# Problem solutions – Chapter 1

# Kepler, Newton, & the mass function

## Problem 1.21. External force on binary; collisions in globular cluster

### (a) Consider external forces on a binary

Assume wide binary, $r_s = 10$ AU $= 1.5 \times 10^{12}$ m. Find ratio of external force difference on the two partners to the force exerted by the binary partners on each other. Each star has mass $M_s = 1$ $M_\odot$.

(i) Effect of galactic center. $M_g \approx 10^6 M_\odot$, point source at $r_g = 25\,000$ LY $= 2.4 \times 10^{20}$ m. The disruptive effect is due to the difference of the external force on the two stars in the binary. It will be maximum when the two stars are aligned with the galactic center.

First consider the ratio of the two forces on one of the two stars,

$$➡ \quad \frac{F_g}{F_s} = \frac{M_g}{r_g^2}\left(\frac{M_s}{r_s^2}\right)^{-1} = \frac{M_g}{M_s}\,\frac{r_s^2}{r_g^2} \qquad (1.21.s1)$$

The radial component of $F_g$ is $kr_g^{-2}$ and its gradient is $dF_g/dr = -2kr^{-3}$. The ratio of the latter to the former is

$$\frac{dF_g/dr}{F_g} = -2\,r_g^{-1}$$

or

$$\frac{dF_g}{F_g} = -2\,\frac{dr}{r_g} = -2\frac{r_s}{r_g} \qquad (1.21.s2)$$

where $dr = r_s$ when all three objects are coaligned.

Multiply (s1) and (s2) to find the desired ratio,

$$➡ \quad \frac{dF_g}{F_s} = -2\,\frac{M_g}{M_s}\,\frac{r_s^3}{r_g^3} = 10^6\left(\frac{1.5\times 10^{12}}{2.4\times 10^{20}}\right)^3 = 4.9\times 10^{-19} \qquad (1.21.s3)$$

*The difference force is negligible* compared to that holding the binary together.

(ii) Effect of nearby star in spherical Globular cluster of $N = 10^6$ stars and radius $R_{glob} = 15$ LY. $= 1.42 \times 10^{17}$ m.

Average distance $d$ between adjacent stars is (volume per star)$^{1/3}$,

$$d = V_s^{1/3} = \left(\frac{4\pi}{3N}\right)^{1/3} R_{glob} = 2.29\times 10^{15}\text{ m} = 15\,000\text{ AU} \qquad (1.21.s4)$$

The force ratio is thus, from (s3),

$$\Rightarrow \quad \frac{F_{\text{n-s}}}{F_{\text{s}}} = \frac{M_\odot}{M_\odot}\left(\frac{1.5 \times 10^{12}}{2.3 \times 10^{15}}\right)^3 = 5.6 \times 10^{-10}$$

Again, the *external difference force* is negligible for our assumption of uniform star density. But, in the more dense center of a globular cluster, occasional close encounters can occur that could disrupt the binary.

**(b) Time for collision, within 10 AU, to occur in globular cluster, on average.**

Find typical speed of star in globular cluster of radius $R_{\text{glob}} = 1.4 \times 10^{17}$ m and mass $M_{\text{glob}} = 2 \times 10^{36}$ kg. From virial theorem (2.14),

$$\frac{GM_{\text{glob}}\, m}{R_{\text{glob}}} \approx mv^2$$

$$v \approx \sqrt{\frac{GM_{\text{glob}}}{R_{\text{glob}}}} = 3.0 \times 10^4 \text{ m/s}$$

Cross section for collision, $\sigma = \pi(10\text{ AU})^2 = 7.0 \times 10^{24}$ m$^2$.

The spatial density of stars, from (s4), is $n = d^{-3} = 8.3 \times 10^{-47}$ m$^{-3}$.

Now the flux of stars is $nv$, and the number of collisions per second with a single binary is $nv\sigma$. Hence the time between collisions is the inverse of this, $\underline{t = (nv\sigma)^{-1} = 5.7 \times 10^{16}\text{ s} = 1.8 \times 10^9\text{ yr}}$, which is ~1/10 the age of the globular cluster.

This is the typical time for a given star (or binary) to suffer such a collision. Since there are $10^6$ stars, one expects an encounter every $\underline{2 \times 10^3\text{ years}}$, for our assumptions. The increased density at the center would markedly decrease the collision time.

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## Problem 1.22 – Finding and observing binary star systems (no solution)

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## Problem 1.23 – Find distance range where binary system is detectable as both a visual and a spectroscopic binary.

Two 1-$M_\odot$ stars in circular binary orbit with $i = 90°$ at distance $D$. Star separation $s$. Each star orbits the barycenter with radius $s/2$ with period $P$. Let $M$ be the system mass, $M = 2M_\odot$.

**Spectroscopy**

Require line shift during the course of an *entire* orbit be at least $(\Delta\lambda/\lambda)_{\min} = 3 \times 10^{-5}$. From classical Doppler shift (2),

$$\frac{\Delta\lambda}{\lambda} = 2\frac{v}{c} = \frac{2}{c}\,\frac{2\pi(s/2)}{P} \qquad (1.23.\text{s}1)$$

From Kepler III (76), $GMP^2 = 4\pi^2 s^3$. Solve for the period, $P = 2\pi\,(GM)^{-1/2}\,s^{3/2}$. Substitute into (s1), to find

$$3 \times 10^{-5} < \frac{(GM)^{1/2}}{c}\, s^{-1/2}$$

Solve for *s,*

$$s < \left(\frac{10^5}{3}\right)^2 \frac{GM}{c^2} = 3.3 \times 10^{12}\text{ m} = 22\text{ AU} \qquad (1.23.s2)$$

Separation *s* must be small so velocities are high enough to get the required $\Delta\lambda/\lambda$.

**Imaging**

Require the angular separation of the images at greatest separation to be $\Delta\theta \geq 1.45 \times 10^{-5}$ rad (=3″). Since $\Delta\theta = s/D$,

$$s > 1.45 \times 10^{-5}\, D \qquad (1.23.s3)$$

The separation *s* must be large so the images are resolved. The requirement is most severe at large *D*. At the largest separation allowed by spectroscopy (s2), the maximum distance for imaging is, equating (s1) and (s2),

$$3.3 \times 10^{12} = 1.45 \times 10^{-5}\, D$$

$$\Rightarrow \quad D = 2.2 \times 10^{17}\text{ m} = 23\text{ LY}$$

Since the nearest stars are about 4 LY distant, the range where it might be observed as both a visual and spectroscopic binary is only 4 to 23 LY. The latter distance is attained only if the separation is at the maximum value allowed by spectroscopy.

Optical interferometry with resolution $10^{-3}$ arc second (1 mas), increases this distance to 23 000 LY, but since interferometry requires bright stars, the distance is much more limited. See Armstrong et al. AJ 104, 2217 (1992) for an example (ϕ Cygni).

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## Problem 1.24 – Doppler shifts – 1st and 2nd order.

**(a) What is range of fractional frequency shift around orbit, for Fig. 6?**

Data: Circular orbit, observer in plane of orbit.

**Star 1:** Max. frequency is when $m_1$ approaches observer, at time $t_1$. Minimum frequency is at $t_3$. From (2)

$$\frac{\nu_{max} - \nu_0}{\nu_0} = -\frac{v_r(t_1)}{c}\,; \quad \frac{\nu_{min} - \nu_0}{\nu_0} = -\frac{v_r(t_3)}{c}$$

Thus,

$$\frac{\nu_{max} - \nu_{min}}{\nu_0} = -\frac{v_r(t_1) - v_r(t_3)}{c}$$

Noting the 30 km/s speed of Star 1 in the diagram and the 50 km/s speed of the barycenter, the radial velocity curve in the figure yields

➡ $$\left(\frac{\Delta\nu}{\nu}\right)_1 = -\frac{(20-80)\times 10^3}{3\times 10^8} = 2.0\times 10^{-4} \qquad (1.24.s1)$$

**Star 2:** Similarly,

➡ $$\left(\frac{\Delta\nu}{\nu}\right)_2 = -\frac{(-40\ -140)\times 10^3}{3\times 10^8} = 6.0\times 10^{-4} \qquad (1.24.s2)$$

**(b)Second order Doppler for Star 2**

Apply (7.40) to Star 2 to find fractional second order frequency shift during its transverse motion,

$$\nu = \left(1-\frac{v^2}{c^2}\right)^{1/2}\nu_0 = \left[1-\left(\frac{90\times 10^3}{3\times\ 10^8}\right)^2\right]^{1/2}\nu_0 \approx 1-\frac{1}{2}(3\times 10^{-4})^2\nu_0$$

➡ $$\left(\frac{\nu-\nu_0}{\nu_0}\right)_{\rm rel} = -4.5\times 10^{-8} \qquad (1.24.s3)$$

This is minuscule compared to the normal Doppler shifts from (a).

**(c) Inclination required so 1/2 the reduced normal Doppler range of Star 2 (s2) matches the transverse relativistic shift (s3)**

At inclinition angle $i$, the maximum radial velocity shift is reduced by the factor sin $i$ (Fig. 3). Thus, from (s2) and (s3),

$$\frac{1}{2}\left(\frac{\Delta\nu}{\nu}\right)_2 \sin i = \left|\left(\frac{\nu-\nu_0}{\nu_0}\right)_{\rm rel}\right|$$

$$i \approx \sin i = \frac{4.5\times 10^{-8}}{3\times 10^{-4}} = 1.5\times 10^{-4}\,\text{rad}$$

The approximation is sufficiently precise for us to write,

➡ $i = 31''$

The orbit must be almost exactly in the plane of the sky, within 31″, to sufficiently reduce the normal Doppler effect to that of the second order. This is highly improbable.

One could ask students to explicitly calculate this latter probability. This would involve finding the solid angle occupied by the normals to the allowed orbital planes and relating it to 4π>

Note: In this question and solution, we sidestep the role of the 50 km/s system recession in masking the second order effect.

