3 DIFFERENTIATION

3.1 Definition of the Derivative

Preliminary Questions

1. Which of the lines in Figure 11 are tangent to the curve?

FIGURE 11

SOLUTION Lines B and D are tangent to the curve.

2. What are the two ways of writing the difference quotient?

SOLUTION The difference quotient may be written either as

$$\frac{f(x) - f(a)}{x - a}$$

or as

$$\frac{f(a+h) - f(a)}{h}.$$

3. Find a and h such that $\frac{f(a+h) - f(a)}{h}$ is equal to the slope of the secant line between $(3, f(3))$ and $(5, f(5))$.

SOLUTION With $a = 3$ and $h = 2$, $\frac{f(a+h) - f(a)}{h}$ is equal to the slope of the secant line between the points $(3, f(3))$ and $(5, f(5))$ on the graph of $f(x)$.

4. Which derivative is approximated by $\frac{\tan\left(\frac{\pi}{4} + 0.0001\right) - 1}{0.0001}$?

SOLUTION $\frac{\tan(\frac{\pi}{4} + 0.0001) - 1}{0.0001}$ is a good approximation to the derivative of the function $f(x) = \tan x$ at $x = \frac{\pi}{4}$.

5. What do the following quantities represent in terms of the graph of $f(x) = \sin x$?

(a) $\sin 1.3 - \sin 0.9$ **(b)** $\frac{\sin 1.3 - \sin 0.9}{0.4}$ **(c)** $f'(0.9)$

SOLUTION Consider the graph of $y = \sin x$.

(a) The quantity $\sin 1.3 - \sin 0.9$ represents the difference in height between the points $(0.9, \sin 0.9)$ and $(1.3, \sin 1.3)$.

(b) The quantity $\frac{\sin 1.3 - \sin 0.9}{0.4}$ represents the slope of the secant line between the points $(0.9, \sin 0.9)$ and $(1.3, \sin 1.3)$ on the graph.

(c) The quantity $f'(0.9)$ represents the slope of the tangent line to the graph at $x = 0.9$.

Exercises

1. Let $f(x) = 5x^2$. Show that $f(3+h) = 5h^2 + 30h + 45$. Then show that

$$\frac{f(3+h) - f(3)}{h} = 5h + 30$$

and compute $f'(3)$ by taking the limit as $h \to 0$.

SOLUTION With $f(x) = 5x^2$, it follows that

$$f(3+h) = 5(3+h)^2 = 5(9 + 6h + h^2) = 45 + 30h + 5h^2.$$

Using this result, we find

$$\frac{f(3+h) - f(3)}{h} = \frac{45 + 30h + 5h^2 - 5 \cdot 9}{h} = \frac{45 + 30h + 5h^2 - 45}{h} = \frac{30h + 5h^2}{h} = 30 + 5h.$$

As $h \to 0$, $30 + 5h \to 30$, so $f'(3) = 30$.

2. Let $f(x) = 2x^2 - 3x - 5$. Show that the secant line through $(2, f(2))$ and $(2+h, f(2+h))$ has slope $2h + 5$. Then use this formula to compute the slope of:

(a) The secant line through $(2, f(2))$ and $(3, f(3))$

(b) The tangent line at $x = 2$ (by taking a limit)

SOLUTION The formula for the slope of the secant line is

$$\frac{f(2+h) - f(2)}{2+h-2} = \frac{[2(2+h)^2 - 3(2+h) - 5] - (8 - 6 - 5)}{h} = \frac{2h^2 + 5h}{h} = 2h + 5$$

(a) To find the slope of the secant line through $(2, f(2))$ and $(3, f(3))$, we take $h = 1$, so the slope is $2(1) + 5 = 7$.

(b) As $h \to 0$, the slope of the secant line approaches $2(0) + 5 = 5$. Hence, the slope of the tangent line at $x = 2$ is 5.

In Exercises 3–8, compute $f'(a)$ in two ways, using Eq. (1) and Eq. (2).

3. $f(x) = x^2 + 9x, \quad a = 0$

SOLUTION Let $f(x) = x^2 + 9x$. Then

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{(0+h)^2 + 9(0+h) - 0}{h} = \lim_{h\to 0} \frac{9h + h^2}{h} = \lim_{h\to 0} (9 + h) = 9.$$

Alternately,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 + 9x - 0}{x} = \lim_{x\to 0} (x + 9) = 9.$$

4. $f(x) = x^2 + 9x, \quad a = 2$

SOLUTION Let $f(x) = x^2 + 9x$. Then

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^2 + 9(2+h) - 22}{h} = \lim_{h\to 0} \frac{13h + h^2}{h} = \lim_{h\to 0} (13 + h) = 13.$$

Alternately,

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{x^2 + 9x - (2^2 + 9(2))}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+11)}{x - 2} = \lim_{x\to 2} (x + 11) = 13.$$

5. $f(x) = 3x^2 + 4x + 2, \quad a = -1$

SOLUTION Let $f(x) = 3x^2 + 4x + 2$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{3(-1+h)^2 + 4(-1+h) + 2 - 1}{h}$$
$$= \lim_{h\to 0} \frac{3h^2 - 2h}{h} = \lim_{h\to 0}(3h - 2) = -2.$$

Alternately,

$$f'(-1) = \lim_{x\to -1} \frac{f(x) - f(-1)}{x - (-1)} = \lim_{x\to -1} \frac{3x^2 + 4x + 2 - 1}{x + 1}$$
$$= \lim_{x\to -1} \frac{(3x+1)(x+1)}{x+1} = \lim_{x\to -1}(3x + 1) = -2.$$

6. $f(x) = x^3, \quad a = 2$

SOLUTION Let $f(x) = x^3$. Then

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$$
$$= \lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \lim_{h\to 0}(12 + 6h + h^2) = 12.$$

Alternately,

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{x^3 - 8}{x - 2}$$
$$= \lim_{x\to 2} \frac{(x-2)(x^2 + 2x + 4)}{x - 2} = \lim_{x\to 2}(x^2 + 2x + 4) = 12.$$

7. $f(x) = x^3 + 2x, \quad a = 1$

SOLUTION Let $f(x) = x^3 + 2x$. Then

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{(1+h)^3 + 2(1+h) - 3}{h}$$
$$= \lim_{h\to 0} \frac{1 + 3h + 3h^2 + h^3 + 2 + 2h - 3}{h} = \lim_{h\to 0}(5 + 3h + h^2) = 5.$$

Alternately,

$$f'(1) = \lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^3 + 2x - 3}{x - 1}$$
$$= \lim_{x\to 1} \frac{(x-1)(x^2 + x + 3)}{x - 1} = \lim_{x\to 1}(x^2 + x + 3) = 5.$$

8. $f(x) = \frac{1}{x}, \quad a = 2$

SOLUTION Let $f(x) = \frac{1}{x}$. Then

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h}$$
$$= \lim_{h\to 0} \frac{2 - (2+h)}{2h(2+h)} = \lim_{h\to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}.$$

Alternately,

$$f'(2) = \lim_{x\to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2}$$
$$= \lim_{x\to 2} \frac{2 - x}{2x(x-2)} = \lim_{x\to 2} \frac{-1}{2x} = -\frac{1}{4}.$$

In Exercises 9–12, refer to Figure 12.

9. Find the slope of the secant line through $(2, f(2))$ and $(2.5, f(2.5))$. Is it larger or smaller than $f'(2)$? Explain.

SOLUTION From the graph, it appears that $f(2.5) = 2.5$ and $f(2) = 2$. Thus, the slope of the secant line through $(2, f(2))$ and $(2.5, f(2.5))$ is

$$\frac{f(2.5) - f(2)}{2.5 - 2} = \frac{2.5 - 2}{2.5 - 2} = 1.$$

From the graph, it is also clear that the secant line through $(2, f(2))$ and $(2.5, f(2.5))$ has a larger slope than the tangent line at $x = 2$. In other words, the slope of the secant line through $(2, f(2))$ and $(2.5, f(2.5))$ is larger than $f'(2)$.

10. Estimate $\dfrac{f(2+h) - f(2)}{h}$ for $h = -0.5$. What does this quantity represent? Is it larger or smaller than $f'(2)$? Explain.

SOLUTION With $h = -0.5$, $2 + h = 1.5$. Moreover, from the graph it appears that $f(1.5) = 1.7$ and $f(2) = 2$. Thus,

$$\frac{f(2+h) - f(2)}{h} = \frac{1.7 - 2}{-0.5} = 0.6.$$

This quantity represents the slope of the secant line through the points $(2, f(2))$ and $(1.5, f(1.5))$. It is clear from the graph that the secant line through the points $(2, f(2))$ and $(1.5, f(1.5))$ has a smaller slope than the tangent line at $x = 2$. In other words, $\dfrac{f(2+h) - f(2)}{h}$ for $h = -0.5$ is smaller than $f'(2)$.

11. Estimate $f'(1)$ and $f'(2)$.

SOLUTION From the graph, it appears that the tangent line at $x = 1$ would be horizontal. Thus, $f'(1) \approx 0$. The tangent line at $x = 2$ appears to pass through the points $(0.5, 0.8)$ and $(2, 2)$. Thus

$$f'(2) \approx \frac{2 - 0.8}{2 - 0.5} = 0.8.$$

12. Find a value of h for which $\dfrac{f(2+h) - f(2)}{h} = 0$.

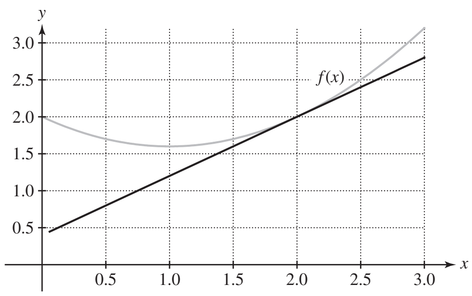


FIGURE 12

SOLUTION In order for

$$\frac{f(2+h) - f(2)}{h}$$

to be equal to zero, we must have $f(2+h) = f(2)$. Now, $f(2) = 2$, and the only other point on the graph with a y-coordinate of 2 is $f(0) = 2$. Thus, $2 + h = 0$, or $h = -2$.

In Exercises 13–16, refer to Figure 13.

13. Determine $f'(a)$ for $a = 1, 2, 4, 7$.

SOLUTION Remember that the value of the derivative of f at $x = a$ can be interpreted as the slope of the line tangent to the graph of $y = f(x)$ at $x = a$. From Figure 13, we see that the graph of $y = f(x)$ is a horizontal line (that is, a line with zero slope) on the interval $0 \le x \le 3$. Accordingly, $f'(1) = f'(2) = 0$. On the interval $3 \le x \le 5$, the graph of $y = f(x)$ is a line of slope $\frac{1}{2}$; thus, $f'(4) = \frac{1}{2}$. Finally, the line tangent to the graph of $y = f(x)$ at $x = 7$ is horizontal, so $f'(7) = 0$.

14. For which values of x is $f'(x) < 0$?

SOLUTION If $f'(x) < 0$, then the slope of the tangent line at x is negative. Graphically, this would mean that the value of the function was decreasing for increasing x. From the graph, it follows that $f'(x) < 0$ for $7 < x < 9$.

15. Which is larger, $f'(5.5)$ or $f'(6.5)$?

SOLUTION The line tangent to the graph of $y = f(x)$ at $x = 5.5$ has a larger slope than the line tangent to the graph of $y = f(x)$ at $x = 6.5$. Therefore, $f'(5.5)$ is larger than $f'(6.5)$.

16. Show that $f'(3)$ does not exist.

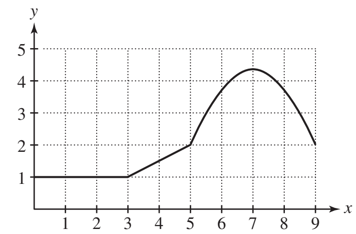


FIGURE 13 Graph of f.

SOLUTION Because

$$\lim_{h\to 0-} \frac{f(3+h) - f(3)}{h} = 0 \qquad \text{but} \qquad \lim_{h\to 0+} \frac{f(3+h) - f(3)}{h} = \frac{1}{2},$$

it follows that

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

does not exist.

In Exercises 17–20, use the limit definition to calculate the derivative of the linear function.

17. $f(x) = 7x - 9$

SOLUTION

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{7(a+h) - 9 - (7a - 9)}{h} = \lim_{h\to 0} 7 = 7.$$

18. $f(x) = 12$

SOLUTION

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{12 - 12}{h} = \lim_{h\to 0} 0 = 0.$$

19. $g(t) = 8 - 3t$

SOLUTION

$$\lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = \lim_{h\to 0} \frac{8 - 3(a+h) - (8 - 3a)}{h} = \lim_{h\to 0} \frac{-3h}{h} = \lim_{h\to 0} (-3) = -3.$$

20. $k(z) = 14z + 12$

SOLUTION

$$\lim_{h\to 0} \frac{k(a+h) - k(a)}{h} = \lim_{h\to 0} \frac{14(a+h) + 12 - (14a + 12)}{h} = \lim_{h\to 0} \frac{14h}{h} = \lim_{h\to 0} 14 = 14.$$

21. Find an equation of the tangent line at $x = 3$, assuming that $f(3) = 5$ and $f'(3) = 2$.

SOLUTION By definition, the equation of the tangent line to the graph of $f(x)$ at $x = 3$ is $y = f(3) + f'(3)(x - 3) = 5 + 2(x - 3) = 2x - 1$.

22. Find $f(3)$ and $f'(3)$, assuming that the tangent line to $y = f(x)$ at $a = 3$ has equation $y = 5x + 2$.

SOLUTION The slope of the tangent line to $y = f(x)$ at $a = 3$ is $f'(3)$ by definition, therefore $f'(3) = 5$. Also by definition, the tangent line to $y = f(x)$ at $a = 3$ goes through $(3, f(3))$, so $f(3) = 17$.

23. Describe the tangent line at an arbitrary point on the "curve" $y = 2x + 8$.

SOLUTION Since $y = 2x + 8$ represents a straight line, the tangent line at any point is the line itself, $y = 2x + 8$.

24. Suppose that $f(2+h) - f(2) = 3h^2 + 5h$. Calculate:

(a) The slope of the secant line through $(2, f(2))$ and $(6, f(6))$

(b) $f'(2)$

SOLUTION Let f be a function such that $f(2+h) - f(2) = 3h^2 + 5h$.

(a) We take $h = 4$ to compute the slope of the secant line through $(2, f(2))$ and $(6, f(6))$:

$$\frac{f(2+4) - f(2)}{(2+4) - 2} = \frac{3(4)^2 + 5(4)}{4} = 17$$

(b) $f'(2) = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \dfrac{3h^2 + 5h}{h} = \lim_{h\to 0} (3h + 5) = 5.$

25. Let $f(x) = \dfrac{1}{x}$. Does $f(-2+h)$ equal $\dfrac{1}{-2+h}$ or $\dfrac{1}{-2} + \dfrac{1}{h}$? Compute the difference quotient at $a = -2$ with $h = 0.5$.

SOLUTION Let $f(x) = \frac{1}{x}$. Then

$$f(-2+h) = \frac{1}{-2+h}.$$

With $a = -2$ and $h = 0.5$, the difference quotient is

$$\frac{f(a+h) - f(a)}{h} = \frac{f(-1.5) - f(-2)}{0.5} = \frac{\frac{1}{-1.5} - \frac{1}{-2}}{0.5} = -\frac{1}{3}.$$

26. Let $f(x) = \sqrt{x}$. Does $f(5+h)$ equal $\sqrt{5+h}$ or $\sqrt{5} + \sqrt{h}$? Compute the difference quotient at $a = 5$ with $h = 1$.

SOLUTION Let $f(x) = \sqrt{x}$. Then $f(5+h) = \sqrt{5+h}$. With $a = 5$ and $h = 1$, the difference quotient is

$$\frac{f(a+h) - f(a)}{h} = \frac{f(5+1) - f(5)}{1} = \frac{\sqrt{6} - \sqrt{5}}{1} = \sqrt{6} - \sqrt{5}.$$

27. Let $f(x) = 1/\sqrt{x}$. Compute $f'(5)$ by showing that

$$\frac{f(5+h) - f(5)}{h} = -\frac{1}{\sqrt{5}\sqrt{5+h}(\sqrt{5+h} + \sqrt{5})}$$

SOLUTION Let $f(x) = 1/\sqrt{x}$. Then

$$\begin{aligned}
\frac{f(5+h) - f(5)}{h} &= \frac{\frac{1}{\sqrt{5+h}} - \frac{1}{\sqrt{5}}}{h} = \frac{\sqrt{5} - \sqrt{5+h}}{h\sqrt{5}\sqrt{5+h}} \\
&= \frac{\sqrt{5} - \sqrt{5+h}}{h\sqrt{5}\sqrt{5+h}}\left(\frac{\sqrt{5} + \sqrt{5+h}}{\sqrt{5} + \sqrt{5+h}}\right) \\
&= \frac{5 - (5+h)}{h\sqrt{5}\sqrt{5+h}(\sqrt{5+h} + \sqrt{5})} = -\frac{1}{\sqrt{5}\sqrt{5+h}(\sqrt{5+h} + \sqrt{5})}.
\end{aligned}$$

Thus,

$$\begin{aligned}
f'(5) &= \lim_{h\to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h\to 0} -\frac{1}{\sqrt{5}\sqrt{5+h}(\sqrt{5+h} + \sqrt{5})} \\
&= -\frac{1}{\sqrt{5}\,\sqrt{5}(\sqrt{5} + \sqrt{5})} = -\frac{1}{10\sqrt{5}}.
\end{aligned}$$

28. Find an equation of the tangent line to the graph of $f(x) = 1/\sqrt{x}$ at $x = 9$.

SOLUTION Let $f(x) = 1/\sqrt{x}$. Then

$$\frac{f(9+h) - f(9)}{h} = \frac{\frac{1}{\sqrt{9+h}} - \frac{1}{3}}{h} = \frac{3 - \sqrt{9+h}}{3h\sqrt{9+h}}$$
$$= \frac{3 - \sqrt{9+h}}{3h\sqrt{9+h}}\left(\frac{3+\sqrt{9+h}}{3+\sqrt{9+h}}\right)$$
$$= \frac{9 - (9+h)}{3h\sqrt{9+h}(\sqrt{9+h}+3)} = -\frac{1}{3\sqrt{9+h}(\sqrt{9+h}+3)}.$$

Thus,

$$f'(9) = \lim_{h\to 0} \frac{f(9+h) - f(9)}{h} = \lim_{h\to 0} -\frac{1}{3\sqrt{9+h}(\sqrt{9+h}+3)}$$
$$= -\frac{1}{9(3+3)} = -\frac{1}{54}.$$

Because $f(9) = \frac{1}{3}$, it follows that an equation of the tangent line to the graph of $f(x) = 1/\sqrt{x}$ at $x = 9$ is

$$y = f'(9)(x-9) + f(9) = -\frac{1}{54}(x-9) + \frac{1}{3}.$$

In Exercises 29–46, use the limit definition to compute $f'(a)$ and find an equation of the tangent line.

29. $f(x) = 2x^2 + 10x, \quad a = 3$

SOLUTION Let $f(x) = 3x^2 + 2x$. Then

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{3(2+h)^2 + 2(2+h) - 16}{h}$$
$$= \lim_{h\to 0} \frac{12 + 12h + 3h^2 + 4 + 2h - 16}{h} = \lim_{h\to 0}(14 + 3h) = 14.$$

At $a = 2$, the tangent line is

$$y = f'(2)(x-2) + f(2) = 14(x-2) + 16 = 14x - 12.$$

30. $f(x) = 4 - x^2, \quad a = -1$

SOLUTION Let $f(x) = 4 - x^2$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{4 - (-1+h)^2 - 3}{h}$$
$$= \lim_{h\to 0} \frac{4 - (1 - 2h + h^2) - 3}{h}$$
$$= \lim_{h\to 0}(2 - h) = 2.$$

At $a = -1$, the tangent line is

$$y = f'(-1)(x+1) + f(-1) = 2(x+1) + 3 = 2x + 5.$$

31. $f(t) = t - 2t^2, \quad a = 3$

SOLUTION Let $f(t) = t - 2t^2$. Then

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{(3+h) - 2(3+h)^2 - (-15)}{h}$$
$$= \lim_{h\to 0} \frac{3 + h - 18 - 12h - 2h^2 + 15}{h}$$
$$= \lim_{h\to 0}(-11 - 2h) = -11.$$

At $a = 3$, the tangent line is

$$y = f'(3)(t-3) + f(3) = -11(t-3) - 15 = -11t + 18.$$

32. $f(x) = 8x^3, \quad a = 1$

SOLUTION Let $f(x) = 8x^3$. Then

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{8(1+h)^3 - 8}{h} \\ &= \lim_{h\to 0} \frac{8 + 24h + 24h^2 + 8h^3 - 8}{h} \\ &= \lim_{h\to 0} (24 + 24h + 8h^2) = 24. \end{aligned}$$

At $a = 1$, the tangent line is

$$y = f'(1)(x-1) + f(1) = 24(x-1) + 8 = 24x - 16.$$

33. $f(x) = x^3 + x, \quad a = 0$

SOLUTION Let $f(x) = x^3 + x$. Then

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{h^3 + h - 0}{h} \\ &= \lim_{h\to 0} (h^2 + 1) = 1. \end{aligned}$$

At $a = 0$, the tangent line is

$$y = f'(0)(x-0) + f(0) = x.$$

34. $f(t) = 2t^3 + 4t, \quad a = 4$

SOLUTION Let $f(t) = 2t^3 + 4t$. Then

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{2(4+h)^3 + 4(4+h) - 144}{h} \\ &= \lim_{h\to 0} \frac{128 + 96h + 24h^2 + 2h^3 + 16 + 4h - 144}{h} \\ &= \lim_{h\to 0} (100 + 24h + 2h^2) = 100. \end{aligned}$$

At $a = 4$, the tangent line is

$$y = f'(4)(t-4) + f(4) = 100(t-4) + 144 = 100t - 256.$$

35. $f(x) = x^{-1}, \quad a = 8$

SOLUTION Let $f(x) = x^{-1}$. Then

$$f'(8) = \lim_{h\to 0} \frac{f(8+h) - f(8)}{h} = \lim_{h\to 0} \frac{\frac{1}{8+h} - \left(\frac{1}{8}\right)}{h} = \lim_{h\to 0} \frac{\frac{8-8-h}{8(8+h)}}{h} = \lim_{h\to 0} \frac{-h}{(64+8h)h} = -\frac{1}{64}$$

The tangent at $a = 8$ is

$$y = f'(8)(x-8) + f(8) = -\frac{1}{64}(x-8) + \frac{1}{8} = -\frac{1}{64}x + \frac{1}{4}.$$

36. $f(x) = x + x^{-1}, \quad a = 4$

SOLUTION Let $f(x) = x + x^{-1}$. Then

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{4 + h + \frac{1}{4+h} - 4 - \frac{1}{4}}{h} = \lim_{h\to 0} \frac{h + \frac{4-4-h}{4(4+h)}}{h} = \lim_{h\to 0} \left(1 - \frac{1}{16+4h}\right) = \frac{15}{16}$$

The tangent at $a = 4$ is

$$y = f'(4)(x-4) + f(4) = \frac{15}{16}(x-4) + \frac{17}{4} = \frac{15}{16}x + \frac{1}{2}.$$

37. $f(x) = \dfrac{1}{x+3}, \quad a = -2$

SOLUTION Let $f(x) = \frac{1}{x+3}$. Then

$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\frac{1}{-2+h+3} - 1}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h\to 0} \frac{-h}{h(1+h)} = \lim_{h\to 0} \frac{-1}{1+h} = -1.$$

The tangent line at $a = -2$ is

$$y = f'(-2)(x+2) + f(-2) = -1(x+2) + 1 = -x - 1.$$

38. $f(t) = \dfrac{2}{1-t}, \quad a = -1$

SOLUTION Let $f(t) = \frac{2}{1-t}$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\frac{2}{1-(-1+h)} - 1}{h} = \lim_{h\to 0} \frac{2-(2-h)}{h(2-h)} = \lim_{h\to 0} \frac{1}{2-h} = \frac{1}{2}.$$

At $a = -1$, the tangent line is

$$y = f'(-1)(x+1) + f(-1) = \frac{1}{2}(x+1) + 1 = \frac{1}{2}x + \frac{3}{2}.$$

39. $f(x) = \sqrt{x+4}, \quad a = 1$

SOLUTION Let $f(x) = \sqrt{x+4}$. Then

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\sqrt{h+5} - \sqrt{5}}{h} = \lim_{h\to 0} \frac{\sqrt{h+5} - \sqrt{5}}{h} \cdot \frac{\sqrt{h+5} + \sqrt{5}}{\sqrt{h+5} + \sqrt{5}}$$
$$= \lim_{h\to 0} \frac{h}{h(\sqrt{h+5} + \sqrt{5})} = \lim_{h\to 0} \frac{1}{\sqrt{h+5} + \sqrt{5}} = \frac{1}{2\sqrt{5}}.$$

The tangent line at $a = 1$ is

$$y = f'(1)(x-1) + f(1) = \frac{1}{2\sqrt{5}}(x-1) + \sqrt{5} = \frac{1}{2\sqrt{5}}x + \frac{9}{2\sqrt{5}}.$$

40. $f(t) = \sqrt{3t+5}, \quad a = -1$

SOLUTION Let $f(t) = \sqrt{3t+5}$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\sqrt{3h+2} - \sqrt{2}}{h} = \lim_{h\to 0} \frac{\sqrt{3h+2} - \sqrt{2}}{h} \cdot \frac{\sqrt{3h+2} + \sqrt{2}}{\sqrt{3h+2} + \sqrt{2}}$$
$$= \lim_{h\to 0} \frac{3h}{h(\sqrt{3h+2} + \sqrt{2})} = \lim_{h\to 0} \frac{3}{\sqrt{3h+2} + \sqrt{2}} = \frac{3}{2\sqrt{2}}.$$

The tangent line at $a = -1$ is

$$y = f'(-1)(t+1) + f(-1) = \frac{3}{2\sqrt{2}}(t+1) + \sqrt{2} = \frac{3}{2\sqrt{2}}t + \frac{7}{2\sqrt{2}}.$$

41. $f(x) = \dfrac{1}{\sqrt{x}}, \quad a = 4$

SOLUTION Let $f(x) = \dfrac{1}{\sqrt{x}}$. Then

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{4+h}} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-\sqrt{4+h}}{2\sqrt{4+h}} \cdot \frac{2+\sqrt{4+h}}{2+\sqrt{4+h}}}{h} = \lim_{h\to 0} \frac{\frac{4-4-h}{4\sqrt{4+h}+2(4+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-1}{4\sqrt{4+h} + 2(4+h)} = -\frac{1}{16}.$$

At $a = 4$ the tangent line is

$$y = f'(4)(x-4) + f(4) = -\frac{1}{16}(x-4) + \frac{1}{2} = -\frac{1}{16}x + \frac{3}{4}.$$

42. $f(x) = \dfrac{1}{\sqrt{2x+1}}, \quad a = 4$

SOLUTION Let $f(x) = \dfrac{1}{\sqrt{2x+1}}$. Then

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{2h+9}} - \frac{1}{3}}{h} = \lim_{h\to 0} \frac{\frac{3-\sqrt{2h+9}}{3\sqrt{2h+9}} \cdot \frac{3+\sqrt{2h+9}}{3+\sqrt{2h+9}}}{h} = \lim_{h\to 0} \frac{\frac{9-2h-9}{9\sqrt{2h+9}+3(2h+9)}}{h}$$

$$= \lim_{h\to 0} \frac{-2}{9\sqrt{2h+9} + 3(2h+9)} = -\frac{1}{27}.$$

At $a = 4$ the tangent line is

$$y = f'(4)(x-4) + f(4) = -\frac{1}{27}(x-4) + \frac{1}{3} = -\frac{1}{27}x + \frac{13}{27}.$$

43. $f(t) = \sqrt{t^2+1}, \quad a = 3$

SOLUTION Let $f(t) = \sqrt{t^2+1}$. Then

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\sqrt{10+6h+h^2} - \sqrt{10}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{10+6h+h^2} - \sqrt{10}}{h} \cdot \frac{\sqrt{10+6h+h^2} + \sqrt{10}}{\sqrt{10+6h+h^2} + \sqrt{10}}$$

$$= \lim_{h\to 0} \frac{6h+h^2}{h(\sqrt{10+6h+h^2} + \sqrt{10})} = \lim_{h\to 0} \frac{6+h}{\sqrt{10+6h+h^2} + \sqrt{10}} = \frac{3}{\sqrt{10}}.$$

The tangent line at $a = 3$ is

$$y = f'(3)(t-3) + f(3) = \frac{3}{\sqrt{10}}(t-3) + \sqrt{10} = \frac{3}{\sqrt{10}}t + \frac{1}{\sqrt{10}}.$$

44. $f(x) = x^{-2}, \quad a = -1$

SOLUTION Let $f(x) = \frac{1}{x^2}$. Then

$$f'(-1) = \lim_{h\to 0} \frac{f(-1+h) - f(-1)}{h} = \lim_{h\to 0} \frac{\frac{1}{(-1+h)^2} - 1}{h} = \lim_{h\to 0} \frac{\frac{h(2-h)}{(-1+h)^2}}{h} = \lim_{h\to 0} \frac{2-h}{(-1+h)^2} = 2.$$

The tangent line at $a = -1$ is

$$y = f'(-1)(x+1) + f(-1) = 2(x+1) + 1 = 2x + 3.$$

45. $f(x) = \dfrac{1}{x^2+1}, \quad a = 0$

SOLUTION Let $f(x) = \dfrac{1}{x^2+1}$. Then

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{1}{(0+h)^2+1} - 1}{h} = \lim_{h\to 0} \frac{\frac{-h^2}{h^2+1}}{h} = \lim_{h\to 0} \frac{-h}{h^2+1} = 0.$$

The tangent line at $a = 0$ is

$$y = f(0) + f'(0)(x - 0) = 1 + 0(x - 1) = 1.$$

46. $f(t) = t^{-3}, \quad a = 1$

SOLUTION Let $f(t) = \dfrac{1}{t^3}$. Then

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(h)}{h} = \lim_{h\to 0} \frac{\frac{1}{(1+h)^3} - 1}{h} = \lim_{h\to 0} \frac{\frac{-h(3+3h+h^2)}{(1+h)^3}}{h} = \lim_{h\to 0} \frac{-(3+3h+h^2)}{(1+h)^3} = -3.$$

The tangent line at $a = 1$ is

$$y = f'(1)(t - 1) + f(1) = -3(t - 1) + 1 = -3t + 4.$$

47. Figure 14 displays data collected by the biologist Julian Huxley (1887–1975) on the average antler weight W of male red deer as a function of age t. Estimate the derivative at $t = 4$. For which values of t is the slope of the tangent line equal to zero? For which values is it negative?

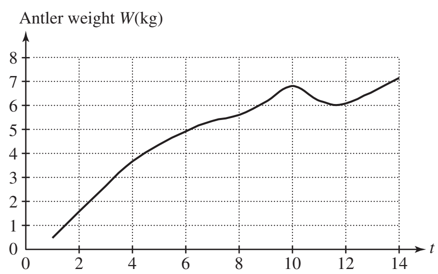
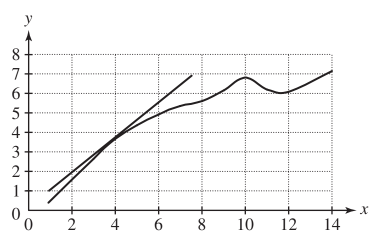


FIGURE 14

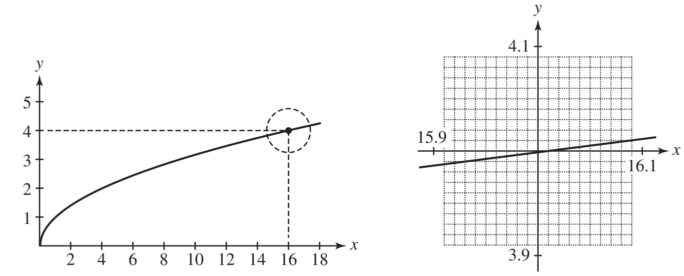
SOLUTION Let $W(t)$ denote the antler weight as a function of age. The “tangent line" sketched in the figure below passes through the points (1, 1) and (6, 5.5). Therefore

$$W'(4) \approx \frac{5.5 - 1}{6 - 1} = 0.9 \text{ kg/year}.$$

If the slope of the tangent is zero, the tangent line is horizontal. This appears to happen at roughly $t = 10$ and at $t = 11.6$. The slope of the tangent line is negative when the height of the graph decreases as we move to the right. For the graph in Figure 14, this occurs for $10 < t < 11.6$.



48. Figure 15(A) shows the graph of $f(x) = \sqrt{x}$. The close-up in Figure 15(B) shows that the graph is nearly a straight line near $x = 16$. Estimate the slope of this line and take it as an estimate for $f'(16)$. Then compute $f'(16)$ and compare with your estimate.



(A) Graph of $y = \sqrt{x}$ (B) Zoom view near (16, 4)

FIGURE 15

SOLUTION From the close-up in Figure 15(B), the line appears to pass through the points (15.92, 3.99) and (16.08, 4.01). Thus,

$$f'(16) \approx \frac{4.01 - 3.99}{16.08 - 15.92} = \frac{0.02}{0.16} = 0.125.$$

With $f(x) = \sqrt{x}$,

$$f'(16) = \lim_{h\to 0} \frac{\sqrt{16+h} - 4}{h} \cdot \frac{\sqrt{16+h} + 4}{\sqrt{16+h} + 4} = \lim_{h\to 0} \frac{16 + h - 16}{h(\sqrt{16+h} + 4)} = \lim_{h\to 0} \frac{1}{\sqrt{16+h} + 4} = \frac{1}{8} = 0.125,$$

which is consistent with the approximation obtained from the close-up graph.

49. GU Let $f(x) = \dfrac{4}{1 + 2^x}$.

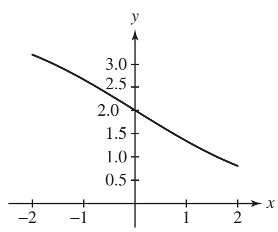
(a) Plot f over $[-2, 2]$. Then zoom in near $x = 0$ until the graph appears straight, and estimate the slope $f'(0)$.

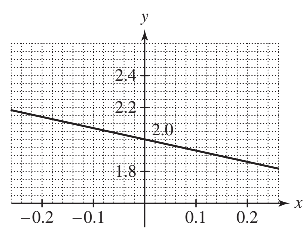
(b) Use (a) to find an approximate equation to the tangent line at $x = 0$. Plot this line and $y = f(x)$ on the same set of axes.

SOLUTION

(a) The figure below at the left shows the graph of $f(x) = \frac{4}{1+2^x}$ over $[-2, 2]$. The figure below at the right is a close-up near $x = 0$. From the close-up, we see that the graph is nearly straight and passes through the points $(-0.22, 2.15)$ and $(0.22, 1.85)$. We therefore estimate

$$f'(0) \approx \frac{1.85 - 2.15}{0.22 - (-0.22)} = \frac{-0.3}{0.44} = -0.68$$

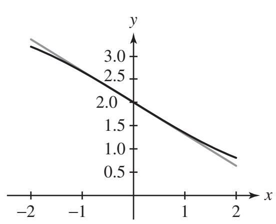




(b) Using the estimate for $f'(0)$ obtained in part (a), the approximate equation of the tangent line is

$$y = f'(0)(x - 0) + f(0) = -0.68x + 2.$$

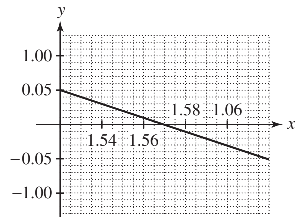
The figure below shows the graph of $f(x)$ and the approximate tangent line.



50. [GU] Let $f(x) = \cot x$. Estimate $f'\left(\frac{\pi}{2}\right)$ graphically by zooming in on a plot of f near $x = \frac{\pi}{2}$.

SOLUTION The figure below shows a close-up of the graph of $f(x) = \cot x$ near $x = \frac{\pi}{2} \approx 1.5708$. From the close-up, we see that the graph is nearly straight and passes through the points $(1.53, 0.04)$ and $(1.61, -0.04)$. We therefore estimate

$$f'\left(\frac{\pi}{2}\right) \approx \frac{-0.04 - 0.04}{1.61 - 1.53} = \frac{-0.08}{0.08} = -1$$



51. Determine the intervals along the x-axis on which the derivative in Figure 16 is positive.

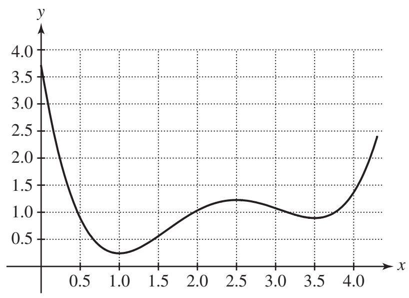
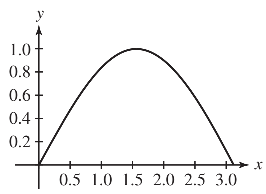


FIGURE 16

SOLUTION The derivative (that is, the slope of the tangent line) is positive when the height of the graph increases as we move to the right. From Figure 16, this appears to be true for $1 < x < 2.5$ and for $x > 3.5$.

52. Sketch the graph of $f(x) = \sin x$ on $[0, \pi]$ and guess the value of $f'\left(\frac{\pi}{2}\right)$. Then calculate the difference quotient at $x = \frac{\pi}{2}$ for two small positive and negative values of h. Are these calculations consistent with your guess?

SOLUTION Here is the graph of $y = \sin x$ on $[0, \pi]$.



At $x = \frac{\pi}{2}$, we're at the peak of the sine graph. The tangent line appears to be horizontal, so the slope is 0; hence, $f'(\frac{\pi}{2})$ appears to be 0.

h	$-.01$	$-.001$	$-.0001$	$.0001$	$.001$	$.01$
$\dfrac{\sin(\frac{\pi}{2} + h) - 1}{h}$	$.005$	$.0005$	$.00005$	$-.00005$	$-.0005$	$-.005$

These numerical calculations are consistent with our guess.

In Exercises 53–58, each limit represents a derivative $f'(a)$. Find $f(x)$ and a.

53. $\displaystyle\lim_{h \to 0} \frac{(5+h)^3 - 125}{h}$

SOLUTION The difference quotient $\dfrac{(5+h)^3 - 125}{h}$ has the form $\dfrac{f(a+h) - f(a)}{h}$ where $f(x) = x^3$ and $a = 5$.

54. $\lim_{x\to 5} \frac{x^3 - 125}{x - 5}$

SOLUTION The difference quotient $\frac{x^3 - 125}{x - 5}$ has the form $\frac{f(x) - f(a)}{x - a}$ where $f(x) = x^3$ and $a = 5$.

55. $\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6} + h\right) - 0.5}{h}$

SOLUTION The difference quotient $\frac{\sin(\frac{\pi}{6} + h) - 0.5}{h}$ has the form $\frac{f(a+h) - f(a)}{h}$ where $f(x) = \sin x$ and $a = \frac{\pi}{6}$.

56. $\lim_{x\to \frac{1}{4}} \frac{x^{-1} - 4}{x - \frac{1}{4}}$

SOLUTION The difference quotient $\frac{\frac{1}{x} - 4}{x - \frac{1}{4}}$ has the form $\frac{f(x) - f(a)}{x - a}$ where $f(x) = \frac{1}{x}$ and $a = \frac{1}{4}$.

57. $\lim_{h\to 0} \frac{5^{2+h} - 25}{h}$

SOLUTION The difference quotient $\frac{5^{(2+h)} - 25}{h}$ has the form $\frac{f(a+h) - f(a)}{h}$ where $f(x) = 5^x$ and $a = 2$.

58. $\lim_{h\to 0} \frac{5^h - 1}{h}$

SOLUTION The difference quotient $\frac{5^h - 1}{h}$ has the form $\frac{f(a+h) - f(a)}{h}$ where $f(x) = 5^x$ and $a = 0$.

59. Apply the method of Example 7 to $f(x) = \sin x$ to determine $f'\left(\frac{\pi}{4}\right)$ accurately to four decimal places.

SOLUTION We know that

$$f'(\pi/4) = \lim_{h\to 0} \frac{f(\pi/4 + h) - f(\pi/4)}{h} = \lim_{h\to 0} \frac{\sin(\pi/4 + h) - \sqrt{2}/2}{h}.$$

Creating a table of values with h close to zero:

h	$-.001$	$-.0001$	$-.00001$	$.00001$	$.0001$	$.001$
$\frac{\sin(\frac{\pi}{4} + h) - (\sqrt{2}/2)}{h}$	.7074602	.7071421	.7071103	.7071033	.7070714	.7067531

Accurate up to four decimal places, $f'(\frac{\pi}{4}) \approx .7071$.

60. Apply the method of Example 7 to $f(x) = \cos x$ to determine $f'\left(\frac{\pi}{5}\right)$ accurately to four decimal places. Use a graph of f to explain how the method works in this case.

SOLUTION We know that

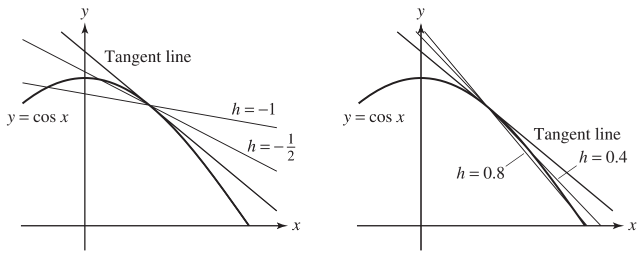
$$f'\left(\frac{\pi}{5}\right) = \lim_{h\to 0} \frac{f(\pi/5 + h) - f(\pi/5)}{h} = \lim_{h\to 0} \frac{\cos(\frac{\pi}{5} + h) - \cos(\frac{\pi}{5})}{h}.$$

We make a chart using values of h close to zero:

h	$-.001$	$-.0001$	$-.00001$
$\frac{\cos(\frac{\pi}{5} + h) - \cos(\frac{\pi}{5})}{h}$	$-.587381$	$-.587745$	$-.587781$
h	$.001$	$.0001$	$.00001$
$\frac{\cos(\frac{\pi}{5} + h) - \cos(\frac{\pi}{5})}{h}$	$-.588190$	$-.587826$	$-.587789$

$f'(\frac{\pi}{5}) \approx -.5878$.

The figures shown below illustrate why this procedure works. From the figure on the left, we see that for $h < 0$, the slope of the secant line is greater (less negative) than the slope of the tangent line. On the other hand, from the figure on the right, we see that for $h > 0$, the slope of the secant line is less (more negative) than the slope of the tangent line. Thus, the slope of the tangent line must fall between the slope of a secant line with $h > 0$ and the slope of a secant line with $h < 0$.



61. For each graph in Figure 17, determine whether $f'(1)$ is larger or smaller than the slope of the secant line between $x = 1$ and $x = 1 + h$ for $h > 0$. Explain.

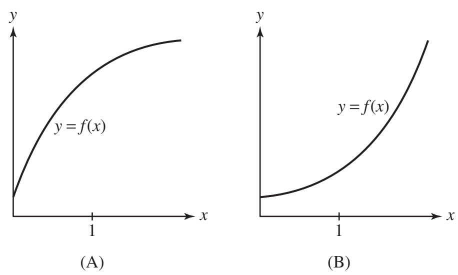


FIGURE 17

SOLUTION

- On curve (A), $f'(1)$ is larger than

$$\frac{f(1+h) - f(1)}{h};$$

the curve is bending downwards, so that the secant line to the right is at a lower angle than the tangent line. We say such a curve is **concave down**, and that its derivative is *decreasing*.

- On curve (B), $f'(1)$ is smaller than

$$\frac{f(1+h) - f(1)}{h};$$

the curve is bending upwards, so that the secant line to the right is at a steeper angle than the tangent line. We say such a curve is **concave up**, and that its derivative is *increasing*.

62. Refer to the graph of $f(x) = 2^x$ in Figure 18.

(a) Explain graphically why, for $h > 0$,

$$\frac{f(-h) - f(0)}{-h} \le f'(0) \le \frac{f(h) - f(0)}{h}$$

(b) Use (a) to show that $0.69314 \le f'(0) \le 0.69315$.

(c) Similarly, compute $f'(x)$ to four decimal places for $x = 1, 2, 3, 4$.

(d) Now compute the ratios $f'(x)/f'(0)$ for $x = 1, 2, 3, 4$. Can you guess an approximate formula for $f'(x)$?

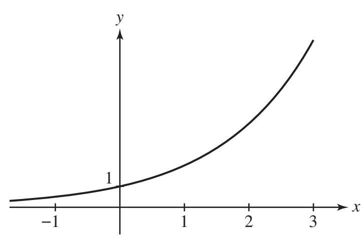


FIGURE 18 Graph of $f(x) = 2^x$.

SOLUTION

(a) In the graph, the inequality

$$f'(0) \leq \frac{f(h) - f(0)}{h}$$

holds for positive values of h, since the difference quotient

$$\frac{f(h) - f(0)}{h}$$

is an increasing function of h. (The slopes of the secant lines between $(0, f(0))$ and a nearby point increase as the nearby point moves from left to right.) Hence the slopes of the secant lines between $(0, f(0))$ and a nearby point to the right, $(h, f(h))$ (where h is positive) exceed $f'(0)$. Similarly, for h negative, since 0 is to the right of h, the slope of the secant line between $(0, f(0))$ and a nearby point to the left, $(h, f(h))$ is less than $f'(0)$. Therefore, the inequality

$$f'(0) \geq \frac{f(h) - f(0)}{h}$$

holds for negative values of h.

(b) For $h = .00001$, we have

$$\frac{f(h) - f(0)}{h} = \frac{2^h - 1}{h} \approx 0.69315,$$

while for $h = -.00001$, we have

$$\frac{f(h) - f(0)}{h} \approx 0.69314.$$

In light of (a), $0.69314 \leq f'(0) \leq 0.69315$.

(c) We'll use the same values of $h = \pm.00001$ and compute difference quotients at $x = 1, 2, 3, 4$.

- Since $1.386290 \leq f'(1) \leq 1.386299$, we conclude that $f'(1) \approx 1.3863$ to four decimal places.
- Since $2.772579 \leq f'(2) \leq 2.772598$, we conclude that $f'(2) \approx 2.7726$ to four decimal places.
- Since $5.545158 \leq f'(3) \leq 5.545197$, we conclude that $f'(3) \approx 5.5452$ to four decimal places.
- With $h = \pm.000001$, $11.090351 \leq f'(4) \leq 11.090359$, so we conclude that $f'(4) \approx 11.0904$ to four decimal places.

(d)

x	1	2	3	4
$f'(x)/f'(0)$	2	4	8	16

Looking at this table, we guess that $f'(x)/f'(0) = 2^x$. In other words, $f'(x) = 2^x f'(0)$.

63. GU Sketch the graph of $f(x) = x^{5/2}$ on $[0, 6]$.

(a) Use the sketch to justify the inequalities for $h > 0$:

$$\frac{f(4) - f(4-h)}{h} \leq f'(4) \leq \frac{f(4+h) - f(4)}{h}$$

(b) Use (a) to compute $f'(4)$ to four decimal places.

(c) Use a graphing utility to plot $y = f(x)$ and the tangent line at $x = 4$, utilizing your estimate for $f'(4)$.

SOLUTION A sketch of the graph of $f(x) = x^{5/2}$ on $[0, 6]$ is shown below in the answer to part (c).

(a) The slope of the secant line between points $(4, f(4))$ and $(4 + h, f(4 + h))$ is

$$\frac{f(4+h) - f(4)}{h}.$$

$x^{5/2}$ is a smooth curve increasing at a faster rate as $x \to \infty$. Therefore, if $h > 0$, then the slope of the secant line is greater than the slope of the tangent line at $f(4)$, which happens to be $f'(4)$. Likewise, if $h < 0$, the slope of the secant line is less than the slope of the tangent line at $f(4)$, which happens to be $f'(4)$.

(b) We know that

$$f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{(4+h)^{5/2}-32}{h}.$$

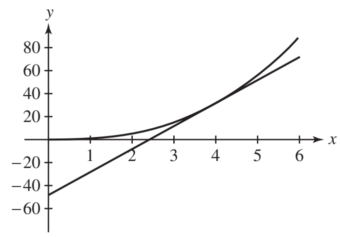
Creating a table with values of h close to zero:

h	$-.0001$	$-.00001$	$.00001$	$.0001$
$\frac{(4+h)^{5/2}-32}{h}$	19.999625	19.99999	20.0000	20.0000375

Thus, $f'(4) \approx 20.0000$.

(c) Using the estimate for $f'(4)$ obtained in part (b), the equation of the line tangent to $f(x) = x^{5/2}$ at $x = 4$ is

$$y = f'(4)(x-4) + f(4) = 20(x-4) + 32 = 20x - 48.$$

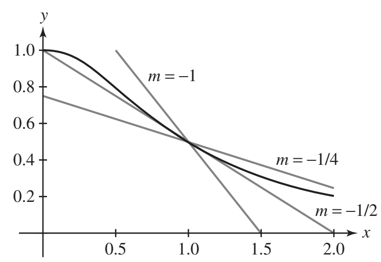


64. GU Verify that $P = \left(1, \frac{1}{2}\right)$ lies on the graphs of both $f(x) = 1/(1+x^2)$ and $L(x) = \frac{1}{2} + m(x-1)$ for every slope m. Plot $y = f(x)$ and $y = L(x)$ on the same axes for several values of m until you find a value of m for which $y = L(x)$ appears tangent to the graph of f. What is your estimate for $f'(1)$?

SOLUTION Let $f(x) = \dfrac{1}{1+x^2}$ and $L(x) = \frac{1}{2} + m(x-1)$. Because

$$f(1) = \frac{1}{1+1^2} = \frac{1}{2} \qquad \text{and} \qquad L(1) = \frac{1}{2} + m(1-1) = \frac{1}{2},$$

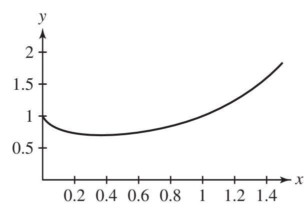
it follows that $P = (1, \frac{1}{2})$ lies on the graphs of both functions. A plot of $f(x)$ and $L(x)$ on the same axes for several values of m is shown below. The graph of $L(x)$ with $m = -\frac{1}{2}$ appears to be tangent to the graph of $f(x)$ at $x = 1$. We therefore estimate $f'(1) = -\frac{1}{2}$.



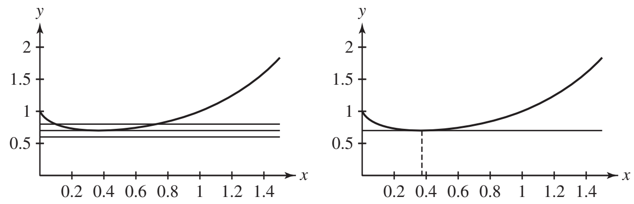
65. GU Use a plot of $f(x) = x^x$ to estimate the value c such that $f'(c) = 0$. Find c to sufficient accuracy so that

$$\left|\frac{f(c+h)-f(c)}{h}\right| \le 0.006 \qquad \text{for} \quad h = \pm 0.001$$

SOLUTION Here is a graph of $f(x) = x^x$ over the interval $[0, 1.5]$.



The graph shows one location with a horizontal tangent line. The figure below at the left shows the graph of $f(x)$ together with the horizontal lines $y = 0.6$, $y = 0.7$ and $y = 0.8$. The line $y = 0.7$ is very close to being tangent to the graph of $f(x)$. The figure below at the right refines this estimate by graphing $f(x)$ and $y = 0.69$ on the same set of axes. The point of tangency has an x-coordinate of roughly 0.37, so $c \approx 0.37$.



We note that

$$\left|\frac{f(0.37 + 0.001) - f(0.37)}{0.001}\right| \approx 0.00491 < 0.006$$

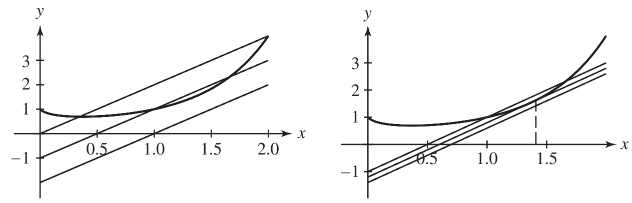
and

$$\left|\frac{f(0.37 - 0.001) - f(0.37)}{0.001}\right| \approx 0.00304 < 0.006,$$

so we have determined c to the desired accuracy.

66. GU Plot $f(x) = x^x$ and $y = 2x + a$ on the same set of axes for several values of a until the line becomes tangent to the graph. Then estimate the value c such that $f'(c) = 2$.

SOLUTION The figure below on the left shows the graphs of the function $f(x) = x^x$ together with the lines $y = 2x$, $y = 2x - 1$, and $y = 2x - 2$; the figure on the right shows the graphs of $f(x) = x^x$ together with the lines $y = 2x - 1$, $y = 2x - 1.2$, and $y = 2x - 1.4$. The graph of $y = 2x - 1.2$ appears to be tangent to the graph of $f(x)$ at $x \approx 1.4$. We therefore estimate that $f'(1.4) = 2$.



In Exercises 67–73, estimate derivatives using the ***symmetric difference quotient*** *(SDQ), defined as the average of the difference quotients at h and −h:*

$$\frac{1}{2}\left(\frac{f(a+h) - f(a)}{h} + \frac{f(a-h) - f(a)}{-h}\right) = \frac{f(a+h) - f(a-h)}{2h} \qquad \boxed{1}$$

The SDQ usually gives a better approximation to the derivative than the difference quotient.

67. The vapor pressure of water at temperature T (in kelvins) is the atmospheric pressure P at which no net evaporation takes place. Use the following table to estimate $P'(T)$ for $T = 303, 313, 323, 333, 343$ by computing the SDQ given by Eq. (1) with $h = 10$.

T (K)	293	303	313	323	333	343	353
P (atm)	0.0278	0.0482	0.0808	0.1311	0.2067	0.3173	0.4754

SOLUTION

(a) Consider the graph of vapor pressure as a function of temperature. If we draw the tangent line at $T = 300$ and another at $T = 350$, it is clear that the latter has a steeper slope. Therefore, $P'(350)$ is larger than $P'(300)$.

(b) Using equation (1),

$$P'(303) \approx \frac{P(313) - P(293)}{20} = \frac{0.0808 - 0.0278}{20} = 0.00265 \text{ atm/K};$$

$$P'(313) \approx \frac{P(323) - P(303)}{20} = \frac{0.1311 - 0.0482}{20} = 0.004145 \text{ atm/K};$$

$$P'(323) \approx \frac{P(333) - P(313)}{20} = \frac{0.2067 - 0.0808}{20} = 0.006295 \text{ atm/K};$$

$$P'(333) \approx \frac{P(343) - P(323)}{20} = \frac{0.3173 - 0.1311}{20} = 0.00931 \text{ atm/K};$$

$$P'(343) \approx \frac{P(353) - P(333)}{20} = \frac{0.4754 - 0.2067}{20} = 0.013435 \text{ atm/K}$$

68. Use the SDQ with $h = 1$ year to estimate $P'(T)$ in the years 2005, 2007, 2009, 2011, where $P(T)$ is the U.S. ethanol production (Figure 19). Express your answer in the correct units.

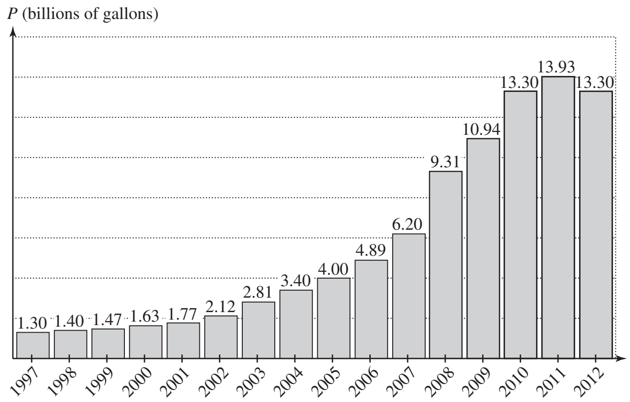


FIGURE 19 U.S. ethanol production.

SOLUTION Using equation (1),

$$P'(2005) \approx \frac{P(2006) - P(2004)}{2} = \frac{4.89 - 3.40}{2} = 0.745 \text{ billions of gallons/yr};$$

$$P'(2007) \approx \frac{P(2008) - P(2006)}{2} = \frac{9.31 - 4.89}{2} = 2.21 \text{ billions of gallons/yr};$$

$$P'(2009) \approx \frac{P(2010) - P(2008)}{2} = \frac{13.30 - 9.31}{2} = 1.995 \text{ billions of gallons/yr};$$

$$P'(2011) \approx \frac{P(2012) - P(2010)}{2} = \frac{13.30 - 13.30}{2} = 0 \text{ billions of gallons/yr}$$

In Exercises 69–70, traffic speed S along a certain road (in kilometers per hour) varies as a function of traffic density q (number of cars per kilometer of road). Use the following data to answer the questions:

q (density)	60	70	80	90	100
S (speed)	72.5	67.5	63.5	60	56

69. Estimate $S'(80)$.

SOLUTION Let $S(q)$ be the function determining S given q. Using equation (1) with $h = 10$,

$$S'(80) \approx \frac{S(90) - S(70)}{20} = \frac{60 - 67.5}{20} = -0.375;$$

with $h = 20$,

$$S'(80) \approx \frac{S(100) - S(60)}{40} = \frac{56 - 72.5}{40} = -0.4125.$$

The mean of these two symmetric difference quotients is -0.39375 kph·km/car.

70. Explain why $V = qS$, called *traffic volume*, is equal to the number of cars passing a point per hour. Use the data to estimate $V'(80)$.

SOLUTION The traffic speed S has units of km/hour, and the traffic density has units of cars/km. Therefore, the traffic volume $V = Sq$ has units of cars/hour. A table giving the values of V follows.

q	60	70	80	90	100
V	4350	4725	5080	5400	5600

To estimate dV/dq, we take the mean of the symmetric difference quotients. With $h = 10$,

$$V'(80) \approx \frac{V(90) - V(70)}{20} = \frac{5400 - 4725}{20} = 33.75;$$

with $h = 20$,

$$V'(80) \approx \frac{V(100) - V(60)}{40} = \frac{5600 - 4350}{40} = 31.25;$$

The mean of the symmetric difference quotients is 32.5. Hence $dV/dq \approx 32.5$ cars per hour when $q = 80$.

Exercises 71–73: The current (in amperes) at time t (in seconds) flowing in the circuit in Figure 20 is given by Kirchhoff's Law:

$$i(t) = Cv'(t) + R^{-1}v(t)$$

where $v(t)$ is the voltage (in volts, V), C the capacitance (in farads, F), and R the resistance (in ohms, Ω).

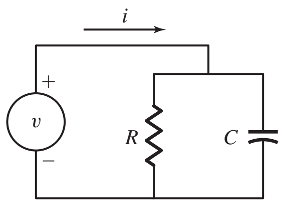


FIGURE 20

71. Calculate the current at $t = 3$ if

$$v(t) = 0.5t + 4 \text{ V}$$

where $C = 0.01$ F and $R = 100\ \Omega$.

SOLUTION Since $v(t)$ is a line with slope 0.5, $v'(t) = 0.5$ volts/s for all t. From the formula, $i(3) = Cv'(3) + (1/R)v(3) = 0.01(0.5) + (1/100)(5.5) = 0.005 + 0.055 = 0.06$ amperes.

72. Use the following data to estimate $v'(10)$ (by an SDQ). Then estimate $i(10)$, assuming $C = 0.03$ and $R = 1000$.

t	9.8	9.9	10	10.1	10.2
$v(t)$	256.52	257.32	258.11	258.9	259.69

SOLUTION Taking $h = 0.1$, we find

$$v'(10) \approx \frac{v(10.1) - v(9.9)}{0.2} = \frac{258.9 - 257.32}{0.2} = 7.9 \text{ volts/s}.$$

Thus,

$$i(10) = 0.03(7.9) + \frac{1}{1000}(258.11) = 0.49511 \text{ amperes}.$$

73. Assume that $R = 200\ \Omega$ but C is unknown. Use the following data to estimate $v'(4)$ (by an SDQ) and deduce an approximate value for the capacitance C.

t	3.8	3.9	4	4.1	4.2
$v(t)$	388.8	404.2	420	436.2	452.8
$i(t)$	32.34	33.22	34.1	34.98	35.86

SOLUTION Solving $i(4) = Cv'(4) + (1/R)v(4)$ for C yields

$$C = \frac{i(4) - (1/R)v(4)}{v'(4)} = \frac{34.1 - \frac{420}{200}}{v'(4)}.$$

To compute C, we first approximate $v'(4)$. Taking $h = 0.1$, we find

$$v'(4) \approx \frac{v(4.1) - v(3.9)}{0.2} = \frac{436.2 - 404.2}{0.2} = 160 \text{ volts/s}.$$

Plugging this in to the equation above yields

$$C \approx \frac{34.1 - 2.1}{160} = 0.2 \text{ farads}.$$

Further Insights and Challenges

74. The SDQ usually approximates the derivative much more closely than does the ordinary difference quotient. Let $f(x) = 2^x$ and $a = 0$. Compute the SDQ with $h = 0.001$ and the ordinary difference quotients with $h = \pm 0.001$. Compare with the actual value, which is $f'(0) = \ln 2$.

SOLUTION Let $f(x) = 2^x$ and $a = 0$.

- The ordinary difference quotient for $h = -.001$ is .69290701 and for $h = .001$ is .69338746.
- The symmetric difference quotient for $h = .001$ is .69314724.
- Clearly the symmetric difference quotient gives a better estimate of the derivative $f'(0) \approx .69314718$.

75. Explain how the symmetric difference quotient defined by Eq. (1) can be interpreted as the slope of a secant line.

SOLUTION The symmetric difference quotient

$$\frac{f(a+h) - f(a-h)}{2h}$$

is the slope of the secant line connecting the points $(a - h, f(a - h))$ and $(a + h, f(a + h))$ on the graph of f; the difference in the function values is divided by the difference in the x-values.

76. Which of the two functions in Figure 21 satisfies the inequality

$$\frac{f(a+h) - f(a-h)}{2h} \leq \frac{f(a+h) - f(a)}{h}$$

for $h > 0$? Explain in terms of secant lines.

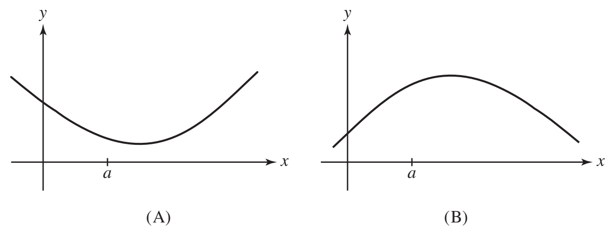


FIGURE 21

SOLUTION Figure (A) satisfies the inequality

$$\frac{f(a+h) - f(a-h)}{2h} \leq \frac{f(a+h) - f(a)}{h}$$

since in this graph the symmetric difference quotient has a larger negative slope than the ordinary right difference quotient. [In figure (B), the symmetric difference quotient has a larger positive slope than the ordinary right difference quotient and therefore does *not* satisfy the stated inequality.]

77. Show that if f is a quadratic polynomial, then the SDQ at $x = a$ (for any $h \neq 0$) is *equal* to $f'(a)$. Explain the graphical meaning of this result.

SOLUTION Let $f(x) = px^2 + qx + r$ be a quadratic polynomial. We compute the SDQ at $x = a$.

$$\frac{f(a+h) - f(a-h)}{2h} = \frac{p(a+h)^2 + q(a+h) + r - (p(a-h)^2 + q(a-h) + r)}{2h}$$
$$= \frac{pa^2 + 2pah + ph^2 + qa + qh + r - pa^2 + 2pah - ph^2 - qa + qh - r}{2h}$$
$$= \frac{4pah + 2qh}{2h} = \frac{2h(2pa + q)}{2h} = 2pa + q$$

Since this doesn't depend on h, the limit, which is equal to $f'(a)$, is also $2pa + q$. Graphically, this result tells us that the secant line to a parabola passing through points chosen symmetrically about $x = a$ is always parallel to the tangent line at $x = a$.

78. Let $f(x) = x^{-2}$. Compute $f'(1)$ by taking the limit of the SDQs (with $a = 1$) as $h \to 0$.

SOLUTION Let $f(x) = x^{-2}$. With $a = 1$, the symmetric difference quotient is

$$\frac{f(1+h) - f(1-h)}{2h} = \frac{\frac{1}{(1+h)^2} - \frac{1}{(1-h)^2}}{2h} = \frac{(1-h)^2 - (1+h)^2}{2h(1-h)^2(1+h)^2} = \frac{-4h}{2h(1-h)^2(1+h)^2} = -\frac{2}{(1-h)^2(1+h)^2}.$$

Therefore,

$$f'(1) = \lim_{h\to 0} -\frac{2}{(1-h)^2(1+h)^2} = -2.$$

3.2 The Derivative as a Function

Preliminary Questions

1. What is the slope of the tangent line through the point $(2, f(2))$ if $f'(x) = x^3$?

SOLUTION The slope of the tangent line through the point $(2, f(2))$ is given by $f'(2)$. Since $f'(x) = x^3$, it follows that $f'(2) = 2^3 = 8$.

2. Evaluate $(f - g)'(1)$ and $(3f + 2g)'(1)$, assuming that $f'(1) = 3$ and $g'(1) = 5$.

SOLUTION $(f - g)'(1) = f'(1) - g'(1) = 3 - 5 = -2$ and $(3f + 2g)'(1) = 3f'(1) + 2g'(1) = 3(3) + 2(5) = 19$.

3. To which of the following does the Power Rule apply?

(a) $f(x) = x^2$
(b) $f(x) = 2^e$
(c) $f(x) = x^e$
(d) $f(x) = e^x$
(e) $f(x) = x^x$
(f) $f(x) = x^{-4/5}$

SOLUTION

(a) Yes. x^2 is a power function, so the Power Rule can be applied.
(b) Yes. 2^e is a constant function, so the Power Rule can be applied.
(c) Yes. x^e is a power function, so the Power Rule can be applied.
(d) No. e^x is an exponential function (the base is constant while the exponent is a variable), so the Power Rule does not apply.
(e) No. x^x is not a power function because both the base and the exponent are variable, so the Power Rule does not apply.
(f) Yes. $x^{-4/5}$ is a power function, so the Power Rule can be applied.

4. Choose (a) or (b). The derivative does not exist if the tangent line is: (a) horizontal (b) vertical.

SOLUTION The derivative does not exist when there is a vertical tangent. At a horizontal tangent, the derivative is zero.

5. Which property distinguishes $f(x) = e^x$ from all other exponential functions $g(x) = b^x$?

SOLUTION The line tangent to $f(x) = e^x$ at $x = 0$ has slope equal to 1.

Exercises

In Exercises 1–6, compute $f'(x)$ using the limit definition.

1. $f(x) = 3x - 7$

SOLUTION Let $f(x) = 3x - 7$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{3(x+h) - 7 - (3x-7)}{h} = \lim_{h\to 0} \frac{3h}{h} = 3.$$

2. $f(x) = x^2 + 3x$

SOLUTION Let $f(x) = x^2 + 3x$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 + 3(x+h) - (x^2+3x)}{h}$$
$$= \lim_{h\to 0} \frac{2xh + h^2 + 3h}{h} = \lim_{h\to 0}(2x + h + 3) = 2x + 3.$$

3. $f(x) = x^3$

SOLUTION Let $f(x) = x^3$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2.$$

4. $f(x) = 1 - x^{-1}$

SOLUTION Let $f(x) = 1 - x^{-1}$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1 - \frac{1}{x+h} - \left(1 - \frac{1}{x}\right)}{h} = \lim_{h\to 0} \frac{\frac{(x+h)-x}{x(x+h)}}{h} = \lim_{h\to 0} \frac{1}{x(x+h)} = \frac{1}{x^2}.$$

5. $f(x) = x - \sqrt{x}$

SOLUTION Let $f(x) = x - \sqrt{x}$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{x + h - \sqrt{x+h} - (x - \sqrt{x})}{h} = 1 - \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$
$$= 1 - \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = 1 - \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = 1 - \frac{1}{2\sqrt{x}}.$$

6. $f(x) = x^{-1/2}$

SOLUTION Let $f(x) = x^{-1/2}$. Then,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}}$$

Multiplying the numerator and denominator of the expression by $\sqrt{x} + \sqrt{x+h}$, we obtain:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} = \lim_{h\to 0} \frac{x - (x+h)}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})}$$
$$= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x}\sqrt{x}(2\sqrt{x})} = \frac{-1}{2x\sqrt{x}}.$$

In Exercises 7–14, use the Power Rule to compute the derivative.

7. $\left.\dfrac{d}{dx}x^4\right|_{x=-2}$

SOLUTION $\dfrac{d}{dx}\left(x^4\right) = 4x^3$ so $\left.\dfrac{d}{dx}x^4\right|_{x=-2} = 4(-2)^3 = -32.$

8. $\left.\dfrac{d}{dt}t^{-3}\right|_{t=4}$

SOLUTION $\dfrac{d}{dt}\left(t^{-3}\right) = -3t^{-4}$ so $\left.\dfrac{d}{dt}t^{-3}\right|_{t=4} = -3(4)^{-4} = -\dfrac{3}{256}.$

9. $\left.\dfrac{d}{dt}t^{2/3}\right|_{t=8}$

SOLUTION $\dfrac{d}{dt}\left(t^{2/3}\right) = \dfrac{2}{3}t^{-1/3}$ so $\left.\dfrac{d}{dt}t^{2/3}\right|_{t=8} = \dfrac{2}{3}(8)^{-1/3} = \dfrac{1}{3}.$

10. $\left.\dfrac{d}{dt}t^{-2/5}\right|_{t=1}$

SOLUTION $\dfrac{d}{dt}\left(t^{-2/5}\right) = -\dfrac{2}{5}t^{-7/5}$ so $\left.\dfrac{d}{dt}t^{-2/5}\right|_{t=1} = -\dfrac{2}{5}(1)^{-7/5} = -\dfrac{2}{5}.$

11. $\dfrac{d}{dx}x^{0.35}$

SOLUTION $\dfrac{d}{dx}\left(x^{0.35}\right) = 0.35(x^{0.35-1}) = 0.35x^{-0.65}.$

12. $\dfrac{d}{dx}x^{14/3}$

SOLUTION $\dfrac{d}{dx}\left(x^{14/3}\right) = \dfrac{14}{3}\left(x^{(14/3)-1}\right) = \dfrac{14}{3}x^{11/3}.$

13. $\dfrac{d}{dt}t^{\sqrt{17}}$

SOLUTION $\dfrac{d}{dt}(t^{\sqrt{17}}) = \sqrt{17}t^{\sqrt{17}-1}$

14. $\dfrac{d}{dt}t^{-\pi^2}$

SOLUTION $\dfrac{d}{dt}(t^{-\pi^2}) = -\pi^2 t^{-\pi^2-1}$

In Exercises 15–18, compute $f'(x)$ and find an equation of the tangent line to the graph at $x = a$.

15. $f(x) = x^4, \quad a = 2$

SOLUTION Let $f(x) = x^4$. Then, by the Power Rule, $f'(x) = 4x^3$. The equation of the tangent line to the graph of $f(x)$ at $x = 2$ is

$$y = f'(2)(x-2) + f(2) = 32(x-2) + 16 = 32x - 48.$$

16. $f(x) = x^{-2}, \quad a = 5$

SOLUTION Let $f(x) = x^{-2}$. Using the Power Rule, $f'(x) = -2x^{-3}$. The equation of the tangent line to the graph of $f(x)$ at $x = 5$ is

$$y = f'(5)(x-5) + f(5) = -\frac{2}{125}(x-5) + \frac{1}{25} = -\frac{2}{125}x + \frac{3}{25}.$$

17. $f(x) = 5x - 32\sqrt{x}, \quad a = 4$

SOLUTION Let $f(x) = 5x - 32x^{1/2}$. Then $f'(x) = 5 - 16x^{-1/2}$. In particular, $f'(4) = -3$. The tangent line at $x = 4$ is

$$y = f'(4)(x-4) + f(4) = -3(x-4) - 44 = -3x - 32.$$

18. $f(x) = \sqrt[3]{x}, \quad a = 8$

SOLUTION Let $f(x) = \sqrt[3]{x} = x^{1/3}$. Then $f'(x) = \frac{1}{3}(x^{1/3-1}) = \frac{1}{3}x^{-2/3}$. In particular, $f'(8) = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$. $f(8) = 2$, so the tangent line at $x = 8$ is

$$y = f'(8)(x-8) + f(8) = \frac{1}{12}(x-8) + 2 = \frac{1}{12}x + \frac{4}{3}.$$

19. Calculate:

(a) $\dfrac{d}{dx}12e^x$ **(b)** $\dfrac{d}{dt}(25t - 8e^t)$ **(c)** $\dfrac{d}{dt}e^{t-3}$

Hint for (c): Write e^{t-3} as $e^{-3}e^t$.

