Chapter 3

The time value of money

Solutions to questions

1. A rate of return is the ratio of net cash inflows to net cash outflows produced by a financial contract. It is often expressed as a percentage. The 'financial contract' involved may be, for example, an investment in shares, land or bonds. An interest rate is a rate of return produced by debt of one form or another. Thus, an interest rate is one type of rate of return.

2. Simple interest is a method of calculating interest in which the interest is computed on the basis of the original sum borrowed. Compound interest is a method of calculating interest in which interest is computed on the basis of the original sum borrowed, plus interest owing but unpaid at the date of calculation.

3. The statement is true. The three different terms—'present value', 'price' and 'principal'—all refer to the value of a financial contract at a given date (typically, today). 'Present value' tends to be used when the valuation is the result of a discounting procedure; 'price' tends to be used when the valuation is the result of a market transaction in a security, and 'principal' tends to be used when the valuation refers to an amount lent by way of a standard loan. However, these circumstances are not independent. For example, a price may be offered and agreed to because a discounting procedure indicates to market participants that the price is the correct valuation.

4. The term 'nominal interest rate' can mean an interest rate where interest is charged more frequently than the quoted period. For example, '6 per cent per annum payable quarterly' is a nominal interest rate because interest is charged quarterly—that is, four times each year—but the quoted period is a year. The term 'nominal interest rate' can also mean an interest rate before taking out the effects of inflation. For example, if a lender receives interest at the rate of 5 per cent per annum over a period of time in which the inflation rate is 7 per cent, the lender's real interest received in this case is negative, but the nominal return is 5 per cent.

5. Successive *arithmetic* rates of return, such as those calculated using Equation 3.1, should be the subject of compounding, which is essentially a multiplicative process. Example 3.13 illustrates that it is incorrect to add such returns. Successive *logarithmic* rates of return, such as those calculated using Equation 3.12, should be added.

6. The statement is true. Having chosen the valuation date, cash flows occurring earlier (later) than the valuation date are accumulated (discounted) at the required rate of return and then summed. Because money has a time value (that is, required rates of return are positive), valuation of a given set of cash inflows must produce a higher value, the later is the valuation date. An important example is that the present value (all cash inflows occur in the future) must be less than the terminal value (all cash inflows occur in the past).

7. Both annuities-due and deferred annuities consist of equal cash flows, equally spaced in time. The distinguishing feature is when the first cash flow is to occur. In an annuity-due, the first cash flow occurs immediately (that is at $t = 0$). In a deferred annuity, the first cash flow occurs more than 1 time period ahead (that is, after $t = 1$); for example, the first cash flow may occur at $t = 5$.

8. If the interest rate on a variable rate loan is increased to such an extent that the periodic repayment is less than the interest accrued since the last periodic repayment was made, then there is an amount of unpaid interest. This amount is, in effect, added to the principal, and therefore the following periodic repayment will fall even further behind the interest accruing. Thus, the principal continues to increase without limit. In short, the loan term is infinite. In the notation used in Equation 3.30, the repayment, C, is less than the interest accrued, $P \times i$. Thus $C - Pi < 0$, and in Equation 3.30, the numerator becomes undefined because the logarithm of a negative number is undefined.

9. In a simple annuity, interest is charged with the same frequency as payments are required. For example, if interest is charged monthly, then payments are required every month. In a general annuity, interest may be charged more (or less) often than payments are required. For example, interest may be charged weekly (or quarterly) even though payments are required monthly.

Solutions to problems

1. Nicholas's interest is:

$$\$2000 \times 0.1325 \times \frac{6}{12}$$
$$= \$132.50$$

2. Nicholas's principal is now \$2 132.50. So the interest is:

$$\$2132.50 \times 0.1325 \times \frac{6}{12}$$
$$= \$141.28$$

3. Using Equation 3.2:

$$S = P\,(1 + rt)$$

$$\$10\,400 = \$10\,000\left[1 + r\,\frac{30}{365}\right]$$

$$\frac{30}{365}r = 0.04$$

$$r = 48.67 \text{ per cent per annum}$$

4. Using Equation 3.2:

$$S = P\,(1 + rt)$$

$$\$7394.70 = \$7250\left[1 + (0.1550)\left(\frac{d}{365}\right)\right]$$

where d is the loan term in days.

Solving,

$$(0.1550)\left(\frac{d}{365}\right) = 0.1995862$$

$$\therefore\ d = 47$$

The loan term was 47 days.

5. Days between 2 April and 16 May = 28 (April) + 16 (May) = 44

Therefore, using simple interest, the repayment is S where:

$$S = \$200\ 000\left[1 + (0.0955)\left(\frac{44}{365}\right)\right]$$
$$= \$200\ 000 \times 1.011512328$$
$$= \$202\ 302.47$$

6. Following the Australian conventions set out in Section 3.3.4, we require the exact number of days in the term of the deposit. There are 23 days remaining in February, then 31 in March, 30 in April and 5 in May; the total is 89 days. Using simple interest, the amount of interest is given by:

$$i = P \times r \times t$$
$$= \$300\ 000 \times 0.044 \times \frac{89}{365}$$
$$= \$3\,218.63$$

7. $$P = \frac{\$500\ 000}{1 + (0.1065)\left(\frac{90}{365}\right)}$$

$$= \frac{\$500\ 000}{1.026260274}$$

$$= \$487\ 205.84$$

8. (a) Using Equation 3.4:

$$S = P(1 + i)^n$$
$$= \$1\,000\,(1.08)^{10}$$
$$= \$2\,158.92$$

The interest component is $1 158.92.

(b) Interest each year is 0.08 × $1 000 = $80. Ten years' interest is 10 × $80 = $800. The interest in part (a) includes ‘interest on interest’, as each year's interest is reinvested. In part (b), interest is withdrawn each year, so there is no compounding effect.