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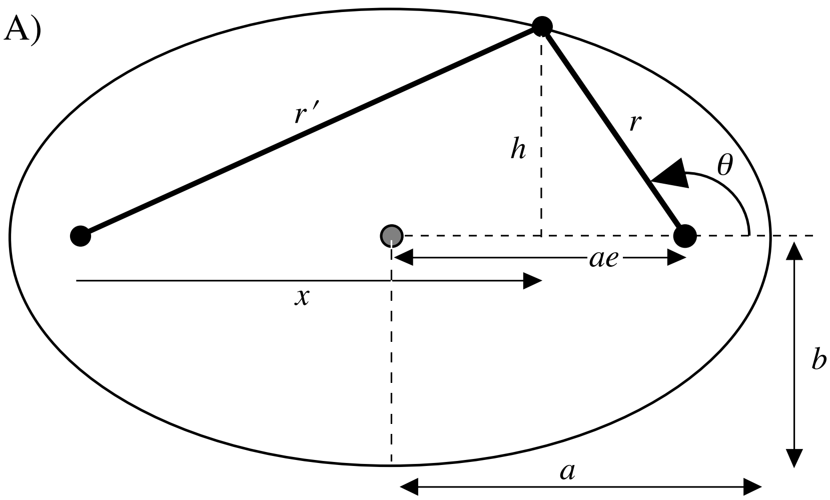
## Problem 1.31 – Ellipse. Prove sum of two radii equals twice semi-major axis.

Sketch A is a simplified version of Fig., 8a.

(1.31A)




Prove that

$$r + r' = 2a \quad (1.31.s1)$$

where the radial coordinate is defined as

$$r(\theta) = a\,\frac{1 - e^2}{1 + e\cos\theta} \quad (1.31.s2)$$

Find $r'$ in terms of ellipse parameters and $r$. From Sketch A,

$$r'^2 = h^2 + x^2 \quad (1.31.s3)$$

where $h = r\sin\theta$ and $x = 2\,ae + r\cos\theta$. Substitute into (s3) to find

$$r'^2 = r^2 + 4aer\cos\theta + 4a^2e^2 \quad (1.31.s4)$$

Now, from the analytical description of an ellipse (s2),

$$e\cos\theta = \frac{a(1-e^2)}{r} - 1 \quad (1.31.s5)$$

Substitute into (s4) to find that $r'^2 = (r - 2a)^2$, or

➡ $r' = \pm(r - 2a)$

The negative root yields the required expression.

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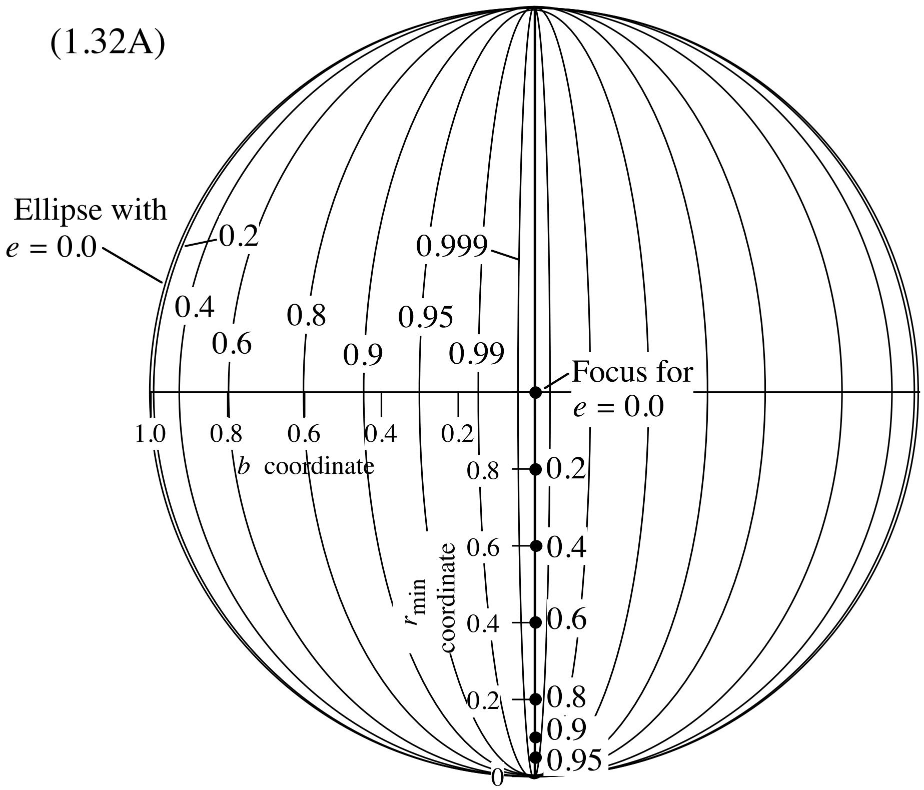
## Problem 1.32 – Shape and focus location of ellipse




**(a) Plot ellipse shape for several eccentricities, using common major axis.**

From (12), $b/a = (1 - e^2)^{1/2}$. We set $a = 1$. Tabulate values in table and plot; see Sketch A.

**(b) Tabulate and plot the positions of the focus distance $r_{min}$ (5).**

The focus distance (5) from one end of the ellipse is $r_{min} = a(1 - e)$. Tabulate and plot positions on the figure.

Eccentricities don't have much effect on appearance until $e \gtrsim 0.4$. In the regime $e \lesssim 0.4$, the focus position is a better indicator of eccentricity than the ration b/a. But, in viewing an ellipse, the focus is not visible!

Table 1.32.s1 *Ellipse parameters*

| Eccentricity $e$ | $b/a$ | $r_{min}$ [a] |
|---|---|---|
| 0.0 | 1.0 | 1.0 |
| 0.2 | 0.98 | 0.80 |
| 0.4 | 0.93 | 0.60 |
| 0.6 | 0.80 | 0.40 |
| 0.8 | 0.60 | 0.20 |
| 0.9 | 0.44 | 0.10 |
| 0.95 | 0.312 | 0.05 |
| 0.99 | 0.141 | 0.01 |
| 0.999 | 0.045 | 0.001 |
| 1.0 | 0.0 | 0.0 |

[a]For $a = 1$

## Problem 1.33 – Radial velocity curves for Phi Cygni (Fig. 7)

**(a) Ratio of masses**

This ratio is the inverse of the ratio of amplitudes in the figure relative to the barycenter velocity (solid horizontal line).

The value from the figure is about unity; the literature value of the mass ratio is: $\mathbf{m_1/m_2 = 1.04}$
Refs: Rach & Herbig ApJ 133 143 (1961). Armstrong et al. AJ 104, 2217 (1992).

**(b) Period of orbit**

Measure length of full cycle. The published value is ***P* = 434.1 d**

**(c) Radial velocity of barycenter**

The solid line indicates the barycenter is receding at **5.0 km/s**

**(d) Fractional frequency shift due to barycenter motion and "resolution" to detect it**

$$\frac{\Delta \nu}{\nu} = -\frac{v_r}{c} = -\frac{5.0 \times 10^3}{3 \times 10^8} = -1.7 \times 10^{-5}$$

Since $\Delta\lambda/\lambda = -\Delta\nu/\nu$, the required resolution is

➡ $$\left|\frac{\lambda}{\Delta\lambda}\right| = \left|\frac{1}{-1.7 \times 10^{-5}}\right| = 6 \times 10^4$$

**(e) Actual speed of barycenter (not just radial component)**

These data **do not yield the actual value**. Neither do they yield of individual masses.

**(f) Do the data points indicate the presence of an eclipse?**

One can not tell from the data of Fig. 7. An eclipse would be expected at radial velocity zero (relative to barycenter) and there are no data points in these regions, which is not surprising because the spectral lines from the two stars must merge as they approach a common wavelength.

The two stars have been resolved with optical interferometry and their separations range from 5 to 21 milliarcsec (mas), so they do not eclipse. At the distance of 234 LY, the 5 mas corresponds to 23 AU.

**(g) Explain the similar appearance of the two light curves.**

Momentum conservation dictates the ratios of radial velocities relative to the barycenter always be in the same ratio,

$$m_1 v_{r,1} + m_2 v_{r,2} = 0$$

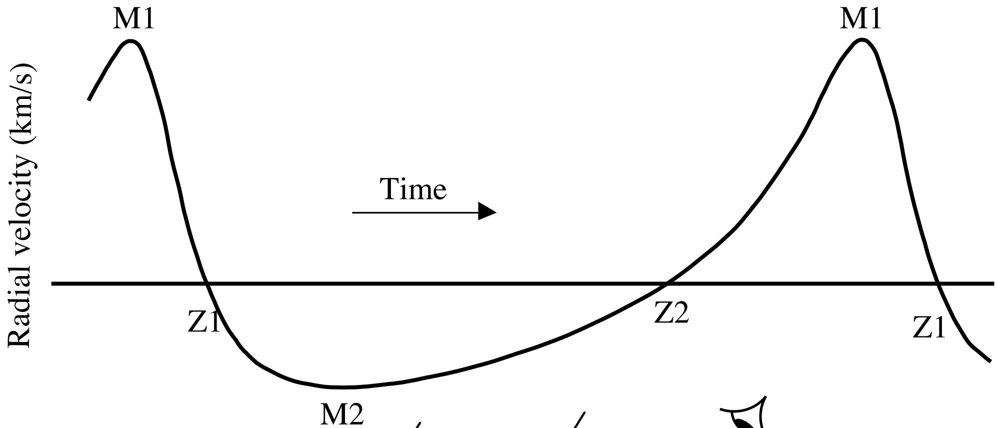
$$\frac{v_{r,2}}{v_{r,1}} = \frac{m_1}{m_2}$$