SOLUTION

(a) $\dfrac{d}{dx}9e^x = 9\dfrac{d}{dx}e^x = 9e^x$.

(b) $\dfrac{d}{dt}(3t - 4e^t) = 3\dfrac{d}{dt}t - 4\dfrac{d}{dt}e^t = 3 - 4e^t$.

(c) $\dfrac{d}{dt}e^{t-3} = e^{-3}\dfrac{d}{dt}e^t = e^{-3} \cdot e^t = e^{t-3}$.

20. Find an equation of the tangent line to $y = 24e^x$ at $x = 2$.

SOLUTION Let $f(x) = 24e^x$. Then $f(2) = 24e^2$, $f'(x) = 24e^x$, and $f'(2) = 24e^2$. The equation of the tangent line is

$$y = f'(2)(x-2) + f(2) = 24e^2(x-2) + 24e^2.$$

In Exercises 21–32, calculate the derivative.

21. $f(x) = 2x^3 - 3x^2 + 5$

SOLUTION $\dfrac{d}{dx}\left(2x^3 - 3x^2 + 5\right) = 6x^2 - 6x$.

22. $f(x) = 2x^3 - 3x^2 + 2x$

SOLUTION $\dfrac{d}{dx}\left(2x^3 - 3x^2 + 2x\right) = 6x^2 - 6x + 2$.

23. $f(x) = 4x^{5/3} - 3x^{-2} - 12$

SOLUTION $\dfrac{d}{dx}\left(4x^{5/3} - 3x^{-2} - 12\right) = \dfrac{20}{3}x^{2/3} + 6x^{-3}$.

24. $f(x) = x^{5/4} + 4x^{-3/2} + 11x$

SOLUTION $\dfrac{d}{dx}\left(x^{5/4} + 4x^{-3/2} + 11x\right) = \dfrac{5}{4}x^{1/4} - 6x^{-5/2} + 11$.

25. $g(z) = 7z^{-5/14} + z^{-5} + 9$

SOLUTION $\dfrac{d}{dz}\left(7z^{-5/14} + z^{-5} + 9\right) = -\dfrac{5}{2}z^{-19/14} - 5z^{-6}$.

26. $h(t) = 6\sqrt{t} + \dfrac{1}{\sqrt{t}}$

SOLUTION $\dfrac{d}{dt}\left(6\sqrt{t} + \dfrac{1}{\sqrt{t}}\right) = \dfrac{d}{dt}\left(6t^{1/2} + t^{-1/2}\right) = 3t^{-1/2} - \dfrac{1}{2}t^{-3/2}$.

27. $f(s) = \sqrt[4]{s} + \sqrt[3]{s}$

SOLUTION $f(s) = \sqrt[4]{s} + \sqrt[3]{s} = s^{1/4} + s^{1/3}$. In this form, we can apply the Sum and Power Rules.

$$\frac{d}{ds}\left(s^{1/4} + s^{1/3}\right) = \frac{1}{4}(s^{(1/4)-1}) + \frac{1}{3}(s^{(1/3)-1}) = \frac{1}{4}s^{-3/4} + \frac{1}{3}s^{-2/3}.$$

28. $W(y) = 6y^4 + 7y^{2/3}$

SOLUTION $\frac{d}{dy}\left(6y^4 + 7y^{2/3}\right) = 24y^3 + \frac{14}{3}y^{-1/3}$.

29. $g(x) = e^2$

SOLUTION Because e^2 is a constant, $\frac{d}{dx}e^2 = 0$.

30. $f(x) = 3e^x - x^3$

SOLUTION $\frac{d}{dx}\left(3e^x - x^3\right) = 3e^x - 3x^2$.

31. $h(t) = 5e^{t-3}$

SOLUTION $\frac{d}{dt}5e^{t-3} = 5e^{-3}\frac{d}{dt}e^t = 5e^{-3}e^t = 5e^{t-3}$.

32. $f(x) = 9 - 12x^{1/3} + 8e^x$

SOLUTION $\frac{d}{dx}\left(9 - 12x^{1/3} + 8e^x\right) = -4x^{-2/3} + 8e^x$.

In Exercises 33–36, calculate the derivative by expanding or simplifying the function.

33. $P(s) = (4s - 3)^2$

SOLUTION $P(s) = (4s - 3)^2 = 16s^2 - 24s + 9$. Thus,

$$\frac{dP}{ds} = 32s - 24.$$

34. $Q(r) = (1 - 2r)(3r + 5)$

SOLUTION $Q(r) = (1 - 2r)(3r + 5) = -6r^2 - 7r + 5$. Thus,

$$\frac{dQ}{dr} = -12r - 7.$$

35. $g(x) = \frac{x^2 + 4x^{1/2}}{x^2}$

SOLUTION $g(x) = \frac{x^2 + 4x^{1/2}}{x^2} = 1 + 4x^{-3/2}$. Thus,

$$\frac{dg}{dx} = -6x^{-5/2}.$$

36. $s(t) = \frac{1 - 2t}{t^{1/2}}$

SOLUTION $s(t) = \frac{1 - 2t}{t^{1/2}} = t^{-1/2} - 2t^{1/2}$. Thus,

$$\frac{ds}{dt} = -\frac{1}{2}t^{-3/2} - t^{-1/2}.$$

In Exercises 37–42, calculate the derivative indicated.

37. $\left.\frac{dT}{dC}\right|_{C=8}$, $T = 3C^{2/3}$

SOLUTION With $T(C) = 3C^{2/3}$, we have $\frac{dT}{dC} = 2C^{-1/3}$. Therefore,

$$\left.\frac{dT}{dC}\right|_{C=8} = 2(8)^{-1/3} = 1.$$

38. $\left.\frac{dP}{dV}\right|_{V=-2}$, $P = \frac{7}{V}$

SOLUTION With $P = 7V^{-1}$, we have $\frac{dP}{dV} = -7V^{-2}$. Therefore,

$$\left.\frac{dP}{dV}\right|_{V=-2} = -7(-2)^{-2} = -\frac{7}{4}.$$

39. $\left.\dfrac{ds}{dz}\right|_{z=2}, \quad s = 4z - 16z^2$

SOLUTION With $s = 4z - 16z^2$, we have $\dfrac{ds}{dz} = 4 - 32z$. Therefore,

$$\left.\frac{ds}{dz}\right|_{z=2} = 4 - 32(2) = -60.$$

40. $\left.\dfrac{dR}{dW}\right|_{W=1}, \quad R = W^{\pi}$

SOLUTION Let $R(W) = W^{\pi}$. Then $dR/dW = \pi W^{\pi-1}$. Therefore,

$$\left.\frac{dR}{dW}\right|_{W=1} = \pi(1)^{\pi-1} = \pi.$$

41. $\left.\dfrac{dr}{dt}\right|_{t=4}, \quad r = t - e^t$

SOLUTION With $r = t - e^t$, we have $\dfrac{dr}{dt} = 1 - e^t$. Therefore,

$$\left.\frac{dr}{dt}\right|_{t=4} = 1 - e^4.$$

42. $\left.\dfrac{dp}{dh}\right|_{h=4}, \quad p = 7e^{h-2}$

SOLUTION With $p = 7e^{h-2}$, we have $\dfrac{dp}{dh} = 7e^{h-2}$. Therefore,

$$\left.\frac{dp}{dh}\right|_{h=4} = 7e^{4-2} = 7e^2.$$

43. Match the functions in graphs (A)–(D) with their derivatives (I)–(III) in Figure 13. Note that two of the functions have the same derivative. Explain why.

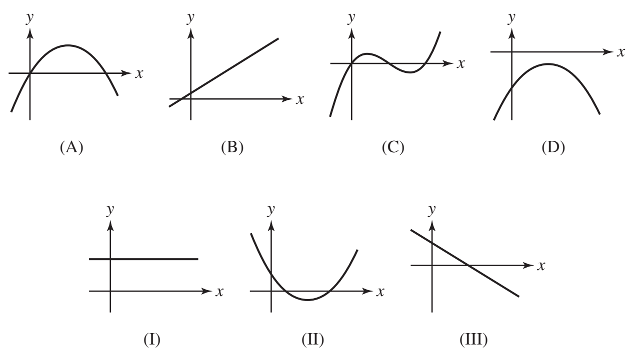


FIGURE 13

SOLUTION

- Consider the graph in (A). On the left side of the graph, the slope of the tangent line is positive but on the right side the slope of the tangent line is negative. Thus the derivative should transition from positive to negative with increasing x. This matches the graph in (III).
- Consider the graph in (B). This is a linear function, so its slope is constant. Thus the derivative is constant, which matches the graph in (I).
- Consider the graph in (C). Moving from left to right, the slope of the tangent line transitions from positive to negative then back to positive. The derivative should therefore be negative in the middle and positive to either side. This matches the graph in (II).

- Consider the graph in (D). On the left side of the graph, the slope of the tangent line is positive but on the right side the slope of the tangent line is negative. Thus the derivative should transition from positive to negative with increasing x. This matches the graph in (III).

Note that the functions whose graphs are shown in (A) and (D) have the same derivative. This happens because the graph in (D) is just a vertical translation of the graph in (A), which means the two functions differ by a constant. The derivative of a constant is zero, so the two functions end up with the same derivative.

44. Of the two functions f and g in Figure 14, which is the derivative of the other? Justify your answer.

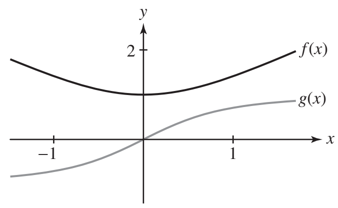


FIGURE 14

SOLUTION $g(x)$ is the derivative of $f(x)$. For $f(x)$ the slope is negative for negative values of x until $x = 0$, where there is a horizontal tangent, and then the slope is positive for positive values of x. Notice that $g(x)$ is negative for negative values of x, goes through the origin at $x = 0$, and then is positive for positive values of x.

45. Assign the labels $y = f(x)$, $y = g(x)$, and $y = h(x)$ to the graphs in Figure 15 in such a way that $f'(x) = g(x)$ and $g'(x) = h(x)$.

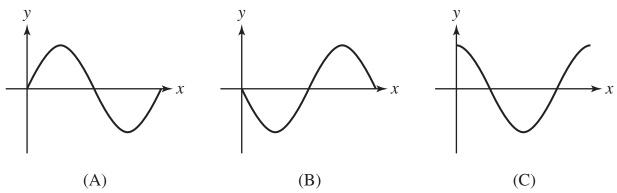


FIGURE 15

SOLUTION Consider the graph in (A). Moving from left to right, the slope of the tangent line is positive over the first quarter of the graph, negative in the middle half and positive again over the final quarter. The derivative of this function must therefore be negative in the middle and positive on either side. This matches the graph in (C).

Now focus on the graph in (C). The slope of the tangent line is negative over the left half and positive on the right half. The derivative of this function therefore needs to be negative on the left and positive on the right. This description matches the graph in (B).

We should therefore label the graph in (A) as $f(x)$, the graph in (B) as $h(x)$, and the graph in (C) as $g(x)$. Then $f'(x) = g(x)$ and $g'(x) = h(x)$.

46. According to the *peak oil theory*, first proposed in 1956 by geophysicist M. Hubbert, the total amount of crude oil $Q(t)$ produced worldwide up to time t has a graph like that in Figure 16.

(a) Sketch the derivative $Q'(t)$ for $1900 \le t \le 2150$. What does $Q'(t)$ represent?
(b) In which year (approximately) does $Q'(t)$ take on its maximum value?
(c) What is $L = \lim_{t\to\infty} Q(t)$? And what is its interpretation?
(d) What is the value of $\lim_{t\to\infty} Q'(t)$?

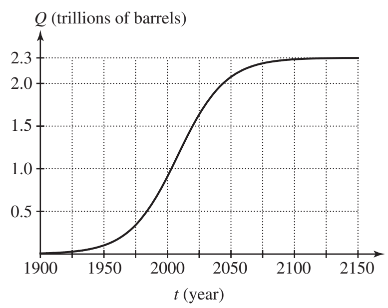
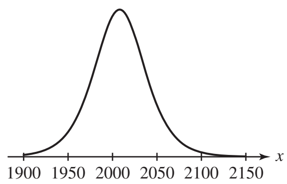


FIGURE 16 Total oil production up to time t.

SOLUTION

(a) One possible derivative sketch is shown below. Because the graph of $Q(t)$ is roughly horizontal around $t = 1900$, the graph of $Q'(t)$ begins near zero. Until roughly $t = 2000$, the graph of $Q(t)$ increases more and more rapidly, so the graph of $Q'(t)$ increases. Thereafter, the graph of $Q(t)$ increases more and more gradually, so the graph of $Q'(t)$ decreases. Around $t = 2150$, the graph of $Q(t)$ is again roughly horizontal, so the graph of $Q'(t)$ returns to zero. Note that $Q'(t)$ represents the rate of change in total worldwide oil production; that is, the number of barrels produced per year.



(b) The graph of $Q(t)$ appears to be increasing most rapidly around the year 2000, so $Q'(t)$ takes on its maximum value around the year 2000.

(c) From Figure 16

$$L = \lim_{t\to\infty} Q(t) = 2.3 \text{ trillion barrels of oil.}$$

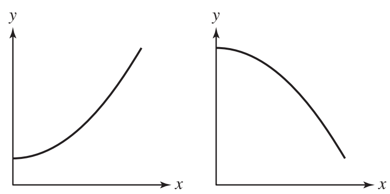
This value represents the total number of barrels of oil that can be produced by the planet.

(d) Because the graph of $Q(t)$ appears to approach a horizontal line as $t \to \infty$, it appears that

$$\lim_{t\to\infty} Q'(t) = 0.$$

47. Use the table of values of f to determine which of (A) or (B) in Figure 17 is the graph of f'. Explain.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$f(x)$	10	55	98	139	177	210	237	257	268



(A) (B)

FIGURE 17 Which is the graph of f'?

SOLUTION The increment between successive x values in the table is a constant 0.5 but the increment between successive $f(x)$ values decreases from 45 to 43 to 41 to 38 and so on. Thus the difference quotients decrease with increasing x, suggesting that $f'(x)$ decreases as a function of x. Because the graph in (B) depicts a decreasing function, (B) might be the graph of the derivative of $f(x)$.

48. Let R be a variable and r a constant. Compute the derivatives:

(a) $\dfrac{d}{dR} R$ **(b)** $\dfrac{d}{dR} r$ **(c)** $\dfrac{d}{dR} r^2R^3$

SOLUTION

(a) $\dfrac{d}{dR} R = 1$, since R is a linear function of R with slope 1.

(b) $\dfrac{d}{dR} r = 0$, since r is a constant.

(c) We apply the Linearity and Power Rules:

$$\frac{d}{dR} r^2R^3 = r^2 \frac{d}{dR} R^3 = r^2\big(3(R^2)\big) = 3r^2R^2.$$

49. Compute the derivatives, where c is a constant.

(a) $\frac{d}{dt}\, ct^3$ **(b)** $\frac{d}{dz}\,(5z + 4cz^2)$ **(c)** $\frac{d}{dy}\,(9c^2y^3 - 24c)$

SOLUTION

(a) $\frac{d}{dt} ct^3 = 3ct^2$.

(b) $\frac{d}{dz}(5z + 4cz^2) = 5 + 8cz$.

(c) $\frac{d}{dy}(9c^2y^3 - 24c) = 27c^2y^2$.

50. Find the points on the graph of $f(x) = 12x - x^3$ where the tangent line is horizontal.

SOLUTION Let $f(x) = 12x - x^3$. Solve $f'(x) = 12 - 2x^2 = 0$ to obtain $x = \pm\sqrt{6}$. Thus, the graph of $f(x) = 12x - x^3$ has a horizontal tangent line at two points: $(\sqrt{6}, 6\sqrt{6})$ and $(-\sqrt{6}, -6\sqrt{6})$.

51. Find the points on the graph of $y = x^2 + 3x - 7$ at which the slope of the tangent line is equal to 4.

SOLUTION Let $y = x^2 + 3x - 7$. Solving $dy/dx = 2x + 3 = 4$ yields $x = \frac{1}{2}$.

52. Find the values of x where $y = x^3$ and $y = x^2 + 5x$ have parallel tangent lines.

SOLUTION Let $f(x) = x^3$ and $g(x) = x^2 + 5x$. The graphs have parallel tangent lines when $f'(x) = g'(x)$. Hence, we solve $f'(x) = 3x^2 = 2x + 5 = g'(x)$ to obtain $x = \frac{5}{3}$ and $x = -1$.

53. Determine a and b such that $p(x) = x^2 + ax + b$ satisfies $p(1) = 0$ and $p'(1) = 4$.

SOLUTION Let $p(x) = x^2 + ax + b$ satisfy $p(1) = 0$ and $p'(1) = 4$. Now, $p'(x) = 2x + a$. Therefore $0 = p(1) = 1 + a + b$ and $4 = p'(1) = 2 + a$; i.e., $a = 2$ and $b = -3$.

54. Find all values of x such that the tangent line to $y = 4x^2 + 11x + 2$ is steeper than the tangent line to $y = x^3$.

SOLUTION Let $f(x) = 4x^2 + 11x + 2$ and let $g(x) = x^3$. We need all x such that $f'(x) > g'(x)$.

$$\begin{aligned} f'(x) &> g'(x) \\ 8x + 11 &> 3x^2 \\ 0 &> 3x^2 - 8x - 11 \\ 0 &> (3x - 11)(x + 1). \end{aligned}$$

The product $(3x - 11)(x + 1) = 0$ when $x = -1$ and when $x = \frac{11}{3}$. We therefore examine the intervals $x < -1$, $-1 < x < \frac{11}{3}$ and $x > \frac{11}{3}$. For $x < -1$ and for $x > \frac{11}{3}$, $(3x - 11)(x + 1) > 0$, whereas for $-1 < x < \frac{11}{3}$, $(3x - 11)(x + 1) < 0$. The solution set is therefore $-1 < x < \frac{11}{3}$.

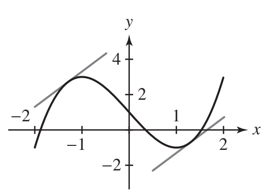
55. Let $f(x) = x^3 - 3x + 1$. Show that $f'(x) \geq -3$ for all x and that, for every $m > -3$, there are precisely two points where $f'(x) = m$. Indicate the position of these points and the corresponding tangent lines for one value of m in a sketch of the graph of f.

SOLUTION Let $P = (a, b)$ be a point on the graph of $f(x) = x^3 - 3x + 1$.

- The derivative satisfies $f'(x) = 3x^2 - 3 \geq -3$ since $3x^2$ is nonnegative.
- Suppose the slope m of the tangent line is greater than -3. Then $f'(a) = 3a^2 - 3 = m$, whence

$$a^2 = \frac{m + 3}{3} > 0 \quad \text{and thus} \quad a = \pm\sqrt{\frac{m + 3}{3}}.$$

- The two parallel tangent lines with slope 2 are shown with the graph of $f(x)$ here.



56. Show that the tangent lines to $y = \frac{1}{3}x^3 - x^2$ at $x = a$ and at $x = b$ are parallel if $a = b$ or $a + b = 2$.

SOLUTION Let $P = (a, f(a))$ and $Q = (b, f(b))$ be points on the graph of $y = f(x) = \frac{1}{3}x^3 - x^2$. Equate the slopes of the tangent lines at the points P and Q: $a^2 - 2a = b^2 - 2b$. Thus $a^2 - 2a - b^2 + 2b = 0$. Now,

$$a^2 - 2a - b^2 + 2b = (a-b)(a+b) - 2(a-b) = (a-2+b)(a-b);$$

therefore, either $a = b$ (i.e., P and Q are the same point) or $a + b = 2$.

57. Compute the derivative of $f(x) = x^{3/2}$ using the limit definition. *Hint:* Show that

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h)^3 - x^3}{h}\left(\frac{1}{\sqrt{(x+h)^3} + \sqrt{x^3}}\right)$$

SOLUTION Once we have the difference of square roots, we multiply by the conjugate to solve the problem.

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^{3/2} - x^{3/2}}{h} = \lim_{h\to 0} \frac{\sqrt{(x+h)^3} - \sqrt{x^3}}{h}\left(\frac{\sqrt{(x+h)^3} + \sqrt{x^3}}{\sqrt{(x+h)^3} + \sqrt{x^3}}\right)$$

$$= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h}\left(\frac{1}{\sqrt{(x+h)^3} + \sqrt{x^3}}\right).$$

The first factor of the expression in the last line is clearly the limit definition of the derivative of x^3, which is $3x^2$. The second factor can be evaluated, so

$$\frac{d}{dx}x^{3/2} = 3x^2\frac{1}{2\sqrt{x^3}} = \frac{3}{2}x^{1/2}.$$

58. Use the limit definition of $m(b)$ to approximate $m(4)$. Then estimate the slope of the tangent line to $y = 4^x$ at $x = 0$ and $x = 2$.

SOLUTION Recall

$$m(4) = \lim_{h\to 0}\left(\frac{4^h - 1}{h}\right).$$

Using a table of values, we find

h	$\frac{4^h-1}{h}$
.01	1.39595
.001	1.38726
.0001	1.38639
.00001	1.38630

Thus $m(4) \approx 1.386$. Knowing that $y'(x) = m(4)\cdot 4^x$, it follows that $y'(0) \approx 1.386$ and $y'(2) \approx 1.386 \cdot 16 = 22.176$.

59. Let $f(x) = xe^x$. Use the limit definition to compute $f'(0)$, and find the equation of the tangent line at $x = 0$.

SOLUTION Let $f(x) = xe^x$. Then $f(0) = 0$, and

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{he^h - 0}{h} = \lim_{h\to 0} e^h = 1.$$

The equation of the tangent line is

$$y = f'(0)(x-0) + f(0) = 1(x-0) + 0 = x.$$

60. The average speed (in meters per second) of a gas molecule is

$$v_{\text{avg}} = \sqrt{\frac{8RT}{\pi M}}$$

where T is the temperature (in kelvins), M is the molar mass (in kilograms per mole), and $R = 8.31$. Calculate dv_{avg}/dT at $T = 300$ K for oxygen, which has a molar mass of 0.032 kg/mol.

SOLUTION Using the form $v_{av} = (8RT/(\pi M))^{1/2} = \sqrt{8R/(\pi M)}T^{1/2}$, where M and R are constant, we use the Power Rule to compute the derivative dv_{av}/dT.

$$\frac{d}{dT}\sqrt{8R/(\pi M)}T^{1/2} = \sqrt{8R/(\pi M)}\frac{d}{dT}T^{1/2} = \sqrt{8R/(\pi M)}\frac{1}{2}(T^{(1/2)-1}).$$

In particular, if $T = 300$ K,

$$\frac{d}{dT}v_{av} = \sqrt{8(8.31)/(\pi(0.032))}\frac{1}{2}(300)^{-1/2} = 0.74234 \text{ m}/(\text{s}\cdot\text{K}).$$

61. Biologists have observed that the pulse rate P (in beats per minute) in animals is related to body mass (in kilograms) by the approximate formula $P = 200m^{-1/4}$. This is one of many *allometric scaling laws* prevalent in biology. Is P an increasing or decreasing function of m? Find an equation of the tangent line at the points on the graph in Figure 18 that represent goat ($m = 33$) and man ($m = 68$).

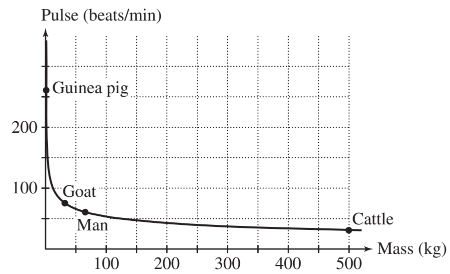


FIGURE 18

SOLUTION We are given that

$$P = 200m^{-1/4} = \frac{200}{m^{1/4}}.$$

As m increases, the denominator of the last expression increases, so the value of P decreases. Thus, P is a decreasing function of m.

For each $m = c$, the equation of the tangent line to the graph of P at m is

$$y = P'(c)(m - c) + P(c).$$

For a goat ($m = 33$ kg), $P(33) = 83.445$ beats per minute (bpm) and

$$\frac{dP}{dm} = -50(33)^{-5/4} \approx -0.63216 \text{ bpm/kg}.$$

Hence, $y = -0.63216(m - 33) + 83.445$.

For a man ($m = 68$ kg), we have $P(68) = 69.647$ bpm and

$$\frac{dP}{dm} = -50(68)^{-5/4} \approx -0.25606 \text{ bpm/kg}.$$

Hence, the tangent line has formula $y = -0.25606(m - 68) + 69.647$.

62. Some studies suggest that kidney mass K in mammals (in kilograms) is related to body mass m (in kilograms) by the approximate formula $K = 0.007m^{0.85}$. Calculate dK/dm at $m = 68$. Then calculate the derivative with respect to m of the relative kidney-to-mass ratio K/m at $m = 68$.

SOLUTION

$$\frac{dK}{dm} = 0.007(0.85)m^{-0.15} = 0.00595m^{-0.15};$$

hence,

$$\left.\frac{dK}{dm}\right|_{m=68} = 0.00595(68)^{-0.15} = 0.00315966.$$

Because

$$\frac{K}{m} = 0.007\frac{m^{0.85}}{m} = 0.007m^{-0.15},$$

we find

$$\frac{d}{dm}\left(\frac{K}{m}\right) = 0.007\frac{d}{dm}m^{-0.15} = -0.00105m^{-1.15},$$

and

$$\frac{d}{dm}\left(\frac{K}{m}\right)\Big|_{m=68} = -8.19981 \times 10^{-6}\ \text{kg}^{-1}.$$

63. The Clausius–Clapeyron Law relates the *vapor pressure* of water P (in atmospheres) to the temperature T (in kelvins):

$$\frac{dP}{dT} = k\frac{P}{T^2}$$

where k is a constant. Estimate dP/dT for $T = 303, 313, 323, 333, 343$ using the data and the approximation

$$\frac{dP}{dT} \approx \frac{P(T+10) - P(T-10)}{20}$$

T (K)	293	303	313	323	333	343	353
P (atm)	0.0278	0.0482	0.0808	0.1311	0.2067	0.3173	0.4754

Do your estimates seem to confirm the Clausius–Clapeyron Law? What is the approximate value of k?

SOLUTION Using the indicated approximation to the first derivative, we calculate

$$P'(303) \approx \frac{P(313) - P(293)}{20} = \frac{0.0808 - 0.0278}{20} = 0.00265\ \text{atm/K};$$
$$P'(313) \approx \frac{P(323) - P(303)}{20} = \frac{0.1311 - 0.0482}{20} = 0.004145\ \text{atm/K};$$
$$P'(323) \approx \frac{P(333) - P(313)}{20} = \frac{0.2067 - 0.0808}{20} = 0.006295\ \text{atm/K};$$
$$P'(333) \approx \frac{P(343) - P(323)}{20} = \frac{0.3173 - 0.1311}{20} = 0.00931\ \text{atm/K};$$
$$P'(343) \approx \frac{P(353) - P(333)}{20} = \frac{0.4754 - 0.2067}{20} = 0.013435\ \text{atm/K}$$

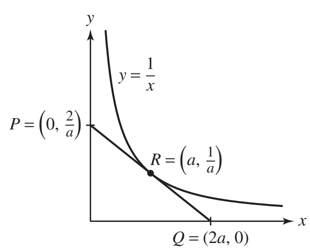
If the Clausius–Clapeyron law is valid, then $\frac{T^2}{P}\frac{dP}{dT}$ should remain constant as T varies. Using the data for vapor pressure and temperature and the approximate derivative values calculated above, we find

T (K)	303	313	323	333	343
$\frac{T^2}{P}\frac{dP}{dT}$	5047.59	5025.76	5009.54	4994.57	4981.45

These values are roughly constant, suggesting that the Clausius–Clapeyron law is valid, and that $k \approx 5000$.

64. Let L be the tangent line to the hyperbola $xy = 1$ at $x = a$, where $a > 0$. Show that the area of the triangle bounded by L and the coordinate axes does not depend on a.

SOLUTION Let $f(x) = x^{-1}$. The tangent line to f at $x = a$ is $y = f'(a)(x-a) + f(a) = -\frac{1}{a^2}(x-a) + \frac{1}{a}$. The y-intercept of this line (where $x = 0$) is $\frac{2}{a}$. Its x-intercept (where $y = 0$) is $2a$. Hence the area of the triangle bounded by the tangent line and the coordinate axes is $A = \frac{1}{2}bh = \frac{1}{2}(2a)\left(\frac{2}{a}\right) = 2$, which is independent of a.



65. In the setting of Exercise 64, show that the point of tangency is the midpoint of the segment of L lying in the first quadrant.

SOLUTION In the previous exercise, we saw that the tangent line to the hyperbola $xy = 1$ or $y = \frac{1}{x}$ at $x = a$ has y-intercept $P = (0, \frac{2}{a})$ and x-intercept $Q = (2a, 0)$. The midpoint of the line segment connecting P and Q is thus

$$\left(\frac{0+2a}{2}, \frac{\frac{2}{a}+0}{2}\right) = \left(a, \frac{1}{a}\right),$$

which is the point of tangency.

66. Match functions (A)–(C) with their derivatives (I)–(III) in Figure 19.

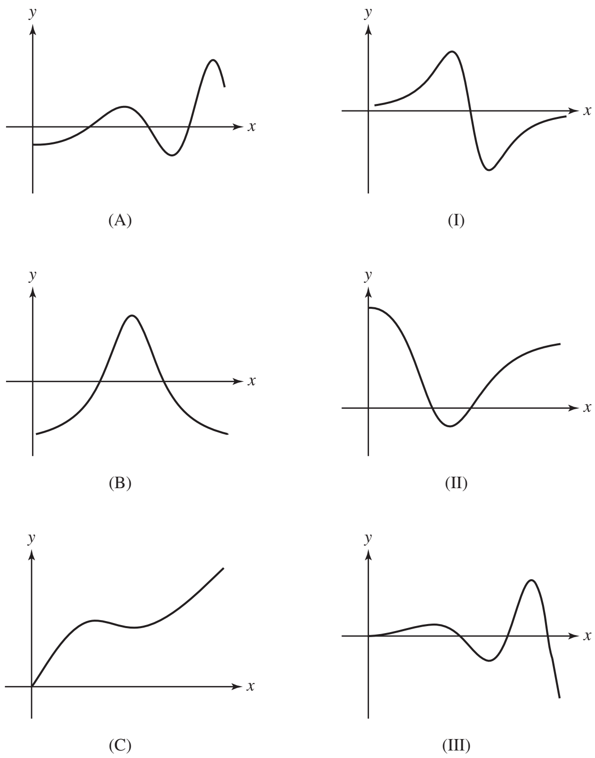


FIGURE 19

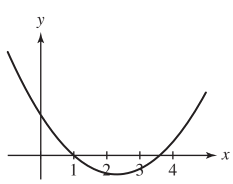
SOLUTION Note that the graph in (A) has three locations with a horizontal tangent line. The derivative must therefore cross the x-axis in three locations, which matches (III).

The graph in (B) has only one location with a horizontal tangent line, so its derivative should cross the x-axis only once. Thus, (I) is the graph corresponding to the derivative of (B).

Finally, the graph in (B) has two locations with a horizontal tangent line, so its derivative should cross the x-axis twice. Thus, (II) is the graph corresponding to the derivative of (C).

67. Make a rough sketch of the graph of the derivative of the function in Figure 20(A).

SOLUTION The graph has a tangent line with negative slope approximately on the interval $(1, 3.6)$, and has a tangent line with a positive slope elsewhere. This implies that the derivative must be negative on the interval $(1, 3.6)$ and positive elsewhere. The graph may therefore look like this:



68. Graph the derivative of the function in Figure 20(B), omitting points where the derivative is not defined.

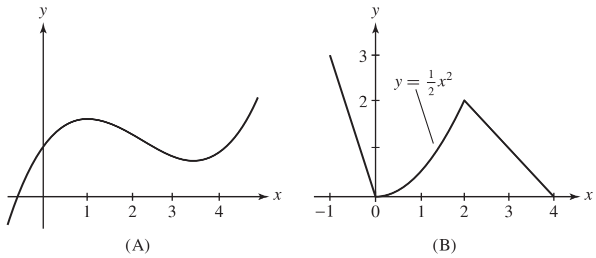
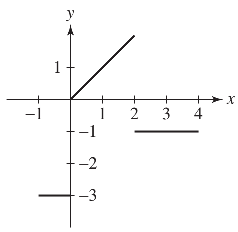


FIGURE 20

SOLUTION On $(-1, 0)$, the graph is a line with slope -3, so the derivative is equal to -3. The derivative on $(0, 2)$ is x. Finally, on $(2, 4)$ the function is a line with slope -1, so the derivative is equal to -1. Combining this information leads to the graph:



69. Sketch the graph of $f(x) = x\,|x|$. Then show that $f'(0)$ exists.

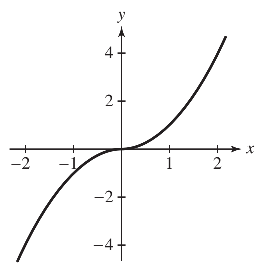
SOLUTION For $x < 0$, $f(x) = -x^2$, and $f'(x) = -2x$. For $x > 0$, $f(x) = x^2$, and $f'(x) = 2x$. At $x = 0$, we find

$$\lim_{h\to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0+} \frac{h^2}{h} = 0$$

and

$$\lim_{h\to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0-} \frac{-h^2}{h} = 0.$$

Because the two one-sided limits exist and are equal, it follows that $f'(0)$ exists and is equal to zero. Here is the graph of $f(x) = x|x|$.



70. Determine the values of x at which the function in Figure 21 is: (a) discontinuous and (b) nondifferentiable.

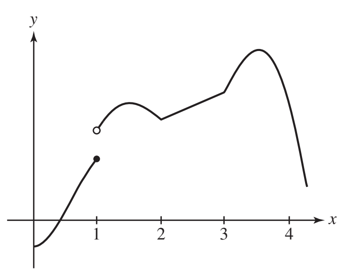


FIGURE 21

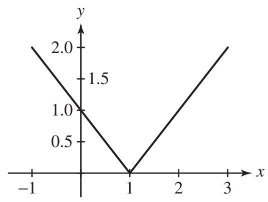
SOLUTION The function is discontinuous at those points where it is undefined or there is a break in the graph. On the interval [0, 4], there is only one such point, at $x = 1$.

The function is nondifferentiable at those points where it is discontinuous or where it has a corner or cusp. In addition to the point $x = 1$ we already know about, the function is nondifferentiable at $x = 2$ and $x = 3$.

In Exercises 71–76, find the points c (if any) such that $f'(c)$ does not exist.

71. $f(x) = |x - 1|$

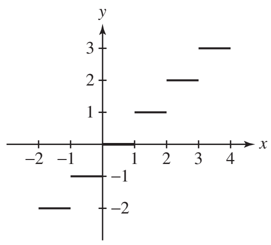
SOLUTION



Here is the graph of $f(x) = |x - 1|$. Its derivative does not exist at $x = 1$. At that value of x there is a sharp corner.

72. $f(x) = \lfloor x \rfloor$

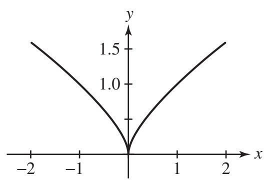
SOLUTION



Here is the graph of $f(x) = \lfloor x \rfloor$. This is the integer step function graph. Its derivative does not exist at all x values that are integers. At those values of x the graph is discontinuous.

73. $f(x) = x^{2/3}$

SOLUTION Here is the graph of $f(x) = x^{2/3}$. Its derivative does not exist at $x = 0$. At that value of x, there is a sharp corner or "cusp".

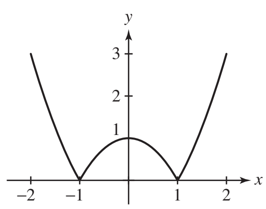


74. $f(x) = x^{3/2}$

SOLUTION The function is differentiable on its entire domain, $\{x : x \geq 0\}$. The formula is $\frac{d}{dx}x^{3/2} = \frac{3}{2}x^{1/2}$.

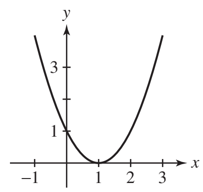
75. $f(x) = |x^2 - 1|$

SOLUTION Here is the graph of $f(x) = \left|x^2 - 1\right|$. Its derivative does not exist at $x = -1$ or at $x = 1$. At these values of x, the graph has sharp corners.



76. $f(x) = |x-1|^2$

SOLUTION

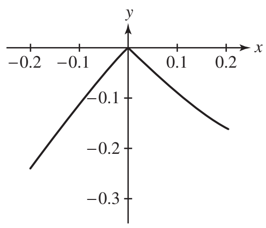


This is the graph of $f(x) = |x-1|^2$. Its derivative exists everywhere.

GU *In Exercises 77–82, zoom in on a plot of f at the point $(a, f(a))$ and state whether or not f appears to be differentiable at $x = a$. If it is nondifferentiable, state whether the tangent line appears to be vertical or does not exist.*

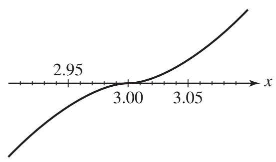
77. $f(x) = (x-1)|x|, \quad a = 0$

SOLUTION The graph of $f(x) = (x-1)|x|$ for x near 0 is shown below. Because the graph has a sharp corner at $x = 0$, it appears that f is not differentiable at $x = 0$. Moreover, the tangent line does not exist at this point.



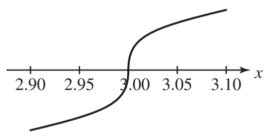
78. $f(x) = (x-3)^{5/3}, \quad a = 3$

SOLUTION The graph of $f(x) = (x-3)^{5/3}$ for x near 3 is shown below. From this graph, it appears that f is differentiable at $x = 3$, with a horizontal tangent line.



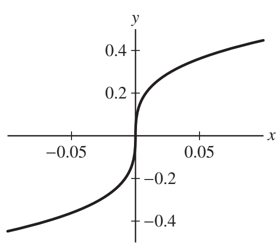
79. $f(x) = (x-3)^{1/3}, \quad a = 3$

SOLUTION The graph of $f(x) = (x-3)^{1/3}$ for x near 3 is shown below. From this graph, it appears that f is not differentiable at $x = 3$. Moreover, the tangent line appears to be vertical.



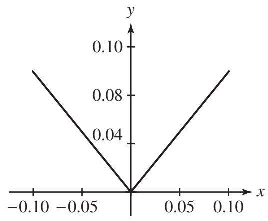
80. $f(x) = \sin(x^{1/3}), \quad a = 0$

SOLUTION The graph of $f(x) = \sin(x^{1/3})$ for x near 0 is shown below. From this graph, it appears that f is not differentiable at $x = 0$. Moreover, the tangent line appears to be vertical.



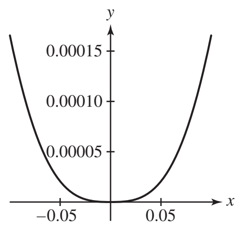
81. $f(x) = |\sin x|, \quad a = 0$

SOLUTION The graph of $f(x) = |\sin x|$ for x near 0 is shown below. Because the graph has a sharp corner at $x = 0$, it appears that f is not differentiable at $x = 0$. Moreover, the tangent line does not exist at this point.



82. $f(x) = |x - \sin x|, \quad a = 0$

SOLUTION The graph of $f(x) = |x - \sin x|$ for x near 0 is shown below. From this graph, it appears that f is differentiable at $x = 0$, with a horizontal tangent line.



83. Find the coordinates of the point P in Figure 22 at which the tangent line passes through $(5, 0)$.

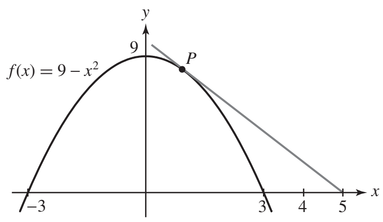


FIGURE 22

SOLUTION Let $f(x) = 9 - x^2$, and suppose P has coordinates $(a, 9 - a^2)$. Because $f'(x) = -2x$, the slope of the line tangent to the graph of $f(x)$ at P is $-2a$, and the equation of the tangent line is

$$y = f'(a)(x - a) + f(a) = -2a(x - a) + 9 - a^2 = -2ax + 9 + a^2.$$

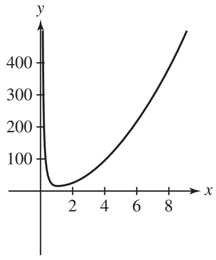
In order for this line to pass through the point $(5, 0)$, we must have

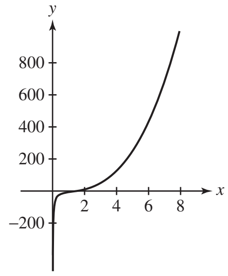
$$0 = -10a + 9 + a^2 = (a - 9)(a - 1).$$

Thus, $a = 1$ or $a = 9$. We exclude $a = 9$ because from Figure 22 we are looking for an x-coordinate between 0 and 5. Thus, the point P has coordinates $(1, 8)$.

84. GU Plot the derivative f' of $f(x) = 2x^3 - 10x^{-1}$ for $x > 0$ (set the bounds of the viewing box appropriately) and observe that $f'(x) > 0$. What does the positivity of $f'(x)$ tell us about the graph of f itself? Plot f and confirm this conclusion.

SOLUTION Let $f(x) = 2x^3 - 10x^{-1}$. Then $f'(x) = 6x^2 + 10x^{-2}$. The graph of $f'(x)$ is shown in the figure below at the left and it is clear that $f'(x) > 0$ for all $x > 0$. The positivity of $f'(x)$ tells us that the graph of $f(x)$ is increasing for $x > 0$. This is confirmed in the figure below at the right, which shows the graph of $f(x)$.





Exercises 85–88 refer to Figure 23. Length QR is called the subtangent *at P, and length RT is called the* subnormal.

85. Calculate the subtangent of

$$f(x) = x^2 + 3x \quad \text{at } x = 2$$

SOLUTION Let $f(x) = x^2 + 3x$. Then $f'(x) = 2x + 3$, and the equation of the tangent line at $x = 2$ is

$$y = f'(2)(x - 2) + f(2) = 7(x - 2) + 10 = 7x - 4.$$

This line intersects the x-axis at $x = \frac{4}{7}$. Thus Q has coordinates $(\frac{4}{7}, 0)$, R has coordinates $(2, 0)$ and the subtangent is

$$2 - \frac{4}{7} = \frac{10}{7}.$$

86. Show that the subtangent of $f(x) = e^x$ is everywhere equal to 1.

SOLUTION Let $f(x) = e^x$. Then $f'(x) = e^x$, and the equation of the tangent line at $x = a$ is

$$y = f'(a)(x - a) + f(a) = e^a(x - a) + e^a.$$

This line intersects the x-axis at $x = a - 1$. Thus, Q has coordinates $(a - 1, 0)$, R has coordinates $(a, 0)$ and the subtangent is

$$a - (a - 1) = 1.$$

87. Prove in general that the subnormal at P is $|f'(x)f(x)|$.

SOLUTION The slope of the tangent line at P is $f'(x)$. The slope of the line normal to the graph at P is then $-1/f'(x)$, and the normal line intersects the x-axis at the point T with coordinates $(x + f(x)f'(x), 0)$. The point R has coordinates $(x, 0)$, so the subnormal is

$$|x + f(x)f'(x) - x| = |f(x)f'(x)|.$$

88. Show that $\overline{PQ}$ has length $|f(x)|\sqrt{1 + f'(x)^{-2}}$.

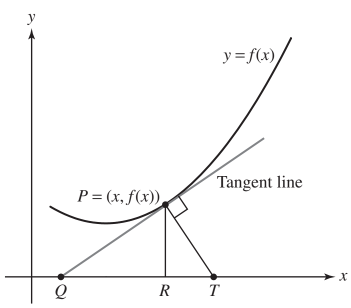


FIGURE 23

SOLUTION The coordinates of the point P are $(x, f(x))$, the coordinates of the point R are $(x, 0)$ and the coordinates of the point Q are

$$\left(x - \frac{f(x)}{f'(x)}, 0\right).$$

Thus, $\overline{PR} = |f(x)|$, $\overline{QR} = \left|\frac{f(x)}{f'(x)}\right|$, and by the Pythagorean Theorem

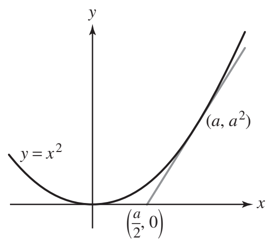
$$\overline{PQ} = \sqrt{\left(\frac{f(x)}{f'(x)}\right)^2 + (f(x))^2} = |f(x)|\sqrt{1 + f'(x)^{-2}}.$$

89. Prove the following theorem of Apollonius of Perga (the Greek mathematician born in 262 BCE who gave the parabola, ellipse, and hyperbola their names): The subtangent of the parabola $y = x^2$ at $x = a$ is equal to $a/2$.

SOLUTION Let $f(x) = x^2$. The tangent line to f at $x = a$ is

$$y = f'(a)(x - a) + f(a) = 2a(x - a) + a^2 = 2ax - a^2.$$

The x-intercept of this line (where $y = 0$) is $\frac{a}{2}$ as claimed.



90. Show that the subtangent to $y = x^3$ at $x = a$ is equal to $\frac{1}{3}a$.

SOLUTION Let $f(x) = x^3$. Then $f'(x) = 3x^2$, and the equation of the tangent line t $x = a$ is

$$y = f'(a)(x - a) + f(a) = 3a^2(x - a) + a^3 = 3a^2x - 2a^3.$$

This line intersects the x-axis at $x = 2a/3$. Thus, Q has coordinates $(2a/3, 0)$, R has coordinates $(a, 0)$ and the subtangent is

$$a - \frac{2}{3}a = \frac{1}{3}a.$$

91. Formulate and prove a generalization of Exercise 90 for $y = x^n$.

SOLUTION The generalized statement is: The subtangent to $y = x^n$ at $x = a$ is equal to $\frac{1}{n}a$.

Proof: Let $f(x) = x^n$. Then $f'(x) = nx^{n-1}$, and the equation of the tangent line t $x = a$ is

$$y = f'(a)(x - a) + f(a) = na^{n-1}(x - a) + a^n = na^{n-1}x - (n - 1)a^n.$$

This line intersects the x-axis at $x = (n - 1)a/n$. Thus, Q has coordinates $((n - 1)a/n, 0)$, R has coordinates $(a, 0)$ and the subtangent is

$$a - \frac{n - 1}{n}a = \frac{1}{n}a.$$

Further Insights and Challenges

92. Two small arches have the shape of parabolas. The first is given by $f(x) = 1 - x^2$ for $-1 \le x \le 1$ and the second by $g(x) = 4 - (x - 4)^2$ for $2 \le x \le 6$. A board is placed on top of these arches so it rests on both (Figure 24). What is the slope of the board? *Hint:* Find the tangent line to $y = f(x)$ that intersects $y = g(x)$ in exactly one point.

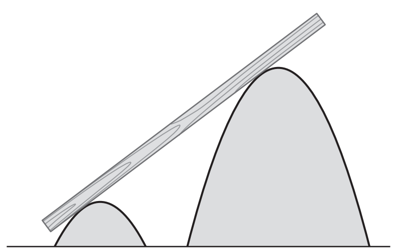


FIGURE 24

SOLUTION At the points where the board makes contact with the arches the slope of the board must be equal to the slope of the arches (and hence they are equal to each other). Suppose $(t, f(t))$ is the point where the board touches the left hand arch. The tangent line here (the line the board defines) is given by

$$y = f'(t)(x - t) + f(t).$$

This line must hit the other arch in exactly one point. In other words, if we plug in $y = g(x)$ to get

$$g(x) = f'(t)(x - t) + f(t)$$

there can only be one solution for x in terms of t. Computing f' and plugging in we get

$$4 - (x^2 - 8x + 16) = -2tx + 2t^2 + 1 - t^2$$

which simplifies to

$$x^2 - 2tx - 8x + t^2 + 13 = 0.$$

This is a quadratic equation $ax^2 + bx + c = 0$ with $a = 1$, $b = (-2t - 8)$ and $c = t^2 + 13$. By the quadratic formula we know there is a unique solution for x iff $b^2 - 4ac = 0$. In our case this means

$$(2t + 8)^2 = 4(t^2 + 13).$$

Solving this gives $t = -3/8$ and plugging into f' shows the slope of the board must be $3/4$.

93. A vase is formed by rotating $y = x^2$ around the y-axis. If we drop in a marble, it will either touch the bottom point of the vase or be suspended above the bottom by touching the sides (Figure 25). How small must the marble be to touch the bottom?

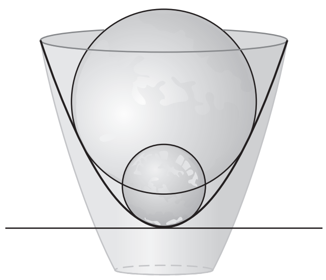


FIGURE 25

SOLUTION Suppose a circle is tangent to the parabola $y = x^2$ at the point (t, t^2). The slope of the parabola at this point is $2t$, so the slope of the radius of the circle at this point is $-\frac{1}{2t}$ (since it is perpendicular to the tangent line of the circle). Thus the center of the circle must be where the line given by $y = -\frac{1}{2t}(x - t) + t^2$ crosses the y-axis. We can find the y-coordinate by setting $x = 0$: we get $y = \frac{1}{2} + t^2$. Thus, the radius extends from $(0, \frac{1}{2} + t^2)$ to (t, t^2) and

$$r = \sqrt{\left(\frac{1}{2} + t^2 - t^2\right)^2 + t^2} = \sqrt{\frac{1}{4} + t^2}.$$

This radius is greater than $\frac{1}{2}$ whenever $t > 0$; so, if a marble has radius $> 1/2$ it sits on the edge of the vase, but if it has radius $\leq 1/2$ it rolls all the way to the bottom.

94. Let f be a differentiable function, and set the function $g(x) = f(x + c)$, where c is a constant. Use the limit definition to show that $g'(x) = f'(x + c)$. Explain this result graphically, recalling that the graph of g is obtained by shifting the graph of f c units to the left (if $c > 0$) or right (if $c < 0$).

SOLUTION

- Let $g(x) = f(x + c)$. Using the limit definition,

$$g'(x) = \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{f((x+h)+c) - f(x+c)}{h}$$
$$= \lim_{h\to 0} \frac{f((x+c)+h) - f(x+c)}{h} = f'(x+c).$$

- The graph of $g(x)$ is obtained by shifting $f(x)$ to the left by c units. This implies that $g'(x)$ is equal to $f'(x)$ shifted to the left by c units, which happens to be $f'(x+c)$. Therefore, $g'(x) = f'(x+c)$.

95. Negative Exponents Let n be a whole number. Use the Power Rule for x^n to calculate the derivative of $f(x) = x^{-n}$ by showing that

$$\frac{f(x+h)-f(x)}{h} = \frac{-1}{x^n(x+h)^n}\,\frac{(x+h)^n - x^n}{h}$$

SOLUTION Let $f(x) = x^{-n}$ where n is a positive integer.

- The difference quotient for f is

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^{-n} - x^{-n}}{h} = \frac{\frac{1}{(x+h)^n} - \frac{1}{x^n}}{h} = \frac{\frac{x^n-(x+h)^n}{x^n(x+h)^n}}{h}$$
$$= \frac{-1}{x^n(x+h)^n}\,\frac{(x+h)^n - x^n}{h}.$$

- Therefore,

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{-1}{x^n(x+h)^n}\,\frac{(x+h)^n - x^n}{h}$$
$$= \lim_{h\to 0}\frac{-1}{x^n(x+h)^n}\,\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = -x^{-2n}\frac{d}{dx}\left(x^n\right).$$

- From above, we continue: $f'(x) = -x^{-2n}\frac{d}{dx}\left(x^n\right) = -x^{-2n}\cdot nx^{n-1} = -nx^{-n-1}$. Since n is a positive integer, $k = -n$ is a negative integer and we have $\frac{d}{dx}\left(x^k\right) = \frac{d}{dx}\left(x^{-n}\right) = -nx^{-n-1} = kx^{k-1}$; i.e. $\frac{d}{dx}\left(x^k\right) = kx^{k-1}$ for negative integers k.

96. Verify the Power Rule for the exponent $1/n$, where n is a positive integer, using the following trick: Rewrite the difference quotient for $y = x^{1/n}$ at $x = b$ in terms of

$$u = (b+h)^{1/n} \quad \text{and} \quad a = b^{1/n}$$

SOLUTION Substituting $x = (b+h)^{1/n}$ and $a = b^{1/n}$ into the left-hand side of equation (5) yields

$$\frac{x^n - a^n}{x-a} = \frac{(b+h)-b}{(b+h)^{1/n} - b^{1/n}} = \frac{h}{(b+h)^{1/n} - b^{1/n}}$$

whereas substituting these same expressions into the right-hand side of equation (5) produces

$$\frac{x^n - a^n}{x-a} = (b+h)^{\frac{n-1}{n}} + (b+h)^{\frac{n-2}{n}}b^{1/n} + (b+h)^{\frac{n-3}{n}}b^{2/n} + \cdots + b^{\frac{n-1}{n}};$$

hence,

$$\frac{(b+h)^{1/n} - b^{1/n}}{h} = \frac{1}{(b+h)^{\frac{n-1}{n}} + (b+h)^{\frac{n-2}{n}}b^{1/n} + (b+h)^{\frac{n-3}{n}}b^{2/n} + \cdots + b^{\frac{n-1}{n}}}.$$

If we take $f(x) = x^{1/n}$, then, using the previous expression,

$$f'(b) = \lim_{h\to 0}\frac{(b+h)^{1/n} - b^{1/n}}{h} = \frac{1}{nb^{\frac{n-1}{n}}} = \frac{1}{n}b^{\frac{1}{n}-1}.$$

Replacing b by x, we have $f'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$.

97. Infinitely Rapid Oscillations Define

$$f(x) = \begin{cases} x\sin\dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Show that f is continuous at $x = 0$ but $f'(0)$ does not exist (see Figure 12).

SOLUTION Let $f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. As $x \to 0$,

$$|f(x) - f(0)| = \left|x\sin\left(\frac{1}{x}\right) - 0\right| = |x|\left|\sin\left(\frac{1}{x}\right)\right| \to 0$$

since the values of the sine lie between -1 and 1. Hence, by the Squeeze Theorem, $\lim_{x\to 0} f(x) = f(0)$ and thus f is continuous at $x = 0$.

As $x \to 0$, the difference quotient at $x = 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{x\sin\left(\frac{1}{x}\right) - 0}{x - 0} = \sin\left(\frac{1}{x}\right)$$

does *not* converge to a limit since it oscillates infinitely through every value between -1 and 1. Accordingly, $f'(0)$ does not exist.

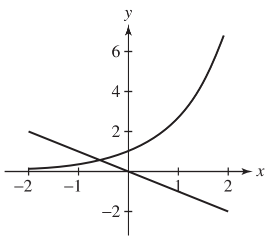
98. For which value of λ does the equation $e^x = \lambda x$ have a unique solution? For which values of λ does it have at least one solution? For intuition, plot $y = e^x$ and the line $y = \lambda x$.

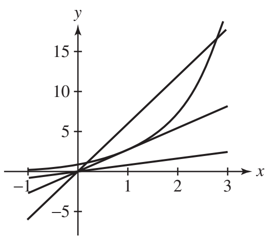
SOLUTION First, note that when $\lambda = 0$, the equation $e^x = 0 \cdot x = 0$ has no real solution. For $\lambda \neq 0$, we observe that solutions to the equation $e^x = \lambda x$ correspond to points of intersection between the graphs of $y = e^x$ and $y = \lambda x$. When $\lambda < 0$, the two graphs intersect at only one location (see the graph below at the left). On the other hand, when $\lambda > 0$, the graphs may have zero, one or two points of intersection (see the graph below at the right). Note that the graphs have one point of intersection when $y = \lambda x$ is the tangent line to $y = e^x$. Thus, not only do we require $e^x = \lambda x$, but also $e^x = \lambda$. It then follows that the point of intersection satisfies $\lambda = \lambda x$, so $x = 1$. This then gives $\lambda = e$.

Therefore the equation $e^x = \lambda x$:

(a) has at least one solution when $\lambda < 0$ and when $\lambda \geq e$;

(b) has a unique solution when $\lambda < 0$ and when $\lambda = e$.





3.3 Product and Quotient Rules

Preliminary Questions

1. Are the following statements true or false? If false, state the correct version.

(a) fg denotes the function whose value at x is $f(g(x))$.

(b) f/g denotes the function whose value at x is $f(x)/g(x)$.

(c) The derivative of the product is the product of the derivatives.

(d) $\left.\dfrac{d}{dx}(fg)\right|_{x=4} = f(4)g'(4) - g(4)f'(4)$

(e) $\left.\dfrac{d}{dx}(fg)\right|_{x=0} = f'(0)g(0) + f(0)g'(0)$

SOLUTION

(a) False. The notation fg denotes the function whose value at x is $f(x)g(x)$.

(b) True.

(c) False. The derivative of a product fg is $f'(x)g(x) + f(x)g'(x)$.

(d) False. $\left.\dfrac{d}{dx}(fg)\right|_{x=4} = f'(4)g(4) + f(4)g'(4)$.

(e) True.

2. Find $(f/g)'(1)$ if $f(1) = f'(1) = g(1) = 2$ and $g'(1) = 4$.

SOLUTION $\frac{d}{dx}(f/g)\big|_{x=1} = [g(1)f'(1) - f(1)g'(1)]/g(1)^2 = [2(2) - 2(4)]/2^2 = -1$.

3. Find $g(1)$ if $f(1) = 0$, $f'(1) = 2$, and $(fg)'(1) = 10$.

SOLUTION $(fg)'(1) = f(1)g'(1) + f'(1)g(1)$, so $10 = 0 \cdot g'(1) + 2g(1)$ and $g(1) = 5$.

Exercises

In Exercises 1–6, use the Product Rule to calculate the derivative.

1. $f(x) = x^3(2x^2 + 1)$

SOLUTION Let $f(x) = x^3(2x^2 + 1)$. Then

$$f'(x) = x^3\frac{d}{dx}(2x^2 + 1) + (2x^2 + 1)\frac{d}{dx}x^3 = x^3(4x) + (2x^2 + 1)(3x^2) = 10x^4 + 3x^2.$$

2. $f(x) = (3x - 5)(2x^2 - 3)$

SOLUTION Let $f(x) = (3x - 5)(2x^2 - 3)$. Then

$$f'(x) = (3x - 5)\frac{d}{dx}(2x^2 - 3) + (2x^2 - 3)\frac{d}{dx}(3x - 5) = (3x - 5)(4x) + (2x^2 - 3)(3) = 18x^2 - 20x - 9.$$

3. $f(x) = x^2e^x$

SOLUTION Let $f(x) = x^2e^x$. Then

$$f'(x) = x^2\frac{d}{dx}e^x + e^x\frac{d}{dx}x^2 = x^2e^x + e^x(2x) = e^x(x^2 + 2x).$$

4. $f(x) = (2x - 9)(4e^x + 1)$

SOLUTION Let $f(x) = (2x - 9)(4e^x + 1)$. Then

$$f'(x) = (2x - 9)\frac{d}{dx}(4e^x + 1) + (4e^x + 1)\frac{d}{dx}(2x - 9) = (2x - 9)(4e^x) + (4e^x + 1)(2) = 8xe^x - 28e^x + 2.$$

5. $\left.\frac{dh}{ds}\right|_{s=4}$, $\quad h(s) = (s^{-1/2} + 2s)(7 - s^{-1})$

SOLUTION Let $h(s) = (s^{-1/2} + 2s)(7 - s^{-1})$. Then

$$\begin{aligned}\frac{dh}{ds} &= (s^{-1/2} + 2s)\frac{d}{dx}(7 - s^{-1}) + (7 - s^{-1})\frac{d}{ds}\left(s^{-1/2} + 2s\right) \\ &= (s^{-1/2} + 2s)(s^{-2}) + (7 - s^{-1})\left(-\frac{1}{2}s^{-3/2} + 2\right) = -\frac{7}{2}s^{-3/2} + \frac{3}{2}s^{-5/2} + 14.\end{aligned}$$

Therefore,

$$\left.\frac{dh}{ds}\right|_{s=4} = -\frac{7}{2}(4)^{-3/2} + \frac{3}{2}(4)^{-5/2} + 14 = \frac{871}{64}.$$

6. $\left.\frac{dy}{dt}\right|_{t=2}$, $\quad y = (t - 8t^{-1})(e^t + t^2)$

SOLUTION Let $y(t) = (t - 8t^{-1})(e^t + t^2)$. Then

$$\begin{aligned}\frac{dy}{dt} &= (t - 8t^{-1})\frac{d}{dt}(e^t + t^2) + (e^t + t^2)\frac{d}{dt}(t - 8t^{-1}) \\ &= (t - 8t^{-1})(e^t + 2t) + (e^t + t^2)(1 + 8t^{-2}).\end{aligned}$$

Therefore,

$$\left.\frac{dy}{dt}\right|_{t=2} = (2 - 4)(e^2 + 4) + (e^2 + 4)(1 + 2) = e^2 + 4.$$

In Exercises 7–12, use the Quotient Rule to calculate the derivative.

7. $f(x) = \dfrac{x}{x-2}$

SOLUTION Let $f(x) = \frac{x}{x-2}$. Then

$$f'(x) = \frac{(x-2)\frac{d}{dx}x - x\frac{d}{dx}(x-2)}{(x-2)^2} = \frac{(x-2)-x}{(x-2)^2} = \frac{-2}{(x-2)^2}.$$

8. $f(x) = \dfrac{x+4}{x^2+x+1}$

SOLUTION Let $f(x) = \frac{x+4}{x^2+x+1}$. Then

$$f'(x) = \frac{(x^2+x+1)\frac{d}{dx}(x+4) - (x+4)\frac{d}{dx}(x^2+x+1)}{(x^2+x+1)^2}$$

$$= \frac{(x^2+x+1)-(x+4)(2x+1)}{(x^2+x+1)^2} = \frac{-x^2-8x-3}{(x^2+x+1)^2}.$$

9. $\left.\dfrac{dg}{dt}\right|_{t=-2}, \quad g(t) = \dfrac{t^2+1}{t^2-1}$

SOLUTION Let $g(t) = \dfrac{t^2+1}{t^2-1}$. Then

$$\frac{dg}{dt} = \frac{(t^2-1)\frac{d}{dt}(t^2+1) - (t^2+1)\frac{d}{dt}(t^2-1)}{(t^2-1)^2} = \frac{(t^2-1)(2t)-(t^2+1)(2t)}{(t^2-1)^2} = -\frac{4t}{(t^2-1)^2}.$$

Therefore,

$$\left.\frac{dg}{dt}\right|_{t=-2} = -\frac{4(-2)}{((-2)^2-1)^2} = \frac{8}{9}.$$

10. $\left.\dfrac{dw}{dz}\right|_{z=9}, \quad w = \dfrac{z^2}{\sqrt{z}+z}$

SOLUTION Let $w(z) = \dfrac{z^2}{\sqrt{z}+z}$. Then

$$\frac{dw}{dz} = \frac{(\sqrt{z}+z)\frac{d}{dz}z^2 - z^2\frac{d}{dz}(\sqrt{z}+z)}{(\sqrt{z}+z)^2} = \frac{2z(\sqrt{z}+z) - z^2((1/2)z^{-1/2}+1)}{(\sqrt{z}+z)^2} = \frac{(3/2)z^{3/2}+z^2}{(\sqrt{z}+z)^2}.$$

Therefore,

$$\left.\frac{dw}{dz}\right|_{z=9} = \frac{(3/2)(9)^{3/2}+9^2}{(\sqrt{9}+9)^2} = \frac{27}{32}.$$

11. $g(x) = \dfrac{1}{1+e^x}$

SOLUTION Let $g(x) = \dfrac{1}{1+e^x}$. Then

$$\frac{dg}{dx} = \frac{(1+e^x)\frac{d}{dx}1 - 1\frac{d}{dx}(1+e^x)}{(1+e^x)^2} = \frac{(1+e^x)(0)-e^x}{(1+e^x)^2} = -\frac{e^x}{(1+e^x)^2}.$$

12. $f(x) = \dfrac{e^x}{x^2+1}$

SOLUTION Let $f(x) = \dfrac{e^x}{x^2+1}$. Then

$$\frac{df}{dx} = \frac{(x^2+1)\frac{d}{dx}e^x - e^x\frac{d}{dx}(x^2+1)}{(x^2+1)^2} = \frac{(x^2+1)e^x - e^x(2x)}{(x^2+1)^2} = \frac{e^x(x-1)^2}{(x^2+1)^2}.$$

In Exercises 13–16, calculate the derivative in two ways. First use the Product or Quotient Rule; then rewrite the function algebraically and apply the Power Rule directly.

13. $f(t) = (2t+1)(t^2-2)$

SOLUTION Let $f(t) = (2t+1)(t^2-2)$. Then, using the Product Rule,

$$f'(t) = (2t+1)(2t) + (t^2-2)(2) = 6t^2 + 2t - 4.$$

Multiplying out first, we find $f(t) = 2t^3 + t^2 - 4t - 2$. Therefore, $f'(t) = 6t^2 + 2t - 4$.

14. $f(x) = x^2(3 + x^{-1})$

SOLUTION Let $f(x) = x^2(3 + x^{-1})$. Then, using the product rule, and then power and sum rules,

$$f'(x) = x^2(-x^{-2}) + (3 + x^{-1})(2x) = 6x + 1.$$

Multiplying out first, we find $f(x) = 3x^2 + x$. Then $f'(x) = 6x + 1$.

15. $h(t) = \dfrac{t^2-1}{t-1}$

SOLUTION Let $h(t) = \frac{t^2-1}{t-1}$. Using the quotient rule,

$$f'(t) = \frac{(t-1)(2t) - (t^2-1)(1)}{(t-1)^2} = \frac{t^2-2t+1}{(t-1)^2} = 1$$

for $t \neq 1$. Simplifying first, we find for $t \neq 1$,

$$h(t) = \frac{(t-1)(t+1)}{(t-1)} = t + 1.$$

Hence $h'(t) = 1$ for $t \neq 1$.

16. $g(x) = \dfrac{x^3 + 2x^2 + 3x^{-1}}{x}$

SOLUTION Let $g(x) = \frac{x^3+2x^2+3x^{-1}}{x}$. Using the quotient rule and the sum and power rules, and simplifying

$$g'(x) = \frac{x(3x^2 + 4x - 3x^{-2}) - (x^3 + 2x^2 + 3x^{-1})1}{x^2} = \frac{1}{x^2}\left(2x^3 + 2x^2 - 6x^{-1}\right) = 2x + 2 - 6x^{-3}.$$

Simplifying first yields $g(x) = x^2 + 2x + 3x^{-2}$, from which we calculate $g'(x) = 2x + 2 - 6x^{-3}$.

In Exercises 17–38, calculate the derivative.

17. $f(x) = (x^3+5)(x^3+x+1)$

SOLUTION Let $f(x) = (x^3+5)(x^3+x+1)$. Then

$$f'(x) = (x^3+5)(3x^2+1) + (x^3+x+1)(3x^2) = 6x^5 + 4x^3 + 18x^2 + 5.$$

18. $f(x) = (4e^x - x^2)(x^3+1)$

SOLUTION Let $f(x) = (4e^x - x^2)(x^3+1)$. Then

$$f'(x) = (4e^x - x^2)(3x^2) + (x^3+1)(4e^x - 2x) = e^x(4x^3 + 12x^2 + 4) - 5x^4 - 2x.$$

19. $\left.\dfrac{dy}{dx}\right|_{x=3}, \quad y = \dfrac{1}{x+10}$

SOLUTION Let $y = \frac{1}{x+10}$. Using the quotient rule:

$$\frac{dy}{dx} = \frac{(x+10)(0) - 1(1)}{(x+10)^2} = -\frac{1}{(x+10)^2}.$$

Therefore,

$$\left.\frac{dy}{dx}\right|_{x=3} = -\frac{1}{(3+10)^2} = -\frac{1}{169}.$$

20. $\left.\dfrac{dz}{dx}\right|_{x=-2}, \quad z = \dfrac{x}{3x^2+1}$

SOLUTION Let $z = \frac{x}{3x^2+1}$. Using the quotient rule:

$$\frac{dz}{dx} = \frac{(3x^2+1)(1) - x(6x)}{(3x^2+1)^2} = \frac{1-3x^2}{(3x^2+1)^2}.$$

Therefore,

$$\left.\frac{dz}{dx}\right|_{x=-2} = \frac{1-3(-2)^2}{(3(-2)^2+1)^2} = -\frac{11}{169}.$$

21. $f(x) = (\sqrt{x}+1)(\sqrt{x}-1)$

SOLUTION Let $f(x) = (\sqrt{x}+1)(\sqrt{x}-1)$. Multiplying through first yields $f(x) = x - 1$ for $x \geq 0$. Therefore, $f'(x) = 1$ for $x \geq 0$. If we carry out the product rule on $f(x) = (x^{1/2}+1)(x^{1/2}-1)$, we get

$$f'(x) = (x^{1/2}+1)\left(\frac{1}{2}(x^{-1/2})\right) + (x^{1/2}-1)\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2} + \frac{1}{2}x^{-1/2} + \frac{1}{2} - \frac{1}{2}x^{-1/2} = 1.$$

22. $f(x) = \dfrac{9x^{5/2}-2}{x}$

SOLUTION Let $f(x) = \frac{9x^{5/2}-2}{x} = 9x^{3/2} - 2x^{-1}$. Then $f'(x) = \frac{27}{2}x^{1/2} + 2x^{-2}$.

23. $\left.\dfrac{dy}{dx}\right|_{x=2}, \quad y = \dfrac{x^4-4}{x^2-5}$

SOLUTION Let $y = \dfrac{x^4-4}{x^2-5}$. Then

$$\frac{dy}{dx} = \frac{\left(x^2-5\right)\left(4x^3\right) - \left(x^4-4\right)(2x)}{\left(x^2-5\right)^2} = \frac{2x^5-20x^3+8x}{\left(x^2-5\right)^2}.$$

Therefore,

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{2(2)^5-20(2)^3+8(2)}{(2^2-5)^2} = -80.$$

24. $f(x) = \dfrac{x^4+e^x}{x+1}$

SOLUTION Let $f(x) = \dfrac{x^4+e^x}{x+1}$. Then

$$\frac{df}{dx} = \frac{(x+1)(4x^3+e^x) - (x^4+e^x)(1)}{(x+1)^2} = \frac{(x+1)(4x^3+e^x) - x^4 - e^x}{(x+1)^2}.$$

25. $\left.\dfrac{dz}{dx}\right|_{x=1}, \quad z = \dfrac{1}{x^3+1}$

SOLUTION Let $z = \frac{1}{x^3+1}$. Using the quotient rule:

$$\frac{dz}{dx} = \frac{(x^3+1)(0) - 1(3x^2)}{(x^3+1)^2} = -\frac{3x^2}{(x^3+1)^2}.$$

Therefore,

$$\left.\frac{dz}{dx}\right|_{x=1} = -\frac{3(1)^2}{(1^3+1)^2} = -\frac{3}{4}.$$

26. $f(x) = \dfrac{3x^3-x^2+2}{\sqrt{x}}$

SOLUTION Let

$$f(x) = \frac{3x^3 - x^2 + 2}{\sqrt{x}} = \frac{3x^3 - x^2 + 2}{x^{1/2}}.$$

Using the quotient rule, and then simplifying by taking out the greatest negative factor:

$$f'(x) = \frac{(x^{1/2})(9x^2 - 2x) - (3x^3 - x^2 + 2)(\frac{1}{2}x^{-1/2})}{x} = \frac{1}{x^{3/2}}\left((9x^3 - 2x^2) - \frac{1}{2}(3x^3 - x^2 + 2)\right)$$
$$= \frac{1}{x^{3/2}}\left(\frac{15}{2}x^3 - \frac{3}{2}x^2 - 1\right).$$

Alternately, since there is a single exponent of x in the denominator, we could also simplify $f(x)$ first, getting $f(x) = 3x^{5/2} - x^{3/2} + 2x^{-1/2}$. Then $f'(x) = \frac{15}{2}x^{3/2} - \frac{3}{2}x^{1/2} - x^{-3/2}$. The two answers are the same.

27. $h(t) = \dfrac{t}{(t+1)(t^2+1)}$

SOLUTION Let $h(t) = \dfrac{t}{(t+1)(t^2+1)} = \dfrac{t}{t^3 + t^2 + t + 1}$. Then

$$h'(t) = \frac{\left(t^3 + t^2 + t + 1\right)(1) - t\left(3t^2 + 2t + 1\right)}{\left(t^3 + t^2 + t + 1\right)^2} = \frac{-2t^3 - t^2 + 1}{\left(t^3 + t^2 + t + 1\right)^2}.$$

28. $f(x) = x^{3/2}\left(2x^4 - 3x + x^{-1/2}\right)$

SOLUTION Let $f(x) = x^{3/2}(2x^4 - 3x + x^{-1/2})$. We multiply through the $x^{3/2}$ to get $f(x) = 2x^{11/2} - 3x^{5/2} + x$. Then $f'(x) = 11x^{9/2} - \frac{15}{2}x^{3/2} + 1$.

