = \frac{1}{t^4 + t^2}$ is an even function.

66. Determine whether the function is even, odd, or neither.

(a) $f(x) = 2x - x^2$ **(b)** $k(w) = (1 - w)^3 + (1 + w)^3$

(c) $f(t) = \frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}$ **(d)** $g(t) = 2^t - 2^{-t}$

SOLUTION

(a) Because $f(-x) = 2(-x) - (-x)^2 = -2x - x^2$ equals neither $f(x)$ nor $-f(x)$, $f(x) = 2x - x^2$ is neither an even nor an odd function.

(b) Because $k(-w) = (1 - (-w))^3 + (1 + (-w))^3 = (1 + w)^3 + (1 - w)^3 = k(w)$, $k(w) = (1 - w)^3 + (1 + w)^3$ is an even function.

(c) Because

$$f(-t) = \frac{1}{(-t)^4 + (-t) + 1} - \frac{1}{(-t)^4 - (-t) + 1} = \frac{1}{t^4 - t + 1} - \frac{1}{t^4 + t + 1}$$

$$= -\left(\frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}\right) = -f(t)$$

$f(t) = \frac{1}{t^4 + t + 1} - \frac{1}{t^4 - t + 1}$ is an odd function.

(d) Because $g(-t) = 2^{-t} - 2^{-(-t)} = 2^{-t} - 2^t = -(2^t - 2^{-t}) = -g(t)$, $g(t) = 2^t - 2^{-t}$ is an odd function.

67. Write $f(x) = 2x^4 - 5x^3 + 12x^2 - 3x + 4$ as the sum of an even and an odd function.

SOLUTION Let $g(x) = 2x^4 + 12x^2 + 4$ and $h(x) = -5x^3 - 3x$. Then

$$g(-x) = 2(-x)^4 + 12(-x)^2 + 4 = 2x^4 + 12x^2 + 4 = g(x)$$

so that g is an even function,

$$h(-x) = -5(-x)^3 - 3(-x) = 5x^3 + 3x = -h(x)$$

so that h is an odd function, and $f(x) = g(x) + h(x)$.

68. Assume that p is a function that is defined for all x.

(a) Prove that if f is defined by $f(x) = p(x) + p(-x)$ then f is even.

(b) Prove that if g is defined by $g(x) = p(x) - p(-x)$ then g is odd.

SOLUTION

(a) Let $f(x) = p(x) + p(-x)$. Then

$$f(-x) = p(-x) + p(-(-x)) = p(-x) + p(x) = f(x)$$

Because $f(-x) = f(x)$, it follows that f is an even function.

(b) Let $g(x) = p(x) - p(-x)$. Then

$$g(-x) = p(-x) - p(-(-x)) = p(-x) - p(x) = -(p(x) - p(-x)) = -g(x)$$

Because $g(-x) = -g(x)$, it follows that g is an odd function.

69. Assume that p is a function that is defined for $x > 0$ and satisfies $p(a/b) = p(b) - p(a)$. Prove that $f(x) = p\left(\frac{2-x}{2+x}\right)$ is an odd function.

SOLUTION Let $f(x) = p\left(\frac{2-x}{2+x}\right)$. Then

$$f(-x) = p\left(\frac{2-(-x)}{2-x}\right) = p\left(\frac{2+x}{2-x}\right) = p(2+x) - p(2-x) = -(p(2-x) - p(2+x)) = -p\left(\frac{2-x}{2+x}\right) = -f(x)$$

Because $f(-x) = -f(x)$, it follows that f is an odd function.

70. State whether the function is increasing, decreasing, or neither.

(a) Surface area of a sphere as a function of its radius

(b) Temperature at a point on the equator as a function of time

(c) Price of an airline ticket as a function of the price of oil

(d) Pressure of the gas in a piston as a function of volume

SOLUTION

(a) Increasing **(b)** Neither **(c)** Increasing **(d)** Decreasing

In Exercises 71–76, let f be the function shown in Figure 28.

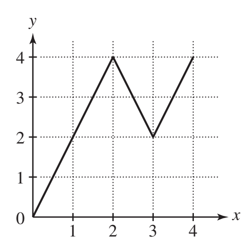


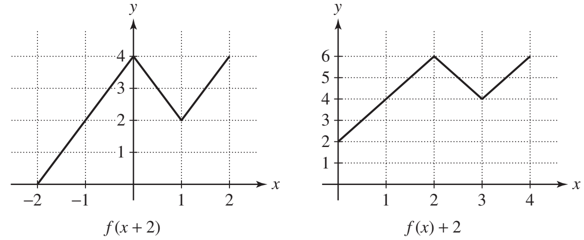
FIGURE 28

71. Find the domain and range of f.

SOLUTION D: $[0, 4]$; R: $[0, 4]$

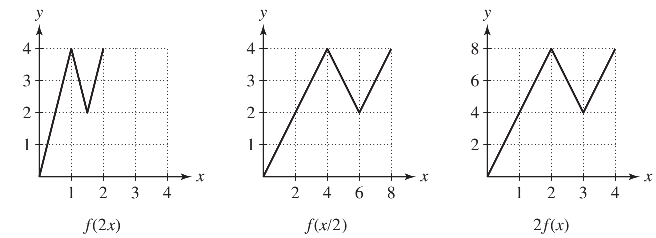
72. Sketch the graphs of $y = f(x + 2)$ and $y = f(x) + 2$.