**(h) About when are stars closest together.**

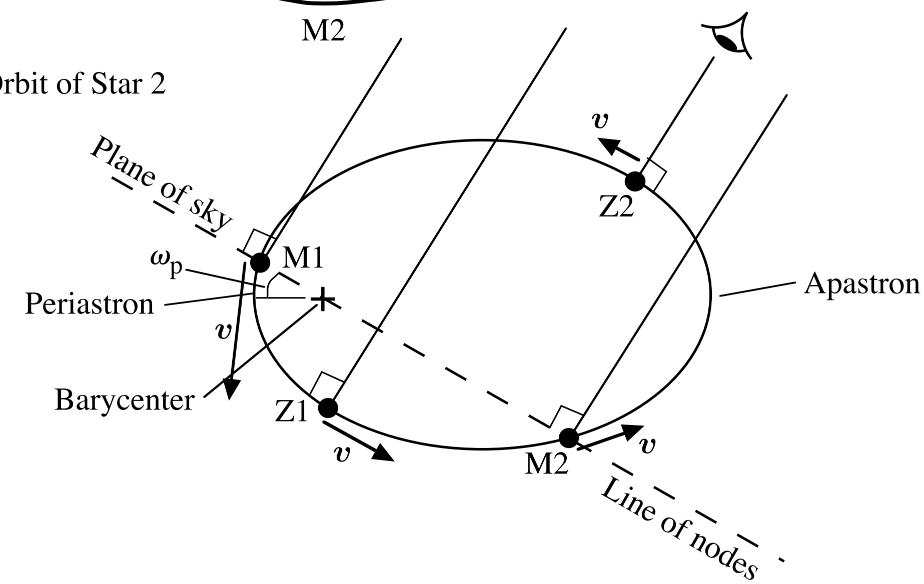
Stars have the maximum speed and greatest acceleration (force is greatest) at the closest point of approach. The radial components of these quantities depend on the view direction. If one is viewing directly along the major axis, the radial component of the velocity will be zero at closest approach with a large rate of change (steep slope or large radial acceleration). If viewing along the minor axis, it would be at maximum with steep rise and fall (zero but rapidly changing radial acceleration). The greatest radial speed and steepest slope are at and just after the first peak. The slope is markedly steeper on the late (right) side of the peak than on the earlier (left) side. Thus the point of closest approach must be **just after the M1 peak**.

**(i) Find direction of observer (roughly).**

(1.33A) Radial velocity Star 2




(1.33B) Orbit of Star 2




Assume the observer in the plane of the orbit. Sketch A defines two zeros Z1,2 and two maxima M1,2. Sketch B shows their postulated approximate locations in a top view of the orbit. The zeros are where the line of sight (los) is normal to the track and the maxima are when the star is on the line of nodes. At these points, the gravitational force (between them) has no radial (los) component. Hence there is no radial (los) acceleration and the velocity curve will have zero slope. The line of nodes, by definition, is the intersection of the sky and orbit that passes through the orbit focus (barycenter in this case) and is normal to the view direction; see Fig. 11.

We have argued in Part h above that M1 must be just before periastron. Also the velocity at M1 is positive and hence receding. The observer must therefore be in the quadrant shown. Note also that the times between zeros are unequal: Z1→Z2 > Z2→Z1. Thus periastron must be in the

Z2→Z1 interval because Kepler II tells us the star velocity is greatest at periastron and least at apastron.

**(j) Longitude of periastron**

The longitude of periastron is defined as the angle from the line of nodes to the periastron measured in the orbital plane in the direction of the receding object; see $\omega_p$ in Sketch B. (The photographic and spectroscopic definitions differ by 180 deg; we quote the latter.) If M1 is to be just prior to periastron as shown, the angle must be on the order of 30° (±15°??) as drawn. The actual value is ~30° (Armstrong et al. AJ 104, 2217 (1992)).

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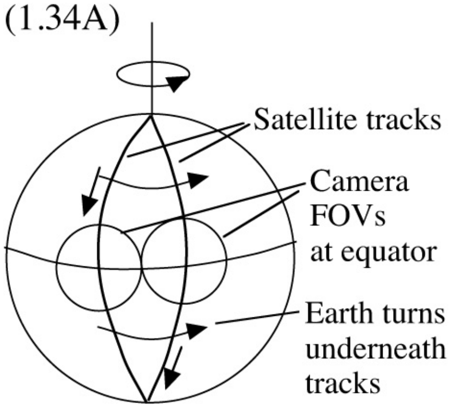
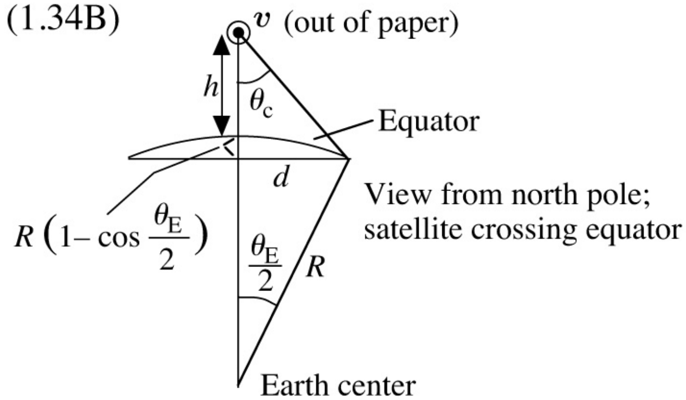
## Problem 1.34 – Satellite in orbit for photographing earth surface in 12 h.

(1.34A)




(1.34B)




**(a) What kind of orbit?**

A polar orbit (inclination 90°) would potentially do it as each bit of the earth' surface would pass under the orbit (not necessarily the satellite) each 12 hours so the scans are along the north-south meridians. (Lower inclinations could work if the field of view (FOV) were sufficiently wide.) The satellite must therefore be launched with an insertion velocity due north *in the frame of the earth center motion about the sun.* The earth's rotation about its own axis carries the earth's surface eastward, so, relative to the surface, the satellite would have to be launched to the northwest. To avoid populated areas, a launch over the ocean from the west coast, e.g. Vandenberg Air Force Base, would be required.

**(b) Altitude required to fully photograph earth surface if camera FOV is circular with angular radius, $\theta_c = 30°$**

As satellite altitude $h$ increases, its orbital period $P$ increases and the earth rotation angle in one satellite orbit increases. To cover the entire surface along the earth's equator with no overlapping at the equator, the physical length along the equator encompassed in the FOV on a given transit must equal the equatorial distance passing under the orbit in the time $P$. The earth rotation angle in the time $P$ is called $\theta_E$. These statements lead us to two expressions relating $h$ and $\theta_E$ which can be solved for the two variables.

(i) Find an expression relating $h$ and $\theta_E$ with the requirement that the time for the earth to rotate $\theta_E$ matches the satellite period.

Apply Kepler III to satellite orbit and solve for $P$: $GMP^2 = 4\pi^2(R + h)^3$

$$P = \frac{2\pi}{(GM)^{1/2}}(R + h)^{3/2} = k\,(R + h)^{3/2} \qquad (1.34.\text{s}1)$$

where $k = 2.285 \times 10^{-11}$ (SI units). The earth rotation angle during a satellite orbit is

$$\theta_E = \frac{P}{T}\,2\pi \qquad (1.34.\text{s}2)$$

where $T$ = 86164 s is the earth sidereal rotation period. Substitute (s1) into (s2) and solve for *h,*

$$h = \left(\frac{\theta_E}{k}\right)^{2/3} - R \qquad (\theta_E \text{ in radians}) \qquad (1.34.\text{s}3)$$

(ii) Find another relation between $h$ and $\theta_E$ with the constraint that the camera FOV just matches the longitudinal angle $\theta_E$ along the equator. From the geometry, Sketch B,

$$\sin\left(\frac{\theta_E}{2}\right) = \frac{d}{R} \qquad (1.34.\text{s}4)$$

$$\tan\theta_c = \frac{d}{h + R\left(1 - \cos\frac{\theta_E}{2}\right)} \qquad (1.34.\text{s}5)$$

Eliminate $d$ from (s4) and (s5) and solve for $h$

$$h = R\left[\frac{\sin(\theta_E/2)}{\tan\theta_c} + \cos\frac{\theta_E}{2} - 1\right] \qquad (1.34.\text{s}6)$$

(iii) Solve, by trial and error, (s3) and (s6) for $h$ and $\theta_E$. Adopt trial values of $\theta_E$ to find that which gives the same value for $h$ from the two expressions. (I programmed my calculator to calculate $h_1 - h_2$ and searched for a zero result.) The result is, for $R$ = 6400 km and $\theta_c = 30°$,

➡ $\theta_E = 40.35°$; $h$ = 3430 km

From (s2), the orbital period of the satellite is 2.68 h which is intermediate between low earth orbit (~1.5 h) and a synchronous orbit (~24 h).