9. Using Equation 3.4:

$$S = P(1 + i)^n$$
$$= \$65\,000\,(1.147)^3$$
$$= \$98\,085.23$$

10. (a) Using Equation 3.4:

$$S = P(1 + i)^n$$
$$= \$87\,000\,(1.0735)^3$$
$$= \$107\,628.03$$

(b) Using Equation 3.4:

$$S = P(1 + i)^n$$
$$= \$87\,000\,(1.0735)^6$$
$$= \$133\,147.05$$

11. (a) (i) The after-tax interest rate is:

$$(1 – 0.45)(12.4 \text{ per cent})$$
$$= 6.82 \text{ per cent}$$

Using Equation 3.4:

$$S = \$10\,000\,(1.0682)^{10}$$
$$= \$19\,343.08$$

(ii) The after-tax interest rate is:

$$= (1 - 0.30)\ (12.4 \text{ per cent})$$
$$= 8.680 \text{ per cent}$$
$$\text{So, } S = \$10\,000\ (1.0868)^{10}$$
$$= \$22\,987.74$$

(iii) The after-tax interest rate is:

$$= (1 - 0.15)\ (12.4 \text{ per cent})$$
$$= 10.54 \text{ per cent}$$
$$\text{So, } S = \$10\,000\ (1.1054)^{10}$$
$$= \$27\,239.22$$

(iv) $S = \$10\,000\ (1.124)^{10}$

$$= \$32\,185.71$$

(b) The accumulated sum at the end of the tenth year (before tax) is $32 185.71. Of this amount, $10 000 is the original principal and $22 185.71 is the accumulated interest. Therefore, the tax owed is 0.45 × $22 185.71 = $9 983.57. Therefore, the after-tax value of the investment is $32 185.71 – $9 983.57 = $22 202.14.

If tax is payable at 45 per cent each year, the after-tax value of the investment is only $19 343.08, as shown in the answer to problem 11(a)(i). Therefore, the tax system in problem 11(b) is better for Frank. Although both systems appear to tax Frank at the rate of 45 per cent, the tax system in part (b) allows Frank to defer the tax payment. Because money has a time value, this is to Frank's benefit. Frank is permitted to invest, and hence benefit from, money that would otherwise have been paid in tax.

12. Using Equation 3.4:

$$S = P(1 + i)^n$$
$$= \$17\,200\ (1.025)^8$$
$$= \$20\,956.53$$

13. Using Equation 3.4:

$$S = P(1 + i)^n$$
$$= \$25\,000\ (1.006)^{36}$$
$$= \$31\,007.54$$

14. All parts in this question are answered using Equation 3.5, which is:

$$P = \frac{S}{(1+i)^n}$$

(a) $P = \dfrac{\$1\,000}{(1.12)^5} = \$567.43$

(b) $P = \dfrac{\$1\,000}{(1.12)^{10}} = \$321.97$

(c) $P = \dfrac{\$1\,000}{(1.06)^5} = \$747.26$

(d) $P = \dfrac{\$16\ 205}{(1.015)^{12}} = \$13\ 553.66$

(e) $P = \dfrac{\$1\ 000\ 000}{(1.15)^{40}} = \$3\,733.24$

(f) $P = \dfrac{\$1\ 000\ 000}{(1.15)^{100}} = \$0.85 = 85\text{ cents}$

15. Interest will be charged each month at one-twelfth of 15 per cent—that is, the monthly interest rate is 1.25 per cent. Therefore, using Equation 3.4:

$$\begin{aligned} S &= P\,(1 + i)^n \\ &= \$8\,000\,(1.0125)^{12} \\ &= \$9\,286.04 \end{aligned}$$

The lump sum repayment is $9 286.04.

16. All parts of this question are answered using Equation 3.5, rearranged to give the interest rate, as follows:

$$S = P\,(1+i)^n$$
$$\therefore (1+i)^n = \frac{S}{P}$$
$$\therefore i = \left(\frac{S}{P}\right)^{1/n} - 1$$

(a) $i = \left(\dfrac{\$92\,000}{\$82\,000}\right)^{1/2} - 1 = 5.9222\text{ per cent}$

(b) $i = \left(\dfrac{\$1\,604\,600}{\$1\,500\,000}\right)^{1/4} - 1 = 1.6995\text{ per cent}$

(c) $i = \left(\dfrac{\$2\,000\,000}{\$1\,307\,600}\right)^{1/3} - 1 = 15.2175\text{ per cent}$

(d) $i = \left(\dfrac{\$10\,000\,000}{\$6\,000\,000}\right)^{1/6} - 1 = 8.8867\text{ per cent}$

(e) $i = \left(\dfrac{\$10\,000\,000}{\$6\,000\,000}\right)^{1/5.5} - 1$

$= \left(\dfrac{10}{6}\right)^{0.18181818} - 1$

$= 9.7327\text{ per cent}$

17. Parts (a) to (d) of this question are answered using Equation 3.6, which is:

$$i=\left(1+\frac{j}{m}\right)^{m}-1$$

(a) $i=\left(1+\frac{0.18}{2}\right)^{2}-1$

$=(1.09)^{2}-1$

$=18.81\text{ per cent}$

(b) $i=\left(1+\frac{0.18}{12}\right)^{12}-1$

$=(1.015)^{12}-1$

$=19.5618\text{ per cent}$

(c) $i=\left(1+\frac{0.18}{365/14}\right)^{365/14}-1$

$=(1.006904109)^{26.07142857}-1$

$=19.6477\text{ per cent}$

(d) $i=\left(1+\frac{0.18}{365}\right)^{365}-1$

$=19.7164\text{ per cent}$

Part (e) is answered using Equation 3.11:

(e) $i=e^{j}-1$

$=e^{0.18}-1$

$=19.7217\text{ per cent}$

18. Parts (a) to (d) of this question are answered using Equation 3.6, which is:

$$i=\left(1+\frac{j}{m}\right)^{m}-1$$

(a)

$i=\left(1+\frac{0.075}{2}\right)^{2}-1$

$=(1.0375)^{2}-1$

$=7.6406\text{ per cent}$

(b)

$$i = \left(1 + \frac{0.075}{12}\right)^{12} - 1$$
$$= (1.00625)^{12} - 1$$
$$= 7.7633 \text{ per cent}$$

(c)

$$i = \left(1 + \frac{0.075}{365/14}\right)^{365/14} - 1$$
$$= (1.0002876712)^{26.07142857} - 1$$
$$= 7.7768 \text{ per cent}$$

(d)

$$i = \left(1 + \frac{0.075}{365}\right)^{365} - 1$$
$$= 7.7876 \text{ per cent}$$

(e) Part (e) is answered using Equation 3.11:

$$i = e^{j} - 1$$
$$= e^{0.075} - 1$$
$$= 7.7888 \text{ per cent}$$

19. The interest rate (achieved over 54 days) is:

$$\frac{\$93\ 323 - \$91\ 107}{\$91\ 107}$$

$$= \frac{\$2\ 216}{\$91\ 107}$$

= 2.4323048 per cent

The number of 54-day periods in one year is:

$$\frac{365}{54} = 6.759259259.$$

Therefore, the effective annual interest rate is:

$$i = (1.024323048)^{6.759259259} - 1$$
$$= 17.6376 \text{ per cent.}$$

The effective annual interest rate is 17.6376 per cent.