SOLUTION The graph of $y = f(x + 2)$ is obtained by shifting the graph of $y = f(x)$ 2 units to the left (see the graph below on the left). The graph of $y = f(x) + 2$ is obtained by shifting the graph of $y = f(x)$ 2 units up (see the graph below on the right).



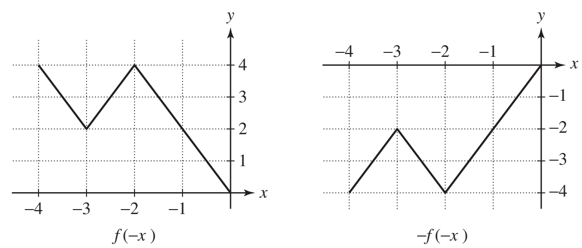
73. Sketch the graphs of $y = f(2x)$, $y = f(\frac{1}{2}x)$, and $y = 2f(x)$.

SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below in the middle). The graph of $y = 2f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 2 (see the graph below on the right).



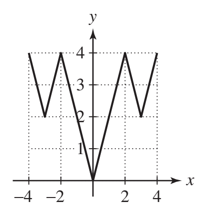
74. Sketch the graphs of $y = f(-x)$ and $y = -f(-x)$.

SOLUTION The graph of $y = f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across the y-axis (see the graph below on the left). The graph of $y = -f(-x)$ is obtained by reflecting the graph of $y = f(x)$ across both the x- and y-axes, or equivalently, about the origin (see the graph below on the right).



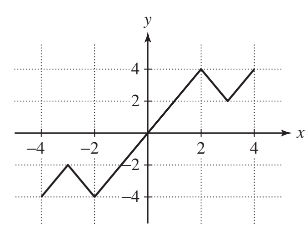
75. Extend the graph of f to $[-4, 4]$ so that it is an even function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an even function, reflect the graph of $f(x)$ across the y-axis (see the graph).



76. Extend the graph of f to $[-4, 4]$ so that it is an odd function.

SOLUTION To continue the graph of $f(x)$ to the interval $[-4, 4]$ as an odd function, reflect the graph of $f(x)$ through the origin (see the graph).



77. Suppose that f has domain $[4, 8]$ and range $[2, 6]$. Find the domain and range of:

(a) $y = f(x) + 3$ **(b)** $y = f(x + 3)$

(c) $y = f(3x)$ **(d)** $y = 3f(x)$

SOLUTION

(a) $f(x) + 3$ is obtained by shifting $f(x)$ upward 3 units. Therefore, the domain remains $[4, 8]$, while the range becomes $[5, 9]$.

(b) $f(x + 3)$ is obtained by shifting $f(x)$ left 3 units. Therefore, the domain becomes $[1, 5]$, while the range remains $[2, 6]$.

(c) $f(3x)$ is obtained by compressing $f(x)$ horizontally by a factor of 3. Therefore, the domain becomes $[\frac{4}{3}, \frac{8}{3}]$, while the range remains $[2, 6]$.

(d) $3f(x)$ is obtained by stretching $f(x)$ vertically by a factor of 3. Therefore, the domain remains $[4, 8]$, while the range becomes $[6, 18]$.

78. Let $f(x) = x^2$. Sketch the graph over $[-2, 2]$ of:

(a) $y = f(x + 1)$

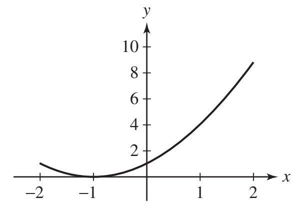
(b) $y = f(x) + 1$

(c) $y = f(5x)$

(d) $y = 5f(x)$

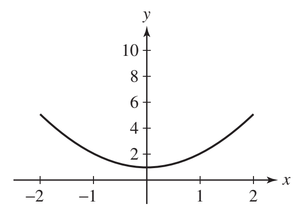
SOLUTION

(a) The graph of $y = f(x + 1)$ is obtained by shifting the graph of $y = f(x)$ 1 unit to the left.



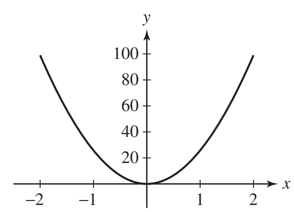
$f(x + 1)$

(b) The graph of $y = f(x) + 1$ is obtained by shifting the graph of $y = f(x)$ 1 unit up.



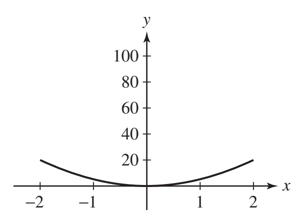
$f(x) + 1$

(c) The graph of $y = f(5x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 5.