29. $f(t) = 3^{1/2} \cdot 5^{1/2}$

SOLUTION Let $f(t) = \sqrt{3}\sqrt{5}$. Then $f'(t) = 0$, since $f(t)$ is a *constant* function!

30. $h(x) = \pi^2(x - 1)$

SOLUTION Let $h(x) = \pi^2(x - 1)$. Then $h'(x) = \pi^2$.

31. $f(x) = (x + 3)(x - 1)(x - 5)$

SOLUTION Let $f(x) = (x + 3)(x - 1)(x - 5)$. Using the Product Rule inside the Product Rule with a first factor of $(x + 3)$ and a second factor of $(x - 1)(x - 5)$, we find

$$f'(x) = (x + 3)\left((x - 1)(1) + (x - 5)(1)\right) + (x - 1)(x - 5)(1) = 3x^2 - 6x - 13.$$

Alternatively,

$$f(x) = (x + 3)\left(x^2 - 6x + 5\right) = x^3 - 3x^2 - 13x + 15.$$

Therefore, $f'(x) = 3x^2 - 6x - 13$.

32. $f(x) = e^x(x^2+1)(x+4)$

SOLUTION Let $f(x) = e^x(x^2+1)(x+4)$. Using the Product Rule inside the Product Rule with a first factor of e^x and a second factor of $(x^2+1)(x+4)$, we find

$$f'(x) = e^x\left((x^2+1)(1) + (x+4)(2x)\right) + (x^2+1)(x+4)e^x = (x^3+7x^2+9x+5)e^x.$$

33. $f(x) = \dfrac{e^x}{x+1}$

SOLUTION Let $f(x) = \dfrac{e^x}{x+1}$. Then

$$f'(x) = \frac{(x+1)e^x - e^x(1)}{(x+1)^2} = \frac{e^x(x+1-1)}{(x+1)^2} = \frac{xe^x}{(x+1)^2}.$$

34. $g(x) = \dfrac{e^{x+1}+e^x}{e+1}$

SOLUTION Let

$$g(x) = \frac{e^{x+1}+e^x}{e+1} = \frac{e^x(e+1)}{e+1} = e^x.$$

Then $g'(x) = e^x$.

35. $g(z) = \left(\dfrac{z^2-4}{z-1}\right)\left(\dfrac{z^2-1}{z+2}\right)$ *Hint:* Simplify first.

SOLUTION Let

$$g(z) = \left(\frac{z^2-4}{z-1}\right)\left(\frac{z^2-1}{z+2}\right) = \left(\frac{(z+2)(z-2)}{z-1}\frac{(z+1)(z-1)}{z+2}\right) = (z-2)(z+1)$$

for $z \neq -2$ and $z \neq 1$. Then,

$$g'(z) = (z+1)(1) + (z-2)(1) = 2z-1.$$

36. $\dfrac{d}{dx}\left((ax+b)(abx^2+1)\right)$ (a, b constants)

SOLUTION Let $f(x) = (ax+b)(abx^2+1)$. Then

$$f'(x) = (ax+b)(2abx) + (abx^2+1)(a) = 3a^2bx^2 + a + 2ab^2x.$$

37. $\dfrac{d}{dt}\left(\dfrac{xt-4}{t^2-x}\right)$ (x constant)

SOLUTION Let $f(t) = \frac{xt-4}{t^2-x}$. Using the quotient rule:

$$f'(t) = \frac{(t^2-x)(x) - (xt-4)(2t)}{(t^2-x)^2} = \frac{xt^2 - x^2 - 2xt^2 + 8t}{(t^2-x)^2} = \frac{-xt^2+8t-x^2}{(t^2-x)^2}.$$

38. $\dfrac{d}{dx}\left(\dfrac{ax+b}{cx+d}\right)$ (a, b, c, d constants)

SOLUTION Let $f(x) = \left(\dfrac{ax+b}{cx+d}\right)$. Using the quotient rule:

$$f'(x) = \frac{(cx+d)a - (ax+b)c}{(cx+d)^2} = \frac{(ad-bc)}{(cx+d)^2}.$$

In Exercises 39–42, calculate the derivative using the values:

$f(4)$	$f'(4)$	$g(4)$	$g'(4)$
10	−2	5	−1

39. $(fg)'(4)$ and $(f/g)'(4)$

SOLUTION Let $h = fg$ and $H = f/g$. Then $h' = fg' + gf'$ and $H' = \frac{gf' - fg'}{g^2}$. Finally,

$$h'(4) = f(4)g'(4) + g(4)f'(4) = (10)(-1) + (5)(-2) = -20,$$

and

$$H'(4) = \frac{g(4)f'(4) - f(4)g'(4)}{(g(4))^2} = \frac{(5)(-2) - (10)(-1)}{(5)^2} = 0.$$

40. $F'(4)$, where $F(x) = x^2 f(x)$

SOLUTION Let $F(x) = x^2 f(x)$. Then $F'(x) = x^2 f'(x) + 2xf(x)$, and

$$F'(4) = 16f'(4) + 8f(4) = (16)(-2) + (8)(10) = 48.$$

41. $G'(4)$, where $G(x) = (g(x))^2$

SOLUTION Let $G(x) = g(x)^2 = g(x)g(x)$. Then $G'(x) = g(x)g'(x) + g(x)g'(x) = 2g(x)g'(x)$, and

$$G'(4) = 2g(4)g'(4) = 2(5)(-1) = -10.$$

42. $H'(4)$, where $H(x) = \dfrac{x}{g(x)f(x)}$

SOLUTION Let $H(x) = \dfrac{x}{g(x)f(x)}$. Then

$$H'(x) = \frac{g(x)f(x) \cdot 1 - x(g(x)f'(x) + f(x)g'(x))}{(g(x)f(x))^2},$$

and

$$H'(4) = \frac{(5)(10) - 4((5)(-2) + (10)(-1))}{((5)(10))^2} = \frac{13}{250}.$$

43. Calculate $F'(0)$, where

$$F(x) = \frac{x^9 + x^8 + 4x^5 - 7x}{x^4 - 3x^2 + 2x + 1}$$

Hint: Do not calculate $F'(x)$. Instead, write $F(x) = f(x)/g(x)$ and express $F'(0)$ directly in terms of $f(0)$, $f'(0)$, $g(0)$, $g'(0)$.

SOLUTION Taking the hint, let

$$f(x) = x^9 + x^8 + 4x^5 - 7x$$

and let

$$g(x) = x^4 - 3x^2 + 2x + 1.$$

Then $F(x) = \frac{f(x)}{g(x)}$. Now,

$$f'(x) = 9x^8 + 8x^7 + 20x^4 - 7 \quad \text{and} \quad g'(x) = 4x^3 - 6x + 2.$$

Moreover, $f(0) = 0$, $f'(0) = -7$, $g(0) = 1$, and $g'(0) = 2$.

Using the quotient rule:

$$F'(0) = \frac{g(0)f'(0) - f(0)g'(0)}{(g(0))^2} = \frac{-7 - 0}{1} = -7.$$

44. Proceed as in Exercise 43 to calculate $F'(0)$, where

$$F(x) = \left(1 + x + x^{4/3} + x^{5/3}\right)\frac{3x^5 + 5x^4 + 5x + 1}{8x^9 - 7x^4 + 1}$$

SOLUTION Write $F(x) = f(x)(g(x)/h(x))$, where

$$f(x) = 1 + x + x^{4/3} + x^{5/3}$$
$$g(x) = 3x^5 + 5x^4 + 5x + 1$$

and

$$h(x) = 8x^9 - 7x^4 + 1.$$

Now, $f'(x) = 1 + \frac{4}{3}x^{\frac{1}{3}} + \frac{5}{3}x^{\frac{2}{3}}$, $g'(x) = 15x^4 + 20x^3 + 5$, and $h'(x) = 72x^8 - 28x^3$. Moreover, $f(0) = 1$, $f'(0) = 1$, $g(0) = 1$, $g'(0) = 5$, $h(0) = 1$, and $h'(0) = 0$. From the product and quotient rules,

$$F'(0) = f(0)\frac{h(0)g'(0) - g(0)h'(0)}{h(0)^2} + f'(0)(g(0)/h(0)) = 1\frac{1(5) - 1(0)}{1} + 1(1/1) = 6.$$

45. Use the Product Rule to calculate $\frac{d}{dx}e^{2x}$.

SOLUTION Note that $e^{2x} = e^x \cdot e^x$. Therefore

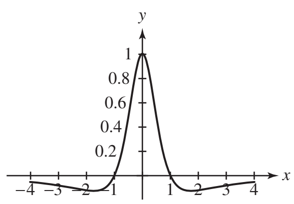
$$\frac{d}{dx}e^{2x} = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot e^x + e^x \cdot e^x = 2e^{2x}.$$

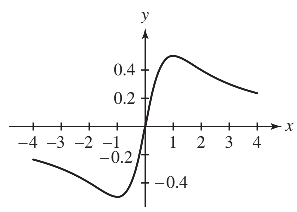
46. GU Plot the derivative of $f(x) = x/(x^2 + 1)$ over $[-4, 4]$. Use the graph to determine the intervals on which $f'(x) > 0$ and $f'(x) < 0$. Then plot f and describe how the sign of $f'(x)$ is reflected in the graph of f.

SOLUTION Let $f(x) = \dfrac{x}{x^2 + 1}$. Then

$$f'(x) = \frac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}.$$

The derivative is shown in the figure below at the left. From this plot we see that $f'(x) > 0$ for $-1 < x < 1$ and $f'(x) < 0$ for $|x| > 1$. The original function is plotted in the figure below at the right. Observe that the graph of $f(x)$ is increasing whenever $f'(x) > 0$ and that $f(x)$ is decreasing whenever $f'(x) < 0$.



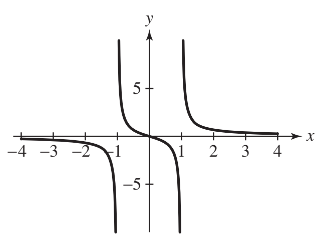


47. GU Plot $f(x) = x/(x^2 - 1)$ (in a suitably bounded viewing box). Use the plot to determine whether $f'(x)$ is positive or negative on its domain $\{x : x \neq \pm 1\}$. Then compute $f'(x)$ and confirm your conclusion algebraically.

SOLUTION Let $f(x) = \dfrac{x}{x^2 - 1}$. The graph of $f(x)$ is shown below. From this plot, we see that $f(x)$ is decreasing on its domain $\{x : x \neq \pm 1\}$. Consequently, $f'(x)$ must be negative. Using the quotient rule, we find

$$f'(x) = \frac{(x^2 - 1)(1) - x(2x)}{(x^2 - 1)^2} = -\frac{x^2 + 1}{(x^2 - 1)^2},$$

which is negative for all $x \neq \pm 1$.



48. Let $P = V^2R/(R+r)^2$ as in Example 7. Calculate dP/dr, assuming that r is variable and R is constant.

SOLUTION Note that V is also constant. Let

$$f(r) = \frac{V^2R}{(R+r)^2} = \frac{V^2R}{R^2 + 2Rr + r^2}.$$

Using the quotient rule:

$$f'(r) = \frac{(R^2+2Rr+r^2)(0) - (V^2R)(2R+2r)}{(R+r)^4} = -\frac{2V^2R(R+r)}{(R+r)^4} = -\frac{2V^2R}{(R+r)^3}.$$

49. Find $a > 0$ such that the tangent line to the graph of

$$f(x) = x^2e^{-x} \quad \text{at } x = a$$

passes through the origin (Figure 4).

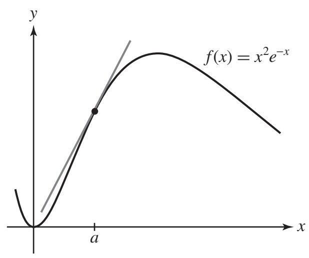


FIGURE 4

SOLUTION Let $f(x) = x^2e^{-x}$. Then $f(a) = a^2e^{-a}$,

$$f'(x) = -x^2e^{-x} + 2xe^{-x} = e^{-x}(2x - x^2),$$

$f'(a) = (2a - a^2)e^{-a}$, and the equation of the tangent line to f at $x = a$ is

$$y = f'(a)(x-a) + f(a) = (2a - a^2)e^{-a}(x-a) + a^2e^{-a}.$$

For this line to pass through the origin, we must have

$$0 = (2a - a^2)e^{-a}(-a) + a^2e^{-a} = e^{-a}\left(a^2 - 2a^2 + a^3\right) = a^2e^{-a}(a-1).$$

Thus, $a = 0$ or $a = 1$. The only value $a > 0$ such that the tangent line to $f(x) = x^2e^{-x}$ passes through the origin is therefore $a = 1$.

50. Current I (amperes), voltage V (volts), and resistance R (ohms) in a circuit are related by Ohm's Law, $I = V/R$.

(a) Calculate $\left.\frac{dI}{dR}\right|_{R=6}$ if V is constant with value $V = 24$.

(b) Calculate $\left.\frac{dV}{dR}\right|_{R=6}$ if I is constant with value $I = 4$.

SOLUTION

(a) According to Ohm's Law, $I = V/R = VR^{-1}$. Thus, using the power rule,

$$\frac{dI}{dR} = -VR^{-2}.$$

With $V = 24$ volts, it follows that

$$\left.\frac{dI}{dR}\right|_{R=6} = -24(6)^{-2} = -\frac{2}{3}\frac{\text{amps}}{\Omega}.$$

(b) Solving Ohm's Law for V yields $V = RI$. Thus

$$\frac{dV}{dR} = I \qquad \text{and} \qquad \left.\frac{dV}{dR}\right|_{I=4} = 4 \text{ amps}.$$

51. The revenue per month earned by the Couture clothing chain at time t is $R(t) = N(t)S(t)$, where $N(t)$ is the number of stores and $S(t)$ is average revenue per store per month. Couture embarks on a two-part campaign: (A) to build new stores at a rate of 5 stores per month, and (B) to use advertising to increase average revenue per store at a rate of \$10,000 per month. Assume that $N(0) = 50$ and $S(0) = \$150{,}000$.

(a) Show that total revenue will increase at the rate

$$\frac{dR}{dt} = 5S(t) + 10{,}000N(t)$$

Note that the two terms in the Product Rule correspond to the separate effects of increasing the number of stores on the one hand, and the average revenue per store on the other.

(b) Calculate $\left.\frac{dR}{dt}\right|_{t=0}$.

(c) If Couture can implement only one leg (A or B) of its expansion at $t = 0$, which choice will grow revenue most rapidly?

SOLUTION

(a) Given $R(t) = N(t)S(t)$, it follows that

$$\frac{dR}{dt} = N(t)S'(t) + S(t)N'(t).$$

We are told that $N'(t) = 5$ stores per month and $S'(t) = 10{,}000$ dollars per month. Therefore,

$$\frac{dR}{dt} = 5S(t) + 10{,}000N(t).$$

(b) Using part (a) and the given values of $N(0)$ and $S(0)$, we find

$$\left.\frac{dR}{dt}\right|_{t=0} = 5(150{,}000) + 10{,}000(50) = 1{,}250{,}000.$$

(c) From part (b), we see that of the two terms contributing to total revenue growth, the term $5S(0)$ is larger than the term $10{,}000N(0)$. Thus, if only one leg of the campaign can be implemented, it should be part A: increase the number of stores by 5 per month.

52. The **tip speed ratio** of a turbine (Figure 5) is the ratio $R = T/W$, where T is the speed of the tip of a blade and W is the speed of the wind. (Engineers have found empirically that a turbine with n blades extracts maximum power from the wind when $R = 2\pi/n$.) Calculate dR/dt (t in minutes) if $W = 35$ km/h and W decreases at a rate of 4 km/h per minute, and the tip speed has constant value $T = 150$ km/h.

FIGURE 5 Turbines on a wind farm

SOLUTION Let $R = T/W$. Then

$$\frac{dR}{dt} = \frac{WT' - TW'}{W^2}.$$

Using the values $T = 150$, $T' = 0$, $W = 35$ and $W' = -4$, we find

$$\frac{dR}{dt} = \frac{(35)(0) - 150(-4)}{35^2} = \frac{24}{49}.$$

53. The curve $y = 1/(x^2 + 1)$ is called the *witch of Agnesi* (Figure 6) after the Italian mathematician Maria Agnesi (1718–1799), who wrote one of the first books on calculus. This strange name is the result of a mistranslation of the Italian word *la versiera*, meaning "that which turns." Find equations of the tangent lines at $x = \pm 1$.

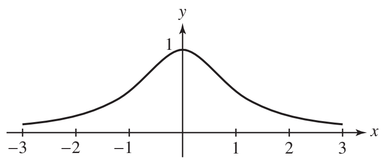


FIGURE 6 The witch of Agnesi.

SOLUTION Let $f(x) = \dfrac{1}{x^2+1}$. Then $f'(x) = \dfrac{(x^2+1)(0) - 1(2x)}{(x^2+1)^2} = -\dfrac{2x}{\left(x^2+1\right)^2}$.

- At $x = -1$, the tangent line is

$$y = f'(-1)(x+1) + f(-1) = \frac{1}{2}(x+1) + \frac{1}{2} = \frac{1}{2}x + 1.$$

- At $x = 1$, the tangent line is

$$y = f'(1)(x-1) + f(1) = -\frac{1}{2}(x-1) + \frac{1}{2} = -\frac{1}{2}x + 1.$$

54. Let $f(x) = g(x) = x$. Show that $(f/g)' \neq f'/g'$.

SOLUTION $(f/g) = (x/x) = 1$, so $(f/g)' = 0$. On the other hand, $(f'/g') = (x'/x') = (1/1) = 1$. We see that $0 \neq 1$.

55. Use the Product Rule to show that $(f^2)' = 2ff'$.

SOLUTION Let $g = f^2 = ff$. Then $g' = \left(f^2\right)' = (ff)' = ff' + ff' = 2ff'$.

56. Show that $(f^3)' = 3f^2 f'$.

SOLUTION Let $g = f^3 = fff$. Then

$$g' = \left(f^3\right)' = [f\,(ff)]' = f\left(ff' + ff'\right) + ff(f') = 3f^2 f'.$$

Further Insights and Challenges

57. Let f, g, h be differentiable functions. Show that $(fgh)'(x)$ is equal to

$$f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

Hint: Write fgh as $f(gh)$.

SOLUTION Let $p = fgh$. Then

$$p' = (fgh)' = f\left(gh' + hg'\right) + ghf' = f'gh + fg'h + fgh'.$$

58. Prove the Quotient Rule using the limit definition of the derivative.

SOLUTION Let $p = \dfrac{f}{g}$. Suppose that f and g are differentiable at $x = a$ and that $g(a) \neq 0$. Then

$$p'(a) = \lim_{h\to 0} \frac{p(a+h) - p(a)}{h} = \lim_{h\to 0} \frac{\dfrac{f(a+h)}{g(a+h)} - \dfrac{f(a)}{g(a)}}{h} = \lim_{h\to 0} \frac{\dfrac{f(a+h)g(a) - f(a)g(a+h)}{g(a+h)g(a)}}{h}$$

$$= \lim_{h\to 0} \frac{f(a+h)g(a) - f(a)g(a) + f(a)g(a) - f(a)g(a+h)}{hg(a+h)g(a)}$$

$$= \lim_{h\to 0} \left(\frac{1}{g(a+h)g(a)} \left(g(a)\,\frac{f(a+h) - f(a)}{h} - f(a)\,\frac{g(a+h) - g(a)}{h} \right) \right)$$

$$= \left(\lim_{h\to 0}\frac{1}{g(a+h)g(a)}\right)\left(\left(g(a)\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right)-\left(f(a)\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}\right)\right)$$

$$= \frac{1}{(g(a))^2}\left(g(a)f'(a)-f(a)g'(a)\right) = \frac{g(a)f'(a)-f(a)g'(a)}{(g(a))^2}$$

In other words, $p' = \left(\frac{f}{g}\right)' = \frac{gf'-fg'}{g^2}$.

59. Derivative of the Reciprocal Use the limit definition to prove

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)}$$ 1

Hint: Show that the difference quotient for $1/f(x)$ is equal to

$$\frac{f(x)-f(x+h)}{hf(x)f(x+h)}$$

SOLUTION Let $g(x) = \frac{1}{f(x)}$. We then compute the derivative of $g(x)$ using the difference quotient:

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{1}{f(x+h)}-\frac{1}{f(x)}\right) = \lim_{h\to 0}\frac{1}{h}\left(\frac{f(x)-f(x+h)}{f(x)f(x+h)}\right)$$

$$= -\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\left(\frac{1}{f(x)f(x+h)}\right).$$

We can apply the rule of products for limits. The first parenthetical expression is the difference quotient definition of $f'(x)$. The second can be evaluated at $h=0$ to give $\frac{1}{(f(x))^2}$. Hence

$$g'(x) = \frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{f^2(x)}.$$

60. Prove the Quotient Rule using Eq. (1) and the Product Rule.

SOLUTION Let $h(x) = \frac{f(x)}{g(x)}$. We can write $h(x) = f(x)\frac{1}{g(x)}$. Applying Eq. (1),

$$h'(x) = f(x)\left(\left(\frac{1}{g(x)}\right)'\right) + f'(x)\left(\frac{1}{g(x)}\right) = -f(x)\left(\frac{g'(x)}{(g(x))^2}\right) + \frac{f'(x)}{g(x)} = \frac{-f(x)g'(x)+f'(x)g(x)}{(g(x))^2}.$$

61. Use the limit definition of the derivative to prove the following special case of the Product Rule:

$$\frac{d}{dx}(xf(x)) = f(x) + xf'(x)$$

SOLUTION First note that because $f(x)$ is differentiable, it is also continuous. It follows that

$$\lim_{h\to 0} f(x+h) = f(x).$$

Now we tackle the derivative:

$$\frac{d}{dx}(xf(x)) = \lim_{h\to 0}\frac{(x+h)f(x+h)-xf(x)}{h} = \lim_{h\to 0}\left(x\frac{f(x+h)-f(x)}{h}+f(x+h)\right)$$

$$= x\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} f(x+h)$$

$$= xf'(x) + f(x).$$

62. Carry out Maria Agnesi's proof of the Quotient Rule from her book on calculus, published in 1748: Assume that f, g, and $h = f/g$ are differentiable. Compute the derivative of $hg = f$ using the Product Rule, and solve for h'.

SOLUTION Suppose that f, g, and h are differentiable functions with $h = f/g$.

- Then $hg = f$ and via the product rule $hg' + gh' = f'$.
- Solving for h' yields $h' = \frac{f'-hg'}{g} = \frac{f'-\frac{f}{g}g'}{g} = \frac{gf'-fg'}{g^2}$.

63. The Power Rule Revisited If you are familiar with *proof by induction*, use induction to prove the Power Rule for all whole numbers n. Show that the Power Rule holds for $n = 1$; then write x^n as $x \cdot x^{n-1}$ and use the Product Rule.

SOLUTION Let k be a positive integer. If $k = 1$, then $x^k = x$. Note that

$$\frac{d}{dx}\left(x^1\right) = \frac{d}{dx}(x) = 1 = 1x^0.$$

Hence the Power Rule holds for $k = 1$. Assume it holds for $k = n$ where $n \geq 2$. Then for $k = n + 1$, we have

$$\frac{d}{dx}\left(x^k\right) = \frac{d}{dx}\left(x^{n+1}\right) = \frac{d}{dx}\left(x \cdot x^n\right) = x\frac{d}{dx}\left(x^n\right) + x^n\frac{d}{dx}(x)$$

$$= x \cdot nx^{n-1} + x^n \cdot 1 = (n+1)x^n = kx^{k-1}$$

Accordingly, the Power Rule holds for all positive integers by induction.

Exercises 64 and 65: A basic fact of algebra states that c is a root of a polynomial f if and only if $f(x) = (x - c)g(x)$ for some polynomial g. We say that c is a ***multiple root*** *if $f(x) = (x - c)^2h(x)$, where h is a polynomial.*

64. Show that c is a multiple root of f if and only if c is a root of both f and f'.

SOLUTION Assume first that $f(c) = f'(c) = 0$ and let us show that c is a multiple root of $f(x)$. We have $f(x) = (x - c)g(x)$ for some polynomial $g(x)$ and so $f'(x) = (x - c)g'(x) + g(x)$. However, $f'(c) = 0 + g(c) = 0$, so c is also a root of $g(x)$ and hence $g(x) = (x - c)h(x)$ for some polynomial $h(x)$. We conclude that $f(x) = (x - c)^2h(x)$, which shows that c is a multiple root of $f(x)$.

Conversely, assume that c is a multiple root. Then $f(c) = 0$ and $f(x) = (x - c)^2g(x)$ for some polynomial $g(x)$. Then $f'(x) = (x - c)^2g'(x) + 2g(x)(x - c)$. Therefore, $f'(c) = (c - c)^2g'(c) + 2g(c)(c - c) = 0$.

65. Use Exercise 64 to determine whether $c = -1$ is a multiple root:

(a) $x^5 + 2x^4 - 4x^3 - 8x^2 - x + 2$

(b) $x^4 + x^3 - 5x^2 - 3x + 2$

SOLUTION

(a) To show that -1 is a multiple root of

$$f(x) = x^5 + 2x^4 - 4x^3 - 8x^2 - x + 2,$$

it suffices to check that $f(-1) = f'(-1) = 0$. We have $f(-1) = -1 + 2 + 4 - 8 + 1 + 2 = 0$ and

$$f'(x) = 5x^4 + 8x^3 - 12x^2 - 16x - 1$$

$$f'(-1) = 5 - 8 - 12 + 16 - 1 = 0$$

(b) Let $f(x) = x^4 + x^3 - 5x^2 - 3x + 2$. Then $f'(x) = 4x^3 + 3x^2 - 10x - 3$. Because

$$f(-1) = 1 - 1 - 5 + 3 + 2 = 0$$

but

$$f'(-1) = -4 + 3 + 10 - 3 = 6 \neq 0,$$

it follows that $x = -1$ is a root of f, but not a multiple root.

66. Figure 7 is the graph of a polynomial with roots at A, B, and C. Which of these is a multiple root? Explain your reasoning using Exercise 64.

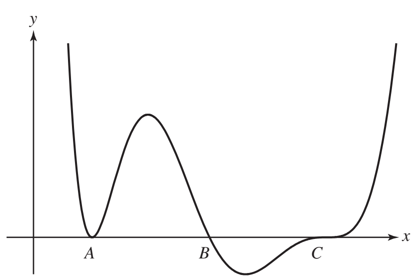


FIGURE 7

SOLUTION A on the figure is a multiple root. It is a multiple root because $f(x) = 0$ at A and because the tangent line to the graph at A is horizontal, so that $f'(x) = 0$ at A. For the same reasons, f also has a multiple root at C.

67. According to Eq. (8) in Section 3.2, $\frac{d}{dx} b^x = m(b)\, b^x$. Use the Product Rule to show that $m(ab) = m(a) + m(b)$.

SOLUTION

$$m(ab)(ab)^x = \frac{d}{dx}(ab)^x = \frac{d}{dx}\left(a^x b^x\right) = a^x \frac{d}{dx} b^x + b^x \frac{d}{dx} a^x = m(b) a^x b^x + m(a) a^x b^x = (m(a) + m(b))(ab)^x.$$

Thus, $m(ab) = m(a) + m(b)$.

3.4 Rates of Change

Preliminary Questions

1. Which units might be used for each rate of change?

(a) Pressure (in atmospheres) in a water tank with respect to depth

(b) The rate of a chemical reaction (change in concentration with respect to time with concentration in moles per liter)

SOLUTION

(a) The rate of change of pressure with respect to depth might be measured in atmospheres/meter.

(b) The reaction rate of a chemical reaction might be measured in moles/(liter·hour).

2. Two trains travel from New Orleans to Memphis in 4 h. The first train travels at a constant velocity of 90 mph, but the velocity of the second train varies. What was the second train's average velocity during the trip?

SOLUTION Since both trains travel the same distance in the same amount of time, they have the same average velocity: 90 mph.

3. Estimate $f(26)$, assuming that

$$f(25) = 43, \qquad f'(25) = 0.75$$

SOLUTION $f(x) \approx f(25) + f'(25)(x - 25)$, so $f(26) \approx 43 + 0.75(26 - 25) = 43.75$.

4. The population $P(t)$ of Freedonia in 2009 was $P(2009) = 5$ million.

(a) What is the meaning of $P'(2009)$?

(b) Estimate $P(2010)$ if $P'(2009) = 0.2$.

SOLUTION

(a) Because $P(t)$ measures the population of Freedonia as a function of time, the derivative $P'(2009)$ measures the rate of change of the population of Freedonia in the year 2009.

(b) $P(2010) \approx P(2009) + P'(2009)$. Thus, if $P'(2009) = 0.2$, then $P(2009) \approx 5.2$ million.

Exercises

In Exercises 1–8, find the rate of change.

1. Area of a square with respect to its side s when $s = 5$

SOLUTION Let the area be $A = f(s) = s^2$. Then the rate of change of A with respect to s is $d/ds(s^2) = 2s$. When $s = 5$, the area changes at a rate of 10 square units per unit increase. (Draw a 5×5 square on graph paper and trace the area added by increasing each side length by 1, excluding the corner, to see what this means.)

2. Volume of a cube with respect to its side s when $s = 5$

SOLUTION Let the volume be $V = f(s) = s^3$. Then the rate of change of V with respect to s is $\dfrac{d}{ds}s^3 = 3s^2$. When $s = 5$, the volume changes at a rate of $3(5^2) = 75$ cubic units per unit increase.

3. Cube root $\sqrt[3]{x}$ with respect to x when $x = 1, 8, 27$

SOLUTION Let $f(x) = \sqrt[3]{x}$. Writing $f(x) = x^{1/3}$, we see the rate of change of $f(x)$ with respect to x is given by $f'(x) = \frac{1}{3}x^{-2/3}$. The requested rates of change are given in the table that follows:

c	ROC of $f(x)$ with respect to x at $x = c$.
1	$f'(1) = \frac{1}{3}(1) = \frac{1}{3}$
8	$f'(8) = \frac{1}{3}(8^{-2/3}) = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$
27	$f'(27) = \frac{1}{3}(27^{-2/3}) = \frac{1}{3}\left(\frac{1}{9}\right) = \frac{1}{27}$

4. The reciprocal $1/x$ with respect to x when $x = 1, 2, 3$

SOLUTION Let $f(x) = x^{-1}$. The rate of change of $f(x)$ with respect to x is given by $f'(x) = -x^{-2}$. The requested rates of change are then -1 when $x = 1$, $-\frac{1}{4}$ when $x = 2$ and $-\frac{1}{9}$ when $x = 3$.

5. The diameter of a circle with respect to radius

SOLUTION The relationship between the diameter d of a circle and its radius r is $d = 2r$. The rate of change of the diameter with respect to the radius is then $d' = 2$.

6. Surface area A of a sphere with respect to radius r ($A = 4\pi r^2$)

SOLUTION Because $A = 4\pi r^2$, the rate of change of the surface area of a sphere with respect to the radius is $A' = 8\pi r$.

7. Volume V of a cylinder with respect to radius if the height is equal to the radius

SOLUTION The volume of the cylinder is $V = \pi r^2 h = \pi r^3$. Thus $dV/dr = 3\pi r^2$.

8. Speed of sound v (in m/s) with respect to air temperature T (in kelvins), where $v = 20\sqrt{T}$

SOLUTION Because, $v = 20\sqrt{T} = 20T^{1/2}$, the rate of change of the speed of sound with respect to temperature is $v' = 10T^{-1/2} = \frac{10}{\sqrt{T}}$.

In Exercises 9–11, refer to Figure 10, the graph of distance s from the origin as a function of time for a car trip.

9. Find the average velocity over each interval.

(a) [0, 0.5] **(b)** [0.5, 1] **(c)** [1, 1.5] **(d)** [1, 2]

SOLUTION

(a) The average velocity over the interval [0, 0.5] is

$$\frac{50-0}{0.5-0} = 100 \text{ km/hour}.$$

(b) The average velocity over the interval [0.5, 1] is

$$\frac{100-50}{1-0.5} = 100 \text{ km/hour}.$$

(c) The average velocity over the interval [1, 1.5] is

$$\frac{100-100}{1.5-1} = 0 \text{ km/hour}.$$

(d) The average velocity over the interval [1, 2] is

$$\frac{50-100}{2-1} = -50 \text{ km/hour}.$$

10. At what time is velocity at a maximum?

SOLUTION The velocity is maximum when the slope of the distance versus time curve is most positive. This appears to happen when $t = 0.5$ hours.

11. Match the descriptions (i)–(iii) with the intervals (a)–(c).
(i) Velocity increasing
(ii) Velocity decreasing
(iii) Velocity negative
(a) [0, 0.5]
(b) [2.5, 3]
(c) [1.5, 2]

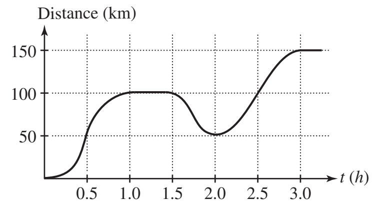


FIGURE 10 Distance from the origin versus time for a car trip.

SOLUTION
(a) (i) : The distance curve is increasing, and is also *bending* upward, so that distance is increasing at an increasing rate.
(b) (ii) : Over the interval [2.5, 3], the distance curve is flattening, showing that the car is slowing down; that is, the velocity is decreasing.
(c) (iii) : The distance curve is decreasing, so the tangent line has negative slope; this means the velocity is negative.

12. Use the data from Table 1 in Example 1 to calculate the average rate of change of Martian temperature T with respect to time t over the interval from 8:36 AM to 9:34 AM.

SOLUTION The time interval from 8:36 AM to 9:34 AM has length 58 minutes, and the change in temperature over this time interval is

$$\Delta T = -42 - (-47.7) = 5.7°\text{C}.$$

The average rate of change is then

$$\frac{\Delta T}{\Delta t} = \frac{5.7}{58} \approx 0.0983°\text{C/min} = 5.897°\text{C/hr}.$$

13. Use Figure 3 from Example 1 to estimate the instantaneous rate of change of Martian temperature with respect to time (in degrees Celsius per hour) at $t = 4$ AM.

SOLUTION The segment of the temperature graph around $t = 4$ AM appears to be a straight line passing through roughly (1:36, −70) and (4:48, −75). The instantaneous rate of change of Martian temperature with respect to time at $t = 4$ AM is therefore approximately

$$\frac{dT}{dt} = \frac{-75 - (-70)}{3.2} = -1.5625°\text{C/hour}.$$

14. The temperature (in degrees Celsius) of an object at time t (in minutes) is $T(t) = \frac{3}{8}t^2 - 15t + 180$ for $0 \le t \le 20$. At what rate is the object cooling at $t = 10$? (Give correct units.)

SOLUTION Given $T(t) = \frac{3}{8}t^2 - 15t + 180$, it follows that

$$T'(t) = \frac{3}{4}t - 15 \quad \text{and} \quad T'(10) = \frac{3}{4}(10) - 15 = -7.5°\text{C/min}.$$

At $t = 10$, the object is cooling at the rate of 7.5°C/min.

15. The velocity (in centimeters per second) of blood molecules flowing through a capillary of radius 0.008 cm is $v = 6.4 \times 10^{-8} - 0.001r^2$, where r is the distance from the molecule to the center of the capillary. Find the rate of change of velocity with respect to r when $r = 0.004$ cm.

SOLUTION The rate of change of the velocity of the blood molecules is $v'(r) = -0.002r$. When $r = 0.004$ cm, this rate is -8×10^{-6} 1/s.

16. Figure 11 displays the voltage V across a capacitor as a function of time while the capacitor is being charged. Estimate the rate of change of voltage at $t = 20$ s. Indicate the values in your calculation and include proper units. Does voltage change more quickly or more slowly as time goes on? Explain in terms of tangent lines.

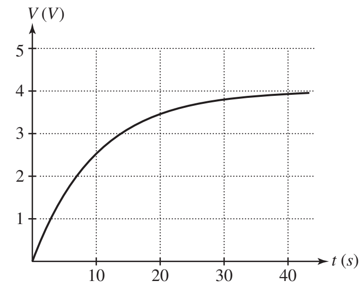
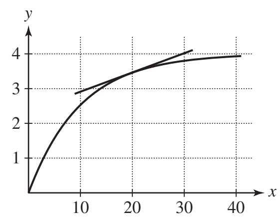


FIGURE 11

SOLUTION The tangent line sketched in the figure below appears to pass through the points (10, 3) and (30, 4). Thus, the rate of change of voltage at $t = 20$ seconds is approximately

$$\frac{4-3}{30-10} = 0.05 \text{ V/s}.$$

As we move to the right of the graph, the tangent lines to it grow shallower, indicating that the voltage changes more slowly as time goes on.



17. Use Figure 12 to estimate dT/dh at $h = 30$ and 70, where T is atmospheric temperature (in degrees Celsius) and h is altitude (in kilometers). Where is dT/dh equal to zero?

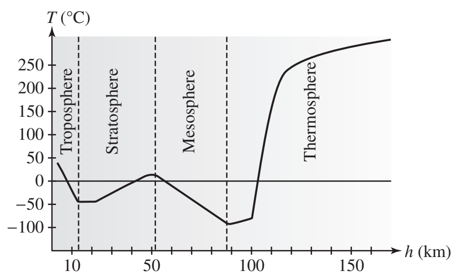


FIGURE 12 Atmospheric temperature versus altitude.

SOLUTION At $h = 30$ km, the graph of atmospheric temperature appears to be linear passing through the points $(23, -50)$ and (40, 0). The slope of this segment of the graph is then

$$\frac{0-(-50)}{40-23} = \frac{50}{17} = 2.94;$$

so

$$\left.\frac{dT}{dh}\right|_{h=30} \approx 2.94°\text{C/km}.$$

At $h = 70$ km, the graph of atmospheric temperature appears to be linear passing through the points (58, 0) and $(88, -100)$. The slope of this segment of the graph is then

$$\frac{-100-0}{88-58} = \frac{-100}{30} = -3.33;$$

so

$$\left.\frac{dT}{dh}\right|_{h=70} \approx -3.33^\circ\text{C/km}.$$

$\frac{dT}{dh} = 0$ at those points where the tangent line on the graph is horizontal. This appears to happen over the interval [13, 23], and near the points $h = 50$ and $h = 90$.

18. The earth exerts a gravitational force of $F(r) = (2.99 \times 10^{16})/r^2$ newtons on an object with a mass of 75 kg located r meters from the center of the earth. Find the rate of change of force with respect to distance r at the surface of the earth.

SOLUTION The rate of change of force is $F'(r) = -5.98 \times 10^{16}/r^3$. Therefore,

$$F'(6.77 \times 10^6) = -5.98 \times 10^{16}/(6.77 \times 10^6)^3 = -1.93 \times 10^{-4} \text{ N/m}.$$

19. Calculate the rate of change of escape velocity $v_{esc} = (2.82 \times 10^7)r^{-1/2}$ m/s with respect to distance r from the center of the earth.

SOLUTION The rate that escape velocity changes is $v'_{esc}(r) = -1.41 \times 10^7 r^{-3/2}$.

20. The power delivered by a battery to an apparatus of resistance R (in ohms) is $P = 2.25R/(R+0.5)^2$ watts (W). Find the rate of change of power with respect to resistance for $R = 3\ \Omega$ and $R = 5\ \Omega$.

SOLUTION

$$P'(R) = \frac{(R+0.5)^2 2.25 - 2.25R(2R+1)}{(R+0.5)^4}.$$

Therefore, $P'(3) = -0.1312$ W/Ω and $P'(5) = -0.0609$ W/Ω.

21. The position of a particle moving in a straight line during a 5-s trip is $s(t) = t^2 - t + 10$ cm. Find a time t at which the instantaneous velocity is equal to the average velocity for the entire trip beginning at $t = 0$.

SOLUTION Let $s(t) = t^2 - t + 10$, $0 \le t \le 5$, with s in centimeters (cm) and t in seconds (s). The average velocity over the t-interval [0, 5] is

$$\frac{s(5) - s(0)}{5 - 0} = \frac{30 - 10}{5} = 4 \text{ cm/s}.$$

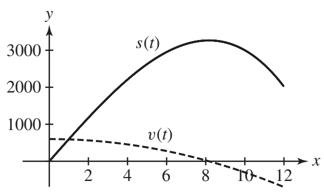
The (instantaneous) velocity is $v(t) = s'(t) = 2t - 1$. Solving $2t - 1 = 4$ yields $t = \frac{5}{2}$ s, the time at which the instantaneous velocity equals the calculated average velocity.

22. The height (in meters) of a helicopter at time t (in minutes) is $s(t) = 600t - 3t^3$ for $0 \le t \le 12$.

(a) Plot s and velocity v as functions of time.
(b) Find the velocity at $t = 8$ and $t = 10$.
(c) Find the maximum height of the helicopter.

SOLUTION

(a) With $s(t) = 600t - 3t^3$, it follows that $v(t) = 600 - 9t^2$. Plots of the position and the velocity are shown below.



(b) From part (a), we have $v(t) = 600 - 9t^2$. Thus, $v'(8) = 24$ meters/minute and $v'(10) = -300$ meters/minute.

(c) From the graph in part (a), we see that the helicopter achieves its maximum height when the velocity is zero. Solving $600 - 9t^2 = 0$ for t yields

$$t = \sqrt{\frac{600}{9}} = \frac{10}{3}\sqrt{6} \text{ minutes}.$$

The maximum height of the helicopter is then

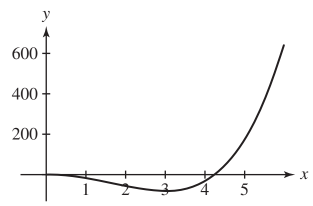
$$s\left(\frac{10}{3}\sqrt{6}\right) = \frac{4000}{3}\sqrt{6} \approx 3266 \text{ meters}.$$

23. A particle moving along a line has position $s(t) = t^4 - 18t^2$ m at time t seconds. At which times does the particle pass through the origin? At which times is the particle instantaneously motionless (i.e., it has zero velocity)?

SOLUTION The particle passes through the origin when $s(t) = t^4 - 18t^2 = t^2(t^2 - 18) = 0$. This happens when $t = 0$ seconds and when $t = 3\sqrt{2} \approx 4.24$ seconds. With $s(t) = t^4 - 18t^2$, it follows that $v(t) = s'(t) = 4t^3 - 36t = 4t(t^2 - 9)$. The particle is therefore instantaneously motionless when $t = 0$ seconds and when $t = 3$ seconds.

24. GU Plot the position of the particle in Exercise 23. What is the farthest distance to the left of the origin attained by the particle?

SOLUTION The plot of the position of the particle in Exercise 23 is shown below. Positive values of position correspond to distance to the right of the origin and negative values correspond to distance to the left of the origin. The most negative value of $s(t)$ occurs at $t = 3$ and is equal to $s(3) = 3^4 - 18(3)^2 = -81$. Thus, the particle achieves a maximum distance to the left of the origin of 81 meters.



25. A bullet is fired in the air vertically from ground level with an initial velocity 200 m/s. Find the bullet's maximum velocity and maximum height.

SOLUTION We employ Galileo's formula, $s(t) = s_0 + v_0t - \frac{1}{2}gt^2 = 200t - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). The velocity is $v(t) = 200 - 9.8t$. The maximum velocity of 200 m/s occurs at $t = 0$. This is the initial velocity. The bullet reaches its maximum height when $v(t) = 200 - 9.8t = 0$; i.e., when $t \approx 20.41$ s. At this point, the height is 2040.82 m.

26. Find the velocity of an air conditioner accidentally dropped from a height of 300 m at the moment it hits the ground.

SOLUTION We employ Galileo's formula, $s(t) = s_0 + v_0t - \frac{1}{2}gt^2 = 300 - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). When the air conditioner hits the ground its height is 0. Solve $s(t) = 300 - 4.9t^2 = 0$ to obtain $t \approx 7.8246$ s. (We discard the negative time, which took place before the air conditioner was dropped.) The velocity at impact is $v(7.8246) = -9.8(7.8246) \approx -76.68$ m/s. This signifies that the air conditioner is *falling* at 76.68 m/s.

27. A ball tossed in the air vertically from ground level returns to earth 4 s later. Find the initial velocity and maximum height of the ball.

SOLUTION Galileo's formula gives $s(t) = s_0 + v_0t - \frac{1}{2}gt^2 = v_0t - 4.9t^2$, where the time t is in seconds (s) and the height s is in meters (m). When the ball hits the ground after 4 seconds its height is 0. Solve $0 = s(4) = 4v_0 - 4.9(4)^2$ to obtain $v_0 = 19.6$ m/s. The ball reaches its maximum height when $s'(t) = 0$, that is, when $19.6 - 9.8t = 0$, or $t = 2$ s. At this time, $t = 2$ s,

$$s(2) = 0 + 19.6(2) - \frac{1}{2}(9.8)(4) = 19.6 \text{ m}.$$

28. Olivia is gazing out a window from the tenth floor of a building when a bucket (dropped by a window washer) passes by. She notes that it hits the ground 1.5 s later. Determine the floor from which the bucket was dropped if each floor is 5 m high and the window is in the middle of the tenth floor. Neglect air friction.

SOLUTION Suppose H is the unknown height from which the bucket fell starting at time $t = 0$. The height of the bucket at time t is $s(t) = H - 4.9t^2$. Let T be the time when the bucket hits the ground (thus $s(T) = 0$). Olivia saw the bucket at time $T - 1.5$. The window is located 9.5 floors or 47.5 m above ground. So we have the equations

$$s(T - 1.5) = H - 4.9(T - 1.5)^2 = 47.5 \quad \text{and} \quad s(T) = H - 4.9T^2 = 0$$

Subtracting the second equation from the first, we obtain $-4.9(-3T + 2.25) = 47.5$, so $T \approx 4$ s. The second equation gives us $H = 4.9T^2 = 4.9(4)^2 \approx 78.4$ m. Since there are 5 m in a floor, the bucket was dropped $78.4/5 \approx 15.7$ floors above the ground. The bucket was dropped from the top of the 15th floor.

29. Show that for an object falling according to Galileo's formula, the average velocity over any time interval $[t_1, t_2]$ is equal to the average of the instantaneous velocities at t_1 and t_2.

SOLUTION The simplest way to proceed is to compute both values and show that they are equal. The average velocity over $[t_1, t_2]$ is

$$\frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{(s_0 + v_0 t_2 - \frac{1}{2} g t_2^2) - (s_0 + v_0 t_1 - \frac{1}{2} g t_1^2)}{t_2 - t_1} = \frac{v_0(t_2 - t_1) + \frac{g}{2}(t_2{}^2 - t_1{}^2)}{t_2 - t_1}$$

$$= \frac{v_0(t_2 - t_1)}{t_2 - t_1} - \frac{g}{2}(t_2 + t_1) = v_0 - \frac{g}{2}(t_2 + t_1)$$

Whereas the average of the instantaneous velocities at the beginning and end of $[t_1, t_2]$ is

$$\frac{s'(t_1) + s'(t_2)}{2} = \frac{1}{2}\Big((v_0 - g t_1) + (v_0 - g t_2)\Big) = \frac{1}{2}(2v_0) - \frac{g}{2}(t_2 + t_1) = v_0 - \frac{g}{2}(t_2 + t_1).$$

The two quantities are the same.

30. An object falls under the influence of gravity near the earth's surface. Which of the following statements is true? Explain.

(a) Distance traveled increases by equal amounts in equal time intervals.

(b) Velocity increases by equal amounts in equal time intervals.

(c) The derivative of velocity increases with time.

SOLUTION For an object falling under the influence of gravity, Galileo's formula gives $s(t) = s_0 + v_0 t - \frac{1}{2} g t^2$.

(a) Since the height of the object varies quadratically with respect to time, it is *not* true that the object covers equal distance in equal time intervals.

(b) The velocity is $v(t) = s'(t) = v_0 - gt$. The velocity varies linearly with respect to time. Accordingly, the velocity decreases (becomes more negative) by equal amounts in equal time intervals. Moreover, its *speed* (the magnitude of velocity) increases by equal amounts in equal time intervals.

(c) Acceleration, the derivative of velocity with respect to time, is given by $a(t) = v'(t) = -g$. This is a *constant*; it does not change with time. Hence it is *not* true that acceleration (the derivative of velocity) increases with time.

31. By Faraday's Law, if a conducting wire of length ℓ meters moves at velocity v m/s perpendicular to a magnetic field of strength B (in teslas), a voltage of size $V = -B\ell v$ is induced in the wire. Assume that $B = 2$ and $\ell = 0.5$.

(a) Calculate dV/dv.

(b) Find the rate of change of V with respect to time t if $v(t) = 4t + 9$.

SOLUTION

(a) Assuming that $B = 2$ and $l = 0.5$, $V = -2(0.5)v = -v$. Therefore,

$$\frac{dV}{dv} = -1.$$

(b) If $v = 4t + 9$, then $V = -2(0.5)(4t + 9) = -(4t + 9)$. Therefore, $\frac{dV}{dt} = -4$.

32. The voltage V, current I, and resistance R in a circuit are related by Ohm's Law: $V = IR$, where the units are volts, amperes, and ohms. Assume that voltage is constant with $V = 12$ volts (V). Calculate (specifying units):

(a) The average rate of change of I with respect to R for the interval from $R = 8$ to $R = 8.1$

(b) The rate of change of I with respect to R when $R = 8$

(c) The rate of change of R with respect to I when $I = 1.5$

SOLUTION Let $V = IR$ or $I = V/R = 12/R$ (since we are assuming $V = 12$ volts).

(a) The average rate of change is

$$\frac{\Delta I}{\Delta R} = \frac{I(8.1) - I(8)}{8.1 - 8} = \frac{\frac{12}{8.1} - \frac{12}{8}}{0.1} \approx -0.185 \text{ A}/\Omega.$$

(b) $dI/dR = -12/R^2 = -12/8^2 = -0.1875$ A/Ω.

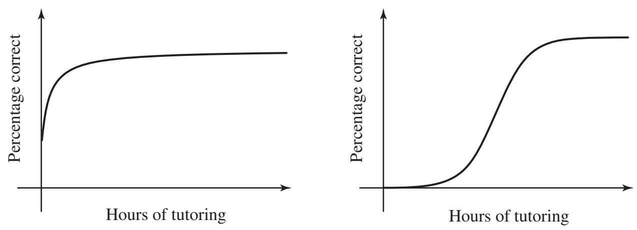
(c) With $R = 12/I$, we have $dR/dI = -12/I^2 = -12/1.5^2 \approx -5.33$ Ω/A.

33. Ethan finds that with h hours of tutoring, he is able to answer correctly $S(h)$ percent of the problems on a math exam. Which would you expect to be larger: $S'(3)$ or $S'(30)$? Explain.

SOLUTION The derivative $S'(h)$ measures the rate at which the percent of problems Ethan answers correctly changes with respect to the number of hours of tutoring he receives.

One possible graph of $S(h)$ is shown in the figure below on the left. This graph indicates that in the early hours of working with the tutor, Ethan makes rapid progress in learning the material but eventually approaches either the limit of his ability to learn the material or the maximum possible score on the exam. In this scenario, $S'(3)$ would be larger than $S'(30)$.

An alternative graph of $S(h)$ is shown below on the right. Here, in the early hours of working with the tutor little progress is made (perhaps the tutor is assessing how much Ethan already knows, his learning style, his personality, etc.). This is followed by a period of rapid improvement and finally a leveling off as Ethan reaches his maximum score. In this scenario, $S'(3)$ and $S'(30)$ might be roughly equal.



34. Suppose $\theta(t)$ measures the angle between a clock's minute and hour hands. What is $\theta'(t)$ at 3 o'clock?

SOLUTION The minute hand makes one full revolution every 60 minutes, so the minute hand moves at a rate of

$$\frac{2\pi}{60} = \frac{\pi}{30} \text{ rad/min.}$$

The hour hand makes one-twelfth of a revolution every 60 minutes, so the hour hand moves with a rate of

$$\frac{\pi}{360} \text{ rad/min.}$$

At 3 o'clock, the movement of the minute hand works to decrease the angle between the minute and hour hands while the movement of the hour hand works to increase the angle. Therefore, at 3 o'clock,

$$\theta'(t) = \frac{\pi}{360} - \frac{\pi}{30} = -\frac{11\pi}{360} \text{ rad/min.}$$

35. To determine drug dosages, doctors estimate a person's body surface area (BSA) (in meters squared) using the formula BSA $= \sqrt{hm}/60$, where h is the height in centimeters and m the mass in kilograms. Calculate the rate of change of BSA with respect to mass for a person of constant height $h = 180$. What is this rate at $m = 70$ and $m = 80$? Express your result in the correct units. Does BSA increase more rapidly with respect to mass at lower or higher body mass?

SOLUTION Assuming constant height $h = 180$ cm, let $f(m) = \sqrt{hm}/60 = \frac{\sqrt{5}}{10}\sqrt{m}$ be the formula for body surface area in terms of weight. The rate of change of BSA with respect to mass is

$$f'(m) = \frac{\sqrt{5}}{10}\left(\frac{1}{2}m^{-1/2}\right) = \frac{\sqrt{5}}{20\sqrt{m}}.$$

If $m = 70$ kg, this is

$$f'(70) = \frac{\sqrt{5}}{20\sqrt{70}} = \frac{\sqrt{14}}{280} \approx 0.0133631\frac{\text{m}^2}{\text{kg}}.$$

If $m = 80$ kg,

$$f'(80) = \frac{\sqrt{5}}{20\sqrt{80}} = \frac{1}{20\sqrt{16}} = \frac{1}{80}\frac{\text{m}^2}{\text{kg}}.$$

Because the rate of change of BSA depends on $1/\sqrt{m}$, it is clear that BSA increases more rapidly at lower body mass.

36. The atmospheric CO_2 level $A(t)$ at Mauna Loa, Hawaii, at time t (in parts per million by volume) is recorded by the Scripps Institution of Oceanography. Reading across, the annual values for the 4-year intervals are

1960	1964	1968	1972	1976	1980	1984
0.54	0.28	1.03	1.69	1.02	1.73	1.36
1988	1992	1996	2000	2004	2008	2012
2.13	0.48	1.25	1.62	1.56	1.60	2.66

(a) Estimate $A'(t)$ in 1962, 1970, 1978, 1986, 1994, 2002, and 2010.

(b) In which of the years from (a) did the approximation to $A'(t)$ take on its largest and smallest values?

(c) In which of these years does the approximation suggest that the CO_2 level was the most constant?

SOLUTION

(a) Using the data in the table, we estimate the values of $A'(t)$ as

$$A'(1962) \approx \frac{A(1964) - A(1960)}{4} = \frac{0.28 - 0.54}{4} = -0.065 \text{ parts per million per year;}$$
$$A'(1970) \approx \frac{A(1972) - A(1968)}{4} = \frac{1.69 - 1.03}{4} = 0.165 \text{ parts per million per year;}$$
$$A'(1978) \approx \frac{A(1980) - A(1976)}{4} = \frac{1.73 - 1.02}{4} = 0.1775 \text{ parts per million per year;}$$
$$A'(1986) \approx \frac{A(1988) - A(1984)}{4} = \frac{2.13 - 1.36}{4} = 0.1925 \text{ parts per million per year;}$$
$$A'(1994) \approx \frac{A(1996) - A(1992)}{4} = \frac{1.25 - 0.48}{4} = 0.1925 \text{ parts per million per year;}$$
$$A'(2002) \approx \frac{A(2004) - A(2000)}{4} = \frac{1.56 - 1.62}{4} = -0.015 \text{ parts per million per year;}$$
$$A'(2010) \approx \frac{A(2012) - A(2008)}{4} = \frac{2.66 - 1.60}{4} = 0.265 \text{ parts per million per year.}$$

(b) From part (a), the largest approximation is $A'(2010)$, and the smallest approximation is $A'(1962)$.

(c) The CO_2 level was most constant in 2002 because the approximate rate of change in that year was nearest to zero.

37. The tangent lines to the graph of $f(x) = x^2$ grow steeper as x increases. At what rate do the slopes of the tangent lines increase?

SOLUTION Let $f(x) = x^2$. The slopes s of the tangent lines are given by $s = f'(x) = 2x$. The rate at which these slopes are increasing is $ds/dx = 2$.

38. Figure 13 shows the height y of a mass oscillating at the end of a spring, through one cycle of the oscillation. Sketch the graph of velocity as a function of time.

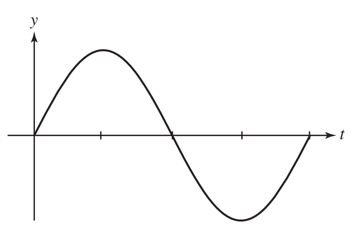
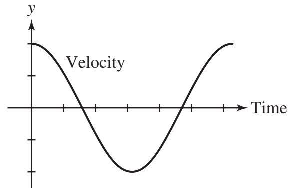


FIGURE 13

SOLUTION The position graph appears to break into four equal-sized components. Over the first quarter of the time interval, the position graph is rising but bending downward, eventually reaching a horizontal tangent. Thus, over the first quarter of the time interval, the velocity is positive but decreasing, eventually reaching 0. Continuing to examine the structure of the position graph produces the following graph of velocity:



In Exercises 39–46, use Eq. (3) to estimate the unit change.

39. Estimate $\sqrt{2}-\sqrt{1}$ and $\sqrt{101}-\sqrt{100}$. Compare your estimates with the actual values.

SOLUTION Let $f(x)=\sqrt{x}=x^{1/2}$. Then $f'(x)=\frac{1}{2}x^{-1/2}$. We are using the derivative to estimate the average rate of change. That is,

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}\approx f'(x),$$

so that

$$\sqrt{x+h}-\sqrt{x}\approx hf'(x).$$

Thus, $\sqrt{2}-\sqrt{1}\approx 1f'(1)=\frac{1}{2}(1)=\frac{1}{2}$. The actual value, to six decimal places, is 0.414214. Also, $\sqrt{101}-\sqrt{100}\approx 1f'(100)=\frac{1}{2}\left(\frac{1}{10}\right)=0.05$. The actual value, to six decimal places, is 0.0498756.

40. Estimate $f(4)-f(3)$ if $f'(x)=2^{-x}$. Then estimate $f(4)$, assuming that $f(3)=12$.

SOLUTION Using the estimate that

$$\frac{f(x+h)-f(x)}{h}\approx f'(x),$$

so that $f(x+h)-f(x)\approx f'(x)h$, with $x=3$ and $h=1$, we get

$$f(4)-f(3)\approx 2^{-3}(1)=\frac{1}{8}.$$

If $f(3)=12$, then $f(4)\approx 12\frac{1}{8}=\frac{97}{8}$.

41. Let $F(s)=1.1s+0.05s^2$ be the stopping distance as in Example 3. Calculate $F(65)$ and estimate the increase in stopping distance if speed is increased from 65 to 66 mph. Compare your estimate with the actual increase.

SOLUTION Let $F(s)=1.1s+0.05s^2$ be as in Example 3. $F'(s)=1.1+0.1s$.

- Then $F(65)=282.75$ ft and $F'(65)=7.6$ ft/mph.
- $F'(65)\approx F(66)-F(65)$ is approximately equal to the change in stopping distance per 1 mph increase in speed when traveling at 65 mph. Increasing speed from 65 to 66 therefore increases stopping distance by approximately 7.6 ft.
- The actual increase in stopping distance when speed increases from 65 mph to 66 mph is $F(66)-F(65)=290.4-282.75=7.65$ feet, which differs by less than one percent from the estimate found using the derivative.

42. According to Kleiber's Law, the metabolic rate P (in kilocalories per day) and body mass m (in kilograms) of an animal are related by a *three-quarter-power law* $P=73.3m^{3/4}$. Estimate the increase in metabolic rate when body mass increases from 60 to 61 kg.

SOLUTION Let $P(m)=73.3m^{3/4}$ be the function relating body mass m to metabolic rate P. Then,

$$P'(m)=\frac{3}{4}(73.3)m^{-1/4}=54.975m^{-1/4}$$
$$P(61)-P(60)\approx P'(60)=54.975(60^{-1/4})=19.7527.$$

As body mass is increased from 60 to 61 kg, metabolic rate is increased by approximately 19.7527 kcal/day.

43. The dollar cost of producing x bagels is given by the function $C(x)=300+0.25x-0.5(x/1000)^3$. Determine the cost of producing 2000 bagels and estimate the cost of the 2001st bagel. Compare your estimate with the actual cost of the 2001st bagel.

SOLUTION Expanding the power of 3 yields

$$C(x) = 300 + 0.25x - 5 \times 10^{-10}x^3.$$

This allows us to get the derivative $C'(x) = 0.25 - 1.5 \times 10^{-9}x^2$. The cost of producing 2000 bagels is

$$C(2000) = 300 + 0.25(2000) - 0.5(2000/1000)^3 = 796$$

dollars. The cost of the 2001st bagel is, by definition, $C(2001) - C(2000)$. By the derivative estimate, $C(2001) - C(2000) \approx C'(2000)(1)$, so the cost of the 2001st bagel is approximately

$$C'(2000) = .25 - 1.5 \times 10^{-9}(2000^2) = \$0.244.$$

$C(2001) = 796.244$, so the *exact* cost of the 2001st bagel is indistinguishable from the estimated cost. The function is very nearly linear at this point.

44. Suppose the dollar cost of producing x video cameras is $C(x) = 500x - 0.003x^2 + 10^{-8}x^3$.

(a) Estimate the marginal cost at production level $x = 5000$ and compare it with the actual cost $C(5001) - C(5000)$.

(b) Compare the marginal cost at $x = 5000$ with the average cost per camera, defined as $C(x)/x$.

SOLUTION Let $C(x) = 500x - 0.003x^2 + 10^{-8}x^3$. Then

$$C'(x) = 500 - 0.006x + (3 \times 10^{-8})x^2.$$

(a) The cost difference is approximately $C'(5000) = 470.75$. The actual cost is $C(5001) - C(5000) = 470.747$, which is quite close to the marginal cost computed using the derivative.

(b) The average cost per camera is

$$\frac{C(5000)}{5000} = \frac{2426250}{5000} = 485.25,$$

which is slightly higher than the marginal cost.

45. Demand for a commodity generally decreases as the price is raised. Suppose that the demand for oil (per capita per year) is $D(p) = 900/p$ barrels, where p is the dollar price per barrel. Find the demand when $p = \$40$. Estimate the decrease in demand if p rises to \$41 and the increase if p declines to \$39.

SOLUTION $D(p) = 900p^{-1}$, so $D'(p) = -900p^{-2}$. When the price is \$40 a barrel, the per capita demand is $D(40) = 22.5$ barrels per year. With an increase in price from \$40 to \$41 a barrel, the change in demand $D(41) - D(40)$ is approximately $D'(40) = -900(40^{-2}) = -0.5625$ barrels a year. With a decrease in price from \$40 to \$39 a barrel, the change in demand $D(39) - D(40)$ is approximately $-D'(40) = +0.5625$. An increase in oil prices of a dollar leads to a decrease in demand of 0.5625 barrels a year, and a decrease of a dollar leads to an *increase* in demand of 0.5625 barrels a year.

46. The reproduction rate f of the fruit fly *Drosophila melanogaster*, grown in bottles in a laboratory, decreases with the number p of flies in the bottle. A researcher has found the number of offspring per female per day to be approximately $f(p) = (34 - 0.612p)p^{-0.658}$.

(a) Calculate $f(15)$ and $f'(15)$.

(b) Estimate the decrease in daily offspring per female when p is increased from 15 to 16. Is this estimate larger or smaller than the actual value $f(16) - f(15)$?

(c) GU Plot f for $5 \le p \le 25$ and verify that $f(p)$ is a decreasing function of p. Do you expect $f'(p)$ to be positive or negative? Plot f' and confirm your expectation.

SOLUTION Let

$$f(p) = (34 - 0.612p)p^{-0.658} = 34p^{-0.658} - 0.612p^{0.342}.$$

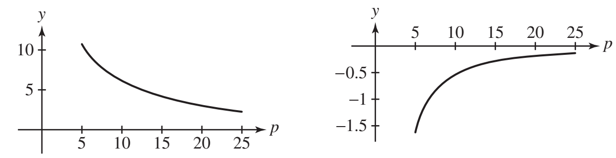
Then

$$f'(p) = -22.372p^{-1.658} - 0.209304p^{-0.658}.$$

(a) $f(15) = 34(15)^{-0.658} - 0.612(15)^{0.342} \approx 4.17767$ offspring per female per day and $f'(15) = -22.372(15)^{-1.658} - 0.209304(15)^{-.658} \approx -.28627$ offspring per female per day per fly.

(b) $f(16) - f(15) \approx f'(15) \approx -0.28627$. The decrease in daily offspring per female is estimated at 0.28627. $f(16) - f(15) = -0.272424$. The actual decrease in daily offspring per female is 0.272424. The actual decrease in daily offspring per female is less than the estimated decrease. This is because the graph of the function bends towards the x axis.

(c) The function $f(p)$ is plotted below at the left and is clearly a decreasing function of p; we therefore expect that $f'(p)$ will be negative. The plot of the derivative shown below at the right confirms our expectation.



47. According to Stevens' Law in psychology, the perceived magnitude of a stimulus is proportional (approximately) to a power of the actual intensity I of the stimulus. Experiments show that the *perceived brightness* B of a light satisfies $B = kI^{2/3}$, where I is the light intensity, whereas the *perceived heaviness* H of a weight W satisfies $H = kW^{3/2}$ (k is a constant that is different in the two cases). Compute dB/dI and dH/dW and state whether they are increasing or decreasing functions. Then explain the following statements:

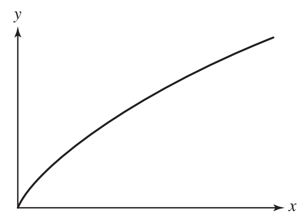
(a) A 1-unit increase in light intensity is felt more strongly when I is small than when I is large.

(b) Adding another pound to a load W is felt more strongly when W is large than when W is small.

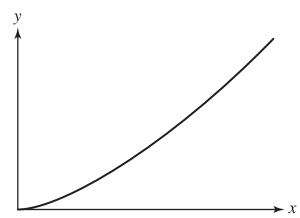
SOLUTION

(a) $dB/dI = \dfrac{2k}{3} I^{-1/3} = \dfrac{2k}{3I^{1/3}}$.

As I increases, dB/dI shrinks, so that the rate of change of perceived intensity decreases as the actual intensity increases. Increased light intensity has a *diminished return* in perceived intensity. A sketch of B against I is shown: See that the height of the graph increases more slowly as you move to the right.



(b) $dH/dW = \frac{3k}{2} W^{1/2}$. As W increases, dH/dW increases as well, so that the rate of change of perceived weight increases as weight increases. A sketch of H against W is shown: See that the graph becomes steeper as you move to the right.



48. Let $M(t)$ be the mass (in kilograms) of a plant as a function of time (in years). Recent studies by Niklas and Enquist have suggested that a remarkably wide range of plants (from algae and grass to palm trees) obey a *three-quarter-power growth law*—that is,

$$\frac{dM}{dt} = CM^{3/4} \quad \text{for some constant } C$$

(a) If a tree has a growth rate of 6 kg/year when $M = 100$ kg, what is its growth rate when $M = 125$ kg?

(b) If $M = 0.5$ kg, how much more mass must the plant acquire to double its growth rate?

SOLUTION

(a) Suppose a tree has a growth rate dM/dt of 6 kg/yr when $M = 100$, then $6 = C(100^{3/4}) = 10C\sqrt{10}$, so that $C = \frac{3\sqrt{10}}{50}$. When $M = 125$,

$$\frac{dM}{dt} = C(125^{3/4}) = \frac{3\sqrt{10}}{50} 25(5^{1/4}) = 7.09306 \text{ kg/yr}.$$

(b) The growth rate when $M = 0.5$ kg is $dM/dt = C(0.5^{3/4})$. To double the rate, we must find M so that $dM/dt = CM^{3/4} = 2C(0.5^{3/4})$. We solve for M.

$$CM^{3/4} = 2C(0.5^{3/4})$$
$$M^{3/4} = 2(0.5^{3/4})$$
$$M = (2(0.5^{3/4}))^{4/3} = 1.25992.$$

The plant must acquire the difference $1.25992 - 0.5 = 0.75992$ kg in order to double its growth rate.

Note that a doubling of growth rate requires *more* than a doubling of mass.

Further Insights and Challenges

Exercises 49–51: The ***Lorenz curve*** $y = F(r)$ *is used by economists to study income distribution in a given country (see Figure 14). By definition,* $F(r)$ *is the fraction of the total income that goes to the bottom* r*th part of the population, where* $0 \le r \le 1$. *For example, if* $F(0.4) = 0.245$, *then the bottom* 40% *of households receive* 24.5% *of the total income. Note that* $F(0) = 0$ *and* $F(1) = 1$.

49. Our goal is to find an interpretation for $F'(r)$. The average income for a group of households is the total income going to the group divided by the number of households in the group. The national average income is $A = T/N$, where N is the total number of households and T is the total income earned by the entire population.

(a) Show that the average income among households in the bottom rth part is equal to $(F(r)/r)A$.

(b) Show more generally that the average income of households belonging to an interval $[r, r + \Delta r]$ is equal to

$$\left(\frac{F(r + \Delta r) - F(r)}{\Delta r}\right) A$$

(c) Let $0 \le r \le 1$. A household belongs to the $100r$th percentile if its income is greater than or equal to the income of $100r$ % of all households. Pass to the limit as $\Delta r \to 0$ in (b) to derive the following interpretation: A household in the $100r$th percentile has income $F'(r)A$. In particular, a household in the $100r$th percentile receives more than the national average if $F'(r) > 1$ and less if $F'(r) < 1$.

(d) For the Lorenz curves L_1 and L_2 in Figure 14(B), what percentage of households have above-average income?

SOLUTION

(a) The total income among households in the bottom rth part is $F(r)T$ and there are rN households in this part of the population. Thus, the average income among households in the bottom rth part is equal to

$$\frac{F(r)T}{rN} = \frac{F(r)}{r} \cdot \frac{T}{N} = \frac{F(r)}{r}A.$$

(b) Consider the interval $[r, r + \Delta r]$. The total income among households between the bottom rth part and the bottom $r + \Delta r$-th part is $F(r + \Delta r)T - F(r)T$. Moreover, the number of households covered by this interval is $(r + \Delta r)N - rN = \Delta rN$. Thus, the average income of households belonging to an interval $[r, r + \Delta r]$ is equal to

$$\frac{F(r + \Delta r)T - F(r)T}{\Delta rN} = \frac{F(r + \Delta r) - F(r)}{\Delta r} \cdot \frac{T}{N} = \frac{F(r + \Delta r) - F(r)}{\Delta r}A.$$

(c) Take the result from part (b) and let $\Delta r \to 0$. Because

$$\lim_{\Delta r \to 0} \frac{F(r + \Delta r) - F(r)}{\Delta r} = F'(r),$$

we find that a household in the $100r$th percentile has income $F'(r)A$.

(d) The point P in Figure 14(B) has an r-coordinate of 0.6, while the point Q has an r-coordinate of roughly 0.75. Thus, on curve L_1, 40% of households have $F'(r) > 1$ and therefore have above-average income. On curve L_2, roughly 25% of households have above-average income.

50. The following table provides values of $F(r)$ for the United States in 2010. Assume that the national average income was $A = \$66{,}000$.

r	0	0.2	0.4	0.6	0.8	1
$F(r)$	0	0.033	0.118	0.264	0.480	1

(a) What was the average income in the lowest 40% of households?

(b) Show that the average income of the households belonging to the interval [0.4, 0.6] was \$48,180.

(c) Estimate $F'(0.5)$. Estimate the income of households in the 50th percentile? Was it greater or less than the national average?

SOLUTION

(a) The average income in the lowest 40% of households is $F'(0.4)A = 0.245(30{,}000) = 7350$ euros.

(b) The average income of the households belonging to the interval [0.4, 0.6] is

$$\frac{F(0.6) - F(0.4)}{0.2}A = \frac{0.423 - 0.245}{0.2}(30{,}000) = (0.89)(30{,}000) = 26700$$

euros.

(c) We estimate

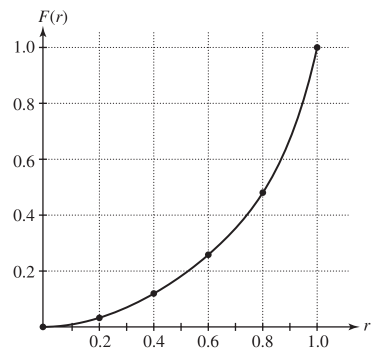
$$F'(0.5) \approx \frac{F(0.6) - F(0.4)}{0.2} = \frac{0.423 - 0.245}{0.2} = 0.89.$$

The income of households in the 50th percentile is then $F'(0.5)A = 0.89(30{,}000) = 26{,}700$ euros, which is less than the national average.