20. The holding period (or, term) is 24 days in January plus 29 days in February, plus 3 days in March, making a total of 56 days.

The interest rate (achieved over 56 days) is:

$$\frac{\$987\,618 - \$976\,751}{\$976\,751}$$

$$= \frac{\$10\,867}{\$976\,751}$$

$$= 1.112566 \text{ per cent}$$

The number of 56-day periods in one year is:

$$\frac{365}{56} = 6.517857143.$$

(a) Therefore, the simple annual interest rate is:

$$j = 6.517857143 \times 1.112566 \text{ per cent}$$

$$= 7.2515 \text{ per cent}$$

The simple annual interest rate is 7.2515 per cent.

(b) Therefore, the effective annual interest rate is:

$$i = (1.0112566)^{6.517857143} - 1$$

$$= 7.4779 \text{ per cent}$$

The effective annual interest rate is 7.5686 per cent.

21. The term of the investment is 15 days in January, plus 29 days in February, plus 11 days in March, making a total of 55 days. Using Australian conventions (see Section 3.3.4), the interest rate that the investor earned over this 55-day period is:

$$0.0615 \times \frac{55}{365}$$

$$= 0.009\ 267\ 123\ 288$$

There are 365/55 periods of length 55 days in a year. Using Equation 3.6, the effective annual interest rate is:

$$i = (1.009\ 267\ 123\ 288)^{365/55} - 1$$

$$= 6.3129 \text{ per cent}$$

The effective annual interest rate is 6.3129 per cent.

22. The discount offered is 0.005 × \$8 465.95 = \$42.33 if payment is made within 7 days. Therefore, if the discount is taken, the purchaser (University Garden Supplies Ltd) must pay \$8465.95 – \$42.33 = \$8423.62. It is assumed that the purchaser will pay as late as possible, within each option. That is, the purchaser will pay either \$8 423.62 on Day 7 or \$8 465.95 on Day 30. An equally attractive investment must therefore earn \$42.33 on an investment of \$8 423.62 over a 23-day period.

    Therefore, the implicit interest rate (to be achieved over 23 days) is:

    $$\frac{\$42.33}{\$8\,423.62} = 0.5025155 \text{ per cent}$$

    The number of 23-day periods in one year is 365/23. Therefore, the effective annual interest rate implicit in the discount offer is:

    $$\begin{aligned} & (1.005025155)^{365/23} - 1 \\ = \ & (1.005025155)^{15.86956522} - 1 \\ = \ & 8.28 \text{ per cent} \end{aligned}$$

23. The interest rates paid are:

    $$\frac{8}{12} \times 9.15 \text{ per cent} = 6.1 \text{ per cent}$$
    $$\frac{6}{12} \times 8.45 \text{ per cent} = 4.225 \text{ per cent}$$
    $$\frac{10}{12} \times 8.16 \text{ per cent} = 6.8 \text{ per cent}$$

    The interest rate earned (over 2 years) is, therefore:

    $$\begin{aligned} & (1.061)\,(1.04225)\,(1.068) - 1 \\ = \ & 18.10235 \text{ per cent} \end{aligned}$$

    The effective annual interest rate is, therefore:

    $$\begin{aligned} & (1.1810235)^{0.5} - 1 \\ = \ & 8.675 \text{ per cent} \end{aligned}$$

24. Using Equation 3.7:

    $$\begin{aligned} j &= m\left[(1+i)^{1/m} - 1\right] \\ &= 12\left[(1.195)^{1/12} - 1\right] \\ &= 12 \times 0.014956257 \\ &= 0.179475 \end{aligned}$$

    The nominal annual interest rate is approximately 17.95 per cent.

25. Using Equation 3.8:

$$i = (1 + i^*)(1 + p) - 1$$
$$= (1.10 \times 1.25) - 1$$
$$= 0.375$$

The nominal interest rate should be 37.5 per cent per annum.

26. Although this question can be answered by manipulating and combining Equations 3.6 and 3.9, the easiest way to answer it is from first principles. Each year the loan principal *P* must 'grow' by enough to cover inflation that year, and to earn a real interest rate of 3.5 per cent. For example, at the end of the first year, the amount owing must be $P \times 1.10 \times 1.035$. At the end of the third year, the amount owing must be:

$$P \times 1.10 \times 1.035 \times 1.06 \times 1.035 \times 1.04 \times 1.035$$
$$= P \times 1.10 \times 1.06 \times 1.04 \times (1.035)^3$$
$$= 1.344\ 475\ 644\ P$$

That is, the nominal interest rate for the 3-year period needs to be 34.4475644 per cent. The effective annual interest rate corresponding to this 3-year interest rate is $(1.344475643)^{1/3} - 1$, which equals 10.369984 per cent. Presumably, in practice, George would round this to, say, 10.37 per cent per annum.

George should set an interest rate of 10.37 per cent per annum.

27. Following the same logic as in the previous question, at the end of the seventh year, the amount owing on an original principal of *P* must be:

$$P \times 1.08 \times 1.05 \times (1.04)^5 \times (1.03)^7$$
$$= 1.696\ 837\ 775\ P$$

That is, the nominal interest rate for the 7-year period needs to be 69.6837775 per cent. The effective annual interest rate corresponding to this 7-year interest rate is $(1.696837775)^{1/7} - 1$, which equals 7.8464268 per cent. Presumably, in practice, Grose Paterson Bank would round this to, say, 7.85 per cent per annum.