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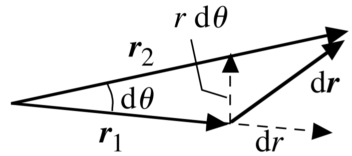
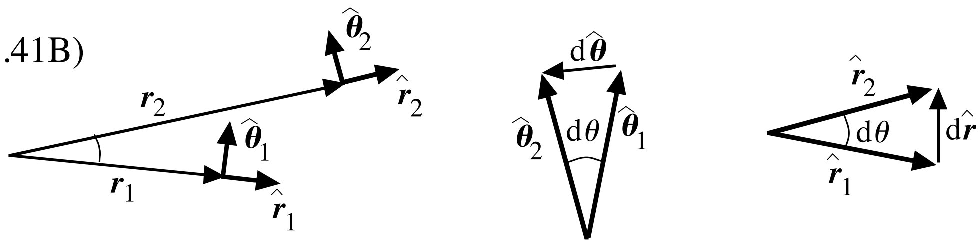
## Problem 1.41 – Acceleration in polar coordinates

(1.41A)




(1.41B)




From Sketch A,

$$d\mathbf{r} = r\, d\theta\ \hat{\boldsymbol{\theta}} + dr\, \hat{\mathbf{r}}$$

where the hatted sybols are unit vectors. Take the time derivative, setting terms multiplied by a differential quantity ($d\theta$ or $dr$) to zero,

$$\frac{d\mathbf{r}}{dt} = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{dr}{dt}\hat{\mathbf{r}}$$

Take second derivative,

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + r\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} + \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr\,d\hat{\mathbf{r}}}{dt\,dt} \qquad (1.41.s1)$$

Now evaluate graphically the quantities $d\hat{\boldsymbol{\theta}}$ and $d\hat{\mathbf{r}}$ (Sketch B),

$$d\hat{\boldsymbol{\theta}} = |\hat{\boldsymbol{\theta}}|\, d\theta\,(-\hat{\mathbf{r}}) = -\, d\theta\ \hat{\mathbf{r}}$$

since the magnitude $|\hat{\boldsymbol{\theta}}|$ is unity. Take the time derivative,

$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\ \hat{\mathbf{r}} \qquad (1.41.s2)$$

Similarly,

$$d\hat{\mathbf{r}} = |\hat{\mathbf{r}}|\, d\theta\ \hat{\boldsymbol{\theta}} = d\theta\ \hat{\boldsymbol{\theta}}$$

so

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \qquad (1.41.s3)$$

Substitute (s2) and (s3) into (s1), collect terms in $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, and let $\omega = d\theta/dt$, the angular velocity,

➡ $$\frac{d^2\boldsymbol{r}}{dt^2} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{\boldsymbol{r}} + \left(r\frac{d^2\theta}{dt^2} + 2\omega\frac{dr}{dt}\right)\hat{\boldsymbol{\theta}}$$

The left parenthesis contains the radial component of the acceleration while the right parenthesis contains the azimuthal component. The expressions are in agreement with (16) and (17).

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## Problem 1.42 –Total energy and KIII for circular orbits.

**(a) Derive Kepler III for circular orbit.**

$$F_r = ma_r \qquad \text{(Newton 2nd law)}$$

$$-\frac{GMm}{r^2} = m(-\omega^2 r) \qquad \text{(radial eq. of motion)} \qquad (1.42.s1)$$

Rearrange terms and use $\omega \equiv 2\pi/P$, where $P$ is the orbital period,

➡ $$GMP^2 = 4\pi^2 r^3 \qquad \text{(KIII)}$$

which agrees with the more general expresswion (45) for $a = r$.

**(b) Find total energy $E_T$ of body in circular orbit**

Sum kinetic and potential energies,

$$E_T = \frac{1}{2}mv^2 - \frac{GMm}{r} \qquad (1.42.s2)$$

From (s1) and $\omega = v/r$, one finds $v^2 = GM/r$ which, when used in (s2), gives

➡ $$E_T = -\frac{1}{2}\frac{GMm}{r}$$

which agrees with the more general expression (52), again for $a = r$.

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## Problem 1.43 – Planets in elliptical orbits

Two planets of masses $m_1$ and $m_2$ orbiting massive object $M$ with radii $r_2 = 9\ r_1$ and identical elliptical orbital shapes.

**(a) Can planets have different angular momentum, and if so what is condition for equality?**

From (36),

$$r(\theta) = \frac{J^2}{G\,M\,m^2}\,\frac{1}{1 + e\cos\theta}$$

In our case, the shapes are identical. Thus the eccentricity is the same for both cases. Thus we have $J^2 \propto rm^2$, so we see that J can indeed differ; it is proportional to the square of the planet mass. For same angular momenta, we would require $r_1 m_1^2 = r_2 m_2^2$ and thus, for our case,

➡ $$m_2 = \sqrt{\frac{r_1}{r_2}}\; m_1 = \frac{1}{3} m_1$$

**(b) Two objects of masses $m_1$ and $m_2$ in same orbit. What are their relative speeds?**

There are three approaches.

(i) The speeds are the same. Gravitational acceleration is independent of mass. Thus if they start out with equal velocities, the changes of velocity will be identical and hence so will the velocities at later times. Think of Galileo and the Leaning Tower of Pisa.

(ii) Kepler III (45) tells us that $P^2 \propto a^3$ with no dependence on mass. Thus with equal semi major axes, the time to complete an orbit is the same for the two masses.

(iii) From Newton's laws we found (37) that

➡ $$J = \left(\frac{G M}{a}\right)^{1/2} m\, b \;\propto\; m \qquad \text{(Angular momentum of } m;\ M >> m) \qquad (1.43.s1)$$

for fixed orbital shape and size (i.e. fixed $a$ and b.) We also know that, at periastron for example, $J = mvr$ for each mass. To satisfy (s1), $J \propto m$, the speed $v$ must be the same for both masses.

**(c) Increase central mass $M$ a factor of 2. By what factor must $v$ change so the orbital track does not change.**

From (s1), $J \propto M^{1/2}$. At periastron, $J = mvr$. For fixed $r$ and $m$, these two expressions tell us that $v \propto M^{1/2}$. In our case then, the speed must be increased a factor of $2^{1/2}$.

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## Problem 1.44. Pluto energy, angular momentum, and period

Parameters:

$a = 39.44$ AU $= 5.92 \times 10^{12}$ m

$e = 0.250$ giving, from (12), $b/a = (1-e^2)^{1/2} = 0.968$

$m = 0.17\, m_E = 1.0 \times 10^{24}$ kg.

$M_\odot = 2 \times 10^{30}$ kg

**(a) Total energy of planet.**

From (52)

➡ $$E_t = -\frac{G M m}{2a} = 1.2 \times 10^{31}\ \text{J}$$

**(b) Angular momentum with respect to sun**

From (37),

$$J = \left(\frac{G\,M}{a}\right)^{1/2} m\,b = (GM)^{1/2}\, m\, a^{1/2}\, b/a$$

➡ $$= 2.7 \times 10^{40}\ \text{kg m}^2\ \text{s}^{-1}$$

**(c) Orbital period**

From (43),

$$\frac{J}{2m} = \frac{\pi a\, b}{P} = \frac{\pi a^2 (1-e^2)^{1/2}}{P}$$

Solve for $P$

$$P = \frac{2m}{J}\, \pi a^2 (1-e^2)^{1/2}$$

➡ $$= 7.9 \times 10^2\ \text{s} = 249\ \text{yr}$$

The actual value is 247.7 yr.

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## Problem 1.45 – Elliptical satellite orbit

A 200 kg satellite is in elliptical orbit with perigee at 400 km altitude and apogee at geosynchronous altitude $r_s$.

**(a) Geosynchronous altitude**

Require period of a circular orbit to be 1.0 sidereal day = 86164 s

From Newton and $\omega = 2\pi/P$,

$$-\frac{GMm}{r^2} = m(-\omega^2 r)$$

$$r_s = \left(\frac{GMP^2}{4\pi^2}\right)^{1/3} \qquad (1.45.s1)$$

➡ $$= 4.16 \times 10^7\ \text{m} = 6.5\ R_E$$

**(b) Eccentricity of orbit**

From (5) and (6),

$$r_p = a\,(1-e);\ \ r_a = a\,(1+e) \qquad (1.45.s2)$$

Solve for eccentricity $e$,

$$\Rightarrow \quad e = \frac{r_p - r_a}{r_p + r_a} = 0.719 \qquad (1.45.s3)$$

where we used $r_p = R_E + 400$ km $= 6.8 \times 10^6$ m and $r_a = r_s = 4.2 \times 10^7$ m (from part (a)).

**(c) Circularize the orbit at $r = r_s$**

Method: give forward boost when satellite at apogee. It will return to same point but the perigee will be lifted. With just the proper amount of boost, it will become circular. The new greater semi major axis represents the greater energy of the orbit, given it by the rocket boost.

Energy required:

From (52), write the final and initial energies of the orbit,

$$E_f = -\frac{G\,M\,m}{2r_s} \quad ;$$

$$E_i = -\frac{G\,M\,m}{2a_i} = -\frac{G\,M\,m}{2r_s/(1+e)}$$

where the initial semi-major axis $a_i = a = r_s/(1+e)$, from (s2). The net energy required for the boost is thus

$$E_f - E_i = +\frac{G\,M\,m}{2r_s}\,e = 7.0 \times 10^8 \text{ J}$$

where $M = 6 \times 10^{24}$ kg is the earth mass, $m = 200$ kg is the satellite mass, $r_s$ from (s1) is the synchronous radius, and $e$ is from (s3).

**(d) Decay of orbit**

Impulses at perigee will lower the apogee without changing the perigee radius. The orbit thus becomes less and less eccentric until it becomes circular at $r = r_p$ (under our idealized impulse model). At this point the entire orbit is embedded in the atmosphere, so atmospheric drag (friction) takes more and more energy from the satellite, always tending to keep the orbit circular because the deepest portions of the orbit experience the greatest frictional forces. It thus spirals lower and lower into the atmosphere (and faster and faster) toward re-entry.

Note that if system angular momentum is to be conserved during circularization, the perigee must rise and the apogee decrease. See Prob. 4.54 and associated discussion in Sect. 4.5.

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## Problem 1.46 – Mass ejection by central object of binary.

Mass $m$ orbits mass $M$ with semimajor axis $a$ and eccentricity $e$. Central mass $M$ suddenly becomes $fM$ just when $m$ is at periastron. The speed of the orbiting object $m$ is maintained at that moment. Also, $M >> m$.