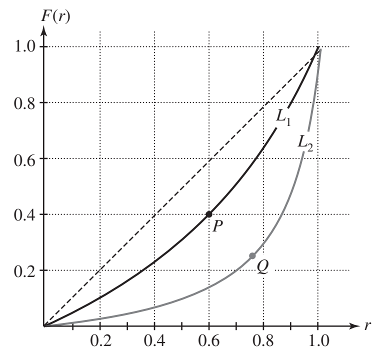
51. Use Exercise 49(c) to prove:

(a) $F'(r)$ is an increasing function of r.

(b) Income is distributed equally (all households have the same income) if and only if $F(r) = r$ for $0 \le r \le 1$.



(A) Lorenz curve for the United States in 2010



(B) Two Lorenz curves: The tangent lines at P and Q have slope 1.

FIGURE 14

SOLUTION

(a) Recall from Exercise 49 (c) that $F'(r)A$ is the income of a household in the $100r$-th percentile. Suppose $0 \le r_1 < r_2 \le 1$. Because $r_2 > r_1$, a household in the $100r_2$-th percentile must have income at least as large as a household in the $100r_1$-th percentile. Thus, $F'(r_2)A \ge F'(r_1)A$, or $F'(r_2) \ge F'(r_1)$. This implies $F'(r)$ is an increasing function of r.

(b) If $F(r) = r$ for $0 \le r \le 1$, then $F'(r) = 1$ and households in all percentiles have income equal to the national average; that is, income is distributed equally. Alternately, if income is distributed equally (all households have the same income), then $F'(r) = 1$ for $0 \le r \le 1$. Thus, F must be a linear function in r with slope 1. Moreover, the condition $F(0) = 0$ requires the F intercept of the line to be 0. Hence, $F(r) = 1 \cdot r + 0 = r$.

52. CAS Studies of Internet usage show that Web site popularity is described quite well by Zipf's Law, according to which the nth most popular Web site receives roughly the fraction $1/n$ of all visits. Suppose that on a particular day, the nth most popular site had approximately $V(n) = 10^6/n$ visitors (for $n \le 15{,}000$).

(a) Verify that the top 50 Web sites received nearly 45% of the visits. *Hint:* Let $T(N)$ denote the sum of $V(n)$ for $1 \le n \le N$. Use a computer algebra system to compute $T(50)$ and $T(15{,}000)$.

(b) Verify, by numerical experimentation, that when Eq. (3) is used to estimate $V(n+1) - V(n)$, the error in the estimate decreases as n grows larger. Find (again, by experimentation) an N such that the error is at most 10 for $n \ge N$.

(c) Using Eq. (3), show that for $n \ge 100$, the nth Web site received at most 100 more visitors than the $(n+1)$st Web site.

SOLUTION

(a) In Mathematica, using the command `Sum[10 ^ 6/n,{n,50}]` yields 4.49921×10^6 and the command `Sum[10 ^ 6/n,{n,15000}]` yields 1.01931×10^7. We see that the first 50 sites get around 4.5 million hits, which is nearly 45% of the 10.19 million hits of the first 15000 sites.

(b) We use `V[n_] := 10^6/n`, and compute the error $V(n+1) - V(n) - V'(n)$ for various values of n. The table of values computed follows:

n	10	20	30	40	50
$(V(n+1) - V(n)) - V'(n)$	909.091	119.048	35.8423	15.2489	7.84314

The error decreases in every entry. Furthermore, for $n > 50$, the error appears to be less than 10.

(c) Since $V(n) = 10^6 n^{-1}$, $V'(n) = -10^6 n^{-2}$. The marginal derivative estimate Eq. (3) tells us that

$$V(n) - V(n+1) \approx -V'(n) = 10^6 n^{-2}.$$

If $n \ge 100$, $-V'(n) \le 10^6(100)^{-2} = 10^6(10^{-4}) = 100$. Therefore $V(n) - V(n+1) < 100$ for $n \ge 100$.

In Exercises 53 and 54, the average cost per unit at production level x is defined as $C_{avg}(x) = C(x)/x$, where $C(x)$ is the cost of producing x units. Average cost is a measure of the efficiency of the production process.

53. Show that $C_{avg}(x)$ is equal to the slope of the line through the origin and the point $(x, C(x))$ on the graph of $y = C(x)$. Using this interpretation, determine whether average cost or marginal cost is greater at points A, B, C, D in Figure 15.

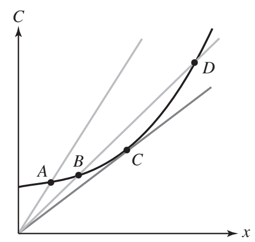


FIGURE 15 Graph of $y = C(x)$.

SOLUTION By definition, the slope of the line through the origin and $(x, C(x))$, that is, between $(0, 0)$ and $(x, C(x))$ is

$$\frac{C(x) - 0}{x - 0} = \frac{C(x)}{x} = C_{av}.$$

At point A, average cost is greater than marginal cost, as the line from the origin to A is steeper than the curve at this point (we see this because the line, tracing from the origin, crosses the curve from below). At point B, the average cost is still greater than the marginal cost. At the point C, the average cost and the marginal cost are nearly the same, since the tangent line and the line from the origin are nearly the same. The line from the origin to D crosses the cost curve from above, and so is less steep than the tangent line to the curve at D; the average cost at this point is less than the marginal cost.

54. The cost in dollars of producing alarm clocks is given by $C(x) = 50x^3 - 750x^2 + 3740x + 3750$, where x is in units of 1000.

(a) Calculate the average cost at $x = 4, 6, 8$, and 10.

(b) Use the graphical interpretation of average cost to find the production level x_0 at which average cost is lowest. What is the relation between average cost and marginal cost at x_0 (see Figure 16)?

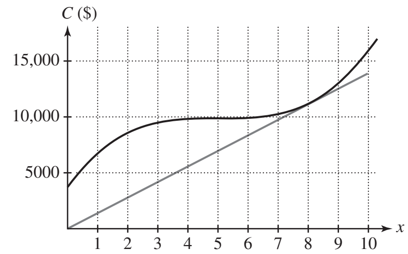


FIGURE 16 Cost function $C(x) = 50x^3 - 750x^2 + 3740x + 3750$.

SOLUTION Let $C(x) = 50x^3 - 750x^2 + 3740x + 3750$.

(a) The slope of the line through the origin and the point $(x, C(x))$ is

$$\frac{C(x) - 0}{x - 0} = \frac{C(x)}{x} = C_{\text{av}}(x),$$

the average cost.

x	4	6	8	10
$C(x)$	9910	9990	11270	16150
$C_{\text{av}}(x)$	2477.5	1665	1408.75	1615

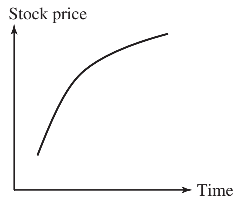
(b) The average cost is lowest at the point P_0 where the angle between the x-axis and the line through the origin and P_0 is lowest. This is at the point (8, 11270), where the line through the origin and the graph of $C(x)$ meet in the figure above. You can see that the line is also tangent to the graph of $C(x)$, so the average cost and the marginal cost are equal at this point.

3.5 Higher Derivatives

Preliminary Questions

1. On September 4, 2003, the *Wall Street Journal* printed the headline "Stocks Go Higher, Though the Pace of Their Gains Slows." Rephrase this headline as a statement about the first and second derivatives of stock prices and sketch a possible graph.

SOLUTION Because stocks are going higher, stock prices are increasing and the first derivative of stock prices must therefore be positive. On the other hand, because the pace of gains is slowing, the second derivative of stock prices must be negative.



2. True or false? The third derivative of position with respect to time is zero for an object falling to Earth under the influence of gravity. Explain.

SOLUTION This statement is true. The acceleration of an object falling to earth under the influence of gravity is constant; hence, the second derivative of position with respect to time is constant. Because the third derivative is just the derivative of the second derivative and the derivative of a constant is zero, it follows that the third derivative is zero.

3. Which type of polynomial satisfies $f'''(x) = 0$ for all x?

SOLUTION The third derivative of all quadratic polynomials (polynomials of the form $ax^2 + bx + c$ for some constants a, b and c) is equal to 0 for all x.

4. What is the millionth derivative of $f(x) = e^x$?

SOLUTION Every derivative of $f(x) = e^x$ is e^x.

Exercises

In Exercises 1–16, calculate y'' and y'''.

1. $y = 14x^2$

SOLUTION Let $y = 14x^2$. Then $y' = 28x$, $y'' = 28$, and $y''' = 0$.

2. $y = 7 - 2x$

SOLUTION Let $y = 7 - 2x$. Then $y' = -2$, $y'' = 0$, and $y''' = 0$.

3. $y = x^4 - 25x^2 + 2x$

SOLUTION Let $y = x^4 - 25x^2 + 2x$. Then $y' = 4x^3 - 50x + 2$, $y'' = 12x^2 - 50$, and $y''' = 24x$.

4. $y = 4t^3 - 9t^2 + 7$

SOLUTION Let $y = 4t^3 - 9t^2 + 7$. Then $y' = 12t^2 - 18t$, $y'' = 24t - 18$, and $y''' = 24$.

5. $y = \dfrac{4}{3}\pi r^3$

SOLUTION Let $y = \frac{4}{3}\pi r^3$. Then $y' = 4\pi r^2$, $y'' = 8\pi r$, and $y''' = 8\pi$.

6. $y = \sqrt{x}$

SOLUTION Let $y = \sqrt{x} = x^{1/2}$. Then $y' = \frac{1}{2}x^{-1/2}$, $y'' = -\frac{1}{4}x^{-3/2}$, and $y''' = \frac{3}{8}x^{-5/2}$.

7. $y = 20t^{4/5} - 6t^{2/3}$

SOLUTION Let $y = 20t^{4/5} - 6t^{2/3}$. Then $y' = 16t^{-1/5} - 4t^{-1/3}$, $y'' = -\frac{16}{5}t^{-6/5} + \frac{4}{3}t^{-4/3}$, and $y''' = \frac{96}{25}t^{-11/15} - \frac{16}{9}t^{-7/3}$.

8. $y = x^{-9/5}$

SOLUTION Let $y = x^{-9/5}$. Then $y' = -\frac{9}{5}x^{-14/5}$, $y'' = \frac{126}{25}x^{-19/5}$, and $y''' = -\frac{2394}{125}x^{-24/5}$.

9. $y = z - \dfrac{4}{z}$

SOLUTION Let $y = z - 4z^{-1}$. Then $y' = 1 + 4z^{-2}$, $y'' = -8z^{-3}$, and $y''' = 24z^{-4}$.

10. $y = 5t^{-3} + 7t^{-8/3}$

SOLUTION Let $y = 5t^{-3} + 7t^{-8/3}$. Then $y' = -15t^{-4} - \frac{56}{3}t^{-11/3}$, $y'' = 60t^{-5} + \frac{616}{9}t^{-14/3}$, and $y''' = -300t^{-6} - \frac{8624}{27}t^{-17/3}$.

11. $y = \theta^2(2\theta + 7)$

SOLUTION Let $y = \theta^2(2\theta + 7) = 2\theta^3 + 7\theta^2$. Then $y' = 6\theta^2 + 14\theta$, $y'' = 12\theta + 14$, and $y''' = 12$.

12. $y = (x^2 + x)(x^3 + 1)$

SOLUTION Since we don't want to apply the product rule to an ever growing list of products, we multiply through first. Let $y = (x^2 + x)(x^3 + 1) = x^5 + x^4 + x^2 + x$. Then $y' = 5x^4 + 4x^3 + 2x + 1$, $y'' = 20x^3 + 12x^2 + 2$, and $y''' = 60x^2 + 24x$.

13. $y = \dfrac{x - 4}{x}$

SOLUTION Let $y = \frac{x-4}{x} = 1 - 4x^{-1}$. Then $y' = 4x^{-2}$, $y'' = -8x^{-3}$, and $y''' = 24x^{-4}$.

14. $y = \dfrac{1}{1 - x}$

SOLUTION Let $y = \frac{1}{1-x}$. Applying the quotient rule:

$$y' = \frac{(1-x)(0) - 1(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} = \frac{1}{1 - 2x + x^2}$$

$$y'' = \frac{(1 - 2x + x^2)(0) - (1)(-2 + 2x)}{(1 - 2x + x^2)^2} = \frac{2 - 2x}{(1-x)^4} = \frac{2}{(1-x)^3} = \frac{2}{1 - 3x + 3x^2 - x^3}$$

$$y''' = \frac{(1 - 3x + 3x^2 - x^3)(0) - 2(-3 + 6x - 3x^2)}{(1 - 3x + 3x^2 - x^3)^2} = \frac{6(x^2 - 2x + 1)}{(1-x)^6} = \frac{6}{(1-x)^4}.$$

15. $y = x^5 e^x$

SOLUTION Let $y = x^5 e^x$. Then

$$\begin{aligned} y' &= x^5 e^x + 5x^4 e^x = (x^5 + 5x^4)e^x \\ y'' &= (x^5 + 5x^4)e^x + (5x^4 + 20x^3)e^x = (x^5 + 10x^4 + 20x^3)e^x \\ y''' &= (x^5 + 10x^4 + 20x^3)e^x + (5x^4 + 40x^3 + 60x^2)e^x = (x^5 + 15x^4 + 60x^3 + 60x^2)e^x. \end{aligned}$$

16. $y = \dfrac{e^x}{x}$

SOLUTION Let $y = \frac{e^x}{x} = x^{-1}e^x$. Then

$$\begin{aligned} y' &= x^{-1}e^x + e^x(-x^{-2}) = (x^{-1} - x^{-2})e^x \\ y'' &= (x^{-1} - x^{-2})e^x + e^x(-x^{-2} + 2x^{-3}) = (x^{-1} - 2x^{-2} + 2x^{-3})e^x \\ y''' &= (x^{-1} - 2x^{-2} + 2x^{-3})e^x + e^x(-x^{-2} + 4x^{-3} - 6x^{-4}) = (x^{-1} - 3x^{-2} + 6x^{-3} - 6x^{-4})e^x. \end{aligned}$$

In Exercises 17–26, calculate the derivative indicated.

17. $f^{(4)}(1), \quad f(x) = x^4$

SOLUTION Let $f(x) = x^4$. Then $f'(x) = 4x^3$, $f''(x) = 12x^2$, $f'''(x) = 24x$, and $f^{(4)}(x) = 24$. Thus $f^{(4)}(1) = 24$.

18. $g'''(-1), \quad g(t) = -4t^{-5}$

SOLUTION Let $g(t) = -4t^{-5}$. Then $g'(t) = 20t^{-6}$, $g''(t) = -120t^{-7}$, and $g'''(t) = 840t^{-8}$. Hence $g'''(-1) = 840$.

19. $\left.\dfrac{d^2y}{dt^2}\right|_{t=1}, \quad y = 4t^{-3} + 3t^2$

SOLUTION Let $y = 4t^{-3} + 3t^2$. Then $\frac{dy}{dt} = -12t^{-4} + 6t$ and $\frac{d^2y}{dt^2} = 48t^{-5} + 6$. Hence

$$\left.\frac{d^2y}{dt^2}\right|_{t=1} = 48(1)^{-5} + 6 = 54.$$

20. $\left.\dfrac{d^4f}{dt^4}\right|_{t=1}, \quad f(t) = 6t^9 - 2t^5$

SOLUTION Let $f(t) = 6t^9 - 2t^5$. Then $\frac{df}{dt} = 54t^8 - 10t^4$, $\frac{d^2f}{dt^2} = 432t^7 - 40t^3$, $\frac{d^3f}{dt^3} = 3024t^6 - 120t^2$, and $\frac{d^4f}{dt^4} = 18144t^5 - 240t$. Therefore,

$$\left.\frac{d^4f}{dt^4}\right|_{t=1} = 17904.$$

21. $\left.\dfrac{d^4x}{dt^4}\right|_{t=16}, \quad x = t^{-3/4}$

SOLUTION Let $x(t) = t^{-3/4}$. Then $\frac{dx}{dt} = -\frac{3}{4}t^{-7/4}$, $\frac{d^2x}{dt^2} = \frac{21}{16}t^{-11/4}$, $\frac{d^3x}{dt^3} = -\frac{231}{64}t^{-15/4}$, and $\frac{d^4x}{dt^4} = \frac{3465}{256}t^{-19/4}$. Thus

$$\left.\frac{d^4x}{dt^4}\right|_{t=16} = \frac{3465}{256}16^{-19/4} = \frac{3465}{134217728}.$$

22. $f'''(4), \quad f(t) = 2t^2 - t$

SOLUTION Since $f(t) = 2t^2 - t$, $f'(t) = 4t - 1$, $f''(t) = 4$, and $f'''(t) = 0$ for all t. In particular, $f'''(4) = 0$.

23. $f'''(-3), \quad f(x) = 4e^x - x^3$

SOLUTION Let $f(x) = 4e^x - x^3$. Then $f'(x) = 4e^x - 3x^2$, $f''(x) = 4e^x - 6x$, $f'''(x) = 4e^x - 6$, and $f'''(-3) = 4e^{-3} - 6$.

24. $f''(1), \quad f(t) = \dfrac{t}{t+1}$

SOLUTION Let $f(t) = \dfrac{t}{t+1}$. Then

$$f'(t) = \frac{(t+1)(1) - (t)(1)}{(t+1)^2} = \frac{1}{(t+1)^2} = \frac{1}{t^2 + 2t + 1}$$

and

$$f''(t) = \frac{(t^2 + 2t + 1)(0) - 1(2t + 2)}{(t^2 + 2t + 1)^2} = -\frac{2(t+1)}{(t+1)^4} = -\frac{2}{(t+1)^3}.$$

Thus, $f''(1) = -\dfrac{1}{4}$.

25. $h''(1), \quad h(w) = \sqrt{w}e^w$

SOLUTION Let $h(w) = \sqrt{w}e^w = w^{1/2}e^w$. Then

$$h'(w) = w^{1/2}e^w + e^w\left(\frac{1}{2}w^{-1/2}\right) = \left(w^{1/2} + \frac{1}{2}w^{-1/2}\right)e^w$$

and

$$h''(w) = \left(w^{1/2} + \frac{1}{2}w^{-1/2}\right)e^w + e^w\left(\frac{1}{2}w^{-1/2} - \frac{1}{4}w^{-3/2}\right) = \left(w^{1/2} + w^{-1/2} - \frac{1}{4}w^{-3/2}\right)e^w.$$

Thus, $h''(1) = \dfrac{7}{4}e$.

26. $g''(0), \quad g(s) = \dfrac{e^s}{s+1}$

SOLUTION Let $g(s) = \dfrac{e^s}{s+1}$. Then

$$g'(s) = \frac{(s+1)e^s - e^s(1)}{(s+1)^2} = \frac{se^s}{s^2 + 2s + 1}$$

and

$$g''(s) = \frac{(s^2 + 2s + 1)(se^s + e^s) - se^s(2s + 2)}{(s^2 + 2s + 1)^2} = \frac{(s^2 + 1)e^s}{(s+1)^3}.$$

Thus, $g''(0) = 1$.

27. Calculate $y^{(k)}(0)$ for $0 \le k \le 5$, where $y = x^4 + ax^3 + bx^2 + cx + d$ (with a, b, c, d the constants).

SOLUTION Applying the power, constant multiple, and sum rules at each stage, we get (note $y^{(0)}$ is y by convention):

k	$y^{(k)}$
0	$x^4 + ax^3 + bx^2 + cx + d$
1	$4x^3 + 3ax^2 + 2bx + c$
2	$12x^2 + 6ax + 2b$
3	$24x + 6a$
4	24
5	0

from which we get $y^{(0)}(0) = d$, $y^{(1)}(0) = c$, $y^{(2)}(0) = 2b$, $y^{(3)}(0) = 6a$, $y^{(4)}(0) = 24$, and $y^{(5)}(0) = 0$.

28. Which of the following satisfy $f^{(k)}(x) = 0$ for all $k \ge 6$?

(a) $f(x) = 7x^4 + 4 + x^{-1}$ **(b)** $f(x) = x^3 - 2$

(c) $f(x) = \sqrt{x}$ **(d)** $f(x) = 1 - x^6$

(e) $f(x) = x^{9/5}$ **(f)** $f(x) = 2x^2 + 3x^5$

SOLUTION Equations (b) and (f) go to zero after the sixth derivative. We don't have to take the derivatives to see this.

- Look at (a). $f'(x) = 28x^3 - x^{-2}$. Every time we take higher derivatives of $f(x)$, the negative exponent will keep decreasing, and will never become zero.
- In the case of (b), we see that every derivative decreases the degree (the highest exponent) of the polynomial by one, so that $f^{(4)}(x) = 0$.
- For (c), $f'(x) = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2}$. Every further derivative of $f(x)$ is going to make the exponent more negative, so that it will never go to zero.
- In the case of (d), like (b), the highest exponent will decrease with every derivative, but 6 derivatives will leave the exponent zero, $f^{(6)}(x)$ will be $-6!$. This is easy to verify.
- (e) is like (c). Since the exponent is not a whole number, successive derivatives will make the exponent "pass over" zero, and go to negative infinity.
- In the case of (f), $f^{(5)}(x)$ is constant, so that $f^{(6)}(x) = 0$ for all x.

29. Use the result in Example 3 to find $\dfrac{d^6}{dx^6}\, x^{-1}$.

SOLUTION The equation in Example 3 indicates that

$$\frac{d^6}{dx^6}\, x^{-1} = (-1)^6 6! x^{-6-1}.$$

$(-1)^6 = 1$ and $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$, so

$$\frac{d^6}{dx^6}\, x^{-1} = 720x^{-7}.$$

30. Calculate the first five derivatives of $f(x) = \sqrt{x}$.

(a) Show that $f^{(n)}(x)$ is a multiple of $x^{-n+1/2}$.

(b) Show that $f^{(n)}(x)$ alternates in sign as $(-1)^{n-1}$ for $n \geq 1$.

(c) Find a formula for $f^{(n)}(x)$ for $n \geq 2$. *Hint:* Verify that the coefficient is $\pm 1 \cdot 3 \cdot 5 \cdots \dfrac{2n-3}{2^n}$.

SOLUTION We use the Power Rule:

$$\frac{df}{dx} = \tfrac{1}{2}\, x^{-1/2} \qquad \frac{d^4 f}{dx^4} = -\tfrac{5}{2}(\tfrac{3}{2})(\tfrac{1}{2})(\tfrac{1}{2})\, x^{-7/2}$$

$$\frac{d^2 f}{dx^2} = -\tfrac{1}{2}(\tfrac{1}{2})\, x^{-3/2} \qquad \frac{d^5 f}{dx^5} = \tfrac{7}{2}(\tfrac{5}{2})(\tfrac{3}{2})(\tfrac{1}{2})(\tfrac{1}{2})\, x^{-9/2}$$

$$\frac{d^3 f}{dx^3} = \tfrac{3}{2}(\tfrac{1}{2})(\tfrac{1}{2})\, x^{-5/2} \qquad \frac{d^6 f}{dx^6} = -\tfrac{9}{2}(\tfrac{7}{2})(\tfrac{5}{2})(\tfrac{3}{2})(\tfrac{1}{2})(\tfrac{1}{2})\, x^{-11/2}$$

The pattern we see here is that the nth derivative is a multiple of $\pm x^{-n+\frac{1}{2}}$. Which multiple? The coefficient is the product of the odd numbers up to $2n-3$ divided by 2^n. Therefore we can write a general formula for the nth derivative as follows:

$$f^{(n)}(x) = (-1)^{n-1}\, \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n}\, x^{-n+\frac{1}{2}}$$ ■

In Exercises 31–36, find a general formula for $f^{(n)}(x)$.

31. $f(x) = x^{-2}$

SOLUTION $f'(x) = -2x^{-3}$, $f''(x) = 6x^{-4}$, $f'''(x) = -24x^{-5}$, $f^{(4)}(x) = 5 \cdot 24x^{-6}, \ldots$. From this we can conclude that the nth derivative can be written as $f^{(n)}(x) = (-1)^n (n+1)! x^{-(n+2)}$.

32. $f(x) = (x+2)^{-1}$

SOLUTION Let $f(x) = (x+2)^{-1} = \frac{1}{x+2}$. Then $f'(x) = -1(x+2)^{-2}$, $f''(x) = 2(x+2)^{-3}$, $f'''(x) = -6(x+2)^{-4}$, $f^{(4)}(x) = 24(x+2)^{-5}, \ldots$ From this we conclude that the nth derivative can be written as

$$f^{(n)}(x) = (-1)^n n!(x+2)^{-(n+1)}.$$

33. $f(x) = x^{-1/2}$

SOLUTION $f'(x) = \frac{-1}{2}x^{-3/2}$. We will avoid simplifying numerators and denominators to find the pattern:

$$f''(x) = \frac{-3}{2}\frac{-1}{2}x^{-5/2} = (-1)^2\frac{3 \times 1}{2^2}x^{-5/2}$$
$$f'''(x) = -\frac{5}{2}\frac{3 \times 1}{2^2}x^{-7/2} = (-1)^3\frac{5 \times 3 \times 1}{2^3}x^{-7/2}$$
$$\vdots$$
$$f^{(n)}(x) = (-1)^n\frac{(2n-1) \times (2n-3) \times \ldots \times 1}{2^n}x^{-(2n+1)/2}.$$

34. $f(x) = x^{-3/2}$

SOLUTION $f'(x) = \frac{-3}{2}x^{-5/2}$. We will avoid simplifying numerators and denominators to find the pattern:

$$f''(x) = \frac{-5}{2}\frac{-3}{2}x^{-7/2} = (-1)^2\frac{5 \times 3}{2^2}x^{-7/2}$$
$$f'''(x) = -\frac{7}{2}\frac{5 \times 3}{2^2}x^{-9/2} = (-1)^3\frac{7 \times 5 \times 3}{2^3}x^{-9/2}$$
$$\vdots$$
$$f^{(n)}(x) = (-1)^n\frac{(2n+1) \times (2n-1) \times \ldots \times 3}{2^n}x^{-(2n+3)/2}.$$

35. $f(x) = xe^{-x}$

SOLUTION Let $f(x) = xe^{-x}$. Then

$$f'(x) = x(-e^{-x}) + e^{-x} = (1-x)e^{-x} = -(x-1)e^{-x}$$
$$f''(x) = (1-x)(-e^{-x}) - e^{-x} = (x-2)e^{-x}$$
$$f'''(x) = (x-2)(-e^{-x}) + e^{-x} = (3-x)e^{-x} = -(x-3)e^{-x}$$

From this we conclude that the nth derivative can be written as $f^{(n)}(x) = (-1)^n(x-n)e^{-x}$.

36. $f(x) = x^2e^x$

SOLUTION Let $f(x) = x^2e^x$. Then

$$f'(x) = x^2e^x + 2xe^x = (x^2+2x)e^x$$
$$f''(x) = (x^2+2x)e^x + e^x(2x+2) = (x^2+4x+2)e^x$$
$$f'''(x) = (x^2+4x+2)e^x + e^x(2x+4) = (x^2+6x+6)e^x$$
$$f^{(4)}(x) = (x^2+6x+6)e^x + e^x(2x+6) = (x^2+8x+12)e^x$$

From this we conclude that the nth derivative can be written as $f^{(n)}(x) = (x^2+2nx+n(n-1))e^x$.

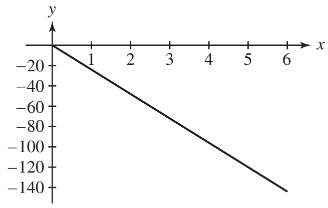
37. **(a)** Find the acceleration at time $t = 5$ min of a helicopter whose height is $s(t) = 300t - 4t^3$ m.

(b) Plot the acceleration s'' for $0 \le t \le 6$. How does this graph show that the helicopter is slowing down during this time interval?

SOLUTION

(a) Let $s(t) = 300t - 4t^3$, with t in minutes and s in meters. The velocity is $v(t) = s'(t) = 300 - 12t^2$ and acceleration is $a(t) = s''(t) = -24t$. Thus $a(5) = -120$ m/min^2.

(b) The acceleration of the helicopter for $0 \le t \le 6$ is shown in the figure below. As the acceleration of the helicopter is negative, the velocity of the helicopter must be decreasing. The velocity is positive for $0 \le t < 5$, so the helicopter is slowing down between 0 and 5 minutes; on the other hand, the velocity is negative for $5 < t \le 6$, so the helicopter is speeding up between 5 and 6 minutes.



38. Find an equation of the tangent to the graph of $y = f'(x)$ at $x = 3$, where $f(x) = x^4$.

SOLUTION Let $f(x) = x^4$ and $g(x) = f'(x) = 4x^3$. Then $g'(x) = 12x^2$. The tangent line to g at $x = 3$ is given by

$$y = g'(3)(x - 3) + g(3) = 108(x - 3) + 108 = 108x - 216.$$

39. Figure 5 shows f, f', and f''. Determine which is which.

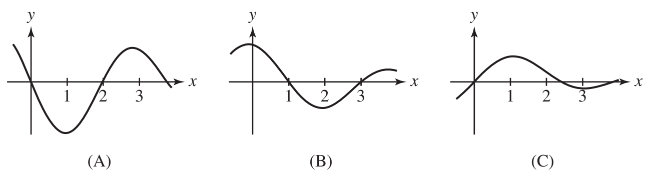


FIGURE 5

SOLUTION (a) f'' (b) f' (c) f.

The tangent line to (c) is horizontal at $x = 1$ and $x = 3$, where (b) has roots. The tangent line to (b) is horizontal at $x = 2$ and $x = 0$, where (a) has roots.

40. The second derivative f'' is shown in Figure 6. Which of (A) or (B) is the graph of f and which is f'?

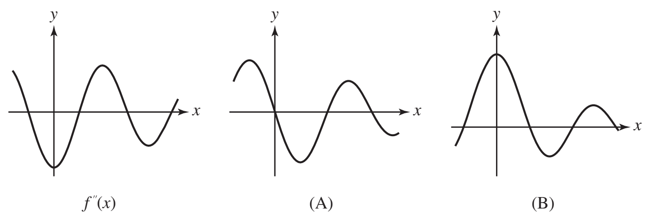


FIGURE 6

SOLUTION $f'(x) = A$ and $f(x) = B$.

41. Figure 7 shows the graph of the position s of an object as a function of time t. Determine the intervals on which the acceleration is positive.

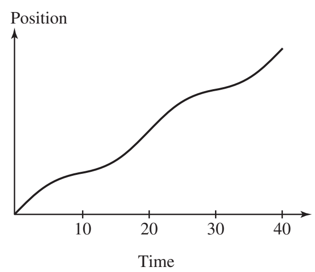


FIGURE 7

SOLUTION Roughly from time 10 to time 20 and from time 30 to time 40. The acceleration is positive over the same intervals over which the graph is bending upward.

42. Find a polynomial $f(x)$ that satisfies the equation $xf''(x) + f(x) = x^2$.

SOLUTION Since $xf''(x) + f(x) = x^2$, and x^2 is a polynomial, it seems reasonable to assume that $f(x)$ is a polynomial of some degree, call it n. The degree of $f''(x)$ is $n - 2$, so the degree of $xf''(x)$ is $n - 1$, and the degree of $xf''(x) + f(x)$ is n. Hence, $n = 2$, since the degree of x^2 is 2. Therefore,let $f(x) = ax^2 + bx + c$.

Then $f'(x) = 2ax + b$ and $f''(x) = 2a$. Substituting into the equation $xf''(x) + f(x) = x^2$ yields $ax^2 + (2a + b)x + c = x^2$, an identity in x. Equating coefficients, we have $a = 1$, $2a + b = 0$, $c = 0$. Therefore, $b = -2$ and $f(x) = x^2 - 2x$.

43. Find all values of n such that $y = x^n$ satisfies

$$x^2y'' - 2xy' = 4y$$

SOLUTION Let $y = x^n$. Then $y' = nx^{n-1}$, $y'' = n(n-1)x^{n-2}$, and

$$x^2y'' - 2xy' = n(n-1)x^n - 2nx^n.$$

In order for this last expression to be equal to $4y = 4x^n$, we must have

$$n(n-1) - 2n = 4 \qquad \text{or} \qquad n^2 - 3n - 4 = (n-4)(n+1) = 0.$$

Thus, $y = x^n$ satisfies the equation $x^2y'' - 2xy' = 4y$ for $n = 4$ and $n = -1$.

44. Which of the following descriptions could *not* apply to Figure 8? Explain.

(a) Graph of acceleration when velocity is constant

(b) Graph of velocity when acceleration is constant

(c) Graph of position when acceleration is zero

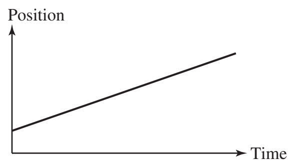


FIGURE 8

SOLUTION

(a) Does NOT apply to the figure because if $v(t) = C$ where C is a constant, then $a(t) = v'(t) = 0$, which is the horizontal line going through the origin.

(b) Can apply because the graph has a constant slope.

(c) Can apply because if we took this as a position graph, the velocity graph would be a horizontal line and thus, acceleration would be zero.

45. According to one model that takes into account air resistance, the acceleration $a(t)$ (in m/s^2) of a skydiver of mass m in free-fall satisfies

$$a(t) = -9.8 + \frac{k}{m}v(t)^2$$

where $v(t)$ is velocity (negative since the object is falling) and k is a constant. Suppose that $m = 75$ kg and $k = 14$ kg/m.

(a) What is the object's velocity when $a(t) = -4.9$?

(b) What is the object's velocity when $a(t) = 0$? This velocity is the object's terminal velocity.

SOLUTION

SOLUTION Solving $a(t) = -9.8 + \frac{k}{m}v(t)^2$ for the velocity and taking into account that the velocity is negative since the object is falling, we find

$$v(t) = -\sqrt{\frac{m}{k}(a(t) + 9.8)} = -\sqrt{\frac{75}{14}(a(t) + 9.8)}.$$

(a) Substituting $a(t) = -4.9$ into the above formula for the velocity, we find

$$v(t) = -\sqrt{\frac{75}{14}(4.9)} = -\sqrt{26.25} = -5.12 \text{ m/s}.$$

(b) When $a(t) = 0$,

$$v(t) = -\sqrt{\frac{75}{14}(9.8)} = -\sqrt{52.5} = -7.25 \text{ m/s}.$$

46. According to one model that attempts to account for air resistance, the distance $s(t)$ (in meters) traveled by a falling raindrop satisfies

$$\frac{d^2s}{dt^2} = g - \frac{0.0005}{D}\left(\frac{ds}{dt}\right)^2$$

where D is the raindrop diameter and $g = 9.8$ m/s^2. Terminal velocity v_{term} is defined as the velocity at which the drop has zero acceleration (one can show that velocity approaches v_{term} as time proceeds).

(a) Show that $v_{\text{term}} = \sqrt{2000gD}$.

(b) Find v_{term} for drops of diameter 10^{-3} m and 10^{-4} m.

(c) In this model, do raindrops accelerate more rapidly at higher or lower velocities?

SOLUTION

(a) v_{term} is found by setting $\frac{d^2s}{dt^2} = 0$, and solving for $\frac{ds}{dt} = v$.

$$0 = g - \frac{0.0005}{D}\left(\frac{ds}{dt}\right)^2$$

$$g = \frac{0.0005}{D}\left(\frac{ds}{dt}\right)^2$$

$$\frac{ds}{dt} = \sqrt{g\frac{D}{0.0005}} = \sqrt{2000gD} = v_{\text{term}}.$$

(b) If $D = 0.001$ m,

$$v_{\text{term}} = \sqrt{2000g(0.001)} = \sqrt{19.6} = 4.4272 \text{ m/s}.$$

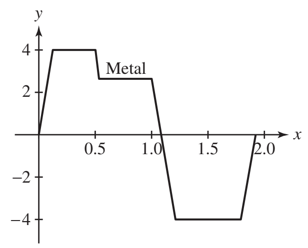
If $D = 0.0001$ m,

$$v_{\text{term}} = \sqrt{2000g(0.0001)} = \sqrt{1.96} = 1.4 \text{ m/s}.$$

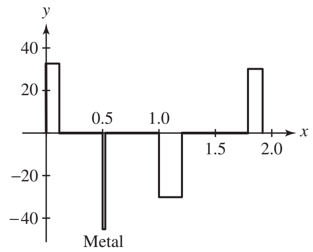
(c) The greater the velocity, the more gets subtracted from g in the formula for acceleration. Therefore, assuming velocity is less than v_{term}, greater velocities correspond to *lower* acceleration.

47. A servomotor controls the vertical movement of a drill bit that will drill a pattern of holes in sheet metal. The maximum vertical speed of the drill bit is 4 in./s, and while drilling the hole, it must move no more than 2.6 in./s to avoid warping the metal. During a cycle, the bit begins and ends at rest, quickly approaches the sheet metal, and quickly returns to its initial position after the hole is drilled. Sketch possible graphs of the drill bit's vertical velocity and acceleration. Label the point where the bit enters the sheet metal.

SOLUTION There will be multiple cycles, each of which will be more or less identical. Let $v(t)$ be the *downward* vertical velocity of the drill bit, and let $a(t)$ be the vertical acceleration. From the narrative, we see that $v(t)$ can be no greater than 4 and no greater than 2.6 while drilling is taking place. During each cycle, $v(t) = 0$ initially, $v(t)$ goes to 4 quickly. When the bit hits the sheet metal, $v(t)$ goes down to 2.6 quickly, at which it stays until the sheet metal is drilled through. As the drill pulls out, it reaches maximum non-drilling upward speed ($v(t) = -4$) quickly, and maintains this speed until it returns to rest. A possible plot follows:



A graph of the acceleration is extracted from this graph:



In Exercises 48 and 49, refer to the following. In a 1997 study, Boardman and Lave related the traffic speed S on a two-lane road to traffic density Q (number of cars per mile of road) by the formula

$$S = 2882Q^{-1} - 0.052Q + 31.73$$

for $60 \leq Q \leq 400$ *(Figure 9).*

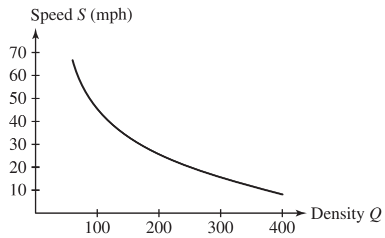


FIGURE 9 Speed as a function of traffic density.

48. Calculate dS/dQ and d^2S/dQ^2.

SOLUTION

$$dS/dQ = -2882Q^{-2} - 0.052$$
$$d^2S/dQ^2 = 5764Q^{-3}.$$

49. **(a)** Explain intuitively why we should expect that $dS/dQ < 0$.

(b) Show that $d^2S/dQ^2 > 0$. Then use the fact that $dS/dQ < 0$ and $d^2S/dQ^2 > 0$ to justify the following statement: *A 1-unit increase in traffic density slows down traffic more when Q is small than when Q is large.*

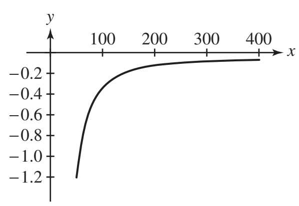
(c) GU Plot dS/dQ. Which property of this graph shows that $d^2S/dQ^2 > 0$?

SOLUTION

(a) Traffic speed must be reduced when the road gets more crowded so we expect dS/dQ to be negative. This is indeed the case since $dS/dQ = -0.052 - 2882/Q^2 < 0$.

(b) The decrease in speed due to a one-unit increase in density is approximately dS/dQ (a negative number). Since $d^2S/dQ^2 = 5764Q^{-3} > 0$ is positive, this tells us that dS/dQ gets larger as Q increases—and a negative number which gets larger is getting closer to zero. So the decrease in speed is smaller when Q is larger, that is, a one-unit increase in traffic density has a smaller effect when Q is large.

(c) dS/dQ is plotted below. The fact that this graph is increasing shows that $d^2S/dQ^2 > 0$.



50. CAS Use a computer algebra system to compute $f^{(k)}(x)$ for $k = 1, 2, 3$ for the following functions:

(a) $f(x) = (1 + x^3)^{5/3}$

(b) $f(x) = \dfrac{1 - x^4}{1 - 5x - 6x^2}$

SOLUTION

(a) Let $f(x) = (1 + x^3)^{5/3}$. Using a computer algebra system,

$$f'(x) = 5x^2(1 + x^3)^{2/3};$$
$$f''(x) = 10x(1 + x^3)^{2/3} + 10x^4(1 + x^3)^{-1/3}; \text{ and}$$
$$f'''(x) = 10(1 + x^3)^{2/3} + 60x^3(1 + x^3)^{-1/3} - 10x^6(1 + x^3)^{-4/3}.$$

(b) Let $f(x) = \dfrac{1 - x^4}{1 - 5x - 6x^2}$. Using a computer algebra system,

$$f'(x) = \frac{12x^3 - 9x^2 + 2x + 5}{(6x - 1)^2};$$
$$f''(x) = \frac{2(36x^3 - 18x^2 + 3x - 31)}{(6x - 1)^3}; \text{ and}$$
$$f'''(x) = \frac{1110}{(6x - 1)^4}.$$

51. CAS Let $f(x) = \dfrac{x + 2}{x - 1}$. Use a computer algebra system to compute the $f^{(k)}(x)$ for $1 \le k \le 4$. Can you find a general formula for $f^{(k)}(x)$?

SOLUTION Let $f(x) = \dfrac{x + 2}{x - 1}$. Using a computer algebra system,

$$f'(x) = -\frac{3}{(x - 1)^2} = (-1)^1 \frac{3 \cdot 1}{(x - 1)^{1+1}};$$
$$f''(x) = \frac{6}{(x - 1)^3} = (-1)^2 \frac{3 \cdot 2 \cdot 1}{(x - 1)^{2+1}};$$
$$f'''(x) = -\frac{18}{(x - 1)^4} = (-1)^3 \frac{3 \cdot 3!}{(x - 1)^{3+1}}; \text{ and}$$
$$f^{(4)}(x) = \frac{72}{(x - 1)^5} = (-1)^4 \frac{3 \cdot 4!}{(x - 1)^{4+1}}.$$

From the pattern observed above, we conjecture

$$f^{(k)}(x) = (-1)^k \frac{3 \cdot k!}{(x - 1)^{k+1}}.$$

Further Insights and Challenges

52. Find the 100th derivative of

$$p(x) = (x + x^5 + x^7)^{10}(1 + x^2)^{11}(x^3 + x^5 + x^7)$$

SOLUTION This is a polynomial of degree $70 + 22 + 7 = 99$, so its 100th derivative is zero.

53. What is $p^{(99)}(x)$ for $p(x)$ as in Exercise 52?

SOLUTION First note that for any integer $n \le 98$,

$$\frac{d^{99}}{dx^{99}} x^n = 0.$$

Now, if we expand $p(x)$, we find

$$p(x) = x^{99} + \text{ terms of degree at most 98};$$

therefore,

$$\frac{d^{99}}{dx^{99}} p(x) = \frac{d^{99}}{dx^{99}}(x^{99} + \text{ terms of degree at most 98}) = \frac{d^{99}}{dx^{99}} x^{99}$$

Using logic similar to that used to compute the derivative in Example (3), we compute:

$$\frac{d^{99}}{dx^{99}}(x^{99}) = 99 \times 98 \times \ldots 1,$$

so that $\frac{d^{99}}{dx^{99}} p(x) = 99!$.

54. Use the Product Rule twice to find a formula for $(fg)''$ in terms of f and g and their first and second derivatives.

SOLUTION Let $h = fg$. Then $h' = f'g + fg'$ and

$$h'' = f'g' + gf'' + fg'' + g'f' = f''g + 2f'g' + fg''.$$

55. Use the Product Rule to find a formula for $(fg)'''$ and compare your result with the expansion of $(a+b)^3$. Then try to guess the general formula for $(fg)^{(n)}$.

SOLUTION Continuing from Exercise 54, we have

$$h''' = f''g' + gf''' + 2(f'g'' + g'f'') + fg''' + g''f' = f'''g + 3f''g' + 3f'g'' + fg'''$$

The binomial theorem gives

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

and more generally

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k,$$

where the binomial coefficients are given by

$$\binom{n}{k} = \frac{k(k-1)\cdots(k-n+1)}{n!}.$$

Accordingly, the general formula for $(fg)^{(n)}$ is given by

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)},$$

where $p^{(k)}$ is the kth derivative of p (or p itself when $k = 0$).

56. Compute

$$\Delta f(x) = \lim_{h \to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2}$$

for the following functions:

(a) $f(x) = x$ **(b)** $f(x) = x^2$ **(c)** $f(x) = x^3$

Based on these examples, what do you think the limit Δf represents?

SOLUTION For $f(x) = x$, we have

$$f(x+h) + f(x-h) - 2f(x) = (x+h) + (x-h) - 2x = 0.$$

Hence, $\Delta(x) = 0$. For $f(x) = x^2$,

$$\begin{aligned} f(x+h) + f(x-h) - 2f(x) &= (x+h)^2 + (x-h)^2 - 2x^2 \\ &= x^2 + 2xh + h^2 + x^2 - 2xh + h^2 - 2x^2 = 2h^2, \end{aligned}$$

so $\Delta(x^2) = 2$. Working in a similar fashion, we find $\Delta(x^3) = 6x$. One can prove that for twice differentiable functions, $\Delta f = f''$. It is an interesting fact of more advanced mathematics that there are functions f for which Δf exists at all points, but the function is not differentiable.

3.6 Trigonometric Functions

Preliminary Questions

1. Determine the sign (+ or −) that yields the correct formula for the following:

(a) $\frac{d}{dx}(\sin x + \cos x) = \pm \sin x \pm \cos x$

(b) $\frac{d}{dx} \sec x = \pm \sec x \tan x$

(c) $\frac{d}{dx} \cot x = \pm \csc^2 x$

SOLUTION The correct formulas are

(a) $\frac{d}{dx}(\sin x + \cos x) = -\sin x + \cos x$

(b) $\frac{d}{dx} \sec x = \sec x \tan x$

(c) $\frac{d}{dx} \cot x = -\csc^2 x$

2. Which of the following functions can be differentiated using the rules we have covered so far?

(a) $y = 3\cos x \cot x$ **(b)** $y = \cos(x^2)$ **(c)** $y = e^x \sin x$

SOLUTION

(a) $3\cos x \cot x$ is a product of functions whose derivatives are known. This function can therefore be differentiated using the Product Rule.

(b) $\cos(x^2)$ is a composition of the functions $\cos x$ and x^2. We have not yet discussed how to differentiate composite functions.

(c) $e^x \sin x$ is a product of functions whose derivatives are known. This function can therefore be differentiated using the Product Rule.

3. Compute $\frac{d}{dx}(\sin^2 x + \cos^2 x)$ without using the derivative formulas for $\sin x$ and $\cos x$.

SOLUTION Recall that $\sin^2 x + \cos^2 x = 1$ for all x. Thus,

$$\frac{d}{dx}(\sin^2 x + \cos^2 x) = \frac{d}{dx}1 = 0.$$

4. How is the addition formula used in deriving the formula $(\sin x)' = \cos x$?

SOLUTION The difference quotient for the function $\sin x$ involves the expression $\sin(x + h)$. The addition formula for the sine function is used to expand this expression as $\sin(x + h) = \sin x \cos h + \sin h \cos x$.

Exercises

In Exercises 1–4, find an equation of the tangent line at the point indicated.

1. $y = \sin x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = \sin x$. Then $f'(x) = \cos x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right).$$

2. $y = \cos x, \quad x = \frac{\pi}{3}$

SOLUTION Let $f(x) = \cos x$. Then $f'(x) = -\sin x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) + \frac{1}{2} = -\frac{\sqrt{3}}{2}x + \frac{1}{2} + \frac{\pi\sqrt{3}}{6}.$$

3. $y = \tan x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = \tan x$. Then $f'(x) = \sec^2 x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = 2\left(x - \frac{\pi}{4}\right) + 1 = 2x + 1 - \frac{\pi}{2}.$$

4. $y = \sec x, \quad x = \frac{\pi}{6}$

SOLUTION Let $f(x) = \sec x$. Then $f'(x) = \sec x \tan x$ and the equation of the tangent line is

$$y = f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right) + f\left(\frac{\pi}{6}\right) = \frac{2}{3}\left(x - \frac{\pi}{6}\right) + \frac{2}{\sqrt{3}} = \frac{2}{3}x + \frac{2\sqrt{3}}{3} + \frac{\pi}{9}.$$

In Exercises 5–24, compute the derivative.

5. $f(x) = \sin x \cos x$

SOLUTION Let $f(x) = \sin x \cos x$. Then

$$f'(x) = \sin x(-\sin x) + \cos x(\cos x) = -\sin^2 x + \cos^2 x.$$

6. $f(x) = x^2 \cos x$

SOLUTION Let $f(x) = x^2 \cos x$. Then

$$f'(x) = x^2(-\sin x) + (\cos x)(2x) = 2x\cos x - x^2 \sin x.$$

7. $f(x) = \sin^2 x$

SOLUTION Let $f(x) = \sin^2 x = \sin x \sin x$. Then

$$f'(x) = \sin x(\cos x) + \sin x(\cos x) = 2\sin x \cos x.$$

8. $f(x) = 9\sec x + 12\cot x$

SOLUTION Let $f(x) = 9\sec x + 12\cot x$. Then $f'(x) = 9\sec x \tan x - 12\csc^2 x$.

9. $H(t) = \sin t \sec^2 t$

SOLUTION Let $H(t) = \sin t \sec^2 t$. Then

$$\begin{aligned} H'(t) &= \sin t \frac{d}{dt}(\sec t \cdot \sec t) + \sec^2 t(\cos t) \\ &= \sin t(\sec t \sec t \tan t + \sec t \sec t \tan t) + \sec t \\ &= 2\sin t \sec^2 t \tan t + \sec t. \end{aligned}$$

10. $h(t) = 9\csc t + t\cot t$

SOLUTION Let $h(t) = 9\csc t + t\cot t$. Then

$$h'(t) = 9(-\csc t \cot t) + t(-\csc^2 t) + \cot t = \cot t - 9\csc t \cot t - t\csc^2 t.$$

11. $f(\theta) = \tan\theta \sec\theta$

SOLUTION Let $f(\theta) = \tan\theta\sec\theta$. Then

$$f'(\theta) = \tan\theta\sec\theta\tan\theta + \sec\theta\sec^2\theta = \sec\theta\tan^2\theta + \sec^3\theta = \left(\tan^2\theta + \sec^2\theta\right)\sec\theta.$$

12. $k(\theta) = \theta^2 \sin^2\theta$

SOLUTION Let $k(\theta) = \theta^2\sin^2\theta$. Then

$$k'(\theta) = \theta^2(2\sin\theta\cos\theta) + 2\theta\sin^2\theta = 2\theta^2\sin\theta\cos\theta + 2\theta\sin^2\theta.$$

Here we used the result from Exercise 7.

13. $f(x) = (2x^4 - 4x^{-1})\sec x$

SOLUTION Let $f(x) = (2x^4 - 4x^{-1})\sec x$. Then

$$f'(x) = (2x^4 - 4x^{-1})\sec x\tan x + \sec x(8x^3 + 4x^{-2}) = (2x^4 - 4x^{-1})\sec x\tan x + (8x^3 + 4x^{-2})\sec x.$$

14. $f(z) = z\tan z$

SOLUTION Let $f(z) = z\tan z$. Then $f'(z) = z(\sec^2 z) + \tan z$.

15. $y = \dfrac{\sec\theta}{\theta}$

SOLUTION Let $y = \dfrac{\sec\theta}{\theta}$. Then

$$y' = \frac{\theta\sec\theta\tan\theta - \sec\theta}{\theta^2}.$$

16. $G(z) = \dfrac{1}{\tan z - \cot z}$

SOLUTION Let $G(z) = \dfrac{1}{\tan z - \cot z}$. Then

$$G'(z) = \frac{(\tan z - \cot z)(0) - 1(\sec^2 z + \csc^2 z)}{(\tan z - \cot z)^2} = -\frac{\sec^2 z + \csc^2 z}{(\tan z - \cot z)^2}.$$

17. $R(y) = \dfrac{3\cos y - 4}{\sin y}$

SOLUTION Let $R(y) = \dfrac{3\cos y - 4}{\sin y}$. Then

$$R'(y) = \frac{\sin y(-3\sin y) - (3\cos y - 4)(\cos y)}{\sin^2 y} = \frac{4\cos y - 3(\sin^2 y + \cos^2 y)}{\sin^2 y} = \frac{4\cos y - 3}{\sin^2 y}.$$

18. $f(x) = \dfrac{x}{\sin x + 2}$

SOLUTION Let $f(x) = \dfrac{x}{\sin x + 2}$. Then

$$f'(x) = \frac{(\sin x + 2)\,(1) - x\cos x}{(\sin x + 2)^2} = \frac{2 + \sin x - x\cos x}{(\sin x + 2)^2}.$$

19. $f(x) = \dfrac{1 + \tan x}{1 - \tan x}$

SOLUTION Let $f(x) = \dfrac{1 + \tan x}{1 - \tan x}$. Then

$$f'(x) = \frac{(1 - \tan x)\sec^2 x - (1 + \tan x)\left(-\sec^2 x\right)}{(1 - \tan x)^2} = \frac{2\sec^2 x}{(1 - \tan x)^2}.$$

20. $f(\theta) = \theta\tan\theta\sec\theta$

SOLUTION Let $f(\theta) = \theta\tan\theta\sec\theta$. Then

$$\begin{aligned} f'(\theta) &= \theta\frac{d}{d\theta}(\tan\theta\sec\theta) + \tan\theta\sec\theta \\ &= \theta(\tan\theta\sec\theta\tan\theta + \sec\theta\sec^2\theta) + \tan\theta\sec\theta \\ &= \theta\tan^2\theta\sec\theta + \theta\sec^3\theta + \tan\theta\sec\theta. \end{aligned}$$

21. $f(x) = e^x\sin x$

SOLUTION Let $f(x) = e^x\sin x$. Then $f'(x) = e^x\cos x + \sin x\,e^x = e^x(\cos x + \sin x)$.

22. $h(t) = e^t\csc t$

SOLUTION Let $h(t) = e^t\csc t$. Then $h'(t) = e^t(-\csc t\cot t) + \csc t\,e^t = e^t\csc t(1 - \cot t)$.

23. $f(\theta) = e^\theta(5\sin\theta - 4\tan\theta)$

SOLUTION Let $f(\theta) = e^\theta(5\sin\theta - 4\tan\theta)$. Then

$$\begin{aligned} f'(\theta) &= e^\theta(5\cos\theta - 4\sec^2\theta) + e^\theta(5\sin\theta - 4\tan\theta) \\ &= e^\theta(5\sin\theta + 5\cos\theta - 4\tan\theta - 4\sec^2\theta). \end{aligned}$$

24. $f(x) = xe^x \cos x$

SOLUTION Let $f(x) = xe^x \cos x$. Then

$$f'(x) = x\frac{d}{dx}(e^x \cos x) + e^x \cos x = x(e^x(-\sin x) + \cos x e^x) + e^x \cos x$$
$$= e^x(x \cos x - x \sin x + \cos x).$$

In Exercises 25–34, find an equation of the tangent line at the point specified.

25. $y = x^3 + \cos x, \quad x = 0$

SOLUTION Let $f(x) = x^3 + \cos x$. Then $f'(x) = 3x^2 - \sin x$ and $f'(0) = 0$. The tangent line at $x = 0$ is

$$y = f'(0)(x - 0) + f(0) = 0(x) + 1 = 1.$$

26. $y = \tan\theta, \quad \theta = \frac{\pi}{6}$

SOLUTION Let $f(\theta) = \tan\theta$. Then $f'(\theta) = \sec^2\theta$ and $f'(\frac{\pi}{6}) = \frac{4}{3}$. The tangent line at $x = \frac{\pi}{6}$ is

$$y = f'\left(\frac{\pi}{6}\right)\left(\theta - \frac{\pi}{6}\right) + f\left(\frac{\pi}{6}\right) = \frac{4}{3}\left(\theta - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{3} = \frac{4}{3}\theta + \frac{\sqrt{3}}{3} - \frac{2\pi}{9}.$$

27. $y = \dfrac{\sin t}{1 + \cos t}, \quad t = \frac{\pi}{3}$

SOLUTION Let $f(t) = \frac{\sin t}{1+\cos t}$. Then

$$f'(t) = \frac{(1 + \cos t)(\cos t) - \sin t(-\sin t)}{(1 + \cos t)^2} = \frac{1 + \cos t}{(1 + \cos t)^2} = \frac{1}{1 + \cos t},$$

and

$$f'\left(\frac{\pi}{3}\right) = \frac{1}{1 + 1/2} = \frac{2}{3}.$$

The tangent line at $x = \frac{\pi}{3}$ is

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = \frac{2}{3}\left(x - \frac{\pi}{3}\right) + \frac{\sqrt{3}}{3} = \frac{2}{3}x + \frac{\sqrt{3}}{3} - \frac{2\pi}{9}.$$

28. $y = \sin x + 3\cos x, \quad x = 0$

SOLUTION Let $f(x) = \sin x + 3\cos x$. Then $f'(x) = \cos x - 3\sin x$ and $f'(0) = 1$. The tangent line at $x = 0$ is

$$y = f'(0)(x - 0) + f(0) = x + 3.$$

29. $y = 2(\sin\theta + \cos\theta), \quad \theta = \frac{\pi}{3}$

SOLUTION Let $f(\theta) = 2(\sin\theta + \cos\theta)$. Then $f'(\theta) = 2(\cos\theta - \sin\theta)$ and $f'(\frac{\pi}{3}) = 1 - \sqrt{3}$. The tangent line at $x = \frac{\pi}{3}$ is

$$y = f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right) + f\left(\frac{\pi}{3}\right) = (1 - \sqrt{3})\left(x - \frac{\pi}{3}\right) + 1 + \sqrt{3}.$$

30. $y = \csc x - \cot x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = \csc x - \cot x$. Then

$$f'(x) = \csc^2 x - \csc x \cot x$$

and

$$f'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2} \cdot 1 = 2 - \sqrt{2}.$$

Hence the tangent line is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = \left(2 - \sqrt{2}\right)\left(x - \frac{\pi}{4}\right) + \left(\sqrt{2} - 1\right)$$
$$= \left(2 - \sqrt{2}\right)x + \sqrt{2} - 1 + \frac{\pi}{4}\left(\sqrt{2} - 2\right).$$

31. $y = e^x \cos x, \quad x = 0$

SOLUTION Let $f(x) = e^x \cos x$. Then

$$f'(x) = e^x(-\sin x) + e^x \cos x = e^x(\cos x - \sin x),$$

and $f'(0) = e^0(\cos 0 - \sin 0) = 1$. Thus, the equation of the tangent line is

$$y = f'(0)(x - 0) + f(0) = x + 1.$$

32. $y = e^x \cos^2 x, \quad x = \frac{\pi}{4}$

SOLUTION Let $f(x) = e^x \cos^2 x$. Then

$$\begin{aligned} f'(x) = e^x \frac{d}{dx}(\cos x \cdot \cos x) + e^x \cos^2 x &= e^x(\cos x(-\sin x) + \cos x(-\sin x)) + e^x \cos^2 x \\ &= e^x(\cos^2 x - 2\sin x \cos x), \end{aligned}$$

and

$$f'\left(\frac{\pi}{4}\right) = e^{\pi/4}\left(\frac{1}{2} - 1\right) = -\frac{1}{2}e^{\pi/4}.$$

The tangent line at $x = \frac{\pi}{4}$ is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = -\frac{1}{2}e^{\pi/4}\left(x - \frac{\pi}{4}\right) + \frac{1}{2}e^{\pi/4}.$$

33. $y = e^t(1 - \cos t), \quad t = \frac{\pi}{2}$

SOLUTION Let $f(t) = e^t(1 - \cos t)$. Then

$$f'(t) = e^t \sin t + e^t(1 - \cos t) = e^t(1 + \sin t - \cos t),$$

and $f'(\frac{\pi}{2}) = 2e^{\pi/2}$. The tangent line at $x = \frac{\pi}{2}$ is

$$y = f'\left(\frac{\pi}{2}\right)\left(t - \frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 2e^{\pi/2}\left(t - \frac{\pi}{2}\right) + e^{\pi/2}.$$

34. $y = e^\theta \sec\theta, \quad \theta = \frac{\pi}{4}$

SOLUTION Let $f(\theta) = e^\theta \sec\theta$. Then

$$f'(\theta) = e^\theta \sec\theta \tan\theta + e^\theta \sec\theta = e^\theta \sec\theta(\tan\theta + 1),$$

and

$$f'\left(\frac{\pi}{4}\right) = e^{\pi/4} \sec\frac{\pi}{4}\left(\tan\frac{\pi}{4} + 1\right) = 2\sqrt{2}e^{\pi/4}.$$

Thus, the equation of the tangent line is

$$y = f'\left(\frac{\pi}{4}\right)\left(x - \frac{\pi}{4}\right) + f\left(\frac{\pi}{4}\right) = 2\sqrt{2}e^{\pi/4}\left(x - \frac{\pi}{4}\right) + \sqrt{2}e^{\pi/4}.$$

In Exercises 35–37, use Theorem 1 to verify the formula.

35. $\frac{d}{dx}\cot x = -\csc^2 x$

SOLUTION $\cot x = \frac{\cos x}{\sin x}$. Using the quotient rule and the derivative formulas, we compute:

$$\frac{d}{dx}\cot x = \frac{d}{dx}\frac{\cos x}{\sin x} = \frac{\sin x(-\sin x) - \cos x(\cos x)}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

36. $\frac{d}{dx}\sec x = \sec x \tan x$

SOLUTION Since $\sec x = \frac{1}{\cos x}$, we can apply the quotient rule and the known derivatives to get:

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x(0) - 1(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\frac{1}{\cos x} = \tan x \sec x.$$

37. $\dfrac{d}{dx}\csc x = -\csc x\cot x$

SOLUTION Since $\csc x = \dfrac{1}{\sin x}$, we can apply the quotient rule and the two known derivatives to get:

$$\frac{d}{dx}\csc x = \frac{d}{dx}\frac{1}{\sin x} = \frac{\sin x(0) - 1(\cos x)}{\sin^2 x} = \frac{-\cos x}{\sin^2 x} = -\frac{\cos x}{\sin x}\frac{1}{\sin x} = -\cot x\csc x.$$

38. Show that both $y = \sin x$ and $y = \cos x$ satisfy $y'' = -y$.

SOLUTION Let $y = \sin x$. Then $y' = \cos x$ and $y'' = -\sin x = -y$. Similarly, if we let $y = \cos x$, then $y' = -\sin x$ and $y'' = -\cos x = -y$.

In Exercises 39–42, calculate the higher derivative.

39. $f''(\theta)$, $\quad f(\theta) = \theta\sin\theta$

SOLUTION Let $f(\theta) = \theta\sin\theta$. Then

$$f'(\theta) = \theta\cos\theta + \sin\theta$$
$$f''(\theta) = \theta(-\sin\theta) + \cos\theta + \cos\theta = -\theta\sin\theta + 2\cos\theta.$$

40. $\dfrac{d^2}{dt^2}\cos^2 t$

SOLUTION

$$\frac{d}{dt}\cos^2 t = \frac{d}{dt}(\cos t\cdot\cos t) = \cos t(-\sin t) + \cos t(-\sin t) = -2\sin t\cos t$$
$$\frac{d^2}{dt^2}\cos^2 t = \frac{d}{dt}(-2\sin t\cos t) = -2(\sin t(-\sin t) + \cos t(\cos t)) = -2(\cos^2 t - \sin^2 t).$$

41. y'', $\quad y'''$, $\quad y = \tan x$

SOLUTION Let $y = \tan x$. Then $y' = \sec^2 x$ and by the Chain Rule,

$$y'' = = \frac{d}{dx}\sec^2 x = 2(\sec x)(\sec x\tan x) = 2\sec^2 x\tan x$$
$$y''' = 2\sec^2 x(\sec^2 x) + (4\sec^2 x\tan x)\tan x = 2\sec^4 x + 4\sec^2 x\tan^2 x$$

42. y'', $\quad y'''$, $\quad y = e^t\sin t$

SOLUTION Let $y = e^t\sin t$. Then

$$y' = e^t\cos t + e^t\sin t = e^t(\cos t + \sin t)$$
$$y'' = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t\cos t$$
$$y''' = 2e^t(-\sin t) + 2e^t\cos t = 2e^t(\cos t - \sin t).$$

43. Calculate the first five derivatives of $f(x) = \cos x$. Then determine $f^{(8)}(x)$ and $f^{(37)}(x)$.

SOLUTION Let $f(x) = \cos x$.

- Then $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x$, $f^{(4)}(x) = \cos x$, and $f^{(5)}(x) = -\sin x$.
- Accordingly, the successive derivatives of f cycle among

  $$\{-\sin x, -\cos x, \sin x, \cos x\}$$

  in that order. Since 8 is a multiple of 4, we have $f^{(8)}(x) = \cos x$.
- Since 36 is a multiple of 4, we have $f^{(36)}(x) = \cos x$. Therefore, $f^{(37)}(x) = -\sin x$.

44. Find $y^{(157)}$, where $y = \sin x$.

SOLUTION Let $f(x) = \sin x$. Then the successive derivatives of f cycle among

$$\{\cos x, -\sin x, -\cos x, \sin x\}$$

in that order. Since 156 is a multiple of 4, we have $f^{(156)}(x) = \sin x$. Therefore, $f^{(157)}(x) = \cos x$.

45. Find the values of x between 0 and 2π where the tangent line to the graph of $y = \sin x \cos x$ is horizontal.

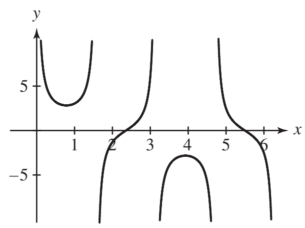
SOLUTION Let $y = \sin x \cos x$. Then

$$y' = (\sin x)(-\sin x) + (\cos x)(\cos x) = \cos^2 x - \sin^2 x.$$

When $y' = 0$, we have $\sin x = \pm\cos x$. In the interval $[0, 2\pi]$, this occurs when $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

46. GU Plot the graph $f(\theta) = \sec\theta + \csc\theta$ over $[0, 2\pi]$ and determine the number of solutions to $f'(\theta) = 0$ in this interval graphically. Then compute $f'(\theta)$ and find the solutions.

SOLUTION The graph of $f(\theta) = \sec\theta + \csc\theta$ over $[0, 2\pi]$ is given below. From the graph, it appears that there are two locations where the tangent line would be horizontal; that is, there appear to be two solutions to $f'(\theta) = 0$. Now $f'(\theta) = \sec\theta\tan\theta - \csc\theta\cot\theta$. Setting $\sec\theta\tan\theta - \csc\theta\cot\theta = 0$ and then multiplying by $\sin\theta\tan\theta$ and rearranging terms yields $\tan^3\theta = 1$. Thus, on the interval $[0, 2\pi]$, there are two solution of $f'(\theta) = 0$: $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.



47. GU Let $g(t) = t - \sin t$.

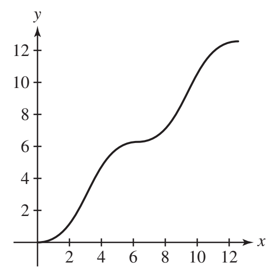
(a) Plot the graph of g with a graphing utility for $0 \le t \le 4\pi$.

(b) Show that the slope of the tangent line is nonnegative. Verify this on your graph.

(c) For which values of t in the given range is the tangent line horizontal?

SOLUTION Let $g(t) = t - \sin t$.

(a) Here is a graph of g over the interval $[0, 4\pi]$.



(b) Since $g'(t) = 1 - \cos t \ge 0$ for all t, the slope of the tangent line to g is always nonnegative. We see that the graph is never decreasing; therefore, the slope of the tangent is never negative.

(c) In the interval $[0, 4\pi]$, the tangent line is horizontal when $t = 0, 2\pi, 4\pi$.

48. CAS Let $f(x) = (\sin x)/x$ for $x \ne 0$ and $f(0) = 1$.

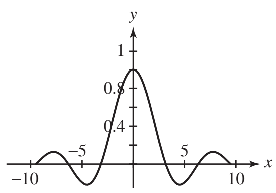
(a) Plot f on $[-3\pi, 3\pi]$.

(b) Show that $f'(c) = 0$ if $c = \tan c$. Use the numerical root finder on a computer algebra system to find a good approximation to the smallest *positive* value c_0 such that $f'(c_0) = 0$.

(c) Verify that the horizontal line $y = f(c_0)$ is tangent to the graph of $y = f(x)$ at $x = c_0$ by plotting them on the same set of axes.

SOLUTION

(a) Here is the graph of $f(x)$ over $[-3\pi, 3\pi]$.



(b) Let $f(x) = \dfrac{\sin x}{x}$. Then

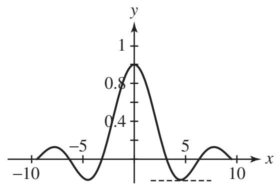
$$f'(x) = \frac{x \cos x - \sin x}{x^2}.$$

To have $f'(c) = 0$, it follows that $c \cos c - \sin c = 0$, or

$$\tan c = c.$$

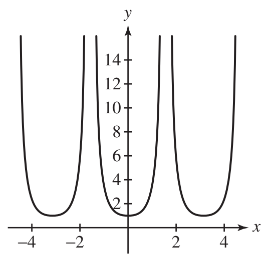
Using a computer algebra system, we find that the smallest positive value c_0 such that $f'(c_0) = 0$ is $c_0 = 4.493409$.

(c) The horizontal line $y = f(c_0) = -0.217234$ and the function $y = f(x)$ are both plotted below. The horizontal line is clearly tangent to the graph of $f(x)$.



49. Show that no tangent line to the graph of $f(x) = \tan x$ has zero slope. What is the least slope of a tangent line? Justify by sketching the graph of $f'(x) = (\tan x)'$.

SOLUTION Let $f(x) = \tan x$. Then $f'(x) = \sec^2 x = \frac{1}{\cos^2 x}$. Note that $f'(x) = \frac{1}{\cos^2 x}$ has numerator 1; the equation $f'(x) = 0$ therefore has no solution. Because the maximum value of $\cos^2 x$ is 1, the minimum value of $f'(x) = \frac{1}{\cos^2 x}$ is 1. Hence, the least slope for a tangent line to $\tan x$ is 1. Here is a graph of f'.



50. The height at time t (in seconds) of a mass, oscillating at the end of a spring, is $s(t) = 300 + 40 \sin t$ cm. Find the velocity and acceleration at $t = \frac{\pi}{3}$ s.

SOLUTION Let $s(t) = 300 + 40 \sin t$ be the height. Then the velocity is

$$v(t) = s'(t) = 40 \cos t$$

and the acceleration is

$$a(t) = v'(t) = -40 \sin t.$$

At $t = \frac{\pi}{3}$, the velocity is $v\left(\frac{\pi}{3}\right) = 20$ cm/sec and the acceleration is $a\left(\frac{\pi}{3}\right) = -20\sqrt{3}$ cm/sec^2.

51. The horizontal range R of a projectile launched from ground level at an angle θ and initial velocity v_0 m/s is $R = (v_0^2/9.8) \sin \theta \cos \theta$. Calculate $dR/d\theta$. If $\theta = 7\pi/24$, will the range increase or decrease if the angle is increased slightly? Base your answer on the sign of the derivative.

SOLUTION Let $R(\theta) = (v_0^2/9.8) \sin \theta \cos \theta$.

$$\frac{dR}{d\theta} = R'(\theta) = (v_0^2/9.8)(-\sin^2 \theta + \cos^2 \theta).$$

If $\theta = 7\pi/24$, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$, so $|\sin \theta| > |\cos \theta|$, and $dR/d\theta < 0$ (numerically, $dR/d\theta = -.0264101 v_0^2$). At this point, increasing the angle will *decrease* the range.

52. Show that if $\frac{\pi}{2} < \theta < \pi$, then the distance along the x-axis between θ and the point where the tangent line intersects the x-axis is equal to $|\tan\theta|$ (Figure 4).

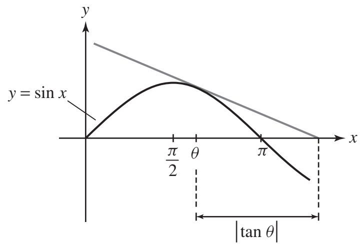


FIGURE 4

SOLUTION Let $f(x) = \sin x$. Since $f'(x) = \cos x$, this means that the tangent line at $(\theta, \sin\theta)$ is $y = \cos\theta(x - \theta) + \sin\theta$. When $y = 0$, $x = \theta - \tan\theta$. The distance from x to θ is then

$$|\theta - (\theta - \tan\theta)| = |\tan\theta|.$$

Further Insights and Challenges

53. Use the limit definition of the derivative and the addition law for the cosine function to prove that $(\cos x)' = -\sin x$.

SOLUTION Let $f(x) = \cos x$. Then

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$
$$= \lim_{h\to 0} \left((-\sin x)\frac{\sin h}{h} + (\cos x)\frac{\cos h - 1}{h} \right) = (-\sin x)\cdot 1 + (\cos x)\cdot 0 = -\sin x.$$

54. Use the addition formula for the tangent

$$\tan(x+h) = \frac{\tan x + \tan h}{1 + \tan x \tan h}$$

to compute $(\tan x)'$ directly as a limit of the difference quotients. You will also need to show that $\lim_{h\to 0} \frac{\tan h}{h} = 1$.