Grose Paterson Bank should set an interest rate of 7.85 per cent per annum.

28. The average annual inflation rate is:

$$\left(\frac{193.8}{147.6}\right)^{1/4} - 1$$

= 7.0451 per cent

The real annual rate of return (using Equation 3.9) is, therefore:

$$\frac{1.114}{1.070451} - 1$$

= 0.0406827

The real annual rate of return is, therefore, slightly more than 4 per cent.

An equivalent solution that may be easier to follow is:
The nominal value of the investment after 4 years is \$50 000 $(1.114)^4$ = \$77 003.55. The real (year zero) value is, therefore:

$$\$77\,003.55 \times \frac{147.6}{193.8} = \$58\,646.67$$

Therefore, the real rate of return (achieved over the 4-year period) is:

$$\frac{\$8\,646.67}{\$50\,000} = 0.1729334$$

Finally, the annualised rate of return this represents is:

$(1.1729334)^{1/4} - 1$
= 4.06827%

29.

Date (end of month)	**Share price**	**Price relative**	***ℓn* of price relative**
May	\$5.50	…	…
June	\$5.85	1.063636	0.061694
July	\$6.12	1.046154	0.045120
August	\$5.75	0.939542	– 0.062362
September	\$5.75	1.000000	0.000000
October	\$6.44	1.120000	0.113329
November	\$6.60	1.024845	0.024541
Sum of log price relatives =			0.182322

The sum of the log price relatives represents the total return of approximately 18.23 per cent made from 31 May to 30 November.

Note that $\ell n\left(\frac{\$6.60}{\$5.50}\right) = \ell n(1.2) = 0.182322$

This is because (using continuous compounding) the return over a period is simply the sum of the returns in each sub-period. Because there are no dividends, the intermediate returns cancel out as shown in the following table.

End of month	**Log price relative**
June	$\ell n(5.85/5.50) = \ell n(5.85) – \ell n(5.50)$
July	$\ell n(6.12/5.85) = \ell n(6.12) – \ell n(5.85)$
August	$\ell n(5.75/6.12) = \ell n(5.75) – \ell n(6.12)$
September	$\ell n(5.75/5.75) = 0$
October	$\ell n(6.44/5.75) = \ell n(6.44) – \ell n(5.75)$
November	$\ell n(6.60/6.44) = \ell n(6.60) – \ell n(6.44)$

When the log price relatives are summed, the intermediate returns cancel, and the total is $– \ell n(5.50) + \ell n(6.60)$, which can be rewritten as $\ell n(6.60/5.50)$.

To put this another way, the November share price equals the May share price after the May share price has grown at the rate of 18.2322 per cent using continuous compounding. That is,

November share price $= \$5.50\ e^{0.182322}$
$= \$5.50 \times 1.2$
$= \$6.60$

30. The value after four years is:
$\$364\ 000 \times 1.07 \times 1.27 \times 0.95 \times 1.11$
$= \$364\ 000 \times 1.432\ 960\ 05$
$= \$521\ 597$

Using Equation 3.4, the average annual rate of return is i, where:

$S = P(1 + i)^n$
$\$521\ 597 = \$364\ 000\ (1 + i)^4$
$\therefore (1 + i)^4 = 1.432\ 958\ 791$
Thus $(1 + i) = 1.094\ 104$
$\therefore i = 0.094\ 104$

The average annual rate of return is approximately 9.41 per cent.

Equivalently, use Equation 3.13, as follows:

$$i = [(1 + r_1)(1 + r_2) \ldots (1 + r_n)]^{1/n} – 1$$
$$= [1.07 \times 1.27 \times 0.95 \times 1.11]^{1/4} – 1$$
$$= (1.432\ 960\ 05)^{0.25} – 1$$
$$= 0.094\ 104$$

31. Present value $= \dfrac{\$7\,601}{(1.07)^{0.5}} + \dfrac{\$9\,900}{(1.07)^{2.5}} + \dfrac{\$18\,522}{(1.07)^{7}}$

$= \$7\,348.164 + \$8\,359.412 + \$11\,534.571$
$= \$27\,242.15$

32. (a) $P_0 = \dfrac{\$8\,000}{(1.085)^2} + \dfrac{\$14\,000}{(1.085)^5}$

$= \$6\,795.64 + \$9\,310.64$
$= \$16\,106.28$

(b) Using the answer to part (a), the value in 5 years' time must be:

$P_5 = P_0\ (1.085)^5$
$= \$16\,106.28 \times 1.503657$
$= \$24\,218.31$

Equivalently, we could accumulate the year 2 cash flow of \$8000 for 3 years, so that it becomes equivalent to a year 5 cash flow. The sum of the year 5 cash flows is then:

$P_5 = \$8000\ (1.085)^3 + \$14\,000$
$= \$10\,218.31 + \$14\,000$
$= \$24\,218.31$

33. If the internal rate of return is 27.2 per cent per annum, then the present value of the cash flows will equal the investment cost of \$50 000 if the present value is calculated using a required rate of return of 27.2 per cent per annum.

This is tested below:

PV (at 27.2% pa) $= \dfrac{\$40\ 000}{1.272} + \dfrac{\$30\ 000}{(1.272)^2}$
$= \$31\,446.54 + \$18\,541.59$
$= \$49\,998.13$
$\approx \$50\,000$

34. To answer each of (a) to (e), we will use Equation 3.15, which is:

$$V_{t^*} = C_t(1+i)^{t^*-t},$$

where t^* is the valuation date and t is the date of the cash flow to be valued.