**(a) Find new semi-major axis, $a'$ in terms of $a, e, f$.**

Write expressions for total energy, $E_t = E_k + E_p$, before and after the change, from (52). Before the mass loss,

$$-\frac{GMm}{2a} = \frac{1}{2}mv_p^2 - \frac{GMm}{a(1-e)} \qquad \text{(Before)} \qquad (1.46.s1)$$

and immediately after the mass loss,

$$-\frac{GfMm}{2a'} = \frac{1}{2}mv_p^2 - \frac{GfMm}{a(1-e)} \qquad \text{(After)} \qquad (1.46.s2)$$

Note that we choose to write the denominator of the rightmost term in terms of $a$ and $e$ rather than $a'$ and $e'$ because the latter is unknown and because it would require we know whether, in the new orbit, this orbital position is the periastron or apastron. Subtract (s2) from (s1) and solve for $a'$,

➡ $$a' = a\,\frac{f(1-e)}{2f-1-e} \qquad (1.46.s3)$$

As a check, for $f = 1$, $a' = a$ as it should. For a sudden increase to a huge mass, $f >> 1$, (s3) yields $a' = a(1\text{–e})/2$. This is 1/2 the original periastron distance, as expected because the new periastron would be right at the large mass.

**(b) Find final eccentricity**

Two cases: (*i*) final state periastron stays periastron and (*ii*) periastron becomes apastron.

(*i*) Periastron remains periastron

Equate the periastron distances before and after, from (5),

$$a(1-e) = a'(1-e')$$

Introduce (s3) for $a'$ and solve for $e'$

➡ $$e' = \frac{1+e-f}{f} \qquad (1.46.s4)$$

Note that limits for $f = 0$, 1, and $\infty$ give the expected results, as follows, respectively: $e' = \infty$ (representing an unbound straight line), $e' = e$ (orbit does not change), and $e' = -1$ (bound infinitely narrow orbit). The latter was mentioned in the discussion above regarding the $f >> 1$ limit. See also (*ii*) below.

(*ii*) Periastron becomes apastron.

This case occurs for large $f$ which can be covered by (s4), but we do it separately. Again, equate distances at time of mass change, from (6)

$$a(1-e) = a'(1+e')$$

Introduce (s3) for $a'$ and solve for $e'$,

➡ $$e' = \frac{f-(1+e)}{f} \qquad (1.46.s5)$$

Again limiting cases yield expected values. Note that (s5) and (s4) are the negative of each other. Thus one can use (s4) for all cases if one interprets $e' < 0$ as indicating the periastron became the apastron.

**(c) Tabulate examples, comment on trends and make sketches.**

Evaluate (s3) and (s4) for the given values of $e$ and $f$.

Table 1.46.s1 Semi major axes and eccentricities ($a'/a$)|$e'$

| $f$ \ $e$ | 0.0 | 0.0 | 0.5 | 0.5 | 0.7 | 0.7 | 0.9 | 0.9 |
|---|---|---|---|---|---|---|---|---|
| | $a'/a$ = | $e'$ = | $a'/a$ = | $e'$ = | $a'/a$ = | $e'$ = | $a'/a$ = | $e'$ = |
| 0.5 | ∞ | 1.0 | <0.0[a] | — | <0.0 | — | <0.0[a] | — |
| 0.9 | 1.125 | 0.11 | 1.5 | 0.67 | 2.7 | 0.89 | <0.0 | — |
| 1.0 | 1.0 | 0.0 | 1.0 | 0.5 | 1.0 | 0.7 | 1.0 | 0.9 |
| 1.1 | 0.92 | –0.09[b] | 0.79 | 0.36 | 0.66 | 0.55 | 0.37 | 0.73 |
| 2.0 | 0.67 | –0.5 | 0.40 | –0.25[b] | 0.26 | –0.15 | 0.095 | –0.05 |
| 100[c] | 0.5 | –0.990 | 0.25 | –0.985 | 0.15 | –0.983 | 0.05 | –0.981 |

[a] $a'/a < 0$ or ∞ signifies unbound orbit.

[b] $e' < 0$ indicates the periastron becomes the apastron.

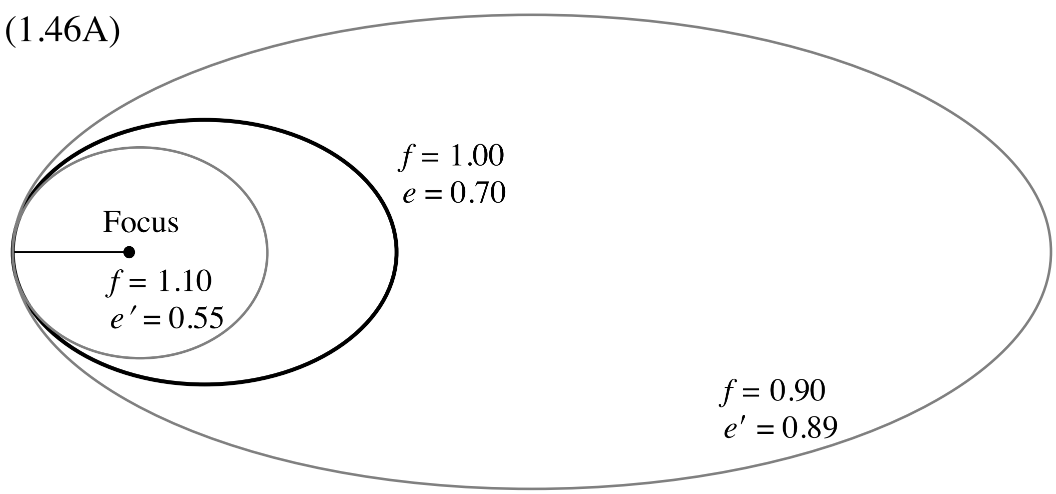
[c] At large $f$, the new semimajor axis is about half the original periastron distance because the orbit just grazes the central mass at the new periastron. Such orbits are very narrow, with $e' \approx 1$ (or –1 in table).

Compare $f = 1.0$ and 0.9. Decreasing the mass (i.e., $f$) at perisastron raises the apastron (thus increasing $a$ and $e$) until eventually the orbit becomes unbound ($e' = 1$).

Compare $f = 1.0$ to $f = 1.1$. Increasing the mass (i.e., $f$) for an eccentric orbit lowers the original apastron, reducing $a$ and reducing eccentricity, until it becomes circular ($e' = 0$). Thereafter, with further mass increase) the apastron becomes the periastron as the orbit becomes increasingly eccentric.

See sketch A of two final orbits for $f = 0.9$ and 1.1 that initially had $e = 0.7$ (dark ellipse).

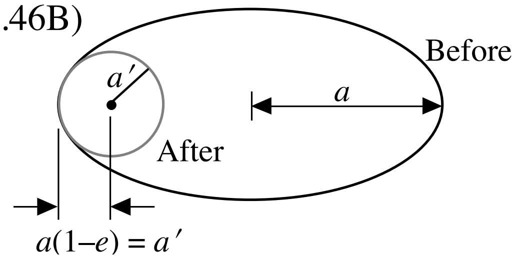
(1.46A)




**(d) Find the condition for circularization of the orbit, $f_c(e)$, and tabulate values of $f_c$ and $a'/a$ for $e$ = 0, 0.5, and 0.9, and comment.**

The requirement is that the periastron distance become $a'$; see Sketch B.

(1.46B)




$a(1-e) = a'$

Substitute $a'$ from (s3),

$$1 - e = \frac{f_c(1-e)}{2f_c - 1 - e}$$

and solve for $f_c$, the circularization factor

➡ $f_c = 1 + e$ (1.46.s6)

The factor is greater than 1, so mass must be added to the central object to circularize the orbit.

Tabulate:

*Table 1.46.s2*. Circularization mass parameter and semimajor axes.

| $e$ | 0 | 0.5 | 0.9 |
|---|---|---|---|
| $f_c$ | 1 | 1.5 | 1.9 |
| $a'/a$ | 1 | 0.5 | 0.1 |

Comment: The more eccentric the original orbit, the more mass has to be added (to the central object) to circularize it. The semi major axis decreases as mass is added.

**(e) Find condition on *f* for orbit to become unbound and tabulate**

This occurs when $f$ is reduced to the point where $a' \to \infty$, or, from (s3), when $2f - 1 - e = 0$, or

➡ $f_{u,per} = \frac{1+e}{2}$ (Unbinding condition) (1.46.s7)

where we add the subscript "per" to indicate the mass loss takes place when $m$ is at periastron.

Tabulate (Table s3).

Comment: If the original orbit is eccentric, it takes less mass decrease to unbind than if it is initially circular.

*Table 1.46.s3*. Mass parameter for unbinding of orbit

| $e$ | 0 | 0.5 | 0.9 |
|---|---|---|---|
| $f_{u,per}$ | 0.5 | 0.75 | 0.95 |
| $f_{u,ap}$ | 0.5 | 0.25 | 0.05 |

Consider the mass loss at apogee. The expressions (s1) and (s2) would have the apogee distance in the rightmost terms, or $a(1+e)$. The changed sign for $e$ carries through the algebra so that, we find

$$f_{\rm u,ap} = \frac{1-e}{2} \qquad (1.46.s8)$$

Tabulate this for the three values of $e$ in Table s3. We find that more mass loss is necessary in this case. A certain fractional mass loss, say 1/2, increases the (negative) potential energy by a factor of 2. For small negative potentials (as at apastron) this is a smaller absolute change of energy than it is for large negative potentials (as at periastron). Thus the total energy is changed less, possibly not enough to unbind the apastron case, so the central mass must be reduced to even smaller values to unbind. The effect is even more pronounced for orbits of large eccentricity, because, for a given $a$, the orbiting mass is even farther removed from the central mass at apogee.