SOLUTION First note that

$$\lim_{h\to 0} \frac{\tan h}{h} = \lim_{h\to 0} \frac{\sin h}{h} \cdot \lim_{h\to 0} \frac{1}{\cos h} = 1(1) = 1.$$

Now, using the addition formula for tangent,

$$\frac{\tan(x+h) - \tan x}{h} = \frac{\frac{\tan x + \tan h}{1 + \tan x \tan h} - \tan x}{h}$$
$$= \frac{\tan h(1 - \tan^2 x)}{h(1 + \tan x \tan h)} = \frac{\tan h}{h} \cdot \frac{\sec^2 x}{1 + \tan x \tan h}.$$

Therefore,

$$\frac{d}{dx}\tan x = \lim_{h\to 0} \frac{\tan h}{h} \cdot \frac{\sec^2 x}{1 + \tan x \tan h}$$
$$= \lim_{h\to 0} \frac{\tan h}{h} \cdot \lim_{h\to 0} \frac{\sec^2 x}{1 + \tan x \tan h}$$
$$= 1(\sec^2 x) = \sec^2 x.$$

55. Verify the following identity and use it to give another proof of the formula $(\sin x)' = \cos x$:

$$\sin(x+h) - \sin x = 2\cos\left(x + \tfrac{1}{2}h\right)\sin\left(\tfrac{1}{2}h\right)$$

Hint: Use the addition formula to prove that $\sin(a+b) - \sin(a-b) = 2\cos a \sin b$.

SOLUTION Recall that

$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

and

$$\sin(a-b) = \sin a \cos b - \cos a \sin b.$$

Subtracting the second identity from the first yields

$$\sin(a+b) - \sin(a-b) = 2\cos a \sin b.$$

If we now set $a = x + \frac{h}{2}$ and $b = \frac{h}{2}$, then the previous equation becomes

$$\sin(x+h) - \sin x = 2\cos\left(x + \frac{h}{2}\right)\sin\left(\frac{h}{2}\right).$$

Finally, we use the limit definition of the derivative of $\sin x$ to obtain

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\\ &= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} = \cos x\cdot 1 = \cos x.\end{aligned}$$

In other words, $\frac{d}{dx}(\sin x) = \cos x$.

56. Show that a nonzero polynomial function $y = f(x)$ *cannot* satisfy the equation $y'' = -y$. Use this to prove that neither $f(x) = \sin x$ nor $f(x) = \cos x$ is a polynomial. Can you think of another way to reach this conclusion by considering limits as $x \to \infty$?

SOLUTION

- Let p be a nonzero polynomial of degree n and *assume* that p satisfies the differential equation $y'' + y = 0$. Then $p'' + p = 0$ for all x. There are exactly three cases.

  (a) If $n = 0$, then p is a constant polynomial and thus $p'' = 0$. Hence $0 = p'' + p = p$ or $p \equiv 0$ (i.e., p is equal to 0 for all x or p is identically 0). This is a contradiction, since p is a *non*zero polynomial.

  (b) If $n = 1$, then p is a linear polynomial and thus $p'' = 0$. Once again, we have $0 = p'' + p = p$ or $p \equiv 0$, a contradiction since p is a nonzero polynomial.

  (c) If $n \geq 2$, then p is at least a quadratic polynomial and thus p'' is a polynomial of degree $n - 2 \geq 0$. Thus $q = p'' + p$ is a polynomial of degree $n \geq 2$. By assumption, however, $p'' + p = 0$. Thus $q \equiv 0$, a polynomial of degree 0. This is a contradiction, since the degree of q is $n \geq 2$.

  CONCLUSION: In all cases, we have reached a contradiction. Therefore the *assumption* that p satisfies the differential equation $y'' + y = 0$ is *false*. Accordingly, a nonzero polynomial *cannot* satisfy the stated differential equation.
- Let $y = \sin x$. Then $y' = \cos x$ and $y'' = -\sin x$. Therefore, $y'' = -y$. Now, let $y = \cos x$. Then $y' = -\sin x$ and $y'' = -\cos x$. Therefore, $y'' = -y$. Because $\sin x$ and $\cos x$ are nonzero functions that satisfy $y'' = -y$, it follows that neither $\sin x$ nor $\cos x$ is a polynomial.
- Alternately, consider limits as $x \to \infty$. For a constant polynomial, this limit would exist, whereas for any polynomial of degree $n \geq 1$, the limit would tend toward either $\pm\infty$. In contrast, neither

  $$\lim_{x\to\infty}\sin x \qquad \text{nor} \qquad \lim_{x\to\infty}\cos x$$

  exists because $\sin x$ and $\cos x$ continuously oscillate between -1 and $+1$. Because the limiting behavior of $\sin x$ and $\cos x$ does not match the limiting behavior of a polynomial, neither $\sin x$ nor $\cos x$ can be a polynomial.

57. Let $f(x) = x\sin x$ and $g(x) = x\cos x$.

(a) Show that $f'(x) = g(x) + \sin x$ and $g'(x) = -f(x) + \cos x$.

(b) Verify that $f''(x) = -f(x) + 2\cos x$ and $g''(x) = -g(x) - 2\sin x$.

(c) By further experimentation, try to find formulas for all higher derivatives of f and g. *Hint:* The kth derivative depends on whether $k = 4n$, $4n+1$, $4n+2$, or $4n+3$.

SOLUTION Let $f(x) = x\sin x$ and $g(x) = x\cos x$.

(a) We examine first derivatives: $f'(x) = x\cos x + (\sin x)\cdot 1 = g(x) + \sin x$ and $g'(x) = (x)(-\sin x) + (\cos x)\cdot 1 = -f(x) + \cos x$; i.e., $f'(x) = g(x) + \sin x$ and $g'(x) = -f(x) + \cos x$.

(b) Now look at second derivatives: $f''(x) = g'(x) + \cos x = -f(x) + 2\cos x$ and $g''(x) = -f'(x) - \sin x = -g(x) - 2\sin x$; i.e., $f''(x) = -f(x) + 2\cos x$ and $g''(x) = -g(x) - 2\sin x$.

(c)
- The third derivatives are $f'''(x) = -f'(x) - 2\sin x = -g(x) - 3\sin x$ and $g'''(x) = -g'(x) - 2\cos x = f(x) - 3\cos x$; i.e., $f'''(x) = -g(x) - 3\sin x$ and $g'''(x) = f(x) - 3\cos x$.
- The fourth derivatives are $f^{(4)}(x) = -g'(x) - 3\cos x = f(x) - 4\cos x$ and $g^{(4)}(x) = f'(x) + 3\sin x = g(x) + 4\sin x$; i.e., $f^{(4)} = f(x) - 4\cos x$ and $g^{(4)}(x) = g(x) + 4\sin x$.
- We can now see the pattern for the derivatives, which are summarized in the following table. Here $n = 0, 1, 2, \ldots$

k	$4n$	$4n+1$	$4n+2$	$4n+3$
$f^{(k)}(x)$	$f(x) - k\cos x$	$g(x) + k\sin x$	$-f(x) + k\cos x$	$-g(x) - k\sin x$
$g^{(k)}(x)$	$g(x) + k\sin x$	$-f(x) + k\cos x$	$-g(x) - k\sin x$	$f(x) - k\cos x$

58. Figure 5 shows the geometry behind the derivative formula $(\sin\theta)' = \cos\theta$. Segments $\overline{BA}$ and $\overline{BD}$ are parallel to the x- and y-axes. Let $\Delta\sin\theta = \sin(\theta + h) - \sin\theta$. Verify the following statements:

(a) $\Delta\sin\theta = BC$

(b) $\angle BDA = \theta$ *Hint:* $\overline{OA} \perp AD$.

(c) $BD = (\cos\theta)AD$

Now explain the following intuitive argument: If h is small, then $BC \approx BD$ and $AD \approx h$, so $\Delta\sin\theta \approx (\cos\theta)h$ and $(\sin\theta)' = \cos\theta$.

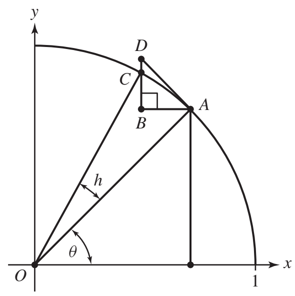


FIGURE 5

SOLUTION

(a) We note that $\sin(\theta + h)$ is the y-coordinate of the point C and $\sin\theta$ is the y-coordinate of the point A, and therefore also of the point B. Now, $\Delta\sin\theta = \sin(\theta + h) - \sin\theta$ can be interpreted as the difference between the y-coordinates of the points B and C; that is, $\Delta\sin\theta = BC$.

(b) From the figure, we note that $\angle OAB = \theta$, so $\angle BAD = \pi - \theta$ and $\angle BDA = \theta$.

(c) Using part (b), it follows that

$$\cos\theta = \frac{BD}{AD} \quad \text{or} \quad BD = (\cos\theta)AD.$$

For h "small," $BC \approx BD$ and AD is roughly the length of the arc subtended from A to C; that is, $AD \approx 1(h) = h$. Thus, using (a) and (c),

$$\Delta\sin\theta = BC \approx BD = (\cos\theta)AD \approx (\cos\theta)h.$$

In the limit as $h \to 0$,

$$\frac{\Delta\sin\theta}{h} \to (\sin\theta)',$$

so $(\sin\theta)' = \cos\theta$.

3.7 The Chain Rule

Preliminary Questions

1. Identify the outside and inside functions for each of these composite functions.

(a) $y = \sqrt{4x + 9x^2}$ **(b)** $y = \tan(x^2 + 1)$

(c) $y = \sec^5 x$ **(d)** $y = (1 + e^x)^4$

SOLUTION

(a) The outer function is $\sqrt{x}$, and the inner function is $4x + 9x^2$.
(b) The outer function is $\tan x$, and the inner function is $x^2 + 1$.
(c) The outer function is x^5, and the inner function is $\sec x$.
(d) The outer function is x^4, and the inner function is $1 + e^x$.

2. Which of the following can be differentiated easily *without* using the Chain Rule?

(a) $y = \tan(7x^2 + 2)$ **(b)** $y = \dfrac{x}{x+1}$

(c) $y = \sqrt{x} \cdot \sec x$ **(d)** $y = \sqrt{x \cos x}$

(e) $y = xe^x$ **(f)** $y = e^{\sin x}$

SOLUTION The function $\frac{x}{x+1}$ can be differentiated using the Quotient Rule, and the functions $\sqrt{x} \cdot \sec x$ and xe^x can be differentiated using the Product Rule. The functions $\tan(7x^2 + 2)$, $\sqrt{x \cos x}$ and $e^{\sin x}$ require the Chain Rule.

3. Which is the derivative of $f(5x)$?

(a) $5f'(x)$ **(b)** $5f'(5x)$ **(c)** $f'(5x)$

SOLUTION The correct answer is **(b)**: $5f'(5x)$.

4. Suppose that $f'(4) = g(4) = g'(4) = 1$. Do we have enough information to compute $F'(4)$, where $F(x) = f(g(x))$? If not, what is missing?

SOLUTION If $F(x) = f(g(x))$, then $F'(x) = f'(g(x))g'(x)$ and $F'(4) = f'(g(4))g'(4)$. Thus, we do not have enough information to compute $F'(4)$. We are missing the value of $f'(1)$.

Exercises

In Exercises 1–4, fill in a table of the following type:

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$

1. $f(u) = u^{3/2}, \quad g(x) = x^4 + 1$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$(x^4 + 1)^{3/2}$	$\frac{3}{2}u^{1/2}$	$\frac{3}{2}(x^4 + 1)^{1/2}$	$4x^3$	$6x^3(x^4 + 1)^{1/2}$

2. $f(u) = u^3, \quad g(x) = 3x + 5$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$(3x + 5)^3$	$3u^2$	$3(3x + 5)^2$	3	$9(3x + 5)^2$

3. $f(u) = \tan u, \quad g(x) = x^4$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$\tan(x^4)$	$\sec^2 u$	$\sec^2(x^4)$	$4x^3$	$4x^3 \sec^2(x^4)$

4. $f(u) = u^4 + u, \quad g(x) = \cos x$

SOLUTION

$f(g(x))$	$f'(u)$	$f'(g(x))$	$g'(x)$	$(f \circ g)'$
$(\cos x)^4 + \cos x$	$4u^3 + 1$	$4(\cos x)^3 + 1$	$-\sin x$	$-4 \sin x \cos^3 x - \sin x$

In Exercises 5 and 6, write the function as a composite $f(g(x))$ and compute the derivative using the Chain Rule.

5. $y = (x + \sin x)^4$

SOLUTION Let $f(x) = x^4$, $g(x) = x + \sin x$, and $y = f(g(x)) = (x + \sin x)^4$. Then

$$\frac{dy}{dx} = f'(g(x))g'(x) = 4(x + \sin x)^3(1 + \cos x).$$

6. $y = \cos(x^3)$

SOLUTION Let $f(x) = \cos x$, $g(x) = x^3$, and $y = f(g(x)) = \cos(x^3)$. Then

$$\frac{dy}{dx} = f'(g(x))g'(x) = -3x^2 \sin(x^3).$$

7. Calculate $\frac{d}{dx} \cos u$ for the following choices of $u(x)$:

(a) $u(x) = 9 - x^2$ **(b)** $u(x) = x^{-1}$ **(c)** $u(x) = \tan x$

SOLUTION

(a) $\cos(u(x)) = \cos(9 - x^2)$.

$$\frac{d}{dx} \cos(u(x)) = -\sin(u(x))u'(x) = -\sin(9 - x^2)(-2x) = 2x \sin(9 - x^2).$$

(b) $\cos(u(x)) = \cos(x^{-1})$.

$$\frac{d}{dx} \cos(u(x)) = -\sin(u(x))u'(x) = -\sin(x^{-1})\left(-\frac{1}{x^2}\right) = \frac{\sin(x^{-1})}{x^2}.$$

(c) $\cos(u(x)) = \cos(\tan x)$.

$$\frac{d}{dx} \cos(u(x)) = -\sin(u(x))u'(x) = -\sin(\tan x)(\sec^2 x) = -\sec^2 x \sin(\tan x).$$

8. Calculate $\frac{d}{dx} f(x^2 + 1)$ for the following choices of $f(u)$:

(a) $f(u) = \sin u$ **(b)** $f(u) = 3u^{3/2}$ **(c)** $f(u) = u^2 - u$

SOLUTION

(a) Let $\sin(u) = \sin(x^2 + 1)$. Then

$$\frac{d}{dx} \sin(x^2 + 1) = \cos(x^2 + 1) \cdot \frac{d}{dx}(x^2 + 1) = \cos(x^2 + 1)2x = 2x \cos(x^2 + 1).$$

(b) Let $3u^{3/2} = 3(x^2 + 1)^{3/2}$. Then

$$\frac{d}{dx} 3(x^2 + 1)^{3/2} = 3 \cdot \frac{3}{2}(x^2 + 1)^{1/2} \frac{d}{dx}(x^2 + 1) = \frac{9}{2}(x^2 + 1)^{1/2}(2x) = 9x(x^2 + 1)^{1/2}.$$

(c) Let $u^2 - u = (x^2 + 1)^2 - (x^2 + 1)$. Then

$$\frac{d}{dx}\left((x^2 + 1)^2 - (x^2 + 1)\right) = [2(x^2 + 1) - 1]\frac{d}{dx}(x^2 + 1) = [2(x^2 + 1) - 1](2x) = 4x^3 + 2x.$$

9. Compute $\frac{df}{dx}$ if $\frac{df}{du} = 2$ and $\frac{du}{dx} = 6$.

SOLUTION Assuming f is a function of u, which is in turn a function of x,

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = 2(6) = 12.$$

10. Compute $\left.\dfrac{df}{dx}\right|_{x=2}$ if $f(u) = u^2$, $u(2) = -5$, and $u'(2) = -5$.

SOLUTION Because $f(u) = u^2$, it follows that $f'(u) = 2u$. Therefore,

$$\left.\frac{df}{dx}\right|_{x=2} = f'(u(2))u'(2) = 2u(2)u'(2) = 2(-5)(-5) = 50.$$

In Exercises 11–22, use the General Power Rule, Exponential Rule, or the Chain Rule to compute the derivative.

11. $y = (x^4 + 5)^3$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}(x^4+5)^3 = 3(x^4+5)^2\frac{d}{dx}(x^4+5) = 3(x^4+5)^2(4x^3) = 12x^3(x^4+5)^2.$$

12. $y = (8x^4 + 5)^3$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}(8x^4+5)^3 = 3(8x^4+5)^2\frac{d}{dx}(8x^4+5) = 3(8x^4+5)^2(32x^3) = 96x^3(8x^4+5)^2.$$

13. $y = \sqrt{7x - 3}$

SOLUTION Using the General Power Rule

$$\frac{d}{dx}\sqrt{7x-3} = \frac{d}{dx}(7x-3)^{1/2} = \frac{1}{2}(7x-3)^{-1/2}(7) = \frac{7}{2\sqrt{7x-3}}.$$

14. $y = (4 - 2x - 3x^2)^5$

SOLUTION Using the General Power Rule,

$$\begin{aligned}\frac{d}{dx}(4-2x-3x^2)^5 &= 5(4-2x-3x^2)^4\frac{d}{dx}(4-2x-3x^2) = 5(4-2x-3x^2)^4(-2-6x)\\ &= -10(1+3x)(4-2x-3x^2)^4.\end{aligned}$$

15. $y = (x^2 + 9x)^{-2}$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}(x^2+9x)^{-2} = -2(x^2+9x)^{-3}\frac{d}{dx}(x^2+9x) = -2(x^2+9x)^{-3}(2x+9).$$

16. $y = (x^3 + 3x + 9)^{-4/3}$

SOLUTION Using the General Power Rule,

$$\begin{aligned}\frac{d}{dx}(x^3+3x+9)^{-4/3} &= -\frac{4}{3}(x^3+3x+9)^{-7/3}\frac{d}{dx}(x^3+3x+9) = -\frac{4}{3}(x^3+3x+9)^{-7/3}(3x^2+3)\\ &= -4(x^2+1)(x^3+3x+9)^{-7/3}.\end{aligned}$$

17. $y = \cos^4\theta$

SOLUTION Using the General Power Rule,

$$\frac{d}{d\theta}\cos^4\theta = 4\cos^3\theta\frac{d}{d\theta}\cos\theta = -4\cos^3\theta\sin\theta.$$

18. $y = \cos(9\theta + 41)$

SOLUTION Using the Chain Rule

$$\frac{d}{d\theta}\cos(9\theta+41) = -9\sin(9\theta+41).$$

19. $y = (2\cos\theta + 5\sin\theta)^9$

SOLUTION Using the General Power Rule,

$$\frac{d}{d\theta}(2\cos\theta + 5\sin\theta)^9 = 9(2\cos\theta + 5\sin\theta)^8 \frac{d}{d\theta}(2\cos\theta + 5\sin\theta) = 9(2\cos\theta + 5\sin\theta)^8(5\cos\theta - 2\sin\theta).$$

20. $y = \sqrt{9 + x + \sin x}$

SOLUTION Using the General Power Rule,

$$\frac{d}{dx}\sqrt{9 + x + \sin x} = \frac{1}{2}(9 + x + \sin x)^{-1/2}\frac{d}{dx}(9 + x + \sin x) = \frac{1 + \cos x}{2\sqrt{9 + x + \sin x}}.$$

21. $y = e^{x-12}$

SOLUTION Using the General Exponential Rule,

$$\frac{d}{dx}e^{x-12} = (1)e^{x-12} = e^{x-12}.$$

22. $y = e^{8x+9}$

SOLUTION Using the General Exponential Rule,

$$\frac{d}{dx}e^{8x+9} = 8e^{8x+9}.$$

In Exercises 23–26, compute the derivative of $f \circ g$.

23. $f(u) = \sin u, \quad g(x) = 2x + 1$

SOLUTION Let $h(x) = f(g(x)) = \sin(2x + 1)$. Then,

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = \cos(2x + 1) \cdot 2 = 2\cos(2x + 1).$$

24. $f(u) = 2u + 1, \quad g(x) = \sin x$

SOLUTION Let $h(x) = f(g(x)) = 2(\sin x) + 1$. Then $h'(x) = 2\cos x$. Alternately,

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = 2\cos x.$$

25. $f(u) = e^u, \quad g(x) = x + x^{-1}$

SOLUTION Let $h(x) = f(g(x)) = e^{x+x^{-1}}$. Then

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = e^{x+x^{-1}}\left(1 - x^{-2}\right).$$

26. $f(u) = \dfrac{u}{u-1}, \quad g(x) = \csc x$

SOLUTION Let $h(x) = f(g(x))$. Then, applying the quotient rule:

$$h'(x) = \frac{(\csc x - 1)(-\csc x \cot x) - (\csc x)(-\csc x \cot x)}{(\csc x - 1)^2} = \frac{\csc x \cot x}{(\csc x - 1)^2}.$$

Alternately,

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = -\frac{1}{(\csc x - 1)^2}(-\csc x \cot x) = \frac{\csc x \cot x}{(\csc x - 1)^2},$$

where we have used the quotient rule to calculate $f'(u) = -\frac{1}{(u-1)^2}$.

In Exercises 27 and 28, find the derivatives of $f(g(x))$ and $g(f(x))$.

27. $f(u) = \cos u, \quad u = g(x) = x^2 + 1$

SOLUTION

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = -\sin(x^2+1)(2x) = -2x\sin(x^2+1).$$
$$\frac{d}{dx} g(f(x)) = g'(f(x))f'(x) = 2(\cos x)(-\sin x) = -2\sin x\cos x.$$

28. $f(u) = u^3, \quad u = g(x) = \dfrac{1}{x+1}$

SOLUTION The derivative of $\frac{1}{x+1}$ is taken using the chain rule.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) = 3\left(\frac{1}{x+1}\right)^2\left(-\frac{1}{(x+1)^2}\right) = -\frac{3}{(x+1)^4}.$$
$$\frac{d}{dx} g(f(x)) = g'(f(x))f'(x) = -\frac{1}{(x^3+1)^2}(3x^2) = -\frac{3x^2}{(x^3+1)^2}.$$

In Exercises 29–42, use the Chain Rule to find the derivative.

29. $y = \sin(x^2)$

SOLUTION Let $y = \sin\left(x^2\right)$. Then $y' = \cos\left(x^2\right)\cdot 2x = 2x\cos\left(x^2\right)$.

30. $y = \sin^2 x$

SOLUTION Let $y = \sin^2 x = (\sin x)^2$. Then $y' = 2\sin x(\cos x)$.

31. $y = \sqrt{t^2+9}$

SOLUTION Let $y = \sqrt{t^2+9} = (t^2+9)^{1/2}$. Then

$$y' = \frac{1}{2}(t^2+9)^{-1/2}(2t) = \frac{t}{\sqrt{t^2+9}}.$$

32. $y = (t^2+3t+1)^{-5/2}$

SOLUTION Let $y = \left(t^2+3t+1\right)^{-5/2}$. Then

$$y' = -\frac{5}{2}\left(t^2+3t+1\right)^{-7/2}(2t+3) = -\frac{5\,(2t+3)}{2\left(t^2+3t+1\right)^{7/2}}.$$

33. $y = (x^4-x^3-1)^{2/3}$

SOLUTION Let $y = \left(x^4-x^3-1\right)^{2/3}$. Then

$$y' = \frac{2}{3}\left(x^4-x^3-1\right)^{-1/3}\left(4x^3-3x^2\right).$$

34. $y = (\sqrt{x+1}-1)^{3/2}$

SOLUTION Let $y = \left((x+1)^{1/2}-1\right)^{3/2}$. Here, we note that calculating the derivative of the inside function, $\sqrt{x+1}-1$, requires the chain rule. After two applications of the chain rule, we have

$$y' = \frac{3}{2}\left((x+1)^{1/2}-1\right)^{1/2}\cdot\left(\frac{1}{2}(x+1)^{-1/2}\cdot 1\right) = \frac{3\sqrt{\sqrt{x+1}-1}}{4\sqrt{x+1}}.$$

35. $y = \left(\dfrac{x+1}{x-1}\right)^4$

SOLUTION Let $y = \left(\dfrac{x+1}{x-1}\right)^4$. Then

$$y' = 4\left(\frac{x+1}{x-1}\right)^3\cdot\frac{(x-1)\cdot 1-(x+1)\cdot 1}{(x-1)^2} = -\frac{8\,(x+1)^3}{(x-1)^5} = \frac{8(1+x)^3}{(1-x)^5}.$$

36. $y = \cos^3(12\theta)$

SOLUTION After two applications of the chain rule,

$$y' = 3\cos^2(12\theta)(-\sin(12\theta))(12) = -36\cos^2(12\theta)\sin(12\theta).$$

37. $y = \sec \dfrac{1}{x}$

SOLUTION Let $f(x) = \sec\left(x^{-1}\right)$. Then

$$f'(x) = \sec\left(x^{-1}\right)\tan\left(x^{-1}\right)\cdot\left(-x^{-2}\right) = -\frac{\sec(1/x)\tan(1/x)}{x^2}.$$

38. $y = \tan(\theta^2 - 4\theta)$

SOLUTION Let $y = \tan(\theta^2 - 4\theta)$. Then

$$y' = \sec^2(\theta^2 - 4\theta)\cdot(2\theta - 4) = (2\theta - 4)\sec^2(\theta^2 - 4\theta).$$

39. $y = \tan(\theta + \cos\theta)$

SOLUTION Let $y = \tan(\theta + \cos\theta)$. Then

$$y' = \sec^2(\theta + \cos\theta)\cdot(1 - \sin\theta) = (1 - \sin\theta)\sec^2(\theta + \cos\theta).$$

40. $y = e^{2x^2}$

SOLUTION Let $y = e^{2x^2}$. Then

$$y' = e^{2x^2}(4x) = 4xe^{2x^2}.$$

41. $y = e^{2-9t^2}$

SOLUTION Let $y = e^{2-9t^2}$. Then

$$y' = e^{2-9t^2}(-18t) = -18te^{2-9t^2}.$$

42. $y = \cos^3(e^{4\theta})$

SOLUTION Let $y = \cos^3(e^{4\theta})$. After two applications of the chain rule, we have

$$y' = 3\cos^2(e^{4\theta})(-\sin(e^{4\theta}))(4e^{4\theta}) = -12e^{4\theta}\cos^2(e^{4\theta})\sin(e^{4\theta}).$$

In Exercises 43–72, find the derivative using the appropriate rule or combination of rules.

43. $y = \tan(x^2 + 4x)$

SOLUTION Let $y = \tan(x^2 + 4x)$. By the chain rule,

$$y' = \sec^2(x^2 + 4x)\cdot(2x + 4) = (2x + 4)\sec^2(x^2 + 4x).$$

44. $y = \sin(x^2 + 4x)$

SOLUTION Let $y = \sin(x^2 + 4x)$. By the chain rule,

$$\frac{dy}{dx} = (2x + 4)\cos(x^2 + 4x).$$

45. $y = x\cos(1 - 3x)$

SOLUTION Let $y = x\cos(1 - 3x)$. Applying the product rule and then the chain rule,

$$y' = x(-\sin(1 - 3x))\cdot(-3) + \cos(1 - 3x)\cdot 1 = 3x\sin(1 - 3x) + \cos(1 - 3x).$$

46. $y = \sin(x^2)\cos(x^2)$

SOLUTION We start by using a trig identity to rewrite

$$y = \sin(x^2)\cos(x^2) = \frac{1}{2}\sin(2x^2).$$

Then, by the chain rule,

$$y' = \frac{1}{2}\cos(2x^2)\cdot 4x = 2x\cos(2x^2).$$

47. $y = (4t+9)^{1/2}$

SOLUTION Let $y = (4t+9)^{1/2}$. By the general power rule,

$$\frac{dy}{dt} = 4\left(\frac{1}{2}\right)(4t+9)^{-1/2} = 2(4t+9)^{-1/2}.$$

48. $y = (z+1)^4(2z-1)^3$

SOLUTION Let $y = (z+1)^4(2z-1)^3$. Applying the product rule and the general power rule,

$$\begin{aligned}\frac{dy}{dz} &= (z+1)^4(3(2z-1)^2)(2) + (2z-1)^3(4(z+1)^3)(1) = (z+1)^3(2z-1)^2(6(z+1)+4(2z-1))\\ &= (z+1)^3(2z-1)^2(14z+2).\end{aligned}$$

49. $y = (x^3+\cos x)^{-4}$

SOLUTION Let $y = (x^3+\cos x)^{-4}$. By the general power rule,

$$y' = -4(x^3+\cos x)^{-5}(3x^2-\sin x) = 4(\sin x - 3x^2)(x^3+\cos x)^{-5}.$$

50. $y = \sin(\cos(\sin x))$

SOLUTION Let $y = \sin(\cos(\sin x))$. Applying the chain rule twice,

$$y' = \cos(\cos(\sin x))\cdot(-\sin(\sin x))\cdot\cos x = -\cos x\sin(\sin x)\cos(\cos(\sin x)).$$

51. $y = \sqrt{\sin x\cos x}$

SOLUTION We start by using a trig identity to rewrite

$$y = \sqrt{\sin x\cos x} = \sqrt{\frac{1}{2}\sin 2x} = \frac{1}{\sqrt{2}}(\sin 2x)^{1/2}.$$

Then, after two applications of the chain rule,

$$y' = \frac{1}{\sqrt{2}}\cdot\frac{1}{2}(\sin 2x)^{-1/2}\cdot\cos 2x\cdot 2 = \frac{\cos 2x}{\sqrt{2\sin 2x}}.$$

52. $y = (9-(5-2x^4)^7)^3$

SOLUTION Let $y = (9-(5-2x^4)^7)^3$. Applying the chain rule twice, we find

$$y' = 3(9-(5-2x^4)^7)^2(-7(5-2x^4)^6)(-8x^3) = 168x^3(5-2x^4)^6(9-(5-2x^4)^7)^2.$$

53. $y = (\cos 6x + \sin x^2)^{1/2}$

SOLUTION Let $y = (\cos 6x + \sin(x^2))^{1/2}$. Applying the general power rule followed by two applications of the chain rule,

$$y' = \frac{1}{2}\left(\cos 6x + \sin(x^2)\right)^{-1/2}\left(-\sin 6x\cdot 6 + \cos(x^2)\cdot 2x\right) = \frac{x\cos(x^2) - 3\sin 6x}{\sqrt{\cos 6x + \sin(x^2)}}.$$

54. $y = \dfrac{(x+1)^{1/2}}{x+2}$

SOLUTION Let $y = \frac{(x+1)^{1/2}}{x+2}$. Applying the quotient rule and the general power rule, we get

$$\frac{dy}{dx} = \frac{(x+2)\frac{1}{2}(x+1)^{-1/2} - (x+1)^{1/2}}{(x+2)^2} = \frac{1}{2\sqrt{x+1}}\frac{(x+2)-2(x+1)}{(x+2)^2} = -\frac{1}{2\sqrt{x+1}}\frac{x}{(x+2)^2}.$$

55. $y = \tan^3 x + \tan(x^3)$

SOLUTION Let $y = \tan^3 x + \tan(x^3) = (\tan x)^3 + \tan(x^3)$. Applying the general power rule to the first term and the chain rule to the second term,

$$y' = 3(\tan x)^2 \sec^2 x + \sec^2(x^3) \cdot 3x^2 = 3\big(x^2 \sec^2(x^3) + \sec^2 x \tan^2 x\big).$$

56. $y = \sqrt{4 - 3\cos x}$

SOLUTION Let $y = (4 - 3\cos x)^{1/2}$. By the general power rule,

$$y' = \frac{1}{2}(4 - 3\cos x)^{-1/2} \cdot 3\sin x = \frac{3\sin x}{2\sqrt{4 - 3\cos x}}.$$

57. $y = \sqrt{\dfrac{z+1}{z-1}}$

SOLUTION Let $y = \left(\dfrac{z+1}{z-1}\right)^{1/2}$. Applying the general power rule followed by the quotient rule,

$$\frac{dy}{dz} = \frac{1}{2}\left(\frac{z+1}{z-1}\right)^{-1/2} \cdot \frac{(z-1)\cdot 1 - (z+1)\cdot 1}{(z-1)^2} = \frac{-1}{\sqrt{z+1}\,(z-1)^{3/2}}.$$

58. $y = (\cos^3 x + 3\cos x + 7)^9$

SOLUTION Let $y = \left(\cos^3 x + 3\cos x + 7\right)^9$. Applying the general power rule followed by the sum rule, with the first term requiring the general power rule,

$$\begin{aligned}\frac{dy}{dx} &= 9\left(\cos^3 x + 3\cos x + 7\right)^8 \left(3\cos^2 x \cdot (-\sin x) - 3\sin x\right)\\ &= -27\sin x\left(\cos^3 x + 3\cos x + 7\right)^8\left(1 + \cos^2 x\right).\end{aligned}$$

59. $y = \dfrac{\cos(1+x)}{1+\cos x}$

SOLUTION Let

$$y = \frac{\cos(1+x)}{1+\cos x}.$$

Then, applying the quotient rule and the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{-(1+\cos x)\sin(1+x) + \cos(1+x)\sin x}{(1+\cos x)^2} = \frac{\cos(1+x)\sin x - \cos x\sin(1+x) - \sin(1+x)}{(1+\cos x)^2}\\ &= \frac{\sin(-1) - \sin(1+x)}{(1+\cos x)^2}.\end{aligned}$$

The last line follows from the identity

$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$

with $A = x$ and $B = 1 + x$.

60. $y = \sec(\sqrt{t^2 - 9})$

SOLUTION Let $y = \sec\left(\sqrt{t^2 - 9}\right)$. Applying the chain rule followed by the general power rule,

$$\frac{dy}{dt} = \sec\left(\sqrt{t^2-9}\right)\tan\left(\sqrt{t^2-9}\right)\cdot\frac{1}{2}\left(t^2-9\right)^{-1/2}\cdot 2t = \frac{t\sec\left(\sqrt{t^2-9}\right)\tan\left(\sqrt{t^2-9}\right)}{\sqrt{t^2-9}}.$$

61. $y = \cot^7(x^5)$

SOLUTION Let $y = \cot^7\left(x^5\right)$. Applying the general power rule followed by the chain rule,

$$\frac{dy}{dx} = 7\cot^6\left(x^5\right)\cdot\left(-\csc^2\left(x^5\right)\right)\cdot 5x^4 = -35x^4\cot^6\left(x^5\right)\csc^2\left(x^5\right).$$

62. $y = \dfrac{\cos(1/x)}{1+x^2}$

SOLUTION Let $y = \frac{\cos(1/x)}{1+x^2} = \frac{\cos(x^{-1})}{1+x^2}$. Then, applying the quotient rule and the chain rule, we get:

$$\frac{dy}{dx} = \frac{(1+x^2)(x^{-2}\sin(x^{-1})) - \cos(x^{-1})(2x)}{(1+x^2)^2} = \frac{\sin(x^{-1}) - 2x\cos(x^{-1}) + x^{-2}\sin(x^{-1})}{(1+x^2)^2}.$$

63. $y = \left(1 + \cot^5(x^4+1)\right)^9$

SOLUTION Let $y = \left(1+\cot^5\left(x^4+1\right)\right)^9$. Applying the general power rule, the chain rule, and the general power rule in succession,

$$\begin{aligned}\frac{dy}{dx} &= 9\left(1+\cot^5\left(x^4+1\right)\right)^8 \cdot 5\cot^4\left(x^4+1\right)\cdot\left(-\csc^2\left(x^4+1\right)\right)\cdot 4x^3\\ &= -180x^3\cot^4\left(x^4+1\right)\csc^2\left(x^4+1\right)\left(1+\cot^5\left(x^4+1\right)\right)^8.\end{aligned}$$

64. $y = 4e^{-x} + 7e^{-2x}$

SOLUTION Let $y = 4e^{-x} + 7e^{-2x}$. Using the chain rule twice, once for each exponential function, we obtain

$$\frac{dy}{dx} = -4e^{-x} - 14e^{-2x}.$$

65. $y = (2e^{3x} + 3e^{-2x})^4$

SOLUTION Let $y = (2e^{3x} + 3e^{-2x})^4$. Applying the general power rule followed by two applications of the chain rule, one for each exponential function, we find

$$\frac{dy}{dx} = 4(2e^{3x}+3e^{-2x})^3(6e^{3x} - 6e^{-2x}) = 24(2e^{3x}+3e^{-2x})^3(e^{3x} - e^{-2x}).$$

66. $y = \cos(te^{-2t})$

SOLUTION Let $y = \cos(te^{-2t})$. Applying the chain rule and the product rule, we have

$$\frac{dy}{dt} = -\sin(te^{-2t})\left(-2te^{-2t} + e^{-2t}\right) = e^{-2t}(2t-1)\sin(te^{-2t}).$$

67. $y = e^{(x^2+2x+3)^2}$

SOLUTION Let $y = e^{(x^2+2x+3)^2}$. By the chain rule and the general power rule, we obtain

$$\frac{dy}{dx} = e^{(x^2+2x+3)^2}\cdot 2(x^2+2x+3)(2x+2) = 4(x+1)(x^2+2x+3)e^{(x^2+2x+3)^2}.$$

68. $y = e^{e^x}$

SOLUTION Let $y = e^{e^x}$. Applying the chain rule, we have

$$\frac{dy}{dx} = e^{e^x}e^x.$$

69. $y = \sqrt{1+\sqrt{1+\sqrt{x}}}$

SOLUTION Let $y = \left(1 + \left(1+x^{1/2}\right)^{1/2}\right)^{1/2}$. Applying the general power rule twice,

$$\frac{dy}{dx} = \frac{1}{2}\left(1+\left(1+x^{1/2}\right)^{1/2}\right)^{-1/2}\cdot\frac{1}{2}\left(1+x^{1/2}\right)^{-1/2}\cdot\frac{1}{2}x^{-1/2} = \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}}.$$

70. $y = \sqrt{\sqrt{x+1}+1}$

SOLUTION Let $y = \left(1 + (x+1)^{1/2}\right)^{1/2}$. Applying the general power rule twice,

$$\frac{dy}{dx} = \frac{1}{2}\left(1 + (x+1)^{1/2}\right)^{-1/2} \cdot \frac{1}{2}(x+1)^{-1/2} \cdot 1 = \frac{1}{4\sqrt{x+1}\sqrt{1+\sqrt{x+1}}}.$$

71. $y = (kx+b)^{-1/3}$; k and b any constants

SOLUTION Let $y = (kx+b)^{-1/3}$, where b and k are constants. By the general power rule,

$$y' = -\frac{1}{3}(kx+b)^{-4/3} \cdot k = -\frac{k}{3}(kx+b)^{-4/3}.$$

72. $y = \dfrac{1}{\sqrt{kt^4+b}}$; k, b constants, not both zero

SOLUTION Let $y = \left(kt^4+b\right)^{-1/2}$, where b and k are constants. By the chain rule,

$$y' = -\frac{1}{2}\left(kt^4+b\right)^{-3/2} \cdot 4kt^3 = -\frac{2kt^3}{\left(kt^4+b\right)^{3/2}}.$$

In Exercises 73–76, compute the higher derivative.

73. $\dfrac{d^2}{dx^2}\sin(x^2)$

SOLUTION Let $f(x) = \sin\left(x^2\right)$. Then, by the chain rule, $f'(x) = 2x\cos\left(x^2\right)$ and, by the product rule and the chain rule,

$$f''(x) = 2x\left(-\sin\left(x^2\right)\cdot 2x\right) + 2\cos\left(x^2\right) = 2\cos\left(x^2\right) - 4x^2\sin\left(x^2\right).$$

74. $\dfrac{d^2}{dx^2}(x^2+9)^5$

SOLUTION Let $f(x) = (x^2+9)^5$. Then, by the general power rule,

$$f'(x) = 5(x^2+9)^4 \cdot 2x = 10x(x^2+9)^4$$

and, by the product rule and the general power rule,

$$f''(x) = 10x \cdot 4(x^2+9)^3 \cdot 2x + 10(x^2+9)^4 = 80x^2(x^2+9)^3 + 10(x^2+9)^4.$$

75. $\dfrac{d^3}{dx^3}(9-x)^8$

SOLUTION Let $f(x) = (9-x)^8$. Then, by repeated use of the general power rule,

$$f'(x) = 8(9-x)^7 \cdot (-1) = -8(9-x)^7$$
$$f''(x) = -56(9-x)^6 \cdot (-1) = 56(9-x)^6,$$
$$f'''(x) = 336(9-x)^5 \cdot (-1) = -336(9-x)^5.$$

76. $\dfrac{d^3}{dx^3}\sin(2x)$

SOLUTION Let $f(x) = \sin(2x)$. Then, by repeated use of the chain rule,

$$f'(x) = 2\cos(2x)$$
$$f''(x) = -4\sin(2x)$$
$$f'''(x) = -8\cos(2x).$$

77. The average molecular velocity v of a gas in a certain container is given by $v(T) = 29\sqrt{T}$ m/s, where T is the temperature in kelvins. The temperature is related to the pressure (in atmospheres) by $T = 200P$. Find $\left.\frac{dv}{dP}\right|_{P=1.5}$.

SOLUTION First note that when $P = 1.5$ atmospheres, $T = 200(1.5) = 300$K. Thus,

$$\left.\frac{dv}{dP}\right|_{P=1.5} = \left.\frac{dv}{dT}\right|_{T=300} \cdot \left.\frac{dT}{dP}\right|_{P=1.5} = \frac{29}{2\sqrt{300}} \cdot 200 = \frac{290\sqrt{3}}{3} \frac{\text{m}}{\text{s} \cdot \text{atmospheres}}.$$

Alternately, substituting $T = 200P$ into the equation for v gives $v = 290\sqrt{2P}$. Therefore,

$$\frac{dv}{dP} = \frac{290\sqrt{2}}{2\sqrt{P}} = \frac{290}{\sqrt{2P}},$$

so

$$\left.\frac{dv}{dP}\right|_{P=1.5} = \frac{290}{\sqrt{3}} = \frac{290\sqrt{3}}{3} \frac{\text{m}}{\text{s} \cdot \text{atmospheres}}.$$

78. The power P in a circuit is $P = Ri^2$, where R is the resistance and i is the current. Find dP/dt at $t = \frac{1}{3}$ if $R = 1000\ \Omega$ and i varies according to $i = \sin(4\pi t)$ (time in seconds).

SOLUTION $\left.\frac{d}{dt}\left(Ri^2\right)\right|_{t=2} = \left.2Ri\frac{di}{dt}\right|_{t=2} = 2(1000)4\pi\ \sin(4\pi t)\cos(4\pi t)|_{t=2} = 0.$

79. An expanding sphere has radius $r = 0.4t$ cm at time t (in seconds). Let V be the sphere's volume. Find dV/dt when (a) $r = 3$ and (b) $t = 3$.

SOLUTION Let $r = 0.4t$, where t is in seconds (s) and r is in centimeters (cm). With $V = \frac{4}{3}\pi r^3$, we have

$$\frac{dV}{dr} = 4\pi r^2.$$

Thus

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \cdot (0.4) = 1.6\pi r^2.$$

(a) When $r = 3$, $\frac{dV}{dt} = 1.6\pi(3)^2 \approx 45.24$ cm^3/s.

(b) When $t = 3$, we have $r = 1.2$. Hence $\frac{dV}{dt} = 1.6\pi(1.2)^2 \approx 7.24$ cm^3/s.

80. A 2005 study by the Fisheries Research Services in Aberdeen, Scotland, suggests that the average length of the species *Clupea harengus* (Atlantic herring) as a function of age t (in years) can be modeled by $L(t) = 32(1 - e^{-0.37t})$ cm for $0 \le t \le 13$. See Figure 1.

(a) How fast is the average length changing at age $t = 6$ years?

(b) At what age is the average length changing at a rate of 5 cm/year?

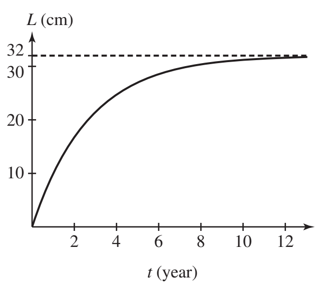


FIGURE 1 Average length of the species *Clupea harengus*.

SOLUTION Let $L(t) = 32(1 - e^{-0.37t})$. Then

$$L'(t) = 32(0.37)e^{-0.37t} = 11.84e^{-0.37t}.$$

(a) At age $t = 6$,

$$L'(t) = 11.84e^{-0.37(6)} = 11.84e^{-2.22} \approx 1.29 \text{ cm/yr}.$$

(b) The length will be changing at a rate of 5 cm/yr when

$$11.84e^{-0.37t} = 5.$$

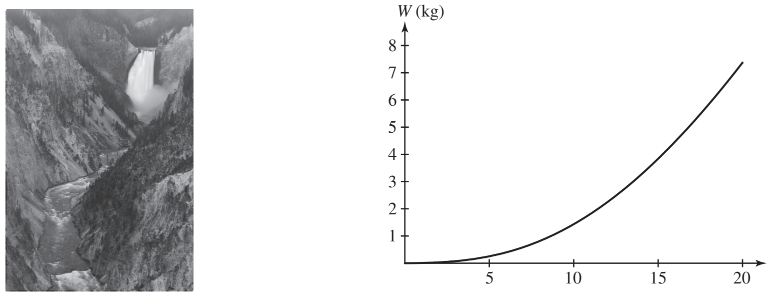
Solving for t yields

$$t = -\frac{1}{0.37} \ln \frac{5}{11.84} \approx 2.33 \text{ years}.$$

81. A 1999 study by Starkey and Scarnecchia developed the following model for the average weight (in kilograms) at age t (in years) of channel catfish in the Lower Yellowstone River (Figure 2):

$$W(t) = (3.46293 - 3.32173e^{-0.03456t})^{3.4026}$$

Find the rate at which average weight is changing at age $t = 10$.



Lower Yellowstone River

FIGURE 2 Average weight of channel catfish at age t.

SOLUTION Let $W(t) = (3.46293 - 3.32173e^{-0.03456t})^{3.4026}$. Then

$$W'(t) = 3.4026(3.46293 - 3.32173e^{-0.03456t})^{2.4026}(3.32173)(0.03456)e^{-0.03456t}$$
$$= 0.3906(3.46293 - 3.32173e^{-0.03456t})^{2.4026}e^{-0.03456t}.$$

At age $t = 10$,

$$W'(10) = 0.3906(1.1118)^{2.4026}(0.7078) \approx 0.3566 \text{ kg/yr}.$$

82. The functions in Exercises 80 and 81 are examples of the **von Bertalanffy growth function**

$$M(t) = \left(a + (b-a)e^{kmt}\right)^{1/m} \qquad (m \neq 0)$$

introduced in the 1930s by Austrian-born biologist Karl Ludwig von Bertalanffy. Calculate $M'(0)$ in terms of the constants a, b, k and m.

SOLUTION Let

$$M(t) = \left(a + (b-a)e^{kmt}\right)^{1/m} \qquad (m \neq 0).$$

Then

$$M'(t) = \frac{1}{m}(a + (b-a)e^{kmt})^{1/m-1} km(b-a)e^{kmt} = k(b-a)e^{kmt}(a + (b-a)e^{kmt})^{1/m-1},$$

and

$$M'(0) = k(b-a)e^0(a + (b-a)e^0)^{1/m-1} = k(b-a)b^{1/m-1}.$$

83. With notation as in Example 8, calculate

(a) $\left.\dfrac{d}{d\theta} \underline{\sin}\,\theta\right|_{\theta=60^\circ}$ **(b)** $\left.\dfrac{d}{d\theta}\left(\theta + \underline{\tan}\,\theta\right)\right|_{\theta=45^\circ}$

SOLUTION

(a) $\left.\dfrac{d}{d\theta} \underline{\sin}\,\theta\right|_{\theta=60^\circ} = \left.\dfrac{d}{d\theta} \sin\left(\dfrac{\pi}{180}\theta\right)\right|_{\theta=60^\circ} = \left(\dfrac{\pi}{180}\right)\cos\left(\dfrac{\pi}{180}(60)\right) = \dfrac{\pi}{180}\dfrac{1}{2} = \dfrac{\pi}{360}.$

(b) $\frac{d}{d\theta}\left(\theta + \underline{\tan}\,\theta\right)\Big|_{\theta=45^\circ} = \frac{d}{d\theta}\left(\theta + \tan\left(\frac{\pi}{180}\theta\right)\right)\Big|_{\theta=45^\circ} = 1 + \frac{\pi}{180}\sec^2\left(\frac{\pi}{4}\right) = 1 + \frac{\pi}{90}.$

84. Assume that

$$f(0) = 2, \quad f'(0) = 3, \quad h(0) = -1, \quad h'(0) = 7$$

Calculate the derivatives of the following functions at $x = 0$:

(a) $(f(x))^3$ **(b)** $f(7x)$ **(c)** $f(4x)h(5x)$

SOLUTION

(a) Let $g(x) = (f(x))^3$. Then

$$g'(0) = 3(f(0))^2(f'(0)) = 12(3) = 36.$$

(b) Let $g(x) = f(7x)$. Then

$$g'(0) = 7f'(7(0)) = 21.$$

(c) Let $F(x) = f(4x)h(5x)$. Then $F'(x) = 4f'(4x)h(5x) + 5f(4x)h'(5x)$ and

$$F'(0) = 4(3)(-1) + 5(2)(7) = 58.$$

85. Compute the derivative of $h(\sin x)$ at $x = \frac{\pi}{6}$, assuming that $h'(0.5) = 10$.

SOLUTION Let $u = \sin x$ and suppose that $h'(0.5) = 10$. Then

$$\frac{d}{dx}(h(u)) = \frac{dh}{du}\frac{du}{dx} = \frac{dh}{du}\cos x.$$

When $x = \frac{\pi}{6}$, we have $u = 0.5$. Accordingly, the derivative of $h(\sin x)$ at $x = \frac{\pi}{6}$ is $10\cos\left(\frac{\pi}{6}\right) = 5\sqrt{3}$.

86. Let $F(x) = f(g(x))$, where the graphs of f and g are shown in Figure 3. Estimate $g'(2)$ and $f'(g(2))$ and compute $F'(2)$.

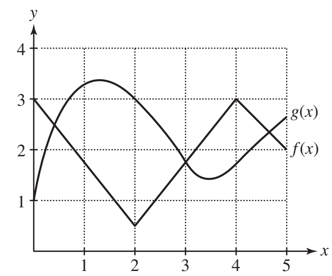
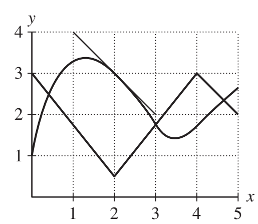


FIGURE 3

SOLUTION After sketching an approximate tangent line to g at $x = 2$ (see the figure below), we estimate $g'(2) = -1$. It appears from the graph that $g(2) = 3$ and $f'(3) = \frac{5}{4}$ (since between $x = 2$ and $x = 4$ the graph of f appears to be linear with slope $\frac{5}{4}$). Thus,

$$F'(2) = f'(g(2))g'(2) = \frac{5}{4}(-1) = -1.25.$$



In Exercises 87–90, use the table of values to calculate the derivative of the function at the given point.

x	1	4	6
$f(x)$	4	0	6
$f'(x)$	5	7	4
$g(x)$	4	1	6
$g'(x)$	5	$\frac{1}{2}$	3

87. $f(g(x)), \quad x = 6$

SOLUTION $\left.\frac{d}{dx} f(g(x))\right|_{x=6} = f'(g(6))g'(6) = f'(6)g'(6) = 4 \times 3 = 12.$

88. $e^{f(x)}, \quad x = 4$

SOLUTION $\left.\frac{d}{dx} e^{f(x)}\right|_{x=4} = e^{f(4)} f'(4) = e^0(7) = 7.$

89. $g(\sqrt{x}), \quad x = 16$

SOLUTION $\left.\frac{d}{dx} g(\sqrt{x})\right|_{x=16} = g'(4)\left(\frac{1}{2}\right)(1/\sqrt{16}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right) = \frac{1}{16}.$

90. $f(2x + g(x)), \quad x = 1$

SOLUTION $\left.\frac{d}{dx} f(2x + g(x))\right|_{x=1} = f'(2(1) + g(1))(2 + g'(1)) = f'(2+4)(7) = 4(7) = 28.$

91. The price (in dollars) of a computer component is $P = 2C - 18C^{-1}$, where C is the manufacturer's cost to produce it. Assume that cost at time t (in years) is $C = 9 + 3t^{-1}$. Determine the rate of change of price with respect to time at $t = 3$.

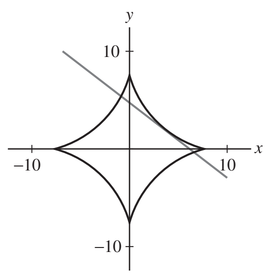
SOLUTION $\frac{dC}{dt} = -3t^{-2}$. $C(3) = 10$ and $C'(3) = -\frac{1}{3}$, so we compute:

$$\left.\frac{dP}{dt}\right|_{t=3} = 2C'(3) + \frac{18}{(C(3))^2} C'(3) = -\frac{2}{3} + \frac{18}{100}\left(-\frac{1}{3}\right) = -0.727 \ \frac{\text{dollars}}{\text{year}}.$$

92. GU Plot the "astroid" $y = (4 - x^{2/3})^{3/2}$ for $0 \le x \le 8$. Show that the part of every tangent line in the first quadrant has a constant length 8.

SOLUTION

- Here is a graph of the astroid.



- Let $f(x) = (4 - x^{2/3})^{3/2}$. Then

$$f'(x) = \frac{3}{2}(4 - x^{2/3})^{1/2}\left(-\frac{2}{3}x^{-1/3}\right) = -\frac{\sqrt{4 - x^{2/3}}}{x^{1/3}},$$

  and the tangent line to f at $x = a$ is

$$y = -\frac{\sqrt{4 - a^{2/3}}}{a^{1/3}}(x - a) + \left(4 - a^{2/3}\right)^{3/2}.$$

  The y-intercept of this line is the point $P = \left(0, 4\sqrt{4 - a^{2/3}}\right)$, its x-intercept is the point $Q = \left(4a^{1/3}, 0\right)$, and the distance between P and Q is 8.

93. According to the U.S. standard atmospheric model, developed by the National Oceanic and Atmospheric Administration for use in aircraft and rocket design, atmospheric temperature T (in degrees Celsius), pressure P (kPa = 1000 pascals), and altitude h (in meters) are related by these formulas (valid in the troposphere $h \le 11{,}000$):

$$T = 15.04 - 0.000649h, \qquad P = 101.29 + \left(\frac{T + 273.1}{288.08}\right)^{5.256}$$

Use the Chain Rule to calculate dP/dh. Then estimate the change in P (in pascals, Pa) per additional meter of altitude when $h = 3000$.

SOLUTION

$$\frac{dP}{dT} = 5.256\left(\frac{T + 273.1}{288.08}\right)^{4.256}\left(\frac{1}{288.08}\right) = 6.21519 \times 10^{-13}\,(273.1 + T)^{4.256}$$

and $\frac{dT}{dh} = -0.000649°\text{C/m}$. $\frac{dP}{dh} = \frac{dP}{dT}\frac{dT}{dh}$, so

$$\frac{dP}{dh} = \left(6.21519 \times 10^{-13}\,(273.1 + T)^{4.256}\right)(-0.000649) = -4.03366 \times 10^{-16}\,(288.14 - 0.000649\,h)^{4.256}.$$

When $h = 3000$,

$$\frac{dP}{dh} = -4.03366 \times 10^{-16}(286.193)^{4.256} = -1.15 \times 10^{-5}\ \text{kPa/m};$$

therefore, for each additional meter of altitude,

$$\Delta P \approx -1.15 \times 10^{-5}\ \text{kPa} = -1.15 \times 10^{-2}\ \text{Pa}.$$

94. Climate scientists use the **Stefan–Boltzmann Law** $R = \sigma T^4$ to estimate the change in the earth's average temperature T (in kelvins) caused by a change in the radiation R (in joules per square meter per second) that the earth receives from the sun. Here, $\sigma = 5.67 \times 10^{-8}\ \text{Js}^{-1}\text{m}^{-2}\text{K}^{-4}$. Calculate dR/dt, assuming that $T = 283$ and $\frac{dT}{dt} = 0.05$ K/year. What are the units of the derivative?

SOLUTION By the Chain Rule,

$$\frac{dR}{dt} = \frac{dR}{dT}\cdot\frac{dT}{dt} = 4\sigma T^3\frac{dT}{dt}.$$

Assuming $T = 283$ K and $\frac{dT}{dt} = 0.05$ K/yr, it follows that

$$\frac{dR}{dt} = 4\sigma(283^3)(0.05) \approx 0.257\ \text{Js}^{-1}\text{m}^{-2}\text{/yr}$$

95. In the setting of Exercise 94, calculate the yearly rate of change of T if $T = 283$ K and R increases at a rate of $0.5\ \text{Js}^{-1}\text{m}^{-2}$ per year.

SOLUTION By the Chain Rule,

$$\frac{dR}{dt} = \frac{dR}{dT}\cdot\frac{dT}{dt} = 4\sigma T^3\frac{dT}{dt}.$$

Assuming $T = 283$ K and $\frac{dR}{dt} = 0.5\ \text{Js}^{-1}\text{m}^{-2}$ per year, it follows that

$$0.5 = 4\sigma(283)^3\frac{dT}{dt} \Rightarrow \frac{dT}{dt} = \frac{0.5}{4\sigma(283)^3} \approx 0.0973\ \text{kelvins/yr}$$

96. CAS Use a computer algebra system to compute $f^{(k)}(x)$ for $k = 1, 2, 3$ for the following functions:

(a) $f(x) = \cot(x^2)$ **(b)** $f(x) = \sqrt{x^3 + 1}$

SOLUTION

(a) Let $f(x) = \cot(x^2)$. Using a computer algebra system,

$$f'(x) = -2x\csc^2(x^2);$$
$$f''(x) = 2\csc^2(x^2)(4x^2\cot(x^2) - 1);\ \text{and}$$
$$f'''(x) = -8x\csc^2(x^2)\left(6x^2\cot^2(x^2) - 3\cot(x^2) + 2x^2\right).$$

(b) Let $f(x) = \sqrt{x^3+1}$. Using a computer algebra system,

$$f'(x) = \frac{3x^2}{2\sqrt{x^3+1}};$$
$$f''(x) = \frac{3x(x^3+4)}{4(x^3+1)^{3/2}}; \text{ and}$$
$$f'''(x) = -\frac{3(x^6+20x^3-8)}{8(x^3+1)^{5/2}}.$$

97. Use the Chain Rule to express the second derivative of $f \circ g$ in terms of the first and second derivatives of f and g.

SOLUTION Let $h(x) = f(g(x))$. Then

$$h'(x) = f'(g(x))g'(x)$$

and

$$h''(x) = f'(g(x))g''(x) + g'(x)f''(g(x))g'(x) = f'(g(x))g''(x) + f''(g(x))\left(g'(x)\right)^2.$$

98. Compute the second derivative of $\sin(g(x))$ at $x = 2$, assuming that $g(2) = \frac{\pi}{4}$, $g'(2) = 5$, and $g''(2) = 3$.

SOLUTION Let $f(x) = \sin(g(x))$. Then $f'(x) = \cos(g(x))g'(x)$ and

$$f''(x) = \cos(g(x))g''(x) + g'(x)(-\sin(g(x)))g'(x) = \cos(g(x))g''(x) - (g'(x))^2\sin(g(x)).$$

Therefore,

$$f''(2) = g''(2)\cos(g(2)) - \left(g'(2)\right)^2\sin(g(2)) = 3\cos\left(\tfrac{\pi}{4}\right) - (5)^2\sin\left(\tfrac{\pi}{4}\right) = -22\cdot\tfrac{\sqrt{2}}{2} = -11\sqrt{2}$$

Further Insights and Challenges

99. Show that if f, g, and h are differentiable, then

$$[f(g(h(x)))]' = f'(g(h(x)))g'(h(x))h'(x)$$

SOLUTION Let f, g, and h be differentiable. Let $u = h(x)$, $v = g(u)$, and $w = f(v)$. Then

$$\frac{dw}{dx} = \frac{df}{dv}\frac{dv}{dx} = \frac{df}{dv}\frac{dg}{du}\frac{du}{dx} = f'(g(h(x)))g'(h(x))h'(x)$$

100. Show that differentiation reverses parity: If f is even, then f' is odd, and if f is odd, then f' is even. *Hint:* Differentiate $f(-x)$.

SOLUTION A function is *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$. By the chain rule, $\frac{d}{dx}f(-x) = -f'(-x)$. Now suppose that f is even. Then $f(-x) = f(x)$ and

$$\frac{d}{dx}f(-x) = \frac{d}{dx}f(x) = f'(x).$$

Hence, when f is even, $-f'(-x) = f'(x)$ or $f'(-x) = -f'(x)$ and f' is odd. On the other hand, suppose f is odd. Then $f(-x) = -f(x)$ and

$$\frac{d}{dx}f(-x) = -\frac{d}{dx}f(x) = -f'(x).$$

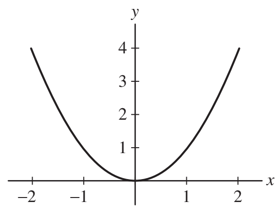
Hence, when f is odd, $-f'(-x) = -f'(x)$ or $f'(-x) = f'(x)$ and f' is even.

101. (a) Sketch a graph of any even function f and explain graphically why f' is odd.

(b) Suppose that f' is even. Is f necessarily odd? *Hint:* Check whether this is true for linear functions.

SOLUTION

(a) The graph of an even function is symmetric with respect to the y-axis. Accordingly, its image in the left half-plane is a mirror reflection of that in the right half-plane through the y-axis. If at $x = a \geq 0$, the slope of f exists and is equal to m, then by reflection its slope at $x = -a \leq 0$ is $-m$. That is, $f'(-a) = -f'(a)$. *Note:* This means that if $f'(0)$ exists, then it equals 0.



(b) Suppose that f' is even. Then f is not necessarily odd. Let $f(x) = 4x + 7$. Then $f'(x) = 4$, an even function. But f is not odd. For example, $f(2) = 15$, $f(-2) = -1$, but $f(-2) \neq -f(2)$.

102. Power Rule for Fractional Exponents Let $f(u) = u^q$ and $g(x) = x^{p/q}$. Assume that g is differentiable.

(a) Show that $f(g(x)) = x^p$ (recall the Laws of Exponents).

(b) Apply the Chain Rule and the Power Rule for whole-number exponents to show that $f'(g(x))\, g'(x) = px^{p-1}$.

(c) Then derive the Power Rule for $x^{p/q}$.

SOLUTION

(a) Let $f(u) = u^q$ and $g(x) = x^{p/q}$, where q is a positive integer and p is an integer. Then

$$f(g(x)) = f\left(x^{p/q}\right) = \left(x^{p/q}\right)^q = x^p.$$

(b) Differentiating both sides of the final expression in part (a), applying the chain rule on the left and the power rule for whole number exponents on the right, it follows that

$$f'(g(x))g'(x) = px^{p-1}.$$

(c) Thus

$$g'(x) = \frac{px^{p-1}}{f'(g(x))} = \frac{px^{p-1}}{q\left(x^{p/q}\right)^{q-1}} = \frac{px^{p-1}}{qx^{p-p/q}} = \frac{p}{q}x^{p/q-1}.$$

103. Prove that for all whole numbers $n \geq 1$,

$$\frac{d^n}{dx^n}\sin x = \sin\left(x + \frac{n\pi}{2}\right)$$

Hint: Use the identity $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

SOLUTION We will proceed by induction on n. For $n = 1$, we find

$$\frac{d}{dx}\sin x = \cos x = \sin\left(x + \frac{\pi}{2}\right),$$

as required. Now, suppose that for some positive integer k,

$$\frac{d^k}{dx^k}\sin x = \sin\left(x + \frac{k\pi}{2}\right).$$

Then

$$\begin{aligned}\frac{d^{k+1}}{dx^{k+1}}\sin x &= \frac{d}{dx}\sin\left(x + \frac{k\pi}{2}\right)\\ &= \cos\left(x + \frac{k\pi}{2}\right) = \sin\left(x + \frac{(k+1)\pi}{2}\right).\end{aligned}$$

104. A Discontinuous Derivative Use the limit definition to show that $g'(0)$ exists but $g'(0) \neq \lim_{x\to 0} g'(x)$, where

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

SOLUTION Using the limit definition,

$$g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h} = \lim_{h\to 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = \lim_{h\to 0} h \sin\left(\frac{1}{h}\right) = 0,$$

where we have used the squeeze theorem in the last step. Now, for $x \neq 0$,

$$g'(x) = x^2\left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right) + 2x \sin\left(\frac{1}{x}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right).$$

Although the first term in g' has a limit of 0 as $x \to 0$ (by the squeeze theorem), the limit as $x \to 0$ of the second term does not exist. Hence, $\lim_{x\to 0} g'(x)$ does not exist, so $g'(0) \neq \lim_{x\to 0} g'(x)$.

105. Chain Rule This exercise proves the Chain Rule without the special assumption made in the text. For any number b, define a new function

$$F(u) = \frac{f(u) - f(b)}{u - b} \quad \text{for all } u \neq b$$

(a) Show that if we define $F(b) = f'(b)$, then F is continuous at $u = b$.

(b) Take $b = g(a)$. Show that if $x \neq a$, then for all u,

$$\frac{f(u) - f(g(a))}{x - a} = F(u)\frac{u - g(a)}{x - a} \qquad \boxed{1}$$

Note that both sides are zero if $u = g(a)$.

(c) Substitute $u = g(x)$ in Eq. (1) to obtain

$$\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x))\,\frac{g(x) - g(a)}{x - a}$$

Derive the Chain Rule by computing the limit of both sides as $x \to a$.

SOLUTION For any differentiable function f and any number b, define

$$F(u) = \frac{f(u) - f(b)}{u - b}$$

for all $u \neq b$.

(a) Define $F(b) = f'(b)$. Then

$$\lim_{u\to b} F(u) = \lim_{u\to b} \frac{f(u) - f(b)}{u - b} = f'(b) = F(b),$$

i.e., $\lim_{u\to b} F(u) = F(b)$. Therefore, F is continuous at $u = b$.

(b) Let g be a differentiable function and take $b = g(a)$. Let x be a number distinct from a. If we substitute $u = g(a)$ into Eq. (1), both sides evaluate to 0, so equality is satisfied. On the other hand, if $u \neq g(a)$, then

$$\frac{f(u) - f(g(a))}{x - a} = \frac{f(u) - f(g(a))}{u - g(a)}\,\frac{u - g(a)}{x - a} = \frac{f(u) - f(b)}{u - b}\,\frac{u - g(a)}{x - a} = F(u)\,\frac{u - g(a)}{x - a}.$$

(c) Hence for all u, we have

$$\frac{f(u) - f(g(a))}{x - a} = F(u)\,\frac{u - g(a)}{x - a}.$$

Substituting $u = g(x)$ in Eq. (1), we have

$$\frac{f(g(x)) - f(g(a))}{x - a} = F(g(x))\,\frac{g(x) - g(a)}{x - a}.$$

Letting $x \to a$ gives

$$\lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x - a} = \lim_{x\to a} \left(F(g(x)) \frac{g(x) - g(a)}{x - a} \right) = F(g(a))g'(a) = F(b)g'(a) = f'(b)g'(a)$$
$$= f'(g(a))g'(a)$$

Therefore $(f \circ g)'(a) = f'(g(a))g'(a)$, which is the Chain Rule.

3.8 Implicit Differentiation

Preliminary Questions

1. Which differentiation rule is used to show $\frac{d}{dx} \sin y = \cos y \frac{dy}{dx}$?

SOLUTION The chain rule is used to show that $\frac{d}{dx} \sin y = \cos y \frac{dy}{dx}$.

2. One of (a)–(c) is incorrect. Find and correct the mistake.

(a) $\frac{d}{dy} \sin(y^2) = 2y \cos(y^2)$

(b) $\frac{d}{dx} \sin(x^2) = 2x \cos(x^2)$

(c) $\frac{d}{dx} \sin(y^2) = 2y \cos(y^2)$

SOLUTION

(a) This is correct. Note that the differentiation is with respect to the variable y.

(b) This is correct. Note that the differentiation is with respect to the variable x.

(c) This is incorrect. Because the differentiation is with respect to the variable x, the chain rule is needed to obtain

$$\frac{d}{dx} \sin(y^2) = 2y \cos(y^2) \frac{dy}{dx}.$$

3. On an exam, Jason was asked to differentiate the equation

$$x^2 + 2xy + y^3 = 7$$

Find the errors in Jason's answer: $2x + 2xy' + 3y^2 = 0$.

SOLUTION There are two mistakes in Jason's answer. First, Jason should have applied the product rule to the second term to obtain

$$\frac{d}{dx}(2xy) = 2x\frac{dy}{dx} + 2y.$$

Second, he should have applied the general power rule to the third term to obtain

$$\frac{d}{dx} y^3 = 3y^2 \frac{dy}{dx}.$$

4. Which of (a) or (b) is equal to $\frac{d}{dx}(x \sin t)$?

(a) $(x \cos t)\frac{dt}{dx}$

(b) $(x \cos t)\frac{dt}{dx} + \sin t$

SOLUTION Using the product rule and the chain rule we see that

$$\frac{d}{dx}(x \sin t) = x \cos t \frac{dt}{dx} + \sin t,$$

so the correct answer is **(b)**.

5. Determine which inverse trigonometric function g has the derivative

$$g'(x) = \frac{1}{x^2 + 1}$$

SOLUTION $g(x) = \tan^{-1} x$ has the derivative

$$g'(x) = \frac{1}{x^2 + 1}.$$

6. What does the following identity tell us about the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$?

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

SOLUTION Differentiating both sides of the idenity with respect to x yields

$$\frac{d}{dx}\sin^{-1} x + \frac{d}{dx}\cos^{-1} x = 0 \qquad \text{or} \qquad \frac{d}{dx}\sin^{-1} x = -\frac{d}{dx}\cos^{-1} x.$$

In other words, the derivatives of $\sin^{-1} x$ and $\cos^{-1} x$ are negatives of each other.

Exercises

1. Show that if you differentiate both sides of $x^2 + 2y^3 = 6$, the result is $2x + 6y^2\frac{dy}{dx} = 0$. Then solve for dy/dx and evaluate it at the point $(2, 1)$.

SOLUTION Let $x^2 + 2y^3 = 6$. Then

$$\frac{d}{dx}(x^2 + 2y^3) = \frac{d}{dx}6$$

$$2x + 6y^2\frac{dy}{dx} = 0$$

Solving for dy/dx yields

$$6y^2\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{6y^2}.$$

At $(2, 1)$, $\frac{dy}{dx} = \frac{-4}{6} = -\frac{2}{3}$.

2. Show that if you differentiate both sides of $xy + 4x + 2y = 1$, the result is $(x + 2)\frac{dy}{dx} + y + 4 = 0$. Then solve for dy/dx and evaluate it at the point $(1, -1)$.

SOLUTION Let $xy + 4x + 2y = 1$. Applying the product rule

$$\frac{d}{dx}(xy + 4x + 2y) = \frac{d}{dx}1$$

$$x\frac{dy}{dx} + y + 4 + 2\frac{dy}{dx} = 0$$

$$(x + 2)\frac{dy}{dx} + y + 4 = 0$$

Solving for dy/dx yields

$$(x + 2)\frac{dy}{dx} = -(y + 4)$$

$$\frac{dy}{dx} = -\frac{y + 4}{x + 2}.$$

At $(1, -1)$, $dy/dx = -3/3 = -1$.

In Exercises 3–8, differentiate the expression with respect to x, assuming that $y = f(x)$.

3. x^2y^3

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}\left(x^2y^3\right) = x^2 \cdot 3y^2y' + y^3 \cdot 2x = 3x^2y^2y' + 2xy^3.$$

4. $\dfrac{x^3}{y^2}$

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}\left(\frac{x^3}{y}\right) = \frac{y(3x^2) - x^3y'}{y^2} = \frac{3x^2}{y} - \frac{x^3y'}{y^2}.$$

5. $(x^2 + y^2)^{3/2}$

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}\left(\left(x^2 + y^2\right)^{3/2}\right) = \frac{3}{2}\left(x^2 + y^2\right)^{1/2}\left(2x + 2yy'\right) = 3\left(x + yy'\right)\sqrt{x^2 + y^2}.$$

6. $\tan(xy)$

SOLUTION Assuming that y depends on x, then $\frac{d}{dx}(\tan(xy)) = \left(xy' + y\right)\sec^2(xy)$.

7. $\frac{y}{y+1}$

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}\left(\frac{y}{y+1}\right) = \frac{(y+1)y' - yy'}{(y+1)^2} = \frac{y'}{(y+1)^2}.$$

8. $e^{y/x}$

SOLUTION Assuming that y depends on x, then

$$\frac{d}{dx}e^{y/x} = e^{y/x}\left(\frac{xy' - y}{x^2}\right).$$

In Exercises 9–26, calculate the derivative of the other variable with respect to x.