(a) The present value (V_0) is the value when $t^* = 0$. Therefore:

$$V_0 = \$1000(1.085)^{-1} + \$8000(1.085)^{-3} + \$12\ 000(1.085)^{-7} + \$10\ 000(1.085)^{-10}$$
$$= \frac{\$1000}{1.085} + \frac{\$8000}{(1.085)^3} + \frac{\$12\ 000}{(1.085)^7} + \frac{\$10\ 000}{(1.085)^{10}}$$
$$= \$921.659 + \$6263.265 + \$6779.116 + \$4422.854$$
$$= \$18\ 386.89$$

(b) The value as at the start of Year 1 is when $t^* = 1$. Therefore:

$$V_1 = \$1000 + \$8000(1.085)^{-2} + \$12\ 000(1.085)^{-6} + \$10\ 000(1.085)^{-9}$$
$$= \$1000 + \frac{\$8000}{(1.085)^2} + \frac{\$12\ 000}{(1.085)^6} + \frac{\$10\ 000}{(1.085)^9}$$
$$= \$1000 + \$6795.642 + \$7355.341 + \$4798.797$$
$$= \$19\ 949.78$$

(c) The value as at the start of Year 3 is when $t^* = 3$. Therefore:

$$V_3 = \$1000(1.085)^2 + \$8000 + \$12\ 000(1.085)^{-4} + \$10\ 000(1.085)^{-7}$$
$$= \$1000(1.085)^2 + \$8000 + \frac{\$12\ 000}{(1.085)^4} + \frac{\$10\ 000}{(1.085)^7}$$
$$= \$1177.225 + \$8000 + \$8658.891 + \$5649.264$$
$$= \$23\ 485.38$$

(d) The value as at the start of Year 7 is when $t^* = 7$. Therefore:

$$V_7 = \$1000(1.085)^6 + \$8000(1.085)^4 + \$12\ 000 + \$10\ 000(1.085)^{-3}$$
$$= \$1000(1.085)^6 + \$8000(1.085)^4 + \$12\ 000 + \frac{\$10\ 000}{(1.085)^3}$$
$$= \$1631.468 + \$11\ 086.870 + \$12\ 000 + \$7829.081$$
$$= \$32\ 547.42$$

(e) The terminal value (V_{10}) is the value when $t^* = 10$. Therefore:

$$V_{10} = \$1000(1.085)^9 + \$8000(1.085)^7 + \$12\ 000(1.085)^3 + \$10\ 000$$
$$= \$2083.856 + \$14\ 161.138 + \$15\ 327.470 + \$10\ 000$$
$$= \$41\ 572.46$$

(f) The value as at successive dates is the value at the earlier date accumulated for the number of years between valuation dates. For example, the value when $t^* = 3$ is \$23 485.38, while the value when $t^* = 7$ is \$32 547.42. There are 4 years between these valuation dates and, because \$23 485.38 × $(1.085)^4$ = \$32 547.42, the later valuation is the accumulated value of the earlier valuation. This relationship holds between any two valuation dates, not only successive dates. For example, $V_1 \times (1.085)^6$ = \$19 949.78 × $(1.085)^6$ = \$32 547.42 = V_7.

35. (a) This is an annuity-due. Using Equation 3.21, the value today is:

$$P = C + \frac{C}{i}\left[1 - \frac{1}{(1+i)^{n-1}}\right]$$

Recall that n is the number of cash flows, which in this question is 6. Therefore:

$$\begin{aligned} P &= \$5000 + \frac{\$5000}{0.07}\left[1 - \frac{1}{(1.07)^5}\right] \\ &= \$5000 + \$20\ 500.99 \\ &= \$25\ 500.99 \end{aligned}$$

(b) This is an ordinary annuity. Using Equation 3.19, the value today is:

$$\begin{aligned} P &= \frac{C}{i}\left[1 - \frac{1}{(1+i)^n}\right] \\ &= \frac{\$5000}{0.07}\left[1 - \frac{1}{(1.07)^6}\right] \\ &= \$23\ 832.70 \end{aligned}$$

Note: Using the solution to (a), we could have calculated this as the value of the annuity-due, discounted 1 year: \$25 500.99/1.07 = \$23 832.70.

(c) This is a deferred annuity. Using Equation 3.24, the value today is:

$$P = \frac{1}{(1+i)^{k-1}} \ \frac{C}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

Recall that n is the number of cash flows and k is the number of time periods until the first cash flow. Hence, $n = 6$ and $k = 4$. Therefore:

$$P=\frac{1}{(1.07)^3}\frac{\$5000}{0.07}\left[1-\frac{1}{(1.07)^6}\right]$$
$$=\frac{1}{1.225\ 043}\times\$23\ 832.70$$
$$=\$19\ 454.58$$

Note: Using the solution to (b), we could have calculated this as the value of the annuity-due, discounted for k – 1 = 3 years: \$23 832.70/(1.07)^3 = \$19 454.58.

36. The timeline is:

	55	56	57	…	65	66	67	…	84	85
	0	1	2		10	11	12		29	30
Deposits	\$*C*	\$*C*	\$*C*	…	\$*C*					
Withdrawals						\$10 000	\$10 000	…	\$10 000	\$10 000

The deposit stream is an annuity-due. The withdrawal stream is a deferred annuity. Because Stanley's plan is 'self-financing', the present value (at time zero) of the deposits must equal the present value (at time zero) of the withdrawals.

There are 11 deposits of \$*C* each to be made, and there are to be 20 withdrawals of \$10 000 each. The interest rate is 10 per cent per annum. Therefore:

$$C+\frac{C}{0.10}\left[1-\frac{1}{(1.10)^{10}}\right]=\frac{1}{(1.10)^{10}}\times\frac{\$10\,000}{0.10}\left[1-\frac{1}{(1.10)^{20}}\right]$$
$$7.144567106\times C=\frac{\$85\,135.6372}{2.59374246}$$
$$C=\frac{\$32\,823.47361}{7.144567106}$$
$$=\$4594.19$$

Stanley needs to set aside \$4594.19 each birthday.

37. Using Equation 3.27:

$$\begin{aligned} P &= \frac{C}{i} \\ &= \frac{\$1000}{0.08} \\ &= \$12\,500 \end{aligned}$$

38. (a) At any given time, there will be six students receiving an annual scholarship payment of $6000 each. Therefore, the fund must produce an income of $36 000 per annum, in perpetuity. Using Equation 3.27:

$$\begin{aligned} P &= \frac{C}{i} \\ &= \frac{\$36\,000}{0.06} \\ &= \$600\,000 \end{aligned}$$

However, this formula assumes that the first scholarship payment will be made a year after the fund is established, whereas Kevin wants to start the scheme immediately—in effect, it is a perpetuity-due payment. Therefore, Kevin needs $36 000 to fund the first year's scholarships, and also has to set up the fund with $600 000. Kevin's total cost is therefore $636 000.