$f(5x)$

(d) The graph of $y = 5f(x)$ is obtained by stretching the graph of $y = f(x)$ vertically by a factor of 5.



$5f(x)$

79. Suppose that the graph of $f(x) = x^4 - x^2$ is compressed horizontally by a factor of 2 and then shifted 5 units to the right.

(a) What is the equation for the new graph?

(b) What is the equation if you first shift by 5 and then compress by 2?

(c) GU Verify your answers by plotting your equations.

SOLUTION

(a) Let $f(x) = x^4 - x^2$. After compressing the graph of f horizontally by a factor of 2, we obtain the function $g(x) = f(2x) = (2x)^4 - (2x)^2$. Shifting the graph 5 units to the right then yields

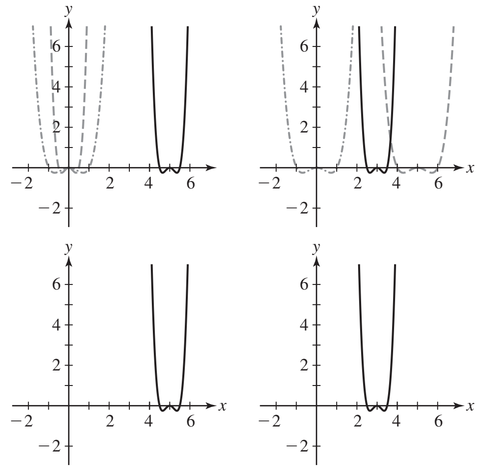
$$h(x) = g(x - 5) = (2(x - 5))^4 - (2(x - 5))^2 = (2x - 10)^4 - (2x - 10)^2$$

(b) Let $f(x) = x^4 - x^2$. After shifting the graph 5 units to the right, we obtain the function $g(x) = f(x - 5) = (x - 5)^4 - (x - 5)^2$. Compressing the graph horizontally by a factor of 2 then yields

$$h(x) = g(2x) = (2x - 5)^4 - (2x - 5)^2$$

(c) The figure below at the top left shows the graphs of $y = x^4 - x^2$ (the dash-dot curve), the graph compressed horizontally by a factor of 2 (the dashed curve), and then shifted right 5 units (the solid curve). Compare this last graph with the graph of $y = (2x - 10)^4 - (2x - 10)^2$ shown at the bottom left.

The figure below at the top right shows the graphs of $y = x^4 - x^2$ (the dash-dot curve), the graph shifted right 5 units (the dashed curve), and then compressed horizontally by a factor of 2 (the solid curve). Compare this last graph with the graph of $y = (2x - 5)^4 - (2x - 5)^2$ shown at the bottom right.



80. Figure 29 shows the graph of $f(x) = |x| + 1$. Match the functions (a)–(e) with their graphs (i)–(v).

(a) $y = f(x - 1)$ **(b)** $y = -f(x)$ **(c)** $y = -f(x) + 2$

(d) $y = f(x - 1) - 2$ **(e)** $y = f(x + 1)$

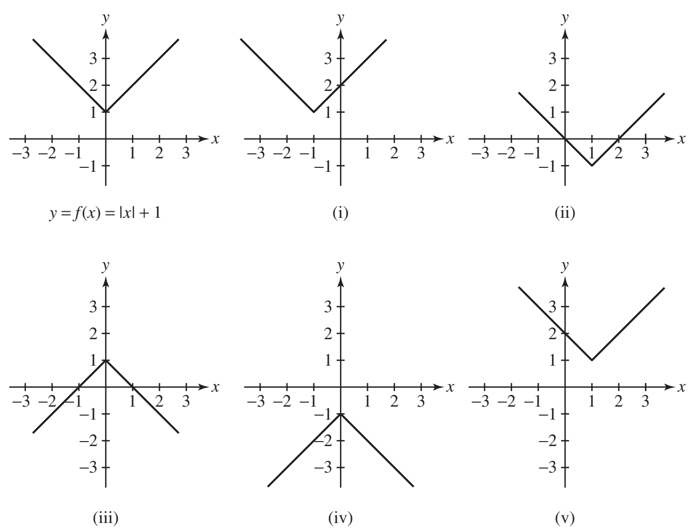


FIGURE 29

SOLUTION

(a) Shift graph to the right 1 unit: (v)

(b) Reflect graph across x-axis: (iv)

(c) Reflect graph across x-axis and then shift up 2 units: (iii)

(d) Shift graph to the right one unit and down 2 units: (ii)

(e) Shift graph to the left 1 unit: (i)

81. Sketch the graph of $y = f(2x)$ and $y = f(\frac{1}{2}x)$, where $f(x) = |x| + 1$ (Figure 29).

SOLUTION The graph of $y = f(2x)$ is obtained by compressing the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the left). The graph of $y = f(\frac{1}{2}x)$ is obtained by stretching the graph of $y = f(x)$ horizontally by a factor of 2 (see the graph below on the right).