9. $3y^3 + x^2 = 5$

SOLUTION Let $3y^3 + x^2 = 5$. Then $9y^2y' + 2x = 0$, and $y' = -\frac{2x}{9y^2}$.

10. $y^4 - 2y = 4x^3 + x$

SOLUTION Let $y^4 - 2y = 4x^3 + x$. Then

$$\begin{aligned}\frac{d}{dx}(y^4 - 2y) &= \frac{d}{dx}(4x^3 + x)\\ 4y^3y' - 2y' &= 12x^2 + 1\\ y'(4y^3 - 2) &= 12x^2 + 1\\ y' &= \frac{12x^2 + 1}{4y^3 - 2}\end{aligned}$$

11. $x^2y + 2x^3y = x + y$

SOLUTION Let $x^2y + 2xy^2 = x + y$. Then

$$\begin{aligned}x^2y' + 2xy + 2x \cdot 2yy' + 2y^2 &= 1 + y'\\ x^2y' + 4xyy' - y' &= 1 - 2xy - 2y^2\\ y' &= \frac{1 - 2xy - 2y^2}{x^2 + 4xy - 1}.\end{aligned}$$

12. $xy^2 + x^2y^5 - x^3 = 3$

SOLUTION Let $xy^2 + x^2y^5 - x^3 = 3$. Then

$$\begin{aligned}2xyy' + y^2 + 5x^2y^4y' + 2xy^5 - 3x^2 &= 0\\ (2xy + 5x^2y^4)y' &= 3x^2 - y^2 - 2xy^5\\ y' &= \frac{3x^2 - y^2 - 2xy^5}{2xy + 5x^2y^4}\end{aligned}$$

13. $x^3R^5 = 1$

SOLUTION Let $x^3R^5 = 1$. Then $x^3 \cdot 5R^4R' + R^5 \cdot 3x^2 = 0$, and $R' = -\frac{3x^2R^5}{5x^3R^4} = -\frac{3R}{5x}$.

14. $x^4 + z^4 = 1$

SOLUTION Let $x^4 + z^4 = 1$. Then $4x^3 + 4z^3 z' = 0$, and $z' = -x^3/z^3$.

15. $\dfrac{y}{x} + \dfrac{x}{y} = 2y$

SOLUTION Let

$$\frac{y}{x} + \frac{x}{y} = 2y.$$

Then

$$\begin{aligned}
\frac{xy' - y}{x^2} + \frac{y - xy'}{y^2} &= 2y' \\
\left(\frac{1}{x} - \frac{x}{y^2} - 2\right) y' &= \frac{y}{x^2} - \frac{1}{y} \\
\frac{y^2 - x^2 - 2xy^2}{xy^2} y' &= \frac{y^2 - x^2}{x^2 y} \\
y' &= \frac{y(y^2 - x^2)}{x(y^2 - x^2 - 2xy^2)}.
\end{aligned}$$

16. $\sqrt{x+s} = \dfrac{1}{x} + \dfrac{1}{s}$

SOLUTION Let $(x+s)^{1/2} = x^{-1} + s^{-1}$. Then

$$\frac{1}{2}(x+s)^{-1/2}\left(1 + s'\right) = -x^{-2} - s^{-2}s'.$$

Multiplying by $2x^2s^2\sqrt{x+s}$ and then solving for s' gives

$$\begin{aligned}
x^2s^2\left(1 + s'\right) &= -2s^2\sqrt{x+s} - 2x^2s'\sqrt{x+s} \\
x^2s^2s' + 2x^2s'\sqrt{x+s} &= -2s^2\sqrt{x+s} - x^2s^2 \\
x^2\left(s^2 + 2\sqrt{x+s}\right)s' &= -s^2\left(x^2 + 2\sqrt{x+s}\right) \\
s' &= -\frac{s^2\left(x^2 + 2\sqrt{x+s}\right)}{x^2\left(s^2 + 2\sqrt{x+s}\right)}.
\end{aligned}$$

17. $y^{-2/3} + x^{3/2} = 1$

SOLUTION Let $y^{-2/3} + x^{3/2} = 1$. Then

$$-\frac{2}{3}y^{-5/3}y' + \frac{3}{2}x^{1/2} = 0 \quad \text{or} \quad y' = \frac{9}{4}x^{1/2}y^{5/3}.$$

18. $x^{1/2} + y^{2/3} = -4y$

SOLUTION Let $x^{1/2} + y^{2/3} = x + y$. Then $\frac{1}{2}x^{-1/2} + \frac{2}{3}y^{-1/3}y' = 1 + y'$, and

$$y' = \frac{1 - \frac{1}{2}x^{-1/2}}{\frac{2}{3}y^{-1/3} - 1} \cdot \frac{6x^{1/2}y^{1/3}}{6x^{1/2}y^{1/3}} = \frac{3y^{1/3}\left(2x^{1/2} - 1\right)}{2x^{1/2}\left(2 - 3y^{1/3}\right)}.$$

19. $y + \dfrac{1}{y} = x^2 + x$

SOLUTION Let $y + \frac{1}{y} = x^2 + x$. Then

$$y' - \frac{1}{y^2}y' = 2x + 1 \quad \text{or} \quad y' = \frac{2x+1}{1 - y^{-2}} = \frac{(2x+1)y^2}{y^2 - 1}.$$

20. $\sin(xt) = t$

SOLUTION In what follows, $t' = \frac{dt}{dx}$. Applying the chain rule and the product rule, we get:

$$\begin{aligned}\frac{d}{dx}\sin(xt) &= \frac{d}{dx}t\\ \cos(xt)(xt' + t) &= t'\\ x\cos(xt)t' + t\cos(xt) &= t'\\ x\cos(xt)t' - t' &= -t\cos(xt)\\ t'(x\cos(xt) - 1) &= -t\cos(xt)\\ t' &= \frac{-t\cos(xt)}{x\cos(xt) - 1}.\end{aligned}$$

21. $\sin(x + y) = x + \cos y$

SOLUTION Let $\sin(x + y) = x + \cos y$. Then

$$\begin{aligned}(1 + y')\cos(x + y) &= 1 - y'\sin y\\ \cos(x + y) + y'\cos(x + y) &= 1 - y'\sin y\\ (\cos(x + y) + \sin y)\, y' &= 1 - \cos(x + y)\\ y' &= \frac{1 - \cos(x + y)}{\cos(x + y) + \sin y}.\end{aligned}$$

22. $\tan(x^2 y) = (x + y)^3$

SOLUTION Let $\tan\left(x^2 y\right) = x + y$. Then

$$\begin{aligned}\sec^2(x^2 y)\cdot(x^2 y' + 2xy) &= 1 + y'\\ x^2\sec^2(x^2 y)y' + 2xy\sec^2(x^2 y) &= 1 + y'\\ \left(x^2\sec^2(x^2 y) - 1\right)y' &= 1 - 2xy\sec^2(x^2 y)\\ y' &= \frac{1 - 2xy\sec^2\left(x^2 y\right)}{x^2\sec^2\left(x^2 y\right) - 1}.\end{aligned}$$

23. $xe^y = 2xy + y^3$

SOLUTION Let $xe^y = 2xy + y^3$. Then $xy'e^y + e^y = 2xy' + 2y + 3y^2 y'$, whence

$$y' = \frac{e^y - 2y}{2x + 3y^2 - xe^y}.$$

24. $e^{xy} = \sin(y^2)$

SOLUTION Let $e^{xy} = \sin(y^2)$. Then $e^{xy}\left(xy' + y\right) = 2y\cos(y^2)y'$, whence

$$y' = \frac{ye^{xy}}{2y\cos(y^2) - xe^{xy}}.$$

25. $e^x + e^y = x - y$

SOLUTION Let $e^x + e^y = x - y$. Then

$$e^x + y'e^y = 1 - y' \quad \text{or} \quad y' = \frac{1 - e^x}{1 + e^y}.$$

26. $e^{x^2+y^2} = x + 4$

SOLUTION Let $e^{x^2+y^2} = x + 4$. Then

$$(2x + 2yy')e^{x^2+y^2} = 1 \quad \text{or} \quad y' = \frac{e^{-(x^2+y^2)} - 2x}{2y}.$$

In Exercises 27–30, compute the derivative at the point indicated without using a calculator.

27. $y = \sin^{-1} x, \quad x = \frac{3}{5}$

SOLUTION Let $y = \sin^{-1} x$. Then $y' = \frac{1}{\sqrt{1-x^2}}$ and

$$y'\left(\frac{3}{5}\right) = \frac{1}{\sqrt{1 - 9/25}} = \frac{1}{4/5} = \frac{5}{4}.$$

28. $y = \tan^{-1} x, \quad x = \frac{1}{2}$

SOLUTION Let $y = \tan^{-1} x$. Then $y' = \frac{1}{x^2+1}$ and

$$y'\left(\frac{1}{2}\right) = \frac{1}{\frac{1}{4} + 1} = \frac{4}{5}.$$

29. $y = \sec^{-1} x, \quad x = 4$

SOLUTION Let $y = \sec^{-1} x$. Then $y' = \frac{1}{|x|\sqrt{x^2-1}}$ and

$$y'(4) = \frac{1}{4\sqrt{15}}.$$

30. $y = \arccos(4x), \quad x = \frac{1}{5}$

SOLUTION Let $y = \cos^{-1}(4x)$. Then $y' = \frac{-4}{\sqrt{1-16x^2}}$ and

$$y'\left(\frac{1}{5}\right) = \frac{-4}{\sqrt{1 - \frac{16}{25}}} = \frac{-4}{\frac{3}{5}} = -\frac{20}{3}.$$

In Exercises 31–44, find the derivative.

31. $y = \sin^{-1}(7x)$

SOLUTION $\frac{d}{dx} \sin^{-1}(7x) = \frac{1}{\sqrt{1 - (7x)^2}} \cdot \frac{d}{dx} 7x = \frac{7}{\sqrt{1 - (7x)^2}}.$

32. $y = \arctan\left(\frac{x}{3}\right)$

SOLUTION $\frac{d}{dx} \tan^{-1}\left(\frac{x}{3}\right) = \frac{1}{(x/3)^2 + 1} \cdot \frac{d}{dx}\left(\frac{x}{3}\right) = \frac{1}{3} \cdot \frac{1}{\left(\frac{x}{3}\right)^2 + 1} = \frac{1}{(x^2/3) + 3}.$

33. $y = \cos^{-1}(x^2)$

SOLUTION $\frac{d}{dx} \cos^{-1}(x^2) = \frac{-1}{\sqrt{1 - x^4}} \cdot \frac{d}{dx} x^2 = \frac{-2x}{\sqrt{1 - x^4}}.$

34. $y = \sec^{-1}(t + 1)$

SOLUTION $\frac{d}{dt} \sec^{-1}(t + 1) = \frac{1}{|t + 1|\sqrt{(t + 1)^2 - 1}} = \frac{1}{|t + 1|\sqrt{t^2 + 2t}}.$

35. $y = x \tan^{-1} x$

SOLUTION $\frac{d}{dx} x \tan^{-1} x = x \left(\frac{1}{1 + x^2}\right) + \tan^{-1} x.$

36. $y = e^{\cos^{-1} x}$

SOLUTION $\frac{d}{dx} e^{\cos^{-1} x} = e^{\cos^{-1} x} \frac{d}{dx} \cos^{-1} x = \frac{-e^{\cos^{-1} x}}{\sqrt{1 - x^2}}.$

37. $y = \arcsin(e^x)$

SOLUTION $\frac{d}{dx} \sin^{-1}(e^x) = \frac{1}{\sqrt{1 - e^{2x}}} \cdot \frac{d}{dx} e^x = \frac{e^x}{\sqrt{1 - e^{2x}}}.$

38. $y = \csc^{-1}(x^{-1})$

SOLUTION $\frac{d}{dx}\csc^{-1}(x^{-1}) = \frac{-1}{|1/x|\sqrt{1/x^2-1}}\left(\frac{-1}{x^2}\right) = \frac{1}{x^2|1/x|\sqrt{1/x^2-1}} = \frac{1}{\sqrt{1-x^2}}.$

39. $y = \sqrt{1-t^2} + \sin^{-1} t$

SOLUTION $\frac{d}{dt}\left(\sqrt{1-t^2}+\sin^{-1} t\right) = \frac{1}{2}(1-t^2)^{-1/2}(-2t) + \frac{1}{\sqrt{1-t^2}} = \frac{-t}{\sqrt{1-t^2}} + \frac{1}{\sqrt{1-t^2}} = \frac{1-t}{\sqrt{1-t^2}}.$

40. $y = \tan^{-1}\left(\frac{1+t}{1-t}\right)$

SOLUTION $\frac{d}{dt}\tan^{-1}\left(\frac{1+t}{1-t}\right) = \frac{1}{\left(\frac{1+t}{1-t}\right)^2+1}\cdot\left(\frac{(1-t)-(1+t)(-1)}{(1-t)^2}\right) = \frac{2}{(1+t)^2+(1-t)^2} = \frac{1}{t^2+1}.$

41. $y = (\tan^{-1} x)^3$

SOLUTION $\frac{d}{dx}\left((\tan^{-1} x)^3\right) = 3(\tan^{-1} x)^2\frac{d}{dx}\tan^{-1} x = \frac{3(\tan^{-1} x)^2}{x^2+1}.$

42. $y = \frac{\cos^{-1} x}{\sin^{-1} x}$

SOLUTION $\frac{d}{dx}\left(\frac{\cos^{-1} x}{\sin^{-1} x}\right) = \frac{\sin^{-1} x\left(\frac{-1}{\sqrt{1-x^2}}\right) - \cos^{-1} x\left(\frac{1}{\sqrt{1-x^2}}\right)}{(\sin^{-1} x)^2} = -\frac{\pi}{2\sqrt{1-x^2}(\sin^{-1} x)^2}.$

43. $y = \cos^{-1} t^{-1} - \sec^{-1} t$

SOLUTION
$$\frac{d}{dt}(\cos^{-1} t^{-1} - \sec^{-1} t) = \frac{-1}{\sqrt{1-(1/t)^2}}\left(\frac{-1}{t^2}\right) - \frac{1}{|t|\sqrt{t^2-1}}$$
$$= \frac{1}{\sqrt{t^4-t^2}} - \frac{1}{|t|\sqrt{t^2-1}} = \frac{1}{|t|\sqrt{t^2-1}} - \frac{1}{|t|\sqrt{t^2-1}} = 0.$$

Alternatively, let $t = \sec\theta$. Then $t^{-1} = \cos\theta$ and $\cos^{-1} t^{-1} - \sec^{-1} t = \theta - \theta = 0$. Consequently,

$$\frac{d}{dx}(\cos^{-1} t^{-1} - \sec^{-1} t) = 0.$$

44. $y = \cos^{-1}(x + \sin^{-1} x)$

SOLUTION $\frac{d}{dx}\cos^{-1}(x+\sin^{-1} x) = \frac{-1}{\sqrt{1-(x+\sin^{-1} x)^2}}\left(1+\frac{1}{\sqrt{1-x^2}}\right).$

45. Use Figure 8 to prove that $(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$.

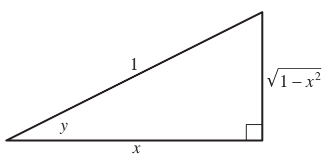


FIGURE 8 Right triangle with $y = \cos^{-1} x$.

SOLUTION Let $y = \cos^{-1} x$. Then $\cos y = x$ and

$$-\sin y\frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sin(\cos^{-1} x)}.$$

From Figure 8, we see that $\sin(\cos^{-1} x) = \sin y = \sqrt{1-x^2}$; hence,

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sin(\cos^{-1} x)} = -\frac{1}{\sqrt{1-x^2}}.$$

46. Show that $(\tan^{-1} x)' = \cos^2(\tan^{-1} x)$ and then use Figure 9 to prove that $(\tan^{-1} x)' = (x^2 + 1)^{-1}$.

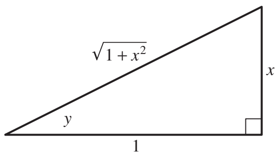


FIGURE 9 Right triangle with $y = \tan^{-1} x$.

SOLUTION Let $y = \tan^{-1} x$. Then $x = \tan y$ and

$$1 = \sec^2 y \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y = \cos^2(\tan^{-1} x).$$

From Figure 9, $\cos y = \dfrac{1}{\sqrt{1+x^2}}$, thus $\cos^2 y = \dfrac{1}{1+x^2}$ and

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

47. Let $y = \sec^{-1} x$. Show that $\tan y = \sqrt{x^2-1}$ if $x \ge 1$ and that $\tan y = -\sqrt{x^2-1}$ if $x \le -1$. *Hint:* $\tan y \ge 0$ on $\left(0, \frac{\pi}{2}\right)$ and $\tan y \le 0$ on $\left(\frac{\pi}{2}, \pi\right)$.

SOLUTION In general, $1 + \tan^2 y = \sec^2 y$, so $\tan y = \pm\sqrt{\sec^2 y - 1}$. With $y = \sec^{-1} x$, it follows that $\sec y = x$, so $\tan y = \pm\sqrt{x^2-1}$. Finally, if $x \ge 1$ then $y = \sec^{-1} x \in [0, \pi/2)$ so $\tan y$ is positive; on the other hand, if $x \le 1$ then $y = \sec^{-1} x \in (\pi/2, \pi]$ so $\tan y$ is negative.

48. Use Exercise 47 to verify the formula

$$(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}$$

SOLUTION Let $y = \sec^{-1} x$. Then $\sec y = x$ and

$$\sec y \tan y \frac{dy}{dx} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{x \tan(\sec^{-1} x)}.$$

By Exercise 47, $\tan(\sec^{-1} x) = \sqrt{x^2-1}$ for $x > 1$ and $\tan(\sec^{-1} x) = -\sqrt{x^2-1}$ for $x < -1$. Hence,

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}}.$$

49. Show that $x + yx^{-1} = 1$ and $y = x - x^2$ define the same curve [except that $(0, 0)$ is not a solution of the first equation] and that implicit differentiation yields $y' = yx^{-1} - x$ and $y' = 1 - 2x$. Explain why these formulas produce the same values for the derivative.

SOLUTION Multiply the first equation by x and then isolate the y term to obtain

$$x^2 + y = x \quad \text{or} \quad y = x - x^2.$$

Implicit differentiation applied to the first equation yields

$$1 - yx^{-2} + x^{-1}y' = 0 \quad \text{or} \quad y' = yx^{-1} - x.$$

From the first equation, we find $yx^{-1} = 1 - x$; upon substituting this expression into the previous derivative, we find

$$y' = 1 - x - x = 1 - 2x,$$

which is the derivative of the second equation.

50. Use the method of Example 4 to compute $\frac{dy}{dx}\Big|_P$ at $P = (2, 1)$ on the curve $y^2x^3 + y^3x^4 - 10x + y = 5$.

SOLUTION Implicit differentiation yields

$$3x^2y^2 + 2x^3yy' + 4x^3y^3 + 3x^4y^2y' - 10 + y' = 0 \quad \text{or} \quad y' = \frac{10 - 3x^2y^2 - 4x^3y^3}{2x^3y + 3x^4y^2 + 1}.$$

Thus, at $P = (2, 1)$,

$$\frac{dy}{dx}\Big|_P = \frac{10 - 3(2)^2(1)^2 - 4(2)^3(1)^3}{2(2)^3(1) + 3(2)^4(1)^2 + 1} = -\frac{34}{65}.$$

In Exercises 51 and 52, find dy/dx at the given point.

51. $(x + 2)^2 - 6(2y + 3)^2 = 3, \quad (1, -1)$

SOLUTION By the chain rule,

$$2(x + 2) - 24(2y + 3)y' = 0.$$

If $x = 1$ and $y = -1$, then

$$2(3) - 24(1)y' = 0,$$

so that $24y' = 6$, or $y' = \frac{1}{4}$.

52. $\sin^2(3y) = x + y, \quad \left(\frac{2 - \pi}{4}, \frac{\pi}{4}\right)$

SOLUTION Taking the derivative of both sides of $\sin^2(3y) = x + y$ yields

$$2\sin(3y)\cos(3y)(3y') = 1 + y'.$$

If $x = \frac{2-\pi}{4}$ and $y = \frac{\pi}{4}$, we get

$$6\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{3\pi}{4}\right)y' = 1 + y'.$$

Using

$$\sin\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2} \quad \text{and} \quad \cos\left(\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

we find

$$\begin{aligned} -6\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)y' &= 1 + y' \\ -3y' &= 1 + y' \\ y' &= -\frac{1}{4}. \end{aligned}$$

In Exercises 53–60, find an equation of the tangent line at the given point.

53. $xy + x^2y^2 = 6, \quad (2, 1)$

SOLUTION Taking the derivative of both sides of $xy + x^2y^2 = 6$ yields

$$xy' + y + 2xy^2 + 2x^2yy' = 0.$$

Substituting $x = 2$, $y = 1$, we find

$$2y' + 1 + 4 + 8y' = 0 \quad \text{or} \quad y' = -\frac{1}{2}.$$

Hence, the equation of the tangent line at $(2, 1)$ is $y - 1 = -\frac{1}{2}(x - 2)$ or $y = -\frac{1}{2}x + 2$.

54. $x^{2/3} + y^{2/3} = 2, \quad (1, 1)$

SOLUTION Taking the derivative of both sides of $x^{2/3} + y^{2/3} = 2$ yields

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' = 0.$$

Substituting $x = 1$, $y = 1$ yields $\frac{2}{3} + \frac{2}{3}y' = 0$, so that $1 + y' = 0$, or $y' = -1$. Hence, the equation of the tangent line at $(1, 1)$ is $y - 1 = -(x - 1)$, or $y = 2 - x$.

55. $x^2 + \sin y = xy^2 + 1, \quad (1, 0)$

SOLUTION Taking the derivative of both sides of $x^2 + \sin y = xy^2 + 1$ yields

$$2x + \cos y y' = y^2 + 2xyy'.$$

Substituting $x = 1$, $y = 0$, we find

$$2 + y' = 0 \quad \text{or} \quad y' = -2.$$

Hence, the equation of the tangent line is $y - 0 = -2(x - 1)$ or $y = -2x + 2$.

56. $\sin(x - y) = x\cos\left(y + \frac{\pi}{4}\right), \quad \left(\frac{\pi}{4}, \frac{\pi}{4}\right)$

SOLUTION Taking the derivative of both sides of $\sin(x - y) = x\cos\left(y + \frac{\pi}{4}\right)$ yields

$$\cos(x - y)(1 - y') = \cos\left(y + \frac{\pi}{4}\right) - x\sin\left(y + \frac{\pi}{4}\right)y'.$$

Substituting $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$, we find

$$1(1 - y') = 0 - \frac{\pi}{4}y' \quad \text{or} \quad y' = \frac{4}{4 - \pi}.$$

Hence, the equation of the tangent line is

$$y - \frac{\pi}{4} = \frac{4}{4 - \pi}\left(x - \frac{\pi}{4}\right).$$

57. $2x^{1/2} + 4y^{-1/2} = xy, \quad (1, 4)$

SOLUTION Taking the derivative of both sides of $2x^{1/2} + 4y^{-1/2} = xy$ yields

$$x^{-1/2} - 2y^{-3/2}y' = xy' + y.$$

Substituting $x = 1$, $y = 4$, we find

$$1 - 2\left(\frac{1}{8}\right)y' = y' + 4 \quad \text{or} \quad y' = -\frac{12}{5}.$$

Hence, the equation of the tangent line is $y - 4 = -\frac{12}{5}(x - 1)$ or $y = -\frac{12}{5}x + \frac{32}{5}$.

58. $x^2e^y + ye^x = 4, \quad (2, 0)$

SOLUTION Taking the derivative of both sides of $x^2e^y + ye^x = 4$ yields

$$x^2e^yy' + 2xe^y + ye^x + e^xy' = 0.$$

Substituting $x = 2$, $y = 0$, we find

$$4y' + 4 + 0 + e^2y' = 0 \quad \text{or} \quad y' = -\frac{4}{4 + e^2}.$$

Hence, the equation of the tangent line is

$$y = -\frac{4}{4 + e^2}(x - 2).$$

59. $e^{2x-y} = \dfrac{x^2}{y}, \quad (2, 4)$

SOLUTION Taking the derivative of both sides of $e^{2x-y} = \dfrac{x^2}{y}$ yields

$$e^{2x-y}(2 - y') = \frac{2xy - x^2y'}{y^2}.$$

Substituting $x = 2$, $y = 4$, we find

$$e^0(2 - y') = \frac{16 - 4y'}{16} \quad \text{or} \quad y' = \frac{4}{3}.$$

Hence, the equation of the tangent line is $y - 4 = \frac{4}{3}(x - 2)$ or $y = \frac{4}{3}x + \frac{4}{3}$.

60. $y^2 e^{x^2-16} - xy^{-1} = 2, \quad (4, 2)$

SOLUTION Taking the derivative of both sides of $y^2 e^{x^2-16} - xy^{-1} = 2$ yields

$$2xy^2 e^{x^2-16} + 2yy' e^{x^2-16} + xy^{-2}y' - y^{-1} = 0.$$

Substituting $x = 4$, $y = 2$, we find

$$32e^0 + 4y'e^0 + y' - \frac{1}{2} = 0 \quad \text{or} \quad y' = -\frac{63}{10}.$$

Hence, the equation of the tangent line is $y - 2 = -\frac{63}{10}(x - 4)$ or $y = -\frac{63}{10}x + \frac{136}{5}$.

61. Find the points on the graph of $y^2 = x^3 - 3x + 1$ (Figure 10) where the tangent line is horizontal.

(a) First show that $2yy' = 3x^2 - 3$, where $y' = dy/dx$.

(b) Do not solve for y'. Rather, set $y' = 0$ and solve for x. This yields two values of x where the slope may be zero.

(c) Show that the positive value of x does not correspond to a point on the graph.

(d) The negative value corresponds to the two points on the graph where the tangent line is horizontal. Find their coordinates.

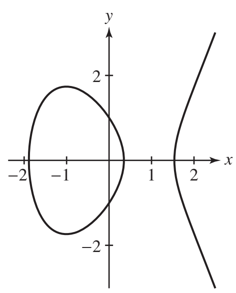


FIGURE 10 Graph of $y^2 = x^3 - 3x + 1$.

SOLUTION

(a) Applying implicit differentiation to $y^2 = x^3 - 3x + 1$, we have

$$2y\frac{dy}{dx} = 3x^2 - 3.$$

(b) Setting $y' = 0$ we have $0 = 3x^2 - 3$, so $x = 1$ or $x = -1$.

(c) If we return to the equation $y^2 = x^3 - 3x + 1$ and substitute $x = 1$, we obtain the equation $y^2 = -1$, which has no real solutions.

(d) Substituting $x = -1$ into $y^2 = x^3 - 3x + 1$ yields

$$y^2 = (-1)^3 - 3(-1) + 1 = -1 + 3 + 1 = 3,$$

so $y = \sqrt{3}$ or $-\sqrt{3}$. The tangent is horizontal at the points $(-1, \sqrt{3})$ and $(-1, -\sqrt{3})$.

62. Show, by differentiating the equation, that if the tangent line at a point (x, y) on the curve $x^2y - 2x + 8y = 2$ is horizontal, then $xy = 1$. Then substitute $y = x^{-1}$ in $x^2y - 2x + 8y = 2$ to show that the tangent line is horizontal at the points $\left(2, \frac{1}{2}\right)$ and $\left(-4, -\frac{1}{4}\right)$.

SOLUTION Taking the derivative on both sides of the equation $x^2y - 2x + 8y = 2$ yields

$$x^2y' + 2xy - 2 + 8y' = 0 \quad \text{or} \quad y' = \frac{2(1 - xy)}{x^2 + 8}.$$

Thus, if the tangent line to the given curve is horizontal, it must be that $1 - xy = 0$, or $xy = 1$. Substituting $y = x^{-1}$ into $x^2y - 2x + 8y = 2$ then yields

$$x - 2x + \frac{8}{x} = 2 \quad \text{or} \quad x^2 + 2x - 8 = (x+4)(x-2) = 0.$$

Hence, the given curve has a horizontal tangent line when $x = 2$ and when $x = -4$. The corresponding points on the curve are thus $\left(2, \frac{1}{2}\right)$ and $\left(-4, -\frac{1}{4}\right)$.

63. Find all points on the graph of $3x^2 + 4y^2 + 3xy = 24$ where the tangent line is horizontal (Figure 11).

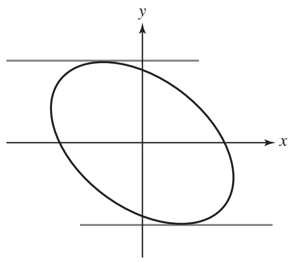


FIGURE 11 Graph of $3x^2 + 4y^2 + 3xy = 24$.

SOLUTION

(a) Differentiating the equation $3x^2 + 4y^2 + 3xy = 24$ implicitly yields

$$6x + 8yy' + 3xy' + 3y = 0,$$

so

$$y' = -\frac{6x + 3y}{8y + 3x}.$$

Setting $y' = 0$ leads to $6x + 3y = 0$, or $y = -2x$.

(b) Substituting $y = -2x$ into the equation $3x^2 + 4y^2 + 3xy = 24$ yields

$$3x^2 + 4(-2x)^2 + 3x(-2x) = 24,$$

or $13x^2 = 24$. Thus, $x = \pm 2\sqrt{78}/13$, and the coordinates of the two points on the graph of $3x^2 + 4y^2 + 3xy = 24$ where the tangent line is horizontal are

$$\left(\frac{2\sqrt{78}}{13}, -\frac{4\sqrt{78}}{13}\right) \quad \text{and} \quad \left(-\frac{2\sqrt{78}}{13}, \frac{4\sqrt{78}}{13}\right).$$

64. Show that no point on the graph of $x^2 - 3xy + y^2 = 1$ has a horizontal tangent line.

SOLUTION Let the implicit curve $x^2 - 3xy + y^2 = 1$ be given. Then

$$2x - 3xy' - 3y + 2yy' = 0,$$

so

$$y' = \frac{2x - 3y}{3x - 2y}.$$

Setting $y' = 0$ leads to $y = \frac{2}{3}x$. Substituting $y = \frac{2}{3}x$ into the equation of the implicit curve gives

$$x^2 - 3x\left(\frac{2}{3}x\right) + \left(\frac{2}{3}x\right)^2 = 1,$$

or $-\frac{5}{9}x^2 = 1$, which has *no* real solutions. Accordingly, there are *no* points on the implicit curve where the tangent line has slope zero.

65. Figure 1 shows the graph of $y^4 + xy = x^3 - x + 2$. Find dy/dx at the two points on the graph with x-coordinate 0 and find an equation of the tangent line at $(1, 1)$.

SOLUTION Consider the equation $y^4 + xy = x^3 - x + 2$. Then $4y^3y' + xy' + y = 3x^2 - 1$, and

$$y' = \frac{3x^2 - y - 1}{x + 4y^3}.$$

- Substituting $x = 0$ into $y^4 + xy = x^3 - x + 2$ gives $y^4 = 2$, which has two real solutions, $y = \pm 2^{1/4}$. When $y = 2^{1/4}$, we have

$$y' = \frac{-2^{1/4} - 1}{4\left(2^{3/4}\right)} = -\frac{\sqrt{2} + \sqrt[4]{2}}{8} \approx -0.3254.$$

When $y = -2^{1/4}$, we have

$$y' = \frac{2^{1/4} - 1}{-4\left(2^{3/4}\right)} = -\frac{\sqrt{2} - \sqrt[4]{2}}{8} \approx -0.02813.$$

- At the point $(1, 1)$, we have $y' = \frac{1}{5}$. At this point the tangent line is $y - 1 = \frac{1}{5}(x - 1)$ or $y = \frac{1}{5}x + \frac{4}{5}$.

66. Folium of Descartes The curve $x^3 + y^3 = 3xy$ (Figure 12) was first discussed in 1638 by the French philosopher-mathematician René Descartes, who called it the folium (meaning "leaf"). Descartes's scientific colleague Gilles de Roberval called it the jasmine flower. Both men believed incorrectly that the leaf shape in the first quadrant was repeated in each quadrant, giving the appearance of petals of a flower. Find an equation of the tangent line at the point $\left(\frac{2}{3}, \frac{4}{3}\right)$.

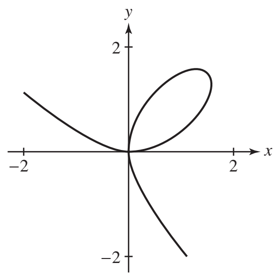


FIGURE 12 Folium of Descartes: $x^3 + y^3 = 3xy$.

SOLUTION Let $x^3 + y^3 = 3xy$. Then $3x^2 + 3y^2y' = 3xy' + 3y$, and $y' = \dfrac{x^2 - y}{x - y^2}$. At the point $\left(\frac{2}{3}, \frac{4}{3}\right)$, we have

$$y' = \frac{\frac{4}{9} - \frac{4}{3}}{\frac{2}{3} - \frac{16}{9}} = \frac{-\frac{8}{9}}{-\frac{10}{9}} = \frac{4}{5}.$$

The tangent line at P is thus $y - \frac{4}{3} = \frac{4}{5}\left(x - \frac{2}{3}\right)$ or $y = \frac{4}{5}x + \frac{4}{5}$.

67. Find a point on the folium $x^3 + y^3 = 3xy$ other than the origin at which the tangent line is horizontal.

SOLUTION Using implicit differentiation, we find

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}(3xy)$$
$$3x^2 + 3y^2y' = 3(xy' + y)$$

Setting $y' = 0$ in this equation yields $3x^2 = 3y$ or $y = x^2$. If we substitute this expression into the original equation $x^3 + y^3 = 3xy$, we obtain:

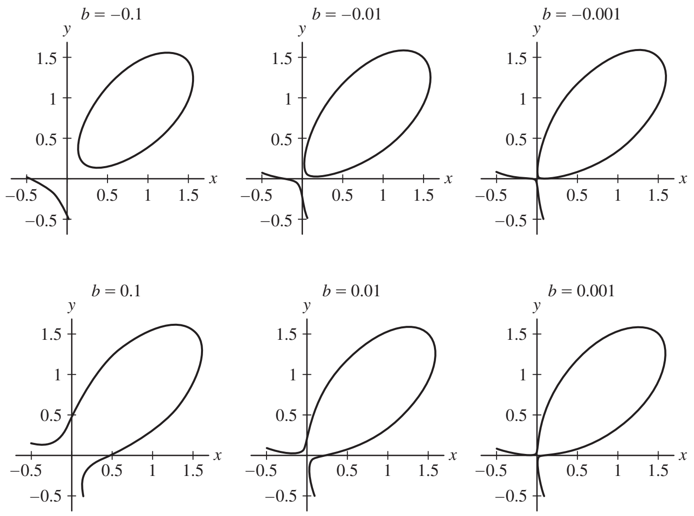
$$x^3 + x^6 = 3x(x^2) = 3x^3 \qquad \text{or} \qquad x^3(x^3 - 2) = 0.$$

One solution of this equation is $x = 0$ and the other is $x = 2^{1/3}$. Thus, the two points on the folium $x^3 + y^3 = 3xy$ at which the tangent line is horizontal are $(0, 0)$ and $(2^{1/3}, 2^{2/3})$.

68. GU Plot $x^3 + y^3 = 3xy + b$ for several values of b and describe how the graph changes as $b \to 0$. Then compute dy/dx at the point $(b^{1/3}, 0)$. How does this value change as $b \to \infty$? Do your plots confirm this conclusion?

SOLUTION

(a) Consider the first row of figures below. When $b < 0$, the graph of $x^3 + y^3 = 3xy + b$ consists of two pieces. As $b \to 0-$, the two pieces move closer to intersecting at the origin. From the second row of figures, we see that the graph of $x^3 + y^3 = 3xy + b$ when $b > 0$ consists of a single piece that has a "loop" in the first quadrant. As $b \to 0+$, the loop comes closer to "pinching off" at the origin.



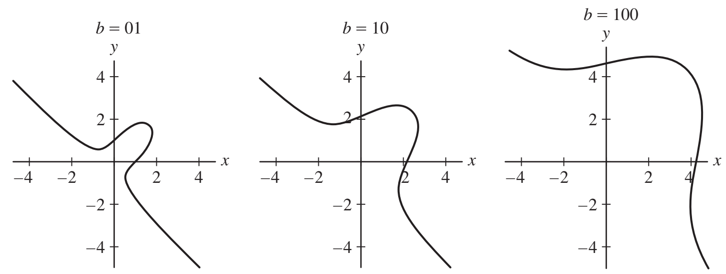
(b) Differentiating the equation $x^3 + y^3 = 3xy + b$ with respect to x yields $3x^2 + 3y^2y' = 3xy' + 3y$, so

$$y' = \frac{y - x^2}{y^2 - x}.$$

At $(b^{1/3}, 0)$, we have

$$y' = \frac{0 - x^2}{0^2 - x} = x = \sqrt[3]{b}.$$

Consequently, as $b \to \infty$, $y' \to \infty$ at the point on the graph where $y = 0$. This conclusion is supported by the figures shown below, which correspond to $b = 1$, $b = 10$, and $b = 100$.



69. Find the x-coordinates of the points where the tangent line is horizontal on the *trident curve* $xy = x^3 - 5x^2 + 2x - 1$, so named by Isaac Newton in his treatise on curves published in 1710 (Figure 13).
Hint: $2x^3 - 5x^2 + 1 = (2x - 1)(x^2 - 2x - 1)$.

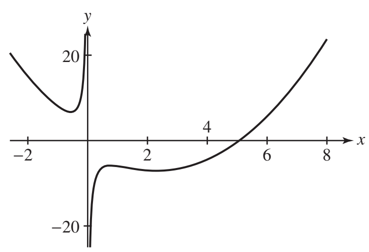


FIGURE 13 Trident curve: $xy = x^3 - 5x^2 + 2x - 1$.

SOLUTION Take the derivative of the equation of a trident curve:

$$xy = x^3 - 5x^2 + 2x - 1$$

to obtain

$$xy' + y = 3x^2 - 10x + 2.$$

Setting $y' = 0$ in (a) gives $y = 3x^2 - 10x + 2$. Substituting this into the equation of the trident, we have

$$xy = x(3x^2 - 10x + 2) = x^3 - 5x^2 + 2x - 1$$

or

$$3x^3 - 10x^2 + 2x = x^3 - 5x^2 + 2x - 1$$

Collecting like terms and setting to zero, we have

$$0 = 2x^3 - 5x^2 + 1 = (2x - 1)(x^2 - 2x - 1).$$

Hence, $x = \frac{1}{2}, 1 \pm \sqrt{2}$.

70. Find an equation of the tangent line at each of the four points on the curve $(x^2 + y^2 - 4x)^2 = 2(x^2 + y^2)$ where $x = 1$. This curve (Figure 14) is an example of a *limaçon of Pascal*, named after the father of the French philosopher Blaise Pascal, who first described it in 1650.

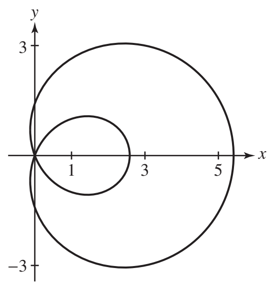


FIGURE 14 Limaçon: $(x^2 + y^2 - 4x)^2 = 2(x^2 + y^2)$.

SOLUTION Plugging $x = 1$ into the equation for the limaçon and solving for y, we find that the points on the curve where $x = 1$ are: $(1, 1)$, $(1, -1)$, $(1, \sqrt{7})$, $(1, -\sqrt{7})$. Using implicit differentiation, we obtain

$$2(x^2 + y^2 - 4x)(2x + 2yy' - 4) = 2(2x + 2yy').$$

We plug in $x = 1$ and get

$$2(1 + y^2 - 4)(2 + 2yy' - 4) = 2(2 + 2yy')$$

or

$$(2y^2 - 6)(2yy' - 2) = 4 + 4yy'.$$

After collecting like terms and solving for y', we have

$$y' = \frac{-2 + y^2}{y^3 - 4y}.$$

At the point $(1, 1)$ the slope of the tangent is $\frac{1}{3}$ and the tangent line is

$$y - 1 = \frac{1}{3}(x - 1) \quad \text{or} \quad y = \frac{1}{3}x + \frac{2}{3}.$$

At the point $(1, -1)$ the slope of the tangent is $-\frac{1}{3}$ and the tangent line is

$$y + 1 = -\frac{1}{3}(x - 1) \quad \text{or} \quad y = -\frac{1}{3}x - \frac{2}{3}.$$

At the point $(1, \sqrt{7})$ the slope of the tangent is $5/3\sqrt{7}$ and the tangent line is

$$y - \sqrt{7} = \frac{5}{3\sqrt{7}}(x - 1) \quad \text{or} \quad y = \frac{5}{3\sqrt{7}}x + \sqrt{7} - \frac{5}{3\sqrt{7}}.$$

At the point $(1, -\sqrt{7})$ the slope of the tangent is $-5/3\sqrt{7}$ and the tangent line is

$$y + \sqrt{7} = -\frac{5}{3\sqrt{7}}(x - 1) \quad \text{or} \quad y = -\frac{5}{3\sqrt{7}}x + \frac{5}{3\sqrt{7}} - \sqrt{7}.$$

71. Find the derivative at the points where $x = 1$ on the folium $(x^2 + y^2)^2 = \frac{25}{4}xy^2$. See Figure 15.

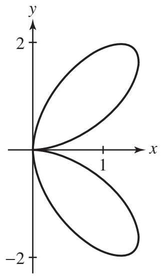


FIGURE 15 Folium curve: $(x^2 + y^2)^2 = \dfrac{25}{4}xy^2$.

SOLUTION First, find the points $(1, y)$ on the curve. Setting $x = 1$ in the equation $(x^2 + y^2)^2 = \frac{25}{4}xy^2$ yields

$$\begin{aligned}(1 + y^2)^2 &= \frac{25}{4}y^2\\ y^4 + 2y^2 + 1 &= \frac{25}{4}y^2\\ 4y^4 + 8y^2 + 4 &= 25y^2\\ 4y^4 - 17y^2 + 4 &= 0\\ (4y^2 - 1)(y^2 - 4) &= 0\\ y^2 &= \frac{1}{4} \text{ or } y^2 = 4\end{aligned}$$

Hence $y = \pm\frac{1}{2}$ or $y = \pm 2$. Taking $\frac{d}{dx}$ of both sides of the original equation yields

$$\begin{aligned}2(x^2 + y^2)(2x + 2yy') &= \frac{25}{4}y^2 + \frac{25}{2}xyy'\\ 4(x^2 + y^2)x + 4(x^2 + y^2)yy' &= \frac{25}{4}y^2 + \frac{25}{2}xyy'\\ (4(x^2 + y^2) - \frac{25}{2}x)yy' &= \frac{25}{4}y^2 - 4(x^2 + y^2)x\\ y' &= \frac{\frac{25}{4}y^2 - 4(x^2 + y^2)x}{y(4(x^2 + y^2) - \frac{25}{2}x)}\end{aligned}$$

- At $(1, 2)$, $x^2 + y^2 = 5$, and

$$y' = \frac{\frac{25}{4}2^2 - 4(5)(1)}{2(4(5) - \frac{25}{2}(1))} = \frac{1}{3}.$$

  Hence, at $(1, 2)$, the equation of the tangent line is $y - 2 = \frac{1}{3}(x - 1)$ or $y = \frac{1}{3}x + \frac{5}{3}$.
- At $(1, -2)$, $x^2 + y^2 = 5$ as well, and

$$y' = \frac{\frac{25}{4}(-2)^2 - 4(5)(1)}{-2(4(5) - \frac{25}{2}(1))} = -\frac{1}{3}.$$

  Hence, at $(1, -2)$, the equation of the tangent line is $y + 2 = -\frac{1}{3}(x - 1)$ or $y = -\frac{1}{3}x - \frac{5}{3}$.
- At $(1, \frac{1}{2})$, $x^2 + y^2 = \frac{5}{4}$, and

$$y' = \frac{\frac{25}{4}\left(\frac{1}{2}\right)^2 - 4\left(\frac{5}{4}\right)(1)}{\frac{1}{2}\left(4\left(\frac{5}{4}\right) - \frac{25}{2}(1)\right)} = \frac{11}{12}.$$

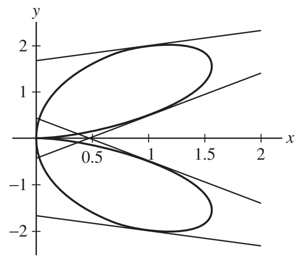
  Hence, at $(1, \frac{1}{2})$, the equation of the tangent line is $y - \frac{1}{2} = \frac{11}{12}(x - 1)$ or $y = \frac{11}{12}x - \frac{5}{12}$.

- At $(1, -\frac{1}{2})$, $x^2 + y^2 = \frac{5}{4}$, and

$$y' = \frac{\frac{25}{4}\left(-\frac{1}{2}\right)^2 - 4\left(\frac{5}{4}\right)(1)}{-\frac{1}{2}\left(4\left(\frac{5}{4}\right) - \frac{25}{2}(1)\right)} = -\frac{11}{12}.$$

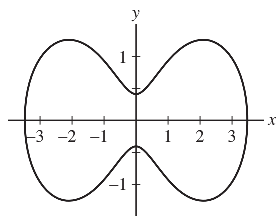
Hence, at $(1, -\frac{1}{2})$, the equation of the tangent line is $y + \frac{1}{2} = -\frac{11}{12}(x - 1)$ or $y = -\frac{11}{12}x + \frac{5}{12}$.

The folium and its tangent lines are plotted below:



72. CAS Plot $(x^2 + y^2)^2 = 12(x^2 - y^2) + 2$ for $-4 \le x \le 4$, $4 \le y \le 4$ using a computer algebra system. How many horizontal tangent lines does the curve appear to have? Find the points where these occur.

SOLUTION A plot of the curve $(x^2 + y^2)^2 = 12(x^2 - y^2) + 2$ is shown below. From this plot, it appears that the curve has a horizontal tangent line at six different locations.



Differentiating the equation $(x^2 + y^2)^2 = 12(x^2 - y^2) + 2$ with respect to x yields

$$2(x^2 + y^2)(2x + 2yy') = 12(2x - 2yy'),$$

so

$$y' = \frac{x(6 - x^2 - y^2)}{y(x^2 + y^2 + 6)}.$$

Thus, horizontal tangent lines occur when $x = 0$ and when $x^2 + y^2 = 6$. Substituting $x = 0$ into the equation for the curve leaves $y^4 + 12y^2 - 2 = 0$, from which it follows that $y^2 = \sqrt{38} - 6$ or $y = \pm\sqrt{\sqrt{38} - 6}$. Substituting $x^2 + y^2 = 6$ into the equation for the curve leaves $x^2 - y^2 = \frac{17}{6}$. From here, it follows that

$$x = \pm\frac{\sqrt{159}}{6} \qquad \text{and} \qquad y = \pm\frac{\sqrt{57}}{6}.$$

The six points at which horizontal tangent lines occur are therefore

$$\left(0, \sqrt{\sqrt{38} - 6}\right), \left(0, -\sqrt{\sqrt{38} - 6}\right)$$

$$\left(\frac{\sqrt{159}}{6}, \frac{\sqrt{57}}{6}\right), \left(\frac{\sqrt{159}}{6}, -\frac{\sqrt{57}}{6}\right), \left(-\frac{\sqrt{159}}{6}, \frac{\sqrt{57}}{6}\right), \left(-\frac{\sqrt{159}}{6}, -\frac{\sqrt{57}}{6}\right)$$

73. Calculate dx/dy for the equation $y^4 + 1 = y^2 + x^2$ and find the points on the graph where the tangent line is vertical.

SOLUTION Let $y^4 + 1 = y^2 + x^2$. Differentiating this equation with respect to y yields

$$4y^3 = 2y + 2x\frac{dx}{dy},$$

so

$$\frac{dx}{dy} = \frac{4y^3 - 2y}{2x} = \frac{y(2y^2-1)}{x}.$$

Thus, $\frac{dx}{dy} = 0$ when $y = 0$ and when $y = \pm\frac{\sqrt{2}}{2}$. Substituting $y = 0$ into the equation $y^4 + 1 = y^2 + x^2$ gives $1 = x^2$, so $x = \pm 1$. Substituting $y = \pm\frac{\sqrt{2}}{2}$, gives $x^2 = 3/4$, so $x = \pm\frac{\sqrt{3}}{2}$. Thus, there are six points on the graph of $y^4 + 1 = y^2 + x^2$ where the tangent line is vertical:

$$(1,0), (-1,0), \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}\right), \left(\frac{\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{\sqrt{2}}{2}\right).$$

74. Show that the tangent lines at $x = 1 \pm \sqrt{2}$ to the *conchoid* with equation $(x-1)^2(x^2+y^2) = 2x^2$ are vertical (Figure 16).

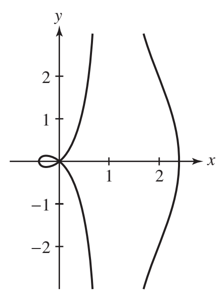


FIGURE 16 Conchoid: $(x-1)^2(x^2+y^2) = 2x^2$.

SOLUTION Consider the equation of a conchoid:

$$(x-1)^2\left(x^2+y^2\right) = 2x^2.$$

Taking the derivative of both sides of this equation gives

$$(x-1)^2\left(2x\frac{dx}{dy} + 2y\right) + \left(x^2+y^2\right)\cdot 2\,(x-1)\,\frac{dx}{dy} = 4x\frac{dx}{dy},$$

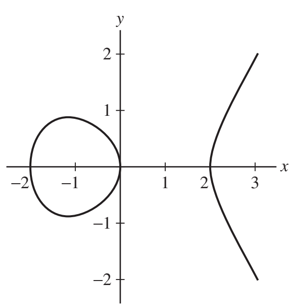
so that

$$\frac{dx}{dy} = \frac{(x-1)^2\,y}{2x + (1-x)\left(x^2+y^2\right) - x\,(x-1)^2}.$$

Setting $dx/dy = 0$ yields $x = 1$ or $y = 0$. We can't have $x = 1$, lest $0 = 2$ in the conchoid's equation. Plugging $y = 0$ into the equation gives $(x-1)^2\,x^2 = 2x^2$ or $x^2\left((x-1)^2 - 2\right) = 0$, which implies $x = 0$ (a double root) or $x = 1 \pm \sqrt{2}$. [Plugging $x = 0$ into the conchoid's equation gives $y^2 = 0$ or $y = 0$. At $(x, y) = (0, 0)$ the expression for dx/dy is undefined (0/0). Via an alternative parametric analysis, the slopes of the tangent lines at the origin turn out to be $\pm\sqrt{3}$.] Accordingly, the tangent lines to the conchoid are vertical at $(x, y) = (1 \pm \sqrt{2}, 0)$.

75. CAS Use a computer algebra system to plot $y^2 = x^3 - 4x$ for $-4 \le x \le 4$, $4 \le y \le 4$. Show that if $dx/dy = 0$, then $y = 0$. Conclude that the tangent line is vertical at the points where the curve intersects the x-axis. Does your plot confirm this conclusion?

SOLUTION A plot of the curve $y^2 = x^3 - 4x$ is shown below.



Differentiating the equation $y^2 = x^3 - 4x$ with respect to y yields

$$2y = 3x^2\frac{dx}{dy} - 4\frac{dx}{dy},$$

or

$$\frac{dx}{dy} = \frac{2y}{3x^2 - 4}.$$

From here, it follows that $\frac{dx}{dy} = 0$ when $y = 0$, so the tangent line to this curve is vertical at the points where the curve intersects the x-axis. This conclusion is confirmed by the plot of the curve shown above.

76. Show that for all points P on the graph in Figure 17, the segments $\overline{OP}$ and $\overline{PR}$ have equal length.

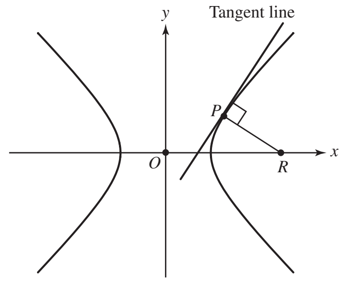


FIGURE 17 Graph of $x^2 - y^2 = a^2$.

SOLUTION Because of the symmetry of the graph, we may restrict attention to any point P in the first quadrant. Suppose P has coordinates $(p, \sqrt{p^2 - a^2})$. Taking the derivative of both sides of the equation $x^2 - y^2 = a^2$ yields $2x - 2yy' = 0$, or $y' = x/y$. It follows that the slope of the line tangent to the graph at P has slope

$$\frac{p}{\sqrt{p^2 - a^2}}$$

and the slope of the normal line is

$$-\frac{\sqrt{p^2 - a^2}}{p}.$$

Thus, the equation of the normal line is

$$y - \sqrt{p^2 - a^2} = -\frac{\sqrt{p^2 - a^2}}{p}(x - p),$$

and the coordinates of the point R are $(2p, 0)$. Finally, the length of the line segment $\overline{OP}$ is

$$\sqrt{p^2 + p^2 - a^2} = \sqrt{2p^2 - a^2},$$

while the length of the segment $\overline{PR}$ is

$$\sqrt{(2p - p)^2 + p^2 - a^2} = \sqrt{2p^2 - a^2}.$$

In Exercises 77–80, use implicit differentiation to calculate higher derivatives.

77. Consider the equation $y^3 - \frac{3}{2}x^2 = 1$.

(a) Show that $y' = x/y^2$ and differentiate again to show that

$$y'' = \frac{y^2 - 2xyy'}{y^4}$$

(b) Express y'' in terms of x and y using part (a).

SOLUTION

(a) Let $y^3 - \frac{3}{2}x^2 = 1$. Then $3y^2y' - 3x = 0$, and $y' = x/y^2$. Therefore,

$$y'' = \frac{y^2 \cdot 1 - x \cdot 2yy'}{y^4} = \frac{y^2 - 2xyy'}{y^4}.$$

(b) Substituting the expression for y' into the result for y'' gives

$$y'' = \frac{y^2 - 2xy\left(x/y^2\right)}{y^4} = \frac{y^3 - 2x^2}{y^5}.$$

78. Use the method of the previous exercise to show that $y'' = -y^{-3}$ on the circle $x^2 + y^2 = 1$.

SOLUTION Let $x^2 + y^2 = 1$. Then $2x + 2yy' = 0$, and $y' = -\frac{x}{y}$. Thus

$$y'' = -\frac{y \cdot 1 - xy'}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3} = -\frac{1}{y^3} = -y^{-3}.$$

79. Calculate y'' at the point $(1, 1)$ on the curve $xy^2 + y - 2 = 0$ by the following steps:
(a) Find y' by implicit differentiation and calculate y' at the point $(1, 1)$.
(b) Differentiate the expression for y' found in (a). Then compute y'' at $(1, 1)$ by substituting $x = 1$, $y = 1$, and the value of y' found in (a).

SOLUTION Let $xy^2 + y - 2 = 0$.

(a) Then $x \cdot 2yy' + y^2 \cdot 1 + y' = 0$, and $y' = -\frac{y^2}{2xy + 1}$. At $(x, y) = (1, 1)$, we have $y' = -\frac{1}{3}$.

(b) Differentiating the expression for y' from (a) yields

$$y'' = -\frac{(2xy + 1)\left(2yy'\right) - y^2\left(2xy' + 2y\right)}{(2xy + 1)^2}.$$

Substituting $x = 1$, $y = 1$, and $y' = -\frac{1}{3}$ gives

$$y'' = -\frac{(3)\left(-\frac{2}{3}\right) - (1)\left(-\frac{2}{3} + 2\right)}{3^2} = -\frac{-6 + 2 - 6}{27} = \frac{10}{27}.$$

80. Use the method of the previous exercise to compute y'' at the point $(1, 1)$ on the curve $x^3 + y^3 = 3x + y - 2$.

SOLUTION Let $x^3 + y^3 = 3x + y - 2$. Then $3x^2 + 3y^2y' = 3 + y'$, and $y' = \frac{3(1 - x^2)}{3y^2 - 1}$. At $(x, y) = (1, 1)$, we find

$$y' = \frac{3(1 - 1)}{3(1) - 1} = 0.$$

Next,

$$y'' = \frac{\left(3y^2 - 1\right)(-6x) - \left(3 - 3x^2\right)\left(6yy'\right)}{\left(3y^2 - 1\right)^2}.$$

When $(x, y) = (1, 1)$ and $y' = 0$, it follows that $y'' = -3$.

In Exercises 81–83, x and y are functions of a variable t and use implicit differentiation to relate dy/dt and dx/dt.

81. Differentiate $xy = 1$ with respect to t and derive the relation $\frac{dy}{dt} = -\frac{y}{x}\frac{dx}{dt}$.

SOLUTION Let $xy = 1$. Then $x\frac{dy}{dt} + y\frac{dx}{dt} = 0$, and $\frac{dy}{dt} = -\frac{y}{x}\frac{dx}{dt}$.

82. Differentiate

$$x^3 + 3xy^2 = 1$$

with respect to t and express dy/dt in terms of dx/dt, as in Exercise 81.

SOLUTION Let $x^3 + 3xy^2 = 1$. Then

$$3x^2\frac{dx}{dt} + 6xy\frac{dy}{dt} + 3y^2\frac{dx}{dt} = 0,$$

and

$$\frac{dy}{dt} = -\frac{x^2 + y^2}{2xy}\frac{dx}{dt}.$$

83. Calculate dy/dt in terms of dx/dt.

(a) $x^3 - y^3 = 1$

(b) $y^4 + 2xy + x^2 = 0$

SOLUTION

(a) Taking the derivative of both sides of the equation $x^3 - y^3 = 1$ with respect to t yields

$$3x^2\frac{dx}{dt} - 3y^2\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = \frac{x^2}{y^2}\frac{dx}{dt}.$$

(b) Taking the derivative of both sides of the equation $y^4 + 2xy + x^2 = 0$ with respect to t yields

$$4y^3\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} + 2x\frac{dx}{dt} = 0,$$

or

$$\frac{dy}{dt} = -\frac{x + y}{2y^3 + x}\frac{dx}{dt}.$$

84. The volume V and pressure P of gas in a piston (which vary in time t) satisfy $PV^{3/2} = C$, where C is a constant. Prove that

$$\frac{dP/dt}{dV/dt} = -\frac{3}{2}\frac{P}{V}$$

The ratio of the derivatives is negative. Could you have predicted this from the relation $PV^{3/2} = C$?

SOLUTION Let $PV^{3/2} = C$, where C is a constant. Then

$$P \cdot \frac{3}{2}V^{1/2}\frac{dV}{dt} + V^{3/2}\frac{dP}{dt} = 0, \quad \text{so} \quad \frac{dP/dt}{dV/dt} = -\frac{3}{2}\frac{P}{V}.$$

If P is increasing (respectively, decreasing), then $V = (C/P)^{2/3}$ is decreasing (respectively, increasing). Hence the ratio of the derivatives ($+/-$ or $-/+$) is negative.

Further Insights and Challenges

85. Show that if P lies on the intersection of the two curves $x^2 - y^2 = c$ and $xy = d$ (c, d constants), then the tangents to the curves at P are perpendicular.

SOLUTION Let $C1$ be the curve described by $x^2 - y^2 = c$, and let $C2$ be the curve described by $xy = d$. Suppose that $P = (x_0, y_0)$ lies on the intersection of the two curves $x^2 - y^2 = c$ and $xy = d$. Since $x^2 - y^2 = c$, the chain rule gives us $2x - 2yy' = 0$, so that $y' = \frac{2x}{2y} = \frac{x}{y}$. The slope to the tangent line to $C1$ is $\frac{x_0}{y_0}$. On the curve $C2$, since $xy = d$, the product rule yields that $xy' + y = 0$, so that $y' = -\frac{y}{x}$. Therefore the slope to the tangent line to $C2$ is $-\frac{y_0}{x_0}$. The two slopes are negative reciprocals of one another, hence the tangents to the two curves are perpendicular.

86. The *lemniscate curve* $(x^2 + y^2)^2 = 4(x^2 - y^2)$ was discovered by Jacob Bernoulli in 1694, who noted that it is "shaped like a figure 8, or a knot, or the bow of a ribbon." Find the coordinates of the four points at which the tangent line is horizontal (Figure 18).

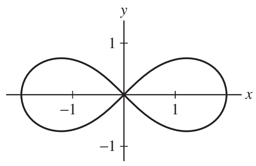


FIGURE 18 Lemniscate curve: $(x^2 + y^2)^2 = 4(x^2 - y^2)$.

SOLUTION Consider the equation of a lemniscate curve: $\left(x^2 + y^2\right)^2 = 4\left(x^2 - y^2\right)$. Taking the derivative of both sides of this equation, we have

$$2\left(x^2 + y^2\right)\left(2x + 2yy'\right) = 4\left(2x - 2yy'\right).$$

Therefore,

$$y' = \frac{8x - 4x\left(x^2 + y^2\right)}{8y + 4y\left(x^2 + y^2\right)} = -\frac{\left(x^2 + y^2 - 2\right)x}{\left(x^2 + y^2 + 2\right)y}.$$

If $y' = 0$, then either $x = 0$ or $x^2 + y^2 = 2$.

- If $x = 0$ in the lemniscate curve, then $y^4 = -4y^2$ or $y^2\left(y^2 + 4\right) = 0$. If y is real, then $y = 0$. The formula for y' in (a) is not defined at the origin (0/0). An alternative parametric analysis shows that the slopes of the tangent lines to the curve at the origin are ± 1.
- If $x^2 + y^2 = 2$ or $y^2 = 2 - x^2$, then plugging this into the lemniscate equation gives $4 = 4\left(2x^2 - 2\right)$ which yields $x = \pm\sqrt{\frac{3}{2}} = \pm\frac{\sqrt{6}}{2}$. Thus $y = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2}$. Accordingly, the four points at which the tangent lines to the lemniscate curve are horizontal are $\left(-\frac{\sqrt{6}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{\sqrt{6}}{2}, -\frac{\sqrt{2}}{2}\right)$, and $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{2}}{2}\right)$.

87. Divide the curve in Figure 19 into five branches, each of which is the graph of a function. Sketch the branches.

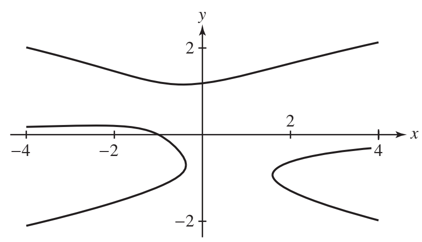
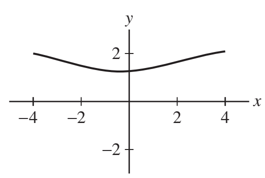


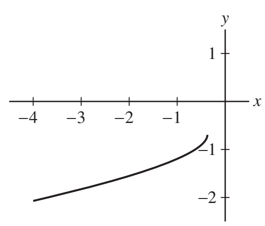
FIGURE 19 Graph of $y^5 - y = x^2 y + x + 1$.

SOLUTION The branches are:

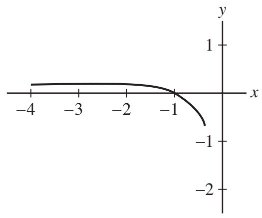
- Upper branch:



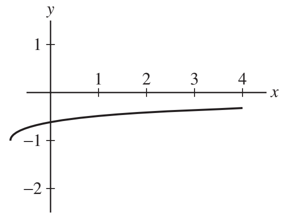
- Lower part of lower left curve:



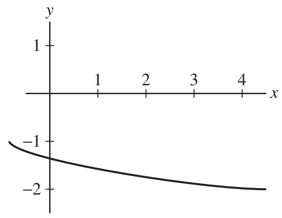
- Upper part of lower left curve:



- Upper part of lower right curve:



- Lower part of lower right curve:



3.9 Derivatives of General Exponential and Logarithmic Functions

Preliminary Questions

1. What is the slope of the tangent line to $y = 4^x$ at $x = 0$?

SOLUTION The slope of the tangent line to $y = 4^x$ at $x = 0$ is

$$\frac{d}{dx}4^x\bigg|_{x=0} = 4^x \ln 4\bigg|_{x=0} = \ln 4.$$

2. What is the rate of change of $y = \ln x$ at $x = 10$?

SOLUTION The rate of change of $y = \ln x$ at $x = 10$ is

$$\frac{d}{dx}\ln x\bigg|_{x=10} = \frac{1}{x}\bigg|_{x=10} = \frac{1}{10}.$$

3. What is $b > 0$ if the tangent line to $y = b^x$ at $x = 0$ has slope 2?

SOLUTION The tangent line to $y = b^x$ at $x = 0$ has slope

$$\frac{d}{dx}b^x\bigg|_{x=0} = b^x \ln b\bigg|_{x=0} = \ln b.$$

This slope will be equal to 2 when

$$\ln b = 2 \quad \text{or} \quad b = e^2.$$

4. What is b if $(\log_b x)' = \dfrac{1}{3x}$?

SOLUTION $(\log_b x)' = \left(\dfrac{\ln x}{\ln b}\right)' = \dfrac{1}{x \ln b}$. This derivative will equal $\frac{1}{3x}$ when

$$\ln b = 3 \quad \text{or} \quad b = e^3.$$

5. What are $y^{(100)}$ and $y^{(101)}$ for $y = \cosh x$?

SOLUTION Let $y = \cosh x$. Then $y' = \sinh x$, $y'' = \cosh x$, and this pattern repeats indefinitely. Thus, $y^{(100)} = \cosh x$ and $y^{(101)} = \sinh x$.

Exercises

In Exercises 1–20, find the derivative.

1. $y = x \ln x$

SOLUTION $\frac{d}{dx} x \ln x = \ln x + \frac{x}{x} = \ln x + 1.$

2. $y = t \ln t - t$

SOLUTION $\frac{d}{dt}(t \ln t - t) = t\left(\frac{1}{t}\right) + \ln t - 1 = \ln t.$

3. $y = 2^{x^3}$

SOLUTION $\frac{d}{dx} 2^{x^3} = 3x^2(\ln 2)2^{x^3}.$

4. $y = \ln(x^5)$

SOLUTION $\frac{d}{dx}(\ln x^5) = \frac{1}{x^5}(5x^4) = \frac{5}{x}.$

5. $y = \ln(9x^2 - 8)$

SOLUTION $\frac{d}{dx} \ln(9x^2 - 8) = \frac{18x}{9x^2 - 8}.$

6. $y = \ln(t5^t)$

SOLUTION Using the rules for logarithms, we write

$$y = \ln(t5^t) = \ln t + \ln(5^t) = \ln t + t \ln 5.$$

Then,

$$\frac{d}{dt} \ln(t5^t) = \frac{1}{t} + \ln 5.$$

7. $y = (\ln x)^2$

SOLUTION $\frac{d}{dx}(\ln x)^2 = (2 \ln x)\frac{1}{x} = \frac{2}{x} \ln x.$

8. $y = x^2 \ln x$

SOLUTION $\frac{d}{dx} x^2 \ln x = 2x \ln x + \frac{x^2}{x} = 2x \ln x + x.$

9. $y = e^{(\ln x)^2}$

SOLUTION $\frac{d}{dx} e^{(\ln x)^2} = e^{(\ln x)^2} \cdot 2 \cdot \frac{\ln x}{x}.$

10. $y = \frac{\ln x}{x}$

SOLUTION $\frac{d}{dx} \frac{\ln x}{x} = \frac{\frac{1}{x}(x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}.$

11. $y = \ln(\ln x)$

SOLUTION $\frac{d}{dx} \ln(\ln x) = \frac{1}{x \ln x}.$

12. $y = \ln(\cot x)$

SOLUTION $\frac{d}{dx} \ln(\cot x) = \frac{1}{\cot x}(-\csc^2 x) = -\frac{1}{\sin x \cos x}.$

13. $y = \left(\ln(\ln x)\right)^3$

SOLUTION $\frac{d}{dx}(\ln(\ln x))^3 = 3(\ln(\ln x))^2 \left(\frac{1}{\ln x}\right)\left(\frac{1}{x}\right) = \frac{3(\ln(\ln x))^2}{x \ln x}.$

14. $y = \ln\left((\ln x)^3\right)$

SOLUTION $\frac{d}{dx}\ln((\ln x)^3) = \frac{3(\ln x)^2}{x(\ln x)^3} = \frac{3}{x\ln x}$.
Alternately, because $\ln((\ln x)^3) = 3\ln(\ln x)$,

$$\frac{d}{dx}\ln((\ln x)^3) = 3\frac{d}{dx}\ln(\ln x) = 3\cdot\frac{1}{x\ln x}.$$

15. $y = \ln\left((x+1)(2x+9)\right)$

SOLUTION

$$\frac{d}{dx}\ln((x+1)(2x+9)) = \frac{1}{(x+1)(2x+9)}\cdot((x+1)2+(2x+9)) = \frac{4x+11}{(x+1)(2x+9)}.$$

Alternately, because $\ln((x+1)(2x+9)) = \ln(x+1)+\ln(2x+9)$,

$$\frac{d}{dx}\ln((x+1)(2x+9)) = \frac{1}{x+1}+\frac{2}{2x+9} = \frac{4x+11}{(x+1)(2x+9)}.$$

16. $y = \ln\left(\frac{x+1}{x^3+1}\right)$

SOLUTION

$$\frac{d}{dx}\ln\left(\frac{x+1}{x^3+1}\right) = \frac{d}{dx}\ln\left(\frac{1}{x^2-x+1}\right) = -\frac{d}{dx}\ln(x^2-x+1) = -\frac{2x-1}{x^2-x+1}.$$

17. $y = 11^x$

SOLUTION $\frac{d}{dx}11^x = \ln 11\cdot 11^x$.

18. $y = 7^{4x-x^2}$

SOLUTION $\frac{d}{dx}7^{4x-x^2} = \ln 7(4-2x)7^{4x-x^2}$.

19. $y = \frac{2^x-3^{-x}}{x}$

SOLUTION $\frac{d}{dx}\frac{2^x-3^{-x}}{x} = \frac{x(2^x\ln 2+3^{-x}\ln 3)-(2^x-3^{-x})}{x^2}$.

20. $y = 16^{\sin x}$

SOLUTION $\frac{d}{dx}16^{\sin x} = \ln 16(\cos x)16^{\sin x}$.

In Exercises 21–24, compute the derivative.

21. $f'(x)$, $f(x) = \log_2 x$

SOLUTION $f(x) = \log_2 x = \frac{\ln x}{\ln 2}$. Thus, $f'(x) = \frac{1}{x}\cdot\frac{1}{\ln 2}$.

22. $f'(3)$, $f(x) = \log_5 x$

SOLUTION $f(x) = \frac{\ln x}{\ln 5}$, so $f'(x) = \frac{1}{x\ln 5}$. Thus, $f'(3) = \frac{1}{3\ln 5}$.

23. $\frac{d}{dt}\log_3(\sin t)$

SOLUTION $\frac{d}{dt}\log_3(\sin t) = \frac{d}{dt}\left(\frac{\ln(\sin t)}{\ln 3}\right) = \frac{1}{\ln 3}\cdot\frac{1}{\sin t}\cdot\cos t = \frac{\cot t}{\ln 3}$.

24. $\frac{d}{dt}\log_{10}(t+2^t)$

SOLUTION $\frac{d}{dt}\log_{10}(t+2^t) = \frac{d}{dt}\left(\frac{\ln(t+2^t)}{\ln 10}\right) = \frac{1}{\ln 10}\cdot\frac{1+2^t\ln 2}{t+2^t}$.

In Exercises 25–36, find an equation of the tangent line at the point indicated.

25. $f(x) = 6^x$, $x = 2$

SOLUTION Let $f(x) = 6^x$. Then $f(2) = 36$, $f'(x) = 6^x \ln 6$ and $f'(2) = 36 \ln 6$. The equation of the tangent line is therefore $y = 36 \ln 6(x - 2) + 36$.

26. $y = (\sqrt{2})^x$, $x = 8$

SOLUTION Let $y = (\sqrt{2})^x$. Then $y(8) = 16$, $y'(x) = (\sqrt{2})^x \ln \sqrt{2}$ and $y'(8) = 16 \ln \sqrt{2} = 8 \ln 2$. The equation of the tangent line is therefore $y = 8 \ln 2(x - 8) + 16$.

27. $s(t) = 3^{9t}$, $t = 2$

SOLUTION Let $s(t) = 3^{9t}$. Then $s(2) = 3^{18}$, $s'(t) = 3^{9t} 9 \ln 3$, and $s'(2) = 3^{18} \cdot 9 \ln 3 = 3^{20} \ln 3$. The equation of the tangent line is therefore $y = 3^{20} \ln 3(t - 2) + 3^{18}$.

28. $y = \pi^{5x-2}$, $x = 1$

SOLUTION Let $y = \pi^{5x-2}$. Then $y(1) = \pi^3$, $y'(x) = \pi^{5x-2} 5 \ln \pi$, and $y'(1) = 5\pi^3 \ln \pi$. The equation of the tangent line is therefore $y = 5\pi^3 \ln \pi(x - 1) + \pi^3$.