(b) The scholarship amount is fixed in nominal dollars at $6000 per annum. But after 5 years of inflation, at 3.5 per cent per annum, the value of $6000 in today's dollars is only:

$$\frac{\$6000}{(1.035)^5} = \$5051.84$$

(c) To preserve the real value of the scholarship, discounting must occur at the real interest rate. The long-term value of the real interest rate is not known with certainty, but 2.5 per cent per annum may well be a reasonable estimate. The ordinary perpetuity formula assumes that the first payment is the one due at the end of the first year, which in this case is $36 000 × 1.035 = $37 260. Using the estimated real interest rate of 2.5 per cent per annum, the amount Kevin should put into the fund today is:

$$\begin{aligned} P &= \frac{C}{i} \\ &= \frac{\$37\,260}{0.025} \\ &= \$1\,490\,400 \end{aligned}$$

In addition, Kevin will need to pay $36 000 for the first year's scholarships, bringing the total cost to $1 526 400.

Note: this problem is equivalent to a dividend growth model problem, which is discussed in the next chapter. Equation 4.8 is:

$$P_0 = \frac{D_0(1+g)}{k_e - g},$$

where P_0 is today's share price, D_0 is the current dividend, g is the annual growth rate in dividends and k_e is the discount rate. In our problem, D_0 = \$36 000, g = 0.035 and k_e = 0.06.

(d) At the end of the first year, the fund has earned interest of 0.06 × \$1 490 400, which amounts to \$89 424. Each scholarship holder must be compensated for inflation for one year at 3.5 per cent, so the amount of each scholarship must be \$6000 × 1.035 = \$6210. With 6 scholarships to be paid, the total cost to the fund is 6 × \$6210 = \$37 260. The amount in the fund is therefore \$1 490 400 *plus* interest of \$89 424 *less* scholarship payments of \$37 260, which amounts to \$1 542 564.

At the end of the second year, the fund has earned interest of 0.06 × \$1 542 564, which amounts to \$92 554. Each scholarship holder must be compensated for inflation for another year at 3.5 per cent, so the amount of each scholarship must be \$6210 × 1.035 = \$6427. With 6 scholarships to be paid, the total cost to the fund is 6 × \$6427 = \$38 564. The amount in the fund is therefore \$1 542 564 *plus* interest of \$92 554 *less* scholarship payments of \$38 564, which amounts to \$1 596 554.

The position is summarised in the following table:

Date	**Transactions**	**Fund balance**
End of Year 0	Fund started	\$1 490 400
End of Year 1	*Plus* interest	
	0.06 × \$1 490 400	
	= \$89 424	
	Less scholarships paid	
	\$36 000 × 1.035	
	= \$37 260	\$1 542 564
End of Year 2	*Plus* interest	
	0.06 × \$1 542 564	
	= \$92 554	
	Less scholarships paid	
	\$37 260 × 1.035	
	= \$38 564	\$1 596 554

Note: \$1 490 400 × 1.035 = \$1 542 564 and
\$1 542 564 × 1.035 = \$1 596 554

That is, the *nominal* value of the fund is growing at 3.5 per cent per annum, which is just sufficient to maintain the *real* value of the fund, given that the inflation rate is 3.5 per cent per annum.

39. (a) Using Equation 3.27:

$$\begin{aligned} P &= \frac{C}{i} \\ &= \frac{\$200\,000}{0.08} \\ &= \$2\,500\,000 \end{aligned}$$

The present value of the cash flows as at the start of the 5th year—that is, at the end of the 4th year—is $2 500 000.

$$\begin{aligned} P &= \frac{\$100\ 000}{(1.08)^3} + \frac{\$150\ 000}{(1.08)^4} + \frac{\$2\ 500\ 000}{(1.08)^4} \\ &= \$79\ 383.22 + \$110\ 254.48 + \$1\ 837\ 574.63 \\ &= \$2\ 027\ 212.33 \end{aligned}$$

The present value of the whole return stream is $2 027 212.33.

40. This is a deferred perpetuity.

The present value of the inflows is:

$$\begin{aligned} P &= \frac{1}{1.05} \times \frac{\$1\,000}{0.05} \\ &= \frac{\$20\ 000}{1.05} \\ &= \$19\ 047.62 \end{aligned}$$

The present value of the outflows is:

$$\begin{aligned} P &= \$10\ 000 + \frac{\$6\,000}{1.05} \\ &= \$15\ 714.29 \end{aligned}$$

The present value of the inflows ($19 047.62) exceeds the present value of the outflows ($15 714.29). Therefore the investment is profitable.

41. The monthly interest rate (i) $= \dfrac{0.102}{12} = 0.0085$

The loan term (n) = 120 months
The principal (P) = $800 000

Using Equation 3.19:

$$P = \frac{C}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

Therefore:

$$\$800\ 000 = \frac{C}{0.0085}\left[1 - \frac{1}{(1.0085)^{120}}\right]$$

$$= \frac{C}{0.0085}\left[1 - \frac{1}{2.761266384}\right]$$

$$= 75.04086209\,C$$

$$\therefore C = \frac{\$800\ 000}{75.04086209}$$

$$= \$10\ 660.86$$

The monthly repayment is $10 660.86.

Interest charged at the end of the first month is 0.0085 × $800 000 = $6800. Therefore, Luke's first repayment reduces the principal by:

$$\$10\ 660.86 - \$6800.00 = \$3860.86$$

At the start of the last month, only one repayment of $10 660.86 remains. The loan principal at the start of the last month is, therefore:

$$\frac{\$10\ 660.86}{1.0085} = \$10\ 571.01.$$

Thus, the last repayment reduces the principal by $10 571.01.

42. When Luke asks for the payout figure, the remaining repayments form an annuity-due. The 22nd repayment of $10 660.86 is due immediately, followed by $10 660.86 every month from Month 23 to Month 120 (inclusive). The principal outstanding is, therefore:

$$P = \$10\ 660.86 + \begin{pmatrix}\text{present value of an} \\ \text{ordinary annuity of} \\ \text{98 payments of \$10 660.86}\end{pmatrix}$$

$$= \$10\ 660.86 + \frac{\$10\ 660.86}{0.0085}\left[1 - \frac{1}{(1.0085)^{98}}\right]$$

$$= \$10\ 660.86 + \$707\ 032.97$$

$$= \$717\ 693.83$$

The payout figure (including the 22nd repayment) should be $717 693.83.