**(f) Consider unbinding for arbitrary masses and comment on relevance to supernovae.**

Examine the expressions for total energy, $E_t = E_k + E_p$, of such a system, given in (77) and (78), for an elliptical orbit with eccentricity $e$ and equate them as in (s1) for periastron passage just before the mass loss (we assume the mass loss takes place at periastron),

$$-\frac{G\,M_T\;\mu}{2a_s} = \frac{1}{2}\,\mu\,v_{s,p}^2 - \frac{G\,M_T\;\mu}{a_s(1-e)} \qquad (1.46.s9)$$

We can treat the $M_T,\mu$ problem exactly as the $M,m$ problem solved above if we specify that the mass change parameter $f$ applies to $M_T$ rather than to only one (central) mass as before. Just after the mass loss and following (s2), we write

$$-\frac{G\,fM_T\;\mu'}{2a_s'} = \frac{1}{2}\,\mu'\,v_{s,p}^2 - \frac{G\,fM_T\;\mu'}{a_s(1-e)} \qquad (1.46.s10)$$

In this case the kinetic energy terms are not identical in the two equations because $\mu' \neq \mu$, but since $\mu'$ and $\mu$ appear in all three terms of their respective equations, they cancel and hence do not enter the algebra. Proceeding as before leads to an expression for the new semimajor axis ratio $a_s'/a_s$ identical to that for $a'/a$ (s3). The unbinding is defined as before by $a_s' \to \infty$, and this leads directly to the unbinding results (s7) and (s8).

These two equations tell us that, for a system with circular orbits, the loss of 1/2 the total system mass will result in the two stars becoming unbound. If the mass is lost in a supernova explosion, our calculation would apply if the material is ejected isotropically in the frame of the collapsing star so as not to change its velocity (via a jet effect). In addition the calculation is valid only if the ejected mass is outside the two orbits and is spherically distributed. Otherwise it would have a residual gravitational effect on the system.

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## Problem 1.47 – Express eccentricity in terms of constants of motion, $E_t$ and $J$ as given in (53).

From the definition of eccentricity (12), $b/a = (1-e^2)^{1/2}$, write eccentricity $e$ in terms of $a$ and $b$,

$$e = \left(1 - \frac{b^2}{a^2}\right)^{1/2} \qquad (1.47.s1)$$

We found that our trial solution (an ellipse) would satisfy the equation of motion only if the following condition was met (35),

$$\frac{b^2}{a^2} = \frac{J^2}{G\,M\,m^2 a} \tag{1.47.s2}$$

where we divided both sides by $a$. Invoke the expression for total energy (52), namely $E_t = -GMm/a$, and solve for $a$,

$$a = \frac{GMm}{2(-E_t)} \tag{1.47.s3}$$

Substitute (s3) into (s2), and then (s2) into (s1), to obtain

➡ $$e = \left[1 + \frac{2E_t\,J^2}{(GM)^2\,m^3}\right]^{1/2}$$

as required. Note that $E_t < 0$ for a bound system.

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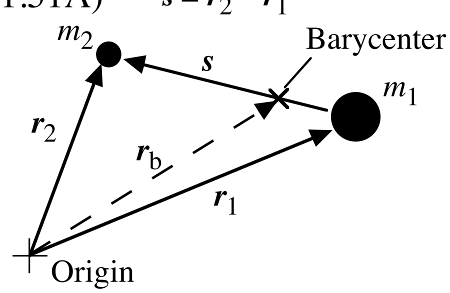
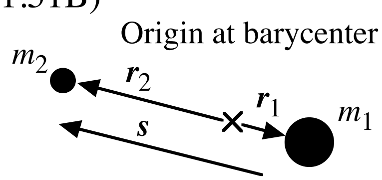
## Problem 1.51 – Relative sizes of orbits from different reference frames.

(1.51A) $\boldsymbol{s} = \boldsymbol{r}_2 - \boldsymbol{r}_1$




(1.51B)




**(a) Show expressions (57) and (58) relating position and separation vectors follow from definition of barycenter, if the origin is chosen to be at the barycenter.**

The definition of the barycenter is (Sketch A),

$$\boldsymbol{r}_b = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2}{m_1 + m_2} \tag{1.51.s1}$$

where the origin is at an arbitrary location. In our case, $\boldsymbol{r}_b = 0$, so we have

$$0 = m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 \tag{1.51.s2}$$

The separation vector is defined as

$$\boldsymbol{s} \equiv \boldsymbol{r}_2 - \boldsymbol{r}_1 \tag{1.51.s3}$$

Eliminate $\boldsymbol{r}_2$ from (s2) and (s3) to obtain

➡ $$\boldsymbol{r}_1 = -\frac{m_2}{m_1 + m_2}\,\boldsymbol{s} \qquad (1.51.s4)$$

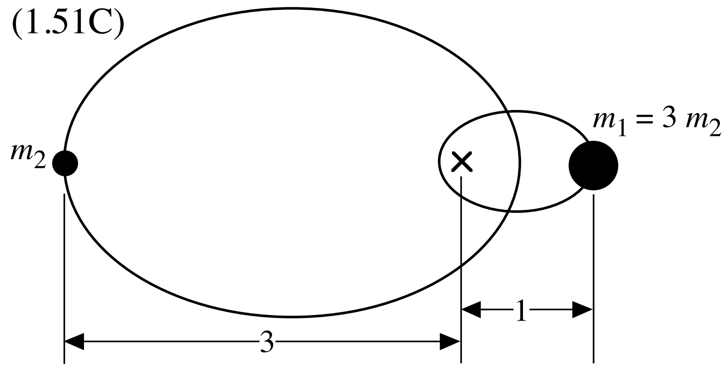
which is (57) and similarly,

➡ $$\boldsymbol{r}_2 = \frac{m_1}{m_1 + m_2}\,\boldsymbol{s} \qquad (1.51.s5)$$

which is (58).

**(b) Compare sizes of orbits**

(1.51C)




The orbits are drawn for an eccentricity $e = 0.74$ in Sketch C. Both have the same eccentricity. The masses are at a ratio of 3:1 from the barycenter at all times, by definition of the barycenter. The orbit sizes for $m_2$ and $m_1$ are thus in the ratio 3:1 in the barycenter frame. In the frame of one of the masses, the orbit size of the other mass relative to that of $m_1$ in the barycenter frame is 4:1.

(*i*) $m_2$ relative to $m_1$: 4 units.

(*ii*) $m_1$ relative to $m_2$: 4 units

(*iii*) $m_1$ relative to barycenter: 1 unit

(*iv*) $m_2$ relative to barycenter: 3 units

The requested relative sizes are thus 4:4:1:3

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## Problem 1.52. Angular momentum and energy.

**(a) Demonstrate that the total angular momentum in the barycenter frame can be written in terms of the reduced mass as $\mu s^2 \omega$. From (72),**

$$J = m_1\, r_1^2\, \omega + m_2\, r_2^2\, \omega \qquad (1.52.s1)$$

Substitute (57) and (58), given above in (1.51.s4) and (1.51.s5), into (s1) to find

➡ $$J = \frac{m_1 m_2}{m_1 + m_2}\, s^2\, \omega = \mu\, s^2 \omega \qquad (1.52.s2)$$

where we introduced the definition of the reduced mass (59), $\mu \equiv m_1\, m_2/(m_1 + m_2)$.

**(b) Repeat for the total energy. Find (77) from (79).**

The total energy in the barycenter in the usual manner is (79)

$$E_t = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{G m_1 m_2}{s} \quad (1.52.s3)$$

The definition of the relative velocity is

$$\boldsymbol{v}_s = \boldsymbol{v}_2 - \boldsymbol{v}_1 \quad (1.52.s4)$$

From the derivatives of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in terms of $\boldsymbol{s}$, (57) and (58), or from the total momentum being zero in the barycentric frame (see text), we have the velocity relations, (83) and (84),

$$\boldsymbol{v}_1 = -\frac{m_2}{m_1 + m_2} \boldsymbol{v}_s \quad (1.52.s5)$$

$$\boldsymbol{v}_2 = \frac{m_1}{m_1 + m_2} \boldsymbol{v}_s \quad (1.52.s6)$$

Substitute (s5) and (s6) in scalar form into (s3), or note that $v_s^2 \equiv \boldsymbol{v} \cdot \boldsymbol{v}$, to find

$$E_t = \frac{1}{2}\left[\frac{m_1 m_2^2}{(m_1 + m_2)^2} + \frac{m_2 m_1^2}{(m_1 + m_2)^2}\right] v_s^2 - \frac{G m_1 m_2}{s}$$

$$E_t = \frac{1}{2} \frac{m_1 m_2}{(m_1 + m_2)} v_s^2 - \frac{G m_1 m_2}{s}$$

Apply the definition of reduced mass (59), which also tells us that $m_1 m_2 = \mu M_T$, to obtain the desired result (77),

$$E_t = \frac{1}{2} \mu v_s^2 - \frac{G M_T \mu}{s}$$

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## Problem 1.53 – Total masses of binary systems

### (a) Kruger 60.

Use data in caption to Fig. 1. $P = 44.6$ yr, $a_s = 9.5$ AU. From (75), Kp III is

$$G M_T P^2 = 4\pi^2 a_s^3 \quad (1.53.s1)$$

Substitute given values for $a_s$ and $P$ in SI units and solve for $M_T$, to find that

$$M_T = \frac{4\pi^2 a_s^3}{GP^2} = 8.58 \times 10^{29}\ \text{kg} = 0.43\ M_\odot \quad (\text{`}1.53.s2)$$

This is consistent with both stars being M stars. An M5V star has mass ~0.2 $M_\odot$ (Allen's Astrophysical Quantities, Fourth Edition, Ed. Cox).

**(b) Find sum of the masses $M_T$ (and the individual masses) of the binary in Fig. 6.**

Given from figure: $P$ = 30 d, $v_2$ = 90 km/s, $v_1$ = 30 km/s with circular orbits.
The velocities can be expressed in terms of the orbit radii,

$$v_1 = \frac{2\pi r_1}{P}\ ;\ \ v_2 = \frac{2\pi r_2}{P} \qquad (1.53.s3)$$

The relative semi-major axis is, from the figure,

$$a_s = r_1 + r_2 = \frac{P}{2\pi}(v_1 + v_2) \qquad (1.53.s4)$$

where we eliminated the radii with (s3). Evaluate (s4) for $a_s$ to find

$$a_s = 4.95 \times 10^{10}\ \text{m} = 0.33\ \text{AU}$$

We thus have the parameters needed to substitute into (s2). Doing so yields

➡ $$M_T = 1.07 \times 10^{31}\ \text{kg} = 5.3\ M_\odot$$

Find $m_1$ and $m_2$. From barycenter definition, $m_1 r_1 = m_2 r_2$, so $m_1/m_2 = 3$, and $M_T/m_2 = 4$, so

➡ $$m_1 = \frac{3}{4} M_T = 8.0 \times 10^{30}\ \text{kg} = 4.0\ M_\odot$$

➡ $$m_2 = \frac{1}{4} M_T = 1.7 \times 10^{30}\ \text{kg} = 1.33\ M_\odot$$

where $M_T = m_1 + m_2$.