29. $f(x) = 5^{x^2-2x}$, $x = 1$

SOLUTION Let $f(x) = 5^{x^2-2x+9}$. Then $f(1) = 5^8$. $f'(x) = \ln 5 \cdot 5^{x^2-2x+9}(2x - 2)$, so $f'(1) = \ln 5(0) = 0$. Therefore, the equation of the tangent line is $y = 5^8$.

30. $s(t) = \ln t$, $t = 5$

SOLUTION Let $s(t) = \ln t$. Then $s(5) = \ln 5$. $s'(t) = 1/t$, so $s'(5) = 1/5$. Therefore the equation of the tangent line is $y = (1/5)(t - 5) + \ln 5$.

31. $s(t) = \ln(8 - 4t)$, $t = 1$

SOLUTION Let $s(t) = \ln(8 - 4t)$. Then $s(1) = \ln(8 - 4) = \ln 4$. $s'(t) = \frac{-4}{8-4t}$, so $s'(1) = -4/4 = -1$. Therefore the equation of the tangent line is $y = -1(t - 1) + \ln 4$.

32. $f(x) = \ln(x^2)$, $x = 4$

SOLUTION Let $f(x) = \ln x^2 = 2 \ln x$. Then $f(4) = 2 \ln 4$. $f'(x) = 2/x$, so $f'(4) = 1/2$. Therefore the equation of the tangent line is $y = (1/2)(x - 4) + 2 \ln 4$.

33. $R(z) = \log_5(2z^2 + 7)$, $z = 3$

SOLUTION Let $R(z) = \log_5(2z^2 + 7)$. Then $R(3) = \log_5(25) = 2$,

$$R'(z) = \frac{4z}{(2z^2 + 7) \ln 5}, \quad \text{and} \quad R'(3) = \frac{12}{25 \ln 5}.$$

The equation of the tangent line is therefore

$$y = \frac{12}{25 \ln 5}(z - 3) + 2.$$

34. $y = \ln(\sin x)$, $x = \frac{\pi}{4}$

SOLUTION Let $f(x) = \ln \sin x$. Then $f(\pi/4) = \ln(\sqrt{2}/2)$. $f'(x) = \cos x / \sin x = \cot x$, so $f'(\pi/4) = 1$. Therefore the equation of the tangent line is $y = (x - \pi/4) + \ln(\sqrt{2}/2)$.

35. $f(w) = \log_2 w$, $w = \frac{1}{8}$

SOLUTION Let $f(w) = \log_2 w$. Then

$$f\left(\frac{1}{8}\right) = \log_2 \frac{1}{8} = \log_2 2^{-3} = -3,$$

$f'(w) = \frac{1}{w \ln 2}$, and

$$f'\left(\frac{1}{8}\right) = \frac{8}{\ln 2}.$$

The equation of the tangent line is therefore

$$y = \frac{8}{\ln 2}\left(w - \frac{1}{8}\right) - 3.$$

36. $y = \log_2(1 + 4x^{-1}), \quad x = 4$

SOLUTION Let $y = \log_2(1 + 4x^{-1})$. Then $y(4) = \log_2(1 + 1) = 1$,

$$y'(x) = -\frac{4x^{-2}}{(1 + 4x^{-1})\ln 2}, \quad \text{and} \quad y'(4) = -\frac{1}{8\ln 2}.$$

The equation of the tangent line is therefore

$$y = -\frac{1}{8\ln 2}(x - 4) - 1.$$

In Exercises 37–44, find the derivative using logarithmic differentiation as in Example 5.

37. $y = (x + 5)(x + 9)$

SOLUTION Let $y = (x + 5)(x + 9)$. Then $\ln y = \ln((x + 5)(x + 9)) = \ln(x + 5) + \ln(x + 9)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{x + 5} + \frac{1}{x + 9}$$

or

$$y' = (x + 5)(x + 9)\left(\frac{1}{x + 5} + \frac{1}{x + 9}\right) = (x + 9) + (x + 5) = 2x + 14.$$

38. $y = (3x + 5)(4x + 9)$

SOLUTION Let $y = (3x + 5)(4x + 9)$. Then $\ln y = \ln((3x + 5)(4x + 9)) = \ln(3x + 5) + \ln(4x + 9)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{3}{3x + 5} + \frac{4}{4x + 9}$$

or

$$y' = (3x + 5)(4x + 9)\left(\frac{3}{3x + 5} + \frac{4}{4x + 9}\right) = (12x + 27) + (12x + 20) = 24x + 47.$$

39. $y = (x - 1)(x - 12)(x + 7)$

SOLUTION Let $y = (x - 1)(x - 12)(x + 7)$. Then $\ln y = \ln(x - 1) + \ln(x - 12) + \ln(x + 7)$. By logarithmic differentiation,

$$\frac{y'}{y} = \frac{1}{x - 1} + \frac{1}{x - 12} + \frac{1}{x + 7}$$

or

$$y' = (x - 12)(x + 7) + (x - 1)(x + 7) + (x - 1)(x - 12) = 3x^2 - 12x - 79.$$

40. $y = \dfrac{x(x + 1)^3}{(3x - 1)^2}$

SOLUTION Let $y = \frac{x(x+1)^3}{(3x-1)^2}$. Then $\ln y = \ln x + 3\ln(x + 1) - 2\ln(3x - 1)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{x} + \frac{3}{x + 1} - \frac{6}{3x - 1},$$

so

$$y' = \frac{(x + 1)^3}{(3x - 1)^2} + \frac{3x(x + 1)^2}{(3x - 1)^2} - \frac{6x(x + 1)^3}{(3x - 1)^3}.$$

41. $y = \dfrac{x(x^2+1)}{\sqrt{x+1}}$

SOLUTION Let $y = \frac{x(x^2+1)}{\sqrt{x+1}}$. Then $\ln y = \ln x + \ln(x^2+1) - \frac{1}{2}\ln(x+1)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{x} + \frac{2x}{x^2+1} - \frac{1}{2(x+1)},$$

so

$$y' = \frac{x(x^2+1)}{\sqrt{x+1}}\left(\frac{1}{x} + \frac{2x}{x^2+1} - \frac{1}{2(x+1)}\right).$$

42. $y = (2x+1)(4x^2)\sqrt{x-9}$

SOLUTION Let $y = (2x+1)(4x^2)\sqrt{x-9}$. Then

$$\ln y = \ln(2x+1) + \ln 4x^2 + \ln(x-9)^{1/2} = \ln(2x+1) + \ln 4 + 2\ln x + \frac{1}{2}\ln(x-9).$$

By logarithmic differentiation

$$\frac{y'}{y} = \frac{2}{2x+1} + \frac{2}{x} + \frac{1}{2(x-9)},$$

so

$$y' = (2x+1)(4x^2)\sqrt{x-9}\left(\frac{2}{2x+1} + \frac{2}{x} + \frac{1}{2(x-9)}\right).$$

43. $y = \sqrt{\dfrac{x(x+2)}{(2x+1)(3x+2)}}$

SOLUTION Let $y = \sqrt{\frac{x(x+2)}{(2x+1)(2x+2)}}$. Then $\ln y = \frac{1}{2}[\ln(x) + \ln(x+2) - \ln(2x+1) - \ln(2x+2)]$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{1}{2}\left(\frac{1}{x} + \frac{1}{x+2} - \frac{2}{2x+1} - \frac{2}{2x+2}\right),$$

so

$$y' = \frac{1}{2}\sqrt{\frac{x(x+2)}{(2x+1)(2x+2)}}\cdot\left(\frac{1}{x} + \frac{1}{x+2} - \frac{2}{2x+1} - \frac{1}{x+1}\right).$$

44. $y = (x^3+1)(x^4+2)(x^5+3)^2$

SOLUTION Let $y = (x^2+1)(x^2+2)(x^2+3)^2$. Then $\ln y = \ln(x^2+1) + \ln(x^2+2) + 2\ln(x^2+3)$. By logarithmic differentiation

$$\frac{y'}{y} = \frac{2x}{x^2+1} + \frac{2x}{x^2+2} + \frac{4x}{x^2+3},$$

so

$$y' = (x^2+1)(x^2+2)(x^2+3)^2\left(\frac{2x}{x^2+1} + \frac{2x}{x^2+2} + \frac{4x}{x^2+3}\right).$$

In Exercises 45–50, find the derivative using either method of Example 6.

45. $f(x) = x^{3x}$

SOLUTION Method 1: $x^{3x} = e^{3x\ln x}$, so

$$\frac{d}{dx}x^{3x} = e^{3x\ln x}(3 + 3\ln x) = x^{3x}(3 + 3\ln x).$$

Method 2: Let $y = x^{3x}$. Then, $\ln y = 3x \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = 3x \cdot \frac{1}{x} + 3 \ln x,$$

so

$$y' = y(3 + 3\ln x) = x^{3x}\,(3 + 3\ln x)\,.$$

46. $f(x) = x^{3^x}$

SOLUTION Method 1: $x^{3^x} = e^{3^x \ln x}$, so

$$\frac{d}{dx} x^{3^x} = e^{3^x \ln x}\left(\frac{3^x}{x} + 3^x \ln 3 \ln x\right) = x^{3^x} 3^x \left(\frac{1}{x} + \ln 3 \ln x\right).$$

Method 2: Let $y = x^{3^x}$. Then, $\ln y = 3^x \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = 3^x \cdot \frac{1}{x} + 3^x \ln 3 \ln x,$$

so

$$y' = y\left(\frac{3^x}{x} + 3^x \ln 3 \ln x\right) = x^{3^x} 3^x \left(\frac{1}{x} + \ln 3 \ln x\right).$$

47. $f(x) = x^{e^x}$

SOLUTION Method 1: $x^{e^x} = e^{e^x \ln x}$, so

$$\frac{d}{dx} x^{e^x} = e^{e^x \ln x}\left(\frac{e^x}{x} + e^x \ln x\right) = x^{e^x}\left(\frac{e^x}{x} + e^x \ln x\right).$$

Method 2: Let $y = x^{e^x}$. Then $\ln y = e^x \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = e^x \cdot \frac{1}{x} + e^x \ln x,$$

so

$$y' = y\left(\frac{e^x}{x} + e^x \ln x\right) = x^{e^x}\left(\frac{e^x}{x} + e^x \ln x\right).$$

48. $f(x) = x^{x^2}$

SOLUTION Method 1: $x^{x^2} = e^{x^2 \ln x}$, so

$$\frac{d}{dx} x^{x^2} = e^{x^2 \ln x}(x + 2x \ln x) = x^{x^2}(x + 2x \ln x) = x^{x^2+1}(1 + 2\ln x).$$

Method 2: Let $y = x^{x^2}$. Then $\ln y = x^2 \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = x + 2x \ln x,$$

so

$$y' = x^{x^2}(x + 2x \ln x) = x^{x^2+1}(1 + 2\ln x).$$

49. $f(x) = x^{\cos x}$

SOLUTION Method 1: $x^{\cos x} = e^{\cos x \ln x}$, so

$$\frac{d}{dx} x^{\cos x} = e^{\cos x \ln x}\left(\frac{\cos x}{x} - \sin x \ln x\right) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

Method 2: Let $y = x^{\cos x}$. Then $\ln y = \cos x \ln x$. By logarithmic differentiation

$$\frac{y'}{y} = \cos x \cdot \frac{1}{x} + \ln x(-\sin x),$$

so

$$y' = y\left(\frac{\cos x}{x} - \sin x \ln x\right) = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

50. $f(x) = e^{x^x}$

SOLUTION Method 1:

$$\frac{d}{dx}e^{x^x} = e^{x^x}\frac{d}{dx}x^x = e^{x^x} \cdot x^x(1 + \ln x),$$

by Example 6 from the text.

Method 2: Let $y = e^{x^x}$. Then $\ln y = x^x \ln e = x^x$. By logarithmic differentiation and Example 6

$$\frac{y'}{y} = x^x(1 + \ln x), \quad \text{so} \quad y' = e^{x^x}(x^x)(1 + \ln x).$$

In Exercises 51–74, calculate the derivative.

51. $y = \sinh(9x)$

SOLUTION $\dfrac{d}{dx}\sinh(9x) = 9\cosh(9x).$

52. $y = \sinh(x^2)$

SOLUTION $\dfrac{d}{dx}\sinh(x^2) = 2x\cosh(x^2).$

53. $y = \cosh^2(9 - 3t)$

SOLUTION $\dfrac{d}{dt}\cosh^2(9 - 3t) = 2\cosh(9 - 3t) \cdot (-3\sinh(9 - 3t)) = -6\cosh(9 - 3t)\sinh(9 - 3t).$

54. $y = \tanh(t^2 + 1)$

SOLUTION $\dfrac{d}{dt}\tanh(t^2 + 1) = 2t\,\text{sech}^2(t^2 + 1).$

55. $y = \sqrt{\cosh x + 1}$

SOLUTION $\dfrac{d}{dx}\sqrt{\cosh x + 1} = \dfrac{1}{2}(\cosh x + 1)^{-1/2}\sinh x.$

56. $y = \sinh x \tanh x$

SOLUTION $\dfrac{d}{dx}\sinh x \tanh x = \cosh x \tanh x + \sinh x\,\text{sech}^2\,x = \sinh x + \tanh x\,\text{sech}\,x.$

57. $y = \dfrac{\coth t}{1 + \tanh t}$

SOLUTION $\dfrac{d}{dt}\dfrac{\coth t}{1 + \tanh t} = -\dfrac{\text{csch}\,t(\text{csch}\,t + 2\,\text{sech}\,t)}{(1 + \tanh t)^2}.$

58. $y = (\ln(\cosh x))^5$

SOLUTION $\dfrac{d}{dx}\ln(\cosh x) = \dfrac{\sinh x}{\cosh x} = \tanh x.$

59. $y = \sinh(\ln x)$

SOLUTION $\dfrac{d}{dx}\sinh(\ln x) = \dfrac{\cosh(\ln x)}{x}.$

60. $y = e^{\coth x}$

SOLUTION $\dfrac{d}{dx}e^{\coth x} = -\text{csch}^2\,x \cdot e^{\coth x}.$

61. $y = \tanh(e^x)$

SOLUTION $\dfrac{d}{dx}\tanh(e^x) = e^x\,\text{sech}^2(e^x).$

62. $y = \sinh(\cosh^3 x)$

SOLUTION $\dfrac{d}{dx}\sinh(\cosh x) = \cosh(\cosh x)(\sinh x).$

63. $y = \operatorname{sech}(\sqrt{x})$

SOLUTION $\dfrac{d}{dx}\operatorname{sech}(\sqrt{x}) = -\dfrac{1}{2}x^{-1/2}\operatorname{sech}\sqrt{x}\tanh\sqrt{x}.$

64. $y = \ln(\coth x)$

SOLUTION $\dfrac{d}{dx}\ln(\coth x) = \dfrac{-\operatorname{csch}^2 x}{\coth x} = \dfrac{-1}{\sinh^2 x\left(\frac{\cosh x}{\sinh x}\right)} = \dfrac{-1}{\sinh x\cosh x}.$

65. $y = \operatorname{sech} x \coth x$

SOLUTION $\dfrac{d}{dx}\operatorname{sech} x\coth x = \dfrac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x\coth x.$

66. $y = x^{\sinh x}$

SOLUTION

$$\frac{d}{dx}x^{\sinh x} = \frac{d}{dx}e^{\ln x\sinh x} = \left(\cosh x\ln x + \frac{\sinh x}{x}\right)e^{\sinh x\ln x} = x^{\sinh x}\left(\cosh x\ln x + \frac{\sinh x}{x}\right).$$

67. $y = \cosh^{-1}(3x)$

SOLUTION $\dfrac{d}{dx}\cosh^{-1}(3x) = \dfrac{3}{\sqrt{9x^2-1}}.$

68. $y = \tanh^{-1}(e^x + x^2)$

SOLUTION $\dfrac{d}{dx}\tanh^{-1}(e^x + x^2) = \dfrac{e^x + 2x}{1-(e^x+x^2)^2}.$

69. $y = (\sinh^{-1}(x^2))^3$

SOLUTION $\dfrac{d}{dx}\sinh^{-1}(x^2) = \dfrac{2x}{\sqrt{x^4+1}}.$

70. $y = (\operatorname{csch}^{-1} 3x)^4$

SOLUTION $\dfrac{d}{dx}(\operatorname{csch}^{-1} 3x)^4 = 4(\operatorname{csch}^{-1} 3x)^3\left(\dfrac{-1}{|3x|\sqrt{1+9x^2}}\right)(3) = \dfrac{-4(\operatorname{csch}^{-1} 3x)^3}{|x|\sqrt{1+9x^2}}.$

71. $y = e^{\cosh^{-1} x}$

SOLUTION $\dfrac{d}{dx}e^{\cosh^{-1} x} = e^{\cosh^{-1} x}\left(\dfrac{1}{\sqrt{x^2-1}}\right).$

72. $y = \sinh^{-1}(\sqrt{x^2+1})$

SOLUTION $\dfrac{d}{dx}\sinh^{-1}(\sqrt{x^2+1}) = \dfrac{1}{\sqrt{x^2+1+1}}\left(\dfrac{1}{2\sqrt{x^2+1}}\right)(2x) = \dfrac{x}{\sqrt{x^2+2}\cdot\sqrt{x^2+1}}.$

73. $y = \tanh^{-1}(\ln t)$

SOLUTION $\dfrac{d}{dt}\tanh^{-1}(\ln t) = \dfrac{1}{t(1-(\ln t)^2)}.$

74. $y = \ln(\tanh^{-1} x)$

SOLUTION $\dfrac{d}{dx}\ln(\tanh^{-1} x) = \dfrac{1}{\tanh^{-1} x}\left(\dfrac{1}{1-x^2}\right).$

In Exercises 75–77, prove the formula.

75. $\dfrac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

SOLUTION $\dfrac{d}{dx}\coth x = \dfrac{d}{dx}\dfrac{\cosh x}{\sinh x} = \dfrac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = \dfrac{-1}{\sinh^2 x} = -\operatorname{csch}^2 x.$

76. $\dfrac{d}{dt}\sinh^{-1} t = \dfrac{1}{\sqrt{t^2+1}}$

SOLUTION Let $x = \sinh^{-1} t$. Then $t = \sinh x$ and

$$1 = \cosh x \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{\cosh x}.$$

Thus,

$$\frac{d}{dt}\sinh^{-1} t = \frac{1}{\cosh(\sinh^{-1} t)}.$$

Now, note

$$\frac{d}{dt}\left[\cosh(\sinh^{-1} t)\right] = \sinh(\sinh^{-1} t)\frac{1}{\sqrt{1+t^2}} = \frac{t}{\sqrt{1+t^2}} = \frac{d}{dt}\sqrt{1+t^2},$$

so the functions $\cosh(\sinh^{-1} t)$ and $\sqrt{1+t^2}$ differ by a constant; substituting $t = 0$ we find that the constant is 0. Therefore,

$$\cosh(\sinh^{-1} t) = \sqrt{t^2+1},$$

and

$$\frac{d}{dt}\sinh^{-1} t = \frac{1}{\cosh(\sinh^{-1} t)} = \frac{1}{\sqrt{t^2+1}}.$$

77. $\dfrac{d}{dt}\cosh^{-1} t = \dfrac{1}{\sqrt{t^2-1}}$ for $t > 1$

SOLUTION Let $x = \cosh^{-1} t$. Then $t = \cosh x$ and

$$1 = \sinh x \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{\sinh x}.$$

Thus, for $t > 1$,

$$\frac{d}{dt}\cosh^{-1} t = \frac{1}{\sinh(\cosh^{-1} t)}.$$

Now, for $t \geq 1$, note

$$\frac{d}{dt}\left[\sinh(\cosh^{-1} t)\right] = \cosh(\cosh^{-1} t)\frac{1}{\sqrt{t^2-1}} = \frac{t}{\sqrt{t^2-1}} = \frac{d}{dt}\sqrt{t^2-1},$$

so the functions $\sinh(\cosh^{-1} t)$ and $\sqrt{t^2-1}$ differ by a constant; substituting $t = 1$ we find that the constant is 0. Therefore, for $t \geq 1$,

$$\sinh(\cosh^{-1} t) = \sqrt{t^2-1},$$

and

$$\frac{d}{dt}\cosh^{-1} t = \frac{1}{\sinh(\cosh^{-1} t)} = \frac{1}{\sqrt{t^2-1}}.$$

78. Use the formula $(\ln f(x))' = f'(x)/f(x)$ to show that $\ln x$ and $\ln(2x)$ have the same derivative. Is there a simpler explanation of this result?

SOLUTION Observe

$$(\ln x)' = \frac{1}{x} \quad \text{and} \quad (\ln 2x)' = \frac{2}{2x} = \frac{1}{x}.$$

As an alternative explanation, note that $\ln(2x) = \ln 2 + \ln x$. Hence, $\ln x$ and $\ln(2x)$ differ by a constant, which implies the two functions have the same derivative.

79. According to one simplified model, the purchasing power of a dollar in the year $2000 + t$ is equal to $P(t) = 0.68(1.04)^{-t}$ (in 1983 dollars). Calculate the predicted rate of decline in purchasing power (in cents per year) in the year 2020.

SOLUTION First, note that

$$P'(t) = -0.68(1.04)^{-t} \ln 1.04;$$

thus, the rate of change in the year 2020 is

$$P'(20) = -0.68(1.04)^{-20} \ln 1.04 = -0.0122.$$

That is, the rate of decline is 1.22 cents per year.

80. The energy E (in joules) radiated as seismic waves by an earthquake of Richter magnitude M satisfies $\log_{10} E = 4.8 + 1.5M$.

(a) Show that when M increases by 1, the energy increases by a factor of approximately 31.5.

(b) Calculate dE/dM.

SOLUTION Solving $\log_{10} E = 4.8 + 1.5M$ for E yields

$$E = 10^{4.8+1.5M}.$$

(a) We find

$$E(M+1) = 10^{4.8+1.5(M+1)} = 10^{1.5}10^{4.8+1.5M} \approx 31.6E(M).$$

(b)

$$\frac{dE}{dM} = (1.5 \ln 10)10^{4.8+1.5M}.$$

81. Show that for any constants M, k, and a, the function

$$y(t) = \frac{1}{2}M\left(1 + \tanh\left(\frac{k(t-a)}{2}\right)\right)$$

satisfies the **logistic equation**: $\dfrac{y'}{y} = k\left(1 - \dfrac{y}{M}\right)$.

SOLUTION Let

$$y(t) = \frac{1}{2}M\left(1 + \tanh\left(\frac{k(t-a)}{2}\right)\right).$$

Then

$$1 - \frac{y(t)}{M} = \frac{1}{2}\left(1 - \tanh\left(\frac{k(t-a)}{2}\right)\right),$$

and

$$\begin{aligned} ky(t)\left(1 - \frac{y(t)}{M}\right) &= \frac{1}{4}Mk\left(1 - \tanh^2\left(\frac{k(t-a)}{2}\right)\right) \\ &= \frac{1}{4}Mk\,\text{sech}^2\left(\frac{k(t-a)}{2}\right). \end{aligned}$$

Finally,

$$y'(t) = \frac{1}{4}Mk\,\text{sech}^2\left(\frac{k(t-a)}{2}\right) = ky(t)\left(1 - \frac{y(t)}{M}\right).$$

82. Show that $V(x) = 2\ln(\tanh(x/2))$ satisfies the **Poisson–Boltzmann** equation $V''(x) = \sinh(V(x))$, which is used to describe electrostatic forces in certain molecules.

SOLUTION Let $V(x) = 2\ln(\tanh(x/2))$. Then

$$V'(x) = 2\frac{1}{\tanh(x/2)} \cdot \frac{1}{2}\,\text{sech}^2(x/2) = \frac{1}{\sinh(x/2)\cosh(x/2)}$$

and

$$V''(x) = -\frac{1}{2}\frac{\sinh^2(x/2) + \cosh^2(x/2)}{\sinh^2(x/2)\cosh^2(x/2)} = -\frac{1}{2}\left(\operatorname{sech}^2(x/2) + \operatorname{csch}^2(x/2)\right).$$

On the other hand,

$$\begin{aligned}\sinh(V(x)) &= \frac{e^{2\ln(\tanh(x/2))} - e^{-2\ln(\tanh(x/2))}}{2} \\ &= \frac{\tanh^2(x/2) - \coth^2(x/2)}{2} \\ &= \frac{(1 - \operatorname{sech}^2(x/2)) - (1 + \operatorname{csch}^2(x/2))}{2} = -\frac{1}{2}\left(\operatorname{sech}^2(x/2) + \operatorname{csch}^2(x/2)\right).\end{aligned}$$

Thus, $V''(x) = \sinh(V(x))$.

83. The Palermo Technical Impact Hazard Scale P is used to quantify the risk associated with the impact of an asteroid colliding with the earth:

$$P = \log_{10}\left(\frac{p_i E^{0.8}}{0.03T}\right)$$

where p_i is the probability of impact, T is the number of years until impact, and E is the energy of impact (in megatons of TNT). The risk is greater than a random event of similar magnitude if $P > 0$.

(a) Calculate dP/dT, assuming that $p_i = 2 \times 10^{-5}$ and $E = 2$ megatons.

(b) Use the derivative to estimate the change in P if T increases from 8 to 9 years.

SOLUTION

(a) Observe that

$$P = \log_{10}\left(\frac{p_i E^{0.8}}{0.03T}\right) = \log_{10}\left(\frac{p_i E^{0.8}}{0.03}\right) - \log_{10} T,$$

so

$$\frac{dP}{dT} = -\frac{1}{T\ln 10}.$$

(b) If T increases to 26 years from 25 years, then

$$\Delta P \approx \left.\frac{dP}{dT}\right|_{T=25} \cdot \Delta T = -\frac{1}{(25 \text{ yr})\ln 10} \cdot (1 \text{ yr}) = -0.017$$

Further Insights and Challenges

84. **(a)** Show that if f and g are differentiable, then

$$\frac{d}{dx}\ln(f(x)g(x)) = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} \qquad \boxed{1}$$

(b) Give a new proof of the Product Rule by observing that the left-hand side of Eq. (1) is equal to $\frac{(f(x)g(x))'}{f(x)g(x)}$.

SOLUTION

(a) $\frac{d}{dx}\ln f(x)g(x) = \frac{d}{dx}(\ln f(x) + \ln g(x)) = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}$.

(b) By part (a),

$$\frac{d}{dx}\ln f(x)g(x) = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} = \frac{f'(x)g(x) + f(x)g'(x)}{f(x)g(x)}.$$

Alternately,

$$\frac{d}{dx}\ln f(x)g(x) = \frac{(f(x)g(x))'}{f(x)g(x)}.$$

Thus,

$$\frac{(f(x)g(x))'}{f(x)g(x)} = \frac{f'(x)g(x) + f(x)g'(x)}{f(x)g(x)},$$

or

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

85. Use the formula $\log_b x = \dfrac{\log_a x}{\log_a b}$ for $a, b > 0$ to verify the formula

$$\frac{d}{dx}\log_b x = \frac{1}{(\ln b)x}$$

SOLUTION $\dfrac{d}{dx}\log_b x = \dfrac{d}{dx}\dfrac{\ln x}{\ln b} = \dfrac{1}{(\ln b)x}.$

3.10 Related Rates

Preliminary Questions

1. Assign variables and restate the following problem in terms of known and unknown derivatives (but do not solve it): How fast is the volume of a cube increasing if its side increases at a rate of 0.5 cm/s?

SOLUTION Let s and V denote the length of the side and the corresponding volume of a cube, respectively. Determine $\frac{dV}{dt}$ if $\frac{ds}{dt} = 0.5$ cm/s.

2. What is the relation between dV/dt and dr/dt if $V = \left(\frac{4}{3}\right)\pi r^3$?

SOLUTION Applying the general power rule, we find $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

In Questions 3 and 4, water pours into a cylindrical glass of radius 4 cm. Let V and h denote the volume and water level respectively, at time t.

3. Restate this question in terms of dV/dt and dh/dt: How fast is the water level rising if water pours in at a rate of 2 cm^3/min?

SOLUTION Determine $\frac{dh}{dt}$ if $\frac{dV}{dt} = 2$ cm^3/min.

4. Restate this question in terms of dV/dt and dh/dt: At what rate is water pouring in if the water level rises at a rate of 1 cm/min?

SOLUTION Determine $\frac{dV}{dt}$ if $\frac{dh}{dt} = 1$ cm/min.

Exercises

In Exercises 1 and 2, consider a rectangular bathtub whose base is 18 ft^2.

1. How fast is the water level rising if water is filling the tub at a rate of 0.7 ft^3/min?

SOLUTION Let h be the height of the water in the tub and V be the volume of the water. Then $V = 18h$ and $\dfrac{dV}{dt} = 18\dfrac{dh}{dt}$. Thus

$$\frac{dh}{dt} = \frac{1}{18}\frac{dV}{dt} = \frac{1}{18}(0.7) \approx 0.039 \text{ ft/min}.$$

2. At what rate is water pouring into the tub if the water level rises at a rate of 0.8 ft/min?

SOLUTION Let h be the height of the water in the tub and V its volume. Then $V = 18h$ and

$$\frac{dV}{dt} = 18\frac{dh}{dt} = 18(0.8) = 14.4 \text{ ft}^3/\text{min}.$$

3. The radius of a circular oil slick expands at a rate of 2 m/min.

(a) How fast is the area of the oil slick increasing when the radius is 25 m?

(b) If the radius is 0 at time $t = 0$, how fast is the area increasing after 3 min?

SOLUTION Let r be the radius of the oil slick and A its area.

(a) Then $A = \pi r^2$ and $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$. Substituting $r = 25$ and $\frac{dr}{dt} = 2$, we find

$$\frac{dA}{dt} = 2\pi\,(25)\,(2) = 100\pi \approx 314.16 \text{ m}^2/\text{min}.$$

(b) Since $\frac{dr}{dt} = 2$ and $r(0) = 0$, it follows that $r(t) = 2t$. Thus, $r(3) = 6$ and

$$\frac{dA}{dt} = 2\pi\,(6)\,(2) = 24\pi \approx 75.40 \text{ m}^2/\text{min}.$$

4. At what rate is the diagonal of a cube increasing if its edges are increasing at a rate of 2 cm/s?

SOLUTION Let s be the length of an edge of the cube and q the length of its diagonal. Two applications of the Pythagorean Theorem (or the distance formula) yield $q = \sqrt{3}s$. Thus $\frac{dq}{dt} = \sqrt{3}\frac{ds}{dt}$. Using $\frac{ds}{dt} = 2$,

$$\frac{dq}{dt} = \sqrt{3} \times 2 = 2\sqrt{3} \approx 3.46 \text{ cm/s}.$$

In Exercises 5–8, assume that the radius r of a sphere is expanding at a rate of 30 cm/min. *The volume of a sphere is $V = \frac{4}{3}\pi r^3$ and its surface area is $4\pi r^2$. Determine the given rate.*

5. Volume with respect to time when $r = 15$ cm

SOLUTION As the radius is expanding at 30 centimeters per minute, we know that $\frac{dr}{dt} = 30$ cm/min. Taking $\frac{d}{dt}$ of the equation $V = \frac{4}{3}\pi r^3$ yields

$$\frac{dV}{dt} = \frac{4}{3}\pi\left(3r^2\frac{dr}{dt}\right) = 4\pi r^2\frac{dr}{dt}.$$

Substituting $r = 15$ and $\frac{dr}{dt} = 30$ yields

$$\frac{dV}{dt} = 4\pi(15)^2(30) = 27000\pi \text{ cm}^3/\text{min}.$$

6. Volume with respect to time at $t = 2$ min, assuming that $r = 0$ at $t = 0$

SOLUTION Taking $\frac{d}{dt}$ of the equation $V = \frac{4}{3}\pi r^3$ yields $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$. Since $\frac{dr}{dt} = 30$ and $r(0) = 0$, it follows that $r(t) = 30t$. From this, $r(2) = 60$, so

$$\frac{dV}{dt} = 4\pi(60^2)(30) = 432000\pi \text{ cm}^3/\text{min}.$$

7. Surface area with respect to time when $r = 40$ cm

SOLUTION Taking the derivative of both sides of $A = 4\pi r^2$ with respect to t yields $\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$. $\frac{dr}{dt} = 30$, so

$$\frac{dA}{dt} = 8\pi(40)(30) = 9600\pi \text{ cm}^2/\text{min}.$$

8. Surface area with respect to time at $t = 2$ min, assuming that $r = 10$ at $t = 0$

SOLUTION Taking $\frac{d}{dt}$ of both sides of $A = 4\pi r^2$ yields $\frac{dA}{dt} = 8\pi r\frac{dr}{dt}$. Since $r = 10$ at $t = 0$ and $\frac{dr}{dt} = 30$, $r = 30t + 10$. Hence, at $t = 2$,

$$\frac{dA}{dt} = 8\pi(30 \cdot 2 + 10)(30) = 16800\pi \text{ cm}^2/\text{min}.$$

In Exercises 9–12, refer to a 5-m ladder sliding down a wall, as in Figures 1 and 2. The variable h is the height of the ladder's top at time t, and x is the distance from the wall to the ladder's bottom.

9. Assume the bottom slides away from the wall at a rate of 0.8 m/s. Find the velocity of the top of the ladder at $t = 2$ s if the bottom is 1.5 m from the wall at $t = 0$ s.

SOLUTION Let x denote the distance from the base of the ladder to the wall, and h denote the height of the top of the ladder from the floor. The ladder is 5 m long, so $h^2 + x^2 = 5^2$. At any time t, $x = 1.5 + 0.8t$. Therefore, at time $t = 2$, the base is $x = 1.5 + 0.8(2) = 3.1$ m from the wall. Furthermore, we have

$$2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0 \quad \text{so} \quad \frac{dh}{dt} = -\frac{x}{h}\frac{dx}{dt}.$$

Substituting $x = 3.1$, $h = \sqrt{5^2 - 3.1^2}$ and $\frac{dx}{dt} = 0.8$, we obtain

$$\frac{dh}{dt} = -\frac{3.1}{\sqrt{5^2 - 3.1^2}}(0.8) \approx -0.632 \text{ m/s}.$$

10. Suppose that the top is sliding down the wall at a rate of 1.2 m/s. Calculate dx/dt when $h = 3$ m.

SOLUTION Let h be the height of the ladder's top and x the distance from the wall of the ladder's bottom. Then $h^2 + x^2 = 5^2$. Thus $2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$, and $\frac{dx}{dt} = -\frac{h}{x}\frac{dh}{dt}$. With $h = 3$, $x = \sqrt{5^2 - 3^2} = 4$, and $\frac{dh}{dt} = -1.2$, we find

$$\frac{dx}{dt} = -\frac{3}{4}(-1.2) = 0.9 \text{ m/s}.$$

11. Suppose that $h(0) = 4$ and the top slides down the wall at a rate of 1.2 m/s. Calculate x and dx/dt at $t = 2$ s.

SOLUTION Let h and x be the height of the ladder's top and the distance from the wall of the ladder's bottom, respectively. After 2 seconds, $h = 4 + 2(-1.2) = 1.6$ m. Since $h^2 + x^2 = 5^2$,

$$x = \sqrt{5^2 - 1.6^2} = 4.737 \text{ m}.$$

Furthermore, we have $2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$, so that $\frac{dx}{dt} = -\frac{h}{x}\frac{dh}{dt}$. Substituting $h = 1.6$, $x = 4.737$, and $\frac{dh}{dt} = -1.2$, we find

$$\frac{dx}{dt} = -\frac{1.6}{4.737}(-1.2) \approx 0.405 \text{ m/s}.$$

12. What is the relation between h and x at the moment when the top and bottom of the ladder move at the same speed?

SOLUTION Let h and x be the height of the ladder's top and the distance from the wall of the ladder's bottom, respectively. When the top and the bottom of the ladder are moving at the same *speed* (say $s > 0$), their *velocities* satisfy $\frac{dh}{dt} = -\frac{dx}{dt} = -s$. Since $h^2 + x^2 = 5^2$, we have $2h\frac{dh}{dt} + 2x\frac{dx}{dt} = 0$ or $-hs + xs = 0$. This implies $h = x$.

13. A conical tank has height 3 m and radius 2 m at the top. Water flows in at a rate of 2 m^3/min. How fast is the water level rising when it is 2 m?

SOLUTION Consider the cone of water in the tank at a certain instant. Let r be the radius of its (inverted) base, h its height, and V its volume. By similar triangles, $\frac{r}{h} = \frac{2}{3}$ or $r = \frac{2}{3}h$ and thus $V = \frac{1}{3}\pi r^2 h = \frac{4}{27}\pi h^3$. Therefore,

$$\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt},$$

and

$$\frac{dh}{dt} = \frac{9}{4\pi h^2}\frac{dV}{dt}.$$

Substituting $h = 2$ and $\frac{dV}{dt} = 2$ yields

$$\frac{dh}{dt} = \frac{9}{4\pi(2)^2} \cdot 2 = \frac{9}{8\pi} \approx 0.36 \text{ m/min}.$$

14. Follow the same set-up as in Exercise 13, but assume that the water level is rising at a rate of 0.3 m/min when it is 2 m. At what rate is water flowing in?

SOLUTION Consider the cone of water in the tank at a certain instant. Let r be the radius of its (inverted) base, h its height, and V its volume. By similar triangles, $\frac{r}{h} = \frac{2}{3}$ or $r = \frac{2}{3}h$ and thus $V = \frac{1}{3}\pi r^2 h = \frac{4}{27}\pi h^3$. Accordingly,

$$\frac{dV}{dt} = \frac{4}{9}\pi h^2 \frac{dh}{dt}.$$

Substituting $h = 2$ and $\frac{dh}{dt} = 0.3$ yields

$$\frac{dV}{dt} = \frac{4}{9}\pi\,(2)^2\,(0.3) \approx 1.68\ \text{m}^3/\text{min}.$$

15. The radius r and height h of a circular cone change at a rate of 2 cm/s. How fast is the volume of the cone increasing when $r = 10$ and $h = 20$?

SOLUTION Let r be the radius, h be the height, and V be the volume of a right circular cone. Then $V = \frac{1}{3}\pi r^2 h$, and

$$\frac{dV}{dt} = \frac{1}{3}\pi\left(r^2\frac{dh}{dt} + 2hr\frac{dr}{dt}\right).$$

When $r = 10$, $h = 20$, and $\frac{dr}{dt} = \frac{dh}{dt} = 2$, we find

$$\frac{dV}{dt} = \frac{\pi}{3}\left(10^2 \cdot 2 + 2 \cdot 20 \cdot 10 \cdot 2\right) = \frac{1000\pi}{3} \approx 1047.20\ \text{cm}^3/\text{s}.$$

16. A road perpendicular to a highway leads to a farmhouse located 2 km away (Figure 8). An automobile travels past the farmhouse at a speed of 80 km/h. How fast is the distance between the automobile and the farmhouse increasing when the automobile is 6 km past the intersection of the highway and the road?

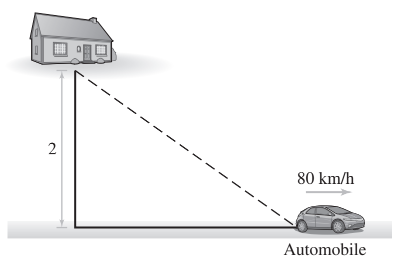


FIGURE 8

SOLUTION Let l denote the distance between the automobile and the farmhouse, and let s denote the distance past the intersection of the highway and the road. Then $l^2 = 2^2 + s^2$. Taking the derivative of both sides of this equation yields $2l\frac{dl}{dt} = 2s\frac{ds}{dt}$, so

$$\frac{dl}{dt} = \frac{s}{l}\frac{ds}{dt}.$$

When the auto is 6 km past the intersection, we have

$$\frac{dl}{dt} = \frac{6 \cdot 80}{\sqrt{2^2 + 6^2}} = \frac{480}{\sqrt{40}} = 24\sqrt{10} \approx 75.89\ \text{km/h}.$$

17. A man of height 1.8 m walks away from a 5-m lamppost at a speed of 1.2 m/s (Figure 9). Find the rate at which his shadow is increasing in length.

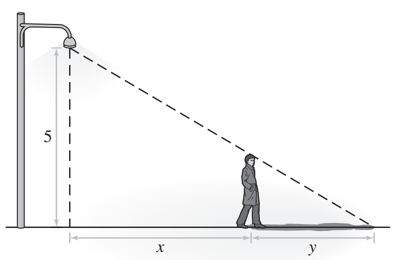


FIGURE 9

SOLUTION Since the man is moving at a rate of 1.2 m/s, his distance from the light post at any given time is $x = 1.2t$. Knowing the man is 1.8 meters tall and that the length of his shadow is denoted by y, we set up a proportion of similar triangles from the diagram:

$$\frac{y}{1.8} = \frac{1.2t + y}{5}.$$

Clearing fractions and solving for y yields

$$y = 0.675t.$$

Thus, $dy/dt = 0.675$ meters per second is the rate at which the length of the shadow is increasing.

18. As Claudia walks away from a 264-cm lamppost, the tip of her shadow moves twice as fast as she does. What is Claudia's height?

SOLUTION Let L be the distance from the base of the lamppost to the tip of Claudia's shadow, let x denote the distance from the base of the lamppost to Claudia's feet, and let h denote Claudia's height. The right triangle with legs $L - x$ and h (formed by Claudia and her shadow) and the right triangle with legs L and 264 (formed by the lamppost and the total distance L) are similar. By similarity

$$\frac{L - x}{h} = \frac{L}{264}.$$

h is constant, so taking the derivative of both sides of this equation yields

$$\frac{dL/dt - dx/dt}{h} = \frac{dL/dt}{264}.$$

The problem states that $\frac{dL}{dt} = 2\frac{dx}{dt}$, so

$$264\left(2\frac{dx}{dt} - \frac{dx}{dt}\right) = 2h\frac{dx}{dt} \quad \text{or} \quad 264 = 2h.$$

Hence, $h = 132$ cm.

19. At a given moment, a plane passes directly above a radar station at an altitude of 6 km.

(a) The plane's speed is 800 km/h. How fast is the distance between the plane and the station changing half a minute later?

(b) How fast is the distance between the plane and the station changing when the plane passes directly above the station?

SOLUTION Let x be the distance of the plane from the station along the ground and h the distance through the air.

(a) By the Pythagorean Theorem, we have

$$h^2 = x^2 + 6^2 = x^2 + 36.$$

Thus $2h\frac{dh}{dt} = 2x\frac{dx}{dt}$, and $\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt}$. After half a minute, $x = \frac{1}{2} \times \frac{1}{60} \times 800 = \frac{20}{3}$ kilometers. With $x = \frac{20}{3}$,

$$h = \sqrt{\left(\frac{20}{3}\right)^2 + 36} = \frac{1}{3}\sqrt{724} = \frac{2}{3}\sqrt{181} \approx 8.969 \text{ km},$$

and $\frac{dx}{dt} = 800$,

$$\frac{dh}{dt} = \frac{20}{3}\,\frac{3}{2\sqrt{181}} \times 800 = \frac{8000}{\sqrt{181}} \approx 594.64 \text{ km/h}.$$

(b) When the plane is directly above the station, $x = 0$, so the distance between the plane and the station is not changing, for at this instant we have

$$\frac{dh}{dt} = \frac{0}{6} \times 800 = 0 \text{ km/h}.$$

20. In the setting of Exercise 19, let θ be the angle that the line through the radar station and the plane makes with the horizontal. How fast is θ changing 12 min after the plane passes over the radar station?

SOLUTION Let the distance x and angle θ be defined as in the figure below. Then

$$\tan\theta = \frac{6}{x} \qquad \text{and} \qquad \sec^2\theta\frac{d\theta}{dt} = -\frac{6}{x^2}\frac{dx}{dt}.$$

Because the plane is traveling at 800 km/h, 12 minutes after the plane passes over the radar station,

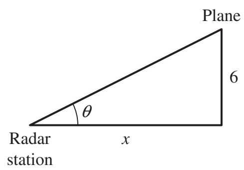
$$x = 160 \qquad \text{and} \qquad \tan\theta = \frac{3}{80}.$$

Furthermore,

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{3^2}{80^2}.$$

Finally,

$$\frac{d\theta}{dt} = -\frac{6}{160^2}\frac{1}{1+\frac{3^2}{80^2}}800 = -\frac{1200}{6409} = -0.187 \text{ rad/hour}.$$



21. A hot air balloon rising vertically is tracked by an observer located 4 km from the lift-off point. At a certain moment, the angle between the observer's line of sight and the horizontal is $\frac{\pi}{5}$, and it is changing at a rate of 0.2 rad/min. How fast is the balloon rising at this moment?

SOLUTION Let y be the height of the balloon (in miles) and θ the angle between the line-of-sight and the horizontal. Via trigonometry, we have $\tan\theta = \frac{y}{4}$. Therefore,

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{4}\frac{dy}{dt},$$

and

$$\frac{dy}{dt} = 4\frac{d\theta}{dt}\sec^2\theta.$$

Using $\frac{d\theta}{dt} = 0.2$ and $\theta = \frac{\pi}{5}$ yields

$$\frac{dy}{dt} = 4\,(0.2)\,\frac{1}{\cos^2(\pi/5)} \approx 1.22 \text{ km/min}.$$

22. A laser pointer is placed on a platform that rotates at a rate of 20 revolutions per minute. The beam hits a wall 8 m away, producing a dot of light that moves horizontally along the wall. Let θ be the angle between the beam and the line through the searchlight perpendicular to the wall (Figure 10). How fast is this dot moving when $\theta = \frac{\pi}{6}$?

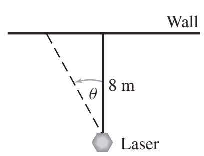


FIGURE 10

SOLUTION Let y be the distance between the dot of light and the point of intersection of the wall and the line through the searchlight perpendicular to the wall. Let θ be the angle between the beam of light and the line. Using trigonometry, we have $\tan\theta = \frac{y}{8}$. Therefore,

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{8}\frac{dy}{dt},$$

and

$$\frac{dy}{dt} = 8\frac{d\theta}{dt}\sec^2\theta.$$

With $\theta = \frac{\pi}{6}$ and $\frac{d\theta}{dt} = 40\pi$, we find

$$\frac{dy}{dt} = 8\,(40\pi)\,\frac{1}{\cos^2(\pi/6)} = \frac{1280}{3}\pi \approx 1340.4\ \text{m/min}.$$

23. A rocket travels vertically at a speed of 1200 km/h. The rocket is tracked through a telescope by an observer located 16 km from the launching pad. Find the rate at which the angle between the telescope and the ground is increasing 3 min after lift-off.

SOLUTION Let y be the height of the rocket and θ the angle between the telescope and the ground. Using trigonometry, we have $\tan\theta = \frac{y}{16}$. Therefore,

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{16}\frac{dy}{dt},$$

and

$$\frac{d\theta}{dt} = \frac{\cos^2\theta}{16}\frac{dy}{dt}.$$

After the rocket has traveled for 3 minutes (or $\frac{1}{20}$ hour), its height is $\frac{1}{20} \times 1200 = 60$ km. At this instant, $\tan\theta = 60/16 = 15/4$ and thus

$$\cos\theta = \frac{4}{\sqrt{15^2 + 4^2}} = \frac{4}{\sqrt{241}}.$$

Finally,

$$\frac{d\theta}{dt} = \frac{16/241}{16}(1200) = \frac{1200}{241} \approx 4.98\ \text{rad/hr}.$$

24. Using a telescope, you track a rocket that was launched 4 km away, recording the angle θ between the telescope and the ground at half-second intervals. Estimate the velocity of the rocket if $\theta(10) = 0.205$ and $\theta(10.5) = 0.225$.

SOLUTION Let h be the height of the vertically ascending rocket. Using trigonometry, $\tan\theta = \dfrac{h}{4}$, so

$$\frac{dh}{dt} = 4\sec^2\theta \cdot \frac{d\theta}{dt}.$$

We are given $\theta(10) = 0.205$, and we can estimate

$$\left.\frac{d\theta}{dt}\right|_{t=10} \approx \frac{\theta(10.5) - \theta(10)}{0.5} = 0.04.$$

Thus,

$$\frac{dh}{dt} \approx 4\sec^2(0.205)\cdot(0.04) \approx 0.166\ \text{km/s},$$

or roughly 600 km/h.

25. A police car traveling south toward Sioux Falls at 160 km/h pursues a truck traveling east away from Sioux Falls, Iowa, at 140 km/h (Figure 11). At time $t = 0$, the police car is 20 km north and the truck is 30 km east of Sioux Falls. Calculate the rate at which the distance between the vehicles is changing:

(a) At time $t = 0$

(b) 5 min later

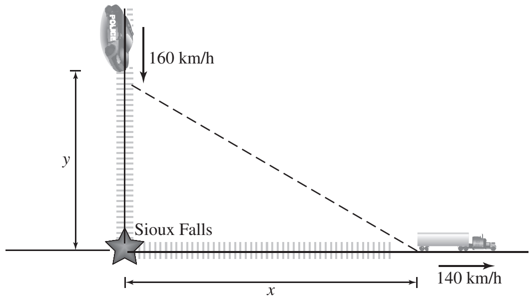


FIGURE 11

SOLUTION Let y denote the distance the police car is north of Sioux Falls and x the distance the truck is east of Sioux Falls. Then $y = 20 - 160t$ and $x = 30 + 140t$. If ℓ denotes the distance between the police car and the truck, then

$$\ell^2 = x^2 + y^2 = (30 + 140t)^2 + (20 - 160t)^2$$

and

$$\ell \frac{d\ell}{dt} = 140(30 + 140t) - 160(20 - 160t) = 1000 + 45200t.$$

(a) At $t = 0$, $\ell = \sqrt{30^2 + 20^2} = 10\sqrt{13}$, so

$$\frac{d\ell}{dt} = \frac{1000}{10\sqrt{13}} = \frac{100\sqrt{13}}{13} \approx 27.735 \text{ km/h}.$$

(b) At $t = 5$ minutes $= \frac{1}{12}$ hour,

$$\ell = \sqrt{\left(30 + 140 \cdot \frac{1}{12}\right)^2 + \left(20 - 160 \cdot \frac{1}{12}\right)^2} \approx 42.197 \text{ km},$$

and

$$\frac{d\ell}{dt} = \frac{1000 + 45200 \cdot \frac{1}{12}}{42.197} \approx 112.962 \text{ km/h}.$$

26. A car travels down a highway at 25 m/s. An observer stands 150 m from the highway.

(a) How fast is the distance from the observer to the car increasing when the car passes in front of the observer? Explain your answer without making any calculations.

(b) How fast is the distance increasing 20 s later?

SOLUTION Let x be the distance (in feet) along the road that the car has traveled and h be the distance (in feet) between the car and the observer.

(a) Before the car passes the observer, we have $dh/dt < 0$; after it passes, we have $dh/dt > 0$. So at the instant it passes we have $dh/dt = 0$, given that dh/dt varies continuously since the car travels at a constant velocity.

(b) By the Pythagorean Theorem, we have $h^2 = x^2 + 150^2$. Thus

$$2h\frac{dh}{dt} = 2x\frac{dx}{dt},$$

and

$$\frac{dh}{dt} = \frac{x}{h}\frac{dx}{dt}.$$

The car travels at 25 m/s, so after 20 seconds, $x = 25(20) = 500$ meters. Therefore,

$$\frac{dh}{dt} = \frac{500}{\sqrt{500^2 + 150^2}}(25) \approx 23.95 \text{ m/s}.$$

27. In the setting of Example 5, at a certain moment, the tractor's speed is 3 m/s and the bale is rising at 2 m/s. How far is the tractor from the bale at this moment?

SOLUTION From Example 5, we have the equation

$$\frac{x\frac{dx}{dt}}{\sqrt{x^2+4.5^2}} = \frac{dh}{dt},$$

where x denotes the distance from the tractor to the bale and h denotes the height of the bale. Given

$$\frac{dx}{dt} = 3 \quad \text{and} \quad \frac{dh}{dt} = 2,$$

it follows that

$$\frac{3x}{\sqrt{4.5^2+x^2}} = 2,$$

which yields $x = \sqrt{16.2} \approx 4.025$ m.

28. Placido pulls a rope attached to a wagon through a pulley at a rate of q m/s. With dimensions as in Figure 12:

(a) Find a formula for the speed of the wagon in terms of q and the variable x in the figure.

(b) Find the speed of the wagon when $x = 0.6$ if $q = 0.5$ m/s.

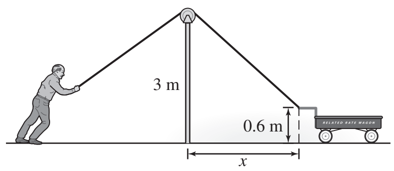


FIGURE 12

SOLUTION Let h be the distance from the pulley to the loop on the wagon. Using the Pythagorean Theorem, we have $h^2 = x^2 + (3-0.6)^2 = x^2 + 2.4^2$.

(a) Thus $2h\frac{dh}{dt} = 2x\frac{dx}{dt}$, and $\frac{dx}{dt} = \frac{h}{x}\frac{dh}{dt}$. Given $dh/dt = q$, it follows that

$$\frac{dx}{dt} = \frac{\sqrt{x^2+2.4^2}}{x}q.$$

(b) As Placido pulls the rope at the rate of $q = 0.5$ m/s and $x = 0.6$

$$\frac{dx}{dt} = \frac{\sqrt{0.6^2+2.4^2}}{0.6}(0.5) \approx 2.06 \text{ m/s}.$$

29. Julian is jogging around a circular track of radius 50 m. In a coordinate system with its origin at the center of the track, Julian's x-coordinate is changing at a rate of -1.25 m/s when his coordinates are (40, 30). Find dy/dt at this moment.

SOLUTION We have $x^2 + y^2 = 50^2$, so

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \quad \text{or} \quad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

Given $x = 40$, $y = 30$ and $dx/dt = -1.25$, we find

$$\frac{dy}{dt} = -\frac{40}{30}(-1.25) = \frac{5}{3} \text{ m/s}.$$

30. A particle moves counterclockwise around the ellipse with equation $9x^2 + 16y^2 = 25$ (Figure 13).

(a) In which of the four quadrants is $dx/dt > 0$? Explain.

(b) Find a relation between dx/dt and dy/dt.

(c) At what rate is the x-coordinate changing when the particle passes the point (1, 1) if its y-coordinate is increasing at a rate of 6 m/s?

(d) Find dy/dt when the particle is at the top and bottom of the ellipse.

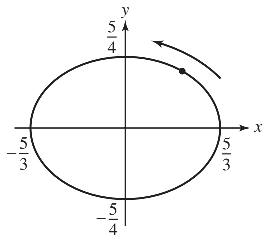


FIGURE 13

SOLUTION A particle moves counterclockwise around the ellipse with equation $9x^2 + 16y^2 = 25$.

(a) The derivative dx/dt is positive in quadrants 3 and 4 since the particle is moving to the right.

(b) From $9x^2 + 16y^2 = 25$ we have $18x\dfrac{dx}{dt} + 32y\dfrac{dy}{dt} = 0$.

(c) From (b), we have $\dfrac{dx}{dt} = -\dfrac{16y}{9x}\dfrac{dy}{dt}$. With $x = y = 1$ and $\frac{dy}{dt} = 6$,

$$\frac{dx}{dt} = -\frac{16 \cdot 1}{9 \cdot 1}(6) = -\frac{32}{3} \text{ m/s}.$$

(d) From (b), we have $\dfrac{dy}{dt} = -\dfrac{9x}{16y}\dfrac{dx}{dt}$. When $(x, y) = \left(0, \pm\dfrac{5}{4}\right)$, it follows that $\dfrac{dy}{dt} = 0$.

In Exercises 31 and 32, assume that the pressure P (in kilopascals) and volume V (in cubic centimeters) of an expanding gas are related by $PV^b = C$, where b and C are constants (this holds in an adiabatic *expansion, without heat gain or loss).*

31. Find dP/dt if $b = 1.2$, $P = 8$ kPa, $V = 100$ cm^2, and $dV/dt = 20$ cm^3/min.

SOLUTION Let $PV^b = C$. Then

$$PbV^{b-1}\frac{dV}{dt} + V^b\frac{dP}{dt} = 0,$$

and

$$\frac{dP}{dt} = -\frac{Pb}{V}\frac{dV}{dt}.$$

Substituting $b = 1.2$, $P = 8$, $V = 100$, and $\frac{dV}{dt} = 20$, we find

$$\frac{dP}{dt} = -\frac{(8)\,(1.2)}{100}(20) = -1.92 \text{ kPa/min}.$$

32. Find b if $P = 25$ kPa, $dP/dt = 12$ kPa/min, $V = 100$ cm^2, and $dV/dt = 20$ cm^3/min.

SOLUTION Let $PV^b = C$. Then

$$PbV^{b-1}\frac{dV}{dt} + V^b\frac{dP}{dt} = 0,$$

and

$$b = -\frac{V}{P}\frac{dP/dt}{dV/dt}.$$

With $P = 25$, $V = 100$, $\frac{dP}{dt} = 12$, and $\frac{dV}{dt} = 20$, we have

$$b = -\frac{100}{25} \times \frac{12}{20} = -\frac{12}{5}.$$

(*Note:* If instead we have $\frac{dP}{dt} = -12$ kPa/min, then $b = \frac{12}{5}$.)

33. The base x of the right triangle in Figure 14 increases at a rate of 5 cm/s, while the height remains constant at $h = 20$. How fast is the angle θ changing when $x = 20$?

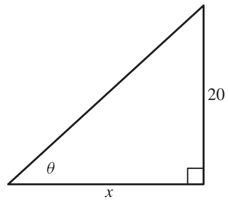


FIGURE 14

SOLUTION We have $\cot\theta = \frac{x}{20}$, from which

$$-\csc^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt}$$

and thus

$$\frac{d\theta}{dt} = -\frac{\sin^2\theta}{20}\frac{dx}{dt}.$$

We are given $\frac{dx}{dt} = 5$ and when $x = h = 20$, $\theta = \frac{\pi}{4}$. Hence,

$$\frac{d\theta}{dt} = -\frac{\sin^2\left(\frac{\pi}{4}\right)}{20}(5) = -\frac{1}{8}\text{ rad/s}.$$

34. Two parallel paths 15 m apart run east–west through the woods. Brooke jogs east on one path at 10 km/h, while Jamail walks west on the other path at 6 km/h. If they pass each other at time $t = 0$, how far apart are they 3 s later, and how fast is the distance between them changing at that moment?

SOLUTION Brooke jogs at 10 km/h $= \frac{25}{9}$ m/s and Jamail walks at 6 km/h $= \frac{5}{3}$ m/s. At time zero, consider Brooke to be at the origin $(0, 0)$ and (without loss of generality) Jamail to be at $(0, 15)$; i.e., due north of Brooke. Then at time t, the position of Brooke is $\left(\frac{25}{9}t, 0\right)$ and that of Jamail is $\left(-\frac{5}{3}t, 15\right)$. The distance between them is

$$L = \sqrt{\left(\frac{25}{9}t + \frac{5}{3}t\right)^2 + (15)^2} = \left(\left(\frac{40}{9}t\right)^2 + 15^2\right)^{1/2}.$$

- When $t = 3$ seconds, the distance between them is

$$L = \sqrt{\left(\frac{40}{3}\right)^2 + 15^2} = \frac{5}{3}\sqrt{145} \approx 20.07\text{ m}.$$

- The distance between them is changing at the rate

$$\frac{dL}{dt} = \frac{1}{2}\left(\left(\frac{40}{9}t\right)^2 + 15^2\right)^{-1/2}\left(2\left(\frac{40}{9}t\right)\frac{40}{9}\right).$$

When $t = 3$, we then have

$$\frac{dL}{dt} = \frac{\frac{1}{9}(40)^2}{\sqrt{40^2 + 45^2}} \approx 2.95\text{ m/s}$$

35. A particle travels along a curve $y = f(x)$ as in Figure 15. Let $L(t)$ be the particle's distance from the origin.

(a) Show that $\frac{dL}{dt} = \left(\frac{x + f(x)f'(x)}{\sqrt{x^2 + f(x)^2}}\right)\frac{dx}{dt}$ if the particle's location at time t is $P = (x, f(x))$.

(b) Calculate $L'(t)$ when $x = 1$ and $x = 2$ if $f(x) = \sqrt{3x^2 - 8x + 9}$ and $dx/dt = 4$.

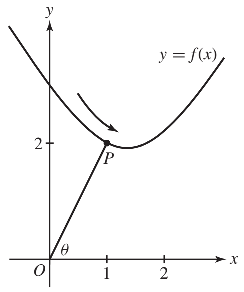


FIGURE 15

SOLUTION

(a) If the particle's location at time t is $P = (x, f(x))$, then

$$L(t) = \sqrt{x^2 + f(x)^2}.$$

Thus,

$$\frac{dL}{dt} = \frac{1}{2}(x^2 + f(x)^2)^{-1/2}\left(2x\frac{dx}{dt} + 2f(x)f'(x)\frac{dx}{dt}\right) = \left(\frac{x + f(x)f'(x)}{\sqrt{x^2 + f(x)^2}}\right)\frac{dx}{dt}.$$

(b) Given $f(x) = \sqrt{3x^2 - 8x + 9}$, it follows that

$$f'(x) = \frac{3x - 4}{\sqrt{3x^2 - 8x + 9}}.$$

Let's start with $x = 1$. Then $f(1) = 2$, $f'(1) = -\frac{1}{2}$ and

$$\frac{dL}{dt} = \left(\frac{1 - 1}{\sqrt{1^2 + 2^2}}\right)(4) = 0.$$

With $x = 2$, $f(2) = \sqrt{5}$, $f'(2) = 2/\sqrt{5}$ and

$$\frac{dL}{dt} = \frac{2 + 2}{\sqrt{2^2 + \sqrt{5}^2}}(4) = \frac{16}{3}.$$

36. Let θ be the angle in Figure 15, where $P = (x, f(x))$. In the setting of the previous exercise, show that

$$\frac{d\theta}{dt} = \left(\frac{xf'(x) - f(x)}{x^2 + f(x)^2}\right)\frac{dx}{dt}$$

Hint: Differentiate $\tan\theta = f(x)/x$ and observe that $\cos\theta = x/\sqrt{x^2 + f(x)^2}$.

SOLUTION If the particle's location at time t is $P = (x, f(x))$, then $\tan\theta = f(x)/x$ and

$$\sec^2\theta\frac{d\theta}{dt} = \frac{xf'(x)\frac{dx}{dt} - f(x)\frac{dx}{dt}}{x^2} = \left(\frac{xf'(x) - f(x)}{x^2}\right)\frac{dx}{dt}.$$

Now

$$\cos\theta = \frac{x}{\sqrt{x^2 + f(x)^2}} \quad \text{so} \quad \sec^2\theta = \frac{x^2 + f(x)^2}{x^2}.$$

Finally,

$$\frac{d\theta}{dt} = \left(\frac{xf'(x) - f(x)}{x^2 + f(x)^2}\right)\frac{dx}{dt}.$$

Exercises 37 and 38 refer to the baseball diamond (a square of side 90 ft*) in Figure 16.*

37. A baseball player runs from home plate toward first base at 20 ft/s. How fast is the player's distance from second base changing when the player is halfway to first base?

SOLUTION Let x be the distance of the player from home plate and h the player's distance from second base. Using the Pythagorean theorem, we have $h^2 = 90^2 + (90 - x)^2$. Therefore,

$$2h\frac{dh}{dt} = 2\,(90 - x)\left(-\frac{dx}{dt}\right),$$

and

$$\frac{dh}{dt} = -\frac{90 - x}{h}\frac{dx}{dt}.$$

We are given $\frac{dx}{dt} = 20$. When the player is halfway to first base, $x = 45$ and $h = \sqrt{90^2 + 45^2}$, so

$$\frac{dh}{dt} = -\frac{45}{\sqrt{90^2 + 45^2}}\,(20) = -4\sqrt{5} \approx -8.94 \text{ ft/s}.$$

38. Player 1 runs to first base at a speed of 20 ft/s, while Player 2 runs from second base to third base at a speed of 15 ft/s. Let s be the distance between the two players. How fast is s changing when Player 1 is 30 ft from home plate and Player 2 is 60 ft from second base?

SOLUTION Let x denote the distance from home plate to Player 1 and y denote the distance from second base to Player 2, both distances measured along the base path. Then

$$s(t) = \sqrt{(90 - x - y)^2 + 90^2},$$

and

$$\frac{ds}{dt} = -\frac{90 - x - y}{\sqrt{(90 - x - y)^2 + 90^2}}\left(\frac{dx}{dt} + \frac{dy}{dt}\right).$$

With $x = 30$ and $y = 60$, it follows that

$$\frac{ds}{dt} = 0.$$

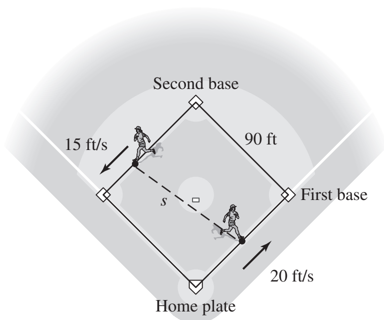


FIGURE 16

39. The conical watering pail in Figure 17 has a grid of holes. Water flows out through the holes at a rate of kA m^3/min, where k is a constant and A is the surface area of the part of the cone in contact with the water. This surface area is $A = \pi r\sqrt{h^2 + r^2}$ and the volume is $V = \frac{1}{3}\pi r^2 h$. Calculate the rate dh/dt at which the water level changes at $h = 0.3$ m, assuming that $k = 0.25$ m.

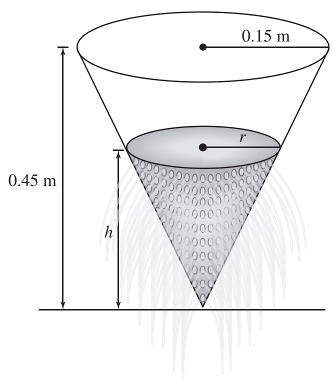


FIGURE 17

SOLUTION By similar triangles, we have

$$\frac{r}{h} = \frac{0.15}{0.45} = \frac{1}{3} \quad \text{so} \quad r = \frac{1}{3}h.$$

Substituting this expression for r into the formula for V yields

$$V = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h = \frac{1}{27}\pi h^3.$$

From here and the problem statement, it follows that

$$\frac{dV}{dt} = \frac{1}{9}\pi h^2 \frac{dh}{dt} = -kA = -0.25\pi r\sqrt{h^2 + r^2}.$$

Solving for dh/dt gives

$$\frac{dh}{dt} = -\frac{9}{4}\frac{r}{h^2}\sqrt{h^2 + r^2}.$$

When $h = 0.3$, $r = 0.1$ and

$$\frac{dh}{dt} = -\frac{9}{4}\frac{0.1}{0.3^2}\sqrt{0.3^2 + 0.1^2} = -0.79 \text{ m/min}.$$

Further Insights and Challenges

40. A bowl contains water that evaporates at a rate proportional to the surface area of water exposed to the air (Figure 18). Let $A(h)$ be the cross-sectional area of the bowl at height h.

(a) Explain why $V(h + \Delta h) - V(h) \approx A(h)\Delta h$ if Δh is small.

(b) Use (a) to argue that $\dfrac{dV}{dh} = A(h)$.

(c) Show that the water level h decreases at a constant rate.

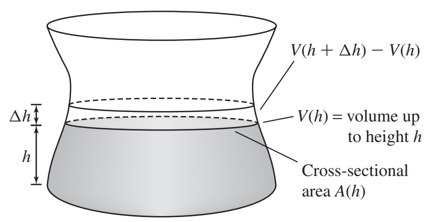


FIGURE 18

SOLUTION

(a) Consider a thin horizontal slice of the water in the cup of thickness Δh at height h. Assuming the cross-sectional area of the cup is roughly constant across this slice, it follows that

$$V(h + \Delta h) - V(h) \approx A(h)\Delta h.$$

(b) If we take the expression from part (a), divide by Δh and pass to the limit as $\Delta h \to 0$, we find

$$\frac{dV}{dh} = A(h).$$

(c) If we take the expression from part (b) and multiply by dh/dt, recognizing that

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt},$$

we find that

$$\frac{dV}{dt} = A(h)\frac{dh}{dt}.$$

We are told that the water in the bowl evaporates at a rate proportional to the surface area exposed to the air; translated into mathematics, this means

$$\frac{dV}{dt} = -kA(h),$$

where k is a positive constant of proportionality. Combining the last two equations yields

$$\frac{dh}{dt} = -k;$$

that is, the water level decreases at a constant rate.

41. A roller coaster has the shape of the graph in Figure 19. Show that when the roller coaster passes the point $(x, f(x))$, the vertical velocity of the roller coaster is equal to $f'(x)$ times its horizontal velocity.

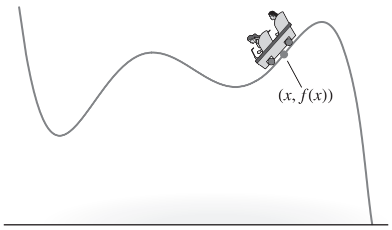


FIGURE 19 Graph of f as a roller coaster track.

SOLUTION Let the equation $y = f(x)$ describe the shape of the roller coaster track. Taking $\frac{d}{dt}$ of both sides of this equation yields $\frac{dy}{dt} = f'(x)\frac{dx}{dt}$. In other words, the vertical velocity of a car moving along the track, $\frac{dy}{dt}$, is equal to $f'(x)$ times the horizontal velocity, $\frac{dx}{dt}$.

42. Two trains leave a station at $t = 0$ and travel with constant velocity v along straight tracks that make an angle θ.

(a) Show that the trains are separating from each other at a rate $v\sqrt{2 - 2\cos\theta}$.

(b) What does this formula give for $\theta = \pi$?

SOLUTION Choose a coordinate system such that

- the origin is the point of departure of the trains;
- the first train travels along the positive x-axis;
- the second train travels along the ray emanating from the origin at an angle of $\theta > 0$.

(a) At time t, the position of the first train is $(vt, 0)$, while that of the second is $(vt\cos\theta, vt\sin\theta)$. The distance between the trains is

$$L = \sqrt{(vt\,(1 - \cos\theta))^2 + (vt\sin\theta)^2} = vt\sqrt{2 - 2\cos\theta}.$$

Thus $dL/dt = v\sqrt{2 - 2\cos\theta}$.

(b) When $\theta = \pi$, we have $dL/dt = 2v$. This is obviously correct since at this angle the trains travel in opposite directions at the same constant speed, having started from the same point.

43. As the wheel of radius r cm in Figure 20 rotates, the rod of length L attached at point P drives a piston back and forth in a straight line. Let x be the distance from the origin to point Q at the end of the rod, as shown in the figure.

(a) Use the Pythagorean Theorem to show that

$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta \qquad \boxed{1}$$

(b) Differentiate Eq. (1) with respect to t to prove that

$$2(x - r\cos\theta)\left(\frac{dx}{dt} + r\sin\theta\frac{d\theta}{dt}\right) + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt} = 0$$

(c) Calculate the speed of the piston when $\theta = \frac{\pi}{2}$, assuming that $r = 10$ cm, $L = 30$ cm, and the wheel rotates at 4 revolutions per minute.

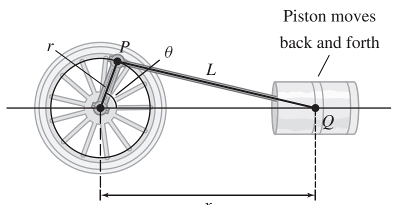


FIGURE 20

SOLUTION From the diagram, the coordinates of P are $(r\cos\theta, r\sin\theta)$ and those of Q are $(x, 0)$.

(a) The distance formula gives

$$L = \sqrt{(x - r\cos\theta)^2 + (-r\sin\theta)^2}.$$

Thus,

$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta.$$

Note that L (the length of the fixed rod) and r (the radius of the wheel) are constants.

(b) From (a) we have

$$0 = 2(x - r\cos\theta)\left(\frac{dx}{dt} + r\sin\theta\frac{d\theta}{dt}\right) + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt}.$$

(c) Solving for dx/dt in (b) gives

$$\frac{dx}{dt} = \frac{r^2\sin\theta\cos\theta\frac{d\theta}{dt}}{r\cos\theta - x} - r\sin\theta\frac{d\theta}{dt} = \frac{rx\sin\theta\frac{d\theta}{dt}}{r\cos\theta - x}.$$

With $\theta = \frac{\pi}{2}$, $r = 10$, $L = 30$, and $\frac{d\theta}{dt} = 8\pi$,

$$\frac{dx}{dt} = \frac{(10)(x)\left(\sin\frac{\pi}{2}\right)(8\pi)}{(10)(0) - x} = -80\pi \approx -251.33 \text{ cm/min}$$

44. A spectator seated 300 m away from the center of a circular track of radius 100 m watches an athlete run laps at a speed of 5 m/s. How fast is the distance between the spectator and athlete changing when the runner is approaching the spectator and the distance between them is 250 m? *Hint:* The diagram for this problem is similar to Figure 20, with $r = 100$ and $x = 300$.