43. The monthly interest rate is 0.168/12 =0.014.

Using Equation 3.30:

$$\begin{aligned} n &= \frac{\log [C/(C-Pi)]}{\log (1+i)} \\ &= \frac{\log [71.07/(71.07-(1999)(0.014))]}{\log (1.014)} \\ &= \frac{\log (71.07/43.084)}{\log (1.014)} \\ &= \frac{\log (1.649568285)}{\log (1.014)} \end{aligned}$$

Using common logarithms (base 10),

$$\begin{aligned} n &= \frac{0.217370298}{0.006037954} \\ &= 36 \text{ months} \end{aligned}$$

John will be repaying the loan over a period of 3 years.

44. Use Equation 3.19. Solve for i:

$$\$10\ 000 = \frac{\$2770}{i}\left[1-\frac{1}{(1+i)^5}\right]$$

At i = 0.1194, the present value is $10 000.09. The rate of return is, therefore. approximately 11.94 per cent per annum.

45. Use Equation 3.19. Solve for i:

$$\$25\,000 = \frac{\$2027.50}{i}\left[1-\frac{1}{(1+i)^{15}}\right]$$

At i = 0.0256, the present value is $24 992.80. Therefore, the monthly interest rate is approximately 2.56 per cent. The nominal annual interest rate is 12 × 2.56 per cent = 30.72 per cent. The effective annual interest rate is $(1.0256)^{12} - 1$ = 35.44 per cent.

46. All parts of this question are answered using Equation 3.30:

$$n = \frac{\log [C/(C - Pi)]}{\log (1+i)}$$

The monthly interest rate is 0.18/12 = 0.015.

(a) $$n = \frac{\log \{\$1100/[\$1100 - (\$70\,000)(0.015)]\}}{\log (1.015)}$$
$$= \frac{\log (22)}{\log (1.015)}$$
$$= 207.61 \text{ months} (= 17.3 \text{ years})$$

It will require 207 monthly repayments of $1100, plus a smaller repayment at the end of the 208th month.

(b) $$n = \frac{\log \{\$1200/[\$1200 - (\$70\,000)(0.015)]\}}{\log (1.015)}$$
$$= \frac{\log (8)}{\log (1.015)}$$
$$= 139.67 \text{ months}. \ (= 11.6 \text{ years})$$

It will require 139 monthly repayments of $1 200, plus a smaller repayment at the end of the 140th month.

(c) $$n = \frac{\log \{\$1\,500/[\$1\,500 - (\$70\,000)(0.015)]\}}{\log (1.015)}$$
$$= \frac{\log (3.33333333)}{\log (1.015)}$$
$$= 80.87 \text{ months} \ (= 6.7 \text{ years})$$

It will require 80 monthly repayments of $1500, plus a smaller repayment at the end of the 81st month.

47. The monthly interest rate is 0.12/12 = 0.01.

Using Equation 3.19:

$$\$50\,000 = \frac{C}{0.01}\left[1 - \frac{1}{(1.01)^{120}}\right]$$
$$= 69.70052203\,C$$
$$\therefore C = \$717.35$$

The monthly repayment is $717.35.

After 3 years, the remaining loan term is 7 × 12 = 84 months. The principal at that time is, therefore:

$$P = \frac{\$717.35}{0.01}\left[1 - \frac{1}{(1.01)^{84}}\right]$$
$$= \$40\,636.77$$

The monthly interest rate is then increased to 0.135/12 = 0.01125. The new monthly repayment is *C*, where:

$$\$40\,636.77 = \frac{C}{0.01125}\left[1 - \frac{1}{(1.01125)^{84}}\right]$$
$$= 54.15682674\,C$$
$$\therefore C = \$750.35$$

Alternatively, if the repayment remains at $717.35 per month, the new loan term is found using Equation 3.30:

$$n = \frac{\log[C / C - Pi)]}{\log(1 + i)}$$

$$= \frac{\log\{\$717.35 / [\$717.35 - (\$40\,636.77)(0.01125)]\}}{\log(1.01125)}$$

$$= \frac{\log(2.757062523)}{\log(1.01125)}$$

= 90.65 months

The new loan term remaining is 90.65 months, rather than the 84 months that would apply if the interest rate had not been increased—that is, 6 further repayments of $717.35 will be required, plus a smaller (7th) repayment.

48. **Step 1: Calculate the monthly repayment**

At initiation, the loan is to be repaid over 25 years (= 300 months) at a monthly interest rate of 0.65 per cent (= 7.8 per cent / 12). Using Equation 3.19:

$$P = \frac{C}{i}\left[1 - \frac{1}{(1 + i)^n}\right]$$

$$\$150\,000 = \frac{C}{0.0065}\left[1 - \frac{1}{1.0065^{300}}\right]$$

So $C = \$1137.92$

Step 2: Calculate the principal remaining after 1 year

After 1 year, there are 288 repayments remaining. Using Equation 3.19:

$$P = \frac{C}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

$$= \frac{\$1137.92}{0.0065}\left[1 - \frac{1}{1.0065^{288}}\right]$$

So $P = \$147\ 973.31$

Step 3: Calculate the time remaining if the monthly repayment decreases by $10

The bank now takes a monthly fee of $10, effectively reducing the monthly repayment to $1 127.92. Using Equation 3.30,

$$n = \frac{\log[C/(C - Pi]}{\log(1+i)}$$

$$= \frac{\log[\$1127.92/(\$1127.92 - \$147\ 973.31 \times 0.0065)]}{\log(1.0065)}$$

Using base 10 logarithms,

$$n = \frac{0.831925699}{0.002813779225}$$

$$= 295.66$$

The loan will now require 7 more full monthly repayments, plus a smaller eighth payment.

49. (a) The monthly interest rate is 0.078/12= 0.0065. Therefore, the effective annual interest rate is $(1.0065)^{12} - 1 = 8.085$ per cent.