**(c) Find $M_T$ for α Cen (Fig. 2)**

Given: the major axis is foreshortened 2/3 due to its projection on the sky. From the caption, we find the distance to the system is $D$ = 4.4 LY and the binary period $P$ = 79.9 years.

First, find $a_s$. On the figure, measure the angle subtended by the projected major axis (periastron to apastron, using the angular scale on the figure); it is about $\Delta\theta = 22.5''$. Multiply by 3/2 to remove the projection effect and by $D$ get the physical major axis $2a_s$. Then divide by 2 to obtain, finally

$$a_s = 3.43 \times 10^{12}\ \text{m} = 23\ \text{AU}.$$

where we took care to use SI units for the angle (rad) and distance (m).

Substitute $a_s$ and $P$ into (s2) to obtain

➡ $$M_T = 3.75 \times 10^{30}\ \text{kg} = 1.9\ M_\odot$$

This result may be accurate to ~5%. It is therefore roughly consistent with the stellar types for G2V and K1V stars. The masses in Table 4.2 for G0V and K0V total 1.84 $M_\odot$ and the total for G5V and K5V is 1.59 (Cox AQ4). We might therefore expect $M_T \approx 1.75$ (by interpolation).

The actual masses are 1.09 $M_\odot$ and 0.90 $M_\odot$ to a precision of about 0.01 $M_\odot$ according to Demarque et al. Ap J 300, 773 (1986). The total, 1.99 $M_\odot$, is also roughly consistent with our result. These masses do not agree with those tabulated for the stated stellar types, probably because the masses for a specific spectral type depend on the evolutionary state of the star; see the Demarque paper.

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## Problem 1.54 – Moon's orbit

Given:

Orbit: $e = 0.0549$, $P_{\rm sidereal} = 27.32$ d $= 2.36 \times 10^6$ s

Masses:

$m_M = 1/(81.3)\, m_E$ where $m_E = 5.974 \times 10^{24}$ kg

$= 7.353 \times 10^{22}$ kg

Physical radii:

$R_{\rm E,mean} = 6371$ km

$R_{\rm M,mean} = 1738$ km

**(a) Find percentage difference from unity:**

(*i*) Ratio of major and minor axis compared to unity

From (12),

$$\Rightarrow \quad \frac{a}{b} = \frac{1}{(1 - e^2)^{1/2}} = 1.0015$$

so the percentage difference is 0.15%.

(*ii*) Ratio of apogee and perigee distances compared to unity

From (5) and (6)

$$\Rightarrow \quad \frac{r_{\rm max}}{r_{\rm min}} = \frac{a(1+e)}{a(1-e)} = 1.116$$

so the percentage difference is 11.6%.

Comment: The orbit is very close to circular, but the focus is substantially displaced.

**(b) Which orbit do these ratios describe?**

They apply to both the orbit about the barycenter and the relative orbit about the earth center. The elliptical orbits have different sizes but they have the same shapes. Hence the ratios based on shape alone are the same for both.

**(c) Absolute values of apogee and perigee distances**

Find relative semi-major axes from Kp III (76). From (76), we have

$$a_s = \left(\frac{GM_T P^2}{4\pi^2}\right)^{1/3} = 3.848 \times 10^8 \text{ m} = 60.40\, R_E \qquad (1.54.s1)$$

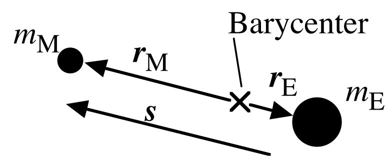
where $M_T = m_E + m_M = m_E(1 + 0.01230)$. The *relative* apogee and perigee distances are thus

➡
$$r_{rel,a} = a_s(1+e) = 4.059 \times 10^8 \text{ m} = 63.71\, R_E \qquad (1.54.s2)$$
$$r_{rel,p} = a_s(1-e) = 3.637 \times 10^8 \text{ m} = 57.09\, R_E$$

These distances are between the centers of the earth and moon. They are relative to the earth (or moon) center because Kepler's third law contains the *relative* semi-major axis $a_s$, not that of the orbit of one or the other body about the barycenter.

**(d) Distance earth center to barycenter at apogee and perigee**

(1.54A)




From definition of barycenter we have (58)

$$r_E = -\frac{m_M}{M_T}\, s$$

where $r_E$ is the varying distance of the earth from the barycenter and $s$ is the relative radial coordinate (distance between moon and earth centers).

At apogee, $s = r_{rel,a} = 63.71\, R_E$ where the latter value is from (s2). Hence

➡
$$|r_{E,a}| = \frac{m_M}{M_T} r_{rel,a} = 0.77\, R_E$$

This is less than $r_E$ but it is a substantial fraction of it. The barycenter is within the earth.

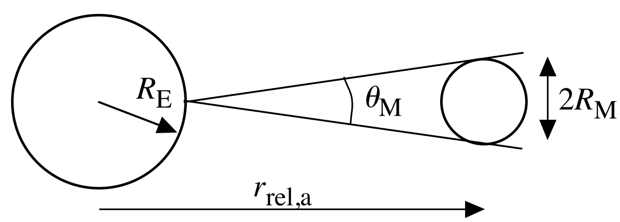
At perigee, $s = r_{p,rel} = 57.09\, R_E$. Hence

➡
$$|r_{E,p}| = \frac{m_M}{M_T} r_{rel,p} = 0.67\, R_E$$

The two differ by ~640 km, a substantial distance.

**(e) Angular size of moon and relation to solar eclipses**

(1.54B)




The full angle subtended by the moon from the closest point on the earth is, at apogee (Sketch B), invoking again (s2),

➡ $$\theta_{\rm M,a} = \frac{2R_{\rm M}}{r_{\rm rel,a} - R_{\rm E}} = 8.701 \times 10^{-3}\ \text{rad} = 1795''$$

At perigee,

➡ $$\theta_{\rm M,p} = \frac{2R_{\rm M}}{r_{\rm rel,p} - R_{\rm E}} = 9.729 \times 10^{-3}\ \text{rad} = 2007''$$

Compare to the sun's diameter (at mean earth distance) $2 \times 960'' = 1920''$. The moon can totally cover the sun at the lunar perigee but not at the lunar apogee. Total solar eclipses can not occur at apogee.

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## Problem 1.61 – Understanding the mass function

**(a) Write the mass function for star #2, similar to that for star #1 (88).**

We interchange the subscripts 1 and 2 in (88) to obtain

$$\frac{m_1^3 \sin^3 i}{(m_1 + m_2)^2} = \frac{4\pi^2}{G P^2}(a_2 \sin i)^3$$

which is also in the text (90).

**(b) Why is it appropriate to call this the mass function for star #2.**

Because the measured quantity, $a_2 \sin i$ is that of the orbit of star #2.

**(c) Show that the measured value of $f_2$ is a lower limit on $m_1$.**

Examine the equivalent of (89)

$$f_2 \equiv \frac{m_1^3 \sin^3 i}{(m_1 + m_2)^2}$$

Hold $f_2$, the measured quantity, constant. The value of $m_1$ depends on the value of sin $i$ and of $m_2$. By inspection, the minimum value of $m_1$ occurs when sin $i$ is maximum (sin $i$ = 1) and when the denominator is at minimum, i.e., when $m_2 = 0$. Hence

$$f_2 = \frac{m_{1,\rm min}^3 \times 1}{(m_{1,\rm min} + 0)^2} = m_{1,\rm min}$$

Hence,

➡ $$m_1 > f_2$$

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## Problem 1.62 – Masses of stars in Fig. 6 binary for $i = 30°$

Use lower curve (data) only, not the upper sketches.

**(a) By inspection, what can one say about the eccentricity of the orbit.**

To the extent that it can be shown to be a pure sine curve, the orbit must be a circular. From the geometry of Fig. 3a and (92), the radial component of velocity for a circular orbit is

$$v_r(t) = v_1 \sin i \sin \omega t$$

which shows that the temporal variation is strictly sinusoidal.

**(b) Evaluate the mass functions for $m_1$ and $m_2$. Are they consistent with the masses given in (98).**

From (98), $m_1 = 4.0\ M_\odot$, $m_2 = 1.3\ M_\odot$. From the plot, $P = 30$ d, $v_1 \sin i = 30 \times 10^3$ m/s, $v_2 \sin i = 90 \times 10^3$ m/s.

The mass function for $m_1$ is given in (98), repeated with more precision here,

$$f_1 = \frac{P}{2\pi G}(v_1 \sin i)^3 = 1.669 \times 10^{29}\ \text{kg} = 0.0839\, M_\odot \qquad (1.62.\text{s1})$$

This is less than the $m_2 = 1.3\ M_\odot$ quoted in (98), so it is consistent with it. (We aergued in the text that the mass function gives a lower limit to the partner star mass.)

The mass function for $m_2$ is

$$f_2 = \frac{P}{2\pi G}(v_2 \sin i)^3 = 4.506 \times 10^{30}\ \text{kg} = 2.266\, M_\odot \qquad (1.62.\text{s2})$$

This is less than the $m_1 = 4\ M_\odot$ quoted in (98), so it too is consistent with it.