SOLUTION From the diagram, the coordinates of P are $(r\cos\theta, r\sin\theta)$ and those of Q are $(x, 0)$.

- The distance formula gives

$$L = \sqrt{(x - r\cos\theta)^2 + (-r\sin\theta)^2}.$$

Thus,

$$L^2 = (x - r\cos\theta)^2 + r^2\sin^2\theta.$$

Note that x (the distance of the spectator from the center of the track) and r (the radius of the track) are constants.

- Differentiating with respect to t gives

$$2L\frac{dL}{dt} = 2(x - r\cos\theta)r\sin\theta\frac{d\theta}{dt} + 2r^2\sin\theta\cos\theta\frac{d\theta}{dt}.$$

Thus,

$$\frac{dL}{dt} = \frac{rx}{L}\sin\theta\frac{d\theta}{dt}.$$

- Recall the relation between arc length s and angle θ, namely $s = r\theta$. Thus $\dfrac{d\theta}{dt} = \dfrac{1}{r}\dfrac{ds}{dt}$. Given $r = 100$ and $\frac{ds}{dt} = -5$, we have

$$\frac{d\theta}{dt} = \frac{1}{100}(-5) = -\frac{1}{20} \text{ rad/s}.$$

(*Note:* In this scenario, the runner traverses the track in a *clockwise* fashion and approaches the spectator from Quadrant 1.)

- Next, the Law of Cosines gives $L^2 = r^2 + x^2 - 2rx\cos\theta$, so

$$\cos\theta = \frac{r^2 + x^2 - L^2}{2rx} = \frac{100^2 + 300^2 - 250^2}{2\,(100)\,(300)} = \frac{5}{8}.$$

Accordingly,

$$\sin\theta = \sqrt{1 - \left(\frac{5}{8}\right)^2} = \frac{\sqrt{39}}{8}.$$

- Finally

$$\frac{dL}{dt} = \frac{(300)\,(100)}{250}\left(\frac{\sqrt{39}}{8}\right)\left(-\frac{1}{20}\right) = -\frac{3\sqrt{39}}{4} \approx -4.68 \text{ m/s}.$$

45. A cylindrical tank of radius R and length L lying horizontally as in Figure 21 is filled with oil to height h.
(a) Show that the volume $V(h)$ of oil in the tank is

$$V(h) = L\left(R^2\cos^{-1}\left(1 - \frac{h}{R}\right) - (R - h)\sqrt{2hR - h^2}\right)$$

(b) Show that $\frac{dV}{dh} = 2L\sqrt{h(2R - h)}$.
(c) Suppose that $R = 1.5$ m and $L = 10$ m and that the tank is filled at a constant rate of 0.6 m^3/min. How fast is the height h increasing when $h = 0.5$?

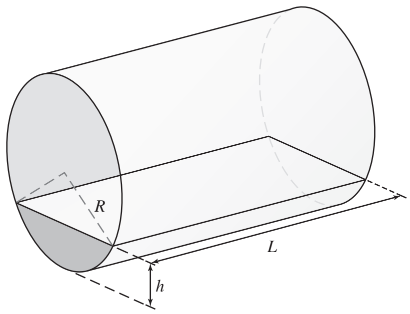


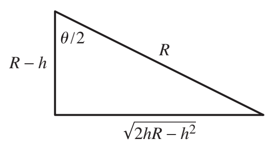
FIGURE 21 Oil in the tank has level h.

SOLUTION
(a) From Figure 21, we see that the volume of oil in the tank, $V(h)$, is equal to L times $A(h)$, the area of that portion of the circular cross section occupied by the oil. Now,

$$A(h) = \text{area of sector} - \text{area of triangle} = \frac{R^2\theta}{2} - \frac{R^2\sin\theta}{2},$$

where θ is the central angle of the sector. Referring to the diagram below,

$$\cos\frac{\theta}{2} = \frac{R - h}{R} \quad \text{and} \quad \sin\frac{\theta}{2} = \frac{\sqrt{2hR - h^2}}{R}.$$



Thus,

$$\theta = 2\cos^{-1}\left(1 - \frac{h}{R}\right),$$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\frac{(R - h)\sqrt{2hR - h^2}}{R^2},$$

and

$$V(h) = L\left(R^2\cos^{-1}\left(1-\frac{h}{R}\right) - (R-h)\sqrt{2hR-h^2}\right).$$

(b) Recalling that $\frac{d}{dx}\cos^{-1} u = -\frac{1}{\sqrt{1-x^2}}\frac{du}{dx}$,

$$\begin{aligned}
\frac{dV}{dh} &= L\left(\frac{d}{dh}\left(R^2\cos^{-1}\left(1-\frac{h}{R}\right)\right) - \frac{d}{dh}\left((R-h)\sqrt{2hR-h^2}\right)\right)\\
&= L\left(-R\frac{-1}{\sqrt{1-(1-(h/R))^2}} + \sqrt{2hR-h^2} - \frac{(R-h)^2}{\sqrt{2hR-h^2}}\right)\\
&= L\left(\frac{R^2}{\sqrt{2hR-h^2}} + \sqrt{2hR-h^2} - \frac{R^2-2Rh+h^2}{\sqrt{2hR-h^2}}\right)\\
&= L\left(\frac{R^2+(2hR-h^2)-(R^2-2Rh+h^2)}{\sqrt{2hR-h^2}}\right)\\
&= L\left(\frac{4hR-2h^2}{\sqrt{2hR-h^2}}\right) = L\left(\frac{2(2hR-h^2)}{\sqrt{2hR-h^2}}\right) = 2L\sqrt{2hR-h^2}.
\end{aligned}$$

(c) $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{1}{dV/dh}\frac{dV}{dt}$. From part (b) with $R=4$, $L=30$ and $h=5$,

$$\frac{dV}{dh} = 2(30)\sqrt{2(5)(4)-5^2} = 60\sqrt{15}\ \text{ft}^2.$$

Thus,

$$\frac{dh}{dt} = \frac{1}{60\sqrt{15}}(10) = \frac{\sqrt{15}}{90} \approx 0.043\ \text{ft/min}.$$

CHAPTER REVIEW EXERCISES

In Exercises 1–4, refer to the function f whose graph is shown in Figure 1.

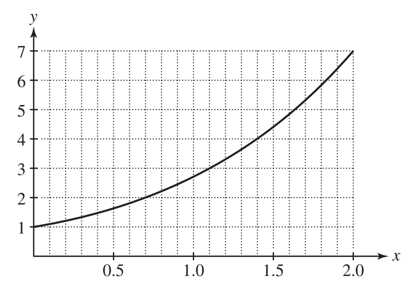


FIGURE 1

1. Compute the average rate of change of $f(x)$ over $[0, 2]$. What is the graphical interpretation of this average rate?

SOLUTION The average rate of change of $f(x)$ over $[0, 2]$ is

$$\frac{f(2)-f(0)}{2-0} = \frac{7-1}{2-0} = 3.$$

Graphically, this average rate of change represents the slope of the secant line through the points $(2, 7)$ and $(0, 1)$ on the graph of $f(x)$.

2. For which value of h is $\frac{f(0.7+h)-f(0.7)}{h}$ equal to the slope of the secant line between the points where $x = 0.7$ and $x = 1.1$?

SOLUTION Because $1.1 = 0.7 + 0.4$, the difference quotient

$$\frac{f(0.7+h) - f(0.7)}{h}$$

is equal to the slope of the secant line between the points where $x = 0.7$ and $x = 1.1$ for $h = 0.4$.

3. Estimate $\dfrac{f(0.7+h) - f(0.7)}{h}$ for $h = 0.3$. Is this number larger or smaller than $f'(0.7)$?

SOLUTION For $h = 0.3$,

$$\frac{f(0.7+h) - f(0.7)}{h} = \frac{f(1) - f(0.7)}{0.3} \approx \frac{2.8 - 2}{0.3} = \frac{8}{3}.$$

Because the curve is concave up, the slope of the secant line is larger than the slope of the tangent line, so the value of the difference quotient should be larger than the value of the derivative.

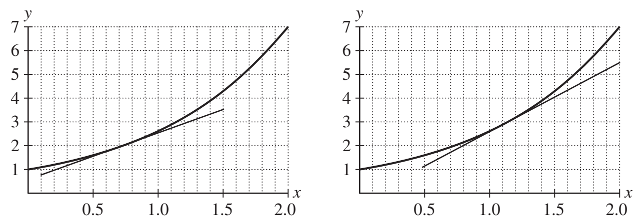
4. Estimate $f'(0.7)$ and $f'(1.1)$.

SOLUTION The tangent line sketched in the graph below at the left appears to pass through the points $(0.2, 1)$ and $(1.5, 3.5)$. Thus,

$$f'(0.7) \approx \frac{3.5 - 1}{1.5 - 0.2} = \frac{2.5}{1.3} = 1.923.$$

The tangent line sketched in the graph below at the right appears to pass through the points $(0.8, 2)$ and $(2, 5.5)$. Thus,

$$f'(1.1) \approx \frac{5.5 - 2}{2 - 0.8} = \frac{3.5}{1.2} = 2.917.$$



In Exercises 5–8, compute $f'(a)$ using the limit definition and find an equation of the tangent line to the graph of f at $x = a$.

5. $f(x) = x^2 - x, \quad a = 1$

SOLUTION Let $f(x) = x^2 - x$ and $a = 1$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{(1+h)^2 - (1+h) - (1^2 - 1)}{h}$$
$$= \lim_{h\to 0} \frac{1 + 2h + h^2 - 1 - h}{h} = \lim_{h\to 0}(1+h) = 1$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = 1(x-1) + 0 = x - 1.$$

6. $f(x) = 5 - 3x, \quad a = 2$

SOLUTION Let $f(x) = 5 - 3x$ and $a = 2$. Then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{5 - 3(2+h) - (5-6)}{h} = \lim_{h\to 0} -3 = -3$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = -3(x-2) - 1 = -3x + 5.$$

7. $f(x) = x^{-1}, \quad a = 4$

SOLUTION Let $f(x) = x^{-1}$ and $a = 4$. Then

$$f'(a) = \lim_{h\to0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to0} \frac{\frac{1}{4+h} - \frac{1}{4}}{h} = \lim_{h\to0} \frac{4-(4+h)}{4h(4+h)}$$
$$= \lim_{h\to0} \frac{-1}{4(4+h)} = -\frac{1}{4(4+0)} = -\frac{1}{16}$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = -\frac{1}{16}(x-4) + \frac{1}{4} = -\frac{1}{16}x + \frac{1}{2}.$$

8. $f(x) = x^3, \quad a = -2$

SOLUTION Let $f(x) = x^3$ and $a = -2$. Then

$$f'(a) = \lim_{h\to0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to0} \frac{(-2+h)^3 - (-2)^3}{h} = \lim_{h\to0} \frac{-8 + 12h - 6h^2 + h^3 + 8}{h}$$
$$= \lim_{h\to0} (12 - 6h + h^2) = 12 - 6(0) + 0^2 = 12$$

and the equation of the tangent line to the graph of $f(x)$ at $x = a$ is

$$y = f'(a)(x-a) + f(a) = 12(x+2) - 8 = 12x + 16.$$

In Exercises 9–12, compute dy/dx using the limit definition.

9. $y = 4 - x^2$

SOLUTION Let $y = 4 - x^2$. Then

$$\frac{dy}{dx} = \lim_{h\to0} \frac{4-(x+h)^2 - (4-x^2)}{h} = \lim_{h\to0} \frac{4 - x^2 - 2xh - h^2 - 4 + x^2}{h} = \lim_{h\to0} (-2x - h) = -2x - 0 = -2x.$$

10. $y = \sqrt{2x+1}$

SOLUTION Let $y = \sqrt{2x+1}$. Then

$$\frac{dy}{dx} = \lim_{h\to0} \frac{\sqrt{2(x+h)+1} - \sqrt{2x+1}}{h} = \lim_{h\to0} \frac{\sqrt{2x+2h+1} - \sqrt{2x+1}}{h} \cdot \frac{\sqrt{2x+2h+1} + \sqrt{2x+1}}{\sqrt{2x+2h+1} + \sqrt{2x+1}}$$
$$= \lim_{h\to0} \frac{(2x+2h+1) - (2x+1)}{h(\sqrt{2x+2h+1} + \sqrt{2x+1})} = \lim_{h\to0} \frac{2}{\sqrt{2x+2h+1} + \sqrt{2x+1}} = \frac{1}{\sqrt{2x+1}}.$$

11. $y = \dfrac{1}{2-x}$

SOLUTION Let $y = \dfrac{1}{2-x}$. Then

$$\frac{dy}{dx} = \lim_{h\to0} \frac{\frac{1}{2-(x+h)} - \frac{1}{2-x}}{h} = \lim_{h\to0} \frac{(2-x) - (2-x-h)}{h(2-x-h)(2-x)} = \lim_{h\to0} \frac{1}{(2-x-h)(2-x)} = \frac{1}{(2-x)^2}.$$

12. $y = \dfrac{1}{(x-1)^2}$

SOLUTION Let $y = \dfrac{1}{(x-1)^2}$. Then

$$\frac{dy}{dx} = \lim_{h\to0} \frac{\frac{1}{(x+h-1)^2} - \frac{1}{(x-1)^2}}{h} = \lim_{h\to0} \frac{(x-1)^2 - (x+h-1)^2}{h(x+h-1)^2(x-1)^2}$$
$$= \lim_{h\to0} \frac{x^2 - 2x + 1 - (x^2 + 2xh + h^2 - 2x - 2h + 1)}{h(x+h-1)^2(x-1)^2} = \lim_{h\to0} \frac{-2x - h + 2}{(x+h-1)^2(x-1)^2}$$
$$= \frac{-2x+2}{(x-1)^4} = -\frac{2}{(x-1)^3}.$$

In Exercises 13–16, express the limit as a derivative.

13. $\lim_{h\to 0} \dfrac{\sqrt{1+h}-1}{h}$

SOLUTION Let $f(x) = \sqrt{x}$. Then

$$\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = f'(1).$$

14. $\lim_{x\to -1} \dfrac{x^3+1}{x+1}$

SOLUTION Let $f(x) = x^3$. Then

$$\lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} \frac{f(x)-f(-1)}{x-(-1)} = f'(-1).$$

15. $\lim_{t\to \pi} \dfrac{\sin t \cos t}{t-\pi}$

SOLUTION Let $f(t) = \sin t \cos t$ and note that $f(\pi) = \sin\pi \cos\pi = 0$. Then

$$\lim_{t\to \pi} \frac{\sin t \cos t}{t-\pi} = \lim_{t\to \pi} \frac{f(t)-f(\pi)}{t-\pi} = f'(\pi).$$

16. $\lim_{\theta\to \pi} \dfrac{\cos\theta - \sin\theta + 1}{\theta-\pi}$

SOLUTION Let $f(\theta) = \cos\theta - \sin\theta$ and note that $f(\pi) = -1$. Then

$$\lim_{\theta\to \pi} \frac{\cos\theta - \sin\theta + 1}{\theta-\pi} = \lim_{\theta\to \pi} \frac{f(\theta)-f(\pi)}{\theta-\pi} = f'(\pi).$$

17. Find $f(4)$ and $f'(4)$ if the tangent line to the graph of f at $x = 4$ has equation $y = 3x - 14$.

SOLUTION The equation of the tangent line to the graph of $f(x)$ at $x = 4$ is $y = f'(4)(x-4) + f(4) = f'(4)x + (f(4) - 4f'(4))$. Matching this to $y = 3x - 14$, we see that $f'(4) = 3$ and $f(4) - 4(3) = -14$, so $f(4) = -2$.

18. Each graph in Figure 2 shows the graph of a function f and its derivative f'. Determine which is the function and which is the derivative.

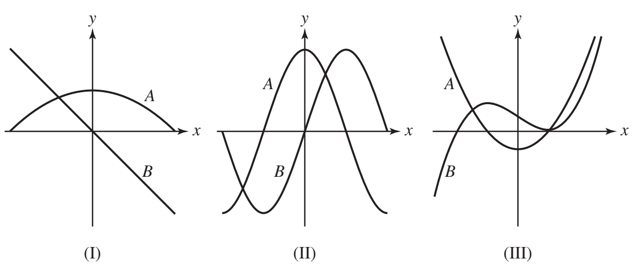


FIGURE 2 Graph of f.

SOLUTION

- In (I), the graph labeled A is increasing when the graph labeled B is positive and is decreasing when the graph labeled B is negative. Therefore, the graph labeled A is the function f and the graph labeled B is the derivative f'.
- In (II), the graph labeled B is increasing when the graph labeled A is positive and is decreasing when the graph labeled A is negative. Therefore, the graph labeled B is the function f and the graph labeled A is the derivative f'.
- In (III), the graph labeled B has horizontal tangent lines at the locations the graph labeled A is zero. Therefore, the graph labeled B is the function f and the graph labeled A is the derivative f'.

19. Is (A), (B), or (C) the graph of the derivative of the function f shown in Figure 3?

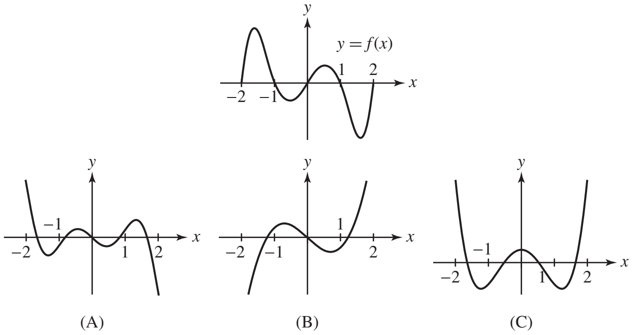


FIGURE 3

SOLUTION The graph of f has four horizontal tangent lines on $[-2, 2]$, so the graph of its derivative must have four x-intercepts on $[-2, 2]$. This eliminates (B). Moreover, f is increasing at both ends of the interval, so its derivative must be positive at both ends. This eliminates (A) and identifies (C) as the graph of f'.

20. Sketch the graph of f' if the graph of f appears as in Figure 4.

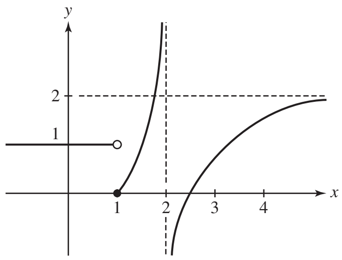
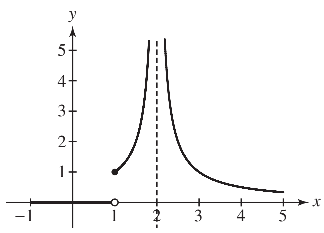


FIGURE 4

SOLUTION Examine Figure 4. For $x < 1$, f is constant, so $f'(x) = 0$. For $1 \le x < 2$ and $x > 2$, f is increasing, so f' must be positive on these intervals. As $x \to 1^+$, the slope of the tangent line appears to approach 1, while as $x \to 2^-$, the slope of the tangent line appears to approach ∞. Moreover, as $x \to 2^+$, the slope of the tangent line appears to approach ∞, while as $x \to \infty$, the slope of the tangent line appears to approach 0. Bringing this information together, one possible graph for f' is shown below.



21. Sketch the graph of a continuous function f if the graph of f' appears as in Figure 5 and $f(0) = 0$.

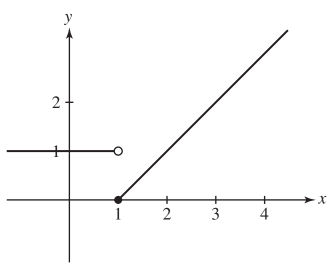
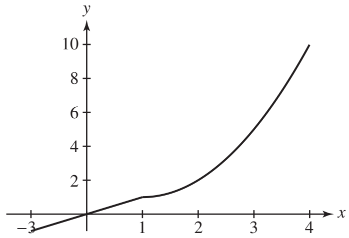


FIGURE 5

SOLUTION Examine Figure 5. For $x < 1$, $f'(x) = 1$, so that the graph of f must be a line with slope 1. Because $f(0) = 0$, it follows that $f(x) = x$ for $x < 1$. For $x \ge 1$, f' is zero and then steadily increases, indicating that the graph of f must “emerge" from the point $(1, 1)$ with zero slope and then curve upward. One possible graph of f is shown below.



22. Let $N(t)$ be the percentage of a state population infected with a flu virus on week t of an epidemic. What percentage is likely to be infected in week 4 if $N(3) = 8$ and $N'(3) = 1.2$?

SOLUTION Because $N(4) - N(3) \approx N'(3)$, we estimate that

$$N(4) \approx N(3) + N'(3) = 8 + 1.2 = 9.2.$$

Thus, 9.2% of the population is likely infected in week 4.

23. A girl's height $h(t)$ (in centimeters) is measured at time t (in years) for $0 \le t \le 14$:

52, 75.1, 87.5, 96.7, 104.5, 111.8, 118.7, 125.2,
131.5, 137.5, 143.3, 149.2, 155.3, 160.8, 164.7

(a) What is the average growth rate over the 14-year period?
(b) Is the average growth rate larger over the first half or the second half of this period?
(c) Estimate $h'(t)$ (in centimeters per year) for $t = 3, 8$.

SOLUTION

(a) The average growth rate over the 14-year period is

$$\frac{164.7 - 52}{14} = 8.05 \text{ cm/year}.$$

(b) Over the first half of the 14-year period, the average growth rate is

$$\frac{125.2 - 52}{7} \approx 10.46 \text{ cm/year},$$

which is larger than the average growth rate over the second half of the 14-year period:

$$\frac{164.7 - 125.2}{7} \approx 5.64 \text{ cm/year}.$$

(c) For $t = 3$,

$$h'(3) \approx \frac{h(4) - h(3)}{4 - 3} = \frac{104.5 - 96.7}{1} = 7.8 \text{ cm/year};$$

for $t = 8$,

$$h'(8) \approx \frac{h(9) - h(8)}{9 - 8} = \frac{137.5 - 131.5}{1} = 6.0 \text{ cm/year}.$$

24. A planet's period P (number of days to complete one revolution around the sun) is approximately $0.199A^{3/2}$, where A is the average distance (in millions of kilometers) from the planet to the sun.
(a) Calculate P and dP/dA for Earth using the value $A = 150$.
(b) Estimate the increase in P if A is increased to 152.

SOLUTION

(a) Let $P = 0.199A^{3/2}$. Then $\frac{dP}{dA} = 0.2985A^{1/2}$. For $A = 150$,

$$P = 0.199(150)^{3/2} \approx 365.6 \text{ days; and}$$
$$\frac{dP}{dA} = 0.2985(150)^{1/2} \approx 3.656 \text{ days/millions of kilometers}.$$

(b) If A is increased to 152, then

$$P(152) - P(150) \approx 2 \left.\frac{dP}{dA}\right|_{A=150} = 7.312 \text{ days}.$$

In Exercises 25 and 26, use the following table of values for the number $A(t)$ of automobiles (in millions) manufactured in the United States in year t.

t	1970	1971	1972	1973	1974	1975	1976
$A(t)$	6.55	8.58	8.83	9.67	7.32	6.72	8.50

25. What is the interpretation of $A'(t)$? Estimate $A'(1971)$. Does $A'(1974)$ appear to be positive or negative?

SOLUTION Because $A(t)$ measures the number of automobiles manufactured in the United States in year t, $A'(t)$ measures the rate of change in automobile production in the United States. For $t = 1971$,

$$A'(1971) \approx \frac{A(1972) - A(1971)}{1972 - 1971} = \frac{8.83 - 8.58}{1} = 0.25 \text{ million automobiles/year.}$$

Because $A(t)$ decreases from 1973 to 1974 and from 1974 to 1975, it appears that $A'(1974)$ would be negative.

26. Given the data, which of (A)–(C) in Figure 6 could be the graph of the derivative A'? Explain.

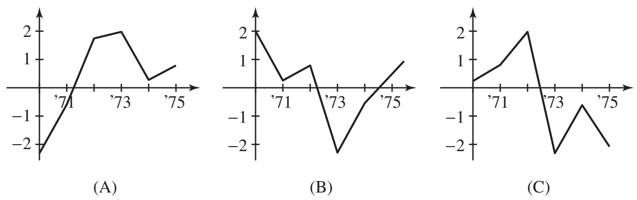


FIGURE 6

SOLUTION The values of $A(t)$ increase, then decrease and finally increase. Thus $A'(t)$ should transition from positive to negative and back to positive. This describes the graph in (B).

27. Which of the following is equal to $\frac{d}{dx}2^x$?

(a) 2^x **(b)** $(\ln 2)2^x$ **(c)** $x2^{x-1}$ **(d)** $\frac{1}{\ln 2}2^x$

SOLUTION The derivative of $f(x) = 2^x$ is

$$\frac{d}{dx}2^x = 2^x \ln 2.$$

Hence, the correct answer is **(b)**.

28. Use the Chain Rule to show that if g is the inverse of f, then $g'(x) = 1/f'(g(x))$ for all x in the domain of g such that $f(g(x)) \neq 0$. Use this to obtain another method for finding the derivative of $\ln x$ using the derivative of e^x.

SOLUTION Let g be the inverse of f. Then $x = f(g(x))$. Upon differentiating both sides of this expression by x, we find

$$1 = f'(g(x)) \cdot g'(x) \qquad \text{or} \qquad g'(x) = \frac{1}{f'(g(x))},$$

provided x is in the domain of g and $f'(g(x)) \neq 0$. Now, let $f(x) = e^x$. The inverse of f is $g(x) = \ln x$, and

$$\frac{d}{dx}\ln x = g'(x) = \frac{1}{f'(g(x))} = \frac{1}{f'(\ln x)} = \frac{1}{e^{\ln x}} = \frac{1}{x}.$$

In Exercises 29–80, compute the derivative.

29. $y = 3x^5 - 7x^2 + 4$

SOLUTION Let $y = 3x^5 - 7x^2 + 4$. Then

$$\frac{dy}{dx} = 15x^4 - 14x.$$

30. $y = 4x^{-3/2}$

SOLUTION Let $y = 4x^{-3/2}$. Then

$$\frac{dy}{dx} = -6x^{-5/2}.$$

31. $y = t^{-7.3}$

SOLUTION Let $y = t^{-7.3}$. Then

$$\frac{dy}{dt} = -7.3t^{-8.3}.$$

32. $y = 4x^2 - x^{-2}$

SOLUTION Let $y = 4x^2 - x^{-2}$. Then

$$\frac{dy}{dx} = 8x + 2x^{-3}.$$

33. $y = \dfrac{x+1}{x^2+1}$

SOLUTION Let $y = \dfrac{x+1}{x^2+1}$. Then

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (x+1)(2x)}{(x^2+1)^2} = \frac{1 - 2x - x^2}{(x^2+1)^2}.$$

34. $y = \dfrac{3t-2}{4t-9}$

SOLUTION Let $y = \dfrac{3t-2}{4t-9}$. Then

$$\frac{dy}{dt} = \frac{(4t-9)(3) - (3t-2)(4)}{(4t-9)^2} = -\frac{19}{(4t-9)^2}.$$

35. $y = (x^4 - 9x)^6$

SOLUTION Let $y = (x^4 - 9x)^6$. Then

$$\frac{dy}{dx} = 6(x^4 - 9x)^5 \frac{d}{dx}(x^4 - 9x) = 6(4x^3 - 9)(x^4 - 9x)^5.$$

36. $y = (3t^2 + 20t^{-3})^6$

SOLUTION Let $y = (3t^2 + 20t^{-3})^6$. Then

$$\frac{dy}{dt} = 6(3t^2 + 20t^{-3})^5 \frac{d}{dt}(3t^2 + 20t^{-3}) = 6(6t - 60t^{-4})(3t^2 + 20t^{-3})^5.$$

37. $y = (2 + 9x^2)^{3/2}$

SOLUTION Let $y = (2 + 9x^2)^{3/2}$. Then

$$\frac{dy}{dx} = \frac{3}{2}(2 + 9x^2)^{1/2} \frac{d}{dx}(2 + 9x^2) = 27x(2 + 9x^2)^{1/2}.$$

38. $y = (x+1)^3(x+4)^4$

SOLUTION Let $y = (x+1)^3(x+4)^4$. Then

$$\begin{aligned}\frac{dy}{dx} &= 4(x+1)^3(x+4)^3 + 3(x+1)^2(x+4)^4 = (x+1)^2(x+4)^3(4x+4+3x+12)\\ &= (7x+16)(x+1)^2(x+4)^3.\end{aligned}$$

39. $y = \dfrac{z}{\sqrt{1-z}}$

SOLUTION Let $y = \dfrac{z}{\sqrt{1-z}}$. Then

$$\frac{dy}{dz} = \frac{\sqrt{1-z} - z\left(-\frac{1}{2\sqrt{1-z}}\right)}{1-z} = \frac{1-z+\frac{z}{2}}{(1-z)^{3/2}} = \frac{2-z}{2(1-z)^{3/2}}.$$

40. $y = \left(1+\dfrac{1}{x}\right)^3$

SOLUTION Let $y = \left(1+\dfrac{1}{x}\right)^3$. Then

$$\frac{dy}{dx} = 3\left(1+\frac{1}{x}\right)^2 \frac{d}{dx}\left(1+\frac{1}{x}\right) = -\frac{3}{x^2}\left(1+\frac{1}{x}\right)^2.$$

41. $y = \dfrac{x^4+\sqrt{x}}{x^2}$

SOLUTION Let

$$y = \frac{x^4+\sqrt{x}}{x^2} = x^2 + x^{-3/2}.$$

Then

$$\frac{dy}{dx} = 2x - \frac{3}{2}x^{-5/2}.$$

42. $y = \dfrac{1}{(1-x)\sqrt{2-x}}$

SOLUTION Let $y = \dfrac{1}{(1-x)\sqrt{2-x}} = \left((1-x)\sqrt{2-x}\right)^{-1}$. Then

$$\frac{dy}{dx} = -\left((1-x)\sqrt{2-x}\right)^{-2} \frac{d}{dx}\left((1-x)\sqrt{2-x}\right) = -\left((1-x)\sqrt{2-x}\right)^{-2}\left(-\frac{1-x}{2\sqrt{2-x}} - \sqrt{2-x}\right)$$
$$= \frac{5-3x}{2(1-x)^2(2-x)^{3/2}}.$$

43. $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$

SOLUTION Let $y = \sqrt{x+\sqrt{x+\sqrt{x}}}$. Then

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\frac{d}{dx}\left(x+\sqrt{x+\sqrt{x}}\right)\\ &= \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left(1+\frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\frac{d}{dx}\left(x+\sqrt{x}\right)\right)\\ &= \frac{1}{2}\left(x+\sqrt{x+\sqrt{x}}\right)^{-1/2}\left(1+\frac{1}{2}\left(x+\sqrt{x}\right)^{-1/2}\left(1+\frac{1}{2}x^{-1/2}\right)\right).\end{aligned}$$

44. $h(z) = \left(z+(z+1)^{1/2}\right)^{-3/2}$

SOLUTION

$$\begin{aligned}\frac{d}{dz}\left(z+(z+1)^{1/2}\right)^{-3/2} &= -\frac{3}{2}\left(z+(z+1)^{1/2}\right)^{-5/2}\frac{d}{dz}\left(z+(z+1)^{1/2}\right)\\ &= -\frac{3}{2}\left(z+(z+1)^{1/2}\right)^{-5/2}\left(1+\frac{1}{2}(z+1)^{-1/2}\right).\end{aligned}$$

45. $y = \tan(t^{-3})$

SOLUTION Let $y = \tan(t^{-3})$. Then

$$\frac{dy}{dt} = \sec^2(t^{-3})\frac{d}{dt}t^{-3} = -3t^{-4}\sec^2(t^{-3}).$$

46. $y = 4\cos(2-3x)$

SOLUTION Let $y = 4\cos(2-3x)$. Then

$$\frac{dy}{dx} = -4\sin(2-3x)\frac{d}{dx}(2-3x) = 12\sin(2-3x).$$

47. $y = \sin(2x)\cos^2 x$

SOLUTION Let $y = \sin(2x)\cos^2 x = 2\sin x\cos^3 x$. Then

$$\frac{dy}{dx} = -6\sin^2 x\cos^2 x + 2\cos^4 x.$$

48. $y = \sin\left(\frac{4}{\theta}\right)$

SOLUTION Let $y = \sin\left(\frac{4}{\theta}\right)$. Then

$$\frac{dy}{d\theta} = \cos\left(\frac{4}{\theta}\right)\frac{d}{d\theta}\left(\frac{4}{\theta}\right) = -\frac{4}{\theta^2}\cos\left(\frac{4}{\theta}\right).$$

49. $y = \dfrac{t}{1+\sec t}$

SOLUTION Let $y = \dfrac{t}{1+\sec t}$. Then

$$\frac{dy}{dt} = \frac{1+\sec t - t\sec t\tan t}{(1+\sec t)^2}.$$

50. $y = z\csc(9z+1)$

SOLUTION Let $y = z\csc(9z+1)$. Then

$$\frac{dy}{dz} = -9z\csc(9z+1)\cot(9z+1) + \csc(9z+1).$$

51. $y = \dfrac{8}{1+\cot\theta}$

SOLUTION Let $y = \dfrac{8}{1+\cot\theta} = 8(1+\cot\theta)^{-1}$. Then

$$\frac{dy}{d\theta} = -8(1+\cot\theta)^{-2}\frac{d}{d\theta}(1+\cot\theta) = \frac{8\csc^2\theta}{(1+\cot\theta)^2}.$$

52. $y = \tan(\cos x)$

SOLUTION Let $y = \tan(\cos x)$. Then

$$\frac{dy}{dx} = \sec^2(\cos x)\frac{d}{dx}\cos x = -\sin x\sec^2(\cos x).$$

53. $y = \tan(\sqrt{1+\csc\theta})$

SOLUTION

$$\frac{dy}{dx} = \sec^2(\sqrt{1+\csc\theta})\frac{d}{dx}\sqrt{1+\csc\theta} = \sec^2(\sqrt{1+\csc\theta})\cdot\frac{1}{2}(1+\csc\theta)^{-1/2}\frac{d}{dx}(1+\csc\theta)$$
$$= -\frac{\sec^2(\sqrt{1+\csc\theta})\csc\theta\cot\theta}{2(\sqrt{1+\csc\theta})}.$$

54. $y = \cos(\cos(\cos(\theta)))$

SOLUTION Let $y = \cos(\cos(\cos(\theta)))$. Then

$$\frac{dy}{d\theta} = -\sin(\cos(\cos(\theta)))\frac{d}{d\theta}\cos(\cos(\theta)) = \sin(\cos(\cos(\theta)))\sin(\cos(\theta))\frac{d}{d\theta}\cos(\theta)$$
$$= -\sin(\cos(\cos(\theta)))\sin(\cos(\theta))\sin(\theta).$$

55. $f(x) = 9e^{-4x}$

SOLUTION $\frac{d}{dx}9e^{-4x} = -36e^{-4x}.$

56. $f(x) = \frac{e^{-x}}{x}$

SOLUTION $\frac{d}{dx}\left(\frac{e^{-x}}{x}\right) = \frac{-xe^{-x} - e^{-x}}{x^2} = -\frac{e^{-x}(x+1)}{x^2}.$

57. $g(t) = e^{4t-t^2}$

SOLUTION $\frac{d}{dt}e^{4t-t^2} = (4-2t)e^{4t-t^2}.$

58. $g(t) = t^2e^{1/t}$

SOLUTION $\frac{d}{dt}t^2e^{1/t} = 2te^{1/t} + t^2\left(-\frac{1}{t^2}\right)e^{1/t} = (2t-1)e^{1/t}.$

59. $f(x) = \ln(4x^2+1)$

SOLUTION $\frac{d}{dx}\ln(4x^2+1) = \frac{8x}{4x^2+1}.$

60. $f(x) = \ln(e^x - 4x)$

SOLUTION $\frac{d}{dx}\ln(e^x-4x) = \frac{e^x-4}{e^x-4x}.$

61. $G(s) = (\ln(s))^2$

SOLUTION $\frac{d}{ds}(\ln s)^2 = \frac{2\ln s}{s}.$

62. $G(s) = \ln(s^2)$

SOLUTION $\frac{d}{ds}\ln(s^2) = 2\frac{d}{ds}\ln s = \frac{2}{s}.$

63. $f(\theta) = \ln(\sin\theta)$

SOLUTION $\frac{d}{d\theta}\ln(\sin\theta) = \frac{\cos\theta}{\sin\theta} = \cot\theta.$

64. $f(\theta) = \sin(\ln\theta)$

SOLUTION $\frac{d}{d\theta}\sin(\ln\theta) = \frac{\cos(\ln\theta)}{\theta}.$

65. $h(z) = \sec(z + \ln z)$

SOLUTION $\frac{d}{dz}\sec(z+\ln z) = \sec(z+\ln z)\tan(z+\ln z)\left(1+\frac{1}{z}\right).$

66. $f(x) = e^{\sin^2 x}$

SOLUTION $\frac{d}{dx}e^{\sin^2 x} = 2\sin x\cos x e^{\sin^2 x} = \sin 2x e^{\sin^2 x}.$

67. $f(x) = 7^{-2x}$

SOLUTION $\frac{d}{dx}7^{-2x} = \big(-2\ln 7\big)\big(7^{-2x}\big).$

68. $h(y) = \dfrac{1+e^y}{1-e^y}$

SOLUTION $\dfrac{d}{dy}\left(\dfrac{1+e^y}{1-e^y}\right) = \dfrac{(1-e^y)e^y - (1+e^y)(-e^y)}{(1-e^y)^2} = \dfrac{e^y(1-e^y+1+e^y)}{(1-e^y)^2} = \dfrac{2e^y}{(1-e^y)^2}.$

69. $g(x) = \tan^{-1}(\ln x)$

SOLUTION $\dfrac{d}{dx}\tan^{-1}(\ln x) = \dfrac{1}{1+(\ln x)^2}\cdot\dfrac{1}{x}.$

70. $G(s) = \cos^{-1}(s^{-1})$

SOLUTION $\dfrac{d}{ds}\cos^{-1}(s^{-1}) = \dfrac{-1}{\sqrt{1-\left(\frac{1}{s}\right)^2}}\left(-\dfrac{1}{s^2}\right) = \dfrac{1}{\sqrt{s^4-s^2}}.$

71. $f(x) = \ln(\csc^{-1} x)$

SOLUTION $\dfrac{d}{dx}\ln(\csc^{-1} x) = -\dfrac{1}{|x|\sqrt{x^2-1}\,\csc^{-1} x}.$

72. $f(x) = e^{\sec^{-1} x}$

SOLUTION $\dfrac{d}{dx}e^{\sec^{-1} x} = \dfrac{1}{|x|\sqrt{x^2-1}}e^{\sec^{-1} x}.$

73. $R(s) = s^{\ln s}$

SOLUTION Rewrite

$$R(s) = \left(e^{\ln s}\right)^{\ln s} = e^{(\ln s)^2}.$$

Then

$$\frac{dR}{ds} = e^{(\ln s)^2}\cdot 2\ln s\cdot\frac{1}{s} = \frac{2\ln s}{s}s^{\ln s}.$$

Alternately, $R(s) = s^{\ln s}$ implies that $\ln R = \ln\left(s^{\ln s}\right) = (\ln s)^2$. Thus,

$$\frac{1}{R}\frac{dR}{ds} = 2\ln s\cdot\frac{1}{s} \quad\text{or}\quad \frac{dR}{ds} = \frac{2\ln s}{s}s^{\ln s}.$$

74. $f(x) = (\cos^2 x)^{\cos x}$

SOLUTION Rewrite

$$f(x) = \left(e^{\ln\cos^2 x}\right)^{\cos x} = e^{2\cos x\ln\cos x}.$$

Then

$$\begin{aligned}\frac{df}{dx} &= e^{2\cos x\ln\cos x}\left(2\cos x\cdot\frac{-\sin x}{\cos x} - 2\sin x\ln\cos x\right)\\ &= -2\sin x(\cos^2 x)^{\cos x}(1+\ln\cos x).\end{aligned}$$

Alternately, $f(x) = (\cos^2 x)^{\cos x}$ implies that $\ln f = \cos x\ln\cos^2 x = 2\cos x\ln\cos x$. Thus,

$$\begin{aligned}\frac{1}{f}\frac{df}{dx} &= 2\cos x\cdot\frac{-\sin x}{\cos x} - 2\sin x\ln\cos x\\ &= -2\sin x(1+\ln\cos x),\end{aligned}$$

and

$$\frac{df}{dx} = -2\sin x(\cos^2 x)^{\cos x}(1+\ln\cos x).$$

75. $G(t) = (\sin^2 t)^t$

SOLUTION Rewrite

$$G(t) = \left(e^{\ln\sin^2 t}\right)^t = e^{2t\ln\sin t}.$$

Then

$$\frac{dG}{dt} = e^{2t\ln\sin t}\left(2t \cdot \frac{\cos t}{\sin t} + 2\ln\sin t\right) = 2(\sin^2 t)^t (t\cot t + \ln\sin t).$$

Alternately, $G(t) = (\sin^2 t)^t$ implies that $\ln G = t\ln\sin^2 t = 2t\ln\sin t$. Thus,

$$\frac{1}{G}\frac{dG}{dt} = 2t \cdot \frac{\cos t}{\sin t} + 2\ln\sin t,$$

and

$$\frac{dG}{dt} = 2(\sin^2 t)^t (t\cot t + \ln\sin t).$$

76. $h(t) = t^{(t^t)}$

SOLUTION Let $h(t) = t^{(t^t)}$. Then $\ln h = t^t \ln t$ and

$$\begin{aligned}\ln(\ln h) &= \ln(t^t \ln t) = \ln t^t + \ln(\ln t)\\ &= t\ln t + \ln(\ln t).\end{aligned}$$

Thus,

$$\frac{1}{h\ln h}\frac{dh}{dt} = t \cdot \frac{1}{t} + \ln t + \frac{1}{t\ln t} = 1 + \ln t + \frac{1}{t\ln t},$$

and

$$\frac{dh}{dt} = t^{(t^t)} t^t \ln t\left(1 + \ln t + \frac{1}{t\ln t}\right).$$

77. $g(t) = \sinh(t^2)$

SOLUTION $\frac{d}{dt}\sinh(t^2) = 2t\cosh(t^2)$.

78. $h(y) = y\tanh(4y)$

SOLUTION $\frac{d}{dy} y\tanh(4y) = \tanh(4y) + 4y\,\text{sech}^2(4y)$.

79. $g(x) = \tanh^{-1}(e^x)$

SOLUTION $\frac{d}{dx}\tanh^{-1}(e^x) = \frac{1}{1-(e^x)^2}e^x = \frac{e^x}{1-e^{2x}}$.

80. $g(t) = \sqrt{t^2-1}\,\sinh^{-1} t$

SOLUTION $\frac{d}{dt}\sqrt{t^2-1}\,\sinh^{-1} t = \frac{t}{\sqrt{t^2-1}}\sinh^{-1} t + \sqrt{t^2-1} \cdot \frac{1}{\sqrt{t^2+1}} = \frac{t\sinh^{-1} t}{\sqrt{t^2-1}} + \sqrt{\frac{t^2-1}{t^2+1}}$.

81. For which values of α is $f(x) = |x|^\alpha$ differentiable at $x = 0$?

SOLUTION Let $f(x) = |x|^\alpha$. If $\alpha < 0$, then $f(x)$ is not continuous at $x = 0$ and therefore cannot be differentiable at $x = 0$. If $\alpha = 0$, then the function reduces to $f(x) = 1$, which is differentiable at $x = 0$. Now, suppose $\alpha > 0$ and consider the limit

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{|x|^\alpha}{x}.$$

If $0 < \alpha < 1$, then

$$\lim_{x\to 0-}\frac{|x|^\alpha}{x} = -\infty \qquad \text{while} \qquad \lim_{x\to 0+}\frac{|x|^\alpha}{x} = \infty$$

and $f'(0)$ does not exist. If $\alpha = 1$, then

$$\lim_{x\to 0-}\frac{|x|}{x} = -1 \qquad \text{while} \qquad \lim_{x\to 0+}\frac{|x|}{x} = 1$$

and $f'(0)$ again does not exist. Finally, if $\alpha > 1$, then

$$\lim_{x\to 0} \frac{|x|^\alpha}{x} = 0,$$

so $f'(0)$ does exist.

In summary, $f(x) = |x|^\alpha$ is differentiable at $x = 0$ when $\alpha = 0$ and when $\alpha > 1$.

82. Find $f'(2)$ if $f(g(x)) = e^{x^2}$, $g(1) = 2$, and $g'(1) = 4$.

SOLUTION We differentiate both sides of the equation $f(g(x)) = e^{x^2}$ to obtain,

$$f'(g(x))\, g'(x) = 2xe^{x^2}.$$

Setting $x = 1$ yields

$$f'(g(1))\, g'(1) = 2e.$$

Since $g(1) = 2$ and $g'(1) = 4$, we find

$$f'(2) \cdot 4 = 2e,$$

or

$$f'(2) = \frac{e}{2}.$$

In Exercises 83 and 84, let $f(x) = xe^{-x}$.

83. Show that f has an inverse on $[1, \infty)$. Let g be this inverse. Find the domain and range of g and compute $g'(2e^{-2})$.

SOLUTION Let $f(x) = xe^{-x}$. Then $f'(x) = e^{-x}(1 - x)$. On $[1, \infty)$, $f'(x) < 0$, so f is decreasing and therefore one-to-one. It follows that f has an inverse on $[1, \infty)$. Let g denote this inverse. Because $f(1) = e^{-1}$ and $f(x) \to 0$ as $x \to \infty$, the domain of g is $(0, e^{-1}]$, and the range is $[1, \infty)$.

To determine $g'(2e^{-2})$, we use the formula $g'(x) = 1/f'(g(x))$. Because $f(2) = 2e^{-2}$, it follows that $g(2e^{-2}) = 2$. Then,

$$g'(2e^{-2}) = \frac{1}{f'(g(2e^{-2}))} = \frac{1}{f'(2)} = \frac{1}{-e^{-2}} = -e^2.$$

84. Show that $f(x) = c$ has two solutions if $0 < c < e^{-1}$.

SOLUTION First note that $f(x) < 0$ for $x < 0$, so we only need to examine $(0, \infty)$ for solutions to $f(x) = c$ when $c > 0$. Next, because $f'(x) = e^{-x}(1 - x)$, f is decreasing on $(1, \infty)$ and increasing on $(0, 1)$. Therefore, f is one-to-one on each of these intervals, which guarantees that the equation $f(x) = c$ can have at most one solution on each of these intervals for any value of c.

Now, let $0 < c < e^{-1}$ and consider the interval $[1, \infty)$. Because

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^x} = 0,$$

it follows that there exists a $d \in (1, \infty)$ such that $f(d) < c$. With $f(1) = e^{-1} > c$, it follows from the Intermediate Value Theorem that the equation $f(x) = c$ has a solution on $[1, \infty)$. Next, consider the interval $[0, 1]$. With $f(0) = 0 < c$ and $f(1) = e^{-1} > c$, it follows from the Intermediate Value Theorem that the equation $f(x) = c$ has a solution on $[0, 1]$.

In summary, the equation $f(x) = c$ has exactly two solutions for any c between 0 and e^{-1}.

In Exercises 85–90, use the following table of values to calculate the derivative of the given function at $x = 2$:

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	5	4	−3	9
4	3	2	−2	3

85. $S(x) = 3f(x) - 2g(x)$

SOLUTION Let $S(x) = 3f(x) - 2g(x)$. Then $S'(x) = 3f'(x) - 2g'(x)$ and

$$S'(2) = 3f'(2) - 2g'(2) = 3(-3) - 2(9) = -27.$$

86. $H(x) = f(x)g(x)$

SOLUTION Let $H(x) = f(x)g(x)$. Then $H'(x) = f(x)g'(x) + f'(x)g(x)$ and

$$H'(2) = f(2)g'(2) + f'(2)g(2) = 5(9) + (-3)(4) = 33.$$

87. $R(x) = \dfrac{f(x)}{g(x)}$

SOLUTION Let $R(x) = f(x)/g(x)$. Then

$$R'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

and

$$R'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{g(2)^2} = \frac{4(-3) - 5(9)}{4^2} = -\frac{57}{16}.$$

88. $G(x) = f(g(x))$

SOLUTION Let $G(x) = f(g(x))$. Then $G'(x) = f'(g(x))g'(x)$ and

$$G'(2) = f'(g(2))g'(2) = f'(4)g'(2) = -2(9) = -18.$$

89. $F(x) = f(g(2x))$

SOLUTION Let $F(x) = f(g(2x))$. Then $F'(x) = 2f'(g(2x))g'(2x)$ and

$$F'(2) = 2f'(g(4))g'(4) = 2f'(2)g'(4) = 2(-3)(3) = -18.$$

90. $K(x) = f(x^2)$

SOLUTION Let $K(x) = f(x^2)$. Then $K'(x) = 2xf'(x^2)$ and

$$K'(2) = 2(2)f'(4) = 4(-2) = -8.$$

91. Find the points on the graph of $f(x) = x^3 - 3x^2 + x + 4$ where the tangent line has slope 10.

SOLUTION Let $f(x) = x^3 - 3x^2 + x + 4$. Then $f'(x) = 3x^2 - 6x + 1$. The tangent line to the graph of f will have slope 10 when $f'(x) = 10$. Solving the quadratic equation $3x^2 - 6x + 1 = 10$ yields $x = -1$ and $x = 3$. Thus, the points on the graph of f where the tangent line has slope 10 are $(-1, -1)$ and $(3, 7)$.

92. Find the points on the graph of $x^{2/3} + y^{2/3} = 1$ where the tangent line has slope 1.

SOLUTION Suppose $x^{2/3} + y^{2/3} = 1$. Differentiating with respect to x leads to

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0,$$

or

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-1/3} = -\left(\frac{y}{x}\right)^{1/3}.$$

Tangents to the curve therefore have slope 1 when $y = -x$. Substituting $y = -x$ into the equation for the curve yields $2x^{2/3} = 1$, so $x = \pm\frac{\sqrt{2}}{4}$. Thus, the points along the curve $x^{2/3} + y^{2/3} = 1$ where the tangent line has slope 1 are:

$$\left(\frac{\sqrt{2}}{4}, -\frac{\sqrt{2}}{4}\right) \quad \text{and} \quad \left(-\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}\right).$$

93. Find a such that the tangent lines to $y = x^3 - 2x^2 + x + 1$ at $x = a$ and $x = a + 1$ are parallel.

SOLUTION Let $f(x) = x^3 - 2x^2 + x + 1$. Then $f'(x) = 3x^2 - 4x + 1$ and the slope of the tangent line at $x = a$ is $f'(a) = 3a^2 - 4a + 1$, while the slope of the tangent line at $x = a + 1$ is

$$f'(a + 1) = 3(a + 1)^2 - 4(a + 1) + 1 = 3(a^2 + 2a + 1) - 4a - 4 + 1 = 3a^2 + 2a.$$

In order for the tangent lines at $x = a$ and $x = a + 1$ to have the same slope, we must have $f'(a) = f'(a + 1)$, or

$$3a^2 - 4a + 1 = 3a^2 + 2a.$$

The only solution to this equation is $a = \frac{1}{6}$.

94. Use the table to compute the average rate of change of Candidate A's percentage of votes over the intervals from day 20 to day 15, day 15 to day 10, and day 10 to day 5. If this trend continues over the last 5 days before the election, will Candidate A win?

Days Before Election	**20**	**15**	**10**	**5**
Candidate A	44.8%	46.8%	48.3%	49.3%
Candidate B	55.2%	53.2%	51.7%	50.7%

SOLUTION The average rate of change of A's percentage for the period from day 20 to day 15 is

$$\frac{46.8 - 44.8}{5} = 0.4\%/\text{day}.$$

For the period from day 15 to day 10, the average rate of change is

$$\frac{48.3 - 46.8}{5} = 0.3\%/\text{day}.$$

Finally, for the period from day 10 to day 5, the average rate of change is

$$\frac{49.3 - 48.3}{5} = 0.2\%/\text{day}.$$

If this trend continues over the last five days before the election, the average rate of change will drop to 0.1 %/day, so A's percentage will increase another 0.5% to 49.8%. Accordingly, A will *not* win the election.

In Exercises 95–100, calculate y''.

95. $y = 12x^3 - 5x^2 + 3x$

SOLUTION Let $y = 12x^3 - 5x^2 + 3x$. Then

$$y' = 36x^2 - 10x + 3 \quad \text{and} \quad y'' = 72x - 10.$$

96. $y = x^{-2/5}$

SOLUTION Let $y = x^{-2/5}$. Then

$$y' = -\frac{2}{5}x^{-7/5} \quad \text{and} \quad y'' = \frac{14}{25}x^{-12/5}.$$

97. $y = \sqrt{2x + 3}$

SOLUTION Let $y = \sqrt{2x + 3} = (2x + 3)^{1/2}$. Then

$$y' = \frac{1}{2}(2x + 3)^{-1/2}\frac{d}{dx}(2x + 3) = (2x + 3)^{-1/2} \quad \text{and} \quad y'' = -\frac{1}{2}(2x + 3)^{-3/2}\frac{d}{dx}(2x + 3) = -(2x + 3)^{-3/2}.$$

98. $y = \dfrac{4x}{x + 1}$

SOLUTION Let $y = \dfrac{4x}{x + 1}$. Then

$$y' = \frac{(x + 1)(4) - 4x}{(x + 1)^2} = \frac{4}{(x + 1)^2} \quad \text{and} \quad y'' = -\frac{8}{(x + 1)^3}.$$

99. $y = \tan(x^2)$

SOLUTION Let $y = \tan(x^2)$. Then

$$y' = 2x\sec^2(x^2) \quad \text{and}$$

$$y'' = 2x\left(2\sec(x^2)\frac{d}{dx}\sec(x^2)\right) + 2\sec^2(x^2) = 8x^2\sec^2(x^2)\tan(x^2) + 2\sec^2(x^2).$$

100. $y = \sin^2(4x + 9)$

SOLUTION Let $y = \sin^2(x + 9)$. Then

$$y' = 2\sin(x + 9)\cos(x + 9) = \sin(2x + 18) \quad \text{and} \quad y'' = 2\cos(2x + 18).$$

In Exercises 101–106, compute $\frac{dy}{dx}$.

101. $x^3 - y^3 = 4$

SOLUTION Consider the equation $x^3 - y^3 = 4$. Differentiating with respect to x yields

$$3x^2 - 3y^2\frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

102. $4x^2 - 9y^2 = 36$

SOLUTION Consider the equation $4x^2 - 9y^2 = 36$. Differentiating with respect to x yields

$$8x - 18y\frac{dy}{dx} = 0.$$

Therefore,

$$\frac{dy}{dx} = \frac{4x}{9y}.$$

103. $y = xy^2 + 2x^2$

SOLUTION Consider the equation $y = xy^2 + 2x^2$. Differentiating with respect to x yields

$$\frac{dy}{dx} = 2xy\frac{dy}{dx} + y^2 + 4x.$$

Therefore,

$$\frac{dy}{dx} = \frac{y^2 + 4x}{1 - 2xy}.$$

104. $\frac{y}{x} = x + y$

SOLUTION Solving $\frac{y}{x} = x + y$ for y yields

$$y = \frac{x^2}{1 - x}.$$

By the quotient rule,

$$\frac{dy}{dx} = \frac{(1 - x)(2x) - x^2(-1)}{(1 - x)^2} = \frac{2x - x^2}{(1 - x)^2}.$$

105. $y = \sin(x + y)$

SOLUTION Consider the equation $y = \sin(x + y)$. Differentiating with respect to x yields

$$\frac{dy}{dx} = \cos(x + y)\left(1 + \frac{dy}{dx}\right).$$

Therefore,

$$\frac{dy}{dx} = \frac{\cos(x+y)}{1-\cos(x+y)}.$$

106. $\tan(x+y) = xy$

SOLUTION Consider the equation $\tan(x+y) = xy$. Differentiating with respect to x yields

$$\sec^2(x+y)\left(1+\frac{dy}{dx}\right) = x\frac{dy}{dx} + y.$$

Therefore,

$$\frac{dy}{dx} = \frac{y - \sec^2(x+y)}{\sec^2(x+y) - x}.$$

107. In Figure 7, label the graphs f, f', and f''.

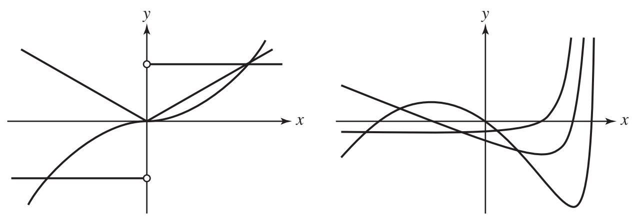


FIGURE 7

SOLUTION First consider the plot on the left. Observe that the green curve is nonnegative whereas the red curve is increasing, suggesting that the green curve is the derivative of the red curve. Moreover, the green curve is linear with negative slope for $x < 0$ and linear with positive slope for $x > 0$ while the blue curve is a negative constant for $x < 0$ and a positive constant for $x > 0$, suggesting the blue curve is the derivative of the green curve. Thus, the red, green and blue curves, respectively, are the graphs of f, f' and f''.

Now consider the plot on the right. Because the red curve is decreasing when the blue curve is negative and increasing when the blue curve is positive and the green curve is decreasing when the red curve is negative and increasing when the red curve is positive, it follows that the green, red and blue curves, respectively, are the graphs of f, f' and f''.

108. Let $f(x) = x^2\sin(x^{-1})$ for $x \neq 0$ and $f(0) = 0$. Show that $f'(x)$ exists for all x (including $x = 0$) but that f' is not continuous at $x = 0$ (Figure 8).

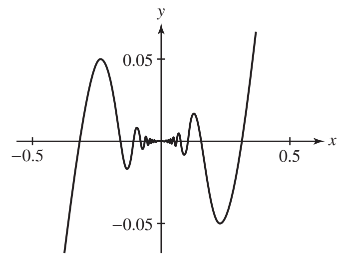


FIGURE 8 Graph of $f(x) = x^2\sin(x^{-1})$.

SOLUTION Let $f(x) = x^2\sin(x^{-1})$ for $x \neq 0$ and $f(0) = 0$. For $x \neq 0$, the product rule and the chain rule give

$$f'(x) = 2x\sin(x^{-1}) - x^2\cos(x^{-1})(x^{-2}) = 2x\sin(x^{-1}) - \cos(x^{-1}),$$

which exists for all $x \neq 0$. At $x = 0$ we use the limit definition of the derivative:

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{1}{h}(h^2\sin(h^{-1})) = \lim_{h\to 0} h\sin(h^{-1}) = 0,$$

by the Squeeze Theorem, since $-h \le h\sin\frac{1}{h} \le h$. Thus, $f'(x)$ exists for all x. However,

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\sin(x^{-1}) - \cos(x^{-1})\right)$$

does not exist, so $f'(x)$ is not continuous at $x = 0$.

In Exercises 109–114, use logarithmic differentiation to find the derivative.

109. $y = \dfrac{(x+1)^3}{(4x-2)^2}$

SOLUTION Let $y = \dfrac{(x+1)^3}{(4x-2)^2}$. Then

$$\ln y = \ln\left(\frac{(x+1)^3}{(4x-2)^2}\right) = \ln\,(x+1)^3 - \ln\,(4x-2)^2 = 3\ln(x+1) - 2\ln(4x-2).$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{3}{x+1} - \frac{2}{4x-2}\cdot 4 = \frac{3}{x+1} - \frac{4}{2x-1},$$

so

$$y' = \frac{(x+1)^3}{(4x-2)^2}\left(\frac{3}{x+1} - \frac{4}{2x-1}\right).$$

110. $y = \dfrac{(x+1)(x+2)^2}{(x+3)(x+4)}$

SOLUTION Let $y = \dfrac{(x+1)(x+2)^2}{(x+3)(x+4)}$. Then

$$\ln y = \ln\left((x+1)(x+2)^2\right) - \ln\,((x+3)(x+4))$$
$$= \ln(x+1) + 2\ln(x+2) - \ln(x+3) - \ln(x+4).$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{1}{x+1} + \frac{2}{x+2} - \frac{1}{x+3} - \frac{1}{x+4},$$

so

$$y' = \frac{(x+1)(x+2)^2}{(x+3)(x+4)}\left(\frac{1}{x+1} + \frac{2}{x+2} - \frac{1}{x+3} - \frac{1}{x+4}\right).$$

111. $y = e^{(x-1)^2}e^{(x-3)^2}$

SOLUTION Let $y = e^{(x-1)^2}e^{(x-3)^2}$. Then

$$\ln y = \ln\left(e^{(x-1)^2}e^{(x-3)^2}\right) = \ln\left(e^{(x-1)^2+(x-3)^2}\right) = (x-1)^2 + (x-3)^2.$$

By logarithmic differentiation,

$$\frac{y'}{y} = 2(x-1) + 2(x-3) = 4x - 8,$$

so

$$y' = 4e^{(x-1)^2}e^{(x-3)^2}(x-2).$$

112. $y = \dfrac{e^x \sin^{-1} x}{\ln x}$

SOLUTION Let $y = \dfrac{e^x \sin^{-1} x}{\ln x}$. Then

$$\ln y = \ln\left(\frac{e^x \sin^{-1} x}{\ln x}\right) = \ln(e^x \sin^{-1} x) - \ln(\ln x)$$
$$= \ln\left(e^x\right) + \ln(\sin^{-1} x) - \ln(\ln x) = x + \ln(\sin^{-1} x) - \ln(\ln x).$$

By logarithmic differentiation,

$$\frac{y'}{y} = 1 + \frac{1}{\sin^{-1} x} \cdot \frac{1}{\sqrt{1-x^2}} - \frac{1}{\ln x} \cdot \frac{1}{x},$$

so

$$y' = \frac{e^x \sin^{-1} x}{\ln x} \left(1 + \frac{1}{\sqrt{1-x^2} \sin^{-1} x} - \frac{1}{x \ln x} \right).$$

113. $y = \dfrac{e^{3x}(x-2)^2}{(x+1)^2}$

SOLUTION Let $y = \dfrac{e^{3x}(x-2)^2}{(x+1)^2}$. Then

$$\ln y = \ln \left(\frac{e^{3x}(x-2)^2}{(x+1)^2} \right) = \ln e^{3x} + \ln (x-2)^2 - \ln (x+1)^2$$
$$= 3x + 2 \ln(x-2) - 2 \ln(x+1).$$

By logarithmic differentiation,

$$\frac{y'}{y} = 3 + \frac{2}{x-2} - \frac{2}{x+1},$$

so

$$y = \frac{e^{3x}(x-2)^2}{(x+1)^2} \left(3 + \frac{2}{x-2} - \frac{2}{x+1} \right).$$

114. $y = x^{\sqrt{x}} (x^{\ln x})$

SOLUTION Let $y = x^{\sqrt{x}} (x^{\ln x})$. Then

$$\ln y = \sqrt{x} \ln x + (\ln x)^2$$

By logarithmic differentiation,

$$\frac{y'}{y} = \frac{1}{2\sqrt{x}} \ln x + \sqrt{x} \cdot \frac{1}{x} + 2(\ln x) \cdot \frac{1}{x} = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} + \frac{2 \ln x}{x},$$

so

$$y' = x^{\sqrt{x}} (x^{\ln x}) \left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} + \frac{2 \ln x}{x} \right).$$

Exercises 115–117: Let q be the number of units of a product (cell phones, barrels of oil, etc.) that can be sold at the price p. The ***price elasticity of demand*** *E is defined as the percentage rate of change of q with respect to p. In terms of derivatives,*

$$E = \frac{p}{q} \frac{dq}{dp} = \lim_{\Delta p \to 0} \frac{(100 \Delta q)/q}{(100 \Delta p)/p}$$

115. Show that the total revenue $R = pq$ satisfies $\dfrac{dR}{dp} = q(1 + E)$.

SOLUTION Let $R = pq$. Then

$$\frac{dR}{dp} = p \frac{dq}{dp} + q = q \frac{p}{q} \frac{dq}{dp} + q = q(E + 1).$$

116. A commercial bakery can sell q chocolate cakes per week at price $\$p$, where $q = 50p(10 - p)$ for $5 < p < 10$.

(a) Show that $E(p) = \dfrac{2p - 10}{p - 10}$.

(b) Show, by computing $E(8)$, that if $p = \$8$, then a 1% increase in price reduces demand by approximately 3%.

SOLUTION

(a) Let $q = 50p(10 - p) = 500p - 50p^2$. Then $q'(p) = 500 - 100p$ and

$$E(p) = \left(\frac{p}{q}\right)\frac{dq}{dp} = \frac{p}{50p(10-p)}(500 - 100p) = \frac{10-2p}{10-p} = \frac{2p-10}{p-10}.$$

(b) From part (a),

$$E(8) = \frac{2(8)-10}{8-10} = -3.$$

Thus, with the price set at \$8, a 1% increase in price results in a 3% decrease in demand.

117. The monthly demand (in thousands) for flights between Chicago and St. Louis at the price p is $q = 40 - 0.2p$. Calculate the price elasticity of demand when $p = \$150$ and estimate the percentage increase in number of additional passengers if the ticket price is lowered by 1%.

SOLUTION Let $q = 40 - 0.2p$. Then $q'(p) = -0.2$ and

$$E(p) = \left(\frac{p}{q}\right)\frac{dq}{dp} = \frac{0.2p}{0.2p - 40}.$$

For $p = 150$,

$$E(150) = \frac{0.2(150)}{0.2(150) - 40} = -3,$$

so a 1% decrease in price increases demand by 3%. The demand when $p = 150$ is $q = 40 - 0.2(150) = 10$, or 10000 passengers. Therefore, a 1% increase in demand translates to 300 additional passengers.

118. How fast does the water level rise in the tank in Figure 9 when the water level is $h = 4$ m and water pours in at 20 m^3/min?

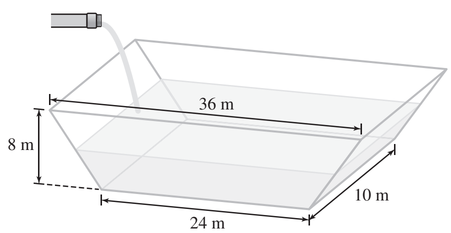


FIGURE 9

SOLUTION When the water level is at height h, the length of the upper surface of the water is $24 + \frac{3}{2}h$ and the volume of water in the trough is

$$V = \frac{1}{2}h\left(24 + 24 + \frac{3}{2}h\right)(10) = 240h + \frac{15}{2}h^2.$$

Therefore,

$$\frac{dV}{dt} = (240 + 15h)\frac{dh}{dt} = 20 \text{ m}^3\text{/min}.$$

When $h = 4$, we have

$$\frac{dh}{dt} = \frac{20}{240 + 15(4)} = \frac{1}{15} \text{ m/min}.$$

119. The minute hand of a clock is 8 cm long, and the hour hand is 5 cm long. How fast is the distance between the tips of the hands changing at 3 o'clock?

SOLUTION Let S be the distance between the tips of the two hands. By the law of cosines

$$S^2 = 8^2 + 5^2 - 2 \cdot 8 \cdot 5\cos(\theta),$$

where θ is the angle between the hands. Thus

$$2S\frac{dS}{dt} = 80\sin(\theta)\frac{d\theta}{dt}.$$

At three o'clock $\theta = \pi/2$, $S = \sqrt{89}$, and

$$\frac{d\theta}{dt} = \left(\frac{\pi}{360} - \frac{\pi}{30}\right) \text{ rad/min} = -\frac{11\pi}{360} \text{ rad/min},$$

so

$$\frac{dS}{dt} = \frac{1}{2\sqrt{89}}(80)(1)\frac{-11\pi}{360} \approx -0.407 \text{ cm/min}.$$

120. Chloe and Bao are in motorboats at the center of a lake. At time $t = 0$, Chloe begins traveling south at a speed of 50 km/h. One minute later, Bao takes off, heading east at a speed of 40 km/h. At what rate is the distance between them increasing at $t = 12$ min?

SOLUTION Take the center of the lake to be origin of our coordinate system. Because Chloe travels at 50 km/h = $\frac{5}{6}$ km/min due south, her position at time $t > 0$ is $(0, \frac{5}{6}t)$; because Bao travels at 40 km/h = $\frac{2}{3}$ km/min due east, her position at time $t > 1$ is $(\frac{2}{3}(t-1), 0)$. Thus, the distance between the two motorboats at time $t > 1$ is

$$s = \sqrt{\frac{4}{9}(t-1)^2 + \frac{25}{36}t^2} = \frac{1}{6}\sqrt{41t^2 - 32t + 16},$$

and

$$\frac{ds}{dt} = \frac{41t - 16}{6\sqrt{41t^2 - 32t + 16}}.$$

At $t = 12$, it follows that

$$\frac{ds}{dt} = \frac{476}{6\sqrt{5536}} \approx 1.066 \text{ km/min}.$$

121. A bead slides down the curve $xy = 10$. Find the bead's horizontal velocity at time $t = 2$ s if its height at time t seconds is $y = 400 - 16t^2$ cm.

SOLUTION Let $xy = 10$. Then $x = 10/y$ and

$$\frac{dx}{dt} = -\frac{10}{y^2}\frac{dy}{dt}.$$

If $y = 400 - 16t^2$, then $\frac{dy}{dt} = -32t$ and

$$\frac{dx}{dt} = -\frac{10}{(400 - 16t^2)^2}(-32t) = \frac{320t}{(400 - 16t^2)^2}.$$

Thus, at $t = 2$,

$$\frac{dx}{dt} = \frac{640}{(336)^2} \approx 0.00567 \text{ cm/s}.$$

122. In Figure 10, x is increasing at 2 cm/s, y is increasing at 3 cm/s, and θ is decreasing such that the area of the triangle has the constant value 4 cm^2.

(a) How fast is θ decreasing when $x = 4$, $y = 4$?

(b) How fast is the distance between P and Q changing when $x = 4$, $y = 4$?

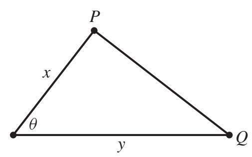


FIGURE 10

SOLUTION

(a) The area of the triangle is

$$A = \frac{1}{2}xy\sin\theta = 4.$$

Differentiating with respect to t, we obtain

$$\frac{dA}{dt} = \frac{1}{2}xy\cos\theta\frac{d\theta}{dt} + \frac{1}{2}y\sin\theta\frac{dx}{dt} + \frac{1}{2}x\sin\theta\frac{dy}{dt} = 0.$$

When $x = y = 4$, we have $\frac{1}{2}(4)(4)\sin\theta = 4$, so $\sin\theta = \frac{1}{2}$. Thus, $\theta = \frac{\pi}{6}$ and

$$\frac{1}{2}(4)(4)\frac{\sqrt{3}}{2}\frac{d\theta}{dt} + \frac{1}{2}(4)\left(\frac{1}{2}\right)(2) + \frac{1}{2}(4)\left(\frac{1}{2}\right)(3) = 0.$$

Solving for $d\theta/dt$, we find

$$\frac{d\theta}{dt} = -\frac{5}{4\sqrt{3}} \approx -0.72 \text{ rad/s}.$$

(b) By the Law of Cosines, the distance D between P and Q satisfies

$$D^2 = x^2 + y^2 - 2xy\cos\theta,$$

so

$$2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2xy\sin\theta\frac{d\theta}{dt} - 2x\cos\theta\frac{dy}{dt} - 2y\cos\theta\frac{dx}{dt}.$$

With $x = y = 4$ and $\theta = \frac{\pi}{6}$,

$$D = \sqrt{4^2 + 4^2 - 2(4)(4)\frac{\sqrt{3}}{2}} = 4\sqrt{2-\sqrt{3}}.$$

Therefore,

$$\frac{dD}{dt} = \frac{16 + 24 - \frac{20}{\sqrt{3}} - 12\sqrt{3} - 8\sqrt{3}}{8\sqrt{2-\sqrt{3}}} \approx -1.50 \text{ cm/s}.$$

123. A light moving at 0.8 m/s approaches a man standing 4 m from a wall (Figure 11). The light is 1 m above the ground. How fast is the tip P of the man's shadow moving when the light is 7 m from the wall?

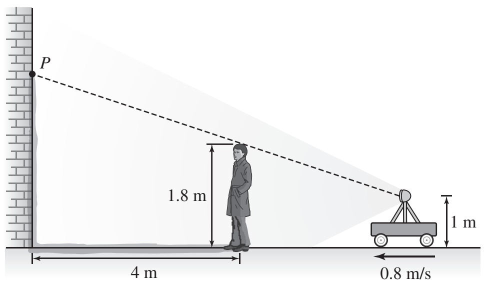


FIGURE 11

SOLUTION Let x denote the distance between the man and the light. Using similar triangles, we find

$$\frac{0.8}{x} = \frac{P-1}{4+x} \quad \text{or} \quad P = \frac{3.2}{x} + 1.8.$$

Therefore,

$$\frac{dP}{dt} = -\frac{3.2}{x^2}\frac{dx}{dt}.$$

When the light is 7 m from the wall, $x = 3$. With $\frac{dx}{dt} = -0.8$, we have

$$\frac{dP}{dt} = -\frac{3.2}{3^2}(-0.8) = 0.284 \text{ m/s}.$$