(b) Using Equation 3.19:

$$\$180\ 000 = \frac{C}{0.0065}\left[1 - \frac{1}{(1.0065)^{240}}\right]$$

$$= 121.3539156\ C$$

$$\therefore C = \$1483.26$$

The monthly repayment is $1483.26.

(c) The first 12 repayments form an ordinary annuity, the next 12 form a deferred annuity, and the next 216 form a second deferred annuity. The sum of the present values of these annuities must equal the loan principal of $180 000. Therefore:

$$\$180\,000 = \frac{\$1100}{0.0065}\left[1 - \frac{1}{(1.0065)^{12}}\right] + \frac{1}{(1.0065)^{12}} \times \frac{\$1250}{0.0065}\left[1 - \frac{1}{(1.0065)^{12}}\right] + \frac{1}{(1.0065)^{24}} \times \frac{\$X}{0.0065}\left[1 - \frac{1}{(1.0065)^{216}}\right]$$

$$\$180\,000 = \$12\,658.813 + \$13\,308.986 + \frac{115.8875412\ \$X}{1.168236313}$$

$$= \$25\,967.80 + 99.19871511\ \$X$$

$$\therefore \$X = \$1552.76$$

The monthly repayment during the final 18 years is $1552.76.

(d) Using Equation 3.30:

$$n = \frac{\log\,[C/(C - Pi)]}{\log\,(1+i)}$$

$$= \frac{\log\,\{\$2\,500/[\$2\,500 - (\$180\,000)\,(0.0065)]\}}{\log\,(1.0065)}$$

$$= \frac{\log\,(1.879699248)}{\log\,(1.0065)}$$

$$= 97.409 \text{ months}$$

It will require 97 monthly repayments of $2 500, plus a smaller (98th) repayment.

The 98th repayment (of $\$C^*$) may be found by solving:

$$\$180\,000 = \frac{\$2500}{0.0065}\left[1 - \frac{1}{(1.0065)^{97}}\right] + \frac{C^*}{(1.0065)^{98}}$$

$$= \$179\,456.62 + \frac{C^*}{1.886906428}$$

$$\therefore C^* = \$1025.31$$

The final (98th) repayment is $1025.31.

50. (a) To calculate the *new* monthly repayment, we need the principal outstanding after 18 months. This is found by calculating the present value of the remaining 162 repayments. To perform this calculation we need to know the amount of each *original* monthly repayment. This is found by using Equation 3.19, with P = \$800 000, i = 0.135/12 = 0.01125, and n = 15 × 12 = 180.

$$\$800\,000 = \frac{C}{0.01125}\left[1-\frac{1}{(1.01125)^{180}}\right]$$
$$= 77.02270031\,C$$
$$C = \$10\,386.55$$

The *original* monthly repayment is \$10 386.55.

The principal outstanding after 18 months (that is, with 162 months remaining) is, therefore:

$$P = \frac{\$10\,386.55}{0.01125}\left[1-\frac{1}{(1.01125)^{162}}\right]$$
$$= \$772\,506.23$$

Therefore, the *new* monthly repayment (C') is found using Equation 3.19 with P = \$772 506.23; i = 0.15/12 = 0.0125 and n = 162:

$$\$722\,506.23 = \frac{C'}{0.0125}\left[1-\frac{1}{(1.0125)^{162}}\right]$$
$$\therefore\ C' = \frac{\$772\,506.23}{69.30711684}$$
$$= \$11\,146.13$$

The new monthly repayment is \$11 146.13.

(b) The new loan term can be found from Equation 3.30:

$$n = \frac{\log\,[C/(C-Pi)]}{\log\,(1+i)}$$
$$= \frac{\log = \{\$10\,386.55/[\$10\,386.55-(\$772\,506.23)\,(0.0125)]\}}{\log\,(1.0125)}$$
$$= \frac{\log\,(14.22382265)}{\log\,(1.0125)}$$
$$= 213.72 \text{ months}$$

The increase in the loan term is therefore 213.72 – 162 = 51.72 months.

51. The loan must be repaid by instalments made every month from Month 6 to Month 24, inclusive. That is, as at the end of the sixth month, the repayments form an annuity-due of 19 repayments. To achieve an effective annual interest rate of 12 per cent requires that the monthly interest rate charged must be $(1.12)^{1/12}-1,$ which solves to give an interest rate of 0.94888 per cent per month. If the monthly repayment is set at *C* dollars, then, using Equation 3.21, the principal outstanding at the end of the sixth month (an instant before the first repayment is made) must be:

$$P = C + \frac{C}{i}\left[1 - \frac{1}{(1+i)^{n-1}}\right]$$

$$P_6 = C + \frac{C}{0.0094888}\left[1 - \frac{1}{(1.0094888)^{18}}\right]$$

$$= 17.47515045\ C$$

The present value of this amount as at date 0 (which is exactly half a year prior to the first repayment being made) is:

$$P_0 = \frac{17.47515045\ C}{(1.12)^{0.5}}$$

$$= \frac{17.47515045}{1.058300524}$$

$$= 16.51246508\ C$$

But the present value as at date 0 must, of course, be equal to the loan principal of \$10 000. Hence:

$$\$10\,000 = 16.51246508\ C,$$

which solves to give *C* = \$605.60.

The required monthly repayment should be set at \$605.60.

52. Within any given year, the four quarterly payments form an ordinary annuity. If these are replaced by annual payments at the *end* of each year, the amount of each annual payment must equal the future value of the annuity. The interest rate applicable to *annual* cash flows is 6 per cent per annum. Therefore, to be consistent, quarterly cash flows should be valued using a quarterly interest rate that is equivalent to an annual effective interest rate of 6 per cent. The quarterly interest rate to be used is, therefore, $(1.06)^{1/4} - 1 = 1.4673846$ per cent per quarter.

The future value of an ordinary annuity (Equation 3.28) is:

$$S = \frac{C}{i}\left[(1+i)^n - 1\right]$$

$$S = \frac{\$300}{0.014673846}\left[(1.014673846)^4 - 1\right]$$

Noting that, by definition, $(1.014673846)^4 = 1.06$,

$$S = \frac{\$300 \times 0.06}{0.014673846}$$
$$= \$1226.67$$

The annual payment is $1 226.67.