**(c) Constraints on the masses if $i < 30°$ (from absence of eclipse).**

We have

$$f_1 = \frac{m_2^3 \sin^3 i}{(m_1 + m_2)^2} \qquad (1.62.\text{s3}$$

and

$$f_2 = \frac{m_1^3 \sin^3 i}{(m_1 + m_2)^2} \qquad (1.62.\text{s4})$$

Consider $f_1$. The smallest value of $m_2$ occurs when $\sin i$ is at its largest allowed value, 0.5, and when $m_1 << m_2$. Thus, invoking the value for $f_1$ (s1),

$$0.0839 = \frac{m_2^3\, 0.5^3}{(\ m_2)^2}$$

giving the minimum $m_2$ as $8\,f_1$, and similarly the minimum $m_1$ as $8\,f_2$,

➡ $$m_2 > 8 \times 0.0839\ M_\odot = 0.67\ M_\odot$$

$$m_1 > 8 \times 2.266\ M_\odot = 18.1\ M_\odot$$

These are the requested new limits. They are much more stringent.

**(d) If the inclination is exactly 30°, what are the masses.**

Solve the mass functions (s3) and (s4) for the two masses, given $i = 30°$.
Take the ratio of the functions to find,

$$\frac{m_1^3}{m_2^3} = \frac{2.266\ M_\odot}{0.0839\ M_\odot} = 27$$

and hence $m_1 = 3\ m_2$. Plug this into (s3), and solve for $m_2$,

$$0.0839\ M_\odot = \frac{m_2^3\,(0.5)^3}{(4\ m_2)^2}$$

➡ $$m_2 = 10.74\ M_\odot$$

while $m_1$ is 3 times this,

➡ $$m_1 = 3\ m_2 = 32.2\ M_\odot$$

Inclination makes a big difference!

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## Problem 1.63. A0620–00 a neutron star or a black hole.

Given: the Doppler curve of the optical partner of a binary which yields $P = 0.323014 \pm 0.000004$ d, and $\Delta\lambda/\lambda = A \sin (2\pi/P)t$ where $A = 1.523\ (\pm 0.027) \times 10^{-3}$, where barycenter motion has already been removed. The optical mass is $m_o > 0.7\ M_\odot$ and the inclination $i < 50°$.

The curve is sinusoidal so the orbit is quite circular. Invoke the mass function equation for a spectroscopic binary with a circular orbit (95),

$$\frac{m_x^3\ \sin^3 i}{(m_o + m_x)^2} = \frac{P}{2\pi G}\,(v_o\ \sin i\,)^3 \qquad (1.63.s1)$$

where the subscripts refer to the x-ray and optical partners. The maximum of the Doppler curve gives the speed modified by the inclination, according to the Doppler relation,

$$\left(\frac{\Delta\lambda}{\lambda}\right)_{max} = \frac{v_o\ \sin i}{c}$$

The constant $A = (\Delta\lambda/\lambda)_{max}$, so one has

$$v_o \sin i = A\,c = 4.57 \times 10^5 \text{ m/s} = 457 \pm 8.1 \text{ km/s} \qquad (1.63.s2)$$

The error introduced by the period $P$ (about 1 part in $10^5$) contributes negligible error to the right side of s1. (Recall that percentage errors get added in quadrature for the error of a product.)

Now examine the left side of (s1). The lowest possible value of $m_x$ is when *sin i* is at maximum, $\sin i_{max} = \sin 50° = 0.766$, and when $m_o$ is at a minimum, $m_{o,min} = 0.7\, M_\odot$. The value of $m_x$ is further lowered when $v_o \sin i$ is $2\sigma$ (= 16 km/s) below the measured value, so

$$(v_o \sin i)_{min} = 457 - 16 = 441 \text{ km/s} \qquad (1.63.s3)$$

The mass function equation for lower limit on $m_x$ is thus

$$\frac{m_x^3 \sin^3 i_{max}}{(m_{o,min} + m_x)^2} = \frac{P}{2\pi G}\,(v_o \sin i)^3_{min}$$

The right hand side, $f_x$, is completely determined from the given period and (s3), giving

$$f = 5.66 \times 10^{30} \text{ kg} = 2.83\, M_\odot$$

so we have, from $\sin i_{max}$ and $m_{o,min}$,

$$\frac{m_x^3\,(0.766)^3}{(0.7 + m_x)^2} = 2.83 \qquad (1.63.s4)$$

where we have dropped the $M_\odot$ symbols so all values and unknowns are in units of solar masses. This equation (s4) is cubic and is easily solved by trial and error, if you program your calculator appropriately. The result turns out to be $m_x = 7.5\, M_\odot$. Since we used values to force $m_x$ to its lowest level consistent with the data, we conclude that

➡ $m_x > 7.5\, M_\odot$

which suggests strongly that it is a black hole according to the given criterion.

Reference: McClintock and Remillard, ApJ 308, 110 (1986).

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## Problem 1.64 – Find mass function for optical partner of Cyg X-1 from the data of Fig. 12, and confirm that the compact partner is $\gtrsim 6\, M_\odot$.

Given from text and fig. caption: Circular orbit, $P = 5.60$ d and amplitude of Doppler curve, $v_{opt} \sin i = 73.8$ km/s and $m_{opt} = 30\, M_\odot$.

The optical mass function for a circular orbit is, from (95),

➡ $$f_{\rm opt} = \frac{P}{2\pi G}(v_{\rm opt}\sin i)^3$$
$$= 4.641 \times 10^{29}\ {\rm kg}$$
$$= 0.233\ M_\odot$$

so that (95) becomes

$$\frac{m_{\rm x}^3 \sin^3 i}{(m_{\rm opt} + m_{\rm x})^2} = 0.233\ M_\odot$$

Set $m_{\rm opt} = 30\ M_\odot$ and $\sin i = 1.0$ to find the lowest allowed value of $m_{\rm x}$. Solve, by trial and error, for $m_{\rm x}$ to find $\underline{m_{\rm x} > 6.8\ M_\odot}$. The black-hole argument is quite strong.

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## Problem 1.71 – Detection limits in exoplanet searches.

Consider a distant star of 1 $M_\odot$ and find the strength of detectable signals for two cases: an earth-like planet at 1 AU and a Jupiter-like planet at 5.2 AU. For each case, consider two detection methods, timing and spectroscopy. In the former, the star is a radio pulsar with well defined pulses and in the latter it has well defined spectral lines. Compare your results to the detectable limits given in the text. Let $i = 90°$ and assume a circular orbit.

**(a) For the pulsing star of 1 $M_\odot$ and earth-like planet at 1 AU, what is the range of timing delays?**

$m_{\rm E} = 6 \times 10^{24}$ kg; 1 $M_\odot = 2 \times 10^{30}$ kg; 1 AU = $1.5 \times 10^{11}$ m.

For the timing delays we need the radius of the orbit of the star about the barycenter. At one instant of time, find the position $x_{\rm B}$ of the barycenter relative to the star. Let $x = 0$ be at the center of the star and $x_{\rm p}$ the position of the planet. From the definition of the barycenter,

$$x_{\rm B} = \frac{0 + m_{\rm p}x_{\rm p}}{M + m_{\rm p}} \approx \frac{m_{\rm E}}{M_\odot}x_{\rm AU} \tag{1.71.s1}$$
$$= 4.5 \times 10^5\ {\rm m}$$

for a sun-earth system. This is radius of the star's orbit about the barycenter, a modest 450 km. The corresponding delay is

➡ $$\Delta\tau = x_{\rm B}/c = \pm 1.5\ {\rm ms} \tag{1.71.s2}$$

This is comparable to the changing delays of PSR 1257+12 (see text) and is much greater than the ~15 μs timing precision possible with a rapidly rotating pulsar such as this (spin period 6 ms).

**(b) Maximum radial velocity of the star**

The velocity is the circumference of the orbit (of radius $x_{\rm B}$) divided by the 1-year period,

$$v = \frac{2\pi x_{\rm B}}{P_{\rm orb}} = \frac{2\pi\ 4.5 \times 10^5\ {\rm m}}{3.16 \times 10^7\ {\rm s}} = 0.089\ {\rm m/s} \tag{1.71.s3}$$

This is much less than the spectroscopic ~3 m/s precision threshold mentioned in the text. An earth-like planet would not be detected through the motions of its parent star with this threshold.

Note: the fractional shifts of the PSR 1257+12 *spin* period are tiny (Fig. 15a) because they are esssentially a measure of the parent pulsar speed in its orbit, which is minute compared to $c$ – as is the star's speed in (s3). The fractional spin period change is

$$\frac{\Delta P}{P} = -\frac{\Delta \nu}{\nu} = \frac{v_r}{c}$$

which in our case, from (s3), equals $3 \times 10^{-10}$.

Each of the spin periods plotted in Fig. 15a was obtained over a ~2-day period, or $3 \times 10^7$ spin periods. The 15-μs timing precision thus yields an average period over two days accurate to (15 μs)/($3 \times 10^7$) = $5 \times 10^{-13}$ s, which is comparable to the error bars in the figure.

**(c) Repeat (a) and (b) for a Jupiter like planet**

$m_J = 318\ m_E$, $P_J = 11.86$, $r_J = 5.2$ AU.

The barycenter distance is, from (s1),

$$x_{B,J} \approx \frac{m_J}{M_\odot} x_{5.2\ AU} = 7.4 \times 10^8 \text{ m}$$

The barycenter is just barely outside the solar surface at $7.0 \times 10^8$ m.

➡ $$\Delta\tau = x_{B,J}/c = \pm 2.5 \text{ s} \qquad (1.71.s2)$$

which is also well above the detection limit for a 6-ms pulsar.

The orbital velocity of the star is

$$v = \frac{2\pi x_{B,J}}{P_J} = \frac{2\pi\ 7.4 \times 10^8 \text{ m}}{3.75 \times 10^8 \text{ s}} = 12.4 \text{ m/s}$$

This is significantly, but not hugely, above the threshold mentioned in the text, ~ 3 m/s. Jupiters can be detected with this technique and they are.

END